Dynamic Equilibrium with Rare Disasters and Heterogeneous Epstein-Zin Investors*

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Abstract

We consider a general equilibrium Lucas (1978) economy with one consumption good and two heterogeneous Epstein-Zin investors. The output is subject to rare disasters or, more generally, can have non-lognormal distribution with higher cumulants. We demonstrate that the heterogeneity in preferences generates excess stock return volatilities, procyclical price-dividend ratios and interest rates, and countercyclical market prices of risk when the elasticity of intertemporal substitution (EIS) is greater than one. We show that the latter results cannot be jointly replicated in a model where investors have $EIS \leq 1$. Our model produces endogenous time-variation in equilibrium processes without assuming time-varying probabilities of disasters, as in the recent literature with homogeneous investors. We propose new approach for finding general equilibrium, and characterize optimal portfolios and consumptions in terms of tractable backward equations. Finally, we extend analysis to the case of heterogeneous beliefs about disaster probabilities.

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Rare economic disasters such as unexpected large decreases of aggregate output lead to significant welfare losses, and mere anticipation of disasters can have significant effect on asset prices in normal times. In particular, growing economic literature argues that accounting for the effects of disasters in general equilibrium models helps reconcile the magnitudes and properties of equity premia, risk free rates and other equilibrium processes with empirical findings [e.g., Rietz (1988); Barro (2006, 2009); Gabaix (2012), among others]. The extant literature primarily studies economies with homogeneous investors. Despite the fact that the heterogeneity in preferences is a salient feature of financial markets, its implications for equilibrium and risk sharing in the presence of rare disasters are relatively unexplored. In this paper, we demonstrate that heterogeneity in Epstein-Zin preferences can generate excess stock return volatility and asset prices dynamics consistent with the empirical literature.

We consider Lucas (1978) economy with one consumption good generated by a Lucas tree and two investors with Epstein-Zin preferences, which differ in their risk aversions and elasticities of intertemporal substitutions. The tree is subject to rare disasters, which lead to large decreases in aggregate consumption. More generally, our model allows for non-lognormal distributions with higher cumulants for the aggregate consumption process. The financial market is complete, and the investors share risks by trading in a riskless bond, one stock, which is a claim to the aggregate consumption, and insurances against rare disasters. In our baseline analysis, we consider the case when investors differ in risk aversions but have the same EIS, and explore the heterogeneity in EIS separately. In particular, we compare the cases when EIS is greater than unity, less than unity, and when both investors have constant relative risk aversion (CRRA) preferences, in which case EIS is a reciprocal of the risk aversion.

The equilibrium processes are derived as functions of the consumption share of the more risk averse investor, which is the only state variable in the model. This variable turns out to be countercyclical in the sense that it is negatively correlated with the changes in aggregate consumption in normal times when there are no disasters. Intuitively, positive shocks to aggregate consumption benefit the less risk averse investors because these investors hold larger fraction of wealth in stocks, and hence the consumption share of the more risk averse investor decreases, and the opposite happens when the dividend shock is negative. The heterogeneity in preferences gives rise to asset prices dynamics similar to those in models with homogeneous investors and time-varying probabilities of disasters [e.g., Gabaix (2012); Wachter (2013)]. However, the time-variation in our model is endogenously generated by risk sharing effects, and hence provides a new channel for explaining the dynamics of asset prices. Below, we discuss our main results.

First, we study the effect of rare events and heterogeneity in preferences for riskless rates and Sharpe ratios in the economy. We demonstrate that when investors have homogeneous EIS parameter the interest rates are procyclical and Sharpe ratios are countercyclical in normal times, consistent with the empirical literature. That is, the latter decreases and the former increases when aggregate consumption shocks are positive, and vice versa when the shocks are
negative. Moreover, the anticipation of disasters makes these processes more sensitive to shocks. For the case of CRRA preferences we observe similar dynamic patterns only in the presence of disasters, whereas without disasters, counterfactually, riskless rates become countercyclical. Furthermore, disaster risk decreases the interest rates and increases Sharpe ratios and bring their magnitudes in line with the empirically observed ones, as in the previous literature [e.g., Rietz (1988); Barro (2006); Gabaix (2012), among others]. Intuitively, as the consumption share of the more risk averse investor increases, Sharpe ratio increases to provide adequate compensation for risk whereas the interest rate decreases due to precautionary savings as this investor “flies to quality”. We also find that the dependence of Sharpe ratios on EIS is very weak, and they look almost the same for different levels of EIS.

Second, we demonstrate that the stock price-dividend ratios are procyclical when EIS > 1, consistent with the literature [e.g., Campbell and Cochrane (1999)], countercyclical when EIS < 1, and have mixed pattern when investors have CRRA preferences. Furthermore, the anticipation of disasters decreases price-dividend ratios when EIS > 1, and increases them when EIS < 1 and when investors have CRRA preferences. Dynamic properties of price-dividend ratios determine those of stock return volatilities. We focus on the volatilities in normal times, that is, between rare disasters. We show that stocks are more volatile than dividends when EIS > 1, consistent with the data [e.g., Shiller (1981)], less volatile when EIS < 1, and the pattern is mixed when investors have CRRA preferences. Intuitively, when EIS > 1 stock return volatility increases because the procyclical price-dividend ratio and the dividend change in the same direction amplifying the effects of each other. The opposite happens when EIS < 1 when price-dividend ratio is countercyclical. The anticipation of disasters strengthens the dynamic patterns and quantitative effects for all processes by increasing their sensitivity to shocks.

Third, we explore the portfolio choice and risk sharing of investors. Less risk averse investor invests more wealth in stocks than the more risk averse investor. The fraction of wealth that the less risk averse investor allocates to stocks depends on the amount of liquidity available for borrowing, similarly to the related literature on heterogeneous investors [e.g., Longstaff and Wang (2012); Chabakauri (2013), among others]. The leverage is highest when the economy is dominated by the more risk averse investor because this investor holds large fraction of wealth in riskless assets, and hence provides liquidity at low cost. We also derive investors’ holdings of insurance contracts and demonstrate that in equilibrium it is the more risk averse investor that sells the insurance to the less risk averse investor. This happens because the less risk averse investor holds a very large fraction of wealth in stocks, and hence needs the insurance the most. Dieckmann and Gallmeyer (2005) find a similar result in a simpler model where one investor is logarithmic while the other has CRRA preferences with risk aversion of either 2 or 0.5.

Fourth, we derive closed-form solutions in the economy where investors have identical risk aversions but different EIS. We demonstrate that differences in EIS affect the interest rates but have no impact on Sharpe ratios even in the presence of disasters. Moreover, all the equilib-
rium processes are deterministic functions of time. Based on our analysis we conclude that the
difference in risk aversions is the main source of risk sharing between the investors. This result
explains the finding that Sharpe ratios only weakly respond to changes in EIS in the general
model, as discussed above. We note that Isaenko (2008) and Gärleanu and Panageas (2012) have
similar result on the irrelevance of heterogeneity in EIS, but only in settings without rare events.

Finally, we consider an extension of the model in which the investors can additionally disagree
about the probability of disasters in the economy. Such disagreement may naturally arise due
to the fact that the probability of disasters is difficult to estimate because these events are rare.
Introducing the heterogeneity of beliefs improves the performance of the model. In particular,
making the more risk averse investor pessimistic further decreases the interest rates, increases
Sharpe ratios and stock return volatilities. When EIS > 1 it also makes the equilibrium processes
more sensitive to economic shocks.

We also contribute to the literature by proposing a new method for solving models with
heterogeneous Epstein-Zin investors and rare events. We consider a discrete-time model with
finite horizon and aggregate consumption following a multinomial process, which converges to a
continuous-time Lévy process as the interval between two dates shrinks to zero. The discreteness
of time allows for distributions with non-lognormal returns and makes the analysis more tractable.
In the calibration we take the horizon equal to 200 years, and the interval between dates is one
trading day. The model also permits taking continuous-time limits, which we derive in some
special cases. First, we solve for the partial equilibrium, in which investors take asset prices as
given. We characterize optimal consumption and portfolio policies in terms of investor’s wealth-
consumption ratio, which satisfies a discrete backward equation. Time-$t$ solution of this equation
is an explicit function of the solution at the previous step, and hence is found simply by backward
iteration without solving any nonlinear equations. In continuous time, the wealth-consumption
ratios satisfy integro-differential equations, which are more difficult to solve.

Then, we solve for general equilibrium in the model. From the condition that the marginal
rates of substitution of investors should be equal when the financial market is complete we derive a
system of equations for consumption shares of the more risk averse investor in all one-period ahead
discrete states of the economy. This system of equation is easily solved by Newton’s algorithm
[e.g., Judd (1998)]. Then, we characterize the state price density and all the equilibrium processes
in terms of these consumption shares. The tractability of our analysis comes from the fact the
multinomial process for aggregate consumption treats normal and disaster shocks equally. The
difference between these shocks emerges only in continuous-time limit. We note that discrete-
time models are commonly employed economics [e.g., Mehra and Prescott (1985); Rietz (1988);
He (1991), among others].

There is growing economic literature that studies the effects of rare disasters in general
equilibrium. In particular, the literature explores the implications of the idea of Rietz (1988)
that the anticipation of rare disasters can explain the equity premium puzzle of Mehra and

Ma (1993) provides a theoretical analysis of models with heterogeneous Epstein-Zin investors and derives conditions for the existence and uniqueness of equilibrium. Dieckmann and Gallmeyer (2005) consider a model similar to ours, but with CRRA investors, where one investor has logarithmic preferences while the other has risk aversion of 2 or 0.5. Dieckmann (2011) considers a similar model with incomplete markets. Chen, Joslin, and Tran (2012) consider an economy where investors have heterogeneous beliefs regarding the intensity of disasters in the economy. They also provide caution for extracting probabilities from asset prices, and demonstrate that ignoring the risk sharing of heterogeneous investors substantially underestimates the disaster risk. Backus, Chernov, and Martin (2011) and Julliard and Ghosh (2012) provide the evidence that the probability of disasters might be smaller than the one estimated in previous literature. The probability of disasters then might not be sufficient to explain equity premia.


The paper is organized as follows. Section 1 discusses the economic setup and defines the equilibrium. In Section 2, discusses optimal consumption and portfolio choice in partial equilibrium, and then provides the characterization of equilibrium processes. In Section 3, we provide the results of calibrations, the analysis of equilibrium, and discuss the economic intuition. Section 4 extends the model to the case of heterogeneous beliefs. Section 5 concludes, and Appendix A provides the proofs.
tion and variance operators conditional on time-
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process (1) converges to a Lévy process when
motion and Poisson processes, respectively. Lemma A.1 in the Appendix demonstrates that
Prob(\omega = \omega_k |\text{disaster}) = \pi_k whereas normal states \omega_{n-1} and \omega_n have conditional probabilities
Prob(\omega = \omega_k |\text{normal}) = 0.5. Processes \omega_t and \omega_t are analogues of continuous-time Brownian motion and Poisson processes, respectively. Lemma A.1 in the Appendix demonstrates that process (1) converges to a Lévy process when \Delta t \to 0. Conveniently, \mathbb{E}_t[\Delta w_t |\text{normal}] = 0 and var\{\Delta w_t |\text{normal}\} = \Delta t, similarly to a Brownian motion, where \mathbb{E}_t[\cdot] and \text{var}_t[\cdot], denote expectation and variance operators conditional on time-t information, respectively.

The discreteness of time has several advantages. First, it is more realistic to assume that
investors make consumption and portfolio choice decisions discretely. Moreover, consumption data are not available at high frequencies. Second, it allows for modeling distributions with non-lognormal consumption processes. Such processes can be easily approximated by multinomial dynamics. Finally, we demonstrate in subsequent sections that the discrete-time model is more tractable, and allows for passing to continuous time. We also note, that discrete-time models with binomial or multinomial processes for consumption and asset prices are widely employed in the literature [e.g., Mehra and Prescott (1985); Rietz (1985); He (1991); Pliska (1997); Buss, Uppal, and Vilkov (2011); Dumas and Lysaoff (2012), among others]. However, the literature typically sets \( \Delta t = 1 \), whereas in our calibration we take much smaller \( \Delta t = 1/250 \) and long horizon \( T = 200 \) and also take continuous-time limits in various special cases.

### 1.1. Securities Markets

The financial market is complete, and the investors can trade \( n \) securities: a riskless bond in zero net supply, one stock in net supply of one unit, which is the claim to the stream of dividends generated by the Lucas tree, and \( n - 2 \) zero net supply catastrophe insurances each paying one unit of consumption in rare disaster states \( \omega_1, \ldots, \omega_{n-2} \), respectively. All trades happen at discrete dates \( t = 0, \Delta t, 2\Delta t, \ldots, T \). We consider Markovian equilibria in which bond prices, \( B \), ex-dividend stock prices, \( S \), and catastrophe insurance prices, \( P_k \), follow dynamics:

\[
\Delta B_t = B_t r_t \Delta t, \tag{3}
\]

\[
\Delta S_t + D_{t+\Delta t} \Delta t = S_t [m_{S,t} \Delta t + \sigma_{S,t} \Delta w_t + J_{S,t}(\omega) \Delta j_t], \tag{4}
\]

\[
\Delta P_{k,t} + 1_{\{\omega=\omega_k\}} = P_{k,t} [m_{P_k,t} \Delta t + \sigma_{P_k,t} \Delta w_t + J_{P_k,t}(\omega) \Delta j_t], \tag{5}
\]

where \( k = 1, \ldots, n - 2 \). The drift and volatility processes \( m_{S,t}, \sigma_{S,t}, m_{P_k,t}, \) and \( \sigma_{P_k,t} \), and jump sizes \( J_{S,t} \) and \( J_{P_k,t} \) are determined in equilibrium.

We denote the vector of drifts by \( m_t = (m_{S,t}, \ldots, m_{P_{n-2},t})^\top \), the vector of risky asset expected returns by \( \mu_t = (\mu_{S,t}, \mu_{P_1,t}, \ldots, \mu_{P_{n-2},t})^\top \), and the volatility matrix by \( \Sigma = (\Sigma_S, \Sigma_{P_k}, \ldots, \Sigma_{P_{n-2}})^\top \in \mathbb{R}^{(n-1)\times(n-1)} \), where \( \Sigma_S = (\sigma_{S,t}, J_{S,t}(\omega_1), \ldots, J_{S,t}(\omega_{n-2}))^\top \), \( \Sigma_{P_k} = (\sigma_{P_k,t}, J_{P_k,t}(\omega_1), \ldots, J_{P_k,t}(\omega_{n-2}))^\top \), for \( k = 2, \ldots, n - 2 \). We note that expected risky asset returns are given by \( \mu_t = m_t + \lambda \Sigma_t (0, \pi_1, \ldots, \pi_{n-2})^\top \). Finally, we define the state price density \( \xi_t \) as a strictly positive process such that asset prices have representations

\[
B_t = E_t \left[ \frac{\xi_{t+\Delta t}}{\xi_t} B_{t+\Delta t} \right], \tag{6}
\]

\[
S_t = E_t \left[ \frac{\xi_{t+\Delta t}}{\xi_t} (S_{t+\Delta t} + D_{t+\Delta t} \Delta t) \right], \tag{7}
\]

\[
P_{k,t} = E_t \left[ \frac{\xi_{t+\Delta t}}{\xi_t} (P_{k,t+\Delta t} + 1_{\{\omega=\omega_k\}}) \right], \tag{8}
\]

\(^1\)We note that the multinomial dynamics for stock prices and catastrophe insurances can always be written as processes (4)-(5) with \( \Delta t, \Delta w, \) and \( \Delta j \) terms because the vector of time-\((t+\Delta t)\) asset returns in states \( \omega_1, \ldots, \omega_n \) can be uniquely decomposed as a linear combination of vectors \( (\Delta t, \ldots, \Delta t)^\top, (\sqrt{\Delta t}, -\sqrt{\Delta t}, 0, \ldots, 0)^\top, \) and \( n - 2 \) linearly independent vectors of the form \((0, 0, \ldots, 1, \ldots, 0)^\top\)\. 

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where \( S_T = 0 \) and \( P_{k,T} = 0 \) because \( D_T = 0 \) and \( D_{k,T} = 0 \) for \( \tau > T \).

### 1.2. Investor Optimization

The investors have recursive utility \( U_t \) over consumption \( c_{i,t} \) [e.g., Epstein and Zin (1989)], which satisfies the following backward equation:

\[
U_{i,t} = \left[ (1 - e^{-\rho \Delta t}) c_{i,t}^{-1/\psi_i} + e^{-\rho \Delta t} \left( E_t[U_{i,t+\Delta t}^{1-\gamma_i}] \right)^{1/(1-\gamma_i)} \right]^{1-1/\psi_i},
\]

where \( i = A, B \), \( \gamma_i \) and \( \psi_i \) denote investor \( i \)'s risk aversion and elasticity of intertemporal substitution (EIS), respectively, and \( \rho > 0 \) is a time-discount parameter. In general, the investors have different risk aversions and EIS. Each period, investor \( i \) allocates fractions \( \alpha_{i,t} \) and \( \theta_{i,t} = (\theta_{i,s,t}, \theta_{i,p_1,t}, \ldots, \theta_{i,p_{n-2},t})^T \) of wealth \( W_{i,t} \) to riskless bonds and risky securities, respectively, so that \( W_{i,t} = \alpha_{i,t} W_{i,t} + \theta_{i,t}^\top W_{i,t} + c_{i,t} \Delta t \), where \( 1 = (1, \ldots, 1)^T \in \mathbb{R}^{n-1} \). At time 0 investor \( A \) is endowed with \( b \) units of bond, \( s \) units of stock, and \( p_k \) units of catastrophe insurance \( k \), whereas investor \( B \) is endowed with \(-b\) units of bond, \(-s\) units of stock, and \(-p_k\) units of insurance \( k \). Investors solve the following dynamic programming problem [e.g., Epstein and Zin (1989)]:

\[
V_{i,t} = \max_{\alpha_{i,t}, \theta_{i,t}} \left[ (1 - e^{-\rho \Delta t}) c_{i,t}^{-1/\psi_i} + e^{-\rho \Delta t} \left( E_t[V_{i,t+\Delta t}^{1-\gamma_i}] \right)^{1/(1-\gamma_i)} \right]^{1-1/\psi_i},
\]

where \( V_{i,t} \) is investor \( i \)'s value function, subject to a self-financing dynamic budget constraint:

\[
\Delta W_{i,t} = W_{i,t} (r_t + \theta_{i,t}^\top (m_t - r_t)) \Delta t + W_{i,t} \theta_{i,t}^\top \sum \Delta \tilde{w}_t - c_{i,t} \Delta t (1 + r_t \Delta t), \quad W_T = c_{i,T} \Delta t,
\]

where \( i = A, B \), \( \Delta \tilde{w} = (\Delta w, 1_{(\omega=\omega_1)}, \ldots, 1_{(\omega=\omega_{n-2})})^T \), and \( 1_{(\omega=\omega_k)} \) is an indicator function.

### 1.3. Equilibrium

**Definition.** An equilibrium is a set of processes \( \{r_t, \mu_t, \Sigma_t\} \), and of consumption and investment policies \( \{c^*_t, \alpha^*_t, \theta^*_t\}_{t \in (A,B)} \) which solve dynamic optimization problem (10) for each investor, given processes \( \{r_t, \mu_t, \Sigma_t\} \), and consumption and securities markets clear, that is,

\[
c_{A,t}^* + c_{B,t}^* = D_t, \quad \alpha_{A,t}^* W_{A,t}^* + \alpha_{B,t}^* W_{B,t}^* = 0, \quad \theta_{A,S,t}^* W_{A,t}^* + \theta_{B,S,t}^* W_{B,t}^* = S_t, \quad \theta_{A,P,t}^* W_{A,t}^* + \theta_{B,P,t}^* W_{B,t}^* = 0,
\]

\(^2\)Gärleanu and Panageas (online appendix, 2012) and Skiadas (2012) consider similar formulations with arbitrarily small interval \( \Delta t \) between dates and derive continuous-time formulations by passing to limits \( \Delta t \to 0 \). Furthermore, the model in Skiadas (2012) allows for rare events, similarly to the present paper.

\(^3\)Investors’ time \( t \) and \( t+\Delta t \) wealths are given by \( W_t = \alpha_t W_t + \theta_{i,t}^\top W_{i,t} + c_t \Delta t \) and \( W_{t+\Delta t} = W_t + \alpha_t W_t + D_t + c_t \Delta t \) for each investor, respectively. Substituting \( \alpha_t W_t + \theta_{i,t}^\top W_{i,t} + c_t \Delta t \) into the latter equation and using asset price dynamics (4)–(5) we obtain budget constraint (11).
where $k = 1, \ldots, n - 2$, and $W_{A,t}^*$ and $W_{B,t}^*$ denote wealths under optimal strategies.

In addition to asset returns $\mu$, we also study their risk premia $\mu - r$. We also derive stock price-dividend and wealth-consumption ratios $\Psi = S/D$ and $\Phi_t = W_t^*/c_t^*$, respectively. We derive a Markovian equilibrium in which the consumption share $y = c_n^*/D$ of investor $B$ in aggregate consumption $D$ is an endogenous state variable, as in the related literature [e.g., Chen, Joslin, and Tran (2012), Gărzeanu and Pedersen (2012); among others]. We demonstrate later that in a Markovian equilibrium consumption share $y_t$ follows a process

$$\Delta y_t = y_t[m_{y,t} \Delta t + \sigma_{y,t} \Delta w_t + J_{y,t}(\omega) \Delta j_t],$$

where the drift $m_y$, volatility $\sigma_y$, and jump sizes $J_{y,t}(\omega)$ are determined in equilibrium.

Throughout the paper we restrict preferences and Lucas tree parameters $\gamma_i$, $\psi_i$, $\rho$, $m_D$, $\sigma_D$, $J_D(\omega)$, $\lambda$, $\pi_k$, and $\Delta t$ to be such that the following two technical conditions are satisfied:

$$g_{i,1} \equiv e^{-\rho \Delta t} \left( E_t \left[ (D_t^{\Delta t})^{-\gamma_i} \right] \right)^{1/(1-\gamma_i)} < 1,$$

$$g_{i,2} \equiv e^{-\rho \Delta t} \left( E_t \left[ (D_t^{\Delta t})^{-\gamma_i} \right] \right)^{\gamma_i/(1-\gamma_i)} \left( \frac{\varpi_t[\Delta \hat{w}_i]}{\varpi_t[\Delta \hat{w}_i]} \right)^{\gamma_i} \left( E_t \left[ (D_t^{\Delta t})^{-\gamma_i} \right] \right) < 1,$$

where $i = A, B$, and $g_{i,1}$ and $g_{i,2}$ are constants. We demonstrate in Section 2.2 that under these conditions the equilibrium processes are bounded as $T \to \infty$ in homogeneous agent economies, and hence are analogous to dividend growth restrictions in continuous-time models.

2. Characterization of Equilibrium

2.1. Consumption and Portfolio Choice

In this section, we derive optimal investment and consumption policies of investors in a partial equilibrium, that is, taking the asset prices dynamics (3)–(5) as given. We propose new method for solving problems with non-lognormal asset returns, which retains the tractability and the structure of asset allocation rules in continuous-time settings. For the time being, we do not take a stand on state variables in the economy, and assume that processes $r_t$, $m_t$, and $\Sigma_t$ are functions of an unspecified Markovian variable $z_t$. We start by deriving the discrete-time process for state price density $\xi_t$ in terms of processes $r_t$, $\mu_t$, and $\Sigma_t$. Lemma 1 reports the result.

**Lemma 1 (State Price Density).** The state price density $\xi_t$ follows a multinomial process

$$\Delta \xi_t = -\frac{\xi_t}{1 + r_t \Delta t} \left[ r_t \Delta t + \left( \Sigma_t^{-1}(\mu_t - r_t \mathbb{1}) \right)^\top \left( \frac{1}{\Delta t} \varpi_t[\Delta \hat{w}_i] \right)^{-1} \left( \Delta \hat{w}_i - E_t[\Delta \hat{w}_i] \right) \right],$$

where $\mu_t = m_t + \Sigma_t E_t[\Delta \hat{w}_i]/\Delta t$ is the vector of expected risky asset returns, $\mathbb{1} = (1, \ldots, 1)^\top \in \mathbb{R}^{n-1}$, and $\Delta \hat{w}_i$, $E_t[\Delta \hat{w}_i]$, and $\varpi_t[\Delta \hat{w}_i]$ are given by

$$\Delta \hat{w}_i = (\Delta \hat{w}_t, 1_{(\omega=\omega_1)}, \ldots, 1_{(\omega=\omega_{n-2})})^\top,$$
\[ E_t[\Delta \tilde{w}_t] = (0, \lambda \pi_1, \ldots, \lambda \pi_{n-2})^\top \Delta t, \]
\[ \text{var}_t[\Delta \tilde{w}_t] = \text{diag}\{1 - \lambda \Delta t, \lambda \pi_1, \ldots, \lambda \pi_{n-2}\} \Delta t - E_t[\Delta \tilde{w}_t]E_t[\Delta \tilde{w}_t]^\top, \]

where \( 1_{\{w=w_k\}} \) is an indicator function and \( \text{diag}\{1 - \lambda \Delta t, \lambda \pi_1, \ldots, \lambda \pi_{n-2}\} \) is a diagonal matrix.

The state price density process (19) preserves the structure of the familiar continuous-time process for \( \xi \) when there is no disaster risk. In particular, as in continuous-time, the drift and volatility terms of process (19) are driven by the interest rate \( r_t \) and the market prices of risk \( \Sigma^{-1}(\mu_t - r_t) \), respectively. In a model without disasters \( \Delta \tilde{w}_t = \Delta w_t, E_t[\Delta \tilde{w}_t] = 0, \text{var}[\Delta \tilde{w}_t] = \Delta t \), and hence from equation (19) we obtain dynamics \( \Delta \xi_t = -\xi_t[r_t \Delta t + (\mu_t - r_t)/\sigma_t \Delta w_t]/(1 + r_t \Delta t) \). Consequently, as \( \Delta t \to 0 \), the dynamics for \( \xi_t \) converges (under some technical assumptions) to the well-known process \( d\xi_t = -\xi_t[r_t \, dt + (\mu_t - r_t)/\sigma_t \, dw_t] \) [e.g., Duffie (2001)], where \( \mu_t \) and \( \sigma_t \) are stock mean-return and volatility, respectively.

Next, we derive optimal consumption and investment policies by solving dynamic programming problem (10). In particular, from the first order conditions we obtain the expressions for stock mean-return and volatility, respectively.

**Proposition 1 (Optimal Consumption and Investment Policies).** The investors’ wealth-consumption ratios \( \Phi_{i,t} = \Phi_i(z_{t,t}, t) \) satisfy backward equations

\[ \Phi_{i,t} = e^{-\rho \psi_{i,t} \Delta t} \left( \mathbb{E}_t \left[ \left( \frac{\xi_t + \Delta t}{\xi_t} \right)^{\gamma_i - 1} \Phi_{i,t+\Delta t} \right] \right)^{\frac{\gamma_i (1-\rho)}{\gamma_i - 1}} + \Delta t, \quad \Phi_{i,T} = \Delta t. \]  

The value functions \( V_{i,t} \), consumption growths \( c_{i,t+\Delta t}/c_{i,t} \) and portfolio weights \( \theta_{i,t}^* \) are given by:

\[ V_{i,t} = \left( \Phi_{i,t} - e^{-\rho \Delta t} \right) \frac{1}{\rho \psi_{i,t}} W_{i,t}, \]
\[ c_{i,t+\Delta t} = e^{-\psi_{i,t} \rho \Delta t} \left( \frac{\xi_t + \Delta t}{\xi_t} \right)^{-\gamma_i} \left( \Phi_{i,t+\Delta t} \right)^{\frac{\gamma_i - 1}{\gamma_i - 1}} \mathbb{E}_t \left[ \left( \frac{\xi_t + \Delta t}{\xi_t} \right)^{\gamma_i - 1} \left( \Phi_{i,t+\Delta t} \right)^{\frac{\gamma_i - 1}{\gamma_i - 1}} \right], \]
\[ \theta_{i,t}^*(z_{t,t}) = (\Sigma_t^{-1})^\top \mathbb{E}_t \left[ \Phi_{i,t+\Delta t} \frac{c_{i,t+\Delta t}}{c_{i,t}^*} \text{var}_t[\Delta \tilde{w}_t]^{-1} \left( \Delta \tilde{w}_t - E_t[\Delta \tilde{w}_t] \right) \right], \]

where \( i = A, B \), \( \Delta \tilde{w}_t \), \( E_t[\Delta \tilde{w}_t] \), and \( \text{var}_t[\Delta \tilde{w}_t] \) are given by equations (20)–(22). Furthermore, the state price density \( \xi_t \) is related to consumption growths \( c_{i,t+\Delta t}/c_{i,t} \) as follows:

\[ \frac{\xi_t + \Delta t}{\xi_t} = e^{-\rho \Delta t} \left( \frac{c_{i,t+\Delta t}}{c_{i,t}^*} \right)^{-\gamma_i} \left( \Phi_{i,t+\Delta t} \right)^{\frac{\gamma_i - 1}{\gamma_i - 1}} \mathbb{E}_t \left[ \left( \frac{c_{i,t+\Delta t}}{c_{i,t}^*} \right)^{1-\gamma_i} \left( \Phi_{i,t+\Delta t} \right)^{\frac{\gamma_i - 1}{\gamma_i - 1}} \right], \quad \Delta t \to 0. \]

and marginal rates of substitution, defined as \( MRS_{i,t+\Delta t}(\omega_k) = \left( \partial U_{i,t}/\partial c_{i,t+\Delta t}(\omega_k) \right) / \left( \partial U_{i,t}/\partial c_{i,t} \right) \), are given by \( MRS_{i,t+\Delta t}(\omega_k) = (1 - \lambda)\pi_k \xi_{t+\Delta t}(\omega_k)/\xi_t \).
Equations (23)-(26) demonstrate that consumption and portfolio choice problem can be solved by backward induction. In particular, wealth-consumption ratio $\Phi_{i,t}$ can be obtained by backward induction starting from the terminal date $t = T$. In particular, $\Phi_{i,t}$ appears to be an explicit function of $\Phi_{i,t+\Delta t}$ from the previous step, and hence its calculation does not require solving any equations. The wealth-consumption ratios can then be used to calculate consumption growths $c_{i,t+\Delta t}/c_{i,t}$ in all states $\omega_1, \ldots, \omega_n$ from equation (25). The growth rates $c_{i,t+\Delta t}/c_{i,t}$ are then used to calculate portfolio weights $\theta_{i,t}$ from equations (26).

We note that all the equations in Proposition 1 significantly simplify when the investors have CRRA preferences. In particular, substituting $c_{i,t+\Delta t}/c_{i,t}$ from (25) into portfolio weight (26), after simple algebra, we obtain the following expression

$$\theta_{i,t}^* = e^{-\rho/\gamma_i \Delta t} \left( \frac{\xi_{t+\Delta t}}{\xi_t} \right)^{-\frac{1}{\gamma_i}} \text{cov}_t \left( \frac{\xi_{t+\Delta t}}{\xi_t}, \text{var}_t[\Delta \tilde{w}_t]^{-1} \left( \Delta \tilde{w}_t - \mathbb{E}_t[\Delta \tilde{w}_t] \right) \right)$$

$$+ e^{-\rho/\gamma_i \Delta t} \left( \frac{\xi_{t+\Delta t}}{\xi_t} \right)^{-\frac{1}{\gamma_i}} \text{var}_t[\Delta \tilde{w}_t]^{-1} \left( \Delta \tilde{w}_t - \mathbb{E}_t[\Delta \tilde{w}_t] \right) .$$

Optimal weight (28) preserves the structure of weights in continuous-time analysis. In particular, the first term in equation (28) can be interpreted as myopic demand and the second term as hedging demand, as in continuous-time portfolio choice [e.g., Merton (1973); Liu (2007)].

### 2.2. General Equilibrium

In this Section, we use the results of Proposition 1 to characterize the equilibrium. Equation (27) provides two expressions for the state price density $\xi_t$ in terms of investor $A$’s and investor $B$’s consumptions, respectively. Equating these expressions and substituting consumptions in terms of consumption share $y_t$, that is, $c_{A,t} = (1 - y_t)D_t$ and $c_{B,t} = y_tD_t$, we obtain the following system of equations from which $y_{t+\Delta t}$ can be obtained as a function of $y_t$ and state $\omega$:

$$\frac{\xi_{t+\Delta t}}{\xi_t} = e^{-\rho \Delta t} \left( \frac{1 - y_{t+\Delta t}}{1 - y_t} \left( \frac{D_{t+\Delta t}}{D_t} \right) \right)^{-\gamma_A} \Phi_{A,t+\Delta t}$$

$$= e^{-\rho \Delta t} \left( \frac{y_{t+\Delta t}}{y_t} \left( \frac{D_{t+\Delta t}}{D_t} \right) \right)^{-\gamma_B} \Phi_{B,t+\Delta t} .$$

Intuitively, equation (29) holds because investors’ marginal rates of substitution, derived in Proposition 1, are equal due to market completeness.

We solve equation (29) using Newton’s method [e.g., Judd (1998)], and derive time-$(t + \Delta t)$ consumption shares $y_{t+\Delta t}(y_t; \omega_k)$ in states $\omega_1, \ldots, \omega_n$. Substituting shares $y_{t+\Delta t}$ back into equation (29) we obtain state price density process $\xi_{t+\Delta t}/\xi_t$ as function of share $y_t$ and time-$(t + \Delta t)$ state $\omega$. Then, we use $\xi_{t+\Delta t}/\xi_t$ to obtain asset prices by solving equations (7)-(8).
backwards from the terminal date $T$. The equilibrium processes $r_t$, $\mu_t - r_t \perp$, and $\Sigma_t$ are determined as drifts and volatilities of processes $\xi_t$, $S_t$, and $P_{k,t}$, and are reported in Proposition 2 below.

**Proposition 2 (Equilibrium Processes).** Investors’ optimal time-$t$ portfolio weights and wealth-consumption ratios are functions of state variable $y_t = c^*_{n,t}/D_t$, and are given by equations (26)–(23) in which $c^*_{n,t} = (1 - y_t)D_t$ and $c_{n,t} = y_tD_t$. The equilibrium interest rate $r_t$, risk premium $\mu_t - r_t \perp$, and volatility matrix $\Sigma_t$ are functions of consumption share $y_t$, given by

$$
\begin{align*}
  r_t &= \left( \frac{1}{\mathbb{E}_t[\xi_t + \Delta \xi_t/\xi_t]} - 1 \right) \frac{1}{\Delta t}, \\
  \mu_t - r_t \perp &= \frac{\Sigma_t \text{cov}_t(\xi_t + \Delta \xi_t/\xi_t, \Delta \tilde{w}_t)}{\mathbb{E}_t[\xi_t + \Delta \xi_t/\xi_t] \Delta t}, \\
  \Sigma_t &= \mathbb{E}_t \left[ R_{t+\Delta t} \left( \text{var}_t[\Delta \tilde{w}_t]^{-1}(\Delta \tilde{w}_t - \mathbb{E}_t[\Delta \tilde{w}_t]) \right)^\top \right],
\end{align*}
$$

where $\Delta \tilde{w}$, $\mathbb{E}_t[\Delta \tilde{w}]$, and $\text{var}_t[\Delta \tilde{w}]$ are given by equations (20)–(22), ratio $\xi_{t+\Delta t}/\xi_t$ is given by equation (29), $1 = (1, \ldots, 1)^\top \in \mathbb{R}^{n-1}$, time-$(t + \Delta t)$ risky assets returns $R_{t+\Delta t}$ are given by

$$
R_{t+\Delta t} = \left( \frac{\Psi_{t+\Delta t} + \Delta t}{\Psi_t} D_{t+\Delta t}, \frac{P_{t+\Delta t} + D_{1,t+\Delta t}}{P_{1,t}}, \ldots, \frac{P_{n-2,t+\Delta t} + D_{n-2,t+\Delta t}}{P_{n-2,t}} \right)^\top - 1.
$$

Asset returns $R_{t+\Delta t}$ are functions of share $y_t$, time $t$, and state $\omega$. Catastrophe insurance prices $P_{k,t}$ and price-dividend ratio $\Psi_t$ are functions of share $y_t$ and time $t$. Furthermore, prices $P_{k,t}$ solve backward equations (8), whereas price-dividend ratio $\Psi_t$ solves backward equation

$$
\Psi_t = \mathbb{E}_t \left[ \xi_{t+\Delta t} \frac{D_{t+\Delta t}}{D_t} \left( \Psi_{t+\Delta t} + \Delta t \right) \right], \quad \Psi_T = 0.
$$

Similarly, consumption share $y_t$ follows process (16) where $m_{y,t} = \mathbb{E}_t[\Delta y_t/y_t|\text{normal}]/\Delta t$ and $(\sigma_{y,1}, J_y(\omega_1), \ldots, J_y(\omega_n))^\top = \mathbb{E}_t[y_{t+\Delta t}/y_t \text{var}_t[\Delta \tilde{w}_t]^{-1}(\Delta \tilde{w}_t - \mathbb{E}_t[\Delta \tilde{w}_t])]$.

Proposition 2 demonstrates how the equilibrium interest rates $r$, risk premia $\mu - r$, volatilities $\Sigma$, and price dividend ratios $\Psi$ can be derived given the state price density $\xi_t$, which is obtained from equation (29). To provide further intuition for the role of rare events, we obtain closed-form expressions for the equilibrium processes when investors have identical risk aversions, and when the economy is populated by homogeneous agents. When $\gamma_A = \gamma_B$ the analysis is simplified by the fact that aggregate consumption growths $D_{t+\Delta t}/D_t$ cancel out from both sides of equation (29) for consumption share $y_{t+\Delta t}$. Consequently, share $y_t$ turns out to be a deterministic function of time. As a result, all the other equilibrium processes are also deterministic processes. To provide tractable expressions, we pass to continuous time limit. Corollary 1 reports the results.

**Corollary 1 (Closed-Form Solutions).** 1) If the investors have the same risk aversion, that is $\gamma_i = \gamma$ for $i = A, B$, the equilibrium risk premium $\mu_{s,t} - r_t$ and the market prices of risk
Contribution share \(y_t\) is a deterministic function of \(t\), which solves the equation

\[
\left(\frac{y_t}{y_0}\right)^{1/\psi_B} = \left(\frac{1 - y_t}{1 - y_0}\right)^{1/\psi_A} \left(\frac{\mathbb{E}_t[\left(\frac{D_t + \Delta_t}{D_t}\right)^{-\gamma}]}{\mathbb{E}_0[\left(\frac{D_0 + \Delta_0}{D_0}\right)^{-\gamma}]}\right)^{1/\psi_A - 1/\psi_B} e^{\frac{\lambda \Delta t}{\gamma}}.
\]

(39)

2) If the economy is populated by investor \(i\) only, price-dividend ratio \(\Psi\) and insurance prices \(P_k\) are given by \(\Psi_i = \frac{1 - g_{i,1}(T-t)/\Delta t}{1 - g_{i,1}}\Psi_i \Delta t, \quad P_{k,t} = \frac{1 - g_{i,2}(T-t)/\Delta t}{1 - g_{i,2}}b_k \lambda \Delta t\)

where \(g_{i,1}\) and \(g_{i,2}\) are given by equations (17)–(18), and \(b_k\) is given in the Appendix. Ratio \(\Psi\) and prices \(P_k\) are bounded functions of \(T\) if \(g_{i,1} < 1\) and \(g_{i,2} < 1\), and as \(\Delta t \to 0\) converge to

\[
\Psi_i = \frac{1 - e^{-(r + (\mu_s - \mu_D)(T-t))/\gamma}}{r + (\mu_s - \mu_D) - \mu_D}, \quad P_{k,t} = \lambda \pi_k (1 + J_k)^{-\gamma} \frac{1 - e^{-(T-t)/\gamma}}{r}.
\]

(40)

where \(r\) is given by equation (35) with \(\psi_A = \psi_B = \psi_i, \quad \mu_D = m_D + \lambda \mathbb{E}_t[J_D(\omega)|\text{disaster}]\) is dividend expected growth rate, and \(\mu_s - \tau\) is given by equation (37).

Corollary 1 demonstrates that the heterogeneity in intertemporal elasticities of substitution affects only interest rates in the economy, whereas the market prices of risk, Sharpe ratio, and stock return volatility are unaffected by EIS, similarly to settings without rare events [e.g., Gårleau and Panageas (2012)]. We observe that risk premium (37) is an increasing function of intensity \(\lambda\), because \(J_\sigma < 0\). In the case of homogeneous-investor economy equilibrium processes (35)–(40) can be interpreted as boundary conditions for the case with heterogeneous investors, and provide good approximation for the equilibrium processes when the economy is dominated by either investor \(A\) (i.e., \(y \approx 0\)) or investor \(B\) (i.e., \(y \approx 1\)). Consequently, the intuition for the effect of EIS on rate \(r\) and price-dividend ratio \(\Psi\) in the homogeneous investor case provides insights about the general case. We provide this intuition below.

Setting \(\psi_A = \psi_B = \psi\) in equation (35) we obtain interest rate in a homogeneous-investor economy, which after some algebra can be rewritten as follows:

\[
r = \rho + \frac{1}{\psi} m_D - \frac{\gamma(1 + \psi)}{2\psi} \sigma_D^2 + \left(\frac{1}{\psi} - 1\right) \lambda \mathbb{E}_t\left[\frac{(1 + J_D)^{-\gamma} - 1}{1 - \gamma}\right] |\text{disaster}|
\]

\[
+ \lambda \mathbb{E}_t\left[(1 + J_D)^{-\gamma} J_D\right] |\text{disaster}|.
\]

(41)
Equation (41) highlights the effect of EIS $\psi$ on interest rate $r$. The second term in (41) captures the consumption smoothing effect. This term decreases with higher $\psi$ because investors with higher EIS tend to save more for consumption smoothing purposes, which pushes down the interest rates. The third term in (41) captures the effect of precautionary savings, and coefficient $\gamma(1+\psi)$ is investor’s prudence parameter for small risks $\Delta w_t$ [e.g., Kimball and Weil (2009)]. The impact of the third term diminishes with higher $\psi$ because the investor with higher EIS saves more for consumption smoothing, and hence has lower demand for precautionary savings. The last two terms in (41) quantify the effect of precautionary savings in the case of large disaster risks. Similarly, higher $\psi$ diminishes the impact of the last two terms. In particular, because $((1+J_\psi)^{1-\gamma}-1)/(1-\gamma) < 0$ and $(1+J_\psi)^{-\gamma}J_\psi < 0$, when $\psi > 1$ the fourth term is positive and the last term is negative, and hence these terms partially offset each other. Finally, from equation (41), after some algebra, we find that rare disasters unambiguously decrease interest rates.\footnote{We note that the sum of the last two terms in equation (41) is negative. This is because $(1/\psi - 1)\mathbb{E}_t[(1 + J_\psi)^{1-\gamma} - 1] < (1 - \gamma)] + \lambda\mathbb{E}_t[(1 + J_\psi)^{-\gamma} J_\psi] < 0$. The latter inequality holds because function $g(J_\psi) = ((1 + J_\psi)^{1-\gamma} - 1)/(1 - \gamma)$ is concave, and hence $g(J_\psi) \leq g(0) + g'(0)J_\psi$. Therefore, the interest rate $r$ in equation (41) is lower than that in the economy without disasters, i.e., when $\lambda = 0$.}

Next, substituting interest rate $r$ and equity premium $\mu_s - r$ from equations (41) and (37) into equation (40) for price-dividend ratio $\Psi$ we obtain:

$$\Psi = \frac{1}{\rho - \left(1 - \frac{1}{\psi}\right)\left(m_D - \frac{\gamma}{2}\sigma_D^2 + \lambda\mathbb{E}_t\left[\frac{(1 + J_\psi)^{1-\gamma} - 1}{1-\gamma}\right]|\text{disaster}\right)}.$$  

Equation (42) analytically demonstrates that the economic uncertainty, captured both by volatility $\sigma_D$ and the probability of disaster $\lambda$, increases (decreases) price-dividend ratio for $\psi < 1$ ($\psi > 1$), as also pointed out in the related literature [e.g., Bansal and Yaron (2004)].

### 3. Analysis of Equilibrium

In this Section, we derive equilibrium processes numerically and study the interaction between investor heterogeneity and rare disasters. Table 1 provides the exogenous parameters for the aggregate dividend process (1). We take dividend growth rates $m_D$ and volatilities $\sigma$ in normal times from Cambell (2003), and the probability of disaster $\lambda$ from Barro (2005). We also fix $\Delta t = 1/250$ and $T = 200$ so that the results are not affected by the discreteness and the horizon. For our calibrations we set investor risk aversions to $\gamma_A = 3$ and $\gamma_B = 6$, and compare the equilibrium processes for different values of EIS $\psi$.

Figure 2 reports equilibrium interest rates $r$, Sharpe ratios $\kappa$, excess volatilities $(\sigma_t - \sigma_D)/\sigma_D$, and price-dividend ratios $\Psi$ as functions of consumption share $y_t$ in the economy without rare disasters (solid lines) and with rare disasters (dashed line). Left, middle and right panels of Figure 2 show the equilibrium processes for the cases $\psi_A = \psi_B = 1.5$, $\psi_A = \psi_B = 0.5$ and CRRA preferences, respectively. For brevity, we do not report drifts and volatilities of process (16) for...
Table 1
Parameters of Aggregate Consumption Process
Table 1 presents the exogenous parameters of the dividend process (1). dividend growth rates $m_D$ and volatilities $\sigma_D$ in normal times from Cambell (2003), and the probability of disaster $\lambda$ from Barro (2005).

<table>
<thead>
<tr>
<th>Description</th>
<th>Notation</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>dividend mean growth</td>
<td>$m_D$</td>
<td>1.8%</td>
</tr>
<tr>
<td>dividend volatility</td>
<td>$\sigma_D$</td>
<td>3.6%</td>
</tr>
<tr>
<td>jump intensity</td>
<td>$\lambda$</td>
<td>1.7%</td>
</tr>
<tr>
<td>dividend jump in state $\omega_1$</td>
<td>$J_D(\omega_1)$</td>
<td>-10%</td>
</tr>
<tr>
<td>dividend jump in state $\omega_2$</td>
<td>$J_D(\omega_2)$</td>
<td>-25%</td>
</tr>
<tr>
<td>dividend jump in state $\omega_3$</td>
<td>$J_D(\omega_3)$</td>
<td>-40%</td>
</tr>
<tr>
<td>conditional probability of state $\omega_1$</td>
<td>$\pi_1$</td>
<td>0.5</td>
</tr>
<tr>
<td>conditional probability of state $\omega_2$</td>
<td>$\pi_2$</td>
<td>0.4</td>
</tr>
<tr>
<td>conditional probability of state $\omega_3$</td>
<td>$\pi_3$</td>
<td>0.1</td>
</tr>
<tr>
<td>time interval</td>
<td>$\Delta t$</td>
<td>1/250</td>
</tr>
<tr>
<td>horizon</td>
<td>$T$</td>
<td>200</td>
</tr>
</tbody>
</table>

consumption share $y_t$. We note, however, that in our model volatility of share $y_t$ in normal times $\sigma_y$ is negative, and hence changes $\Delta y_t$ are negatively correlated with dividend changes $\Delta D_t$, conditional on being in normal times. Therefore, following the related literature [e.g., Gärleanu and Panageas (2012), Longstaff and Wang (2012)], we call process $y_t$ countercyclical. Intuitively, negative shocks to dividend process $D_t$ shift consumption and wealth from the less risk averse investor $A$ to more risk averse investor $B$ because the former holds larger fraction of wealth in stocks and hence is more sensitive to dividend shocks. As a result, $B$’s consumption share $y_t$ increases after negative shocks and decreases after positive shocks to dividends.

Panels (A.L), (A.M), and (A.R) of Figure (2) show interest rates for the cases $\psi = 1.5$, $\psi = 0.5$, and CRRA preferences, respectively. Comparing the results on these panels, we observe that higher EIS decreases interest rates because investors with higher EIS save more for consumption smoothing purposes. Furthermore, the fear of rare disasters increases investors’ precautionary savings and decreases interest rates, consistent with the literature [e.g., Barro (2006)]. The latter effect is stronger for more risk averse investor, and hence the interest rates on panels (A.L) and (A.M) are lower when consumption share $y_t$ of risk averse investor $B$ is high in the economy, which makes rates $r$ procyclical. However, when both investors have CRRA preferences, interest rates become countercyclical because investor $B$ has lower EIS given by $\psi_B = 1/\gamma_B$ and hence saves less for consumption smoothing, which counterbalances the precautionary savings motive and pushes the interest rates up when investor $B$’s share $y_t$ is large.

Panels (B.L), (B.M), and (B.R) show stock Sharpe ratios $\kappa = (\mu_{S,t} - r_t)\Delta t/\left(\Sigma_{S,t}^\top\text{var}_t[\Delta \tilde{w}_t]\Sigma_{S,t}\right)$, defined as ratios of risk premia $(\mu_{S,t} - r_t)\Delta t$ and stock return volatilities, given by $\Sigma_{S,t}^\top\text{var}_t[\Delta \tilde{w}_t]\Sigma_{S,t}$. 

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Figure 2
Equilibrium Processes

Solid lines show the processes for the case of no disaster risk, whereas dashed lines show the same processes in the economy with rare disasters. Investors have risk aversions $\gamma_A = 3$ and $\gamma_B = 6$, and time discount parameter $\rho = 2\%$. Left, middle, and right panels show the equilibrium processes when investors have $\psi_A = \psi_B = 1.5$, $\psi_A = \psi_B = 0.5$, and $\psi_A = 1/\gamma_A$ and $\psi_B = 1/\gamma_B$, respectively.
We observe that EIS has only small impact on $\kappa$, consistent with the results of Section 2. The fear of rare disasters leads to significant increases in Sharpe ratios, bringing them in line with the empirical estimates of 36% [e.g., Campbell (2003)]. Furthermore, the Sharpe ratios are countercyclical, consistent with the empirical findings [e.g., Ferson and Harvey (1991)]. Similarly, stock risk premia $\mu_{S,t} - r_t$ are also countercyclical, although they are not shown for brevity. We note that despite large Sharpe ratios stock risk premia remain low in the economy because of low stock return volatilities, as discussed below.

The dynamic behavior of interest rates and Sharpe ratios and the EIS determine the dynamic properties of price-dividend ratios $\Psi$, shown on Panels (C.L), (C.M), and (C.R). The fear of rare disasters decreases wealth-consumption ratios when $\psi > 1$ and increases them when $\psi < 1$, consistent with equation (42) in Section 2, derived for the homogeneous-investor economy. Furthermore, we observe that $\Psi$ is procyclical when $\psi > 1$, consistent with empirical evidence [e.g., Campbell and Cochrane (1999)]. However, for $\psi < 1$ and CRRA preferences $\Psi$ is countercyclical. We also observe that rare disasters make price-dividend ratios more procyclical for $\psi > 1$ and more countercyclical for $\psi < 1$.

Panels (D.L), (D.M), and (D.R) show the excess volatilities of stock returns over the volatilities of dividends conditional on being in normal times, $(\sigma_t - \sigma_D)/\sigma_D$. The dynamic properties of volatilities are determined by those of price-dividend ratios because stock price is given by $S_t = \Psi_t D_t$. Consequently, when $\psi > 1$, and hence $\Psi_t$ is procyclical [see Panel (C.L)], both $\Psi_t$ and $D_t$ move in the same direction, which gives rise to positive excess volatility, consistent with the data [e.g., Shiller (1981); Schwert (1989); Campbell and Cochrane (1999)]. Furthermore, volatilities turn out to be countercyclical over a large interval of consumption shares $y_t$. When $\psi < 1$ ratio $\Psi_t$ is countercyclical, and hence its variation cancels the variation in dividends, leading to lower volatility. As noted above, the fear of disasters enhances countercyclical or procyclicity of price-dividend ratios, depending on the EIS. Therefore, the presence of disaster risk leads to larger excess volatilities in absolute terms. We note, that the results for total volatilities $\sum_t\text{var}[\Delta \tilde{w}_t] \Sigma_{S,t}$ are qualitatively the same, and hence are not reported for brevity.

As demonstrated on Panels (D.L), (D.M), and (D.R) disasters and the heterogeneity in risk aversions generates only small increases of volatilities relative to the volatilities of dividends. We note that the difficulty of matching the magnitudes of stock return volatility is common for general equilibrium models [e.g., Heaton and Lucas (1996), Barro (2006)]. In a model with homogeneous investors and rare disasters Barro (2006) proposes to use exogenous levered claims on consumption to increase stock return volatilities. However, lest to complicate the model and to focus on the dynamic properties of asset prices, we adhere to the classic Lucas (1978) model.

Finally, we look at portfolio strategies of investors for the case when investors have risk aversions $\gamma_A = 3$ and $\gamma_B = 6$ and EIS $\psi_A = \psi_B = 1.5$. Panels (A.L) and (A.R) of Figure 3 show the fractions of wealth that investors $A$ and $B$ invest in stocks, respectively, and Panels (B.L) and (B.R) show the fractions of wealth that investors $A$ and $B$ invest in rare disaster insurance.
Figure 3
Portfolio Strategies
Solid lines show the processes for the case of no disaster risk, whereas dashed lines show the same processes in the economy with rare disasters. Investors have risk aversions $\gamma_A = 3$ and $\gamma_B = 6$, EIS $\psi_A = \psi_B = 1.5$, and time discount parameter $\rho = 2\%$.

Dashed and solid lines show the strategies in the economies with and without rare disasters, respectively. The results on Figure 3 demonstrate that investor $A$ increases the investment in stocks to take advantage of high risk premia in the economy whereas more risk averse investor $B$ decreases the investment in stocks. The increase in the stockholding of investor $A$ is financed by leverage, and hence the share of wealth invested in stocks exceeds unity, $\theta_{A,S} > 1$, as in models with heterogeneous investors and no disaster risks [e.g., Longstaff and Wang (2012); Chabakauri (2013), among others]. Moreover, as investor $A$’s consumption share decreases, that is, $1 - y_t$ goes down, investor $A$ decreases investment in stocks and increases the investment in insurance contracts. This is because when $1 - y_t$ is low, investor $A$’s consumption and wealth are low, and hence the investor becomes more sensitive to disaster risk.

Panels (B.L) and (B.R) demonstrate that, surprisingly, in the economy with rare disasters more risk averse investor $B$ sells insurance to less risk averse investor $A$ because the latter has very high exposure to disaster risk. Dieckmann and Gallmeyer (2005) find a similar result in economies with two CRRA investors with risk aversions $\gamma_A = 1$ and $\gamma_B = 0.5$ and $\gamma_A = 1$ and $\gamma_B = 2$. We note that investor $B$ allocates only a small fraction of wealth to short positions in insurance securities. The fraction invested by investor $A$ in long positions in insurance securities
is larger, but only in states where their consumption share \(1 - y_t\) small. Consequently, the insurance trading has small impact on risk sharing in the economy.

4. Extension with Heterogeneous Beliefs

In this Section, we study an extension of the model in Section 2 in which investors agree on observed prices and dividends but disagree on the intensity \(\lambda\) of disasters, because the latter are difficult to estimate due to insufficient number of observations [e.g., Chen, Joslin, Tran (2012)]. We assume that investor \(A\) has correct estimate of intensity \(\lambda\), whereas investor \(B\) believes that the intensity is \(\lambda_B\). For the sake of tractability, we assume that investor \(B\) does not update intensity \(\lambda_B\). Due to the heterogeneity in beliefs each investor prices the assets using investor-specific state price density \(\xi_i\), where \(i = A, B\), and hence equation (29), which equates the marginal rates of substitution, is not satisfied. Proposition 3 below generalizes equation (29) to the case of heterogeneous beliefs and derives the equilibrium processes.

Proposition 3 (Equilibrium Processes Under Heterogeneous Beliefs). Investors’ state price densities satisfy equation \(\xi_{A,t + \Delta t}/\xi_{A,t} = \eta(\omega_k)\xi_{B,t + \Delta t}/\xi_{B,t}\) in states \(\omega_k\), where \(\eta(\omega_k)\) is Radon-Nikodym derivative of investor \(B\)’s subjective probability measure \(Q\) with respect to the correct measure \(\mathbb{P}\). Investor \(B\)’s consumption shares \(y_{t + \Delta t}(y_t; w_k)\) at time \(t + \Delta t\) in states \(w_k\) as functions of time-\(t\) share \(y_t\) satisfy the following system of equations:

\[
\left(\frac{1 - y_{t + \Delta t}(y_t; \omega_k)}{1 - y_t} \frac{D_{t + \Delta t}}{D_t}\right)^{-\gamma_A} (\frac{\gamma_A}{\gamma_A - 1}) \Phi_{A,t + \Delta t} = \eta(\omega_k) \left(\frac{y_{t + \Delta t}(y_t; \omega_k)}{y_t} \frac{D_{t + \Delta t}}{D_t}\right)^{-\gamma_B} (\frac{\gamma_B}{\gamma_B - 1}) \Phi_{B,t + \Delta t},
\]

(43)

where \(\eta(\omega_1) = \ldots = \eta(\omega_{n{-1}}) = \lambda_B/\lambda\), \(\eta(\omega_{n{-1}}) = (1 - \lambda_B)/(1 - \lambda)\), and \(\mathbb{E}_t[B]\) is expectation under investor \(B\)’s probability measure. The equilibrium interest rate \(r_t\), risk premia under correct beliefs \(\mu_t - r_1\), and the volatility matrix \(\Sigma\) are given by equations (30)–(32), in which all expectations are under the correct beliefs of investor \(A\), and the state price density is that of investor \(A\), and \(\xi_A\) is given by the first equality in equation (29).

We solve the system of equations (43) numerically, using Newton’s method [e.g., Judd (1998)] and obtain the dynamics of consumption shares \(y_{t + \Delta t}\). We obtain these consumption shares in all time-\((t + \Delta t)\) states \(\omega_1, \ldots, \omega_n\), then, we obtain state price density \(\xi_A\) from equation (29), and finally, derive the equilibrium processes from the expressions in Proposition 2. The processes for interest rate \(r_t\), Sharpe ratios \(\kappa\), stock return excess volatilities \((\sigma - \sigma_o)/\sigma_o\), and price-dividend ratios \(\Psi\), are given on Figure 4.

Figure 4 shows the equilibrium processes for calibrated model parameters as in Section 3. We assume that investor \(A\) has correct beliefs so that \(\lambda_A = \lambda = 1.7\%). We study the equilibrium in
Figure 4
Equilibrium Processes under Heterogeneous Beliefs

This Figure shows equilibrium processes for different ratios of investors’ estimates of the intensity of disasters $\lambda_B/\lambda_A$. Investors have risk aversions $\gamma_A = 3$ and $\gamma_B = 6$, time discount parameter $\rho = 2\%$, and EIS $\psi_A = \psi_B = 1.5$, the parameters for process $D_t$ are given in Table 1, and $\lambda_A = 1.7\%$.

three cases, when investor $B$ is pessimistic (i.e., $\lambda_B = 1.5\lambda$), has correct beliefs (i.e., $\lambda_B = \lambda$), and is optimistic (i.e., $\lambda_B = 0.5\lambda$). We provide the analysis only for the case where both investors have the same EIS $\psi_A = \psi_B = 1.5$, because only $\psi_i > 1$ generates realistic dynamic properties of asset prices, as demonstrated in Section 3.

We find that making investor $B$ 50% more pessimistic or optimistic than investor $A$ has significant effect on equilibrium. In particular, making investor $B$ pessimistic, that is, increasing $\lambda_B$, decreases interest rates, increases Sharpe ratios and volatilities, and decreases wealth-consumption-ratios. Making investor $B$ pessimistic has the opposite effects. However, the dynamic properties of equilibrium are not affected by the ratio of intensities $\lambda_B/\lambda_A$. In particular, interest rates and price-dividend ratios are procyclical, whereas Sharpe ratios are countercyclical, consistent with the empirical evidence, which is discussed in Section 3. The intuition for these results can be analyzed similarly to Section 3.

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5. Conclusion

In this paper we study the asset pricing implications of rare disasters and investor heterogeneity in pure exchange Lucas (1978) economy. We demonstrate that EIS has significant impact on asset prices. Based on our analysis we conclude that the model with EIS > 1 provides the best match with the data. This model generates low procyclical interest rates, large countercyclical Sharpe ratios, procyclical price-dividend ratios, and excess volatility. We show that the anticipation of rare events decreases price dividend ratios and increases stock price volatilities when EIS > 1 and has the opposite effects when EIS < 1. We also find that in equilibrium the more risky investor provides the insurance to the less risky investor, because the latter holds a very large fraction of wealth in stocks.

We develop new methodology which provides tractable approach for finding optimal consumptions, portfolio strategies, and other equilibrium processes. The tractability of the solution method allows us to obtain closed-form expressions for the equilibrium processes in the setting where both investors have identical risk aversions but different EIS. In particular, we demonstrate analytically that the heterogeneity in EIS does not affect asset risk premia when investors have identical risk aversions. We note that our methodology can be extended to solve models with heterogeneous beliefs, two Lucas trees, and with portfolio constraints.
Appendix A

Lemma A.1 (Convergence of Multinomial Processes). In the continuous time limit $\Delta t \to 0$ the distribution of consumption variable $D_t$ which follows process (1) converges to the distribution of a variable following a continuous time Lévy process, given by

$$dD_t = D_t[m_D dt + \sigma_D dw_t + J_D(\omega)\Delta j_t], \quad (A1)$$

where $w_t$ is a Brownian motion and $j_t$ is a Poisson process with intensity $\lambda$.

Proof of Lemma A.1. Consider a characteristic function $\varphi_{\Delta t}(p) = \mathbb{E}[e^{ip \ln(D_t/D_0)}]$ of random variable $\ln(D_t)$, where $D_t$ follows process (1). From the fact that $\Delta D_t$ are i.i.d. we obtain

$$\varphi_{\Delta t}(p) = \left(\mathbb{E}\left[(1 + m_D \Delta t + \sigma_D \Delta w_t + J_D(\omega)\Delta j_t)^p\right]\right)^{\frac{1}{\Delta t}}$$

$$= \left((1 - \lambda \Delta t)\mathbb{E}\left[(1 + m_D \Delta t + \sigma_D \Delta w_t)^p|\text{normal}\right] + \lambda \Delta t \mathbb{E}\left[(1 + J_D(\omega))^p|\text{disaster}\right]\right)^{\frac{1}{\Delta t}}$$

$$= \left(1 + ip m_D \Delta t + \frac{ip(ip - 1)}{2} \sigma_D^2 \Delta t + \lambda \Delta t \mathbb{E}_t[(1 + J_D(\omega))^p - 1|\text{disaster}] + o(\Delta t)\right)^{\frac{1}{\Delta t}}. \quad (A2)$$

Taking the limit $\Delta t \to 0$ we obtain that $\varphi_{\Delta t}(p)$ point-wise converges to function $\varphi(p)$, given by

$$\varphi(p) = \exp(ip tm_D + \frac{ip(ip - 1)}{2} t \sigma_D^2 + \lambda t \mathbb{E}_t[(1 + J_D(\omega))^p - 1]). \quad (A3)$$

It can be easily verified that function (A3) is a characteristic function for Lévy process (A1) [e.g., Shreve (2004)]. Therefore, the distribution function for the discrete-time process $D_t$ converges to the distribution of Lévy process (A1) by Lévy’s continuity theorem [e.g., Shiryaev (1996)].

Proof of Lemma 1. Suppose, state price density follows process $\Delta \xi_t = \xi_t[a_t \Delta t + b^\top_t (\Delta \tilde{w}_t - \mathbb{E}_t[\Delta \tilde{w}_t])$, where $\Delta \tilde{w}_t$ is given by equation (20). Next, we find coefficients $a_t$ and $b_t$ from the condition that equations (6)–(8) for asset returns are satisfied. The vector of time-$(t + \Delta t)$ risky asset returns can be written as $R_{t+\Delta t} = 1 + \mu_t \Delta t + \Sigma_t(\Delta \tilde{w}_t - \mathbb{E}_t[\Delta \tilde{w}_t])$, where $\mu_t = m_t + \Sigma_t \mathbb{E}_t[\Delta \tilde{w}_t]/\Delta t$ is the vector of risky assets expected returns. The equations (6)–(8) for asset prices imply that $\mathbb{E}_t[\xi_{t+\Delta t}/\xi_t B_{t+\Delta t}/B_t] = 1$ and $\mathbb{E}_t[\xi_{t+\Delta t}/\xi_t R_{t+\Delta t}] = 1$. Substituting $B_{t+\Delta t}/B_t$ and $R_{t+\Delta t}$ into the latter equations, we obtain two equations for $a_t$ and $b_t$:

$$(1 + a_t \Delta t)(1 + r_t \Delta t) = 1, \quad (A4)$$

$$(1 + a_t \Delta t)(1 + \mu_t \Delta t) + \Sigma_t \text{var}_t[\Delta \tilde{w}_t]b_t = 1. \quad (A5)$$

Solving the system of equations (A4)–(A5) we obtain process (19) for the state price density. □
Proof of Proposition 1. Suppose, all processes are functions of a Markovian state variable $z_t$. The investor solves the following dynamic programming problem:

$$V_i(W_t, z_t, t) = \max_{c_{i,t}, \theta_{i,t}} \left[ (1 - e^{-\rho \Delta t})c_{i,t}^{1-1/\gamma} + e^{-\rho \Delta t} \left( \mathbb{E}_t[V_i(W_{t+\Delta t}, z_{t+\Delta t}, t+\Delta t)^{1-\gamma} \right) \right]^{1-\gamma/\rho}.$$  

(A6)

For simplicity, we omit subscript $i$ for the rest of the proof. Next, we substitute $W_{t+\Delta t}$ from budget constraint (11) into optimization (A6), and taking derivatives with respect to $c_{i,t}$ and $\theta_{i,t}$ we obtain the following first order conditions:

$$e^{-\rho \Delta t} \left( \mathbb{E}_t \left[ \frac{V_{t+\Delta t}^{1-\gamma}}{V_t^{1-\gamma}} \right] \right)^{\gamma-1/\gamma} \mathbb{E}_t \left[ \frac{\partial V_{t+\Delta t}^{1-\gamma}}{\partial W_{t+\Delta t}} V_t^{-\gamma} \right] (1 + r_t \Delta t) \Delta t = (1 - e^{-\rho \Delta t}) \left( \frac{V_t}{c_t} \right)^{1/\rho},$$  

(A7)

$${\mathbb{E}}_t \left[ \frac{\partial V_{t+\Delta t}}{\partial W_{t+\Delta t}} V_t^{-\gamma} \right] \left( (m_t - r_t) \Delta t + \Sigma_t \Delta \tilde{w}_t \right) = 0.$$  

(A8)

To proceed further, we conjecture that $c_t^* = W_t/\Phi_t(z_t, t)$ and that $\theta_t^*$ does not depend on $W_t$, which can be verified by backward induction starting at terminal date $T$, where $W_T = c_T^* \Delta t$, and hence $\Phi_t(z_t, T) = \Delta t$. To find $\partial J/\partial W_t$ we substitute $c_t^*$ and $\theta_t^*$ into equation (A6), and differentiating $V_t$ in (A6) with respect to $W_t$ we obtain:

$$\frac{\partial V_t}{\partial W_t} = V_t^{1/\rho} \left( 1 - e^{-\rho \Delta t} \right) c_t^{1-1/\gamma} \frac{d}{d t} + e^{-\rho \Delta t} \left( \mathbb{E}_t \left[ V_{t+\Delta t}^{1-\gamma} \right] \right)^{\gamma-1/\gamma} \mathbb{E}_t \left[ \frac{\partial V_{t+\Delta t}}{\partial W_{t+\Delta t}} V_t^{-\gamma} \right] (1 + r_t \Delta t) \Delta t \times \left( 1 + r_t \Delta t + (\theta_t^*)^T (m_t - r_t) \Delta t + (\theta_t^*)^T \Sigma_t \Delta \tilde{w}_t - (1 + r_t \Delta t) (\Delta t) \right).$$  

(A9)

Using the first order conditions (A7)–(A8) to simplify equation (A9) we find that

$$\frac{\partial V_t}{\partial W_t} = \frac{1 - e^{-\rho \Delta t}}{\Delta t} \left( \frac{V_t}{c_t} \right)^{1/\rho}.$$  

(A10)

Substituting equation (A10) back into equations (A7)–(A8), after some algebra, we obtain:

$$e^{-\rho \Delta t} \left( \mathbb{E}_t \left[ \left( \frac{V_t^{1+\Delta t}}{V_t} \right)^{1-\gamma} \right] \right)^{\gamma-1/\gamma} \mathbb{E}_t \left[ \left( \frac{V_t^{1+\Delta t}}{V_t} \right)^{1-\gamma} \right]^{\gamma-1/\gamma} \mathbb{E}_t \left[ \left( \frac{V_t}{c_t} \right)^{1-1/\gamma} \right] = 1,$$  

(A11)

$$e^{-\rho \Delta t} \left( \mathbb{E}_t \left[ \left( \frac{V_t^{1+\Delta t}}{V_t} \right)^{1-\gamma} \right] \right)^{\gamma-1/\gamma} \mathbb{E}_t \left[ \left( \frac{V_t^{1+\Delta t}}{V_t} \right)^{1-\gamma} \right]^{\gamma-1/\gamma} \mathbb{E}_t \left[ \left( \frac{c_t^{1+\Delta t}}{c_t} \right)^{1-1/\gamma} \right] = 1.$$  

(A12)

Substituting $1 + R_t + \Delta t = 1 + \mu_t \Delta t + \Sigma_t \Delta \tilde{w}_t$ and $B_t + \Delta t = 1 + r_t \Delta t$ into equations (A11)–(A12), where $R_t + \Delta t$ is the vector of risky asset returns, and comparing the resulting equations (6)–(8) with the representations for asset prices in terms of state price density $\xi_t$, we obtain that

$$\frac{\xi_t + \Delta t}{\xi_t} = e^{-\rho \Delta t} \left( \mathbb{E}_t \left[ V_t^{1+\Delta t} \right] \right)^{\gamma-1/\gamma} \left( \frac{c_t^{1+\Delta t}}{c_t} \right)^{1-1/\gamma} V_t^{1+\Delta t}.$$  

(A13)

Next, we prove the equation (24) for the value function $V_t$. Multiplying both sides of equation (A13) by $(V_t^{1+\Delta t})^{1-1/\gamma}(c_t^{1+\Delta t}/c_t^{1/\gamma})$ and taking expectation $\mathbb{E}_t[\cdot]$ on both sides we obtain that

$$e^{-\rho \Delta t} \left( \mathbb{E}_t \left[ V_t^{1+\Delta t} \right] \right)^{\gamma-1/\gamma} = \mathbb{E}_t \left[ \frac{\xi_t + \Delta t}{\xi_t} \left( \frac{c_t^{1+\Delta t}}{c_t^{1/\gamma}} \right)^{1/\gamma} V_t^{1+\Delta t} \right].$$  

(A14)
Rewriting equation (A6) for $V_t$ in terms of $(V_t/c_t)^{1-1/\psi}$ and using equation (A14) we find that $(V_t/c_t)^{1-1/\psi}$ solves the equation

$$
\left(\frac{V_t}{c_t}\right)^{1-1/\psi} = 1 - e^{-\rho\Delta t} + \frac{1}{\left(c_t^{*}\right)^{1-1/\psi}} \left(\mathbb{E}_t \left[V_{t+\Delta t}^{1-\gamma}\right]\right)^{1-1/\psi} = 1 - e^{-\rho\Delta t} + \mathbb{E}_t \left[\frac{\xi_{t+\Delta t}}{c_t^*} \frac{c_{t+\Delta t}^*}{c_t^*} \left(\frac{V_{t+\Delta t}}{c_{t+\Delta t}^*}\right)^{1-1/\psi}\right].
$$

(A15)

Furthermore, because the market is complete, wealth $W_t$ is given by the martingale representation $W_t = c_t\Delta t + \mathbb{E}_t[(\xi_{t+\Delta t}/\xi_t)W_{t+\Delta t}]$. Rewriting the latter equation in terms of wealth-consumption ratio $\Phi_t = W_t/c_t$ we obtain a recursive equation for $\Phi_t$:

$$
\Phi_t = \Delta t + \mathbb{E}_t \left[\frac{\xi_{t+\Delta t}}{c_t^*} \frac{c_{t+\Delta t}^*}{c_t^*} \Phi_{t+\Delta t}\right].
$$

(16)

Comparing the latter equation with equation (A15) we conclude that $(V_t/c_t)^{1-1/\psi} = (1 - e^{\rho\Delta t})\Phi_t/\Delta t$. Substituting consumption $c_t^* = W_t/\Phi_t$, after simple algebra, we obtain expression (24) for the value function. Next, substituting equation (24) for $V_t$ into equation (A13) for state price density $\xi_t$, after simple algebra, we prove expression (27) for $\xi_t$ in Proposition 1. Optimal consumption growths (25) can be obtained by solving equation (A13), which provides $\xi_t$ in terms of $c_{t+\Delta t}^*/c_t^*$. We omit the details, but note that it can be directly verified by substitution that $c_{t+\Delta t}^*/c_t^*$ in equation (25) satisfies equation (A13). Backward equation (23) for $\Phi_t$ can be obtained by substituting $c_{t+\Delta t}^*/c_t^*$ given by equation (25) into equation (16) for $\Phi_t$.

It remains to prove expression (26) for $\theta^*$. First, we rewrite budget constraint (11) under optimal strategies $\theta^*$ and $c^*$ as $\Delta W_t = (\ldots)\Delta t + (\theta^*_t)^\top\Sigma(\hat{w}_t - \mathbb{E}_t[\Delta \hat{w}_t])$. Multiplying both sides by $(\hat{w}_t - \mathbb{E}_t[\Delta \hat{w}_t])^\top$ and then taking expectations, we obtain that $\mathbb{E}_t[(W_{t+\Delta t}/W_t)(\hat{w}_t - \mathbb{E}_t[\Delta \hat{w}_t])^\top] = (\theta^*_t)^\top\Sigma \text{var}_t[\Delta \hat{w}_t]$. Next, replacing $W_{t+\Delta t}$ and $W_t$ by $\Phi_{t+\Delta t}c_{t+\Delta t}^*$ and $\Phi_tc_t^*$, respectively, and solving for $\theta^*_t$ we obtain expression (26) in Proposition 1.

Finally, we find the marginal rate of substitution $MRS_{t+\Delta t}(\omega_k) = \left(\partial U_t/\partial c_{t+\Delta t}(\omega_k)\right)/\left(\partial U_t/\partial c_t\right)$:

$$
MRS_{t+\Delta t}(\omega_k) = \frac{\partial U_t}{\partial U_{t+\Delta t}} \frac{\partial U_{t+\Delta t}/\partial c_{t+\Delta t}}{U_{t+\Delta t}/\partial c_t} = e^{-\rho\Delta t} \left(\mathbb{E}_t \left[U_{t+\Delta t}^{1-\gamma}\right]\right)^{\frac{1-\psi}{\psi}} \left(U_{t+\Delta t}^{1-\gamma} - \gamma \frac{c_{t+\Delta t}^*}{c_t^*}\right)^{-\psi} \text{Prob}_t(\omega_k).
$$

(A17)

Under optimal strategies $\theta^*_t$ and $c^*_t$, we obtain that $U_t = V_t$, and hence, from equation (A13) we obtain that $MRS_{t+\Delta t}(\omega_k) = (1 - \lambda)\pi_k \xi_{t+\Delta t}(\omega_k)/\xi_t$.

**Proof of Proposition 2.** Taking expectation $\mathbb{E}_t[\cdot]$ on both sides of equation (19) for $\xi_t$, we find that $\mathbb{E}_t[\xi_{t+\Delta t}] = 1/(1 + r_t\Delta t)$. Solving the latter equation we obtain $\gamma$ in equation (30). Next, multiplying both sides of equation (19) by $(\Delta \hat{w}_t - \mathbb{E}_t[\Delta \hat{w}_t])^\top$ and taking expectations, we obtain that $\mathbb{E}_t[\xi_{t+\Delta t}/\xi_t(\Delta \hat{w}_t - \mathbb{E}_t[\Delta \hat{w}_t])^\top] = - (\Sigma^{-1}(\mu_t - r_t\underline{1}))^\top/(1 + r_t\Delta t)/\Delta t$. Solving for $(\mu_t - r_t\underline{1})$, we obtain equation (31) for the risk premia.
To obtain \( \Sigma_t \), from the dynamics of asset prices (4)–(5), we observe that asset returns \( R_{t+\Delta t} \), defined by equation (33), are given by \( R_{t+\Delta t} = \mu_t \Delta t + \Sigma_t [\Delta \tilde{w}_t - E_t [\Delta \tilde{w}_t]] \). Multiplying both sides by \((\Delta \tilde{w}_t - E_t [\Delta \tilde{w}_t])^\top\) and taking expectations, we obtain \( E_t [R_{t+\Delta t} (\Delta \tilde{w}_t - E_t [\Delta \tilde{w}_t])^\top] = \Sigma_t \text{var}_t [\Delta \tilde{w}_t] \). Solving the latter equation, we obtain equation (32) for \( \Sigma_t \). Next, we derive backward equation (34) for the price-dividend ratio by substituting \( S_t = \Psi_t D_t \) into equation (7). Finally, we note that if equation (29) for consumption share \( y_{t+\Delta t} \) has solution \( y_{t+\Delta t}(y_t; \omega) \), then \( \xi_{t+\Delta t}/\xi_t \) is also a function of \( y_t \) and \( \omega \). Consequently, from equations (30)–(34) we obtain that all the equilibrium processes are functions of \( y_t \), and returns \( R_{t+\Delta t} \) are also functions of state \( \omega \). The drift and volatility of consumption share \( y_t \) are found analogously.

**Proof of Corollary 1.** 1) From equation (29) for consumption share \( y_{t+\Delta t} \) we note that when the risk aversions are the same, \( \gamma_t = \gamma \), terms involving \( D_{t+\Delta t}/D_t \) cancel out from both sides of the equation. As a result, \( y_{t+\Delta t} \), and hence all the equilibrium processes, are deterministic functions of time. Factoring out terms with \( y_{t+\Delta t} \), \( \Phi_{t,t} \) from the expectation operators in equation (29) and canceling terms, we obtain that \( y_{t+\Delta t} \) satisfies equation

\[
\left( \frac{y_{t+\Delta t}}{y_t} \right)^{1/\psi_B} = \left( \frac{1 - y_{t+\Delta t}}{1 - y_t} \right)^{1/\psi_A} \left( \frac{D_{t+\Delta t}}{D_t} \right)^{1-\gamma} \left( \frac{E_t \left[ \left( \frac{D_{t+\Delta t}}{D_t} \right)^{1-\gamma} \right]}{\psi A} \right)^{1/(1-\gamma)}. \tag{A18}
\]

Next, we write equation (A18) for all \( t = 0, \Delta t, \ldots, T \), and multiplying right and left sides of these equations, respectively, we obtain equation (39) for \( y_t \). Using similar algebra, from equation (29) we find that \( \xi_t \) is given by:

\[
\left( \frac{\xi_{t+\Delta t}}{\xi_t} \right) = e^{-\rho \Delta t} \left( \frac{y_{t+\Delta t}}{y_t} \right)^{-1/\psi_B} \left( \frac{D_{t+\Delta t}}{D_t} \right)^{-\gamma} \left( \frac{E_t \left[ \left( \frac{D_{t+\Delta t}}{D_t} \right)^{1-\gamma} \right]}{\psi A} \right)^{1/(1-\gamma)}. \tag{A19}
\]

Now, we pass to the limit when \( \Delta t \to 0 \). First, we substitute \( D_{t+\Delta t}/D_t \) from the aggregate consumption process (1) into \( E_t [(D_{t+\Delta t}/D_t)^\alpha] \), and obtain the following expansion:

\[
E_t \left[ \left( \frac{D_{t+\Delta t}}{D_t} \right)^\alpha \right] = E_t \left[ (1 + m_D \Delta t + \sigma_D \Delta w_t + j_D (\omega) \Delta j_t)^\alpha \right]
\]

\[
= 1 - \frac{\lambda \Delta t}{2} \left( 1 + m_D \Delta t + \sigma_D \sqrt{\Delta t} \right)^\alpha + \left( 1 + m_D \Delta t - \sigma_D \sqrt{\Delta t} \right)^\alpha \tag{A20}
+ \lambda \Delta t E \left[ (1 + m_D \Delta t + j_D (\omega))^{\alpha} \right]_{\text{disaster}}
= 1 + \left( \alpha m_D + \frac{\alpha (\alpha - 1)}{2} \sigma_D^2 + \lambda \left( E_t \left[ (1 + j_D (\omega))^{\alpha} \right] - 1 \right) \right) \Delta t + o(\Delta t).
\]

Next, substituting expansions \((y_{t+\Delta t}/y_t)^{1/\psi_B} = 1 + (1/\psi_B)(\Delta y_t/y_t)(\Delta t + o(\Delta t))\) and \((1 - y_{t+\Delta t}/(1 - y_t))^{1/\psi_A} = 1 - (1/\psi_A)(\Delta y_t/(1 - y_t))(\Delta t + o(\Delta t))\) into equation (A18), we obtain a linear equation for \( \Delta y_t \). Using expansion (A20), after some algebra, we obtain expansion:

\[
\Delta y_t = \left( \frac{\psi_B - \psi_A}{\psi_B y_t + \psi_A (1 - y_t)} \right) \left( m_D - \frac{\gamma}{2} \sigma_D^2 + \frac{\lambda}{1 - \gamma} \left( E_t \left[ (1 + j_D (\omega))^{1-\gamma} \right]_{\text{disaster}} - 1 \right) \right) \Delta t + o(\Delta t). \tag{A21}
\]
Using expansions (A20) and (A21), we obtain expansion for \( \mathbb{E}_t[\xi_{t+\Delta t}/\xi_t] \), where \( \xi_{t+\Delta t}/\xi_t \) is given by equation (A19). Then, we derive an expansion for interest \( r_t \), given by (30), and passing to the limit \( \Delta t \to 0 \), after some algebra, we obtain closed-form solution (35). The expression for the market price of risk (36) is obtained similarly, using the same expansions, and equation (31) for the risk premia in Proposition 1.

Finally, we derive the stock risk premium. Writing down the dynamics for stock prices (7) in states \( \omega_{n-1} \) and \( \omega_n \), and using the fact that price-dividend ratio \( \Psi_{t+\Delta t} \) is deterministic, after some algebra we obtain expressions for the drift of the stock price:

\[
m_{s,t} \Delta t = \frac{\Psi_{t+\Delta t} + \Delta t}{2\Psi_t} \left( \frac{D_{t+\Delta t}(\omega_{n-1})}{D_t} + \frac{D_{t+\Delta t}(\omega_n)}{D_t} \right),
\]

where \( D_{t+\Delta t}(\omega_{n-1}) \) and \( D_{t+\Delta t}(\omega_n) \) denote time-\( t+\Delta t \) dividend in states \( \omega_{n-1} \) and \( \omega_n \), respectively. Moreover, from equation (30) \( 1+r_t \Delta t = 1/\mathbb{E}_t[\xi_{t+\Delta t}/\xi_t] \) and from equation (34) \( (\Psi_{t+\Delta t} + \Delta t)/\Psi_t = 1/\mathbb{E}_t[(\xi_{t+\Delta t}/\xi_t)(D_{t+\Delta t}/D_t)] \). Using all the above equations, we obtain

\[
m_{s,t} - r = \frac{1}{2} \left( \frac{D_{t+\Delta t}(\omega_{n-1})}{D_t} + \frac{D_{t+\Delta t}(\omega_n)}{D_t} \right)/\mathbb{E}_t[\xi_{t+\Delta t}/\xi_t] - 1/\mathbb{E}_t[(\xi_{t+\Delta t}/\xi_t)].
\]

Risk premium is then found as \( \mu_{s,t} - r_t = m_{s,t} - r_t + \Sigma_{s,t} \mathbb{E}_t[\tilde{w}_i]/\Delta t \). We also note that because \( \Psi_t \) is deterministic, the volatility \( \sigma_{s,t} \) and jump sizes \( J_s(t,\omega) \) of stock prices are the same as those of dividend process (1). Therefore, \( \Sigma_{s,t} = (\sigma_D, J_D(\omega_1), \ldots, J_D(\omega_{n-2}))^\top \). Substituting \( \xi_{t+\Delta t}/\xi_t \) from equation (A19) into equation (A23) and noting from the dividend dynamics (1) that \( D_{t+\Delta t}(\omega_{n-1})/D_t + D_{t+\Delta t}(\omega_n)/D_t = 2 + 2m_D \Delta t \), using expansions (A20) and (A21), after some algebra we obtain risk premium (37).

2) Now, we consider the case of homogeneous investors, that is, \( \psi_A = \psi_B = \psi \), \( \gamma_A = \gamma_B = \gamma \). From the equation (34) for price-dividend ratio \( \Psi_t \) and the fact that it is deterministic, we obtain

\[
\Psi_t = (\Psi_{t+\Delta t} + \Delta t) \mathbb{E}_t \left[ \frac{\xi_{t+\Delta t}/\xi_t}{D_{t+\Delta t}/D_t} \right] = (\Psi_{t+\Delta t} + \Delta t)e^{-\rho \Delta t} \mathbb{E}_t \left[ \left( \frac{D_{t+\Delta t}}{D_t} \right)^{1-\gamma} \right]^{1/1-\psi},
\]

where the second equality is obtained by substituting \( \xi_t \) from equation (A19) into equation (A24) and noting that \( y_{t+\Delta t} = y_t \) in homogeneous investor economy. Solving backward equation (A24) we obtain that \( \Psi_t = (1 - g_{t,1}(T-t+\Delta t)/\Delta t)/(1 - g_{t,1})g_{t,1}\Delta t \) where \( g_{t,1} \) is given by equation (17). As \( T \to \infty \), the solution converges to a stationary one if \( g_{t,1} < 1 \).

Next, we obtain another representation for \( \Psi_t \) in terms of rate \( r \) and risk premium \( \mu_s - r \). Using the expression for \( \xi_{t+\Delta t}/\xi_t \) from equation (19) we obtain:

\[
\mathbb{E}_t \left[ \frac{\xi_{t+\Delta t} \Delta t}{\xi_t} \right] = \frac{1}{1 + r\Delta t} \mathbb{E}_t \left[ \left( 1 - (\Sigma^{-1}(\mu - r))^\top (\text{var}_t[\Delta \tilde{w}_i]/\Delta t)^{-1}(\Delta \tilde{w}_i - \mathbb{E}_t[\Delta \tilde{w}_i]) \right) \times \left( 1 + m_D \Delta t + \Sigma_D^\top \mathbb{E}_t[\Delta \tilde{w}_i] + \Sigma_D^\top (\Delta \tilde{w}_i - \mathbb{E}_t[\Delta \tilde{w}_i]) \right) \right]
\]

\[
= \frac{1 + (m_D + \Sigma_D J^\top \mathbb{E}_t[\Delta \tilde{w}_i]/\Delta t - (\Sigma^{-1}(\mu - r))^\top J) \Delta t}{1 + r\Delta t},
\]

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where $\Sigma^\ast = (\sigma^\ast, J^\ast(\omega_1), \ldots, J^\ast(\omega_{n-2}))^\top$. Using formula (36) for $\Sigma^{-1}(\mu - r)$, after some algebra, as $\Delta t \to 0$, we obtain $(\Sigma^{-1}(\mu - r))^\top \Sigma^\ast = \gamma \sigma^2_p - \lambda \mathbb{E}_t[(1 + J^\ast(\omega) - \gamma)J^\ast(\omega)|\text{disaster}] + \lambda \mathbb{E}_t[J^\ast(\omega)|\text{disaster}]$. Furthermore, it can be shown by straightforward algebra that $\Sigma^\ast \mathbb{E}_t[\Delta \tilde{w}_t] = \lambda \Delta t \mathbb{E}_t[J^\ast(\omega)|\text{disaster}]$. Substituting the latter expressions back into equation (A25) we obtain

$$
\mathbb{E}_t \left[ \frac{\xi_{t+\Delta t} D_{t+\Delta t}}{\xi_t D_t} \right] = 1 - (r + (\mu_s - r) - m_S) \Delta t + o(\Delta t). \tag{A26}
$$

Substituting (A26) back into equation (A24) we obtain that in continuous time $\Psi'(t) - (r + (\mu_s - r) - m_S) \Psi(t) + 1 = 0$, subject to $\Psi(T) = 0$. Solving the ODE we obtain $\Psi_t$ in equation (40).

Similarly, given that prices $P_{k,t}$ are deterministic, from equation (8) we obtain:

$$
P_{k,t} = P_{k,t+\Delta t} \mathbb{E}_t \left[ \frac{\xi_{t+\Delta t}}{\xi_t} \right] + \lambda \pi_k \Delta t \mathbb{E}_t \left[ \frac{\xi_{t+\Delta t} \omega_k}{\xi_t} \right] = P_{k,t+\Delta t} \frac{1}{1 + \rho \Delta t} + \lambda \pi_k \Delta t e^{-\rho \Delta t} \left( \frac{D_{t+\Delta t}}{D_t} \right)^{\gamma} \mathbb{E}_t \left[ \left( \frac{D_{t+\Delta t}}{D_t} \right)^{1-\gamma} \right] \frac{\gamma^{-\frac{1}{\gamma}}}{1-\gamma}.
$$

Iterating backward it can be demonstrated that $P_{k,t} = \left( 1 - g_{k,2}^{(T-t)/\Delta t} \right) \big/ \left( 1 - g_{k,2} \right) b_k \lambda \Delta t$, where $g_{k,2}$ is given by equation (18), and $b_k$ is given by:

$$
b_k = \lambda \pi_k e^{-\rho \Delta t} \left( \frac{D_{t+\Delta t}}{D_t} \right)^{-\gamma} \mathbb{E}_t \left[ \left( \frac{D_{t+\Delta t}}{D_t} \right)^{1-\gamma} \right] \frac{\gamma^{-\frac{1}{\gamma}}}{1-\gamma}.
$$

Passing to continuous time limit in the second equality in equation (A27), similarly to price-dividend ratios $\Psi_t$, we obtain the insurance prices in equation (40).

**Proof of Proposition 3.** Because the investors agree on observed asset prices, using equations (6)–(8) for asset prices in terms of the state price density, we obtain:

$$
B_t = \mathbb{E}_t \left[ \frac{\xi_{s,t+\Delta t}}{\xi_{s,t}} B_{t+\Delta t} \right] \tag{A29}
$$

$$
S_t = \mathbb{E}_t \left[ \frac{\xi_{s,t+\Delta t}}{\xi_{s,t}} (S_{t+\Delta t} + D_{t+\Delta t} \Delta t) \right] \tag{A30}
$$

$$
P_{k,t} = \mathbb{E}_t \left[ \frac{\xi_{s,t+\Delta t}}{\xi_{s,t}} (P_{k,t+\Delta t} + D_{k,t+\Delta t}) \right] \tag{A31}
$$

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The expectations under investor $B$’s subjective probability measure in equations (A32)–(A32)
can be rewritten in terms of the expectations under the correct measure of investor $A$ and Radon-
Nikodym derivative $\eta_{t+\Delta t}(\omega)$ to obtain:

$$
E_t \left[ \frac{\xi_{A,t+\Delta t}}{\xi_{A,t}} B_{t+\Delta t} \right] = E_t \left[ \eta_{t+\Delta t} \frac{\xi_{B,t+\Delta t}}{\xi_{B,t}} B_{t+\Delta t} \right],
$$

$$
E_t \left[ \frac{\xi_{A,t+\Delta t}}{\xi_{A,t}} (S_{t+\Delta t} + D_{t+\Delta t} \Delta t) \right] = E_t \left[ \eta_{t+\Delta t} \frac{\xi_{B,t+\Delta t}}{\xi_{B,t}} (S_{t+\Delta t} + D_{t+\Delta t} \Delta t) \right],
$$

$$
E_t \left[ \frac{\xi_{A,t+\Delta t}}{\xi_{A,t}} (P_{k,t+\Delta t} + D_{k,t+\Delta t}) \right] = E_t \left[ \eta_{t+\Delta t} \frac{\xi_{B,t+\Delta t}}{\xi_{B,t}} (P_{k,t+\Delta t} + D_{k,t+\Delta t}) \right].
$$

From the latter equations and from the uniqueness of the state price density under the correct
expectations, demonstrated in Lemma 1, we obtain that $\frac{\xi_{A,t+\Delta t}}{\xi_{A,t}} = \frac{\eta_{t+\Delta t} \xi_{B,t+\Delta t}}{\xi_{B,t}}$. Next,
using the latter equality and equation (27) for the state price density in terms of investors
consumptions, similarly to equation (29) we obtain a system of equations (43) for consumption
shares $y_{t+\Delta t}(y_t; w_k)$. Because the time is discrete, the Radon-Nikodym derivative is simply given
by the ratio of subjective investor $B$’s and real probabilities of states $\omega_1, \ldots, \omega_n$. Therefore, the
Radon-Nikodym derivative does not depend on time, and hence can be written as $\eta(\omega)$. ■
References


