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Abstract

We investigate the comparative statics of "more ambiguity aversion" as defined by Klibanoff, Marinacci and Mukerji (2005) in the context of the static two-asset portfolio problem. It is not true in general that more ambiguity aversion reduces the demand for the uncertain asset. We exhibit some sufficient conditions to guarantee that, ceteris paribus, an increase in ambiguity aversion reduces the demand for the ambiguous asset, and raises the equity premium. For example, this is the case when the set of plausible distributions of returns can be ranked according to the monotone likelihood ratio order. We also show how ambiguity aversion distorts the price kernel in the alternative portfolio problem with complete markets for contingent claims.

Keywords: Smooth ambiguity aversion, monotone likelihood ratio, equity premium, portfolio choice, price kernel, central dominance.

1 Introduction

In this paper, we examine the standard static portfolio problem with one safe asset and one uncertain asset. The investor perceives ambiguity about what the true probability distribution for the excess return of the uncertain asset is. This ambiguity is expressed by a second order prior probability distribution over the set of plausible (first order) distributions of the excess return. Following Segal (1987) and Klibanoff, Marinacci and Mukerji (2005) (hereafter KMM), we introduce ambiguity attitude by relaxing reduction of first and second order probabilities. In words, the investor does not evaluate in the same way asset 1 yielding a return of 20% with probability p and asset 2 yielding a return of 20% with an unknown probability whose expectation is p, unlike in the standard Bayesian expected utility framework. Following KMM, we assume that the investor is ambiguity averse in the sense that he dislikes any mean-preserving spread in the space of first order probability distributions of excess returns. For example, he prefers asset 1 to asset 2. KMM proposes a nice decision criteria called "smooth ambiguity aversion" to characterize such preferences under uncertainty. For a given portfolio allocation, the ex ante welfare of the investor is measured by computing the (second order) expectation of a concave function ϕ of the (first order) expected utility u of final consumption conditional to each plausible distribution of the excess return. As usual, the concavity of the utility function u expresses risk aversion in the special case of risky acts, i.e. acts entailing consequences whose plausible probability distribution is unique. When ϕ is linear, we are back to the standard expected utility model in which the uncertain context can be reduced to a single compound probability distribution. When ϕ is concave, the investor is ambiguity averse, and the reduction of compound distributions does not hold.

Interestingly enough, KMM also define the comparative notion of "more ambiguity aversion". Consider two agents, respectively with function ϕ_1 and ϕ_2, who have the same beliefs expressed by the set of first and second order probability distributions. Suppose also that they have the same utility function u to evaluate risky acts. Agent ϕ_2 is more ambiguity averse than agent ϕ_1 if ϕ_1 prefers an uncertain act over a pure risky one whenever ϕ_2 does so. This is true if and only if function ϕ_2 is more concave than function ϕ_1, in the sense of Arrow-Pratt: $-\phi_2''/\phi_2'$ is uniformly larger than $-\phi_1''/\phi_1'$. In this paper, we examine the effect of such an increase in ambiguity aversion on the optimal demand for the uncertain asset. By doing so, we fix the first and second order beliefs, and the utility function u. We examine the effect of a concave transformation of function ϕ on the optimal portfolio.

The KMM model has two attractive features in comparison to other models of ambiguity such as the pioneering maxmin expected utility (or multiple-prior) model of Gilboa and Schmeidler (1989). First, it organizes a nice separation between ambiguity aversion and ambiguity, i.e., between tastes and beliefs. Without this feature, we could not perform the comparative statics of more ambiguity aversion. Second, the KMM model applies the expected utility machinery sequentially, first on order probability distributions, and then on the second order distribution. This allows us to exploit the huge armory of techniques amassed over the years to tackle questions involving risk under the expected utility framework to the analysis of problems involving non-expected utility involving ambiguity. This paper illustrates this point.

This question of the comparative statics of ambiguity aversion for the portfolio problem is parallel to the one of risk aversion. Since Arrow (1963) and Pratt (1964), it is well known that an increase in risk aversion reduces the demand for the risky asset. It is therefore quite surprising that, as we show in this paper, it is not true in general that more ambiguity aversion reduces the demand for the ambiguous asset. For cleverly chosen - but still not spurious - set of priors for the return of the risky asset, we show that the introduction of ambiguity aversion *increases* the investor's demand for the ambiguous asset. The intuition for why such counterexamples may exist can be explained as follows. The first-order condition of the portfolio problem with ambiguity aversion can be rewritten to take the form that it would take under expected utility, but with a distortion in the way the different first order probability distributions on excess return are compounded. Under expected utility, i.e., ambiguity neutrality, the compounding is made by using the true second order probability distribution. Under ambiguity aversion, this second order probability distribution is distorted by putting more weight on the plausible second order distributions yielding a smaller expected utility, as first observed by Taboga (2005). In spite of the fact that the ambiguity averse investor's beliefs cannot be reduced to a single compound probability distribution over excess returns, the introduction of ambiguity aversion is observationally equivalent to the effect of distorting the compound distribution used by the ambiguity-neutral investor. This distortion is pessimistic. It is well known from expected utility theory that pessimistic deteriorations in beliefs do not always reduce the demand for the risky asset.[1] As for Giffen goods in consumer theory, this deterioration in the terms of trade yields a wealth effect that may raise the asset demand.

The main objective of the paper is to characterize conditions under which more ambiguity aversion reduces the optimal exposure to uncertainty. This can be done by restricting either the set of utility functions and/or the set of possible priors. If we assume that the set of priors can be ranked according to the first-degree or second-degree stochastic dominance orders (FSD/SSD),

[1] See Rothschild and Stiglitz (1971), Fishburn and Porter (1976), Meyer and Ormiston (1985), Hadar and Seo (1990), Gollier (1995), Eeckhoudt and Gollier (1995), Abel (2002) and Athey (2002), and the bibliographical references in these papers.

we exhibit some simple sufficient conditions on the utility function to obtain an unambiguous comparative static property of the introduction of ambiguity aversion. These results are derived by using the following technique. It happens that any increase in ambiguity aversion deteriorates the observationally equivalent second order distribution in a very specific way. It transfers more weight on the worse priors in the sense of the monotone likelihood ratio (MLR) order. This puts a very specific structure to the notion of pessimism entailed by more ambiguity aversion. For example, if the plausible priors can be order according to FSD, their compound through the MLR deteriorated second order probability distribution generated by more ambiguity aversion entails a FSD deterioration of the behaviorally equivalent changes of beliefs under expected utility. This implies that, under this assumption, the following two problems are linked : 1) under the EU model, what are the conditions on the utility function u that guarantee that any FSD deterioration in the distribution of excess return of the risky asset reduces the demand for it; and 2) in the KMM model, what are the conditions that guarantee that any increase in ambiguity aversion reduces the demand for the ambiguous asset? The same property holds when replacing FSD by SSD. More sophisticated methods are required when considering other stochastic orders to rank plausible priors for the excess return. Let us just mention at this stage the result that is easiest to express: if the plausible priors can be ranked according to MLR (a special case of FSD), then it is always true that more ambiguity aversion reduces the demand for the ambiguous asset.

It is easy to translate these results about the effect of comparative ambiguity aversion on the demand for the ambiguous asset into its effect on the equity premium. Thus, our work is related to the recent developments about the effect of ambiguity aversion on the equity premium. Ju and Miao (2009) and Collard, Mukerji, Sheppard and Tallon (2009) examine a dynamic infinite-horizon portfolio problem in which the representative investor exhibits smooth ambiguity aversion and faces time-varying ambiguity about the second order distribution of the plausible probability distributions of consumption growth. These two papers consider different sets of risk-ambiguity attitudes (u, ϕ), and different stochastic processes for the first and second order probability distributions. They both use numerical analyses to solve the calibrated dynamic portfolio problem. They both conclude that ambiguity aversion raises the equity premium. Our paper demonstrates that these results are specific to the calibration under scrutiny. Other papers conclude in the same direction, but using other decision criterion for ambiguity aversion, either maxmin expected utility, Choquet expected utility, or robust control theory.[2]

The portfolio problem and two illustrations are presented in Section 2. Our main results are presented in Section 3 in which we derive sufficient conditions for the comparative statics of more ambiguity aversion. In Section 4, we examine a Lucas economy with a representative agent facing ambiguous state probabilities. We show how the ambiguity aversion of the representative agent affects the equity premium, the price kernel of the economy, and individual asset prices.

2 The smooth ambiguity model applied to the portfolio problem

Our model is static with two assets. The first asset is safe with a rate of return that is normalized to zero. The risky asset has a return x whose distribution is ambiguous in the sense that it is sensitive to some parameter θ whose true

[2]See for example Dow and Werlang (1992), Chen and Epstein (2002), Epstein and Wang (1994), Hansen, Sargent and Tallarini (1999), Maenhout (2004), Hansen and Sargent (2008).

value is unknown. The investor is initially endowed with wealth w_0. If he invests α in the risky asset, his final wealth will be $w_0 + \alpha x$ conditional to a realized return x of the risky asset.

The ambiguity of the uncertain asset is characterized by a set $\Pi = \{F_1, ..., F_n\}$ of plausible cumulative probability distributions for $\widetilde{x}$. Let $\widetilde{x}_\theta$ denote the random variable distributed as F_θ. We suppose that the support of all priors are bounded in $[x_-, x_+]$, with $x_- < 0 < x_+$. Based on his subjective information, the investor associates a second order probability distribution $(q_1, ..., q_n)$ over the set of priors Π, with $\Sigma_{\theta=1}^n q_\theta = 1$, where $q_\theta \geq 0$ is the probability that F_θ be the true probability distribution of excess returns. We hereafter denote $\widetilde{\theta}$ for the random variable $(1, q_1; 2; q_2; ...; n, q_n)$. Following Klibanoff, Marinacci and Mukerji (2005), we assume that the preferences of the investor exhibit smooth ambiguity aversion. For each plausible probability distribution F_θ, the investor computes the expected utility $U(\alpha, \theta) = Eu(w_0 + \alpha\widetilde{x}_\theta) = \int u(w_0 + \alpha x)dF_\theta(x)$ conditional to F_θ being the true distribution. We assume that u is increasing and concave, so that $U(., \theta)$ is concave in the investment α in the ambiguous asset, for all θ. Ex ante, for a given portfolio allocation α, the welfare of the agent is measured by $V(\alpha)$ with

$$V(\alpha) = \phi^{-1}\left(\sum_{\theta=1}^{n} q_\theta \phi(U(\alpha, \theta))\right) = \phi^{-1}\left(\sum_{\theta=1}^{n} q_\theta \phi\left(Eu(w_0 + \alpha\widetilde{x}_\theta)\right)\right),$$

$V(\alpha)$ can be interpreted as the certainty equivalent of the uncertain conditional expected utility $U(\alpha, \widetilde{\theta})$. The shape of ϕ describes the investor's attitude towards ambiguity. A linear ϕ means that the investor is neutral to ambiguity, and that he can reduce compound probability distributions to a single one $\Sigma_\theta q_\theta F_\theta$. On the contrary, a concave ϕ is synonymous of ambiguity aversion in the sense that the DM dislikes any mean-preserving spread of the conditional expected utility $U(\alpha, \widetilde{\theta})$.

An interesting particular case arises when the absolute ambiguity aversion $\eta(U) = -\phi''(U)/\phi'(U)$ is constant, so that $\phi(U) = -\eta^{-1}\exp(-\eta U)$. As proved by Klibanoff, Marinacci and Mukerji (2005), the ex ante welfare $V(\alpha)$ essentially exhibits a maxmin expected utility functional $V^{MEU}(\alpha) = \min_\theta Eu(w_0 + \alpha\widetilde{x}_\theta)$ à la Gilboa and Schmeidler (1989) when the degree η of absolute ambiguity aversion tends to infinity.

The optimal portfolio allocation α^* maximizes the ex ante welfare of the investor $V(\alpha)$. Because ϕ is increasing, α^* is the solution of the following program:

$$\alpha^* \in \arg\max_\alpha \quad \sum_{\theta=1}^{n} q_\theta \phi\left(Eu(w_0 + \alpha\widetilde{x}_\theta)\right). \tag{1}$$

If ϕ and u are strictly concave, the objective function is concave in α and the solution to program (1), when it exists, is unique. Let us observe that the demand for the ambiguous asset shares its sign with the equity premium $E\widetilde{x} = \Sigma_\theta q_\theta E\widetilde{x}_\theta$. All proofs are relegated to the Appendix.

Lemma 1 *The demand for the ambiguous asset is positive (zero/negative) if the equity premium is positive (zero/negative).*

This means that ambiguity aversion, as risk aversion, has a second order nature, as defined by Segal and Spivak (1990). As soon as the equity premium is positive, the demand for the ambiguous asset is positive, independent of the degree of ambiguity surrounding the distribution of returns. We hereafter assume that the equity premium is positive, so that α^* is positive.

In the remainder of this section, we examine two illustrations. Consider first the following special case in which an analytical solution can be found for α^*:

- Priors are normally distributed with the same variance σ^2, and with $E\widetilde{x}_\theta = \theta : \widetilde{x}_\theta \sim N(\theta, \sigma)$;[3]

- The ambiguity on the equity premium θ is itself normally distributed: $\widetilde{\theta} \sim N(\mu, \sigma_0)$;

- The investor's preferences exhibit constant absolute risk aversion: $u(z) = -A^{-1} \exp -Az \in \mathbb{R}_-$;

- The investor's preferences exhibit constant relative ambiguity aversion: $\phi(U) = -(-U)^{1+\gamma}/(1+\gamma)$. This function is increasing in $\mathbb{R}_-$ and is concave in this domain if γ is positive.

As is well-known, the normality of the priors and the constancy of absolute risk aversion implies that the Arrow-Pratt approximation is exact.[4] This implies in turn that

$$U(\alpha, \theta) = -A^{-1} \exp -A\left(w_0 + \alpha\theta - 0.5A\alpha^2\sigma^2\right). \qquad (2)$$

Because $\phi(U)$ is an exponential function of θ, and because $\widetilde{\theta}$ is normally distributed, the same trick can be used to compute $E\phi$. It yields

$$V(\alpha) = -A^{-1} \exp -A(w_0 + \alpha\mu - 0.5A\alpha^2(\sigma^2 + (1+\gamma)\sigma_0^2)). \qquad (3)$$

The optimal demand for the risky asset is thus equal to

$$\alpha^* = \frac{\mu}{A(\sigma^2 + (1+\gamma)\sigma_0^2)}. \qquad (4)$$

We see that under ambiguity ($\sigma_0^2 > 0$), the demand for the risky asset is decreasing in the relative degree γ of ambiguity aversion of the investor. This exponential-power specification for (u, ϕ) differs from the other three papers

[3] It is easy to extend this to the case of an ambiguous variance.

[4] For a simple proof, see for example Gollier (2001, page 57).

existing on this topic. Taboga (2005) examines an exponential-exponential specification. Ju and Miao (2009) used a power-power specification, whereas Collard, Mukerji, Sheppard and Tallon (2009) used a power-exponential specification. None of these three alternative problems can be solved analytically.

Consider alternatively the following counterexample such that

- $n = 2$, $\widetilde{x}_1 \sim (-1, 2/10; -0.25, 3/20; 0.75, 7/20; 1.25, 3/10)$ and $\widetilde{x}_2 \sim (-1, 1/5; 0, 1/5; 1, 3/5)$;
- $q_1 = 5\%$ and $q_2 = 95\%$;
- $u(z) = \min(z, 3 + 0.3(z - 3))$ and $w_0 = 2$;
- $\phi(U) = -\eta^{-1} \exp(-\eta U)$.

It is easy to check that $\widetilde{x}_1$ is riskier than $\widetilde{x}_2$ in the sense of Rothschild and Stiglitz (1970). We solved this problem numerically. Below a minimum threshold around 20 for the degree η of ambiguity aversion, the optimal holding of the ambiguous asset equals $\alpha^* = 1$. However, above this threshold, the introduction of ambiguity aversion *increases* α^* above the optimal investment of the ambiguity-neutral investor. For example, α^* equals 1.204 when $\eta = 50$. When η tends to infinity, the optimal investment in the risky asset tends to $\alpha^* = 4/3$, which is the optimal holding of the ambiguous asset for an ambiguity-neutral investor who believes that the distribution of excess return is $\widetilde{x}_1$ with certainty. In terms of portfolio allocation, it is observationally equivalent to increase absolute ambiguity aversion from zero to infinity or to replace beliefs $\widetilde{y}_1 \sim (\widetilde{x}_1, 5\%; \widetilde{x}_2, 95\%)$ by $\widetilde{y}_2 \sim (\widetilde{x}_1, 100\%)$ under expected utility. Notice that because $\widetilde{x}_1$ is riskier than $\widetilde{x}_2$, the extreme belief $\widetilde{y}_2$ is riskier than $\widetilde{y}_1$ in the sense of Rothschild and Stiglitz. This example illustrates the fact – first observed by Rothschild and Stiglitz (1971) – that it is not true in general that a riskier distribution of excess return reduces the demand for the risky asset in the expected utility model.

3 Effect of an increase of ambiguity aversion

The beliefs are represented by the set of priors $(\widetilde{x}_1, ..., \widetilde{x}_n)$ of the excess return of the risky asset, together with the second order distribution $(q_1, ..., q_n)$ on these priors. We compare two agents with the same beliefs and the same concave utility function u, but with different attitudes toward ambiguity represented by concave functions ϕ_1 and ϕ_2. The demand for the risky asset by agent ϕ_1 is expressed by α_1^* which must satisfy the following first-order condition:

$$\sum_{\theta=1}^{n} q_\theta \phi_1'(U(\alpha_1^*, \theta)) E\widetilde{x}_\theta u'(w_0 + \alpha_1^* \widetilde{x}_\theta) = 0. \tag{5}$$

Following Klibanoff, Marinacci and Mukerji (2005), we assume that the agent with function ϕ_2 is more ambiguity averse than agent ϕ_1 in the sense that there exists an increasing and concave transformation function k such that $\phi_2(U) = k(\phi_1(U))$ for all U in the relevant domain. We would like to characterize conditions under which the more ambiguity averse agent ϕ_2 has a smaller demand for the risky asset than agent ϕ_1: $\alpha_2^* \leq \alpha_1^*$. This would be the case if and only if

$$\sum_{\theta=1}^{n} q_\theta \phi_2'(U(\alpha_1^*, \theta)) E\widetilde{x}_\theta u'(w_0 + \alpha_1^* \widetilde{x}_\theta) \leq 0. \tag{6}$$

We would like to find conditions under which it is always true that (5) implies (6). Notice that this condition can be rewritten as

$$E\widetilde{y}_1 u'(w_0 + \alpha_1^* \widetilde{y}_1) = 0 \implies E\widetilde{y}_2 u'(w_0 + \alpha_1^* \widetilde{y}_2) \leq 0, \tag{7}$$

where $\widetilde{y}_i$ is a compound random variable which equals $\widetilde{x}_\theta$ with probability $\widehat{q}^i_\theta$, $\theta = 1, ..., n$, such that

$$\widehat{q}^i_\theta = \frac{q_\theta \phi'_i(U(\alpha^*_1, \theta))}{\sum_{t=1}^n q_t \phi'_i(U(\alpha^*_1, t))}. \tag{8}$$

Notice that the left equality in (7) can be interpreted as the first-order condition of the problem $\max_\alpha Eu(w_0 + \alpha\widetilde{y}_1)$ of an expected-utility-maximizing investor whose beliefs are represented by $\widetilde{y}_1 \sim (\widetilde{x}_1, \widehat{q}^1_1; ...; \widetilde{x}_n, \widehat{q}^1_n)$. Thus, the ambiguity averse agent ϕ_1 behaves as an EU-maximizing agent who would distort his second order beliefs from $(q_1, ..., q_n)$ to the "observationally equivalent probability distribution" $\widehat{q}^1 = (\widehat{q}^1_1, ..., \widehat{q}^1_n)$. The distortion factor $\phi'_1(U(\alpha^*_1, \theta))/\Sigma_t q_t \phi'_1(U(\alpha^*_1, t))$ is a Radon-Nikodym derivative, and the probability distribution $\widehat{q}^1$ is analogous to the risk-neutral probability distribution used in the theory of finance. Notice that the distortion functional described by equation (8) is endogenous, as it depends upon the portfolio allocation α^*_1 selected by the agent. The right inequality in (7) just means that shifting beliefs from $\widetilde{y}_1$ to $\widetilde{y}_2$ reduces the ambiguity-neutral investor's holding of the asset. These findings are summarized in the following lemma.

Lemma 2 *The change in preferences from (u, ϕ_1) to (u, ϕ_2) reduces the demand for the ambiguous asset if the EU agent with utility function u reduces his demand for the risky asset when his beliefs about the excess return shift from $\widetilde{y}_1 \sim (\widetilde{x}_1, \widehat{q}^1_1; ...; \widetilde{x}_n, \widehat{q}^1_n)$ to $\widetilde{y}_2 \sim (\widetilde{x}_1, \widehat{q}^2_1; ...; \widetilde{x}_n, \widehat{q}^2_n)$, where $\widehat{q}^i_\theta$ is defined by (8).*

This result was initially due to Taboga (2005). It precisely expresses the observational equivalence property that we already encountered in the counterexample presented in the previous section. It provides a test to determine whether more ambiguity aversion reduces demand. Observe that this test relies on two reduced probability distribution $\widetilde{y}_1$ and $\widetilde{y}_2$. However, it is not true that the ambiguity averse investor (u, ϕ_1) uses the corresponding reduced probability distribution $\widetilde{y}_1$ to evaluate the optimality of the different feasible portfolios. If he would do so, he would reevaluate the distribution of $\widetilde{y}_1$ for each portfolio, since vector $\widehat{q}_1$ is a function of α. In the smooth ambiguity aversion model, beliefs cannot be reduced to a single probability distribution on the state payoffs. But this lemma builds a bridge between the comparative statics of increased ambiguity aversion and the one of changes in risk in the classical EU model.

Let us examine how does changing function ϕ_1 into ϕ_2 modify the observationally equivalent probability distribution of the excess return. A first answer to this question is provided by the following lemma.

Lemma 3 *The following two conditions are equivalent:*

1. *Agent ϕ_2 is more ambiguity averse than agent ϕ_1;*
2. *Beliefs $\widehat{q}^2$ is dominated by $\widehat{q}^1$ in the sense of the monotone likelihood ratio order, i.e., $\widehat{q}^2_\theta/\widehat{q}^1_\theta$ is decreasing in θ, whenever $U(\alpha_1^*, 1) \leq U(\alpha_1^*, 2) \leq ... \leq U(\alpha_1^*, n)$.*

An increase in ambiguity aversion has an effect on demand that is observationally equivalent to a MLR-dominated shift in the prior beliefs. In other words, it distorts beliefs by favoring the worse priors in a very specific sense: if agent ϕ_1 prefers prior $\widetilde{x}_\theta$ over prior $\widetilde{x}_{\theta'}$, then, compared to agent ϕ_1, the more ambiguity averse agent ϕ_2 increases the distorted probability $\widehat{q}^2_{\theta'}$ relatively more than the probability $\widehat{q}^2_\theta$. Lemma 3 provides a justification to say that, in the case of the portfolio problem, more ambiguity aversion is observationally equivalent to more pessimism, i.e., to a MLR deterioration of beliefs. This result is central to prove our next proposition, in which we consider three dominance orders: first-degree stochastic dominance (FSD), second-degree stochastic dominance (SSD), and Rothschild and Stiglitz's increase in risk (IR).

Proposition 1 *Let D be one of the following three stochastic orders: FSD, SSD or IR. Suppose that $E\widetilde{x} > 0$, and that $(\widetilde{x}_1, ..., \widetilde{x}_n)$ can be ranked according to the stochastic order D . It implies that an increase in ambiguity aversion deteriorates the observationally equivalent probability distribution of the excess return in the sense of the stochastic order D: If $\exists k$ concave: $\phi_2 = k(\phi_1)$ and $\widetilde{x}_1 \preceq_D \widetilde{x}_2 \preceq_D ... \preceq_D \widetilde{x}_n$, then*

$$\widetilde{y}_2 \sim (\widetilde{x}_1, \widehat{q}_1^2; ...; \widetilde{x}_n, \widehat{q}_n^2) \preceq_D (\widetilde{x}_1, \widehat{q}_1^1; ...; \widetilde{x}_n, \widehat{q}_n^1) \sim \widetilde{y}_1.$$

Thus, if priors can be ranked according first-degree stochastic dominance, the increase in ambiguity aversion modifies the demand for the asset in the same direction as a FSD deterioration of the excess return in the expected utility model. The problem is that the comparative statics of an FSD deterioration in the excess return is in general ambiguous in the expected utility model. The intuition for this negative result is that a reduction in the return of an asset has a substitution effect and a wealth effect. As for the existence of Giffen goods in consumption theory, the wealth effect may induce an increase in the asset demand. Technically, it is not true in general that condition (7) holds when $\widetilde{y}_2 \preceq_{FSD} \widetilde{y}_1$. It is easy to see why: By definition of FSD, this would be true if and only if function $f(y) = yu'(w_0 + \alpha y)$ would be increasing, which is not true in general. As observed by Fishburn and Porter (1976), a sufficient condition for f to be increasing is that relative risk aversion $R(z) = -zu''(z)/u'(z)$ be smaller than unity.[5] It implies that this is also a sufficient condition for an increase of ambiguity aversion to reduce the demand for the ambiguous asset when priors can be ranked according to FSD. The same strategy can be used to examine the case when

[5] This is because $f'(y) = 1 - R(w_0 + \alpha y) + w_0 A(w_0 + \alpha y)$, with $A(z) = -u''(z)/u'(z)$.

priors can be ranked in the sense of Rothschild-Stiglitz increase in risk. In that case, the above proposition tells us that the observationally equivalent probability distribution $\widetilde{y}_2$ is an increase in risk compared to $\widetilde{y}_1$. Because f is not concave, this does not in general imply that condition (7) holds. As initially shown by Rothschild and Stiglitz (1971), it is not true in general that an increase in risk of the excess return of the risky asset reduces its demand. Hadar and Seo (1990) provided a sufficient condition, which guarantees that f is concave. This condition is that relative prudence is positive and less than 2, where relative prudence is defined by $P(z) = -zu'''(z)/u''(z)$ (Kimball (1990)). This proves the following result.

Proposition 2 *Suppose that* $u \in C^3$ *and* $E\widetilde{x} > 0$*. Any increase in ambiguity aversion reduces the demand for the risky asset if one of the following two conditions is satisfied:*

1. $(\widetilde{x}_1, ..., \widetilde{x}_n)$ *can be ranked according first-degree stochastic dominance, and* $R \leq 1$*;*

2. $(\widetilde{x}_1, ..., \widetilde{x}_n)$ *can be ranked according to the Rothschild and Stiglitz's riskiness order, and* $0 \leq P^r \leq 2$.

More generally, if the set of marginals can be ranked according to the SSD order, an increase in ambiguity aversion reduces the demand for the risky asset if relative risk aversion is less than unity, and relative prudence is positive and less than two. In the case of power utility function, relative prudence equals relative risk aversion plus one. This implies that when relative risk aversion is constant, and when priors can be ranked according to SSD, any increase in ambiguity aversion reduces the demand for the ambiguous asset if relative risk aversion is less than unity. This condition is not very convincing, since relative risk aversion is usually assumed to be larger than unity. Arguments have been provided based on introspection (Drèze (1981), Kandel and Stambaugh (1991), Gollier (2001)) or on the equity premium puzzle that can be solved in the canonical model only with a degree of relative risk aversion exceeding 40.

Rather than limiting the set of utility functions yielding an unambiguous effect, an alternative approach consists in restricting the set of priors. To do this, let us first introduce the following concepts, which rely on the location-weighted-probability function T_θ that is defined as follows:

$$T_\theta(x) = \int_{x_-}^{x} t dF_\theta(t). \tag{9}$$

Following Gollier (1995), we say that $\widetilde{x}_2$ is dominated by $\widetilde{x}_1$ in the sense of Central Dominance if there exists a nonnegative scalar m such that $T_2(x) \leq mT_1(x)$ for all $x \in [x_-, x_+]$.[6] Gollier (1995) showed that $\widetilde{x}_2 \preceq_{CD} \widetilde{x}_1$ is necessary and sufficient to guarantee that all risk-averse investors reduce their demand for the risky asset whose distribution of excess return goes from $\widetilde{x}_1$ to $\widetilde{x}_2$. SSD-dominance is not sufficient for CD-dominance. Notice that $\widetilde{x}_1$ and $\widetilde{x}_2$ in the counterexample of the previous section violate the CD condition. It implies that there exists a concave utility function such that the demand for the asset is increased when beliefs go from $(\widetilde{x}_1, 5\%; \widetilde{x}_2, 95\%)$ to the riskier $\widetilde{x}_1$.

Here is a partial list of stochastic orders that have been shown to belong to the wide set of CD:

[6] There is no simple interpretation of this stochastic order in the literature. However, observe that replacing $\widetilde{x}_1$ by $\widetilde{x}_2 \sim (\widetilde{x}_1, m; 0, 1-m)$ implies that $T_2 = mT_1$. This proportional probability transfer to the zero excess return has no effect on the risk-averse investor's demand for the risky asset. This explains the presence of the arbitrary scalar m in the definition of CD. Moreover a CD shift with $m = 1$ requires a reduction of the location-weighted-probability function T. For example, if one divides a probability mass p at some return $x = r < 0$ in two equal masses $(p/2, p/2)$, the transfer of mass to the left must be to the left of $2r$, which is a strong condition.

- Monotone Likelihood Ratio order (MLR) (Ormiston and Schlee (1993)). Notice that MLR is a subset of FSD.

- Strong Increase in Risk (Meyer and Ormiston (1985)): The excess return $\widetilde{x}_2$ is a strong increase in risk with respect to $\widetilde{x}_1$ if they have the same mean and if any probability mass taken out of some of the realizations of $\widetilde{x}_1$ is transferred out of the support of this random variable.

- Simple Increase in Risk (Dionne and Gollier (1992)): Random variable $\widetilde{x}_2$ is a simple increase in risk with respect to $\widetilde{x}_1$ if they have the same mean and $x(F_1(x) - F_2(x))$ is nonnegative for all x.

- Monotone Probability Ratio order (MPR) (Eeckhoudt and Gollier (1995), Athey (2002)): When the two random variables have the same support, we say that $\widetilde{x}_2$ is dominated by $\widetilde{x}_1$ in the sense of MPR if the cumulative probability ratio $F_2(x)/F_1(x)$ is nonincreasing. It can be shown that MPR is more general than MLR, but is still a subset of FSD.

The next result allows us to relax the conditions on u, but at the cost of restricting the set of priors.

Proposition 3 *Suppose that $E\widetilde{x} > 0$. Any increase in ambiguity aversion reduces the demand for the risky asset if the set of priors $(\widetilde{x}_1, ..., \widetilde{x}_n)$ can be ranked according to both SSD dominance and central dominance, that is, if $\widetilde{x}_\theta \preceq_{SSD} \widetilde{x}_{\theta+1}$ and $\widetilde{x}_\theta \preceq_{CD} \widetilde{x}_{\theta+1}$ for all $\theta = 1, ..., n-1$.*

To illustrate, because we know that MLR yields both first-degree stochastic dominance and central dominance, we directly obtain the following corollary.

Corollary 1 *Suppose that $E\widetilde{x} > 0$ and that $(\widetilde{x}_1, ..., \widetilde{x}_n)$ can be ranked according to the monotone likelihood ratio order. Then, any increase in ambiguity aversion reduces the demand for the risky asset.*

In this case, we conclude that ambiguity aversion and risk aversion goes into the same direction. A more general corollary holds where the MLR order is replaced by the more general MPR order.

It is noteworthy that the comparative statics of ambiguity aversion is much simpler when considering market participation. Of course, as observed in Lemma 1, our basic model is not well fitted to examine this question, since all agents should have a positive demand for equity as soon as the equity premium is positive (second order risk aversion).[7] Let us introduce a fixed cost C for market participation, so that the new model is with $U(\alpha, \theta) = Eu(w_0 - C + \alpha \widetilde{x}_\theta)$, $V(\alpha) = \phi^{-1}\left(\sum_\theta q_\theta \phi(U(\alpha, \theta))\right)$, $\overrightarrow{\alpha} = \arg\max V(\alpha)$, and $\alpha^* = \overrightarrow{\alpha}$ if $V(\overrightarrow{\alpha}) \geq u(w_0)$, and $\alpha^* = 0$ otherwise. Obviously, because V is the certainty equivalent of $U(\alpha, \widetilde{\theta})$ under function ϕ, an increase in the concavity of ϕ reduces $V(\alpha)$ for all α. Thus the condition for market participation $V(\overrightarrow{\alpha}) \geq u(w_0)$ is less likely to hold when ambiguity aversion is increased. This means that ambiguity aversion may explain the market participation puzzle (Haliassos and Bertaut (1995)).

4 Asset prices with complete markets

In this section, we extend the focus of our analysis to the effect of ambiguity aversion to the price of contingent claims. Consider a Lucas tree economy with a risk-averse and ambiguity averse representative agent whose preferences are characterized by increasing and concave functions (u, ϕ_i). Each agent is endowed with a tree producing an uncertainty quantity of fruits at the end of the period. There are S possible states of nature, with c_s denoting the number of fruits produced by trees in state s, $s = 1, ..., S$. The distribu-

[7]Thus, our story of the role ambiguity aversion to explain the market participation puzzle differs from the one by Dow and Werlang (1992) and Epstein and Schneider (2007), who consider a MEU model without participation cost.

tion of states is subject to some parameter uncertainty. Parameter θ can take value $1, ..., n$ with probabilities $(q_1, ..., q_n)$, and $p_{s|\theta}$ is the probability of state s conditional to θ. Let $p_s = \Sigma_\theta q_\theta p_{s|\theta}$ denote the unconditional probability of state s. Ex ante, there is a market for contingent claims. Agents trade claims of fruits contingent on the harvest. Assuming complete markets, the ambiguity averse and risk-averse agent whose preferences are given by the pair (u, ϕ_i) solves the following problem:

$$\max_{(x_1,...,x_S)} \quad \sum_{\theta=1}^{n} q_\theta \phi_i \left(\sum_{s=1}^{S} p_{s|\theta} u(x_s) \right), \text{ s.t. } \quad \sum_{s=1}^{S} \Pi_s (x_s - c_s) = 0, \tag{10}$$

where $x_s - c_s$ is the demand for the Arrow-Debreu security associated to state s, and Π_s is the price of that contingent claim. The first-order conditions for this program are written as

$$u'(x_s) \left[\sum_{\theta=1}^{n} q_\theta \phi_i' \left(\sum_{s'=1}^{S} p_{s'|\theta} u(x_{s'}) \right) p_{s|\theta} \right] = \lambda \Pi_s, \tag{11}$$

for all s, where λ is the Lagrange multiplier associated to the budget constraint. The market-clearing conditions impose that $x_s = c_s$ for all s, which implies the following equilibrium state prices :

$$\Pi_s^i = \widehat{p}_s^i u'(c_s), \tag{12}$$

for all s, where the distorted state probability $\widehat{p}_s^i$ is defined as follows:

$$\widehat{p}_s^i = \sum_{\theta=1}^{n} \widehat{q}_\theta^i p_{s|\theta} \text{ with } \widehat{q}_\theta^i = \frac{q_\theta \phi_i' \left(Eu(\widetilde{c}_\theta) \right)}{\sum_{t=1}^{n} q_t \phi_i' \left(Eu(\widetilde{c}_t) \right)}, \tag{13}$$

where $\widetilde{c}_\theta$ is distributed as $(c_1, p_{1|\theta}; ..., c_S, p_{S|\theta})$. Under ambiguity neutrality, we have that $\widehat{q}_\theta^i = q_\theta$, and $\widehat{p}_s^i$ is the true probability of state s computed from the compound first and second order probabilities. The aversion to ambiguity of the representative agent affects the equilibrium state prices in a way that is observationally equivalent to a distortion of beliefs in the EU model. This distortion takes the form of a transformation of the subjective prior distribution from $(q_1, ..., q_n)$ to $(\widehat{q}_1^i, ..., \widehat{q}_n^i)$ that is equivalent to the previous section with $\widetilde{c}_\theta = w_0 + \alpha_1^* \widetilde{x}_\theta$. Lemma 3 implies that $\widehat{q}^2$ is dominated by $\widehat{q}^1$ in the sense of MLR when ϕ_2 is more ambiguity averse than agent ϕ_1. The next proposition is a direct consequence of this observation.

Proposition 4 *Suppose that the set of priors $(\widetilde{c}_1, ..., \widetilde{c}_n)$ can be ranked according the stochastic order D (D =FSD, SSD or IR) . It implies that an increase in ambiguity aversion deteriorates the observationally equivalent probability distribution of consumption in the sense of the stochastic order D : If $\exists k$ concave: $\phi_2 = k(\phi_1)$ and $\widetilde{c}_1 \preceq_D \widetilde{c}_2 \preceq_D ... \preceq_D \widetilde{c}_n$, then $(\widetilde{c}_1, \widehat{q}_1^2; ...; \widetilde{c}_n, \widehat{q}_n^2) \preceq_D (\widetilde{c}_1, \widehat{q}_1^1; ...; \widetilde{c}_n, \widehat{q}_n^1)$.*

It is easy to reinterpret this result in terms of the impact of ambiguity aversion on the price kernel $\pi_s = \Pi_s/p_s$. Suppose that $c_s \neq c_{s'}$ for all (s, s'), so that we can substitute index $s = 1, ..., n$ by another index $s = c_1, ..., c_S$. In Figure 1, we have drawn the state price $\pi_s = \widehat{p}_s^i u'(c_s)/p_s$ as a function of c_s. Under ambiguity neutrality, this is a decreasing function, because u' is decreasing. The slope of the curve $\pi(c)$ describes the degree of risk aversion of the agent. From Proposition 4, ambiguity aversion tends to reinforce risk aversion. Indeed, if the priors can be ranked by FSD, an increase in ambiguity aversion has an effect on asset prices that is observationally equivalent to a FSD-deteriorating shift in beliefs, i.e., it tends to transfer the distorted probability mass $\widehat{p}$ from the good states to the bad ones. The corresponding shift in $\pi_s = \widehat{p}_s u'(c_s)/p_s$ is described in Figure 1a. If the priors can be ranked according to their riskiness, an increase in ambiguity aversion tends to transfer the distorted probability mass to the extreme states. This implies convexifying the price kernel, as depicted in Figure 1b.

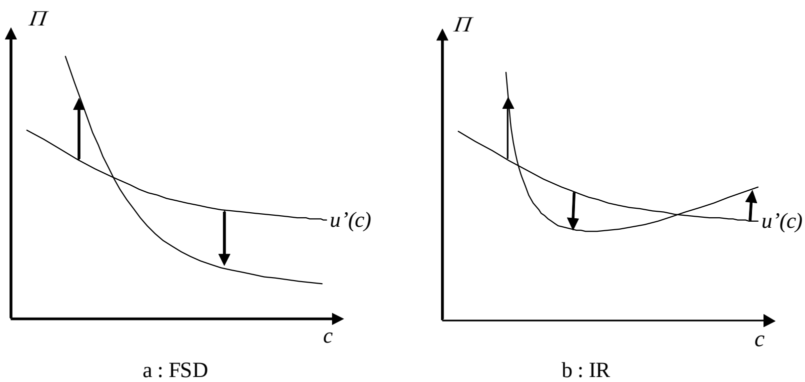

Figure 1: The effect of an increase in ambiguity aversion on the price kernel, when the priors can be ranked by the FSD order (a), or by the Rothschild-Stiglitz riskiness order (b).

The equilibrium price of trees equals $P_i = \Sigma_s \Pi_s^i c_s / \Sigma_s \Pi_s^i$. It is easily checked that this implies that

$$\sum_{\theta=1}^{n} q_\theta \phi_i'(Eu(\widetilde{c}_\theta))E\left[(\widetilde{c}_\theta - P_i)u'(\widetilde{c}_\theta)\right] = 0.$$

It is intuitive that more ambiguity aversion should reduce the price of equity, thereby increasing the equity premium. We obtain that P_2 is smaller than P_1 if and only the following inequality holds:

$$\sum_{\theta=1}^{n} q_\theta \phi_2'(Eu(\widetilde{c}_\theta))E\left[(\widetilde{c}_\theta - P_1)u'(\widetilde{c}_\theta)\right] \leq 0. \tag{14}$$

Technically, this condition is equivalent to condition (6) with $\widetilde{c}_\theta = w_0 + \alpha_1^* \widetilde{x}_\theta$, $\alpha_1^* = 1$ and $P_1 = w_0$. We conclude this section with the following proposition, which is a direct consequence of this observation together with the results presented in the previous section.

Proposition 5 *An increase in ambiguity aversion raises the equity premium if one of the following conditions is satisfied:*

1. $(\widetilde{c}_1, ..., \widetilde{c}_n)$ *can be ranked according first-degree stochastic dominance, and* $R \leq 1$*;*

2. $(\widetilde{c}_1, ..., \widetilde{c}_n)$ *can be ranked according to the Rothschild and Stiglitz's riskiness order, and* $0 \leq P^r \leq 2$*;*

3. $(\widetilde{c}_1 - P_1, ..., \widetilde{c}_n - P_1)$ *can be ranked according to both Second-degree Stochastic Dominance and Central Dominance, where* P_1 *is the initial price of equity.*

Property 3 implies for example that the equity premium is increasing in the degree of ambiguity aversion of the representative agent if the set of priors $(\widetilde{c}_1, ..., \widetilde{c}_n)$ can be ranked according to the MLR/MPR order.

5 Conclusion

In this paper, we explored the determinants of the demand for risky assets and of asset prices when investors are ambiguity averse. We have shown that, contrary to the intuition, ambiguity aversion may yield an increase in the demand for the risky and ambiguous asset, and a reduction in the demand for the safe one. In the same fashion, it is not true in general that ambiguity aversion raises the equity premium in the economy. We have first shown that the qualitative effect of an increase in ambiguity aversion in these settings is observationally equivalent to that of a shift in the beliefs of the investor in the standard EU model. If the set of plausible priors can be ranked according to the first degree stochastic dominance order, this shift is first degree stochastic deteriorating, whereas it is risk-increasing if these priors can be ranked according to the Rothschild-Stiglitz risk order. The problem originates from the observation already made by Rothschild and Stiglitz (1971) and Fishburn and Porter (1976) that a FSD/SSD deteriorating shift in the distribution of the return of the risky asset has an ambiguous effect on the demand for that asset in the EU framework. We heavily relied on the literature that emerged from this negative result to provide some sufficient conditions for any increase in ambiguity aversion to yield a reduction in the demand for the risky asset and an increase in the equity premium.

In most cases however, an increase in ambiguity aversion reduces the demand for the ambiguous asset, and it raises the equity premium. Two sets of findings confirm this view. First, the numerical analyses in the existing literature all go in that direction. We also showed that this is always true when the first and second order probability distributions are normal, and the pair (u, ϕ) are exponential-power functions. Second, some of our sufficient conditions cover a wide set of realistic situations. For example, if the set of priors can be ranked according to the well-known monotone likelihood ratio order, then it is always true that an increase in ambiguity aversion raises the equity premium. Our conclusion is that the potential existence of a counterintuitive effect of ambiguity aversion plays a role similar to the potential existence of Giffen goods in consumption theory. The observationally equivalent FSD deterioration of more ambiguity aversion has a wealth effect on the demand for the asset that may dominate the substitution effect. This is a rare event, but theoretical progress can rarely be made without understanding the mechanism that generates it. After all, the existence of Giffen goods is taught in Microeconomics 101.

References

Arrow, K.J., (1963), Liquidity preference, Lecture VI in "*Lecture Notes for Economics 285, The Economics of Uncertainty*", pp 33-53, undated, Stanford University.

Athey, S.C., (2002), Monotone Comparative Statics Under Uncertainty, *Quarterly Journal of Economics*, 117(1) : 187-223.

Abel, A.B., (2002), An exploration of the effects of optimism and doubt on asset returns, *Journal of Economic Dynamics and Control*, 26, 1075-1092.

Chen, Z;, and L. Epstein, (2002), Ambiguity, risk, and asset returns in continuous time, *Econometrica*, 70, 1403-1443.

Cecchetti, S.G., P.-S. Lam and N.C. Mark, (2000), Asset pricing with distorted beliefs: Are equity returns too good to be true?, *American Economic Review*, 90, 787-805.

Collard, F., S. Mukerji, K. Sheppard, and J.-M. Tallon, (2009), Ambiguity and historical equity premium, unpublished manuscript, Oxford University.

Dionne, G., and C. Gollier, (1992), Comparative Statics Under Multiple Sources of Risk with Applications to Insurance Demand, *The Geneva Papers on Risk and Insurance Theory*, 17.

Dow, J., and S. Werlang, (1992), Uncertainty aversion, risk aversion and the optimal choice of portfolio, *Econometrica,* 60, 197-204.

Drèze, J. H. (1981). "Inferring Risk Tolerance from Deductibles in Insurance Contracts", *The Geneva Papers* 6, 48-52.

Eeckhoudt, L., and C. Gollier, (1995), Demand for Risky Assets and the Monotone Probability Ratio Order, *Journal of Risk and Uncertainty*, 11, 113-122.

Epstein, L. G. and M. Schneider, (2007), Learning Under Ambiguity, *Review of Economic Studies*, 74, 1275-1303.

Epstein, L.G., and T. Wang, (1994), Intertemporal asset pricing under Knightian uncertainty, *Econometrica*, 62, 283-322.

Fishburn, P., and B. Porter, (1976), Optimal portfolios with one safe and one risky asset: Effects of changes in rate of return and risk, *Management Science*, 22, 1064-73.

Gilboa, I. and D. Schmeidler, (1989), Maximin expected utility with non-unique prior, *Journal of Mathematical Economics*, 18, 141, 153.

Gollier, C., (1995), The Comparative Statics of Changes in Risk Revisited, *Journal of Economic Theory*, 66, 522-535.

Gollier, C., (2001), *The economics of risk and time*, MIT Press, Cambridge.

Hadar, J. and T.K. Seo, (1990), The Effects of Shifts in a Return Distribution on Optimal Portfolios, *International Economic Review*, 31, 721-736.

Haliassos, M., and C.C. Bertaut, (1995), Why do so few hold stocks?, *Economic Journal*, 105, 1110-1129.

Hansen, L.P., and T.J. Sargent, (2008), *Robustness*, Princeton University Press, Princeton and Oxford.

Hansen, L.P., T.J. Sargent, and T.D. Tallarini, (1999), Robust permanent income and pricing, *Review of Economic Studies*, 66.

Ju, N., and J. Miao, (2009), Ambiguity, learning and asset returns, unpublished manuscript, Boston University.

Kandel, S., and R.F. Stambaugh, (1991), Assets returns and intertemporal preferences, *Journal of Monetary Economics*, 27, 39-71.

Kimball, M.S., (1990), Precautionary savings in the small and in the large, *Econometrica*, 58, 53-73.

Klibanoff, P., M. Marinacci and S. Mukerji, (2005), A smooth model of decision making under ambiguity, *Econometrica* 73(6), 1849-1892.

Maenhout, P., (2004), Robust portfolio rules and asset pricing, *Review of Financial Studies*, 17, 951-983.

Meyer, J. and M.B. Ormiston, (1985), Strong Increase in Risk and Their Comparative Statics, *International Economic Review*, 26, 425-437.

Ormiston, M. and E. Schlee, (1993), Comparative Statics Under Uncertainty for a Class of Economic Agents, *Journal of Economic Theory,* 61, 412-422.

Rothschild, M. and J. Stiglitz, (1970), Increasing Risk: I. A Definition, *Journal of Economic Theory,* 2, 225-243.

Rothschild, M. and Stiglitz, J.E., (1971), Increasing Risk II: Its Economic Consequences, *Journal of Economic Theory*, 3, pp. 225-243.

Segal, U., (1987), The Ellsberg paradox and risk aversion: An anticipated utility approach, *International Economic Review*, 87, 175-202.

Segal, U. and A. Spivak, (1990), First order versus second order risk aversion, *Journal of Economic Theory*, 51, 111-125.

Taboga, M., (2005), Portfolio selection with two-stage preferences, *Finance Research Letters*, 2, 152-164.

Appendix: Proofs

Proof of Lemma 1

By concavity of the objective function in (1) with respect to α, we have that α^* is positive if the derivative of this objective function with respect to α evaluated at $\alpha = 0$ is positive. This derivative is written as

$$\sum_{\theta=1}^{n} q_\theta \phi'(u(w_0)) E\widetilde{x}_\theta u'(w_0) = u'(w_0)\phi'(u(w_0))E\widetilde{x},$$

where $E\widetilde{x} = \Sigma_\theta q_\theta E x_\theta$ is the equity premium. This concludes the proof. ■

Proof of Lemma 3

Because ϕ_1 and ϕ_2 are increasing in U, there exists an increasing function k such that $\phi_2(U) = k(\phi_1(U))$, or $\phi_2'(U) = k'(\phi_1(U))\phi_1'(U)$ for all U. Using definition (8), we obtain that

$$\frac{\widehat{q}_\theta^2}{\widehat{q}_\theta^1} = k'(\phi_1(U(\alpha_1^*, \theta))\frac{\sum_{t=1}^{n} q_t \phi_1'\left(U(\alpha_1^*, t)\right)}{\sum_{t=1}^{n} q_t \phi_2'\left(U(\alpha_1^*, t)\right)} \tag{15}$$

for all $\theta = 1, ..., n$. The Lemma is a direct consequence of (15), in the sense that the likelihood ratio $\widehat{q}_\theta^2/\widehat{q}_\theta^1$ is decreasing in θ if k' is decreasing in ϕ_1. ■

Proof of Proposition 1

Suppose that $\widetilde{x}_1 \preceq_D \widetilde{x}_2 \preceq_D ... \preceq_D \widetilde{x}_n$. It implies that $U(\alpha_1^*, 1) \leq U(\alpha_1^*, 2) \leq ... \leq U(\alpha_1^*, n)$. We have to prove that $(\widetilde{x}_1, \widehat{q}_1^1; ....; \widetilde{x}_n, \widehat{q}_n^1)$ is preferred to $(\widetilde{x}_1, \widehat{q}_1^2; ....; \widetilde{x}_n, \widehat{q}_n^2)$ by all utility functions v in C, that is

$$\sum_{\theta=1}^{n} \widehat{q}_\theta^2 Ev(\widetilde{x}_\theta) \leq \sum_{\theta=1}^{n} \widehat{q}_\theta^1 Ev(\widetilde{x}_\theta),$$

where C is the set of increasing functions if D=FSD, C is the set of increasing and concave functions if D=SSD, and C is the set of concave functions if D=IR. Combining the conditions that $\widetilde{x}_\theta \preceq_D \widetilde{x}_{\theta+1}$ and that $v \in C$ implies that $Ev(\widetilde{x}_\theta)$ is increasing in θ. The above inequality is obtained by combining this property with the fact that $\widehat{q}^2$ is dominated by $\widehat{q}^1$ in the sense of MLR (Lemma 3), a special case of FSD. ■

Proof of Proposition 3

The following lemma is useful to prove Proposition 3. Let K denote interval $[\min_\theta \alpha^*_\theta, \max_\theta \alpha^*_\theta]$, where α^*_θ is the maximand of $Eu(w_0 + \alpha\widetilde{x}_\theta)$.

Lemma 4 *Consider a specific set of priors $(\widetilde{x}_1, ..., \widetilde{x}_n)$ and a concave utility function u. They characterize function U defined by $U(\alpha, \theta) = Eu(w_0 + \alpha\widetilde{x}_\theta)$. Consider a specific scalar α^*_1 in K. The following two conditions are equivalent:*

1. *Any agent ϕ_2 that is more ambiguity averse than agent ϕ_1 with demand α^*_1 for the ambiguous asset will have a demand for it that is smaller than α^*_1;*

2. *There exists $\overline{\theta} \in \{1, ..., n\}$ such that*
$$U(\alpha^*_1, \theta)U_\alpha(\alpha^*_1, \theta) \geq U(\alpha^*_1, \overline{\theta})U_\alpha(\alpha^*_1, \theta) \tag{16}$$
for all $\theta \in \{1, ..., n\}$.

Proof: We first prove that condition 2 implies condition 1. Consider an agent $\phi_2 = k(\phi_1)$ that is more ambiguity averse than agent ϕ_1, so that the transformation function k is concave. The condition thus implies that

$$k'(\phi_1(U(\alpha^*_1, \theta)))U_\alpha(\alpha^*_1, \theta) \leq k'(\phi_1(U(\alpha^*_1, \overline{\theta})))U_\alpha(\alpha^*_1, \theta)$$

for all θ. Multiplying both side of this inequality by $q_\theta\phi'_1(U(\alpha^*_1, \theta)) \geq 0$ and summing up over all θ yields

$$\sum_{\theta=1}^{n} q_\theta\phi'_2(U(\alpha^*_1, \theta))U_\alpha(\alpha^*_1, \theta) \leq k'(\phi_1(U(\alpha^*_1, \overline{\theta}))) \sum_{\theta=1}^{n} q_\theta\phi'_1(U(\alpha^*_1, \theta))U_\alpha(\alpha^*_1, \theta) = 0.$$

The last equality comes from the assumption that agent ϕ_1 selects portfolio α_1^*. Thus, condition (6) is satisfied, thereby implying that α_2^* is less than α_1^*.

We then prove that condition 1 implies condition 2. Without loss of generality, rank the θs such that $U(\alpha_1^*, \theta)$ is increasing in θ. By contradiction, suppose that there exists a $\theta_0 < n$ such that $U_\alpha(\alpha_1^*, \theta_0) \geq 0$ and $U_\alpha(\alpha_1^*, \theta_0 + 1) \leq 0$. Select a prior distribution $(q_1, ..., q_n)$ so that $q_\theta = 0$ for all θ except for θ_0 and $\theta_0 + 1$. Select $q_{\theta_0} = q \in [0, 1]$ so that

$$q\phi_1'(U(\alpha_1^*, \theta_0))U_\alpha(\alpha_1^*, \theta_0) + (1-q)\phi_1'(U(\alpha_1^*, \theta_0 + 1))U_\alpha(\alpha_1^*, \theta_0 + 1) = 0, \tag{17}$$

so that agent ϕ_1 selects portfolio α_1^*. Consider any concave transformation function k. It implies that

$$\begin{aligned} & \sum_{\theta=1}^{n} q_\theta \phi_2'(U(\alpha_1^*, \theta))U_\alpha(\alpha_1^*, \theta) \\ = \; & qk'(\phi_1(U(\alpha_1^*, \theta_0)))\phi_1'(U(\alpha_1^*, \theta_0))U_\alpha(\alpha_1^*, \theta_0) \\ & +(1-q)k'(\phi_1(U(\alpha_1^*, \theta_0 + 1)))\phi_1'(U(\alpha_1^*, \theta_0 + 1))U_\alpha(\alpha_1^*, \theta_0 + 1). \end{aligned}$$

Because $U_\alpha(\alpha_1^*, \theta_0 + 1) \leq 0$ and $k'(\phi_1(U(\alpha_1^*, \theta_0 + 1))) \leq k'(\phi_1(U(\alpha_1^*, \theta_0)))$, this is larger than

$$k'(\phi_1(U(\alpha_1^*, \theta_0)))\left[q\phi_1'(U(\alpha_1^*, \theta_0))U_\alpha(\alpha_1^*, \theta_0) + (1-q)\phi_1'(U(\alpha_1^*, \theta_0 + 1))U_\alpha(\alpha_1^*, \theta_0 + 1)\right] = 0.$$

It implies that condition (6) is violated, implying in turn that α_2^* is larger than α_1^*, a contradiction. ■

If we rank the θ in such a way that $U(\alpha_1^*, \theta)$ is monotone in θ, condition 2 is essentially a single-crossing property of function $U_\alpha(\alpha_1^*, \theta)$. To illustrate, suppose that $u(z) = -A^{-1}\exp(-Az)$ and $\widetilde{x}_\theta \sim N(\theta, \sigma^2)$, which implies that $U(\alpha, \theta)$ is increasing in θ and is given by equation (2). It implies that $U_\alpha(\alpha, \theta)$ has the same sign as $\theta - \alpha A\sigma^2$. It implies in turn that condition 2 in Lemma 4 is satisfied with $\overline{\theta} = \alpha A\sigma^2$. Our Lemma implies that ambiguity aversion reduces the demand for the risky asset in the exponential/normal case. This was shown in Section 2 in the special case of power ϕ functions.

We need to prove a second lemma in order to prepare for the proof of Proposition 3.

Lemma 5 *Suppose that* $\widetilde{x}_2$ *is centrally dominated by* $\widetilde{x}_1$. *Then,* $E\widetilde{x}_2 u'(w_0 + \alpha\widetilde{x}_2) \leq 0$ *for any* $\alpha \geq 0$ *such that* $E\widetilde{x}_1 u'(w_0 + \alpha\widetilde{x}_1) \leq 0$.

Proof: By assumption, there exists a positive scalar m such that $T_2(x) \leq mT_1(x)$. Integrating by part, we have that

$$\begin{aligned} E\widetilde{x}_2 u'(w_0 + \alpha\widetilde{x}_2) &= \int_{x_-}^{x_+} u'(w_0 + \alpha x) x dF_2(x) \\ &= u'(w_0 + \alpha x_+) T_2(x_+) - \alpha \int_{x_-}^{x_+} u''(w_0 + \alpha x) T_2(x) dx. \end{aligned}$$

This implies that

$$\begin{aligned} E\widetilde{x}_2 u'(w_0 + \alpha\widetilde{x}_2) &\leq m\left[u'(w_0 + \alpha x_+) T_1(x_+) - \alpha \int_{x_-}^{x_+} u''(w_0 + \alpha x) T_1(x) dx\right] \\ &= mE\widetilde{x}_1 u'(w_0 + \alpha\widetilde{x}_1). \end{aligned}$$

By assumption, this is nonpositive. ■

We can now prove Proposition 3. Condition $\widetilde{x}_\theta \preceq_{SSD} \widetilde{x}_{\theta+1}$ implies that $U(\alpha, \theta+1) \geq U(\alpha, \theta)$, whereas, by Lemma 5, condition $\widetilde{x}_\theta \preceq_{CD} \widetilde{x}_{\theta+1}$ implies that $U_\alpha(\alpha, \theta) \leq 0$ whenever $U_\alpha(\alpha, \theta+1) \leq 0$. This latter result implies that there exists a $\overline{\theta}$ such that $(\theta - \overline{\theta})U_\alpha(\alpha, \theta) \leq 0$ for all θ. This immediately yields condition 2 in Lemma 4, which is sufficient for our comparative static property. ■