Memento of algebra

Adrien Blanchet

23 août 2024

Table des matières

1 Vector spaces

1.1 Vector spaces

Definition 1.1 (Vector spaces) *Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. A $\mathbb{K}$-* vector space *is a set V together with two binary operations,*

- vector addition *: $V \times V \to V$ denoted $v + w$, where v, w are in V, and*
- scalar multiplication *: $\mathbb{R} \times V \to V$ denoted $a\,v$, where $a \in \mathbb{R}$ and $v \in V$,*

satisfying the axioms below.

1. *Vector addition is* associative *:*
$$\forall (u, v, w) \in V^3, \quad u + (v + w) = (u + v) + w.$$
2. *Vector addition is* commutative *:*
$$\forall (u, v, w) \in V^3, \quad v + w = w + v.$$
3. *Vector addition has an* identity element *: There exists an element $0 \in V$ such that*
$$\forall v \in V, \quad v + 0 = v.$$
4. *Vector addition has* inverse elements *:*
$$\forall v \in V, \, \exists w \in V : \quad v + w = 0.$$
5. Distributivity *holds for scalar multiplication over vector addition :*
$$\forall a \in \mathbb{K}, \, \forall (v, w) \in V^2, \quad a(v + w) = av + aw.$$
6. Distributivity *holds for scalar multiplication over field addition :*
$$\forall (a, b) \in \mathbb{K}^2, \, \forall v \in V, \quad (a + b)v = av + bv.$$
7. *Scalar multiplication is* compatible *with multiplication in the field of scalars :*
$$\forall (a, b) \in \mathbb{K}^2, \, \forall v \in V, \quad a(bv) = (ab)v.$$
8. *Scalar multiplication has an* identity element *:*
$$\forall v \in V, \quad 1v = v.$$

Example 1 — *Let $n \in \mathbb{N}$. The set $\mathbb{R}^n$ with the usual addition of vectors and the scalar multiplication is a vector space.*
— *The set of matrices,*
— *Let $(a, b) \in \mathbb{R} \cup \{-\infty, +\infty\}$. The set of maps from a (a, b) to $\mathbb{R}$ is a vector space.*

Definition 1.2 (Vector sub-space) *Let $(V, +, \cdot)$ be a vector space. A non-empty sub-space F of E is a* vector sub-space *equipped with the operation "+" and "·" if it satisfies*
$$\forall (u, v) \in F^2, \forall \lambda \in \mathbb{R} \quad u + \lambda \cdot v \in F.$$

Example 2 — *The set on polynomial functions of degree less than n,*
— *The set on polynomial functions of any degree.*

Exercise 1 *In $\mathbb{R}^3$, the plan defined by $\{(x, y, z) \in \mathbb{R}^3 \, : \, y = 2x\}$ is a vector sub-space of $\mathbb{R}^3$ (see Section 1.3 for other examples).*

Exercise 2 *In $\mathbb{R}^2$, let $F_1 = \{(x, y) \in \mathbb{R}^2 \, : \, y = x\}$ and $F_2 = \{(x, y) \in \mathbb{R}^2 \, : \, y = x\}$. Is $F_1 \cup F_2$ is a vector sub-space of $\mathbb{R}^2$.*

Proposition 1.3 *Let E be a vector space and F_1 and F_2 be two vector sub-spaces of E. The set $F_1 \cap F_2$ is a vector sub-space of E.*

1.2 Linear combinaison of vectors

Definition 1.4 (Linear combinaison of vectors) *Let E be a vector space and F be a vector sub-space of E. Consider $(u_i)_{i\in[[1,p]]} = (u_1, \dots, u_p) \in F^p$. We call* linear combinaison of $(u_i)_{i\in[[1,p]]}$ *any vector $\sum_{i=1^p} \lambda_i\, u_i$.*

The set of all the linear combinaison of $(u_i)_{i\in[[1,p]]}$ is denoted $\mathrm{Span}\{u_1, u_2, \dots, u_n\}$.

The family $(u_i)_{i\in[[1,p]]}$ spans F is $F \in \mathrm{Span}\{u_1, u_2, \dots, u_n\}$.

Exercise 3 *In $\mathbb{R}^3$, under which condition a vector $(x, y, z) \in \mathbb{R}^3$ is a linear combinaison of $(1, 0, -1)$ and $(0, 1, -1)$.*

Exercise 4 *Determine $\{(x, y, z) \in \mathbb{R}^3 : (x, y, z) \in \mathrm{Span}\{(1, 0, -1), (0, 1, -1)\}$.*

Proposition 1.5 *Let V be a vector space and $(u_i)_{i\in[[1,p]]} \in V^p$. The set $\mathrm{Span}\{u_1, u_2, \dots, u_n\}$ is a vector sub-space of V.*

Definition 1.6 (Linear independance) *Let V be a vector space and $(u_i)_{i\in[[1,p]]} \in V^p$. The family $(u_i)_{i\in[[1,p]]}$ is* linearly independent *if*

$$\sum_{i=1}^{p} \lambda_i u_i = 0 \Rightarrow \lambda_i = 0, \quad \forall i \in [[1, p]].$$

The family $(u_i)_{i\in[[1,p]]}$ is linearly dependent *if it is not linearly independent.*

Example 3 *The family $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ is linearly independant.*

Exercise 5 *(i) In $\mathbb{R}^3$, prove that $\{(0, 1, 2), (1, 0, 2), (1, 2, 0)\}$ is a linearly independant family.*
(ii) In $\mathbb{R}^3$, prove that $\{(0, 1, 2), (1, 0, 2), (3, 2, 10)\}$ is a linearly dependant family.

Exercise 6

Let $u = (2, 1, 0)$, $v = (1, -1, 2)$ and $w = (0, 3, -4)$. Determine an equation of $F = \mathrm{Vect}\{u, v, w\}$.

Let $P = X + 1$, $Q = X^2$ end $R = X - 1$. Determine $\mathrm{Vect}\{P, Q, R\}$.

Let $P = X - 2$ and $Q = 3X - 6$. Determine $\mathrm{Vect}\{P, Q\}$.

Find a basis of $\mathcal{G} = \{(x, y, z) \in \mathbb{R}^4, x + y = 0$ et $x + 3y - 2z = 0\}$.

Are the following families linearly independent ?

1. $v_1 = (1, 0, 1)$, $v_2 = (0, 2, 2)$ *and* $v_3 = (3, 7, 1)$ *in* $\mathbb{R}^3$.
2. $v_1 = (1, 0, 0)$, $v_2 = (0, 1, 1)$ *and* $v_3 = (1, 1, 1)$ *in*$\mathbb{R}^3$.
3. $v_1 = (1, 2, 1, 2)$, $v_2 = (2, 1, 2, 1)$, $v_3 = (1, 0, 1, 1)$ *and* $v_4 = (0, 1, 0, 0)$ *in* $\mathbb{R}^4$.
4. $u = 2 - 5X + 6X^2 - X^3$, $v = 3 + 2X - 4X^2 + 5X^3$ *in* $\mathbb{R}[X]$.
5. $f : \mathbb{R}^2 \to \mathbb{R}, (x, y) \mapsto (x + y)$ *and* $f : \mathbb{R}^2 \to \mathbb{R}, (x, y) \mapsto (x - y)$
6. $\cos$ *and* $\sin$

Example 4 *The family $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ is a spanning family of $\mathbb{R}^3$.*

Exercise 7 *(i) In $\mathbb{R}^2$, $\{(1, 1), (-1, 1)\}$ is a spanning family of $\mathbb{R}^2$.*
(ii) In $\mathbb{R}^3$, $\{(1, 0, 0), (0, 0, 1)\}$ is not a spanning family of $\mathbb{R}^3$.
(iii) In $\mathbb{R}^2$, $\{(1, 1), (-1, 1), (0, 1)\}$ is a spanning family of $\mathbb{R}^2$.

Remark 1 *Any spanning family of $\mathbb{R}^n$ contains at least n vectors.*

Definition 1.7 (Basis) *Let E be a vector sub-space. If a family $(u_i)_{i\in[[1,p]]}$ is linearly independant and spanning of E we say that $(u_i)_{i\in[[1,p]]}$ is a* basis *of E. We say that E is of* dimension *p.*

Example 5 *The family $\{(1,0,0),(0,1,0),(0,0,1)\}$ is a basis of $\mathbb{R}^3$.*

Exercise 8 *In $\mathbb{R}^2$, $\{(1,1),(-1,1),(0,1)\}$ is not a basis of $\mathbb{R}^2$.*

Example 6 (Canonic base of $\mathbb{R}^n$) *Let $(e_i)_{i\in[[1,n]]}$ be the vector whose component are 0 exept the i-th component wich is 1. The family $(e_i)_{i\in[[1,n]]}$ is called the* canonic base of $\mathbb{R}^n$*.*

Theorem 1.8 *Any base of $\mathbb{R}^n$ contains exactly n vectors.*

Exercise 9
1. *Prove that $u=(1,2,3)$, $v=(0,1,2)$ and $w=(0,0,1)$ is a basis of $\mathbb{R}^3$.*
2. *Determine a basis of $\{(x,y,z,t)\in\mathbb{R}^4 : x+y-z+t=0$ and $y-2z+t=0\}$.*
3. *Let $F:=\operatorname{Span}\{(1,1,0,1),(2,2,0,1),(0,2,2,0),(1,0,0,0)\}$. Complete $\{(1,3,2,1),(2,6,4,1)\}$ to obtain a basis of F.*
4. *Let $G:=\{(x,y,z,t)\in\mathbb{R}^4 : x-y+z+t=0\}$.*
   - *(a) Prove that $\mathcal{F}:=\{(1,1,1,-1),(0,1,2,-1)\}$ is a linearly independent family of $\mathbb{R}^4$.*
   - *(b) Complete $\mathcal{F}$ to obtain a basis of G.*

1.3 Geometry

Definition 1.9 (Geometric vector space) *Let E be a vector space and F be a vector subspace of E.*

- *If $\dim(F)=1$ then F is called a* (vectorial) straigh line*,*
- *If $\dim(F)=2$ then F is called a* (vectorial) plan*.*

Proposition 1.10 *Let F and G be vector spaces. $F=G$ is and only if $F\subset G$ and $\dim(F)=\dim(G)$.*

Exercise 10
1. *Determine $\dim(E\cap F)$ where*
$$E=\Big\{(x,y,z,t)\in\mathbb{R}^4 : x+y+z+t=0\Big\} \quad \text{and} \quad F=\Big\{(x,y,z,t)\in\mathbb{R}^4 : x+y=z+t\Big\}\ .$$
2. *Determine the geometric nature of $F\cap G$ where*
$$F=\operatorname{Span}\{(1,0,1,-1),(2,1,2,1),(3,1,2,0)\} \quad \text{and} \quad G=\operatorname{Span}\{(1,1,3,-1),(2,-1,-4,4)\}$$
3. *Let $F:=\operatorname{Span}\{(-1,1,1),(1,0,1)\}$ and $G:=\Big\{(x,y,z)\in\mathbb{R}^3 : x+2y-z=0\Big\}$. Prove that $F=G$.*
4. *Prove that $\operatorname{Span}\{(1,-1,-2),(2,3,-1)\}=\operatorname{Span}\{(3,7,0),(5,0,-7)\}$.*
5. *Prove that $\operatorname{Span}\{X,X^2\}=\operatorname{Span}\{X(X-1),X^2\}$.*

2 Matrix

2.1 Definition

Definition 2.1 (Matrix) *Let n and p in $\mathbb{N}$. We call* (real coefficient) matrix*, the collection of $n \times p$ reals denoted $(a_{ij})_{i\in[[1,n]],\, j\in[[1,p]]}$. We say that a_{ij} is the general term of the matrix A. We represente the matrix in a table :*

$$\begin{bmatrix} a_{11} & \dots & a_{1j} & \dots & a_{1p} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \dots & a_{ij} & \dots & a_{ip} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \dots & a_{nj} & \dots & a_{np} \end{bmatrix}$$

The first index is the index of the line, the second is the index of the column.

We denoted by $\mathcal{M}_{n,p}(\mathbb{R}^n)$ the set of all the matrix with n lines and p column.

Example 7 *In $\mathbb{R}^n$, a vector "is" a matrix in $\mathcal{M}_{n,1}(\mathbb{R}^n)$.*

Remark 2 (Representation of a family of vectors) *Let $S := (u_i)_{i\in[[1,p]]}$ a family of p vectors of $\mathbb{R}^n$. For any $j \in [[1,p]]$, there exists n reals $(a_{ij})_{i\in[[1,n]]}$ such that $u_j = \sum_{i=1}^{n} a_{ij}\, e_i$ (where $(u_i)_{i\in[[1,p]]}$ is the canonical base of $\mathbb{R}^n$). The matrix $A := (a_{ij})_{i\in[[1,n]],\, j\in[[1,p]]}$ is a representation of the family S in the canonic base of $\mathbb{R}^n$.*

The j-th column of A is made of the coordinates of u_j in the canonic base.

Exercise 11 *(i) In $\mathbb{R}^3$, determine the matrix of the canonic base ?*

(ii) Determine is the matricial representation of $\{(1,0,0,0),(0,1,2,0),(1,0,0,3)\}$.

Definition 2.2 *Let $A = (a_{ij})_{i\in[[1,n]],\, j\in[[1,p]]}$ and $B = (b_{ij})_{i\in[[1,n]],\, j\in[[1,p]]}$ be two matrix of $\mathcal{M}_{n,p}(\mathbb{R}^n)$.*

- *A is* egal *to B if for all $i \in [[1,n]]$ and $j \in [[1,p]]$, $a_{ij} = b_{ij}$,*
- *The sum $A + B$ is a matrix of $\mathcal{M}_{n,p}(\mathbb{R}^n)$ of general term $a_{ij} + b_{ij}$,*
- *Let $\lambda \in \mathbb{R}$. The* scalar product *of the matrix A by A denoted, λA is the matrix of general term $\lambda\, a_{ij}$.*

Exercise 12 *Define*

$$A := \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 0 \end{bmatrix} \quad \text{and} \quad B := \begin{bmatrix} 0 & 3 & 4 \\ 2 & 1 & 6 \end{bmatrix}$$

Compute $3A$, $A + B$ and $3A - B$.

Proposition 2.3 *The space $\mathcal{M}_{n,p}(\mathbb{R}^n)$ with the addition and the scalar product as defined above is a vector space.*

Definition 2.4 (Matricial product) *Let $A = (a_{ij})_{i\in[[1,n]],\, j\in[[1,p]]}$ be a matrix in $\mathcal{M}_{n,p}(\mathbb{R}^n)$ and $B = (b_{ij})_{i\in[[1,p]],\, j\in[[1,q]]}$ be a matrix of $\mathcal{M}_{p,q}(\mathbb{R}^n)$. The product of $A\,B$ is the matrix in $\mathcal{M}_{n,q}(\mathbb{R}^n)$ of general term $c_{ij} := \sum_{k=1}^{n} a_{ik}\, b_{kj}$ for any $i \in [[1,n]]$ and $j \in [[1,q]]$.*

Exercise 13 *(i) Consider*

$$A := \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 0 \end{bmatrix} \quad \text{and} \quad B := \begin{bmatrix} 0 & 2 & 2 & 1 \\ 2 & 1 & 0 & 2 \\ 1 & 1 & 0 & 2 \end{bmatrix}$$

Compute if possible AB and BA.

(ii) Consider

$$C := \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \textit{and} \quad D := \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

Compute if possible CD and DC.

Remark 3 (Block product)

$$A \begin{bmatrix} C_1 & \dots & C_q \end{bmatrix} = \begin{bmatrix} AC_1 & \dots & AC_q \end{bmatrix}$$

and

$$\begin{bmatrix} L_1 \\ \vdots \\ L_p \end{bmatrix} B = \begin{bmatrix} L_1 B \\ \vdots \\ L_p B \end{bmatrix}$$

Definition 2.5 (Transpose) *Let A of general term a_{ij} be a matrix in $\mathcal{M}_{n,p}$. The transpose of A, denoted A^t, is the matrix in $\mathcal{M}_{p,n}$ of general term a_{ji}.*

Exercise 14 *Consider A defined in Exercise 12. Compute A^t.*

2.2 Application to a vector family

Remark 4 (Linear combinaison of vectors) *Let $u_i = \sum_{i=1}^{n} a_{ij} e_i$. The relation $(y_1, \dots, y_n) = \sum_{i=1}^{p} \lambda_i \, u_i$ can be written as*

$$\begin{cases} y_1 & = & a_{11}\lambda_1 + \ldots + a_{1j}\lambda_j + \ldots + a_{1p}\lambda_p \\ & \vdots & \\ y_i & = & a_{i1}\lambda_1 + \ldots + a_{ij}\lambda_j + \ldots + a_{ip}\lambda_p \\ & \vdots & \\ y_n & = & a_{n1}\lambda_1 + \ldots + a_{nj}\lambda_j + \ldots + a_{np}\lambda_p \end{cases}$$

or thanks to matrix by $Y = A\,X$ where

$$Y := \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \quad \textit{and} \quad X := \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix}$$

and A is the matrix representation of the family of vectors $(u_i)_{i \in [[1,p]]}$ (see Remark 2).

Exercise 15 *What is the matricial representation of the family $\{u_1, u_2, u_3\}$ where $u_1 := (1, 0, 1, -1)$, $u_2 := (0, 1, 2, 1)$ and $u_3 := (1, 2, 2, 1)$? What are the coordinates of $u_1 - 2\,u_2 + u_3$?*

2.3 Square matrix

Definition 2.6 *A matrix in $\mathcal{M}_{n,p}$ is said to be a* square matrix *if $n = p$. We denote this space $\mathcal{M}_n$.*

Notation (Kronecker symbol) We denote δ_{ij} the real defined by $\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ii} = 1$.

Definition 2.7 (Identity matrix) *In $\mathbb{R}^n$, the square matrix in $\mathcal{M}_n$ of general term δ_{ij}. We denote it I_n.*

Example 8 *In* $\mathbb{R}^2$*, the matrix*

$$I_2 := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

is the identity matrix of $\mathbb{R}^2$*.*

Proposition 2.8 *For any matrix* $A \in \mathcal{M}_n$*, we have* $A\,I_n = I_n\,A$*.*

Definition 2.9 *In* $\mathbb{R}^n$*, let* I *be the identity matrix. We say that a square matrix* $A \in \mathcal{M}_n$ *is* inversible *is there exists a matrix* $B \in \mathcal{M}_n$ *such that* $AB = BA = I$*.*

If the matrix is not inversible we say that it is singular*.*

Proposition 2.10 *Let* A *and* B *be two inversible (square) matrix.* $(A\,B)^{-1} = B^{-1}A^{-1}$*.*

Theorem 2.11 (Criterion for inversibility (Part 1)) $A \in \mathcal{M}_n$ *inversible* $\Leftrightarrow$ *the family of the column vectors of* A *is linearly independant* $\Leftrightarrow$ *the family of the lines of* A *is linearly independant* $\Leftrightarrow$ *the family of the column vectors of* A *is a basis of* $\mathbb{R}^n$ $\Leftrightarrow$ *the family of the lines of* A *is a basis of* $\mathbb{R}^n$*.*

3 Linear systems

3.1 Definitions

Definition 3.1 (Elementary operation) *We call* elementary operation on the lines *one of the following transformation :*

- *Multiply the* i*-th line by a* **non-nul** *real* α *(we write* $L_i \leftarrow \alpha\,L_i$*),*
- *Exchange the lines* i*-th and* j*-th (we write* $L_i \leftrightarrow L_j$*),*
- *Add to the* i*-th line the* j*-line multiplied by* α *(we write* $L_i \leftarrow L_i + \alpha L_j$*),.*
- *Add to* α *times the* i*-th line the* j*-line multiplied by* β *where* $\alpha \neq 0$ *and* $i \neq j$ *(we write* $L_i \leftarrow \alpha L_i + \beta L_j$*),.*

Proposition 3.2 *Let* $\mathcal{S}$ *be a system. We obtain a system equivalent to* $\mathcal{S}$ *by doing on one line one of the above elementary operation.*

Definition 3.3 (Echelon form) *A matrix is in* echelon form *if*

- *All nonzero rows are above any rows of all zeroes, and*
- *The leading coefficient of a row is always strictly to the right of the leading coefficient of the row above it.*

The first non-zero entry in each row is called a pivot*.*

3.2 Resolution

Let $\mathcal{S}$ be a system. The aim of the Gauss method is to use successively the elementary operations on the lines of $\mathcal{S}$ to obtain a equivalent system in echelon form.

Gauss algorithm :
For $k = 1$ to n
If there exists a line $i \geq k$ such that $a_{ik}^{k-1} \neq 0$ do $L_i \leftrightarrow L_k$
$L_k^k \leftarrow \frac{1}{a_{kk}^{k-1}} L_k^{k-1}$
for $i = 1$ to n and $i \neq k$
$L_i^k \leftarrow L_i^{k-1} - a_{ik}^{k-1} \times L_k^k$
$k = k + 1$.

The echelon system obtained after the Gauss algorithm :

$$R\begin{pmatrix} x_1 \\ \vdots \\ x_p \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_p \end{pmatrix}$$

is equivalent to the initial system $\mathcal{S}$ (R is in echelon form).

Resolution algorithm :

Let q be the number of pivots.

- If ($q < n$ and $\exists i \in [[q+1, n]]$ such that $\beta_i \neq 0$) then $\mathcal{S}$ is **impossible**
- If ($q = n$) or (($q < n$ and $\forall i \in [[q+1, n]]$, $\beta_i = 0$)

we solve the echelon system (by transfering if $q < p$ the auxilary unknown in the right hand side)

if ($p = q$), $\mathcal{S}$ admits a unique solution. We say that $\mathcal{S}$ is a **Cramer system**,

if ($p < q$), $\mathcal{S}$ is **undetermined**.

Exercise 16 *(i) Solve*

$$\begin{cases} 2x + y + z = 7 \\ x + 2y + z = 8 \\ x + y + 2z = 9 \end{cases}$$

(ii) Solve

$$\begin{cases} x + y + z = 3 \\ 2x + y + 3z = 1 \\ x - y + 3z = 5 \end{cases}$$

(iii) Solve

$$\begin{cases} x + 2y + 3z = 0 \\ 3x + 4y + 5z = 0 \\ 5x + 8y + 11z = 0 \\ x + y + z = 0 \end{cases}$$

Remark 5 (Application to linearly independent family) *Let $(u_i)_{i\in[[1,p]]}$ be a family of vectors of $\mathbb{R}^n$ and A its matrix representation in the canonic base. The family is linearly independent if and only if the Gauss reduction of A has p pivots.*

Exercise 17 *Is the family $\{(1,3,5,1),(2,4,8,1),(3,5,11,1)\}$ linearly independant ?*

Consequence

- A family of p vectors of $\mathbb{R}^n$ **cannot be linearly independent** if $p > n$,
- If F is a family of n vectors of $\mathbb{R}^n$. F linearly independent $\Leftrightarrow$ F base $\Leftrightarrow$ F is a spanning family of $\mathbb{R}^n$.

3.3 Application to the inverse of a matrix

Theorem 3.4 (Criterion for inversibility (Part 2)) *Let $A \in \mathcal{M}_n$. The matrix A is inversible if and only if it has n pivots.*

Gauss-Jordan method : See TD.

Exercise 18 *What is the inverse of*

$$\begin{bmatrix} 1 & 2 & 5 \\ -1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}.$$

4 Linear map

Definition 4.1 (Linear map) *Let E and F be vector spaces. A map $f : E \to F$ is linear if*

- $\forall (x, y) \in E^2 \quad f(x + y) = f(x) + f(y)$,
- $\forall x \in E, \forall \lambda \in \mathbb{R} \quad f(\lambda x) = \lambda f(x)$

We denote $\mathcal{L}(E, F)$ the sert of linear maps from E to F.

Matrix representation : Reciproquely, let A be a matrix of $\mathcal{M}_{n,p}$.The relation $X \mapsto AX$ is isomorph to a linear map $f : \mathbb{R}^p \to \mathbb{R}^n$.

Let E and F be vector spaces of finite dimnsion and $f \in \mathcal{L}(E, F)$. Let $\mathcal{E} := \{e_i\}_{\{1,\cdots,n\}}$ be basis of E and $\mathcal{F}$ e a basis of F. The matricial representation of f in the basis $\mathcal{E}$, FF denoted $\mathrm{Mat}(f, \mathcal{E}, \mathcal{F})$, is the matrix which i-th column is made of the coordinates of $f(e_i)$ in the basis $\mathcal{F}$.

Exercise 19 *(i) What is the matricial representation of the application f defined by $f(x, y, z) = (x, z)$?*

(ii) What is the application corresponding to

$$\begin{bmatrix} 1 & 2 & 0 \\ 3 & 0 & -1 \end{bmatrix} ?$$

Exercise 20 *Let f which matricial representation in the canonical basis is*

$$\begin{pmatrix} 3 & 1 & -3 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

Set $u_1 := (1, 1, 1)_{\mathcal{C}}$, $u_2 := (1, -1, 0)_{\mathcal{C}}$ and $u_3 := (1, 0, 1)_{\mathcal{C}}$. Let $\mathcal{B}' := \{u_1, u_2, u_3\}$.

1. *Prove that $\mathcal{B}'$ is a basis of $\mathbb{R}^3$.*
2. *Determine the matricial representation of f in the basis $\mathcal{B}'$.*

Definition 4.2 (Image, Kernel) *Let E and F be two sets and f a map from E to F.*

- *We call* kernel of f *the sub-set of E defined by*

$$\mathrm{Ker}\, f := \{x \in E \,:\, f(x) = 0\}\ .$$

- *We call* image of f *the sub-set of F defined by*

$$\mathrm{Im}\, f := f(E) = \{y \in F \,:\, \exists x \in E \text{ such that } y = f(x)\}\ .$$

Proposition 4.3 *Let E and F be two vector spaces. Consider $f \in LL(E, F)$. The kernel of f is a vector sub-space of E and the image of f is a vector sub-space of F.*

Proof Left to the reader.

Proposition 4.4 *Let E and F be two vector spaces. Consider $f \in LL(E, F)$. The image of f is spanned by the columns of any matricial representation of f*

Application to the notions of injectivity, surjectivity and bijections

Definition 4.5 (Injectivity) *Let $f : E \to F$. If*

$$\forall (x, x') \in E^2\ ,\ f(x) = f(x') \Rightarrow x = x'$$

then f is injective.

Proposition 4.6 *Let $f \in \mathcal{L}(E, F)$. f is injective if and only if* $\ker(f) = \{0\}$.

proof Left to the reader.

Definition 4.7 (Surjectivity) *Let $f : E \to F$. If*

$$\forall y \in F \;,\; \exists x \in E \quad \textit{such that} \quad f(x) = y$$

then f is surjective on F.

By definition of the image we have

Proposition 4.8 *Let $f : E \to F$. f is surjective if and only if* $\mathrm{Im}(f) = F$.

Definition 4.9 (Bijectivity) *Let $f : E \to F$. If then f is injective and surjective on F then f is bijective from E to F.*

Exercise 21 *Let f which matricial representation in the canonical basis is*

$$\begin{pmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}$$

1. *Determine a basis of B of* $\mathrm{Im}(f)$ *and B' of* $\mathrm{Ker}(f)$.
2. *Determine the matricial representation of f in the basis which first vectoers are the vectors of B and the last vectors the vectors of B' .*
3. *Describe f in this basis.*
4. *Describe f in the canonical basis.*

Exercise 22 *Let f which matricial representation in the canonical basis is*

$$\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

1. *Determine a basis of B of* $\mathrm{Im}(f)$ *and B' of* $\mathrm{Ker}(f)$.
2. *Determine the matricial representation of f in the basis which first vectoers are the vectors of B and the last vectors the vectors of B' .*
3. *Describe f in this basis.*
4. *Describe f in the canonical basis.*

Application to linear systems Let $A \in \mathcal{M}_{n,p}$ and $B \in \mathcal{M}_{n,1}$. The system $AX = B$ of unknowns $X \in \mathcal{M}_{p,1}$ has a solution if and only if $B \in \mathrm{Im}(A)$. Moreover if X and X' are two solutions to this system $X - X' \in \ker(A)$. Indeed, substracting the equalities $AX = B$ and $AX' = B$ we obtain $A(X - X') = 0$. As a consequence any solution to $AX = B$ is of the form $X_P + \ker(A)$ where X_P is any particular solution.

Exercise 23 *Consider*

$$f : \begin{array}{ccc} \mathbb{R}^3 & \to & \mathbb{R}^3 \\ (x, y, z) & \mapsto & (y + z, x + y, x + 2y + z) \; . \end{array}$$

1. *Prove that f is a linear map from $\mathbb{R}^3$ to $\mathbb{R}^3$.*

2. *Determine the matricial representation of f in the canonical basis.*
3. *Determine a basis of* $\text{Ker}\,(f)$.
4. *Determine a basis of* $\text{Im}\,(f)$.
5. *Determine* $\text{Im}\,(f) \cap \text{Ker}\,(f)$.
6. *Discuss the numer of solutions to* $f(x, y, z) = (1, 1, 1)$.
7. *Discuss the numer of solutions to* $f(x, y, z) = (1, 1, 2)$.
8. *Discus, depending on* $(a, b) \in \mathbb{R}^2$*, the number of solutions to* $f(x, y, z) = (1, a, b)$.

Definition 4.10 (Rank) *The rank of f is the dimension of the image of f. It is denoted* $\text{rk}f$.

Exercise 24 *Consider*

$$\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

Determine the rank of A. Find a basis of $\text{Im}\,A$ *and* $\text{Ker}\,A$. *Prove that* $\text{Im}\,A \cap \text{Ker}\,A = 0$.

Theorem 4.11 (Rank-nullity theorem) *Let E and F be two vector space and f a linear mapping from E to F. We have*

$$\dim E = \text{rk}\ f + \dim\ \text{Ker}\ f\ .$$

Remark 6 *These quantity are also defined at the matrix level thanks to the matricial representation of a linear mapping.*

As a conquequence of the rank-nullity theorem we have

Proposition 4.12 *Let E anf F be vector spaces of same dimension n. Consider $f \in \mathcal{L}(E, F)$. The following statements are equivalent*

- *f is a one-to-one correspondance,*
- $\ker(f) = \{0\}$,
- $\dim\ker(f) = 0$,
- $\text{Im}(f) = F$,
- $\dim \text{Im}(f) = n$.

All these statements are equivalent to $\text{Det}(f) \neq 0$. In term of matrix this is equivalent to say that the matricial representation of f is inversible.

5 Diagonalisation

5.1 Change of basis

Definition 5.1 (▶ Passage matrix) *Let E be a vector space and $\mathcal{E}$ and $\mathcal{E}'$ be two basis of E. Wecall* passage matrix from $\mathcal{E}$ to $\mathcal{E}'$*, and denote, $P_{\mathcal{E}}^{\mathcal{E}'}$, the matrix*

$$P_{\mathcal{E}}^{\mathcal{E}'} := \text{Mat}(\text{Id}, \mathcal{E}', \mathcal{E})\ .$$

Example 9 *(▶) Determine $P_{\mathcal{C}_3}^{\mathcal{E}}$ where $\mathcal{E} := \{(1, 1, 0), (0, 1, 0), (1, 1, 1)\}$.*

▷

$$P_{\mathcal{C}_3}^{\mathcal{E}} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Example 10 *(▶) In $\mathbb{R}_2[X]$, determine $P_{\mathcal{C}_3}^{\mathcal{E}}$ where $\mathcal{E} := \{1, X-1, 2X^2 - X + 1\}$.*
▷

$$P_{\mathcal{C}_3}^{\mathcal{E}} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix}$$

Example 11 *(▶) Determine $P_{\mathcal{E}}^{\mathcal{C}_3}$ where $\mathcal{E} := \{(1,1,0),(0,1,0),(1,1,1)\}$.*
▷ *We have :*

$$\begin{aligned} (1,0,0) &= 1*(1,1,0) + (-1)*(0,1,0) + 0*(1,1,1) = (1,-1,0)_{\mathcal{E}} \\ (0,1,0) &= (0,1,0)_{\mathcal{E}} \\ (0,0,1) &= (-1,0,1)_{\mathcal{E}} \end{aligned}$$

So that

$$P_{\mathcal{C}_3}^{\mathcal{E}} = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Proposition 5.2 (▶ Inverse of a passage matrix) *Let E be a vector space and $\mathcal{E}$ and $\mathcal{E}'$ be two basis of E. We have $P_{\mathcal{E}'}^{\mathcal{E}} = \left[P_{\mathcal{E}}^{\mathcal{E}'}\right]^{-1}$.*

We have

$$P_{\mathcal{E}}^{\mathcal{E}'} \cdot P_{\mathcal{E}'}^{\mathcal{E}} = \mathrm{Mat}(\mathrm{Id}, \mathcal{E}', \mathcal{E}) \cdot \mathrm{Mat}(\mathrm{Id}, \mathcal{E}, \mathcal{E}') = \mathrm{Mat}(\mathrm{Id}, \mathcal{E}', \mathcal{E}') = I_n \ .$$

5.1.1 Change of basis : vector

Let E and F be a vector space and $\mathcal{E}$ and $\mathcal{E}'$ be two basis of E and $\mathcal{F}$ and $\mathcal{F}'$ be two basis of F. Consider $f \in \mathcal{L}(E, F)$. We have

$$\mathrm{Mat}(f, \mathcal{E}, \mathcal{F}) = P_{F}^{\mathcal{F}'} \mathrm{Mat}(f, \mathcal{E}', \mathcal{F}') P_{\mathcal{E}'}^{\mathcal{E}} \ .$$

▶ In the case of an endomorphisme we have

$$\mathrm{Mat}(f, \mathcal{E}) = P \, \mathrm{Mat}(f, \mathcal{E}') \, P^{-1} \quad \text{où} \quad P := P_{\mathcal{E}}^{\mathcal{E}'} \ . \tag{5.1}$$

5.2 Diagonalisation and eigen elements

Proposition 5.3 (▶ Change of basis : vector Let E be a vector space and $\mathcal{E}$ and $\mathcal{E}'$ be two ba
Let $(A, B) \in_N (\mathbb{K})^2$. A and B are similar *if there exists an invertible $P \in_N (\mathbb{K})$ such that*

$$A = PBP^{-1} \ .$$

Definition 5.6 (▶ Diagonalisable matrix) *Let $A \in_N$. A is* diagonalisable *in $\mathbb{K}$ if it is similar in $\mathbb{K}$ to a diagonal matrix.*

elle est diagonalisable.

▶ We look for a basis $\mathcal{E} := \{_i\}_{i \in \{1, \cdots, N}$ such that : $\mathrm{Mat}(f, \mathcal{E}) = (\lambda_i)_{i \in \{1, \cdots, N}$. It means

$$\mathrm{Mat}(f, \mathcal{E}) = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_N \end{pmatrix}$$

Hence

$$\forall i \in \{1, \cdots, N\}, \quad f(_i) = \lambda_i \, _i \ .$$

Definition 5.7 (▶ Eigenvalue, eigenvector) *Let $A \in_N (\mathbb{K})$. A real number λ is an* eigenvalue *of A in $\mathbb{K}$ if there exists $X_\lambda \neq 0$ in $\mathbb{K}^N$ such that $AX_\lambda = \lambda X_\lambda$. Such a X_λ is called an* eigenvector *associated to the eigenvalue λ. The set of all the eigenvlues of A is called the* spectrum *of A and is denoted* $\mathrm{Sp}(A)$.

Example 12 *Let*

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

1. *Prove that $(1, 1)$ is an eigenvalue of A.*
2. *Prove that $(-1, 1)$ is an eigenvalue of A.*

▷ *We compute*

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

hence $(1, 1)$ is an eigenvector associated to 3. Similarly

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = - \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

hence $(-1, 1)$ is an eigenvector associated to -1.

▶ To determine such a basis we are looking for $(\lambda_{i,i}) \in \mathbb{R} \times E \setminus \{0_E\}$ such that

$$f(_i) = \lambda_{i\,i} \,.$$

or

$$\text{we are looking for } _i \neq 0_E \text{ such that } f(_i) - \lambda_{i\,i} = 0_E$$

ou de façon équivalente

$$\text{we are looking for } _i \neq 0_E \text{ such that } (f - \lambda_i \,\mathrm{Id})_i = 0_E$$

ou encore

$$\text{we are looking for } _i \neq 0_E \text{ in } \ker(f - \lambda_i \,\mathrm{Id})$$

Thi is possible if $\dim(\ker(f - \lambda_i \,\mathrm{Id}))$. Or $\mathrm{Det}(f - \lambda_i \,\mathrm{Id}) = 0$. We then have to find the zeros of $\mathrm{Det}(f - \lambda_i \,\mathrm{Id})$ or of $\mathrm{Det}(A - \lambda_i \, I)$.

Definition 5.8 (▶ Characteristic polynomial function) *Let $A \in_N$. The polynomial function*

$$\chi_A : \quad \begin{array}{ccc} E & \to & \mathbb{R} \\ X & \mapsto & \mathrm{Det}(A - X\, I) \,. \end{array}$$

is the haracteristic polynomial function *of A.*

Proposition 5.9 (▶ Characterisation of an eigenvalue) *Let $A \in_N$. A scalar $\lambda \in \mathbb{K}$ is an eigenvalue if and only if $\chi_A(\lambda) = 0$.*

Example 13 *Determine the spetrum of*

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

Definition 5.10 (▶ Eigenspace) *Let $A \in_N$ and $\lambda \in \mathbb{K}$. The* eigenspave *is $E_\lambda := \ker(A - \lambda I)$.*

Proposition 5.11 (▶ Eigenvector) *Let $A \in_N$ and $\lambda \in \mathrm{Sp}(A)$. Any vector of E_λ is an eigenvectro associated to λ.*

Example 14 *Determine the eigenvector associated to -1 and 3 of*

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

[▶] We have

$$\chi_A = (-1)^n X^n + (-1)^{n-1} \mathrm{Tr}(A) X^{n-1} + \cdots + \mathrm{Det}(A) \,.$$

so that

- $\mathrm{Tr}(A)$ has to be egal to the sum of the eigenvalues counted as many times as their multiplicity,
- $\mathrm{Det}(A)$ has to be egal to the product of the eigenvalues counted as many times as their multiplicity.

Example 15 *Let*

$$A = \begin{pmatrix} 1 & -4 \\ -4 & 7 \end{pmatrix}$$

1. *Determine χ_A.*
2. *Determine $\mathrm{Sp}(A)$.*
3. *Determine a basis of eigenvectors of A.*

▷

1. *We compute*

$$\begin{aligned} \chi_A(\lambda) &= \mathrm{Det}\left(\begin{pmatrix} 1 & -4 \\ -4 & 7 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = \mathrm{Det}\left(\begin{pmatrix} 1-\lambda & -4 \\ -4 & 7-\lambda \end{pmatrix}\right) \\ &= \lambda^2 - 8\lambda - 9 = (\lambda - 9)(\lambda + 1) \,. \end{aligned}$$

2. $\mathrm{Sp}(A) = \{-1, 9\}$.
3. *We compute*

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in E_{-1} = \ker(A + I) \Leftrightarrow \begin{pmatrix} 2 & -4 \\ -4 & 8 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow x - 2y = 0$$

so that $E_{-1} = \mathrm{Span}\{(2,1)\}$. We can check

$$\begin{pmatrix} 1 & -4 \\ -4 & 7 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \end{pmatrix} = (-1) * \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

hece $(2,1)$ is an eigenvector associated to -1.

We compute

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in E_9 \Leftrightarrow \begin{pmatrix} -8 & -4 \\ -4 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow 2x + y = 0$$

so $E_9 = \mathrm{Span}\{(1,-2)\}$.

4. *$\{(2,1),(1,-2)\}$ is a basis of $\mathbb{R}^2$ made of eigenvectors.*

Example 16 *Let*

$$A = \begin{pmatrix} 2 & 1 & -2 \\ 2 & 3 & -4 \\ 1 & 1 & -1 \end{pmatrix}$$

1. *Determine* $\mathrm{Sp}(A)$.
2. *Determine a basis of eigenvectors of* A.

▷

1. *We compute* $\chi_A(\lambda) = -\lambda^3 + 4\lambda^2 - 5\lambda + 2 = -(\lambda-2)(\lambda-1)^2$. *Hence* $\mathrm{Sp}(A) = \{1, 2\}$.
2. *We compute* $E_2 = \mathrm{Span}\{(1,2,1)\}$ *and* $E_{-} = \mathrm{Span}\{(-1,1,0),(2,0,1)\}$. $\{(1,2,1),(-1,1,0),(2,0,1)\}$ *is a basis of* $\mathbb{R}^3$.

Example 17 *Let*

$$A = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 4 & 6 \\ 3 & 0 & 1 \end{pmatrix}$$

1. *Determine* $\mathrm{Sp}(A)$.
2. *can we determine a basis of eigenvectors of* A.

▷

1. *We compute* $\chi_A(\lambda) = -\lambda^3 + 6\lambda^2 - 32 = -(\lambda+2)(\lambda-4)^2$.
2. *We compute* $E_4 = \mathrm{Span}\{(0,1,0)\}$ *and* $E_{-2} = \{(-3,-2,3)\}$. *There are ony two eigenvectors so they cannot form a basis of* $\mathbb{R}^3$.

5.3 Dimension criterion

Theorem 5.12 (▶ Dimension criterion) *Let* $A \in_N$ *and* $\mathrm{Sp}(A) = \{\lambda_i\}_{i\in\{1,\cdots,p\}}$. *A is diagonalisable if and only if*

$$\sum_{i=1}^{p} \dim(E_i) = N\ .$$

5.4 Multiplicity criterion

Theorem 5.13 (▶ Multiplicity criterion) *Let* $A \in_N (\mathbb{K})$.

$$A \text{ is diagonalisable} \Leftrightarrow \begin{cases} \chi_A \text{ is spilt on } \mathbb{K} \\ \text{for all } \lambda \in \mathrm{Sp}(A) \text{ we have } \dim E_\lambda \text{ is egal to the multiplicity of } \lambda \text{ in } \chi_A \end{cases}.$$

Corollary 5.14 (▶ Polynôme caractéristique scindé à racines simples) *Let* $A \in_N (\mathbb{K})$. *If* A *has* N *distinct eigenvalues in* $\mathbb{K}$ *then* A *is diagonalisable in* $\mathbb{K}$.

5.5 Applications

5.5.1 Application to iteration of linear maps

Definition 5.15 (iteration of linear maps) *Let* $f \in \mathcal{L}(E)$. *We define* th iteration *of* f *for all* $k \in \mathbb{N}^*$

$$\begin{cases} f^{(1)} = f \\ f^{(k+1)} = f \circ f^{(k)} \end{cases}.$$

Proposition 5.16 (▶ Power of a diagonlisable matrix) *Let* $M \in_n (\mathbb{R})$ *similar to a diagonl marix* D. *For all* $k \in \mathbb{N}^*$,

$$M^k = PD^kP^{-1}\ .$$

5.5.2 Application to the geometry of linear maps

Example 18 *describe geometrically*

$$f: \begin{array}{ccc} \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (x,y) & \mapsto & (2x,-2y) \end{array}$$

▷ *The matricial representation of f is*

$$A = \begin{pmatrix} 2 & 0 \\ -2 & 0 \end{pmatrix}$$

We have $\mathrm{Sp}(A) = \{0,2\}$ *and* $E_2 = \mathrm{Span}(1,-1)$, $E_0 = \mathrm{Span}\{(0,1)\}$. *Hene in the basis* $\{(1,-1),(0,1)\}$ *the matricila representation of f is*

$$\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$$

in this basis the linear map is the compose of a projection on the second eigenvector and an homothetia of power 2.

In the original basis f is the compose of a projection on the axis $\mathrm{Span}\{(1,-1)\}$ *and an homothetia of power 2.*

6 Bilinear and quadratic forms

Definition 6.1 (▶ Bilinear form) *Let E be a vector space. We consider the map $b : E \times E \to \mathbb{K}$. b is* a bilinear map *if*

— *For all $y \in E$*

$$\begin{array}{rcl} b(\cdot,y) : E & \to & \mathbb{K} \\ x & \mapsto & b(x,y) \end{array}$$

is a linear map on E.

— *For all $x \in E$*

$$\begin{array}{rcl} b(x,\cdot) : E & \to & \mathbb{K} \\ y & \mapsto & b(x,y) \end{array}$$

is a linear map on E.

We denote $\mathcal{B}(E)$ the set of biliear forms on E.

Example 19 *Prove that*

$$b: \begin{array}{ccc} \mathbb{R}^3 \times \mathbb{R}^3 & \to & \mathbb{K} \\ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} & \mapsto & 2x_1y_1 + x_2y_2 + 3x_1y_2 - x_1y_3 + 5x_2y_1 + 2x_2y_3 + x_3y_1 - 6x_3y_2 - 3x_3y_3 \end{array}$$

is a biliear form.

▷ *y being fixed*

$$\begin{array}{ccc} \mathbb{R}^3 & \to & \mathbb{K} \\ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} & \mapsto & x_1(2y_1 + 3y_2 - y_3) + x_2(5y_1 + y_2 + 2y_3) + x_3(y_1 - 6y_2 - 3y_3) \end{array}$$

is a linear form.

Similarly

$$\begin{array}{rcl} \mathbb{R}^3 & \to & \mathbb{K} \\ \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} & \mapsto & y_1(2x_1+5x_2+x_3)+y_2(3x_1+x_2-6x_3)+y_3(-x_1+2x_2-3y_3) \end{array}$$

is a linear form.

Hence b is a bilinear form on $\mathbb{R}^3$.

Definition 6.2 (▶ Symmetric bilinear form) *Let E be a vector space and $b \in \mathcal{B}(E)$. b is* symmetric *if*

$$\forall (x,y) \in E^2, \quad b(x,y) = b(y,x) \ .$$

Example 20 *(▶) Consider*

$$\begin{array}{rcl} \phi : \mathcal{C}([0,1])^2 & \to & \mathbb{R} \\ (f,g) & \mapsto & \int_0^1 f(x)g(x)x \end{array}$$

1. *Prove that ϕ is well defined.*
2. *Prove that ϕ is a bilinear form.*
3. *Prove that ϕ is symmetric.*

▷

1. *Any continuous function defined on $[0,1]$ is integrable on $[0,1]$.*
2. *Let us first prove that ϕ is symmetric*

$$\phi(f,g) = \int_0^1 f(x)g(x)x = \int_0^1 g(x)f(x)x = \phi(g,f) \ .$$

   Remains to prove that $f \mapsto \int_0^1 f(x)g(x)x$ is linear which is trivial.

▶ Let $\{e_i\}_{i\in\{1,\cdots,N\}}$ be a basis of E. Any $x \in E$ can be written $x = \sum_{i=1}^N x_i\, e_i$ and any $y \in E$ can be written $y = \sum_{j=1}^N y_j\, e_j$. Let $b \in \mathcal{B}(E)$. By bilinearity of b we have

$$b(x,y) = b\left(\sum_{i=1}^N x_i\, e_i, \sum_{j=1}^N y_j\, e_j\right) = \sum_{i=1}^N \sum_{j=1}^N x_i\, y_j b(e_i,e_j) \ .$$

So that any biklinear form can be written

$$\sum_{i=1}^N \sum_{j=1}^N a_{ij}\, x_i\, y_j \ .$$

where a_{ij} ae real numbers. Moreover b is characterised by $a_{ij} = b(e_i,e_j)$ for all $(i,j) \in \{1,\cdots,N\}$.

Definition 6.3 (▶ Matricial representation) *Let E be a vector space and $\mathcal{E} := \{e_i\}_{i\in\{1,\cdots,N\}}$ be a basis of E. Consider $b \in \mathcal{B}(E)$. The* matricial representation of b in the basis $\mathcal{E}$ *is the matrix of general term $b(e_i,e_j)$.*

Example 21 *determine the matricial representation of b defined in exercise 19.*
▷ *We have*

$$\begin{pmatrix} 2 & 3 & -1 \\ 5 & 1 & 2 \\ 1 & -6 & -3 \end{pmatrix}$$

▶ Reciproquely to any $A \in_N (\mathbb{K})$ we can associate the bilinear form

$$\begin{array}{ccc} {}_N(\mathbb{K})^2 & \to & \mathbb{R} \\ (X,Y) & \mapsto & X^T AY \end{array}$$

Example 22 *Determine the biliear map associated to*

$$\begin{pmatrix} 2 & 3 & -1 \\ 5 & 1 & 2 \\ 1 & -6 & -3 \end{pmatrix}$$

▷ *We compute*

$$\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} 2 & 3 & -1 \\ 5 & 1 & 2 \\ 1 & -6 & -3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$
$$= 2x_1y_1 + x_2y_2 + 3x_1y_2 - x_1y_3 + 5x_2y_1 + 2x_2y_3 + x_3y_1 - 6x_3y_2 - 3x_3y_3 \; .$$

[▶ Change of basis in a bilinear form] Let E be a vector space and $\mathcal{E}$ and $\mathcal{E}'$ be two basis of E. Consider $b \in \mathcal{B}(E)$. We have

$$\mathrm{Mat}(b, \mathcal{E}) = (P_{\mathcal{E}'}^{\mathcal{E}})^T \mathrm{Mat}(b, \mathcal{E}') P_{\mathcal{E}'}^{\mathcal{E}} \; .$$

6.1 Quadratic forms

Definition 6.4 (▶ Quadratic forms) *Let E be a vector space. A* uadratic form *on $\mathbb{K}$ is a map $q : E \to \mathbb{K}$ such that*

- *for all $\lambda \in \mathbb{K}$ and all $v \in E$, $q(\lambda v) = \lambda^2 q(v)$,*
- *the map $\phi : (u,v) \mapsto [q(u+v) - q(u) - q(v)]/2$ (called the polar form of q) is a bilinear form*

We denote (E) the set of quadratic forms on E.

Example 23 *(▶) Consider*

$$q : \begin{array}{ccc} \mathbb{R}^3 & \to & \mathbb{K} \\ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} & \mapsto & 2x_1^2 + x_2^2 + 8x_1x_2 - 4x_3x_2 - 3x_3^2 \end{array}$$

Prove that q is a quadratic form.
▷ *Let us determine the polar form associatd to q : for all $u = (x_1, x_2, x_3)$ and $v = (y_1, y_2, y_3)$ in $\mathbb{R}^3$ we have*

$$q(u+v) - q(u) - q(v) = 2(x_1+y_1)^2 + (x_2+y_2)^2 + 8(x_1+y_1)(x_2+y_2) - 4(x_3+y_3)(x_2+y_2) - 3(x_3+y_3)^2$$
$$- (2x_1^2 + x_2^2 + 8x_1x_2 - 4x_3x_2 - 3x_3^2) - (2y_1^2 + y_2^2 + 8y_1y_2 - 4y_3y_2 - 3y_3^2)$$

We obtain

$$b(u,v) := \frac{1}{2}\left[q(u+v)-q(u)-q(v)\right] = 2x_1y_1 + x_2y_2 + 4x_1y_2 + 4x_2y_1 - 2x_3y_2 - 2x_2y_3 - 3x_3y_3 \; .$$

b is a bilinear form because it is of the form $\sum a_{ij}x_i\,y_j$. Moreover $b(u,u) = q(u)$ so that for all $\lambda \in \mathbb{K}$ and all $v \in E$, $q(\lambda\, v) = b(\lambda\, v, \lambda\, v) = \lambda^2 b(v,v) = \lambda^2 q(v)$.

Example 24 *(▶) Consider*

$$\begin{array}{rrcl} \psi: & \mathcal{C}([0,1]) & \to & \mathbb{R} \\ & f & \mapsto & \displaystyle\int_0^1 f(x)^2 x \end{array}$$

Prove that ψ is a quadratic form.
▷ *For all $f \in \mathcal{C}([0,1])$ and $\lambda \in \mathbb{R}$, we have*

$$\psi(\lambda f) = \int_0^1 \left[\lambda\, f(x)\right]^2 x = \lambda^2 \int_0^1 f(x)^2 x = \lambda^2 \psi(f) \; .$$

Let us determine the polar form associated to ψ :

$$\psi(f+g) - \psi(f) - \psi(g) = \int_0^1 \left[f(x)+g(x)\right]^2 x - \int_0^1 f(x)^2 x - \int_0^1 g(x)^2 x = 2\int_0^1 f(x)\, g(x) x \; .$$

Hence

$$\begin{array}{rrcl} b: & \mathcal{C}([0,1])^2 & \to & \mathbb{R} \\ & (f,g) & \mapsto & \displaystyle\int_0^1 f(x)\, g(x) x \end{array}$$

For any given g

$$\begin{array}{rcl} \mathcal{C}([0,1]) & \to & \mathbb{R} \\ f & \mapsto & \displaystyle\int_0^1 f(x)\, g(x) x \end{array}$$

is linear. Moreover b is symmetric hence b is a bilinear form and ψ is a quadratic form.

Definition 6.5 (▶ Quadratic form associated to a bilinear form) *Let E be a vector space. Consider $b \in \mathcal{B}(E)$. We call* quadratic form associated to a bilinear formf*b the form*

$$\begin{array}{rcl} E & \to & \mathbb{R} \\ x & \mapsto & b(x,x) \end{array}$$

Trick to find the polar form associted to a quadratic form in $\mathbb{R}^n$: x_i^2 becomes $x_i\, y_i$ and $x_i\, x_j$ becomes $(x_i\, y_j + x_j\, y_i)/2$.

Definition 6.6 (▶ Matricial representation of a quadratic form) *The* matricial representation of a quadratic form *in a basis is the atricial representation of the polar form associated in the sam basis.*

6.2 Sign of a quadratic form

Definition 6.7 (▶ Positive and negative quadratic forms) *Let E be a vector space. Consider $b \in \mathcal{B}(E)$.*

- *b is* positive semi-definite *if for all $x \in E$, $b(x,x) \geq 0$.*
- *b is* negative semi-definite *if for all $x \in E$, $b(x,x) \leq 0$.*

- b *is* positive definite *if for all* $x \in E$,

$$\begin{cases} b(x,x) \geq 0 \\ b(x,x) = 0 \Leftrightarrow x = 0 \end{cases}$$

- b *is* negative definite *if for all* $x \in E$,

$$\begin{cases} b(x,x) \leq 0 \\ b(x,x) = 0 \Leftrightarrow x = 0 \end{cases}$$

Example 25 *Consider*

1.
$$q_1 : \begin{array}{ccc} \mathbb{R}^2 & \to & \mathbb{R} \\ (x_1, x_2) & \mapsto & x_1^2 + x_2^2 \end{array}$$

2.
$$q_2 : \begin{array}{ccc} \mathbb{R}^2 & \to & \mathbb{R} \\ (x_1, x_2) & \mapsto & x_1^2 + x_2^2 - 2x_1x_2 \end{array}$$

3.
$$q_3 : \begin{array}{ccc} \mathbb{R}^2 & \to & \mathbb{R} \\ (x_1, x_2) & \mapsto & x_1^2 + 5x_2^2 - 4x_1x_2 \end{array}$$

Determine if the quadratic forms are positive definite.

▷

1. *For all* $(x_1, x_2) \in \mathbb{R}^2$ *we have*

$$q_1(x_1, x_2) \geq 0 \quad and \quad q_1(x_1, x_2) = 0 \Leftrightarrow x_1^2 + x_2^2 = 0 \Leftrightarrow \begin{cases} x_1 = 0 \\ x_2 = 0 \end{cases}$$

Hence q_1 *is positive definite.*

2. *Pour* q_2 *on pourrait s'attendre naïvement à ce que le terme en* $-x_1x_2$ *compense les termes positifs lorsque* x_1 *et* x_2 *sont grands. Ça n'est évidemment pas le cas puisque* $q_2(x_1, x_2) = (x_1 + x_2)^2 \geq 0$. *Par contre*

$$q_2(x_1, x_2) = 0 \Leftrightarrow (x_1 + x_2)^2 = 0 \Leftrightarrow x_1 + x_2 = 0$$

mais ceci n'implique pas que $x_1 = x_2 = 0$. *En effet,* $q_2(1, -1) = 0$ *ce qui prouve que* q_2 *n'est pas définie positive.*

3. *Pour* q_3*, on ne peut pas appliquer directement une identité remarquable. Par contre on peut faire apparaître des carrés. Cette méthode de réduction en carrés est due à Gauss et sera présentée en détail dans la section qui va suivre.*

$$q_3(x_1, x_2) = x_1^2 + x_2^2 - 4x_1x_2 = (x_1 - 2x_2)^2 - (-2x_2)^2 + 5x_2^2 = (x_1 - 2x_2)^2 + x_2^2 \,.$$

Comme q_3 *est une somme de carrés elle est au moins semi-définie positive. Pour ce qui est de la positivité on a*

$$q_3(x_1, x_2) = 0 \Leftrightarrow (x_1 - 2x_2)^2 + x_2^2 = 0 \Leftrightarrow \begin{cases} x_1 - 2x_2 = 0 \\ x_2 = 0 \end{cases} \Leftrightarrow \begin{cases} x_1 = 0 \\ x_2 = 0 \end{cases}$$

Donc q_3 *est définie positive.*

6.3 Méthode de réduction en carrés de Gauss

Cette méthode permet toujours de déterminer la signature d'une forme quadratique. Cette méthode ouvre aussi le champs à de multiples développements. Nous en présenterons une dans la section à venir. Cette méthode est donc à maîtriser sans négociation possible.

D'après Remarque **??**, toute forme quadratique sur $\mathbb{R}^N$ s'écrit

$$q((x_i)_{\{1,\cdots,N\}}) = \sum_{i=1}^{N} c_{ii}x_i^2 + \sum_{(i,j)=1,i<j}^{N} c_{ij}x_ix_j \ .$$

On va considérer deux cas :

► **Cas où il y a au moins un terme x_i^2 :** Sans perte de généralité, quitte à renommer les variables, on peut supposer que $c_{11} \neq 0$. L'idée est de choisir une variable qui a un terme carré, ici on va prendre x_1 et de faire apparaître le début d'un carré afin de faire disparaître cette variable du reste de la forme quadratique :

$$\begin{aligned}
\sum_{i=1}^{N} c_{ii}x_i^2 + \sum_{(i,j)=1,i<j}^{N} c_{ij}x_ix_j &= c_{11}\left[\sum_{i=1}^{N}\frac{c_{ii}}{c_{11}}x_i^2 + \sum_{(i,j)=1,i<j}^{N}\frac{c_{ij}}{c_{11}}x_ix_j\right] \\
&= c_{11}\left[x_1^2 + x_1\sum_{j=2}^{N}\frac{c_{ij}}{c_{11}}x_j + \sum_{i=2}^{N}\frac{c_{ii}}{c_{11}}x_i^2 + \sum_{(i,j)=2,i<j}^{N}\frac{c_{ij}}{c_{11}}x_ix_j\right] \\
&= c_{11}\left[x_1 + \frac{1}{2}\sum_{j=2}^{N}\frac{c_{ij}}{c_{11}}x_j\right]^2 - \frac{c_{11}}{4}\left[\sum_{j=2}^{N}\frac{c_{ij}}{c_{11}}x_j\right]^2 \\
&\qquad + c_{11}\left[\sum_{i=2}^{N}\frac{c_{ii}}{c_{11}}x_i^2 + \sum_{(i,j)=2,i<j}^{N}\frac{c_{ij}}{c_{11}}x_ix_j\right]
\end{aligned}$$

La première ligne de la dernière égalité correspond exactement à tous les termes qui contiennent x_1. On voit que la partie

$$c_{11}\left\{-\left[\frac{1}{2}\sum_{j=2}^{N}\frac{c_{ij}}{c_{11}}x_j\right]^2 + \left[\sum_{i=2}^{N}\frac{c_{ii}}{c_{11}}x_i^2 + \sum_{(i,j)=2,i<j}^{N}\frac{c_{ij}}{c_{11}}x_ix_j\right]\right\}$$

est une forme quadratique en $(x_i)_{\{2,\cdots,N\}}$ (donc qui ne contient plus de terme en x_1). Donc à chaque étape un élimine une variable ce qui assure qu'en au plus N étapes la forme quadratique sera décomposée sous forme de somme de carrés.

► **Cas où il n'y a pas de terme x_i^2 :** Sans perte de généralité, quitte à renommer les variables, on peut supposer que $c_{12} \neq 0$. L'idée est de choisir deux variables, ici on va prendre x_1 et x_2, et de faire apparaître deux carrés tout en faisant disparaître ces deux variables du reste de la forme quadratique :

$$\sum_{(i,j)=1,i<j}^{N} c_{ij}x_ix_j = c_{12}\left\{x_1x_2 + \sum_{(i,j)=2,i<j}^{N} \frac{c_{ij}}{c_{12}}x_ix_j\right\}$$
$$= c_{12}\left\{x_1x_2 + x_1\sum_{j=3}^{N}\frac{c_{ij}}{c_{12}}x_j + x_2\sum_{j=3}^{N}\frac{c_{ij}}{c_{12}}x_j + \sum_{(i,j)=3,i<j}^{N}\frac{c_{ij}}{c_{12}}x_ix_j\right\}$$
$$= c_{12}\left\{\underbrace{\left(x_1 + \sum_{j=3}^{N}\frac{c_{ij}}{c_{12}}x_j\right)}_{=:A}\underbrace{\left(x_2 + \sum_{j=3}^{N}\frac{c_{ij}}{c_{12}}x_j\right)}_{B} + \sum_{(i,j)=3,i<j}^{N}\frac{c_{ij}}{c_{12}}x_ix_j\right\}$$

En utilisant l'identité

$$AB = \frac{(A+B)^2}{4} - \frac{(A-B)^2}{4}$$

On obtient

$$q((x_i)_{\{1,\cdots,N\}}) = c_{12}\left\{\frac{1}{4}\left(x_1 + \sum_{j=3}^{N}\frac{c_{ij}}{c_{12}}x_j + x_2 + \sum_{j=3}^{N}\frac{c_{ij}}{c_{12}}x_j\right)^2 - \frac{1}{4}\left(x_1 + \sum_{j=3}^{N}\frac{c_{ij}}{c_{12}}x_j - x_2 - \sum_{j=3}^{N}\frac{c_{ij}}{c_{12}}x_j\right)^2 + \sum_{(i,j)=3,i<j}^{N}\frac{c_{ij}}{c_{12}}x_ix_j\right\}.$$

Pour la suite de l'algorithme, ou bien on a un terme carré et on utilise la technique vue dans le cas 1 ou bien il n'y a pas de terme carré et on utilise de nouveau cette technique vue dans ce cas 2.

En au plus N étapes l'algorithme s'arrête puisque la cas 1 fait apparaître un carré en faisant disparaître une variable du reste de la forme quadratique, alors que le cas 2 fait disparaître deux variables en faisant apparaître deux carrés. On a donc au plus N terme carrés.

Finalement cette technique permet d'écrire toute forme quadratique sous la forme

$$q(x) = \sum_{i=1}^{p}\alpha_i l_i^2(x)$$

où les α_i sont des scalaires l_i sont des formes linéaires. De plus, comme on obtient des formes linéaires échelonnées ou de la forme $\{(A,B)\mapsto A+B, (A,B)\mapsto A-B\}$, la famille $\{l_i\}_{i\in\{1,\cdots,N}$ est libre. Il est noter qu'il est possible d'utiliser une autre méthode pour faire apparaître des carrés mais cette méthode assure la liberté de la famille des formes linéaires. Et la liberté ça compte. Le liberalisme compte aussi mais sans cœur..

Example 26 *Considérons*

$$\begin{array}{rcl} f:\mathbb{R}^3 & \to & \mathbb{R} \\ (x,y,z) & \to & x^2+2y^2+3z^2+4xy-6xz-4yz \end{array}$$

Est-ce que f est définie positive ?

▷ *On peut commencer par choisir x car il est précédé du coefficient 1 :*

$$f(x,y,z) = \underbrace{(x^2+2x(2y-3z))}_{tous\ les\ termes\ en\ x} + \underbrace{2y^2+3z^2-4yz}_{pas\ de\ terme\ en\ x} \quad (6.2)$$

On reconnaît le début d'un carré dans le terme comportant tous les termes en x :

$$x^2+2x(2y-3z) = [x+(2y-3z)]^2 - \underbrace{(2y-3z)^2}_{\text{terme carré à enlever pour corriger}} = (x+2y-3z)^2-4y^2+12yz-9z^2 \ .$$

Si on réintroduit dans l'expression 6.2 on obtient

$$f(x,y,z) = (x+2y-3z)^2-4y^2+12yz-9z^2+2y^2+3z^2-4yz = (x+2y-3z)^2 \underbrace{-2y^2-6z^2+8yz}_{\text{forme quadratique en } y \text{ et } z} \ .$$

Puis on recommence

$$\begin{aligned} f(x,y,z) &= (x+2y-3z)^2 - 2(y^2+4z^2-4yz) \\ &= (x+2y-3z)^2 - 2[(y^2-4yz)+3z^2] \\ &= (x+2y-3z)^2 - 2[(y-2z)^2-4z^2+3z^2] \\ &= (x+2y-3z)^2 - 2(y-2z)^2 - z^2 \ . \end{aligned}$$

Il y a au moins un terme qui est précédé d'un coefficient négatif, nous verrons Section **??** *que cela assure que f n'est pas même semi-définie positive, pour le moment il suffit d'exhiber un vecteur en lequel f n'est pas positive. Par exemple en résolvant le système échelonné :*

$$\begin{cases} x+2y-3z = 0 \\ y-2z = 0 \\ z = 1 \end{cases}$$

qui a pour solution $(-1,2,1)$. Donc $f(-1,2,1) = -1$ et f n'est pas positive. On peut remarquer qu'elle n'est pas non plus semi-définie négative, parce qu'il y a un carré précédé d'un coefficient positif, en choisissant par exemple un vecteur tel que

$$\begin{cases} x+2y-3z = 1 \\ y-2z = 0 \\ z = 0 \end{cases} \Leftrightarrow \begin{cases} x = 1 \\ y = 0 \\ z = 0 \end{cases}$$

en lequel f vaut +1.

Example 27 *Considérons la forme quadratique*

$$\begin{array}{rccc} f: & \mathbb{R}^4 & \to & \mathbb{R} \\ & (x,y,z,t) & \to & xy+xz+2xt+yz-4yt \end{array}$$

Quelle est la signature de f ?
▷ *On a*

$$f(x,y,z,t) = (x + \underbrace{z-4t}_{\text{les termes multipliés par } y})(y + \underbrace{z+2t}_{\text{les termes multipliés par } x}) - \underbrace{(z-4t)(z+2t)}_{\text{termes de correction, sans } x \text{ ni } y}$$

On utilise $AB = (A+B)^2/4 - (A-B)^2/4$ et on obtient

$$\begin{aligned} f(x,y,z,t) = \frac{(x+z-4t+y+z+2t)^2}{4} - \frac{(x+z-4t-y-z-2t))^2}{4} - z^2 + 8t^2 + 2zt \\ = \frac{(x+y+2z-2t)^2}{4} - \frac{(x-y-6t))^2}{4} - z^2 + 8t^2 + 2zt \ . \end{aligned}$$

Le terme de reste, sans x ni y, comporte des carrés donc on fait apparaître, comme dans le premier cas, le début d'un carré :

$$\begin{aligned} f(x,y,z,t) &= \frac{(x+y+2z-2t)^2}{4} - \frac{(x-y-6t))^2}{4} - [z^2 - 8t^2 - 2zt] \\ &= \frac{(x+y+2z-2t)^2}{4} - \frac{(x-y-6t))^2}{4} - [(z-t)^2 - t^2 - 8t^2] \\ &= \frac{(x+y+2z-2t)^2}{4} - \frac{(x-y-6t))^2}{4} - (z-t)^2 + 9t^2 \ . \end{aligned}$$

Ne pas oublier que le signe - dans la dernière égalité porte sur tout le terme entre parenthèses. C'est la raison pour laquelle cela facilite les calculs de commencer par les termes précédés d'un coefficient le plus simple possible.

7 Scalar product

7.1 Definition

Definition 7.1 (Scalar product) *Let E be a vector space. Consider $b \in \mathcal{B}(E)$. The map b is a* scalar product *if*

- *b is symmetric*
- *b is positive definite.*

We call norm *of u, and denote $|u|$, the quantity $\|u\| := \sqrt{\langle u, u\rangle}$.*

Definition 7.2 (Orthogonality) *Let E be a vector space and u and v be two vectors of E. u and v are orthogonal if $\langle u, v\rangle = 0$.*

Definition 7.3 (Orthogonal basis) *Let E be a vector sub-space and $(u_i)_{i\in[[1,p]]}$ a basis of E. If $\langle u_i, u_j\rangle = 0$, for all $i \neq j$ in $[[1,p]]$ then $(u_i)_{i\in[[1,p]]}$ is a* orthogonal basis.

If moreover $\|u_i\| = 1$, for all i in $[[1,p]]$ then the basis is said to be orthonormal.

7.2 Geometric interpretation

The norm denotes the length of a vector. We can define the angle between two vectors u and v as

$$\theta := \arccos\left(\frac{u \cdot v}{\|u\|\,\|v\|}\right).$$

With this definition we recover the usual identity

$$u \cdot v = \|u\|\,\|v\|\,\cos(u,v).$$

Two vectors are orthogonal if the angle between them is 0.

7.3 Properties

Proposition 7.4 (Pythagore theorem) *Let E be a vector space and u and v be two vectors of E. u and v are orthogonal if and only if $\|x+y\|^2 = \|x\|^2 + \|y\|^2$.*

Proof

$$\begin{aligned} \|v+w\|^2 &= \langle v+w, v+w\rangle = \langle v,v\rangle + \langle v,w\rangle + \langle w,v\rangle + \langle w,w\rangle \\ &= \|v\|^2 + \|w\|^2 \end{aligned}$$

Q.E.D.

Proposition 7.5 (Cauchy-Schwarz inequality) *Let E be a vector space and u and v be two vectors of E. Then $|\langle u,v\rangle| \leq \|u\|.\|v\|$.*

Proof
Step 1 : if $v = 0$, we have

$$\langle 0,w\rangle = \langle 0+0,w\rangle = \langle 0,w\rangle + \langle 0,w\rangle = 2\,\langle 0,w\rangle.$$

Whence $\langle 0,w\rangle = 0$.
Step 2 : We write $w = \alpha v + u$ with , avec $\alpha \in \mathbb{R}$ and u orthogonal to v. Observe that que $w = \alpha v + u Leftrightarrow u = w - \alpha v$. Whence,

$$0 = \langle v,u\rangle = \langle v, w - \alpha v\rangle = \langle v,w\rangle - \alpha\langle v,v\rangle$$

Choose $\alpha = \frac{\langle v,w\rangle}{\|v\|^2}$. We can apply Pythagore theorem (Proposition 7.4) :

$$\|w\|^2 = \|\frac{\langle v,w\rangle}{\|v\|^2}v\| + \|u\|^2 \geq \frac{\|\langle v,w\rangle\|^2}{\|v\|^4}\|v\|^2 = \frac{\|\langle v,w\rangle\|^2}{\|v\|^2}.$$

We obtain

$$\|v\|^2\|w\|^2 \geq \|\langle v,w\rangle\|^2$$

and so $\|v\|\|w\| \geq \|\langle v,w\rangle\|$. Q.E.D.

Remark 7 *There is equality in the Cauchy-Scwartz inequality if and only if there exists β such that $v = \beta w$.*

Proposition 7.6 (Minkowski inequality, also known as triangular inequality) *Let E be a vector space and u and v be two vectors of E. Then $\|x+y\| \leq \|x\| + \|y\|$.*

Proof Let us compute

$$\begin{aligned}\|v+w\|^2 &= \langle v+w, v+w\rangle = \langle v,v\rangle + 2\,\langle v,w\rangle + \langle w,w\rangle \\ &\leq \|v\|^2 + 2\,\langle v,w\rangle + \|w\|^2 \leq \|v\|^2 + 2\,\|v\|\|w\| + \|w\|^2 = (\|v\| + \|w\|)^2.\end{aligned}$$

Whence $\|v+w\| \leq \|v\| + \|w\|$. Q.E.D.

Proposition 7.7 (Parallelogramme rule) *Let E be a vector space and u and v be two vectors of E. Then $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$.*

Proof Left to the reader.

Proposition 7.8 (Polar identity) *Let E be a vector space and u and v be two vectors of E. Then $\|x+y\|^2 - \|x-y\|^2 = 2\langle x,y\rangle + 2\langle y,x\rangle$.*

Proof Left to the reader.

Exercise 25 *Let*

$$\begin{aligned} f : \mathbb{R}^3 \times \mathbb{R}^3 &\rightarrow \mathbb{R} \\ (x_1,x_2,x_3),(y_1,y_2,y_3) &\mapsto x_1y_1 + 5x_2y_2 + 6x_3y_3 - x_1y_2 - x_2y_1 + x_3y_1 + x_1y_3 - 3x_3y_2 - 3x_2y_3 \end{aligned}$$

1. *Prove that f is a bilinear form.*
2. *Determine the matricial representation of f in the canonical basis.*
3. *Determine the matricial representation of f in the basis $\{(1,1,0),(1,0,1),(0,0,1)\}$.*

4. *Determine the quadratic form Q associated to f.*
5. *Determine the matricial representation of f in the canonical basis.*
6. *Determine the matricial representation of f in the basis $\{(1,1,0),(1,0,1),(0,0,1)\}$.*

Exercise 26 *Let*

$$\begin{array}{rcl} f : \mathbb{R}^2 \times \mathbb{R}^2 & \to & \mathbb{R} \\ (x_1,x_2),(y_1,y_2) & \mapsto & x_1y_1 + 5x_2y_2 - x_1y_2 + x_2y_1 \end{array}$$

1. *Prove that f is a bilinear form.*
2. *Determine the matricial representation of f in the canonical basis.*
3. *Determine the matricial representation of f in the basis $\{(1,1),(1,0)\}$.*
4. *Determine the quadratic form Q associated to f.*
5. *Determine the matricial representation of f in the canonical basis.*
6. *Determine the matricial representation of f in the basis $\{(1,1),(1,0)\}$.*

Exercise 27 *Let*

$$g : x_1y_1 + x_2y_2 + 4x_4y_4 + x_1y_2 + x_2y_1 - 2x_1y_4 - 2x_4y_1 + x_2y_3 + x_3y_2 - 2x_3y_4 - 2x_4y_3\ .$$

1. *Prove that g is a bilinear form.*
2. *Determine the matricial representation of g in the canonical basis.*
3. *What is the signture of g ?*

Exercise 28 *Are the folowing scalar products (positive definite symmetric bilinear form) ?*

1.
$$(x_1,x_2,x_3),(y_1,y_2,y_3) \mapsto 2x_1y_1 + 5x_2y_2 + x_3y_3 - 3x_1y_2 - 3x_2y_1$$

2.
$$(x_1,x_2,x_3),(y_1,y_2,y_3) \mapsto x_1y_1+x_2y_2+x_3y_3-2x_1y_2-2x_2y_1-2x_1y_3-2x_3y_1-2x_2y_3-2x_3y_2$$

3.
$$(x_1,x_2,x_3,x_4),(y_1,y_2,y_3,y_4) \mapsto x_1y_2+x_2y_1-2x_1y_3-2x_3y_1+x_2y_3+x_3y_2-2x_3y_4-2x_4y_3$$

7.4 Orthogonal projection

Definition 7.9 (Projection) *Let E be a vector space. A* projection *is a linear mapping $p : E \to E$ such that $p^2 = p$.*

Definition 7.10 (Orthogonal projection) *Let E be a vector space, p be a projection and P its matricial representation. The projection p is an* orthogonal projection *if $P = P^T$.*

Projection on a line : If u is a vector, then the projection on the line $\mathrm{Span}(u)$ is given by

$$P_u = \frac{1}{\|u\|^2} u\, u^T.$$

This operator leaves u invariant, and it annihilates all vectors orthogonal to u.
Projection on a vector sub-space : If $(u_i)_{i\in[[1,p]]}$ is a basis, and A is the matrix representing theses vectors, then the projection on the space $\mathrm{Span}(\mathrm{u}_1,\cdots,\mathrm{u}_\mathrm{p})$ is given by $P_A = (A^tA)^{-1}AA^T$.
Distance to a straigh line : Let E be a vector space, F a vector sub-space, p be a projection on F and u be a vector of E. The orthonormal projection $p(x)$ of x satisfies the following minimising property :

$$\forall y \in F, \|x - p(x)\| \leq \|x - y\|$$

and there is egality if and only if $x = y$. $p(x)$ is thus the closest point of x which belongs to F. The distance $\|x - p(x)\|$ is called the distance from x to F.