1 Lax-Milgram theorem

Theorem 1 (Lax-Milgram theorem) *Let H be a Hilbert space. Consider b a coninuous linear form on H and a bilinear form a on H such that*

- *there exists $M > 0$ such that*

$$\forall (u,v) \in H^2, \quad a(u,v) \leq M\|u\|\|v\| \,.$$

(a is continuous)

- *there exists $\alpha > 0$ such that*

$$\forall u \in H, \quad a(u,u) \geq \alpha\|u\|^2 \,.$$

(a is coercive)

Then there is a unique $u \in H$ such that

$$\forall v \in V, \quad a(u,v) = b(v) \,. \tag{1}$$

Corollary 1 *Under the above assumptions if we moreover assume that a is symmetric then the unique solution $u \in H$ to (1) is also the unique solution of the following minimisation problem*

$$\inf_{v \in H} J(v) \tag{2}$$

where the energy J is defined by

$$J(v) := \frac{1}{2}a(u,v) - b(v) \,.$$

1.1 Proof of the theorem

1. Prove that there exists a unique $w \in H$ such that

$$\forall v \in H, \qquad b(v) = \langle w, v\rangle$$

2. Prove that for all $u \in H$, there exists a unique $A_u \in H$, such that

$$\forall v \in H, \quad a(u,v) = \langle A_u, v\rangle \,.$$

3. We would like to prove that there exists a unique $u \in H$ such that $A_u = w$.
   (a) Prove that
   $$\begin{array}{rccc} A: & H & \to & H \\ & u & \mapsto & A_u \end{array}$$
   is linear.

(b) Prove that A is continuous.

4. Introduce now

$$\begin{array}{rccl} T: & H & \to & H \\ & u & \mapsto & u - \dfrac{\alpha}{M^2}(A_u - w) \end{array}$$

We will prove that T has a unique fixed point.

(a) Prove that T is a contraction

(b) Prove that there exists a unique $u \in H$ such that $A_u = w$.

(c) Conclude

1.2 Proof of the corollary

1. Prove that a is a scalar product.
2. Prove that b is continuous for the norm $\|\cdot\|_a$ associated to a.
3. Deduce that there exists c such that $b(u) = a(u, c)$.
4. Prove that
$$J(u) = \frac{1}{2}\|u - c\|_a - \frac{1}{2}a(c, c)\ .$$
5. Deduce that there is a unique minimiser of J.
6. Conclude.

2 Application to partial differential equation

We are interested in anlysing mathematically *linear elliptic problems* made of :

- a partial differential equation (PDE),
- a boundary condition,
- a functional framework

The last point is more subtle but fundamental. Indeed, even the meaning of the sign "=", depends on the functional framework we consider. For the purpose of this part, the good framework is the Sobolev space which we present in the following section.

To go further 1 *In research, it is still a very exciting question. P.-L. Lions was awarded the Fields medal in 1994 for his concept of viscosity solutions.*

2.1 Introduction to Sobolev spaces

Set I be an open interval of $\mathbb{R}$. Introduce

$$H^1(I) = \left\{ u \in L^2(I) \ :\ \exists g \in L^2(I) \text{ such that } \forall \varphi \in \mathcal{C}^1_c(I),\ \int_I u\varphi' = -\int_I g\varphi \right\}$$

where $\mathcal{C}^1_c(I)$ denotes the set of $\mathcal{C}^1$-functions on I which are compactly supported.

To go further 2 *A better way to introduce $H^1(I)$ would have been to introduce the notion of derivative in the distribution sense and the theory of distributions. The idea, as above, is to transfer the derivative on the test function φ by integration by parts. We can remark that if $u \in \mathcal{C}^1(I) \cap L^p(I)$ then u' also is in $L^p(I)$, where here u' stands for the usual derivative. This distribution derivatives is egal to the usual derivative in this situation. See Schwartz "Théorie des distributions"*

To go further 3 *The set $H^1(I)$ is called a Sobolev space. For more general Sobolev spaces we can define, for $p \in (1,\infty)$*

$$W^{1,p}(I) = \left\{ u \in L^p(I) \,:\, \exists g \in L^p(I) \text{ such that } \forall \varphi \in \mathcal{C}_c^1(I),\ \int_I u\varphi' = -\int_I g\varphi \right\}$$

The set $H^1(I)$ corresponds to $W^{1,2}(I)$.

We can even recursively define the set

$$W^{m,p}(I) = \left\{ u \in W^{m-1,p}(I) \,:\, u' \in W^{m-1,p}(I) \right\}$$

The space $H^1(I)$ is equipped with the norm $\|\cdot\|_{H^1}$ defined by

$$\|u\|_{H^1}^2 = \|u\|_{L^2} + \|u'\|_{L^2} = \int_0^1 \left(|u(x)|^2 + |u'|^2 \right) \mathrm{d}x$$

To go further 4 *This norm comes from the scalar product*

$$\langle u, v \rangle_{H^1} = \langle u, v \rangle_{L^2} + \langle u', v' \rangle_{L^2}$$

2.2 Application to partial differential equations

We denote $H = H_1^0(I) = \overline{\mathcal{C}_c^\infty(I)}$ in $H^1(I)$ where $\mathcal{C}_c^\infty(I)$ is the set of smooth functions which are compatcly supported in I (the adherence of the support is in I) . This space is equipped with the norm and scalar product of $H^1(I)$.

2.2.1 The Poisson problem

Theorem 2 (The Poisson problem) *Let $f \in L^2(I)$ be given. There is a unique weak solution $u \in H$ to*

$$\begin{cases} -u'' + u = f & \text{on } (0,1) \\ u(0) = u(1) = 0 \end{cases} \tag{3}$$

in the sense that

$$\forall v \in H, \qquad \int_0^1 u'(x)\, v'(x) \mathrm{d}x + \int_0^1 u(x)\, v(x) \mathrm{d}x = \int_0^1 f(x)\, v(x) \mathrm{d}x$$

Introduce for all $(u, v) \in H^2$

$$a(u, v) := \int_0^1 u'(x)\, v'(x) \mathrm{d}x + \int_0^1 u(x)\, v(x) \mathrm{d}x$$

and for all $v \in H$

$$l(v) = \int_0^1 f(x)\, v(x) \mathrm{d}x$$

1. Prove that l is continuous in $L^2(I)$.
2. Prove that a is continuous in H.
3. Prove that a is coercive in H.
4. Conclude that there exists a unique weak solution to (3).
5. Prove that, if $f \in \mathcal{C}^0$ the weak solution is indeed a strong solution in the usual sense.

2.2.2 The Sturm-Liouville problem

Theorem 3 (The Sturm-Liouville problem) *Let $p \in \mathcal{C}^1(\bar{I})$, $(r, q) \in \mathcal{C}^0(\bar{I})$ and $f \in L^2(I)$ be given. We assume that $q \geq 1$ and there is $\alpha > 0$ such that $p \geq \alpha$ and $r^2 \leq \alpha$. There is a unique weak solution $u \in H$ to*

$$\begin{cases} -(pu')' + ru' + qu = f & \text{on } (0,1) \\ u(0) = u(1) = 0 \end{cases} \tag{4}$$

in the sense that

$$\forall v \in H, \qquad \int_0^1 p(x)\, u'(x)\, v'(x)\mathrm{d}x + \int_0^1 r(x)\, u'(x)\, v(x)\mathrm{d}x + \int_0^1 q(x)\, u(x)\, v(x)\mathrm{d}x = \int_0^1 f(x)\, v(x)\mathrm{d}x$$

The proof relies on the following very important theorem :

Theorem 4 (Poincaré inequality) *There exists C_P such that*

$$\forall u \in H, \qquad \|u\|_{H_1} \leq C_P \|u'\|_2 \,.$$

1. Prove that for all $u \in H$
$$\|u\|_2^2 \leq \|u'\|_2 \,.$$
2. Conclude.

Introduce for all $(u, v) \in H^2$

$$a(u,v) := \int_0^1 p(x)\, u'(x)\, v'(x)\mathrm{d}x + \int_0^1 r(x)\, u'(x)\, v(x)\mathrm{d}x + \int_0^1 q(x)\, u(x)\, v(x)\mathrm{d}x$$

and for all $v \in H$

$$l(v) = \int_0^1 f(x)\, v(x)\mathrm{d}x$$

1. Prove that l is continuous in $L^2(I)$.
2. Prove that a is continuous in H.
3. Prove that a is coercive in H.
4. Conclude that there exists a unique weak solution to (4).
5. Prove that, if $f \in \mathcal{C}^0$ the weak solution is indeed a strong solution in the usual sense.