# A Theory of Disappointment

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**Abstract:**

We develop an axiomatic model of decision making under risk based on the concept of disappointment aversion. Disappointment is measured using the expected utility of the lottery as a reference point. From an axiomatics, which is supported by many studies on psychology and emotions research, we derive a general class of model that is lottery dependent and which can be viewed as the theoretical ground of the work of Loomes and Sugden (1986). We then impose some restrictions on the preference functional in order to obtain practical implications in finance. We thus obtain a subjective utility model where decision weights are lottery dependent and depend on behavior towards disappointment. Using a weak restriction on the change of probability measure, we avoid stochastic dominance inconsistency. The model is consistent with the Allais paradox and embeds expected utility theory as a special case. It also allows a strict separation between the concepts of risk aversion and that of disappointment aversion. Moreover, it justifies the CAPM without requiring normality of asset returns or quadratic utility function. Finally, it can provide new insights on the equity premium puzzle.

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# 1 Introduction

Outcomes of decisions often give rise to the experience of emotions. We experience positive emotions when a decision turns out favorably and we experience negative emotions when a decision turns out unfavorably. Psychologists have long recognized the importance of anticipatory emotions. Two of the emotions that attracted most attention from those researchers are regret and disappointment (Mellers 2000, Zeelenberg *et alii* 2000). Disappointment is experienced when the chosen option turns out to be worse than expected. In the psychology field, research has shown that disappointment has a negative impact on the utility that is derived from decision outcomes (Mellers 2000) and on consumer satisfaction (Inman *et alii* 1997). These ideas about the relevance of disappointment for decision making are consistent with those in emotions research. As Frijda (1994) pointed out, "actual emotion, affective response, anticipation of future emotion can be regarded as the primary source of decisions". Research on emotions has shown that disappointment is a frequent and intense emotion. Schimmack and Diener (1997) analyzed the frequency and intensity of emotions experienced in real-life events. They show that disappointment is the first most negative emotion. Moreover, Weiner *et alii* (1979) found that disappointment is one of the most frequently experienced emotions.

The role of disappointment in decision making was first formalized independently by Bell (1985) and Loomes and Sugden (1986). In these theories, individuals not only experience disappointment and elation as a consequence of making decisions, but also anticipate them and take them into account when making decisions. Thus, decisions are partly based on disappointment aversion or, in other words, the tendency to make choices in such a way as to minimize the future experience of disappointment.

Although these models constitute very appealing alternatives to the Expected Utility hypothesis (henceforth EU) for describing choice behavior, they have been widely neglected by the applied literature.[1,2] A reason for this relative ignorance may be the lack of an axiomatic foundation of the models and, consequently, the impossibility of conducting experimental tests of the axioms. A second reason may be that the functional used remains somewhat too general for providing direct applications to business. The aim of this paper is to remedy to this situation.

We thus develop an axiomatic model of decision making under risk grounded on a concept of disappointment aversion which can be viewed as the theoretical foundation of Loomes and Sugden's model. The expected utility of a lottery is then the reference point for measuring disappointment and elation. Some other possible reference points have been considered in the literature, such as the best possible outcome (Grant and Kajii 1998), the certainty equivalent of the lottery (Gul 1991),[3] the other possible outcomes of the lottery (Delquie and Cillo 2005) and the decision maker's aspiration level (Diecidue and van de Ven 2004). Sugden (2003) provides an axiomatic model with a reference lottery.

As defined by Loomes and Sugden, disappointment is a psychological reaction to an outcome that does not match up against prior expectations. Consequently, an individual compares the outcomes within a given prospect, giving rise to the possibility of disappointment (elation) when the outcome compares unfavorably (favorably) with what it might have been. The satisfaction that an individual is assumed to feel after a lottery has been run can be split into two elements: (i) the satisfaction due to the ownership of the realized prize, which is generally identified to the utility of wealth and (ii) elation (or disappointment) which depends on the difference between the level actually reached by the utility of wealth and its expected value. Basically, disappointment is assumed to be in direct proportion to the difference between what was expected and what has actually been got. There is a lot of empirical evidence that support this assumption in the psychological literature (Van Dijk and Van der Pligt 1996, Zeelenberg *et alii* 2002, Van Dijk *et alii* 2003).

Our model differs from Gul's theory in that it assumes that disappointment can be defined ex post and independently of the certainty equivalent (henceforth CE) of the lottery being considered. Unlike

[1] Exceptions to this statement are papers by Inman et alii (1997) who study the impact of disappointment on post-choice valuation in the marketing area, and by Jia and Dyer (1996) who develop a standard measure of risk based on a closely related concept of disappointment in the management science area.

[2] For instance, unlike Gul's model of disappointment aversion, the theories of Bell (1985) and Loomes and Sugden (1986) have not been considered in the two prominent empirical investigations of non-expected utility theories by Harless and Camerer (1994) and Hey and Horme (1994).

[3] To our knowledge, this model constitutes the only competitive axiomatic theory of disappointment aversion. It has given rise to numerous applications. Considering the finance field, it has been used for either describing portfolio choice (Ang *et alii* 2004) or providing solutions to financial puzzles (Routledge and Zin 2004).

Gul's model it no longer implies betweenness and so it doesn't belong to the implicit expected utility class (Starmer 2000).[4] Alternatively, our approach preserves a fully choice-based foundation and may be classified as a lottery dependent utility model (Becker and Sarin 1987, Schmidt 2001). Moreover, unlike Gul's theory, it allows for disappointment aversion to be wealth dependent.[5]

The paper is organized as follows. After having reviewed the original formulation of the disappointment concept from Loomes and Sugden (1986), we develop the axiomatics of this general model of disappointment aversion (Section 2). In Section 3, we focus on a particular specification of this theory, called disappointment weighted utility theory, in order to get a more testable model and derive significant implications in finance. We also explore the properties of this model. In Section 4, we first derive a new risk measure based on disappointment aversion. We then establish that an appealing justification of using mean and variance as choice criteria dwells from the existence of disappointment averse decision-makers with constant marginal utility. Next, we study the equilibrium of a financial market populated by disappointment averse investors and show the existence of a CAPM relationship. Finally, considering a two assets economy, the Euler equations derived from the single-period financial market equilibrium exhibit properties that mitigate the equity premium puzzle. Section 5 concludes.

# 2 A Disappointment Theory

This section is devoted to presenting our theory of disappointment. Some introductory definitions and denominations are first given. From now on, we shall consider a set of lotteries whose prizes are monetary outcomes belonging to an interval $[a, b]$ of $\mathbb{R}$. It is denoted $\mathbb{L}\,[a, b]$ or, in short, $\mathbb{L}$. The cumulative distribution function of lottery $L$ will be labelled $F_L$. Lotteries may be discrete or not.[6] The density function of $F_L$ (if it exists) will be denoted $f_L$[7] and its support $supp(L)$. $\delta_w$ denotes a degenerate lottery whose outcome is $w$. If $\alpha$ belongs to $[0, 1]$ and if $L_1$ and $L_2$ belong to $\mathbb{L}$, then there exists a lottery $L$ whose cumulative distribution function $F_L$ is equal to $\alpha F_{L_1} + (1-\alpha)F_{L_2}$ and which belongs to $\mathbb{L}$. Lottery $L$ is called the $\alpha$-mixing of $L_1$ and $L_2$ and will be denoted $\alpha L_1 \oplus (1-\alpha)L_2$. If the supports of $L_1$ and $L_2$ are finite, then lottery $L$ may be viewed as a two-stage lottery.[8] Using conventional notations, preferences over prospects are denoted $\succcurlyeq$, with $\succ$ (strict preference) and $\sim$ (indifference).

## 2.1 A brief review of the Loomes and Sugden's disappointment model

According to Loomes and Sugden (1986) the ex post satisfaction of an agent facing a lottery includes two components: $(a)$ the ex post elementary utility $u(w)$ of winning the realized prize $w$ and $(b)$ the disappointment −or the elation− felt by the investor once the uncertainty is resolved. The ex post elementary utility $u(w)$ is the satisfaction that the considered agent would have felt if she had faced the degenerated lottery $\delta_w$. Elation (disappointment) occurs, once the uncertainty is resolved, if and only if the ex post elementary utility $u(w)$ is greater (lower) than its ex ante expected value $E_L\,[u(\widetilde{w})]$. Hence, the realized prize will be considered as the result of a "good or lucky drawing" or of a "bad or unlucky one" according to the fact that the satisfaction $u(w)$ got from the certain ownership of the realized prize $w$ is higher or lower than its prior expectation. Moreover, the intensity of the elation/disappointment felt, $ED\,(w)$, is assumed to depend on the gap between the elementary utility $u(w)$ and its expected value $E_L\,[u(\widetilde{w})]$. Formally:

$$ED\,(w) \overset{def}{=} D\,(u(w) - E_L\,[u(\widetilde{w})]) \tag{1}$$

[4] Implicit expected utility theories are based on the betweenness axiom (Dekel 1986) which is not considered as a suitable property. For instance, Camerer and Ho (1994) find extensive violations of betweenness.

[5] A property that is shared with the traditional Arrow-Pratt measure of risk aversion.

[6] In the first case, we shall use the usual denomination: $L = \left[w_1, ..., w_K; p_1^L, ..., p_K^L\right]$ with $p_k^L \geq 0$ and the following equivalence: $k < l \Leftrightarrow\ w_k < w_l$.

[7] *i.e.*: $f_L\,(w) = \frac{dF_L}{dw}\,(w)\ ,\ \forall w \in [a, b]\,.$

[8] A lottery is first randomly drawn from the set $\{L_1, L_2\}$ (with probability $\alpha$ $(1-\alpha)$ for $L_1(L_2)$) and the selected lottery is then run. As usual we assume that (i) getting the prize with probability one is the same as getting the prize for certain; (ii) the individual does not care about the order in which the lottery is described and (iii) the individual's perception of a lottery depends only on the net probabilities of receiving the various prizes.

where $u(.)$ is a strictly increasing function and if $D(.)$ is assumed to meet the following requirements:

$$D(0) = 0 \text{ and } \frac{dD}{dg}(.) > 0$$

The ex post satisfaction of an agent who is sensitive to elation or disappointment may be identified to her conditional satisfaction $U(L \mid w)$ whose value is:

$$\mathcal{U}(L \mid w) = u(w) + D(u(w) - E_L[u(\widetilde{w})])$$

Now, the corresponding ex ante satisfaction may be expressed as the expected ex post satisfaction:

$$\mathcal{U}(L) = E_L[\mathcal{U}(L \mid \widetilde{w})] = E_L[u(\widetilde{w})] + E_L[D(u(w) - E_L[u(\widetilde{w})])] = E_L[u(\widetilde{w})] + E_L[ED(\widetilde{w})] \tag{2}$$

The ex ante utility of a lottery is then the sum of the expected value of both its elementary utility and its corresponding elation/disappointment.

The aim of this section is to provide an axiomatic foundation to that intuition. We now depart from Loomes and Sugden's analysis and define the notion of zero disappointment prize. This reference point will allow us to separate the support of the lottery under review into two subsets on which both disappointment and elation prizes will be clearly defined.[9]

## 2.2 The notion of Zero Disappointment Prize

**Definition (Zero Disappointment Prize)** *Given an economic agent facing a lottery $L$ belonging to the set $\mathbb{L}[a,b]$ we shall call Zero Disappointment Prize (henceforth ZDP) the value of the prize implying neither elation nor disappointment. It will be denoted $z_L$.*

$$ED(z_L) \overset{def}{=} D(u(z_L) - E_L[u(\widetilde{w})]) = 0 \tag{3}$$

Such a definition will make sense if the ZDP of any lottery exists and is unique. The proof of the existence and uniqueness of the ZDP is postponed until subsection 2.4., once the necessary assumptions will have been introduced. From Equation (3), we get:

$$u(z_L) - E_L[u(\widetilde{w})] = 0 \Leftrightarrow D(u(z_L) - E_L[u(\widetilde{w})]) = 0$$

or, equivalently:

$$z_L = u^{-1}(E_L[u(\widetilde{w})]) = 0$$

Finally the ZDP of a lottery is defined by the above equation. An important feature of this approach is that the certainty equivalent (henceforth denoted CE) of a lottery does not generally coincide with the ZDP. Indeed, the link between these two notions can be exhibited noticing that:

$$CE(L) = u^{-1}(\mathcal{U}(L)) = u^{-1}(E_L[\mathcal{U}(L \mid \widetilde{w})]) \Leftrightarrow u(CE(L)) = u(z_L) - E_L[ED(\widetilde{w})]$$

This feature is worth being stressed because it makes one of the fundamental difference between the approach developed in this paper and that of Gul (1991), who provided the first attempt to axiomatize a disappointment averse behavior. Actually, in Gul's model, the ZDP of a lottery is identical to the CE of the lottery being considered. In our approach, the value of the CE of a particular lottery is above (below) its ZDP if elation (disappointment) is expected to occur.

[9] Applied in different contexts, this approach is similar to those adopted by Gul (1991) and Diecidue and van de Ven (2004).

## 2.3 Understanding disappointment effects and the ZDP

In this subsection, we present a general setting for highlighting the source of disappointment that will serve as a basis of the axiomatics of our model in the next subsection. Gul (1991) and Grant and Kajii (1998) use a very similar background in their analysis of disappointment effects. For the sake of simplicity, we work with the following binary lottery:

$$L \overset{def}{=} [a, b; 1-\pi, \pi]$$

whose expected gain is denoted $E_L[\widetilde{w}]$.

We now address the problem of learning agents ZDP's through observing their choices. We shall establish in this subsection that an investor will reveal what is the ZDP of the lottery which she faces, if she is asked to compare the two following compound lotteries:

$$\mathcal{L}_{\mathcal{L}}(\alpha, w) \overset{def}{=} \alpha L \oplus \ (1-\alpha)\,\delta_w \text{ and } \mathcal{L}_{CE(L)}(\alpha, w) \overset{def}{=} \alpha\delta_{CE(L)} \oplus (1-\alpha)\,\delta_w$$

where $L$ corresponds to the elementary lottery defined above, $\alpha$ is a real number belonging to $[0, 1\ ]$, $\delta_w$ is a degenerated lottery whose prize is $w$ and $CE(L)$ denotes the CE of $L$.

A first case is when the EU theory holds: then, there exists an infinity of ZDPs since we have:

$$\begin{aligned} \forall w &\in [a, b]\,,\ \forall \alpha \in [0, 1]\,,\ \forall L \in \mathbb{L}[a, b]\,, \\ \mathcal{L}_{CE(L)}(\alpha, w) &\sim \mathcal{L}_L(\alpha, w) \Leftrightarrow CE\left(\mathcal{L}_{CE(L)}(\alpha, w)\right) - CE\left(\mathcal{L}_L(\alpha, w)\right) = 0 \end{aligned}$$

We now assume that the EU theory no longer holds, giving rise to phenomena such as the Allais paradox. Compounding $L$ and $\delta_{CE(L)}$ with the degenerated lottery $\delta_a$ will have some interesting consequences. Since we have ruled out the case wherein subjects are indifferent between $\mathcal{L}_L(\alpha, a)$ and $\mathcal{L}_{CE(L)}(\alpha, a)$, two possible behaviors may occur. Both of them contradict the independence axiom which represents the cornerstone of the EU theory. They can be characterized as follows:

(**A**) subjects prefer $\mathcal{L}_L(\alpha, a)$ to $\mathcal{L}_{CE(L)}(\alpha, a)$, although they are indifferent between $L$ and $\delta_{CE(L)}$. Formally, we get :

$$L \backsim \delta_{CE(L)} \textit{ and } \mathcal{L}_{CE(L)}(\alpha, a) \prec \mathcal{L}_L(\alpha, a) \tag{4}$$

(**B**) subjects prefer $\mathcal{L}_{CE(L)}(\alpha, a)$ to $\mathcal{L}_L(\alpha, a)$, although they are indifferent between $L$ and $\delta_{CE(L)}$. Formally, we get :

$$L \backsim \delta_{CE(L)} \textit{ and } \mathcal{L}_L(\alpha, a) \prec \mathcal{L}_{CE(L)}(\alpha, a)$$

To spare space we only discuss in this subsection the case (**A**) that corresponds to the most frequently observed behavior.[10] It constitutes the basis of the Gul's discussion about the role of disappointment in accounting for the common ratio effect.[11] According to him, lottery $\delta_{CE(L)}$ is likely to suffer more than lottery $L$ when it is mixed with an *inferior* lottery such as lottery $\delta_a$ since the increase of the probability of disappointment when shifting from $\delta_{CE(L)}$ to $\mathcal{L}_{CE(L)}(\alpha, a)$ is higher than the corresponding variation when shifting from $L$ to $\mathcal{L}_L(\alpha, a)$.[12]

What happens, now, when lotteries $L$ and $\delta_{CE(L)}$ are mixed with a *superior* lottery? The increase of the probability of disappointment when shifting from $\delta_{CE(L)}$ to $\mathcal{L}_{CE(L)}(\alpha, b)$ is, here again, higher than the corresponding variation when shifting from $L$ to $\mathcal{L}_L(\alpha, b)$.[13] Apparently, compounding $L$ and $\delta_{CE(L)}$ with the superior lottery $\delta_b$ should imply the same preference reversal as before. In other words, we should observe the following preferences:

$$L \backsim \delta_{CE(L)} \textit{ and } \mathcal{L}_{CE(L)}(\alpha, b) \prec \mathcal{L}_L(\alpha, b)$$

[10] This is the case when the decision maker is risk averse in the usual sense.

[11] As Gul pointed out, "the lottery with the lower probability of disappointment suffers more than that with the higher when it is mixed with an inferior lottery. That is, if the lotteries were nearly indifferent initially, the lottery with the higher probability of disappointment becomes preferred after being mixed with the inferior lottery". In Gul, F. 1991. A theory of disappointment aversion. *Econometrica* **59** p. 668.

[12] See Appendix 1 for more details.

[13] See Appendix 1 for more details.

However, as pointed out by Grant and Kajii (1998) using a slightly different setting,[14] this result is not consistent with the natural intuition. Following them, we claim that agents are more likely to exhibit the following preferences:

$$L \backsim \delta_{CE(L)} \text{ and } \mathcal{L}_L(\alpha,b) \prec \mathcal{L}_{CE(L)}(\alpha,b) \quad (5)$$

This is precisely our point of departure from Gul's analysis who only focuses on the overall probability of getting disappointed. The way of reasoning in our model is as follows: in order to allow for obtaining two opposite results when lottery $L$ is mixed alternatively either with an inferior lottery or with a superior lottery, the values of the disappointing and/or the elating outcomes will be taken into account, together with their probability of occurring.[15] In the above example, when lottery $\mathcal{L}_{CE(L)}(\alpha,b)$ and $\mathcal{L}_L(\alpha,b)$ are considered, the disappointing outcomes are respectively $CE(L)$ and $a$, whereas the elating ones are both equal to $b$. Hence, disappointment should be much stronger in the second case than in the first one since $CE(L)$ is clearly greater than $a$. Moreover, the intensity of disappointment may not only compensate partially but overcompensate the effect of the increase in the probabilities of disappointment. Finally preferences such as (4) and (5) are likely to be observed.

Although our approach is connected with that of Grant and Kajii (1998), we adopt a different point of view. In their model, they identify the disappointment reference level to that of the best outcome of the lottery. Recall that Gul (1991) refers to the certainty equivalent of the lottery. In our case, following the spirit of Loomes and Sugden (1986), we will provide in the next subsection axioms that support the expected utility of the lottery under review as the reference level for measuring disappointment. Finally, we are led to draw the following conclusions from the previous discussion. It seems likely that $(i)$ there exists one real number $z_L$ (namely the ZDP) for which $\mathcal{L}_L(\alpha,z_L)$ is equivalent to $\mathcal{L}_{CE(L)}(\alpha,z_L)$ and that $(ii)$ when an agent is risk averse, she may be endowed with either of the following preferences:

$$L \backsim \delta_{CE(L)} \text{ and } \mathcal{L}_{CE(L)}(\alpha,a) \prec \mathcal{L}_L(\alpha,a) \text{ and } \mathcal{L}_L(\alpha,b) \prec \mathcal{L}_{CE(L)}(\alpha,b)$$

$$L \backsim \delta_{CE(L)} \text{ and } \mathcal{L}_L(\alpha,a) \prec \mathcal{L}_{CE(L)}(\alpha,a) \text{ and } \mathcal{L}_{CE(L)}(\alpha,b) \prec \mathcal{L}_L(\alpha,b)$$

Now consider the function $\Delta_L(\alpha,w) = CE\left(\mathcal{L}_{CE(L)}(\alpha,w)\right) - CE\left(\mathcal{L}_L(\alpha,w)\right)$ which maps interval $[a,b]$ on to a subset of $\mathbb{R}$. This function exhibits the following property:

(P1) the product $\Delta_L(\alpha,a) * \Delta_L(\alpha,b)$ is negative

Moreover, it is likely that $\Delta_L(\alpha,w)$ is continuous.[16] As a result there exists at least one value of the prize such that $\Delta_L(\alpha,w) = 0$. A case of interest is obviously that when $\Delta_L(\alpha,w)$ is monotonous. Intuitively the reversal of preferences should occur slowly. We are thus led to consider the second property of the function $\Delta_L(\alpha,w)$:

(P2) there exists one unique real number $z_L$ (namely the ZDP) for which $\Delta_L(\alpha,z_L) = 0$.

To sum up this discussion, we will consider in the axiomatics of our theory that $\Delta_L(\alpha,w)$ is a continuous function endowed with properties (P1) and (P2). Finally, we may also consider the case where the EU theory holds. No reversal then occurs and the function $\Delta_L(\alpha,w)$ is constant and null over $[a,b]$. All that discussion constitutes the basis of our first axiom which will be presented in the next subsection.

## 2.4 Axiomatics of our theory

We now turn to the axioms of our theory. We first consider ZDPs.

[14] For supporting their views, these authors have adapted the Problems 3 and 4 of Kahneman and Tversky (1979) that emphasize the famous common ratio effect. To make the link between their presentation and ours, consider the following preference reversal: $L \prec \delta_m$ and $\mathcal{L}_m(\alpha,a) \prec \mathcal{L}_L(\alpha,a)$ ($\mathcal{L}_m(\alpha,a) \overset{def}{=} \alpha\delta_m \oplus (1-\alpha)\delta_a$) with the following numerical values: $a = 0$ ; $b = 4000$ ; $m = 3000$ ; $\pi = .80$ ; $\alpha = .25$. They claim that the preference reversal may not occur if you mix both $L$ and $\partial_m$ with the superior lottery $\delta_b$ ( $L \prec \delta_m$ and $\mathcal{L}_L(\alpha,b) \prec \mathcal{L}_m(\alpha,b)$). Obviously this reversal of preferences is an another view of the same phenomenon since lowering $m$ will lead to the preferences exhibited in (4) and (5).

[15] Focusing on aspiration levels, Diecidue and van de Ven (2004) use similar arguments.

[16] If $w$ and $w'$ are close enough, then $(i)$ $CE\left(\mathcal{L}_{CE(L)}(\alpha,w)\right)$ should be close to $\left(\mathcal{L}_{CE(L)}(\alpha,w')\right)$ ; $(ii)$ $CE\left(\mathcal{L}_L(\alpha,w)\right)$ should be close to $\left(\mathcal{L}_L(\alpha,w')\right)$ and $(iii)$, consequently, $\Delta_L(\alpha,w)$ should be close to $\Delta_L(\alpha,w')$.

### 2.4.1 Properties of the ZDPs

Two axioms will be set. The first one is labelled ZDP1.

**Axiom ZDP1** *Given an economic agent facing the set $\mathbb{L}[a,b]$ of the available lotteries, given any lottery $L$ belonging to $\mathbb{L}[a,b]$ and its certainty equivalent $CE(L)$, given two compound lotteries $\mathcal{L}_L(\alpha,w)$ and $\mathcal{L}_{CE(L)}(\alpha,w)$ which are, respectively, the $\alpha-$mixing of lottery $L$ and of lottery $\delta_w$, and the $\alpha-$mixing of lottery $\delta_{CE(L)}$ and that of lottery $\delta_w$, and given any real number $\alpha$ belonging to $]0,1[$, then:*

*- either the function $\Delta_L(\alpha,w) = CE\left(\mathcal{L}_{CE(L)}(\alpha,w)\right) - CE\left(\mathcal{L}_L(\alpha,w)\right)$ is continuous and strictly monotonous and the product $\Delta_L(a,\alpha) * \Delta_L(b,\alpha)$ is negative*

*- or the function $\Delta_L(\alpha,w)$ is constant and null over $[a,b]$.*

**Corollary 1 (existence and uniqueness of the ZDP)** *Either there exists exactly one value $z_L \in [a,b]$ such that $\mathcal{L}_L(\alpha,z_L)$ is equivalent to $\mathcal{L}_{CE(L)}(\alpha,z_L)$ or the independence axiom holds, implying that the two lotteries $\mathcal{L}_L(\alpha,w)$ and $\mathcal{L}_{CE(L)}(\alpha,w)$ are equivalent for any value of $w$.*

*Proof: If $\Delta_L(\alpha,w)$ is continuous it maps $[a,b]$ on to a closed interval $[c,d]$ of $\mathbb{R}$. If $\Delta_L(a,\alpha)*\Delta_L(b,\alpha)$ is negative the continuity of $\Delta_L(\alpha,w)$ also implies that there exists at least a one real number $z \in [c,d]$ such that $\Delta_L(\alpha,z) = 0$. The uniqueness of $z_L$ derives from the monotonicity of $\Delta_L(\alpha,w)$.*

We can now give the characteristics of the binary relationship $(\preceq_D)$ which is induced by the weak order over the ZDPs. One can easily establish the two following propositions :

**Proposition 1** *The binary relationship $\preceq_D$ is a complete weak order over $\mathbb{L}[a,b]$.*

*Proof: the weak order $(\preceq_D)$ is induced by the weak order $(\leq)$.*

**Proposition 2** *The binary relationship $\preceq_D$ is continuous in the topology of weak convergence.*

*Proof: the topology on $\mathbb{L}[a,b]$ is induced by the topology on $\mathbb{R}$.*

Although the independence axiom may be violated as far as the preference relationship $(\preceq)$ is considered, it will be assumed that it holds when the binary relationship $(\preceq_D)$ induced by the weak order over the ZDP's is under review: more precisely, we assume that if the ZDP of lottery $L''$ is lower than that of lottery $L'$, then compounding either lottery with a third one $(L)$ will not modify the ranking of the ZDPs. Hence we set the following axiom:

**Axiom ZDP2** *Given any triple $(L, L', L'')$ of lotteries belonging to $\mathbb{L}[a,b] \times \mathbb{L}[a,b] \times \mathbb{L}[a,b]$ and any real number $\alpha$ belonging to $[0,1]$ we have the following implication:*

$$z_{L''} \leq z_{L'} \Rightarrow z_{\alpha L" \oplus\ (1-\alpha)L} \leq z_{\alpha L' \oplus\ (1-\alpha)L}$$

**Corollary 2 (independence property for $\preceq_D$)** *Given any triple $(L, L', L'')$ of lotteries belonging to $\mathbb{L}[a,b] \times \mathbb{L}[a,b] \times \mathbb{L}[a,b]$ and any real number $\alpha$ belonging to $[0,1]$ we have the following implication:*

$$L" \preceq_D L' \Rightarrow \alpha L" \oplus\ (1-\alpha)L \preceq_D \alpha L' \oplus\ (1-\alpha)L$$

*Proof: the corollary is a direct consequence of Axiom ZDP2.*

A well-known consequence of the above corollary is the following representation theorem:

**Proposition 3 (representation theorem for $\preceq_D$)** *The weak order $\preceq_D$ admits a VNM representation i.e. there exists a continuous increasing function $u(.)$, defined up to a positive affine transformation, such that:*

$$\forall (L_1, L_2) \in \mathbb{L}[a,b] \times \mathbb{L}[a,b]\ ;\ L_1 \preceq_D L_2 \Leftrightarrow \int_a^b u(w)dF_{L_1} \leq \int_a^b u(w)dF_{L_2}$$

*Proof: see Fishburn (1970).*

**Corollary 3** *The following equivalence holds:*

$$\forall (L_1, L_2) \in \mathbb{L}[a,b] \times \mathbb{L}[a,b] \ ; z_{L_1} \leq z_{L_2} \Leftrightarrow \int_a^b u(w)dF_{L_1} \leq \int_a^b u(w)dF_{L_2}$$

*Proof: the corollary is a direct consequence of the above proposition.*

Without loss of generality, $u(.)$ will be, from now on, normalized as follows:

$$u(a) = 0 \ ; \ u(b) = 1$$

Moreover the subset of lotteries exhibiting the same ZDP, whose utility is $\lambda$, will be denoted $\mathbb{L}_\lambda$. The common value of the ZDP will be denoted $z_\lambda$. Hence, we have the following equivalence:

$$E_L [u(\widetilde{w})] = u(z_\lambda) = \lambda \ \Leftrightarrow \ L \in \mathbb{L}_\lambda$$

Finally there exists a partition of the set of the available lotteries which consists in the union of the family of subsets defined as indicated below:

$$\mathbb{L}[a,b] = \bigcup_{\lambda \in [0,1]} \mathbb{L}_\lambda \ \ \textit{with} \ \lambda \neq \mu \Rightarrow \mathbb{L}_\lambda \cap \mathbb{L}_\mu = \varnothing$$

This partition is the analog of that used by Schmidt (2001). However, it has been given a psychological grounding through Axioms ZDP1 and ZDP2. We then assume that the independence axiom will hold only for lotteries exhibiting the same ZDP.

### 2.4.2 Axiomatics of preferences over lotteries

We now turn to preferences over lotteries. Any individual will be assumed to have preferences over the set $\mathbb{L}[a,b]$ of the available lotteries. Her preferences will be assumed to obey the two following well-known axioms:

**Axiom PRE1 (total ordering of $\preceq$)** *The binary relation $\preceq$ is a complete weak order.*

**Axiom PRE2 (continuity of $\preceq$)** *For any lottery $L \in \mathbb{L}[a,b]$ the sets $\{ M \in \mathbb{L}[a,b] \mid M \preceq L \}$ and $\{ M \in \mathbb{L}[a,b] \mid L \preceq M \}$ are closed in the topology of weak convergence.*

Axioms PRE1 and PRE2 are those of the EU theory. They allow for defining a numerical representation for the preference relation such that there exists a continuous utility function $U(.)$ mapping $\mathbb{L}[a,b]$ onto $\mathbb{R}$. Function $U(.)$ is then defined up to a continuous and increasing transformation. The EU theory restricts the utility function further and requires that it have the well known form: $U(L) = \int_a^b u(w)dF_L(w)$ where $u(.)$ is a continuous and increasing utility function defined up to an affine positive transformation. Such a result is obtained under the independence axiom.

Since we want to depart from the EU theory, we must set a substitute to the independence axiom which can be viewed, as in competitive axiomatized non-expected utility theories,[17] as a weak independence axiom.

**Axiom PRE3 (weak independence axiom)** *If lotteries $L_\lambda^1$ and $L_\lambda^2$ belong to $\mathbb{L}_\lambda$ and if lotteries $L_\mu^1$ and $L_\mu^2$ belong to $\mathbb{L}_\mu$, if $L_\mu^1$ is preferred to $L_\lambda^1$ and if $L_\mu^2$ is preferred to $L_\lambda^2$, then lottery $\alpha L_\mu^1 + (1-\alpha)L_\mu^2$ is preferred to lottery $\alpha L_\lambda^1 + (1-\alpha)L_\lambda^2$, whatever $\alpha$ belonging to $[0,1]$:*

$$\begin{aligned} \forall \left(L_\lambda^1, L_\lambda^2\right) &\in \mathbb{L}_\lambda \times \mathbb{L}_\lambda, \forall \left(L_\mu^1, L_\mu^2\right) \in \mathbb{L}_\mu \times \mathbb{L}_\mu, \forall \alpha \in [0,1], \\ L_\lambda^1 &\preceq L_\mu^1 \ \textit{and} \ L_\lambda^2 \preceq L_\mu^2 \Rightarrow \alpha L_\lambda^1 + (1-\alpha)L_\lambda^2 \preceq \alpha L_\mu^1 + (1-\alpha)L_\mu^2 \end{aligned}$$

[17] Among others: Yaari (1987), Gul (1991), Schmidt (2001).

Axiom PRE3 reduces to the traditional independence axiom for lotteries exhibiting the same ZDP, since this axiom is also valid when three lotteries belonging to $\mathbb{L}_\lambda$ are considered (make $\lambda = \mu$ and choose $L_\mu^2 = L_\mu^1$). Consequently, there exists a preference functional that numerically represents preferences over lotteries which display the same ZDP. This preference functional takes the form of the expected utility of a VNM utility function for every subset $(\mathbb{L}_\lambda)_{\lambda\in[0,1]}$in $\mathbb{L}[a,b]$ as indicated in the following proposition:

**Proposition 4 (Representation theorem for $\preceq$ over any subset $\mathbb{L}_\lambda$)** *Under Axioms ZDP1 and ZDP2 and Axioms PRE1 to PRE3, the weak order of preferences $\preceq$ admits a VNM representation on any subset $\mathbb{L}_\lambda$, i.e., there exists a continuous increasing function $v_\lambda(.)$, defined up to a positive affine transformation, such that:*

$$\forall\lambda \in ]0,1[\,,\ \forall (L_1, L_2) \in \mathbb{L}_\lambda \times \mathbb{L}_\lambda,\ L_1 \preceq L_2 \Leftrightarrow V_\lambda(L_1) \equiv \int_a^b v_\lambda(w)dF_{L_1} \leq \int_a^b v_\lambda(w)dF_{L_2} \equiv V_\lambda(L_2)$$

*Proof: Axioms PRE1, PRE2 and the independence axiom are valid over each subset $\mathbb{L}_\lambda$. Hence the standard result holds (see Fishburn 1970).*

We shall say that $v_\lambda(.)$ is the criterion used by the individual to compare lotteries belonging to $L_\lambda$. From now on and without loss of generality, the following normalization will be used:

$$v_\lambda(a) = u(a) = 0\ ;\ v_\lambda(w_\lambda) = u(w_\lambda) = \lambda$$

and we shall denote $\mathcal{F} = (\{v_\lambda(.)\}, \lambda \in [0,1])$ the family of the criteria $v_\lambda(.)$.

Any complete theory of decision making under risk should be able to describe choices made by individual over the complete set of lotteries, namely $\mathbb{L}$. Comparisons in terms of preferences between lotteries that display different ZDPs are possible since the weak independence axiom PRE3 implies the following result:

**Proposition 5 (Representation theorem for $\preceq$ over $\mathbb{L}[a,b]$)** *Under Axioms ZDP1 and ZDP2 and Axioms PRE1 to PRE3, the following preference functional represents the weak order of preferences $\preceq$ over $\mathbb{L}[a,b]$:*

$$\mathcal{V}(L) \overset{def}{=} V_\lambda(L) \overset{def}{=} \int_a^b v_\lambda(w)dF_L(w) \text{ with } \lambda = E_L[u(w)] \equiv \int_a^b u(w)dF_L(w)$$

*Proof: see Appendix 2.*

Equivalently, the following corollary holds:

**Corollary 4** *The weak order of preferences $\preceq$ can be represented over $\mathbb{L}[a,b]$, using the following lottery dependent utility function:*

$$\mathcal{V}(L) = \int_a^b \upsilon(E_L[u(\widetilde{w})], u(w))dF_L \text{ where } \upsilon(\lambda, u) \overset{def}{=} v_\lambda(u) \tag{6}$$

*Proof: the corollary is a direct consequence of the first representation theorem.*

The latter corollary shows that the satisfaction of the individual depends on the expected cardinal utility of the lottery $E_L[u(\widetilde{w})]$ and on the elementary utility $u(w)$. It can also be viewed as depending on $E_L[u(\widetilde{w})]$ and on elation/disappointment terms $(u(w) - E_L[u(\widetilde{w})])$. Indeed equation (6) expresses as:

$$\mathcal{V}(L) = \int_a^b \upsilon(E_L[u(\widetilde{w})], (u(w) - E_L[u(\widetilde{w})]) + E_L[u(\widetilde{w})])dF_L \tag{7}$$

### 2.4.3 Link with Loomes and Sugden's work

At this stage, we need to characterize further the function $\upsilon(.,.)$ in order to make this theory fully applicable for business uses. Two ways of thinking have so far been adopted in the literature.

The first one corresponds to a purely descriptive theory that imposes some *ad-hoc* definition of that function which may be justified by the fact that it correctly describes the actual decision making process used by subjects. This is typically the way adopted by Loomes and Sugden (1986). Our model can then be viewed as an axiomatics of the theory they developed in their article. The link between function $\upsilon(.,.)$ and function $D(.)$, which is defined by (1) and (2) in subsection 2.1 now expresses as:

$$\upsilon(E_L[u(\widetilde{w})], u(w)) = u(w) + D(u(w) - E_L[u(\widetilde{w})])$$

As it is shown in the Loomes and Sugden's study, several restrictions on the shape of $D$ allow to predict both the common ratio effect and the isolation effect[18] together with the preservation of the first order stochastic dominance principle. Indifference curves in the Marschak-Machina triangle might also have a mixed fanning shape[19] what is considered as a desirable property (Starmer 2000).

An alternative point of view is to adopt a normative approach. This is the choice which will be made in the following section by setting some restrictions on function $\upsilon(.,.)$. Hence, we present a particular case of our general theory of disappointment, namely the disappointment weighted utility theory.

# 3 A particular case of disappointment theory: the disappointment weighted utility theory

Two kind of arguments can be brought over to deem the compelling nature of those restrictions. We may try to build up an experimental design in order to directly test whether those restrictions (or axioms) represent well actual choices made by the subjects. An another approach is to justify those restrictions (or axioms) *ex-post* by checking whether they are compatible with stylized facts from the real world. To make this discussion more apparent, we might say, for instance, that the common ratio effect represents a clear sign of the poor predictive power of the VNM independence axiom. But, we might also argue that the equity premium puzzle (Mehra and Prescott, 1985) epitomizes the failure of that axiom. In the final part of this article, we confront the disappointment weighted utility theory by analyzing its financial implications.

We now present the disappointment weighted utility theory (DWU theory). It will avert useful to define a new function based on the one defined in (6):[20]

$$\gamma(\lambda, u(w)) \overset{def}{=} \frac{\upsilon_\lambda(w) - u(w)}{u(w)}$$

We can thus rewrite the preference functional $V(L)$ under the following form:

$$V(L) = \int_a^b u(w)\,(1 + \gamma(\lambda, u(w)))\, dF_L(w) \text{ with } \lambda = E_L[u(\widetilde{w})]$$

We are then led to identify $(1 + \gamma(\lambda, u(w)))$ to a change of measure of probability depending on the level of the disappointment the individual will feel once the lottery is run. Note that, since, by definition, $1 + \gamma(\lambda, u(w))$ is positive,[21] the requirement reduces to the fact that the expected value of the distortion $\gamma(\lambda, u(w))$ is worth zero (if the expectation is taken with $dF_L(\widetilde{w})$ where $L$ belongs to $\mathbb{L}_\lambda$). We now consider the following lemma:

**Lemma 1** *It is equivalent to state:*

*(a) The compound lottery* $L^u_\lambda = \frac{u(w)}{\lambda} L_\lambda \oplus \left[1 - \frac{u(w)}{\lambda}\right] L^*_\lambda$ *is equivalent*

[18] For a description of those effects, you may refer to Kahneman and Tversky (1979).

[19] Namely, a fanning out shape in the lower right corner of the triangle diagram and a fanning in shape in the upper left of the triangle diagram.

[20] Since $u(.)$ is strictly increasing on $[a, b]$, and since, by definition, $\lambda$ coincides with $E_L[u(\widetilde{w})]$, it is more convenient to express the distortion as being a function of the two variables $E_L[u(\widetilde{w})]$ and $u(w)$ than as a function of $\lambda$ and of $w$.

[21] Since both $u(.)$ and $\upsilon_\lambda(.)$ are positive, $1 + \gamma(\lambda, u(.))$ must be positive.

*to lottery* $L^1_\lambda(w) = \left[a, w; \frac{u(w)-\lambda}{u(w)}, \frac{\lambda}{u(w)}\right]$ *when* $w \in [w_\lambda, b]$*;*

*to lottery* $L^2_\lambda(w) = \left[w, b; \frac{1-\lambda}{1-u(w)}, \frac{\lambda-u(w)}{1-u(w)}\right]$ *when* $w \in [a, w_\lambda]$.

*(b)* $\gamma(\lambda, w)$ *has a zero expected value i.e.:*

$$\forall L \in \mathbb{L}_\lambda, \quad E_L\left[\gamma(\lambda, u(w))\right] = \int_a^b \gamma(E_L\left[u(\widetilde{w})\right], u(w)) dF_L(w) = 0$$

*(c)* $(1 + \gamma(\lambda, u(w)))$ *is a change of measure of probability i.e. there exists a cumulative distribution function* $\Phi_\lambda(.)$ *over* $[a, b]$*, such that:*

$$\forall L \in \mathbb{L}_\lambda, \quad \frac{d\ \Phi_\lambda(w)}{dF_L(w)} = (1 + \gamma(\lambda, u(w)))$$

*(d)* $\gamma(E_L\left[u(w)\right], u(w))$*is linear with respect to* $u(w)$*, i.e.:*

$$\gamma(E_L\left[u(\widetilde{w})\right], u(w)) = A\left[E_L\left[u(\widetilde{w})\right]\right]\left(E_L\left[u(\widetilde{w})\right] - u(w)\right)$$

*Proof: see Appendix 3.*

Hence, we make the following self-explanatory assumption:

**Assumption 1** *The compound lottery* $L^u_\lambda$ *is equivalent to lottery* $L^1_\lambda(w)$ *when* $w \in [a, w_\lambda]$ *and to lottery* $L^2_\lambda(w)$ *when* $w \in [w_\lambda, b]$.

This assumption allows us to establish the preference functional of the disappointment weighted utility theory which is exposed in the next corollary. Indeed, the weak order $\preceq$ admits the following lottery dependent representation:

**Corollary 5** **(second representation theorem for** $\preceq$**)** *Under Axioms ZDP1 and ZDP2 and Axioms PRE1 to PRE3 and Assumption 1, the following equivalence holds:*

$$\begin{aligned} L_1 \quad \preceq \quad & L_2 \Leftrightarrow \int_a^b u(w) d\Phi_{L_1} \leqslant \int_a^b u(w) d\Phi_{L_2} \qquad (8) \\ \Leftrightarrow \quad & \int_a^b u(w)\left[1 - A\left(E_{L_1}\left[u(\widetilde{w})\right]\right)\left(u(w) - E_{L_1}\left[u(\widetilde{w})\right]\right)\right] dF_{L_1} \\ \leqslant \quad & \int_a^b u(w)\left[1 - A\left(E_{L_2}\left[u(\widetilde{w})\right]\right)\left(u(w) - E_{L_2}\left[u(\widetilde{w})\right]\right)\right] dF_{L_2} \end{aligned}$$

*Proof: See previous discussion.*

It is worth to be noticed that the agent satisfaction $\mathcal{V}(L)$ can now be expressed as the sum of a standard VNM utility and of a penalty which is equal to the covariance between the disappointment effects and the utility function. Mathematically, we have:

$$\begin{aligned} \mathcal{V}(L) \quad & \equiv \quad \int_a^b u(w) dF_L + \int_a^b u(w)\gamma(\lambda, u(w)) dF_L \\ & = \quad E_L\left[u(w)\right] - COV_L\left[u(w), A\left(E_L\left[u(\widetilde{w})\right]\right)\left(u(w) - E_L\left[u(\widetilde{w})\right]\right)\right] \\ & = \quad E_L\left[u(w)\right] - A\left(E_L\left[u(\widetilde{w})\right]\right) Var_L\left[u(w)\right] \end{aligned}$$

Allais (1979) argues that a positive theory of choice should contain two basics elements: $(i)$ the existence of a cardinal utility function that is independent of risk attitudes and $(ii)$ a valuation functional of risky lotteries that depends on the second moment of the probability distribution of uncertain utility. In the EU theory, only the first moment is relevant to determine the attitude towards risk. In contrast, risk attitudes are determined by the second moment of the probability distribution of utility in the Allais theory. Hence, our model described in (8) provides an axiomatization of Allais analysis. When considering two lotteries with the same expected utility, as Allais does, we argue that a risk averse subject (in our terminology, we call her a disappointment averse subject) prefers the lottery with the smallest utility variance. This statement will have interesting implications in finance which will be explored in Section 4.

## 3.1 Properties of the model

The properties of this preference functional are brought together in the following seven remarks:

**Remark 1** *As stated by the last equivalence relation, the decision maker whose preferences are represented by the disappointment weighted utility theory is an expected utility maximizer who uses transformed probabilities that convey either disappointment averse or disappointment loving behavior.*

**Remark 2** *The term* $u(w) - E_L[u(\widetilde{w})]$ *represents a measure of the intensity of disappointment (elation) felt by the individual when the drawn outcome* $w$ *is less (greater) than the ZDP* $u^{-1}(E_L[u(\widetilde{w})])$ *of the lottery* $L$ *under review.*

**Remark 3** *The function* $A(E_L[u(\widetilde{w})])$ *can be then interpreted as a disappointment aversion measure. If the decision maker is disappointment averse (loving), the function must be positive (negative). The theory then implies that her satisfaction is worth less (greater) than the one of a VNM individual (or in other words, a decision maker who is disappointment neutral* $(A(E_L[u(\widetilde{w})]) = 0)$

**Remark 4** *Our theory embeds the EU theory since choices made by any disappointment neutral decision maker are represented by a pure VNM utility function.*

**Remark 5** *Any disappointment averse decision maker is pessimistic since he simultaneously overweights (underweights) probabilities of unfavorable (favorable) outcomes.*[22] *He will then use a new cumulative distribution function* $\Phi_L$ *that is stochastically dominated by the objective probability distribution* $F_L$*. We shall explore in the next section some theoretical implications of that property in finance.*

**Remark 6** *If* $A(.)$ *is a constant function, the disappointment weighted utility theory falls into the quadratic in probability class of models proposed Chew et alii (1991). Indeed, borrowing their notations, we get:*
$T(x,y) = \frac{1}{2}\left[u(x) + u(y) - A\left(u^2(x) + u^2(y)\right) + 2Au(x)u(y)\right]$

**Remark 7** *The model can handle the common ratio effect (see Appendix 4).*

## 3.2 Consistency with the stochastic dominance principle

We now address the issue of the stochastic dominance principle. We establish a sufficient condition for which the use of distributions $\Phi_L$ instead of $F_L$ does not lead to the violation of that principle.

**Proposition 6** *A sufficient condition for the change in measure, defined by* $\frac{d\Phi_L}{dF_L}(w) = 1 - A(E_L[u(\widetilde{w})])*$ $(u(w) - E_L[u(w)])$*, to be compatible with the first-order stochastic dominance principle is that the absolute disappointment aversion* $A(E_L[u(\widetilde{w})])$ *is a non-increasing function of* $E_L[u(\widetilde{w})]$ *and that the relative disappointment aversion* $A(E_L[u(\widetilde{w})])\,E_L[u(\widetilde{w})]$ *is a non-decreasing function of* $E_L[u(\widetilde{w})]$*:*

$$\frac{dA}{dw}(w) \leq 0 \text{ and } \frac{d}{dw}(wA(w)) \geq 0$$

*Proof: see Appendix 5.*

# 4 The financial implications of the disappointment weighted utility theory

This section focuses on the financial implications of the disappointment weighted utility theory. Following Pratt (1964), we first investigate the notions of risk aversion and risk premium in the case of disappointment averse individuals.

[22] Every disappointing outcome (those whose amounts fall below the expected utility of the lottery) makes the term $u(w) - E_L[u(\widetilde{w})]$ being negative. Then, for every disappointment averse decision maker, the term $-A(E_L[u(\widetilde{w})])(u(w) - E_L[u(\widetilde{w})])$ becomes positive. Hence, the probability of all disappointing outcomes are overweighted. Symmetric arguments can be used to show that the probabilities of elating outcomes are underweighted in the case of a disappointment averse individual.

## 4.1 Two definitions for risk premia and risk aversion

We first study risk "in the small" that is infinitesimal risks. We define as usual the risk premium of an infinitesimal risk as the difference between the expected outcome of the lottery and its certainty equivalent. From standard calculations we get the following equality:

$$TARP(\widetilde{w}) = ADP(\widetilde{w}) + ARP(\widetilde{w}) = \left[ Au^{'}(\overline{w}) - \frac{u^{''}(\overline{w})}{2u^{'}(\overline{w})} \right] \sigma^2$$

It states that the total (absolute) risk premium $TARP(\widetilde{w})$ can be split into two parts : an (absolute) disappointment premium $ADP(\widetilde{w})$ and the usual (absolute) Arrow-Pratt risk premium $ARP(\widetilde{w})$ (which is a concavity premium). Each premium is proportional to the variance of the lottery under review which may be identified to the infinitesimal risk. When the first premium is considered, the coefficient of proportionality only depends on the tastes of the investor since it is equal to $A[u(\overline{w})]\, u^{'}(\overline{w})$, which characterizes disappointment aversion. Hence, we also can write:

$$TARA(\overline{w}) = ADA(\overline{w}) + ARA(\overline{w}) = 2Au^{'}(\overline{w}) - \frac{u^{''}(\overline{w})}{u^{'}(\overline{w})}$$

It expresses that total (absolute) risk aversion $TARA(\overline{w})$ is the sum of (absolute) disappointment aversion and (absolute) risk aversion in the sense of Arrow-Pratt premium. The reason for coefficient 2 to appear is that, according to tradition, we have defined risk aversion as twice the ratio of the premium to the variance of the payoffs. All the preceding results hold "in the small", that is for infinitesimal risks. We could generalize Pratt's theorem and cope with risk in the large. Such a generalization is beyond the scope of this paper.

## 4.2 The mean-variance criterion as "dual" theory

We now come to the particular case where individuals display constant marginal utility.[23] As pointed out by Yaari (1987), this hypothesis is suitable to model the behavior of institutional investors. It means in the setting of our theory that the ZDP of any lottery corresponds exactly to its expected value. Hence using (8), the overall satisfaction of that individual expresses as:

$$U(\widetilde{w}) = \int_a^b w(1 + A(E_L[\widetilde{w}])(E_L[\widetilde{w}] - w)\, dF_L = E_L[\widetilde{w}] - A(E_L[\widetilde{w}])\, Var_L[\widetilde{w}] \qquad (9)$$

where $A(.)$ is a decreasing function of $E_L[\widetilde{w}]$.

Equation (9) means that any disappointment averse investor with constant marginal utility uses the mean-variance criterion as a basis for valuing risky prospects. (9) shares traditional interpretations since $U(\widetilde{w})$ is well an increasing function of $E_p[\widetilde{w}]$. and a decreasing function of $Var_L[\widetilde{w}]$. Hence, our model have provided an axiomatic foundation of the use of the mean-variance criterion as a risk-value model.

## 4.3 Market Equilibrium: Some Illustrations

We now address the issue of market equilibrium with disappointment averse individuals. Our analysis is fourfold. We first characterize a static market equilibrium. We then show that the CAPM can be viewed as a model corresponding to the dual case the disappointment weighted utility theory for individuals with both constant marginal utility and constant absolute disappointment aversion. Next, we interpret the transformed probabilities used by a disappointment averse agent in terms of risk-neutral probabilities. Finally, in a two assets exchange economy, we explore the ability of our model to cope with the equity premium puzzle.

[23] The VNM axiomatics cannot suitably deal with this case except for admitting the risk neutral assumption of all investors.

### 4.3.1 Static market equilibrium

We consider a very simple economy with both a representative agent who has the preferences in equation (8) and a single consumption good taken as the numéraire. For the sake of simplicity, choices are assumed to be made over one period. By assumption, there exists a finite set of states of the world with $K$ elements. The probability that state $k$ occurs is $p_k$. $E_p[\widetilde{x}]$ will denote the expected value of the random variable $\widetilde{x}$ under the original probability measure $p$. There are $N+1$ assets traded on financial markets. They are labelled with the superscript $n$ $(n = 0, 1, ..., N)$. $S_0^n$ will denote the time-0 price of security $n$, and, $S_k^n$ $(k = 1, ..., K)$ the time-1 price of security $n$ in the $k$th state of world. The asset labelled 0 corresponds to the risk-free asset. Consequently $S_0^0 = 1$ and for any $k$, $S_k^0 = (1 + r^f)$ where $r^f$ expresses the risk-free rate return. The disappointment averse investor has preferences which are represented by an additive and separable utility function. Using $\alpha^{-1}$ as subjective discount factor, her utility can be written as follows:[24]

$$U(c_0, \widetilde{c}) = u(c_0) + \alpha^{-1} \sum_{k=1}^{K} p_k \left(1 - A\left(E_p[u(\widetilde{c})]\right)\left(u(c_k) - E_p[u(\widetilde{c})]\right)\right) u(c_k) \tag{10}$$

where $c_0$ $(\widetilde{c})$ represents her present (future random) consumption depending on both the state of nature to be occurring and asset returns.

To spare space, we assume the existence and the uniqueness of a static equilibrium. Given (10), Appendix 6 provides the equilibrium prices in the financial market:

$$S_0^n = \frac{1}{\alpha u'(c_0)} \left( E_p\left[u'(\widetilde{c})\widetilde{S}^n\right](1 - A' Var_p[u(\widetilde{c})]) - 2ACov_p\left(u'(\widetilde{c})\widetilde{S}^n, u(\widetilde{c})\right)\right) \text{ for all } \ n = 0, ..., N \tag{11}$$

### 4.3.2 Disappointment weighted utility and CAPM

We now turn to the interesting case of both constant marginal utility $(u(w) = w)$ and constant absolute disappointment aversion $(A' = 0)$. By noticing that the final consumption corresponds to the final wealth $(W)$ in a one period financial market equilibrium. We can obtain the expected return of the $n$th risky asset using (11):

$$E_p[\widetilde{r}^n] = r^f + 2ACov_p\left[\widetilde{W}, \widetilde{r}^n\right] \tag{12}$$

At the equilibrium, the risk premium of any risky asset is equal to the product between the disappointment aversion and the covariance between the rate return of the risky asset and the final wealth of the investor. To go on with our analysis, we now suppose there exists a representative consumer. Let $M_0$ ( $M$) denote the value of the market portfolio at date 0 (1) and $\theta^n$ the agent's asset $n$ holdings at date 0. Appendix 6 provides us with the final wealth value of the representative investor:

$$\widetilde{W} = \left(1 + r^f\right)\left(W_0 - c_0 - \sum_{j=1}^{N} \theta^j S_0^j\right) + \sum_{j=1}^{N} \theta^j \widetilde{S}^j = \left(1 + r^f\right)(W_0 - c_0 - M_0) + M \tag{13}$$

If $\tilde{r}^M$ is the rate of return of the market portfolio, we then have $M = M_0\left(1 + \tilde{r}^M\right)$. Using (12) and (13) we obtain the equilibrium return for any asset:

$$E_P[\tilde{r}^n] = r^f + 2ACov_p\left(\sum_{j=1}^{N} \theta^j \widetilde{S}^j, \tilde{r}^n\right) = r^f + 2AM_0 Cov\left(\tilde{r}^M, \tilde{r}^n\right) \quad \text{for all } n = 1, ..., N \tag{14}$$

In particular for the market portfolio, we get:

$$E_P\left[\tilde{r}^M\right] = r^f + 2AM_0 Var\left[\tilde{r}^M\right] \tag{15}$$

By noticing that $Cov\left(\tilde{r}^M, \tilde{r}^n\right) = \beta_n Var\left[\tilde{r}^M\right]$, using Equations (14) and (15) lead to the well-known CAPM equation:

[24] The individual can be viewed as facing $N$ lotteries of the form: $\left(L^n = \left\{S_1^n, ..., S_K^n; p_1, ..., p_K\right\}\right)_{n=1,...,N}$

$$E_P\left[\widetilde{r}^n\right] = r^f + \beta_n\left(E_P\left[\tilde{r}^M\right] - r^f\right)$$

It is well-known that, in the VNM universe, the CAPM relation as well as the underlined hypothesis stating that investors are mean-variance maximizers can be established if and only if at least one of the two following conditions is verified: $(i)$ the asset returns follow laws which are fully defined from their two first moments; $(ii)$ the investors utility functions are quadratic.

Nevertheless, it is well-documented that the second assumption implies a result which is invalidated by empirical works: the absolute risk aversion in the Arrow-Pratt sense is not an increasing function of wealth. The first assumption leads to consider the normality of the asset returns. But it is contradicted by the empirical observations even if we consider the interesting Lévy-stable laws.

Hence, it is somewhat paradoxical that the CAPM has been so popular, that it remains the reference for any empirical work on the stock markets and that fund managers still continue to use the mean-variance criterion. Such a paradox can be addressed, if one assumes that investors, especially institutional investors, who play the most important role on financial markets, have both a constant marginal utility and a constant absolute disappointment aversion; if so, the market equilibrium must be in accordance with the CAPM. This justification of the use of the CAPM appears to us more convincing than that of the one traditionally made, in terms of quadratic utility functions or normal distributions of asset returns.

#### 4.3.3 Transformed probabilities as risk-neutral probabilities

In this subsection, we provide an interpretation of the transformed probabilities induced by the change of probability measure in terms of risk-neutral probabilities. Combining together the first and the two last equations of the system (35) in Appendix 6, we get:

$$S_0^n = \frac{1}{1+r^f}\frac{\sum\limits_{k=1}^{K}\lambda_k S_k^n}{\sum\limits_{k=1}^{K}\lambda_k} = \frac{1}{1+r^f}\sum_{k=1}^{K}\mu_k S_k^n \text{ with } \mu_k = \frac{\lambda_k}{\sum\limits_{j=1}^{K}\lambda_j} \geq 0 \text{ and } \sum_{k=1}^{K}\mu_k = 1 \tag{16}$$

The equilibrium price of any risky asset is then defined by the condition (16) stating that the $(\mu_k)_{k=1,\dots,K}$ correspond to the risk-neutral probabilities. Assuming both constant marginal utility and constant absolute disappointment aversion, we can provide a simple interpretation of those risk-neutral probabilities in function of the behavior towards disappointment. Using the equations in the system (35) of Appendix 6 and (11), we get:

$$S_0^n = \frac{1}{1+r^f}\sum_{k=1}^{K}p_k\left(1-2A\left(c_k - E_P\left[\widetilde{c}\right]\right)\right)S_k^n \text{ for all } n=1,\dots,N \tag{17}$$

where $p_k$ corresponds to the objective probability of state $k$ occurrence.

From (17), the risk-neutral probability of the state $k$ is worth $p_k\left(1-2A\left(c_k - E_P\left[\widetilde{c}\right]\right)\right)$. Therefore, in bad states of nature, the greater the agent's disappointment aversion, the greater is the associated risk-neutral probability.

In the VNM universe, pricing derivative securities requires the existence of risk averse investors in the usual sense. As advocated by Yaari (1987) and the discussion introduced in the previous subsection, the most natural intuition for modelling institutional investors' preferences would be to consider constant marginal utility ; or in other words investors' risk-neutrality in the usual sense. In our setting, (17) shows that it remains possible to exhibit risk-neutral probabilities as a non-identical function of the original probability measure together with investors equipped with constant marginal utility. We are thus able to price derivative securities even in the case of investors' risk-neutrality.

#### 4.3.4 Equity premium puzzle

To examine the role of disappointment on asset returns, we assume the joint lognormality of asset returns with consumption growth, what constitutes a commonly adopted hypothesis in studies of asset pricing. To isolate the effect of disappointment from that of standard risk aversion, we study the case of an individual who displays constant relative disappointment aversion[25] and constant marginal utility. Considering a two assets economy, from (11) and with the approximation $\log(1+x) = x$, we directly obtain the equilibrium values of both the risk-free rate and the equity risk premium:

$$r^f \approx -\ln\left(\alpha^{-1}\right) - aVar\left[\Delta c\right] \tag{18}$$

$$E\left[r\right] - r^f = \frac{2aCov\left[\Delta c, r\right]}{1 + aVar\left[\Delta c\right]} \tag{19}$$

where $\Delta c$ denotes the growth rate of consumption.

The equation (18) shows that the risk free rate is a decreasing function of $a$, namely the relative disappointment aversion coefficient, whereas (19) illustrates that the risk premium is an increasing function of that coefficient. Those properties attest that our model can mitigate the equity premium puzzle.

# 5 Conclusion

In this paper, a fully choice-based theory of disappointment has been developed as an alternative to the implicit expected utility theory based on disappointment aversion of Gul (1991). It can be viewed as an axiomatic basis of Loomes and Sugden (1986). A particular case of this theory leads us to define a flexible parametrization called disappointment weighted utility theory.

Two different concepts for describing the behavior towards risk have been distinguished in this model: Arrow-Pratt risk aversion and disappointment aversion. Four main results from this disappointment weighted utility theory have been obtained:

*(i)* the risk premium is the sum of the Arrow-Pratt premium, which is generally viewed as a concavity premium, and of a disappointment premium which characterizes the intensity of the probabilities transformation.

*(ii)* an appealing justification for using the mean-variance criterion, and, consequently for referring to the CAPM is the existence of investors having both a constant marginal utility and a behavior towards disappointment.

*(iii)* risk-neutral probabilities may differ from the historical ones even when institutional investors are equipped with constant marginal utility.

*(iv)* the model exhibits properties which mitigate the equity premium puzzle.

Further empirical investigations should be undertaken concerning both the theoretical and the applied parts of the paper. In particular, direct experimental tests of the axioms could be designed. The building of an intertemporal asset pricing model based on disappointment averse investors would allow to monitor the ability of the model to cope with the observed second moments of both the equity risk premium and the risk-free rate as well as the persistence and predictability of excess returns found in the data.

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# APPENDIX 1

The corresponding values of that increase in probabilities of disappointment respectively are $(1-\alpha)-0 = (1-\alpha)$ and $(1-\alpha\pi)-(1-\pi) = \pi(1-\alpha))$

More precisely, the probability of feeling elation (disappointment), once lottery $\mathcal{L}_{ce(L)}(\alpha, b)$ has been run, is now $1-\alpha$ $(\alpha)$. Similarly, lottery $\mathcal{L}_L(\alpha, b)$ exhibits $1-\alpha(1-\pi)$ $(\alpha(1-\pi))$ as probability of elation (disappointment). Hence, the increase of the probability of disappointment when shifting from $\delta_{ce(L)}$ to $\mathcal{L}_{ce(L)}(\alpha, b)$ is now $\alpha - 0 = \alpha$ whereas the corresponding variation, when shifting from $L$ to $\mathcal{L}_L(\alpha, b)$, is now negative (equal to $(1-\pi)\alpha-(1-\pi) = -(1-\pi)(1-\alpha)$). All the results are summed up on Table 1, where the value of the probabilities of disappointment of the four compound lotteries ($\mathcal{L}_{CE(L)}(\alpha, a)$, $\mathcal{L}_{CE(L)}(\alpha, b)$, $\mathcal{L}_L(\alpha, a)$ and $\mathcal{L}_L(\alpha, b)$) are reported together with the value of the difference between these probabilities and that of the initial lottery ($\delta_{CE(L)}$ or $L$).

**Table 1: Probabilities of disappointment**

| | $\mathcal{L}_{CE(L)}(\alpha, a)$ | $\mathcal{L}_L(\alpha, a)$ | $\mathcal{L}_{CE(L)}(\alpha, b)$ | $\mathcal{L}_L(\alpha, b)$ |
|---|---|---|---|---|
| $(Level)$ | $(1-\alpha)$ | $1-\alpha\pi$ | $\alpha$ | $\alpha(1-\pi)$ |
| $(Variations)$ | $(1-\alpha)$ | $\pi(1-\alpha)$ | $\alpha$ | $-(1-\alpha)(1-\pi)$ |

# APPENDIX 2

**Proof of Proposition 5**

The proof is given assuming that we have the following equivalences:

$$\lambda < (\leq)(=)\,\mu \;\Leftrightarrow\; \delta_{w_\lambda} \prec (\preceq)(\sim)\,\delta_{w_\mu} \tag{20}$$

$$\lambda < (\leq)(=)\,\mu \;\Leftrightarrow \underline{L}_\lambda \prec (\preceq)(\sim)\,\underline{L}_\mu \tag{21}$$

$$\underline{L}_\lambda \prec (\preceq)(\sim)\,\delta_{w_\lambda} \Leftrightarrow \underline{L}_\mu \prec (\preceq)(\sim)\,\delta_{w_\mu} \tag{22}$$

where $\underline{L}_\lambda$ ($\underline{L}_\mu$) is the binary lottery $[a, b; 1-\lambda, \lambda]$ ($[a, b; 1-\mu, \mu]$).

The first one expresses that the degenerated lottery $\partial_{w_\mu}$ will be (strictly) preferred to the degenerated lottery $\partial_{w_\lambda}$ if and only if $\lambda$ is (strictly) less than $\mu$ that is if and only if $w_\lambda$ is (strictly) less than $w_\mu$. The second one expresses that the binary lottery $[a, b; 1-\lambda, \lambda]$ will be (strictly) preferred to the binary lottery $[a, b; 1-\mu, \mu]$ if and only if $\lambda$ is (strictly) less than $\mu$. The third one expresses that the degenerated lottery $\partial_{w_\lambda}$ will be (strictly) preferred to the binary lottery $[a, b; 1-\lambda, \lambda]$ if and only if the degenerated lottery $\partial_{w_\mu}$ is (strictly) preferred to the binary lottery $[a, b; 1-\mu, \mu]$. The two first equivalences hold if the stochastic dominance property is met. The third one means that an agent is either risk averse (in the usual acceptation) or a risk lover but cannot be both.

Consider a pair $(\lambda, \mu)$ of positive real numbers (with $0 < \lambda < \mu < 1$). From Proposition 4, we know that there exists functions $\nu_\lambda(.)$ and $\nu_\mu(.)$ representing preferences over $\mathbb{L}_\lambda$ and $\mathbb{L}_\mu$ respectively. Formally:

$$\mathcal{V}_\lambda(L) = \int_a^b \nu_\lambda(w)\, dF_L(w) \text{ and } \mathcal{V}_\mu(L') = \int_a^b \nu_\mu(w)\, dF_{L'}(w) \tag{23}$$

with:

$$\begin{array}{rcl} \forall (L, L^*) & \in & \mathbb{L}_\lambda \times \mathbb{L}_\lambda \; L \preceq L^* \Leftrightarrow \mathcal{V}_\lambda(L) \leq \mathcal{V}_\mu(L^*) \\ \forall (L,' L'^*) & \in & \mathbb{L}_\mu \times \mathbb{L}_\mu \; L' \preceq L'^* \Leftrightarrow \mathcal{V}_\mu(L') \leq \mathcal{V}_\mu(L'^*) \end{array}$$

Functions $\nu_\lambda(.)$ and $\nu_\mu(.)$ are increasing and continuous over $[a, b]$ and they map $[a, b]$ on to $[\nu_\lambda(a), \nu_\lambda(b)]$ and $[\nu_\mu(a), \nu_\mu(b)]$ respectively. They are defined up to an affine and positive transformation. Hence, we can select the following normalization conditions:

$$\begin{array}{rcl} v_\lambda(a) & = & u(a) = 0 \; ; \; v_\lambda(w_\lambda) = u(w_\lambda) = \lambda \\ v_\mu(a) & = & u(a) = 0 \; ; \; v_\mu(w_\mu) = u(w_\mu) = \mu \end{array}$$

Now, consider four lotteries. Two of them belong to the subset $\mathbb{L}_\lambda$ ($\mathbb{L}_\mu$). They are the degenerated lottery $\delta_{w_\lambda}$ ($\delta_{w_\mu}$) and the binary lottery $\underline{L}_\lambda = [a, b; 1-\lambda, \lambda]$.($\underline{L}_\mu = [a, b; 1-\mu, \mu]$). We have:

$$\mathcal{V}_\lambda(\delta_{w_\lambda}) = \lambda \; ; \mathcal{V}_\lambda(\underline{L}_\lambda) = \lambda\nu_\lambda(b) \text{ and } \mathcal{V}_\mu(\delta_{w_\mu}) = \mu \; ; \mathcal{V}_\mu(\underline{L}_\mu) = \mu\nu_\mu(b)$$

If the stochastic dominance property is met, the lotteries must rank as:

$$\underline{L}_\lambda \prec \underline{L}_\mu \prec \delta_{w_\lambda} \prec \delta_{w_\mu} \qquad (Case\ 1)$$

or as:

$$\underline{L}_\lambda \prec \delta_{w_\lambda} \prec \underline{L}_\mu \prec \delta_{w_\mu} \qquad (Case\ 2)$$

We, first, consider Case 1. From Axiom PRE2 we know that there exists a lottery $L_\mu^\lambda$ belonging to $\mathbb{L}_\mu$ which is equivalent to $\delta_{w_\lambda}$ and which is a compound lottery mixing $\delta_{w_\mu}$ and $\underline{L}_\mu$(with weights $\alpha_\mu^\lambda$ and $1-\alpha_\mu^\lambda$). Formally:

$$\exists\ \alpha_\mu^\lambda \in ]0,1[\,,\ L_\mu^\lambda \underset{def}{=} \alpha_\mu^\lambda \delta_{w_\mu} \oplus \left(1-\alpha_\mu^\lambda\right)\underline{L}_\mu \sim \delta_{w_\lambda} \tag{24}$$

Since lotteries $\delta_{w_\mu}$ and $\underline{L}_\mu$ both belong to $\mathbb{L}_\mu$, lottery $L_\mu^\lambda$ well belongs to $\mathbb{L}_\mu$. Similarly there exists a lottery $L_\lambda^\mu$ belonging to $\mathbb{L}_\lambda$ which is equivalent to $\underline{L}_\mu$ and which is a compound lottery mixing $\delta_{w_\lambda}$ and $\underline{L}_\lambda$(with weights $\alpha_\lambda^\mu$ and $1-\alpha_\lambda^\mu$). Formally:

$$\exists\ \alpha_\lambda^\mu \in ]0,1[\ :\ L_\lambda^\mu \underset{def}{=} \alpha_\lambda^\mu \delta_{w_\lambda} \oplus (1-\alpha_\lambda^\mu)\,\underline{L}_\lambda \sim \underline{L}_\mu \tag{25}$$

Now, we claim that the following equivalence holds:

$$\forall L \in \mathbb{L}_\lambda,\ \forall L' \in \mathbb{L}_\mu \quad L \preceq L' \Leftrightarrow \mathcal{V}_\lambda(L) \leq \mathcal{V}_\mu(L') \tag{26}$$

To establish this result, we provisionally assume that the two following inequalities are satisfied:

$$L_\lambda^\mu \preceq L \preceq \delta_{w_\lambda} \text{ and } \underline{L}_\mu \preceq L' \preceq L_\mu^\lambda \qquad (subcase\ (i))$$

The proof of relation (26) is threefold:

1. Axiom PRE2 implies that there exists a lottery $L_\lambda^\alpha$ belonging to $\mathbb{L}_\lambda$ which is equivalent to $L$ and which is a compound lottery mixing $\delta_{w_\lambda}$ and $\underline{L}_\lambda$ (with weights $\alpha$ and $1-\alpha$). Similarly there exists a lottery $L_\mu^\beta$ belonging to $\mathbb{L}_\mu$ which is equivalent to $L'$ and which is a compound lottery mixing $\delta_{w_\mu}$ and $\underline{L}_\mu$ (with weights $\beta$ and $1-\beta$). Formally:

$$\begin{aligned}\exists\ \alpha &\in\ ]0,1[\,,\ L_\lambda^\alpha \underset{def}{=} \alpha L_\lambda^\mu \oplus (1-\alpha)\,\underline{L}_\lambda \sim L \\ \exists\ \beta &\in\ ]0,1[\,,\ L_\mu^\beta \underset{def}{=} \beta\delta_{w_\mu} \oplus (1-\beta)\,\underline{L}_\mu \sim L'\end{aligned}$$

2. If we impose the following conditions:

$$\mathcal{V}_\lambda(L_\lambda^\mu) = \mathcal{V}_\mu\left(\underline{L}_\mu\right) \text{ and } \mathcal{V}_\mu\left(L_\mu^\lambda\right) = \mathcal{V}_\lambda(\delta_{w_\lambda}) \tag{27}$$

   then we get:

$$\mathcal{V}_\lambda(L) = \alpha v_\mu^\lambda + (1-\alpha)\,v_\lambda^\mu \text{ and } \mathcal{V}_\mu(L') = \beta v_\mu^\lambda + (1-\beta)\,v_\lambda^\mu$$

   and, consequently, the following equivalence holds:

$$\mathcal{V}_\lambda(L) \leq \mathcal{V}_\mu(L') \Leftrightarrow \alpha \leq \beta \tag{28}$$

   Moreover, from Axiom PRE3, we get:

$$\alpha \leq \beta \Leftrightarrow L \preceq L' \tag{29}$$

   and finally, equivalence (26) holds.

3. The last step consists in establishing that equations (27) are compatible with the chosen normalization. In other words we must check that equations (27), yield the same couple of real numbers $\left(\overline{\alpha}_{\lambda}^{\mu}, \overline{\alpha}_{\mu}^{\lambda}\right)$ as the couple deriving from equation (24) and (25):

$$\begin{aligned} \alpha_{\lambda}^{\mu}\lambda + (1-\alpha_{\lambda}^{\mu})\,\lambda\nu_{\lambda}(b) &= \mu\nu_{\mu}(b) \\ \alpha_{\mu}^{\lambda}\mu + \left(1-\alpha_{\mu}^{\lambda}\right)\mu\nu_{\mu}(b) &= \lambda \end{aligned} \qquad (30)$$

The above set of equations is a Cramer system whose solutions $\left(\overline{\alpha}_{\lambda}^{\mu}, \overline{\alpha}_{\mu}^{\lambda}\right)$ express as:

$$\overline{\alpha}_{\lambda}^{\mu} = \frac{\mu\lambda^{-1}\nu_{\mu}(b) - \nu_{\lambda}(b)}{(1-\nu_{\lambda}(b))} \text{ and } \overline{\alpha}_{\mu}^{\lambda} = \frac{\lambda\mu^{-1} - \nu_{\mu}(b)}{(1-\nu_{\mu}(b))}$$

Recall that the following equivalences hold:

$$\begin{aligned} \mathcal{V}_{\lambda}(\underline{L}_{\lambda}) &\prec \mathcal{V}_{\lambda}(\delta_{w_{\lambda}}) \Leftrightarrow \lambda\nu_{\lambda}(b) < \lambda \Leftrightarrow \nu_{\lambda}(b) < 1 \\ \mathcal{V}_{\mu}\left(\underline{L}_{\mu}\right) &\prec \mathcal{V}_{\mu}\left(\delta_{w_{\mu}}\right) \Leftrightarrow \mu\nu_{\mu}(b) < \mu \Leftrightarrow \nu_{\mu}(b) < 1 \\ 0 &< \lambda < \mu < 1 \Rightarrow \mu\lambda^{-1} > 1 \Leftrightarrow \lambda\mu^{-1} < 1 \\ \underline{L}_{\mu} &\prec \delta_{w_{\lambda}} \Leftrightarrow \mathcal{V}(\underline{L}_{\mu}) \leq \mathcal{V}(\delta_{w_{\lambda}}) \Leftrightarrow \mathcal{V}_{\mu}\left(\underline{L}_{\mu}\right) \prec \mathcal{V}_{\lambda}(\delta_{w_{\lambda}}) \Longleftrightarrow \mu\nu_{\mu}(b) < \lambda \end{aligned}$$

Consequently, the solution of the Cramer system (30) is a couple of real numbers $\left(\overline{\alpha}_{\lambda}^{\mu}, \overline{\alpha}_{\mu}^{\lambda}\right)$ belonging to $[0,1]$. Since the decompositions (24) and (25) are unique, the considered solutions $\left(\overline{\alpha}_{\lambda}^{\mu}, \overline{\alpha}_{\mu}^{\lambda}\right)$ coincide with the couple of real numbers $\left(\alpha_{\lambda}^{\mu}, \alpha_{\mu}^{\lambda}\right)$ characterizing the compound lotteries $L_{\mu}^{\lambda}$ and $L_{\lambda}^{\mu}$.

A complete proof should include the five following subcases which may also occur:

$$\begin{aligned} .\underline{L}_{\lambda} &\preceq L \preceq L_{\lambda}^{\mu} \text{ and } \underline{L}_{\mu} \preceq L' \preceq L_{\mu}^{\lambda} && (subcase\ (ii)) \\ L_{\lambda}^{\mu} &\preceq L \preceq \delta_{w_{\lambda}} \text{ and } L_{\mu}^{\lambda} \preceq L' \preceq \delta_{w_{\mu}} && (subcase\ (iii)) \\ \underline{L}_{\lambda} &\preceq L \preceq L_{\lambda}^{\mu} \text{ and } L_{\mu}^{\lambda} \preceq L' \preceq \delta_{w_{\mu}} && (subcase\ (iv)) \\ \underline{L}_{\lambda} &\preceq L \preceq L_{\lambda}^{\mu} \text{ and } L' \preceq \underline{L}_{\mu} && (subcase\ (v)) \\ L &\preceq \underline{L}_{\lambda} \text{ and } L' \preceq \underline{L}_{\mu} && (subcase\ (vi)) \end{aligned}$$

Consider the first three subcases ($(ii)$ to $(iv)$): equivalence (26) holds since the following inequalities are met by assumption:

$$\begin{aligned} L_{\lambda}^{\mu} &\backsim \underline{L}_{\mu} \Rightarrow L \preceq L' \text{ and } \mathcal{V}_{\lambda}(L) \leq \mathcal{V}_{\lambda}(L_{\lambda}^{\mu}) = \mathcal{V}_{\mu}\left(\underline{L}_{\mu}\right) \leq \mathcal{V}_{\mu}(L') && (subcase\ (ii)) \\ \delta_{w_{\lambda}} &\backsim L_{\mu}^{\lambda} \Rightarrow L \preceq L' \text{ and } \mathcal{V}_{\lambda}(L) \leq \mathcal{V}_{\lambda}(\delta_{w_{\lambda}}) = \mathcal{V}_{\mu}\left(L_{\mu}^{\lambda}\right) \leq \mathcal{V}_{\mu}(L') && (subcase\ (iii)) \\ L_{\lambda}^{\mu} &\preceq L_{\mu}^{\lambda} \Rightarrow L \preceq L' \text{ and } \mathcal{V}_{\lambda}(L) \leq \mathcal{V}_{\lambda}(L_{\lambda}^{\mu}) \leq \mathcal{V}_{\mu}\left(L_{\mu}^{\lambda}\right) \leq \mathcal{V}_{\mu}(L') && (subcase\ (iv)) \end{aligned}$$

The last two subcases ($(v)$ to $(vi)$) are analogous to the first one, although $\alpha$, $\beta$, $\alpha_{\lambda}^{\mu}$ and $\alpha_{\mu}^{\lambda}$ may then be negative. Last, consider Case 2. Equivalence (26) again holds since the following inequalities are met by assumption:

$$\underline{L}_{\lambda} \prec \delta_{w_{\lambda}} \prec \underline{L}_{\mu} \prec \delta_{w_{\mu}} \Rightarrow L \preceq L' \text{ and } \mathcal{V}_{\lambda}(L) \leq \mathcal{V}_{\lambda}(\delta_{w_{\lambda}}) < \mathcal{V}_{\mu}\left(\underline{L}_{\mu}\right) \leq \mathcal{V}_{\mu}(L')$$

Finally, we have established that it is equivalent to say that a lottery $L'$ belonging to $\mathbb{L}_{\mu}$ is preferred (not preferred) to a lottery $L$ belonging to $\mathbb{L}_{\lambda}$.if and only if $\mathcal{V}_{\mu}(.) \geq \mathcal{V}_{\lambda}(L.)$ $(\mathcal{V}_{\mu}(.) \leq \mathcal{V}_{\lambda}(L.))$.

***Q.E.D.***

# APPENDIX 3

**Proof of Lemma 1**

Since statement *(b)* is trivially equivalent to statement *(c)*, we limit our discussion by proving that statement *(a)* is equivalent to statement *(d)* and that statement *(b)* is equivalent to statement *(d)*.

**A. Equivalence between statement *(a)* and statement *(d)*:**

From Appendix 2 we get the following results:

$$\begin{aligned}\frac{v_\lambda(w)}{u(w)} &= v_\lambda^* + (1-v_\lambda^*)\alpha_{L_\lambda^1}(w) && \text{for all } w \in [w_\lambda, b]\\ \frac{v_\lambda(w)}{u(w)} &= v_\lambda^* + \frac{(1-u(w))}{u(w)}\frac{\lambda}{(1-\lambda)}(1-v_\lambda^*)\alpha_{L_\lambda^2}(w) && \text{for all } w \in [a, w_\lambda]\end{aligned}$$

and, consequently:

$$\begin{aligned}E_{L_\lambda^1}\left[\frac{v_\lambda(z)}{u(z)}\right] &= \frac{\lambda}{u(w)}\left[v_\lambda^* + (1-v_\lambda^*)\alpha_{L_\lambda^1}(w)\right] + \frac{u(b)-\lambda}{u(b)}\left[v_\lambda^* + (1-v_\lambda^*)\alpha_{L_\lambda^1}(b)\right]\\ &= v_\lambda^* + \frac{\lambda}{u(w)}(1-v_\lambda^*)\alpha_{L_\lambda^1}(w)\end{aligned}$$

or:

$$\begin{aligned}E_{L_\lambda^2}\left[\frac{v_\lambda(z)}{u(z)}\right] &= \frac{1-\lambda}{1-u(w)}\left[\frac{(1-u(w))}{u(w)}\frac{\lambda}{(1-\lambda)}(1-v_\lambda^*)\alpha_{L_\lambda^2}(w) + v_\lambda^*\right]\\ &\quad + \frac{\lambda-u(w)}{1-u(w)}\left[\frac{(1-u(b))}{u(b)}\frac{\lambda}{(1-\lambda)}(1-v_\lambda^*)\alpha_{L_\lambda^2}(a) + v_\lambda^*\right]\\ &= v_\lambda^* + \frac{\lambda}{u(w)}(1-v_\lambda^*)\alpha_{L_\lambda^2}(w)\end{aligned}$$

hence:

$$E_{L_\lambda^I}\left[\frac{v_\lambda(z)}{u(z)}\right] = 1 \Leftrightarrow 1 = v_\lambda^* + \left[\frac{\lambda}{u(w)}(1-v_\lambda^*)\alpha_{L_\lambda^2}(w)\right] \Leftrightarrow \alpha_{L_\lambda^I}(w) = \frac{u(w)}{\lambda}$$

and

$$\begin{aligned}\gamma_{L_\lambda^i}(w) &= \frac{v_\lambda(w)}{u(w)} - 1 = (1-v_\lambda^*)(\alpha_{L_\lambda^I}(w) - 1)\\ &= (1-v_\lambda^*)(\frac{u(w)}{\lambda} - 1) = \frac{(1-v_\lambda^*)}{\lambda}(u(w)-\lambda) = A_\lambda(u(w)-\lambda)\end{aligned}$$

We have thus established the equivalence between **statement *(a)*** and **statement** *(d)*.

**B. Equivalence between statement *(b)* and statement *(d)*:**

It is trivial to establish the following implication:

$$\gamma(E_L[u(w)], u(w)) = A[E_L[u(w)]](E_L[u(w)] - u(w)) \Rightarrow E_L[\gamma(E_L[u(w)], u(w))] = 0$$

To establish the converse, one can consider the following lottery $L_w^\alpha = \{1-\alpha, \alpha; a, w\}$ which, by assumption, belongs to $\mathbb{L}_\lambda$. Consequently, the following equation holds:[26]

$$\lambda = E_L[u(w)] = \alpha u(w) + (1-\alpha)u(a) = \alpha u(w)$$

or, equivalently:[27]

$$\frac{\lambda}{\alpha} = u(w) \Leftrightarrow \alpha = \frac{\lambda}{u(w)}$$

[26] Recall that, by assumption, $u(0) = 0$.

[27] Which implies that $\frac{\lambda}{\alpha} < 1 \Leftrightarrow \lambda < \alpha$.

If the preceding equation holds, the condition $E_L\left[\gamma(E_L\left[u(w)\right], u(w))\right] = 0$ now reads:

$$0 = E_L\left[\gamma(E_L\left[u(w)\right], u(w))\right] = \alpha\gamma\left(\lambda, u(w)\right) + (1-\alpha)\gamma\left(\lambda, 0\right)$$

Hence:

$$\begin{aligned}\gamma\left(\lambda, u(w)\right) &= -((1-\alpha)/\alpha)\gamma\left(\lambda, 0\right) = -(1-\frac{u(w)}{\lambda})\gamma\left(\lambda, 0\right) = -\left(\frac{\gamma\left(\lambda, 0\right)}{\lambda}\right)(\lambda - u(w)) \\ &= \left(-\frac{\gamma(E_L\left[u(w)\right], 0)}{E_L\left[u(w)\right]}\right)(E_L\left[u(w)\right] - u(w))\end{aligned}$$

We have then shown that $\gamma\left(\lambda, w\right)$ is an affine function of $u(.)$ over $\left[\frac{\lambda}{\alpha}, 1\right]$. A similar argument can be used to establish that $\gamma\left(\lambda, w\right)$ is an affine function of $u(.)$ over $\left[0, \frac{\lambda}{\alpha}\right]$. To do so, let us consider the lottery $L_w^\alpha = \{\alpha, 1-\alpha; w, b\}$, which, by assumption, belongs to $\mathbb{L}_\lambda$. Consequently, the following equation holds:

$$\lambda = E_L\left[u(w)\right] = \alpha u(w) + (1-\alpha)u(b) = \alpha u(w) + (1-\alpha)$$

or, equivalently:[28]

$$u(w) = \frac{\lambda - (1-\alpha)}{\alpha} \Leftrightarrow \alpha = \frac{1-\lambda}{1-u(w)}$$

If the preceding equation holds, the condition $E_L\left[\gamma(E_L\left[u(w)\right], u(w))\right] = 0$ now reads:

$$0 = E_L\left[\gamma(E_L\left[u(w)\right], u(w))\right] = \alpha\gamma\left(\lambda, u(w)\right) + (1-\alpha)\gamma\left(\lambda, 1\right)$$

Hence:

$$\begin{aligned}\gamma\left(\lambda, u(w)\right) &= -((1-\alpha)/\alpha)\gamma\left(\lambda, 1\right) = -(1-\frac{1-u(w)}{1-\lambda})\gamma\left(\lambda, 1\right) = \left(-\frac{\gamma\left(\lambda, 1\right)}{1-\lambda}\right)(\lambda - u(w)) \\ &= -\left(\frac{\gamma(E_L\left[u(w)\right], 1)}{1-E_L\left[u(w)\right]}\right)(E_L\left[u(w)\right] - u(w))\end{aligned}$$

Finally the proposition is established if and only if the following equation holds:

$$\left(\frac{\gamma(E_L\left[u(w)\right], 0)}{E_L\left[u(w)\right]}\right) = \left(\frac{\gamma(E_L\left[u(w)\right], 1)}{1-E_L\left[u(w)\right]}\right)$$

It is the case because if we consider the lottery $L_\lambda = \{a, b; 1-\lambda, \lambda\}$, which belongs to $\mathbb{L}_\lambda$, we have by assumption that:

$$E_L\left[\gamma(E_L\left[u(w)\right], u(w))\right] = \lambda\gamma(E_L\left[u(w)\right], 1) + (1-\lambda)\gamma(E_L\left[u(w)\right], 0) = 0$$

*Q.E.D.*

# APPENDIX 4

**Common Ratio Effect**

We borrow the problems 3 and 4 from Kahneman and Tversky (1979).
Problem 3:
A: (0.2,0.8;0,4000) or B: (1;3000)
Problem 4:
C: (0.8,0.2;0,4000) or D: (0.75,0.25;0,3000)

[28] Which implies that $\frac{\lambda}{\alpha} < 1 \Leftrightarrow \lambda < \alpha$.

The majority of the respondents of this experience shows preferences for the lottery B in problem 3 and the lottery C in problem 4. The paradox is that for an arbitrary utility function $u(.)$ normalized to $u(0) = 0$, B preferred to A implies that $u(3000) > 0.8u(4000)$ whereas C preferred to D implies that $0.2u(4000) > 0.25u(3000) \Rightarrow 0.8u(4000) > u(3000)$. So, that pattern of preferences is not compatible with the EU theory.

The following inequalities must hold if we want to offer a good representation of the revealed preferences:

$$U(A)/U(B) \leq 1 \text{ and } U(C)/U(D) \geq 1 \tag{31}$$

Without loss of generality, we normalize $u(.)$ as follows:

$$u(0) = 0 \text{ and } u(4000) = 1$$

Using our axiomatics, we get:
$U(A) = 0.8(1 - 0.2A\left[0.8\right])$ and $U(B) = u(3000)$;
$U(C) = 0.2(1 - 0.8A\left[0.2\right])$ and $U(D) = 0.25(1 - 0.75u(3000)A\left[0.25u(3000)\right])u\left(3000\right)$
We can rewrite the inequalities (31) above as follows:

$$0.8(1 - 0.2A\left[0.8\right]) \leq u(3000) \leq 0.8\frac{(1 - 0.8A\left[0.2\right])}{(1 - 0.75u(3000)A\left[0.25u(3000)\right])}$$

or, alternatively as the two conditions

$$\begin{aligned} A\left[0.8\right] &\geq \frac{0.8 - u(3000)}{0.16} \quad \text{and} \\ 1.25u(3000) - 1 &\leq 0.9375u^2(3000)A\left[0.25u(3000)\right] - 0.8A\left[0.2\right] \end{aligned}$$

By assumption, $A\left[.\right]$ is positive and it is a decreasing function of $E\left[u\left(w\right)\right]$. Hence the first condition is automatically reached, whereas the second condition can always be achieved if $A(.)$ is sufficiently decreasing. For instance, with a linear utility function ($u(3000) = 0.75$), the Allais paradox is solved with the following choice of parameters which fully satisfy the sufficient condition for stochastic dominance consistency :
$A(0.8) = 0.3125$ ; $A(0.2) = 0.35$ and $A(0.25u(3000)) = A(0.1875) = 0.5425$

# APPENDIX 5

**First Order Stochastic Dominance**

In this appendix we shall use the following denominations:

$$\begin{aligned} A(E_k\left[u\left(\widetilde{w}\right)\right]) &= A_k \quad k = p, q \\ E_k\left[u\left(\widetilde{w}\right)\right] &= E_k \quad k = p, q \end{aligned}$$

Let us consider the following difference:

$$U\left(L_p\right) - U\left(L_q\right) = \int_a^b u\left(w\right)\left(1 + A_p\left(E_p - u\left(w\right)\right)\right)dF_p\left(w\right) - \int_a^b u\left(w\right)\left(1 + A_q\left(E_q - u\left(w\right)\right)\right)dF_q\left(w\right)$$

Or,equivalently:

$$\begin{aligned} U\left(L_p\right) - U\left(L_q\right) &= \int_a^b u\left(w\right)\left(dF_p\left(w\right) - dF_q\left(w\right)\right) + \int_a^b u(w)\left[A_pE_pdF_p\left(w\right) - A_qE_qdF_q\left(w\right)\right] \\ &\quad - \int_a^b u^2(w)\left[A_pdF_p\left(w\right) - A_qdF_q\left(w\right)\right] \end{aligned} \tag{32}$$

The dominance principle consists in the following statement: if a lottery $L_p$ first-order stochastically dominates a lottery $L_q$, then any rational individual will prefer $L_p$ to $L_q$. If our preference functional satisfies the latter property, it means that the difference $U(L_p) - U(L_q)$ should be positive. We will obtain below some sufficient conditions that insure the positivity of the 3 terms in the right hand side of (32).

If $(A_pE_p - A_qE_q)$ is positive and if $(A_p - A_q)$ is negative, from (32) we get:

$$U(L_p) - U(L_q) \geq \int_a^b u(w)\left(1 + A_q(E_q - u(w))\right)\left[dF_p(w) - dF_q(w)\right] \tag{33}$$

If we integrate by part the right hand side of (33), we obtain:

$$\begin{aligned} U(L_p) - U(L_q) &\geq u(w)\left(1 + A_q(E_q - u(w))\right)\left(F_p(w) - F_q(w)\right)\Big|_a^b \\ &\quad - \int_a^b \left[u(w)\left(1 + A_q(E_q - u(w))\right)\right]'\left(F_p(w) - F_q(w)\right)dw \\ &= 0 + \int_a^b \left[u(w)\left(1 + A_q(E_q - u(w))\right)\right]'\left(F_q(w) - F_p(w)\right)dw > 0 \end{aligned} \tag{34}$$

From (34), since $u(w)\left(1 + A_q(E_q - u(w))\right)$ is an increasing function over $[a, b]$ and by hypothesis $L_p$ first-order stochastically dominates $L_q$, or more formally $F_q(w) \geq F_p(w)\ \forall w \in [a, b]$, it comes that the right hand side of (33) is positive. Finally a sufficient condition for having $U(L_p) - U(L_q) \geq 0$ is that:

$$A_p - A_q \leq 0 \text{ and } (A_pE_p - A_qE_q) \geq 0$$

*Q.E.D.*

# APPENDIX 6

**Static maximization program with probabilities alteration**

We consider the following program:
$MAX\ E_\pi\left[U(c_0, \widetilde{c})\right]$
subject to the constraints:

i) $c_0 + \theta^0 + \sum\limits_{n=1}^{N} \theta^n S_0^n - W_0 = 0 \Rightarrow c_0 = W_0 - \left(\theta^0 + \sum\limits_{n=1}^{N} \theta^n S_0^n\right)$

ii) $c_k - \left(1 + r^f\right)\theta^0 - \sum\limits_{n=1}^{N} \theta^n S_k^n = 0 \Rightarrow c_k = \left(1 + r^f\right)\theta^0 + \sum\limits_{n=1}^{N} \theta^n S_k^n \quad , k = 1, ..., K$

Where $E_\pi[.]$ denotes the expectation operator under the transformed probability measure $\pi$. $W_0$ represents the consumer's initial wealth. $c_k$ denotes the date 1 consumption of the state $k$.

$$\begin{aligned} L &= E_\pi\left[U(c_0, \widetilde{c})\right] - \lambda_0\left(c_0 + \theta^0 + \sum_{n=1}^{N} \theta^n S_0^n - W_0\right) \\ &\quad - \sum_{k=1}^{K} \lambda_k\left(c_k - \left(1 + r^f\right)\theta^0 - \sum_{n=1}^{N} \theta^n S_k^n\right) \end{aligned}$$

with:

$$
\begin{aligned}
E_\pi\left[U\left(c_0,\widetilde{c}\right)\right] &= u(c_0)+\alpha^{-1}\left[\sum_{j=1}^{K}p_j(1-A\left(u\left(c_j\right)-E_p\left[u\left(\widetilde{c}\right)\right]\right))u\left(c_j\right)\right] \\
&= u(c_0)+\alpha^{-1}\left[\sum_{j=1}^{K}p_j\Gamma_j u\left(c_j\right)\right]
\end{aligned}
$$

The first order conditions are:

$$
\left\{
\begin{array}{c}
\frac{\partial L}{\partial c_0}=u^{'}\left(c_0\right)-\lambda_0=0\Rightarrow\lambda_0=u^{'}\left(c_0\right) \\
\frac{\partial L}{\partial c_k}=\alpha^{-1}\left[p_k\Gamma_k u^{'}\left(c_k\right)+\sum\limits_{j=1}^{K}p_j u\left(c_j\right)\frac{\partial\Gamma_j}{\partial c_k}\right]-\lambda_k=0 \\
\Rightarrow\lambda_k=\alpha^{-1}\left[p_k\Gamma_k u^{'}\left(c_k\right)+G\left(k\right)\right]\qquad ,k=1,...,K \\
\frac{\partial L}{\partial\theta^0}=-\lambda_0+\left(\sum\limits_{k=1}^{K}\lambda_k\right)\left(1+r^f\right)=0\Rightarrow\sum\limits_{k=1}^{K}\lambda_k=\frac{1}{1+r^f}\lambda_0 \\
\frac{\partial L}{\partial\theta^n}=-\lambda_0 S_0^n+\left(\sum\limits_{k=1}^{K}\lambda_k S_k^n\right)=0\ ,\ n=1,...,N
\end{array}
\right. \tag{35}
$$

Combining together the first and the two last equations of the system (35), we get the solution of our program:

$$
S_0^n=\frac{1}{\alpha u^{'}\left(c_0\right)}\sum_{k=1}^{K}\left(p_k\Gamma_k u^{'}\left(c_k\right)+G\left(k\right)\right)S_k^n
$$

Since we have:

$$
\sum_{k=1}^{K}p_k\Gamma_k u^{'}\left(c_k\right)S_k^n=\sum_{k=1}^{K}p_k u^{'}\left(c_k\right)S_k^n\left(\left(1-A\left(u(c_k)-E_p\left[u(\widetilde{c})\right]\right)\right)\right)=E_p\left[u^{'}\left(\widetilde{c}\right)\widetilde{S}^n\right]-ACov_p\left(u^{'}\left(\widetilde{c}\right)\widetilde{S}^n,u(\widetilde{c})\right)
$$

And:

$$
\begin{aligned}
\frac{\partial\Gamma_j}{\partial c_k} &= p_k u'\left(c_k\right)\left[A'\left(E_p\left[u\left(\widetilde{c}\right)\right]-u\left(c_j\right)\right)+A\right]-Au'\left(c_k\right)\mathbf{1}_{j=k} \\
\Rightarrow &\quad G\left(k\right)=\sum_{j=1}^{K}p_j u\left(c_j\right)\frac{\partial\Gamma_j}{\partial c_k}=p_k u'\left(c_k\right)\left[-A'Var_p\left[u\left(\widetilde{c}\right)\right]+AE_P\left[u\left(\widetilde{c}\right)\right]\right]-Ap_k u\left(c_k\right)u'\left(c_k\right)
\end{aligned}
$$

Therefore:

$$
\sum_{k=1}^{K}G\left(k\right)S_k^n=-A'E_p\left[u^{'}\left(\widetilde{c}\right)\widetilde{S}^n\right]Var_p\left[u\left(\widetilde{c}\right)\right]-ACov_p\left(u^{'}\left(\widetilde{c}\right)\widetilde{S}^n,u\left(\widetilde{c}\right)\right)
$$

We finally get:

$$
S_0^n=\frac{1}{\alpha u^{'}\left(c_0\right)}\left(E_p\left[u^{'}\left(\widetilde{c}\right)\widetilde{S}^n\right]\left(1-A'Var_p\left[u\left(\widetilde{c}\right)\right]\right)-2ACov_p\left(u^{'}\left(\widetilde{c}\right)\widetilde{S}^n,u\left(\widetilde{c}\right)\right)\right)\ \text{with}\ A=A\left(W_0,E_p\left[u\left(\widetilde{c}\right)\right]\right)
$$