Toulouse School of Economics

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1 Fourier transform

There are different functional frameworks to define the Fourier transform : $\mathcal{L}^1$, $\mathcal{L}^2$, etc. Here we will introduce a space with is much more adpated to the definition of Fourier transforms :

1.1 The Schwartz space

Definition 1 (The Schwartz space) *The Schwartz space, denoted $\mathcal{S}(\mathbb{R})$, is the set of $\mathcal{C}^\infty(\mathbb{R})$ functions such that, for all $k \in \mathbb{N}$,*

$$\|u\|_{k,\mathcal{S}} := \sup_{x\in\mathbb{R},\alpha\leq k,\beta\leq k} |x^\beta u^{(\alpha)}(x)| < \infty$$

The space $\mathcal{S}(\mathbb{R})$ is also called the space of functions rapidly decreasing, in the sense that f and all its derivatives go to zero as $x \to \infty$ faster than any reciproqual power of x.

Exercice 1

1. *Prove that a $\mathcal{C}^\infty(\mathbb{R})$-function which is compactly supported is in the Schwartz space.*
2. *Prove that the gaussian functions, $x \mapsto \exp(-ax^2)$, $a > 0$, are in the Schwartz space.*

Exercice 2 *Prove that the space $\mathcal{S}(\mathbb{R})$ is a vector space.*

To go further 1 *The vector space $\mathcal{S}(\mathbb{R})$ can be seen as the intersection for all $k \in \mathbb{N}$ of the Banach spaces (exo) of the functions such that $\|u\|_{k,\mathcal{S}}$ is bounded. As the intersection of a non-increasing family of a countable Banach spaces, we can define a distance on $\mathcal{S}(\mathbb{R})$ by*

$$d(u,v) := \sum_{k\in\mathbb{N}} 2^{-k} \min\{1, \|u-v\|_{k,\mathcal{S}}\} .$$

The topology of $(\mathcal{S}(\mathbb{R}), d)$ is not the topology of a normed vector space.

Exercice 3 *Let $f \in \mathcal{S}(\mathbb{R})$.*

1. *Prove that for any $l \in \mathbb{N}$, $f^{(l)}$ is in $\mathcal{S}(\mathbb{R})$.*
2. *Prove that for any $k \in \mathbb{N}$, $x^k f$ is in $\mathcal{S}(\mathbb{R})$.*

1.2 Definition of the Fourier transform

Definition 2 *For $f \in \mathcal{S}(\mathbb{R})$ we define its Fourier transform as the function defined for all $\xi \in \mathbb{R}$, by*

$$\hat{f}(\xi) := \int_{-\infty}^{+\infty} f(x)\, e^{-2i\pi x\xi} \mathrm{d}x \,.$$

Exercice 4 *Let $f \in \mathcal{S}(\mathbb{R})$. Prove the following assertions :*

1. *Linearity : let $a \in \mathcal{C}$ and define $h(x) = af(x) + g(x)$. We have $\hat{h}(\xi) = a\hat{f}(\xi) + \hat{g}(\xi)$.*
2. *Translation : let $h \in \mathbb{R}$ and define $h(x) = f(x - x_0)$. We have $\hat{h}(\xi) = e^{-2i\pi h\xi}\hat{f}(\xi)$.*
3. *Modulation : let $\xi_0 \in \mathbb{R}$ and define $h(x) = e^{2i\pi x\xi_0} f(x)$. We have $\hat{h}(\xi) = \hat{f}(\xi - \xi_0)$.*
4. *Scaling : let $a \in \mathbb{R}^*$ and define $h(x) = f(ax)$. We have*

$$\hat{h}(\xi) = \frac{1}{|a|}\hat{f}\left(\frac{\xi}{a}\right) \,.$$

5. *Derivatives : Let $\alpha \in \mathbb{N}$ and define $h(x) = f^{(\alpha)}(x)$. We have $\hat{h}(\xi) = (2\pi i\xi)^\alpha \hat{f}(\xi)$.*
6. *Define $h(x) = - = x^n f(x)$. We have $\hat{h}(\xi) = \frac{i}{2\pi})^n \frac{d\hat{f}}{d\xi^n}(\xi)$.*
7. *(*) Conjugaison : define $h(x) = \overline{f(x)}$. We have $\hat{h}(\xi) = \overline{\hat{f}(-\xi)}$.*
8. *(*) Real part : define $h(x) = \mathcal{R}e(f(x))$ then*

$$\hat{h}(\xi) = \frac{1}{2}\left(\hat{f}(\xi) + \overline{\hat{f}(-\xi)}\right)$$

9. *(*) Imaginary part : define $h(x) = \mathcal{I}m(f(x))$ then*

$$\hat{h}(\xi) = \frac{1}{2}\left(\hat{f}(\xi) - \overline{\hat{f}(-\xi)}\right)$$

Exercice 5 *Prove $\mathcal{S}(\mathbb{R})$ is stable by Fourier transform i.e. if $f \in \mathcal{S}(\mathbb{R})$ then $\hat{f} \in \mathcal{S}(\mathbb{R})$.*

Exercice 6 (Fourier transform of the Gaussian) *Let α be such that $(\alpha) > 0$. Consider $f : x \mapsto e^{-\alpha x^2}$. Compute $\hat{f}$.*
Hint : complete the square.

Exercice 7 (Fourier transform of an exponential) *Let α be such that $(\alpha) > 0$. Consider $f : x \mapsto e^{-a|x|}$. Compute $\hat{f}$.*

1.3 Inversion formula

Exercice 8 (Mollifiers) *Consider for $\delta > 0$,*

$$K_\delta : x \mapsto \frac{1}{\sqrt{\delta}} e^{-\pi x^2/\delta} \,.$$

1. *Prove that $\hat{K}_\delta(\xi) = e^{-\pi\delta\xi^2}$.*
2. *Prove that for any $\delta \in \mathbb{N}$,*

$$\int_{\mathbb{R}} K_\delta(x) \mathrm{d}x = 1 \,,$$

3. *Prove that ,*

$$\sup_{\delta\in\mathbb{N}} \int_{\mathbb{R}} |K_\delta(x)|\,\mathrm{d}x < \infty\,,$$

4. *Prove that for all* $\gamma > 0$*,*

$$\lim_{\delta\to\infty} \int_{\gamma\le|x|<+\infty} |K_\delta(x)|\,\mathrm{d}x = 0\ .$$

5. *Let* f *and* g *be in* $\mathcal{S}(\mathbb{R})$*. Define*

$$(f*g)(x) = \int_{\mathbb{R}} f(x-y)g(y)\mathrm{d}y\ .$$

Prove that if $f \in \mathcal{S}(\mathbb{R})$

$$\lim_{\delta\to+\infty} \|f*K_\delta - f\|_\infty = 0\ .$$

Exercice 9 (Multiplication formula (*)) *Let* f *and* g *be in* $\mathcal{S}(\mathbb{R})$*. Prove that*

$$\int_{-\infty}^{+\infty} f(x)\,\hat{g}(x)\mathrm{d}x = \int_{-\infty}^{+\infty} \hat{f}(y)\,g(y)\mathrm{d}y$$

Hint : Use Fubini's theorem

Exercice 10 (The Fourier transform is a one-to-one correspondence on $\mathcal{S}(\mathbb{R})$) *Let* $f \in \mathcal{S}(\mathbb{R})$*.*

1. *Fourier inversion formula : Prove that*

$$f(x) = \int_{-\infty}^{+\infty} \hat{f}(\xi)e^{2i\pi x\xi}\mathrm{d}\xi$$

Hint : Use the above mollifier sequence to approximate f *and pass to the limit.*

2. *Prove that the Fourier transform is a one-to-one correspondence on* $\mathcal{S}(\mathbb{R})$*.*

1.4 The Plancherel formula

Exercice 11 (Convolution) 1. *Prove that* $f*g$ *is in* $\mathcal{S}(\mathbb{R})$*.*

2. *Prove that* $(f \hat{*} g) = \hat{f}\,\hat{g}$*.*
Hint : Use Fubini's theorem

Exercice 12 (The Parceval formula (*)) *Let* $f \in \mathcal{S}(\mathbb{R})$*. Prove that*

$$\langle f, g\rangle_{L^2} = \int_{-\infty}^{+\infty} f(x)\,\overline{g(x)}\mathrm{d}x = \int_{-\infty}^{+\infty} \hat{f}(\xi)\,\overline{\hat{g}(\xi)}\mathrm{d}\xi$$

Exercice 13 (The Plancherel formula) *Let* $f \in \mathcal{S}(\mathbb{R})$*. Prove that* $\|f\| = \|\hat{f}\|$*.*

2 Application to the Heat equation

The heat equation is a partial differential equation describing the distribution of heat over time. In one spatial dimension, we denote $u(x,t)$ as the temperature at time t in x. The function u obeys the relation

$$\frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t) \tag{1}$$

supplemented with the initial condition $u(x,0) = \phi(x)$.

To go further 2 *The transition density for Brownian motion satisfies the heat equation. See* `http://stat.math.uregina.ca/~kozdron/Research/UgradTalks/BM_and_Heat/heat_and_BM.pdf`

Exercice 14 *Let u be a solution to* (1)*.*

1. *Prove that*

$$\frac{\partial \hat{u}}{\partial t}(\xi,t) = -4\pi^2\xi^2\hat{u}(\xi,t) \tag{2}$$

2. *Solve* (2)*.*

3. *Using the initial condition prove that* $\hat{u}(\xi,t) = \hat{\phi}(\xi,t)\exp(-4\pi^2\xi^2 t)$*.*
4. *Find the Gaussian G such that* $\hat{G}(\xi,t) = \exp(-4\pi^2\xi^2 t)$*.*
5. *Prove that* $\hat{u} = (\widehat{\phi * G})$*.*
6. *Prove that* $u = \phi * G$*.*
7. *Conclude.*
8. *Consider now ϕ to be defined by*

$$\phi : x \mapsto \begin{cases} 1 & \text{if } |x| \leq 1/2 \\ 0 & \text{if } |x| > 1/2 \end{cases}$$

Determine the solution to (1) *with ϕ as initial data.*