Discounted and Finitely Repeated Minority Games with Public Signals

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December 2007

*Partially supported by the Agence Nationale de la Recherche (ANR), under grants ATLAS and Croyances, and the "Chaire de la Fondation du Risque", Dauphine-ENSAE-Groupama: Les Particuliers Face aux Risques

†Partially supported by MIUR-COFIN.

‡Partially supported by MIUR-COFIN. The nice hospitality that this author received at CEREMADE is gratefully acknowledged.

Abstract

We consider a repeated game where at each stage players simultaneously choose one of two rooms. The players who choose the less crowded room are rewarded with one euro. The players in the same room do not recognize each other, and between the stages only the current majority room is publicly announced, hence the game has imperfect public monitoring. An undiscounted version of this game was considered by Renault et al. (2005), who proved a folk theorem. Here we consider a discounted version and a finitely repeated version of the game, and we strengthen our previous result by showing that the set of equilibrium payoffs Hausdorff-converges to the feasible set as either the discount factor goes to one or the number of repetition goes to infinity. We show that the set of public equilibria for this game is strictly smaller than the set of private equilibria.

AMS 2000 Subject Classification: Primary 91A20. Secondary 91A18.

JEL Classification: C72.

Keywords: Repeated games, imperfect monitoring, public equilibria, private equilibria, Pareto-efficiency, discount factor.

1 Introduction

Following a suggestion in Arthur (1994) many authors have considered a class of games called *minority games*, where each player repeatedly acts trying to choose the action that is less popular among all players. In the original problem considered by Arthur (1994) customers like to go to *El Farol* bar on Thursday night, when it offers Irish music, but they deem it enjoyable only if it is not too crowded. In many real life situations it is preferable to be in the minority. Think for instance of a residential suburban area that is linked to downtown by two main roads. Commuters have to decide every morning which road to take and, for obvious reasons, they all want to avoid traffic. Since the commuter typically do not recognize their fellow commuters on the road, but only perceive the existence of traffic, this phenomenon can be modelled as a game with imperfect public monitoring. Even if the analysis of minority games started considering situations involving a huge number of players, strategically it is often more interesting to consider models with a small number of players. For instance consider the case where each of three agents can satisfactorily carry out a procedure only if a certain minimal throughput is obtained via a communication link. They can choose one of two links and the minimal throughput is guaranteed only if one agent uses that link alone. This is often the case when downloading data in P2P systems (see, e.g., Suri et al. (2005)).

In general this class of games is interesting when several agents must take decentralized decision on whether to access a scarce resource, knowing that at most a fixed number of them will be able to enjoy its benefits. Similar situations have been analyzed by the empirical economic literature on market entry games (see e.g., Selten and Güth (1982); Ochs (1990, 1995); Rapoport et al. (2002); Erev and Rapoport (1998)). In these models players must decide independently whether to enter a market (and incur an entry cost). Since capacity is limited, the entrants will reap a reward only if their number is smaller than a fixed threshold. This clearly creates a problem of coordination.

Similar ideas have been used to analyze speculative behavior in financial markets, and minority games have been used as a formalization of concepts, like the *contrarian investment strategy*, previously considered in empirical studies (see e.g., De Bondt and Thaler (1985); Chan (1988)). Most of the literature on the topic can be found in theoretical physics journals and has a non-strategic approach where agents have bounded memory and act inductively, i.e., they adjust their behavior according to their past experience. More precisely, at the beginning each agent is endowed with a finite set of simple decision rules or strategies, which are kept fixed throughout the game. At all subsequent stages each player evaluates the performance of all her strategies in the last k outcomes and plays the best predictor, i.e., the strategy with the highest fitting score. Physicists then study the degree of coordination that a population of N players exibihts when N is large, mainly exploiting concepts and methodologies used in statistical mechanics of disordered systems. The reader is referred to the recent books by Challet et al. (2005) and Coolen (2005) for a history of the problem, its statistical-mechanics analysis, and some applications to financial markets, and to http://www.unifr.ch/econophysics/minority/ for an extensive list of references.

An analysis of minority games from the learning viewpoint can be found in Bottazzi et al. (2003, 2002); Bottazzi and Devetag (2004).

Renault et al. (2005) are the first to consider a minority game from a traditional strategic viewpoint. In their model an odd number of players have to choose simultaneously one of two rooms. The players who choose the less crowded room receive a reward of one euro. The others receive nothing. The game is repeated over time. At each step, after the players' actions, only a public signal (the majority room) is announced to everybody, so players do not observe the actions or the payoffs of the other players. Notice that this is equivalent to announcing to each player her own reward at each time, but not the reward of the other players. Renault et al. (2005) prove that an undiscounted folk theorem holds for this game, and characterize the set of uniform equilibrium payoffs, i.e. they show that any feasible payoff is an equilibrium payoff. In particular, they construct a uniform equilibrium where the payoff of each player is zero. This equilibrium can be considered as particularly inefficient, since all feasible payoffs are non-negative.

A minority game is basically a repeated coordination game, the specific feature being that players want to coordinate negatively and be where their fellow players are not. The stage game used in a repeated minority game is strongly related to congestion games introduced by Rosenthal (1973) and to crowding games studied by Milchtaich (1998, 2000).

The paper by Renault et al. (2005) is in the tradition of repeated games with imperfect public monitoring. Examples of such games go back to Rubinstein (1979), Rubinstein and Yaari (1983), and Radner (1985) with reference to principal-agent models and by Green and Porter (1984) with reference to oligopoly. More systematic analyses of games with imperfect public monitoring have been provided by Abreu et al. (1990) Fudenberg et al. (1994), Fudenberg and Levine (1994), and Tomala (1998). These papers focus on perfect public equilibria, namely, equilibria in which each player uses only strategies that depend on the public signal and not on her private history.

Imperfect private monitoring was studied by Lehrer (1989, 1990, 1992a,b). The issue of *Journal of Economic Theory* dedicated to this topic, with the introduction by Kandori (2002), is a good source of reference on this literature (see also the book by Mailath and Samuelson (2006)).

Our model deals with an intermediate situation where the signal is public, but strategies are private, namely, they depend on the public signal *and* on the private history of each player. Games with public monitoring and private strategies have been studied for instance by Mailath et al. (2002) and Kandori and Obara (2006). Mailath et al. (2002) deal with finitely repeated games and show three examples of substantially different behavior of private versus public strategies in games with imperfect public monitoring. Kandori and Obara (2006) consider infinitely repeated games and provide a method to construct private strategies that are more efficient than public strategies.

In this paper we consider a discounted version and a finitely repeated version of the minority game studied by Renault et al. (2005). We are able to prove that a version of the folk theorem holds also for these games. In the discounted game, as the discount factor grows, the set of equilibrium payoffs converges (in the Hausdorff metric) to the feasible set, namely, to the set of uniform equilibrium payoffs of the undiscounted game. The same convergence result holds for the finite game, as the number of repetitions goes to the infinity. The convergence theorems in this paper cannot be obtained by known results in the literature.

For instance Lehrer (1989, 1992a,b) deals with two-player games only, Lehrer (1990) assumes semi-standard information. Furthermore all these papers consider repeated games without discounting. Abreu et al. (1990) consider pure strategy equilibria, keep the discoungt factor fixed, and do not analyze convergence theorem. Fudenberg et al. (1994) impose a pair-wise identifiability condition that is not satisfied by our game.

In the undiscounted case studied by Renault et al. (2005) the folk theorem was proved by constructing specific equilibrium strategies whose payoff is one of the extreme points of the feasible payoff set. In this paper we merely prove that all points in the feasible set can be arbitrarily approximated by equilibrium payoffs if either the discount factor is big enough or the game is repeated enough times.

In order to achieve this approximation we will make use of a strategy that allows the players to punish a possibly deviating player with high probability, without even knowing her identity. In games with imperfect monitoring identifying a deviator is an issue. It is therefore important here to use a strategy that avoids this issue, namely, a strategy that involves an effective punishment even if the deviator is not identified. A similar idea appears already in Renault et al. (2005), although the strategy used in that paper is different. In that paper the equilibrium strategy was based on a statistical test aimed at detecting possible deviations from the equilibrium path. Once a likely deviation is detected, all players enact a punishment that involves replaying the actions played at the periods that produced an unexpected signal. This creates a situation where, with very high probability, the non-deviating players are equally split between the two rooms, so the deviator is automatically punished without any need to know her identity.

Folk theorems for finitely repeated games with perfect monitoring have been proved by Benoit and Krishna (1985) in pure strategies and by Gossner (1995) in mixed strategies.

As we mentioned before, although in this game signals are public, we consider equilibria in private strategies. We will show that if we consider only public equilibria the equilibrium payoff set is strictly smaller, in particular, in the case of three players, it coincides with the convex envelop of the Nash equilibrium payoffs of the one-shot game.

The paper is organized as follows. Section 2 describes the model. Section 3 examines the discounted game. Section 4 studies the finitely repeated game. Section 5 considers public equilibria. All the proofs are contained in Section 6.

2 The model

There are two rooms: L(eft) and R(ight), and an odd number of players. At each stage, the players have to choose simultaneously one of the two rooms. The player who finds herself in the less crowded room (if any) gains a positive payoff of 1, and the most crowded room is publicly announced before going to the next stage.

2.1 The stage game

The set of players is $N = \{1, \ldots, 2n+1\}$. For all $i \in N$ denote by $A^i = \{L, R\}$ the set of actions for player i, and put $A = \times_{i \in N} A^i$. For $a = (a^1, \ldots, a^{2n+1}) \in A$ define the payoff function $g^i : A \to \mathbb{R}$ of player i as

$$g^i(a) = \begin{cases} 0 & \text{if } \mathrm{card}\{j \in N \setminus \{i\} | a^j = a^i\} \geq n, \\ 1 & \text{otherwise}, \end{cases}$$

where, given a set B, $\mathrm{card}(B)$ is its cardinality.

For each subset $S \subset N$ define e_S as the payoff in $\mathbb{R}^N$ where each player in S gets 1, and each player not in S gets 0. If $S = \varnothing$, then e_S is just the null vector.

The set of feasible vectors is

$$\mathcal{S} = \mathrm{conv}\{e_S, S \subset N \text{ such that } \mathrm{card}(S) \leq n\},$$

where, if D is a subset of a Euclidean space, $\mathrm{conv}\, D$ is the convex hull of D. Looking at extreme points, it is not difficult to show that:

$$\mathcal{S} = \{(x^1, \ldots, x^{2n+1}) \in [0,1]^{2n+1} : \sum_{i=1}^{2n+1} x^i \leq n\}.$$

If $n = 1$, there are three players and it is easy to compute the set of Nash equilibria of the stage game. These are the action profiles where one player plays L with probability 1 and one other player plays R with probability 1, plus the strategy profile where every player plays the randomized action $\frac{1}{2}L \oplus \frac{1}{2}R$. In the general case where n is a positive integer, there exist many more Nash equilibria of the stage game. They can all be described as follows:

- the equilibria where n players play L, and n players play R. In particular every e_S, with $S \subset N$ such that $\mathrm{card}(S) = n$, is an equilibrium payoff of the stage game.

- for every a and b in $\{0, \ldots, n-1\}$, there is an equilibrium where: a players play L, b players play R, and all other players play the mixed action $\beta\, L \oplus (1-\beta)\, R$, where $\beta \in [0,1]$ is uniquely determined by n, a and b (e.g., $\beta = 1/2$ if $a = b$).

Since the stage game will be repeated, we also need notations about what the players observe. We define the set of public signals as $U = \{L, R\}$. The signalling function $\ell : A \to U$, giving the most crowded room, is formally defined by

$$\ell(a) = \begin{cases} L, & \text{if } \mathrm{card}\{i \in N | a^i = L\} > \mathrm{card}\{i \in N | a^i = R\}, \\ R, & \text{if } \mathrm{card}\{i \in N | a^i = L\} < \mathrm{card}\{i \in N | a^i = R\}. \end{cases}$$

2.2 The repeated game

At each stage $t \geq 1$, each player i (simultaneously with the other players) selects an action $a_t^i \in A^i$. If $a_t = (a_t^1, \ldots, a_t^{2n+1}) \in A$ is chosen, the stage payoff of player i is $g^i(a_t)$, and the signal $u_t = \ell(a_t)$ is publicly announced. Then the play proceeds to stage $t+1$. All the players have perfect recall and the whole description of the game is common knowledge.

The game has imperfect monitoring, in that the players do not observe the actions of their opponents, but only a public signal (the majority room). This is equivalent to assuming that each player observes only her own payoff, but not the payoff of the other players.

A behavioral strategy of player i is an element $\sigma^i = (\sigma_t^i)_{t\geq 1}$, where for all t

$$\sigma_t^i : (A^i \times U)^{t-1} \to \Delta(A^i).$$

Therefore, for each $t \geq 1$, $\sigma_t^i(a_1^i, u_1, a_2^i, u_2, \dots, a_{t-1}^i, u_{t-1})$ is the lottery played by player i at stage t if she played a_1^i at stage 1, …, a_{t-1}^i at stage $t-1$, and the signal was u_1 at stage 1, …, u_{t-1} at stage $t-1$.

We denote by Σ^i the set of behavioral strategies of player i, and $\Sigma = \times_{i\in N}\Sigma^i$. A strategy profile $\sigma = (\sigma^1, \dots, \sigma^{2n+1}) \in \Sigma$ induces a probability measure $\mathbb{P}_\sigma$ over the set of plays $\Omega = (A \times U)^\infty = \{(a_1, u_1, a_2, u_2, \dots), \forall t \geq 1, a_t \in A, u_t \in U)\}$. With an abuse of notation we will denote by a_t the random variable of the joint action profile in A played at stage t.

2.2.1 The discounted game

For λ in $]0, 1]$, the payoff of player i ($i \in N$) in the λ-discounted game is

$$\gamma_\lambda^i(\sigma) := \sum_{t=1}^{\infty} \lambda(1-\lambda)^{t-1}\mathbb{E}_\sigma[g^i(a_t)] = \mathbb{E}_\sigma\left(\sum_{t=1}^{\infty} \lambda(1-\lambda)^{t-1} g^i(a_t)\right),$$

where $\mathbb{E}_\sigma$ denotes the expectation computed according to $\mathbb{P}_\sigma$. We denote by $E_\lambda \subset \mathcal{S}$ the set of Nash equilibrium payoffs of the λ-discounted game.

2.2.2 The finitely repeated game

For $T \in \mathbb{N}_+$ the payoff of player i ($i \in N$) in the T-repeated game is

$$\gamma_T^i(\sigma) := \sum_{t=1}^{T} \frac{1}{T}\mathbb{E}_\sigma[g^i(a_t)].$$

We denote by $E_T \subset \mathcal{S}$ the set of Nash equilibrium payoffs of the T-repeated game.

3 Convergence of E_λ

The main result in this section is the following, which shows that, as the discount factor $(1-\lambda)$ grows, the set E_λ of equilibrium payoffs converges (in the Hausdorff metric) to the feasible set $\mathcal{S}$.

Theorem 3.1.

$$\lim_{\lambda\to 0} E_\lambda = \mathcal{S}.$$

The idea behind the proof of Theorem 3.1 is the following. By standard convexification properties, attention may be restricted to the extreme points of the feasible set $\mathcal{S}$. For each extreme point e_S we will find, for λ small enough, a point in E_λ that is arbitrarily close to e_S. If $|S| = n$, $e_S \in E_1$ so *a fortiori* $e_S \in E_\lambda$ for all $\lambda \in (0,1)$. Consider now S such that $|S| = s \leq n-1$. By symmetry we may restrict attention to the case where $e_S = (1,\ldots,1,0,\ldots,0)$.

In order to find an equilibrium payoff that is arbitrarily close to e_S, we proceed as follows: fix λ, and start with an equilibrium payoff $z = (q,\ldots,q,y,\ldots,y,w,\ldots,w) \in E_\lambda$, where s players have payoff q, $(n+1-s)$ players have payoff y, and n players have payoff w. The quantities w and y will be required to satisfy some conditions, that are actually always met by some equilibrium payoffs in E_λ. Given z, we get a new payoff $z' \in E_\lambda$ such that the distance between z' and e_S is significantly smaller than the distance between z and e_S. This payoff z' is obtained by the following strategy profile:

- at stage 1 the first s players play R, the last n players play L and the remaining players play L with probability $1-\alpha$ and R with some probability α (denote this mixed action as $(1-\alpha)L \oplus \alpha R$),
- at stage 2, if the signal was L, then all players play forever the equilibrium with payoff z; if the signal was R, then each player plays forever the pure action taken at stage 1.

We want to prove that, for a suitable α, this strategy is an equilibrium.

If α is small, then, at stage 1, the last $2n+1-s$ players all play L with high probability, hence the first s players are in the minority, the payoff is e_S, and the public signal is L.

Consider that, given that the last n players play L, according to the above strategy profile, at most $n+1$ players can play R at stage 1. The first s players have no incentive to deviate at stage 1 because they already play R and have a good chance to get a payoff of 1 at this stage. The remaining $2n+1-s$ players do not mind playing L with (at least) positive probability, because if one of them plays R at stage 1, then with non-negligible probability $n+1$ players will be in room R and n players will be in room L. Since the strategy then recommends to repeat forever the action of stage 1, the players in room R will have a null payoff at every stage. So, no deviation at the following stages is profitable, either.

Intuitively, repeating forever the action of stage 1, if the public signal was R, may be considered as a punishment phase having the following peculiarity: the players do not know who did deviate, but still, with high probability, they are able to punish the deviating player, if any, by staying in the same room. This idea of punishing a deviating player without knowing its identity is an important feature of our strategy.

If α is small, then with high probability the payoff vector is $\lambda e_S + (1-\lambda)z$. This implies that the equilibrium payoff z' is closer to e_S than the original payoff z was.

The above argument is formalized by the following lemma.

Lemma 3.2. *Fix $\lambda \in (0,1)$, and*

$$z = (\underbrace{q,\ldots,q}_{s},\underbrace{y,\ldots,y}_{n+1-s},\underbrace{w,\ldots,w}_{n}) \in E_\lambda \tag{3.1}$$

such that $w \geq y \geq \lambda^{1/(m+1)}$, *where*

$$m = n - s \geq 1. \tag{3.2}$$

Let

$$\alpha = \left(\frac{\lambda}{\lambda + (1-\lambda)y} \right)^{1/m}. \tag{3.3}$$

Then

$$z' = (\underbrace{q', \dots, q'}_{s}, \underbrace{y', \dots, y'}_{n+1-s}, \underbrace{w', \dots, w'}_{n}) \in E_\lambda$$

if

$$\begin{aligned}
y' &= (1-\lambda)y, \\
w' &= (1-\lambda)w + \alpha^{m+1}(1-(1-\lambda)w), \\
1 - q' &= (1-\lambda)(1-q) + \alpha^{m+1}(1-(1-\lambda)(1-q)) \\
&= (1-\lambda)(1-q) + \alpha^{m+1}(\lambda + (1-\lambda)q).
\end{aligned}$$

The rest of the proof of Theorem 3.1 is based on the following steps:

- We use Lemma 3.2 to prove that for λ small enough, there exist a payoff $z \in E_\lambda$ such that $\|z - e_S\|$ can be made arbitrarily small.
- We show that if $(1-\mu) = (1-\lambda)^d$ for some integer d, then $E_\mu \subset E_\lambda$.
- We consider that any point in $\mathcal{S}$ can be approximated with a convex combination of its extreme points having rational coefficients with denominator d.
- We prove that the convex combination of any $z_1, \dots, z_L \in E_\mu$ with weights $d_1/d, \dots, d_L/d$ is close to the payoff obtained by cycling over z_1 d_1 times, z_2 d_2 times, ..., z_L d_L times.
- We combine the last two approximations to achieve the result.

Following a request of a referee, we give a heuristic idea of why the proof works. Since the most efficient outcome in the stage game (say, n players choose R and $n+1$ players choose L) is already a stage-game Nash equilibrium, the difficulty lies in supporting inefficient outcomes, such as $(1, 0, \dots, 0)$.

It is important to notice that we prove our result by contradiction (proof of Lemma 6.2): we don't construct an appropriate equilibrium of the λ discounted game, but, rather, we prove its existence.

Given an equilibrium payoff z, the proof of Lemma 3.2 shows the way to support a payoff z' that is closer to $(1, 0, \dots, 0)$. At stage 1 player 1 plays R, players $2, \dots, n+1$ play $\alpha R \oplus (1-\alpha)L$, and players $n+2, \dots, 2n+1$ play L.

After that, if the public signal of stage 1 is L, then all players go on playing an equilibrium with payoff z. This, together with the action at stage 1, will produce a payoff z', which is closer to $(1, 0, \dots, 0)$); if the public signal is R, then the players repeat forever their action played at stage 1.

The calibration of α is very important. On the one hand it has to be big enough to imply that player 1 is in best reply by playing R at stage 1 (the probability α^n of all players $2, \dots, n+1$ playing R is small enough). On the other hand it has to be small enough so that if player 2 imitates player 1 and plays R at stage 1, then player 2 is not better off because the probability α^{n-1} that all players $3, \dots, n+1$ play R is not that small. So no player has an incentive to imitate player 1, even if with high probability player 1 will be alone at stage 1. This is because of the possible punishment, i.e., of what happens if the signal is R: this punishment lasts forever, so has to be feared, even if a cheater only has a small probability to get caught.

Lemma 6.2 proves that this way we can get as close as we want to the payoff $(1, 0, \dots, 0)$.

4 Convergence of E_T

In this section we will show that, as the number of times the game is repeated grows, the set of equilibrium payoffs E_T converges to the feasible set $\mathcal{S}$.

Theorem 4.1.

$$\lim_{T\to\infty} E_T = \mathcal{S}.$$

Theorem 4.1 will be proved with a technique that resembles the one used for the proof of Theorem 3.1. We will start with an equilibrium payoff in E_T and we will find an equilibrium payoff in E_{T+1} that is closer to an extreme point of $\mathcal{S}$.

The key lemma is the following.

Lemma 4.2. *Fix $T \in \mathbb{N} \setminus \{0\}$, $s \in \{0, \dots, n-1\}$ and*

$$z = (\underbrace{q, \dots, q}_{s}, \underbrace{y, \dots, y}_{n+1-s}, \underbrace{w, \dots, w}_{n}) \in E_T \tag{4.1}$$

such that

$$w \geq y \geq \left(\frac{1}{T+1}\right)^{1/(m+1)}. \tag{4.2}$$

Then

$$z' = (\underbrace{q', \dots, q'}_{s}, \underbrace{y', \dots, y'}_{n+1-s}, \underbrace{w', \dots, w'}_{n}) \in E_{T+1},$$

with

$$\begin{aligned} y' &= \frac{T}{T+1} y, \\ w' &= \frac{T}{T+1} w + \alpha^{m+1}\left(1 - \frac{T}{T+1} w\right), \\ 1 - q' &= \frac{T}{T+1}(1-q) + \alpha^{m+1}\left(1 - \frac{T}{T+1}(1-q)\right), \end{aligned}$$

and

$$\alpha = \left(\frac{1}{1+Ty}\right)^{1/m}. \tag{4.3}$$

The strategy used for the proof of Lemma 4.2 is similar to the one used for the proof of Lemma 3.2, except that now the calibration of the mixed strategy is in terms of T rather than λ.

5 Public equilibria

A strategy σ^i of player i is called *public* if it depends only on the past public signal, i.e., $\sigma^i = (\sigma^i_t)_{t\geq 1}$ is a behavioral strategy ($\sigma^i_t : (A^i \times U)^{t-1} \to \Delta(A^i)$), such that for each t $\sigma^i_t(a^i_1, u_1, a^i_2, u_2, \ldots, a^i_{t-1}, u_{t-1})$ depends only on $u_1, \ldots, u_{t-1}$, and not on $a^i_1, \ldots, a^i_{t-1}$. A *public equilibrium* of the λ-discounted game (resp. of the T stage finite game) is a profile of public strategies which is an equilibrium of the λ-discounted game (resp. of the T stage finite game). We denote by E^{p}_λ (resp. by E^{p}_T) the set of public equilibria of the λ-discounted game (resp. of the T stage finite game). The case of public equilibria is studied by Fudenberg et al. (1994) who prove a folk theorem based on a pair-wise identifiability condition, that implies that deviations by different players induce different probability distributions of public signals. In our model this condition is not valid, since the deviation of any player in a room produces the same effect on the signal. Below we prove an *anti-folk theorem*.

5.1 Three players

Here we assume that $n = 1$, so that $N = \{1, 2, 3\}$. We start with the three-player game because in this case we have a complete characterization of the set of public equilibria in the discounted game for small λ and an asymptotic characterization for the finitely repeated game.

The set of Nash equilibrium payoffs of the stage game is

$$E_1 = \{(1/4, 1/4, 1/4)\} \cup \{(\beta, 1-\beta, 0) : \beta \in [0,1]\} \\ \cup \{(\beta, 0, 1-\beta) : \beta \in [0,1]\} \cup \{(0, \beta, 1-\beta) : \beta \in [0,1]\}.$$

Its convex hull is

$$\begin{aligned} \mathcal{C} &= \operatorname{conv} E_1 \\ &= \operatorname{conv}\{(1/4, 1/4, 1/4), (1,0,0), (0,1,0), (0,0,1)\} \\ &= \{(x^1, x^2, x^3) \in \mathbb{R}^3_+ : x^1 + x^2 + x^3 \leq 1, 2x^1 + x^2 + x^3 \geq 1, x^1 + 2x^2 + x^3 \geq 1, \\ &\qquad x^1 + x^2 + 2x^3 \geq 1\}. \end{aligned}$$

Proposition 5.1. *For all* $0 < \lambda \leq 1$, $E^{\mathrm{p}}_\lambda \subset \mathcal{C}$, *and for all* $0 < \lambda \leq 1/4$, $E^{\mathrm{p}}_\lambda = \mathcal{C}$.

The above proposition shows that in the discounted game with three players the set of public equilibrium payoffs coincides with the convex hull of stage-Nash payoffs if λ is small enough (smaller than 1/4). The next proposition shows that a similar property holds asymptotically for the finitely repeated game.

Proposition 5.2. *For all* $T \geq 1$, $E^{\mathrm{p}}_T \subset \mathcal{C}$, *and* $\lim_{T\to\infty} E^{\mathrm{p}}_T = \mathcal{C}$.

5.2 $2n + 1$ players

Now $N = \{1, \dots, 2n+1\}$ for some integer n. In this case, due to computational complexity, we cannot obtain characterizations similar to Propositions 5.1 and 5.2. We just give a bound, which is probably far from being optimal, but is sufficient to show that the public equilibrium payoffs are bounded away from the null vector.

Proposition 5.3. *For any* $0 < \lambda \leq 1$ *and* $z \in E^{\mathrm{p}}_\lambda$, *we have*

$$\sum_{i\in N} z^i \geq \frac{1}{3}.$$

Proposition 5.4. *For all* $T \geq 1$ *and* $z \in E^{\mathrm{p}}_T$, *we have*

$$\sum_{i\in N} z^i \geq \frac{1}{3}.$$

The two above propositions prove that also in this case private strategies can achieve equilibrium payoffs that public strategies cannot.

6 Proofs

Section 3. Convergence of E_λ

The symbols $\alpha, \lambda, m, n, s, z, q, y, w, z', q', y', w'$ were defined in Section 3.

Proof of Lemma 3.2. By (3.3) we have

$$\lambda^{1/m} \leq \alpha \leq \left(\frac{\lambda}{y}\right)^{1/m}, \tag{6.1}$$

hence

$$(1-\lambda)w \leq w' \leq (1-\lambda)w + \left(\frac{\lambda}{y}\right)^{(m+1)/m}, \tag{6.2}$$

$$(1-\lambda)(1-q) \leq 1-q' \leq (1-\lambda)(1-q) + \left(\frac{\lambda}{y}\right)^{(m+1)/m}. \tag{6.3}$$

When z is as in (4.1), call

$$W(z) = s(1-q) + (n+1-s)y + nw. \tag{6.4}$$

The value $W(z)$ represents the L_1 distance between z and e_S. By definition $W(z) \geq 0$. Eventually we would like $W(z)$ to be close to 0. Inequalities (6.2) and (6.3) imply

$$(1-\lambda)W(z) \leq W(z') \leq (1-\lambda)W(z) + (n+s)\left(\frac{\lambda}{y}\right)^{(m+1)/m}, \tag{6.5}$$

hence

$$W(z') - W(z) \leq \lambda\left(-W(z) + (n+s)\frac{\lambda^{1/m}}{y^{(m+1)/m}}\right). \tag{6.6}$$

Using (6.1) we obtain

$$\begin{aligned}
\frac{\lambda}{1-\lambda}\frac{1-\alpha^{m+1}}{\alpha^{m+1}} &\geq \frac{\lambda}{1-\lambda}\frac{1-\left(\frac{\lambda}{y}\right)^{(m+1)/m}}{\left(\frac{\lambda}{y}\right)^{(m+1)/m}} \\
&= \frac{1}{1-\lambda}\frac{y^{(m+1)/m}}{\lambda^{1/m}}\left(1-\left(\frac{\lambda}{y}\right)^{(m+1)/m}\right) \\
&= \frac{1}{1-\lambda}\left(y\left(\frac{y}{\lambda}\right)^{1/m} - \lambda\right).
\end{aligned}$$

So since

$$y \geq \lambda^{1/(m+1)}, \tag{6.7}$$

then

$$y\left(\frac{y}{\lambda}\right)^{1/m} - \lambda \geq 1-\lambda,$$

therefore we obtain the following inequality:

$$\frac{\lambda}{1-\lambda}\left(\frac{1-\alpha^{m+1}}{\alpha^{m+1}}\right) \geq 1. \tag{6.8}$$

Now construct a new equilibrium where players play as follows.

At stage 1

$$\left.\begin{array}{c}\text{player } 1\\ \vdots\\ \text{player } s\end{array}\right\}\text{ play } R,$$

$$\left.\begin{array}{c}\text{player } s+1\\ \vdots\\ \text{player } n+1\end{array}\right\}\text{ play } (1-\alpha)L \oplus \alpha R,$$

$$\left.\begin{array}{c}\text{player } n+2\\ \vdots\\ \text{player } 2n+1\end{array}\right\}\text{ play } L.$$

If the signal was L at stage 1, then, from stage 2 on, all players play the equilibrium with payoff z; if the signal was R at stage 1, then each player plays forever what she played at stage 1.

In order to show that this is indeed an equilibrium, we will show that every player is in best reply.

Consider player $(s+1)$ (or any other player in her group). If she plays L at stage 1, her payoff in the discounted game is $(1-\lambda)y$. If she plays R at stage 1, her payoff is

$$\mathbb{P}(L)(\lambda + (1-\lambda)y) + \mathbb{P}(R)0,$$

where $\mathbb{P}(R)$ indicates the probability that the signal is R at stage 1, and in this case $\mathbb{P}(R) = \alpha^{n-s}$.

Since $\alpha \in (0,1)$, player $(s+1)$ is in best response iff

$$(1-\lambda)y = (1-\alpha^{n-s})(\lambda + (1-\lambda)y)$$

iff

$$1-\alpha^{n-s} = \frac{(1-\lambda)y}{\lambda + (1-\lambda)y}$$

iff

$$\alpha^m = \frac{\lambda}{\lambda + (1-\lambda)y},$$

which is equality (3.3). And then the payoff of player $(s+1)$ is $y' = (1-\lambda)y$.

Consider player 1 (or any other player in her group). If she plays L at stage 1, her payoff is $(1-\lambda)q$. If she plays R at stage 1, her payoff is

$$\mathbb{P}(L)(\lambda + (1-\lambda)q) + \mathbb{P}(R)0,$$

where in this case $\mathbb{P}(R) = \alpha^{n+1-s}$.

Therefore player 1 is in best response iff

$$(1-\alpha^{n+1-s})(\lambda + (1-\lambda)q) \geq (1-\lambda)q$$

iff

$$\lambda(1-\alpha^{m+1}) \geq (1-\lambda)q\alpha^{m+1}$$

By (6.8), we know that this inequality is always satisfied.

Consider player $(n+2)$ (or any other player in her group). If she plays L she gets

$$\begin{aligned}\mathbb{P}(L)(1-\lambda)w + \mathbb{P}(R)1 &= (1-\alpha^{m+1})(1-\lambda)w + \alpha^{m+1} \\ &= (1-\lambda)w + \alpha^{m+1}(1-(1-\lambda)w),\end{aligned}$$

since here $\mathbb{P}(R) = \alpha^{n+1-s}$.

If she plays R, she gets at most

$$\begin{aligned}&\mathbb{P}(L)(\lambda + (1-\lambda)w) + \mathbb{P}(\text{all players } s+1,\ldots,n+1 \text{ play } R)(\lambda 0 + (1-\lambda)1) \\ &\qquad + \mathbb{P}(\text{exactly } n-s \text{ players among } s+1,\ldots,n+1 \text{ play } R)(\lambda 0 + (1-\lambda)0) \\ &\qquad\qquad = \mathbb{P}(L)(\lambda + (1-\lambda)w) + \alpha^{n+1-s}(1-\lambda),\end{aligned}$$

where the last 1 in the first line of the above formula is an upper bound assuming that the deviator switches to the minority in all future periods, and the 0 in the following line is the case where players are equally split and the deviator is always in the majority. Now

$$\begin{aligned}\mathbb{P}(L) &= 1 - \alpha^{n+1-s} - (n+1-s)(\alpha^{n-s} - \alpha^{n+1-s}) \\ &= 1 + \alpha^{n-s}(-(n+1-s)) + \alpha^{n+1-s}(-1+n+1-s) \\ &= 1 - \alpha^{n-s}(n+1-s) + (n-s)\alpha^{n+1-s}.\end{aligned}$$

Hence playing R gives player $(n+2)$ at most

$$(\lambda + (1-\lambda)w)(1-(n+1-s)\alpha^{n-s} + (n-s)\alpha^{n+1-s}) + (1-\lambda)\alpha^{n+1-s}.$$

Therefore player $(n+2)$ is in best reply iff

$$\begin{aligned}&(1-\lambda)w + \alpha^{m+1}(1-(1-\lambda)w) \\ &\qquad \geq (\lambda + (1-\lambda)w)(1-(n+1-s)\alpha^{n-s} + (n-s)\alpha^{n+1-s}) + (1-\lambda)\alpha^{n+1-s}\end{aligned}$$

iff

$$\begin{aligned}&(1-\lambda)w + \alpha^{m+1}(1-(1-\lambda)w) \\ &\qquad \geq (\lambda + (1-\lambda)w)(1-(m+1)\alpha^{m} + m\alpha^{m+1}) + (1-\lambda)\alpha^{n+1-s}\end{aligned}$$

iff

$$\begin{aligned}&w(1-\lambda)(1-\alpha^{m+1} - 1 + (m+1)\alpha^m - m\alpha^{m+1}) + \alpha^{m+1} - (1-\lambda)\alpha^{m+1} \\ &\qquad - \lambda(1-(m+1)\alpha^m + m\alpha^{m+1}) \geq 0\end{aligned}$$

iff

$$F(w) := w(1-\lambda)(m+1)\alpha^m(1-\alpha) + \lambda(-1 + (m+1)\alpha^m + (1-m)\alpha^{m+1}) \geq 0.$$

The function F is increasing, so $F(w) \geq F(y)$, since $w \geq y$ by assumption. By (3.3) we have

$$\alpha^m(1-\lambda)y = \lambda(1-\alpha^m),$$

hence

$$\begin{aligned} F(y) &= y(1-\lambda)(m+1)\alpha^m(1-\alpha) + \lambda(-1+(m+1)\alpha^m + (1-m)\alpha^{m+1}) \\ &= \lambda(1-\alpha^m)(m+1)(1-\alpha) + \lambda(-1+(m+1)\alpha^m + (1-m)\alpha^{m+1}) \\ &= \lambda\left[(m+1)(1-\alpha)(1-\alpha^m) - 1 + (m+1)\alpha^m + (1-m)\alpha^{m+1}\right] \\ &= \lambda\left[(m+1)(1-\alpha-\alpha^m+\alpha^{m+1}) - 1 + (m+1)\alpha^m + (1-m)\alpha^{m+1}\right] \\ &= \lambda\left[m+1-\alpha(m+1)+2\alpha^{m+1}-1\right] \\ &= \lambda\left[m-\alpha(m+1)+2\alpha^{m+1}\right], \end{aligned}$$

One can study the mapping $\alpha \mapsto m - \alpha(m+1) + 2\alpha^{m+1}$, and show that it is positive for any value of α in $[0,1]$. So $F(y) > 0$, and player $n+2$ is in best reply. Her payoff is $w' = (1-\lambda)w + \alpha^{m+1}(1-(1-\lambda)w)$.

To conclude the proof, just consider that the payoff of player 1 is

$$q' = (1-\alpha^{m+1})(\lambda + (1-\lambda)q),$$

hence

$$\begin{aligned} 1-q' &= 1-(1-\alpha^{m+1})(\lambda+(1-\lambda)q) \\ &= 1-(\lambda+(1-\lambda)q) + \alpha^{m+1}(\lambda+(1-\lambda)q) \\ &= 1-\lambda-(1-\lambda)q + \alpha^{m+1}(1-(1-\lambda)(1-q)) \\ &= (1-\lambda)(1-q) + \alpha^{m+1}(1-(1-\lambda)(1-q)). \end{aligned}$$

□

We will use the following notations in the sequel:

$$k(n,m) = \frac{m^m(m+1)^{m+1}}{(2n-m)^m(2n+1)^{m+1}(2m+1)^{2m+1}}, \tag{6.9}$$

$$C = \frac{m}{(2n+1)(2m+1)} > 0. \tag{6.10}$$

Lemma 6.1. *For $a \in (0,1]$, the condition:*

$$\lambda \leq a^{2m+1}k(n,m) \tag{6.11}$$

implies

$$\lambda \leq \left(a\left(\frac{1}{2n+1} - C\right)\right)^{m+1}, \tag{6.12}$$

which in turns implies

$$\lambda \leq \frac{1}{2n+1}. \tag{6.13}$$

Proof. (6.11) implies (6.12): first consider that (6.11) becomes

$$\lambda \leq \frac{a^{2m+1}}{(n+s)^m} \cdot \frac{m^m}{(2n+1)^{m+1}} \cdot \frac{(m+1)^{m+1}}{(2m+1)^{2m+1}}$$

where the equality stems from (3.2) and (6.9). If (6.11) holds, then

$$\begin{aligned}\lambda \left(\frac{(2n+1)(2m+1)}{a(m+1)}\right)^{m+1} &\leq \frac{a^{2m+1}m^m(m+1)^{m+1}(2n+1)^{m+1}(2m+1)^{m+1}}{(2n+1)^{m+1}(2m+1)^{2m+1}(n+s)^m a^{m+1}(m+1)^{m+1}} \\ &= \frac{a^{2m+1}m^m}{(2m+1)^m(n+s)^m a^{m+1}} \\ &\leq \frac{a^m}{(2m+1)^m} \\ &\leq 1.\end{aligned}$$

This implies (6.12).

(6.12) implies (6.13): this is immediate, since $a \leq 1$. □

Lemma 6.2. *Fix* $0 < a \leq 1$. *If*

$$\lambda \leq k(n,m)a^{2m+1}, \tag{6.14}$$

then there exists $z \in E_\lambda$ *such that* $W(z) \leq a$, *where* $W(\cdot)$ *is defined as in* (6.4).

Proof. Fix $0 < a \leq 1$, and λ that satisfies (6.14). Consider the following condition

$$w \geq y \geq \frac{W(z)}{2n+1} - Ca. \tag{6.15}$$

Define

$$Z_{\lambda,a} = \{z = (\underbrace{q,\ldots,q}_{s}, \underbrace{y,\ldots,y}_{n+1-s}, \underbrace{w,\ldots,w}_{n}) \in E_\lambda \quad \text{such that } w \geq y \geq \frac{W(z)}{2n+1} - Ca \quad \text{holds}\}$$

First we prove that $Z_{\lambda,a} \neq \varnothing$. In fact for all $Q \subset \{1,\ldots,2n+1\}$ with $\{1,\ldots,s\} \subset Q$ and $\text{card}(Q) = n$ there exists a Nash equilibrium of the stage game with payoff e_Q.

By Lemma 6.1

$$\lambda \leq \frac{1}{2n+1},$$

So it is possible to use time to convexify and obtain exactly the uniform convex combination of $(e_Q)_{Q\supset S}$ as a payoff vector in E_λ (see for example Mertens et al. (1994, pp. 194–195)):

$$z = (\underbrace{1,\ldots,1}_{s}, \underbrace{y,\ldots,y}_{2n+1-s}),$$

with $n = s + (2n+1-s)y$.

This vector z is such that $w = y$, and $W(z) = (2n + 1 - s)y$, therefore

$$y \geq \frac{W(z)}{2n+1} \geq \frac{W(z)}{2n+1} - Ca,$$

hence $z \in Z_{\lambda,a}$.

The set E_λ is a compact set since the strategy sets are compact and the discounted payoff functions are continuous. Since W is continuous, we obtain that the set $Z_{\lambda,a}$ is compact.

Fix $z \in Z_{\lambda,a}$ that minimizes $W(z)$. Assume that

$$W(z) > a. \tag{6.16}$$

We will prove that this leads to a contradiction.

If $W(z) > a$, then, by (6.15),

$$y \geq a\left(\frac{1}{2n+1} - C\right), \tag{6.17}$$

and, by Lemma 6.1,

$$a\left(\frac{1}{2n+1} - C\right) \geq \lambda^{1/(m+1)},$$

hence Lemma 3.2 can be applied to z.

We will verify that (6.15) holds for z'. Recall (6.5).

$$W(z') \leq (1-\lambda)W(z) + (n+s)\left(\frac{\lambda}{y}\right)^{(m+1)/m}.$$

Hence

$$\begin{aligned}
y' - \frac{W(z')}{2n+1} + Ca &\geq (1-\lambda)y - \frac{1-\lambda}{2n+1}W(z) - \frac{n+s}{2n+1}\left(\frac{\lambda}{y}\right)^{(m+1)/m} + Ca \\
&\geq -(1-\lambda)Ca + Ca - \frac{n+s}{2n+1}\left(\frac{\lambda}{y}\right)^{(m+1)/m} \\
&= \lambda Ca - \frac{n+s}{2n+1}\left(\frac{\lambda}{y}\right)^{(m+1)/m} \\
&\geq \lambda Ca - \frac{n+s}{2n+1} \cdot \frac{\lambda^{(m+1)/m}}{a^{(m+1)/m}\left(\frac{1}{2n+1} - C\right)^{(m+1)/m}},
\end{aligned} \tag{6.18}$$

where the second inequality stems from (6.15), and the last one from (6.17).

Expression (6.18) is non-negative iff

$$\lambda^{1/m} \leq \frac{Ca^{(2m+1)/m}\left(\frac{1}{2n+1} - C\right)^{(m+1)/m}(2n+1)}{n+s},$$

iff

$$\lambda \leq \frac{a^{2m+1} C^m \left(\frac{1}{2n+1} - C \right)^{m+1} (2n+1)^m}{(n+s)^m},$$

iff

$$\lambda \leq \frac{a^{2m+1} C^m \left(1 - (2n+1)C\right)^{m+1}}{(2n+1)(n+s)^m},$$

which is just (6.11). Therefore $z' \in Z_{\lambda,a}$.

To find a contradiction we need to show that $W(z') < W(z)$. By (6.6), (6.16), and (6.17) we have

$$\begin{aligned} W(z') - W(z) &\leq -\lambda W(z) + (n+s) \left(\frac{\lambda}{y} \right)^{(m+1)/m} \\ &< -\lambda a + (n+s) \left(\frac{\lambda}{y} \right)^{(m+1)/m} \\ &< -\lambda a + \frac{(n+s)\lambda^{(m+1)/m}}{a^{(m+1)/m} \left(\frac{1}{2n+1} - C \right)^{(m+1)/m}}. \end{aligned} \tag{6.19}$$

This last expression is non-positive iff

$$\frac{(n+s)\lambda^{(m+1)/m}}{a^{(m+1)/m} \left(\frac{1}{2n+1} - C \right)^{(m+1)/m}} \leq \lambda a,$$

iff

$$\lambda^{1/m} \leq \frac{a^{(2m+1)/m}}{n+s} \left(\frac{1}{2n+1} - C \right)^{(m+1)/m},$$

iff

$$\lambda \leq \frac{a^{2m+1}(1-(2n+1)C)^{m+1}}{(n+s)^m(2n+1)^{m+1}}. \tag{6.20}$$

By (6.11) we have:

$$\lambda \leq \frac{a^{2m+1} C^m \left(1 - (2n+1)C\right)^{m+1}}{(2n+1)(n+s)^m},$$

which holds iff

$$\lambda \leq \frac{C^m(2n+1)^m a^{2m+1} \left(1 - (2n+1)C\right)^{m+1}}{(n+s)^m(2n+1)^{m+1}}.$$

Since $C(2n+1) < 1$, we have that (6.11) implies (6.20). $\square$

Lemma 6.2 gives a bound of the order of a^{2m+1} for the speed of convergence of E_λ to a set containing the vector e_S. Notice that $m = n - s$ represents the distance between e_S and a Nash equilibrium payoff of the one-shot game.

Lemma 6.3. *Fix $\lambda \in (0,1)$ and $d \geq 1$ integer. Let*

$$\mu = 1-(1-\lambda)^d \geq \lambda. \tag{6.21}$$

If $x_1, \ldots, x_d \in E_\mu$, then

$$\sum_{\ell=1}^{d} \frac{\lambda(1-\lambda)^{\ell-1}}{1-(1-\lambda)^d} x_\ell \in E_\lambda,$$

hence in particular $E_\mu \subset E_\lambda$.

Proof. Assume that $x_1, \ldots, x_d \in E_\mu$. For $\ell \in \{1, \ldots, d\}$ let σ^ℓ be an equilibrium of the μ-discounted game that achieves the payoff x_ℓ. We now define a strategy τ which will be an equilibrium of the λ-discounted game.

Divide the set of stages $\{1, 2, \ldots, \}$ into d equivalence classes B_1, $B_2, \ldots$, B_d, with $B_\ell = \{\ell + t\, d,\ t \in \mathbb{N}\}$ for every ℓ in $\{1, \ldots, d\}$. The strategy τ plays independently on each class. At class B_ℓ, τ forgets all the stages not in B_ℓ and plays according to σ^ℓ. For i in N, the payoff of player i in the λ-discounted game if τ is played is:

$$\begin{aligned}
\gamma^i_\lambda(\tau) &= \sum_{t=1}^{\infty} \lambda(1-\lambda)^{t-1} \mathbb{E}_\tau[g^i(a_t)] \\
&= \lambda \sum_{\ell=1}^{d} \sum_{t'=0}^{\infty} (1-\lambda)^{l+t'd-1} \mathbb{E}_\tau[g^i(a_{\ell+t'd})] \\
&= \sum_{\ell=1}^{d} \lambda(1-\lambda)^{\ell-1} \left(\sum_{t'=0}^{\infty} \left((1-\lambda)^d\right)^{t'} \mathbb{E}_\tau[g^i(a_{\ell+t'd})] \right) \\
&= \sum_{\ell=1}^{d} \lambda(1-\lambda)^{\ell-1} \left(\sum_{t'=0}^{\infty} (1-\mu)^{t'} \mathbb{E}_\tau[g^i(a_{\ell+t'd})] \right) \\
&= \sum_{\ell=1}^{d} \frac{\lambda(1-\lambda)^{\ell-1}}{\mu} \left(\sum_{t'=1}^{\infty} \mu\,(1-\mu)^{t'-1} \mathbb{E}_\tau[g^i(a_{\ell+(t'-1)d})] \right) \\
&= \sum_{\ell=1}^{d} \frac{\lambda(1-\lambda)^{\ell-1}}{\mu} x_\ell \\
&= \sum_{\ell=1}^{d} \frac{\lambda(1-\lambda)^{\ell-1}}{1-(1-\lambda)^d} x_\ell
\end{aligned}$$

Hence this is the payoff achieved by strategy τ. In order to see that τ is an equilibrium in the λ-discounted game, consider that it requires to cycle over d equilibria of the λ-discounted game. Relation (6.21) and a similar computation ensure that the discounting is properly taken into account. ☐

Remark 6.4. Since $\mathcal{S} = \text{conv}\{e_S, S \subset N \text{ with } \text{card}(S) \leq n\}$, the number of extreme points of $\mathcal{S}$ is

$$L = \sum_{k=0}^{n} \binom{2n+1}{k} = 4^n.$$

Proof of Theorem 3.1. Since $\mathcal{S}$ is the set of feasible payoffs, we have, for all $\lambda \in (0,1)$, $E_\lambda \subset \mathcal{S}$.

We want to prove that for all $\varepsilon > 0$ there exists $\lambda_0 > 0$ such that for all $\lambda \leq \lambda_0$, for all $z \in \mathcal{S}$ there exists $z'' \in E_\lambda$ such that $\|z - z''\| \leq \varepsilon$. In all the computations we use $\|\cdot\| = \|\cdot\|_1$.

Write $\mathcal{S} = \operatorname{conv}\{v_1, \ldots, v_L\}$. Fix $\varepsilon \in (0,1)$. There exists $d \in \mathbb{N}$ such that every element $z \in \mathcal{S}$ is at a distance of at most ε from another element $z' \in \mathcal{S}$ that can be written as a convex combination of $v_1, \ldots, v_L$ with rational coefficients having all d as a denominator.

Define, for every $d', d'' \in \{0, \ldots, d\}$

$$\zeta(\lambda, d', d'') = \left| \frac{d'}{d} - \frac{(1-\lambda)^{d''}(1-(1-\lambda)^{d'})}{1-(1-\lambda)^d} \right|.$$

We have $\lim_{\lambda \to 0} \zeta(\lambda, d', d'') = 0$. So we can find $\lambda_0 > 0$ such that for all $\lambda \in (0, \lambda_0]$,

$$1 - (1-\lambda)^d \leq k\varepsilon^{2n+1}, \tag{6.22}$$

where $k = \min_{m \in \{1,\ldots,n\}} k(n,m)$, and $k(n,m)$ is defined as in (6.9). Furthermore for all $d', d'' \in \{0, \ldots, d\}$ we have

$$\zeta(\lambda, d', d'') \leq \varepsilon. \tag{6.23}$$

Fix now λ in $(0, \lambda_0]$. Take $z \in \mathcal{S}$. Consider $z' \in \mathcal{S}$ such that $\|z - z'\| \leq \varepsilon$ and there exist integers $d_1, \ldots, d_L \geq 0$ such that

$$z' = \sum_{\ell=1}^{L} \frac{d_\ell}{d} v_\ell \quad \text{and} \quad \sum_{\ell=1}^{L} d_\ell = d.$$

Let μ be as in (6.21). By (6.22) we have $\mu \leq k\varepsilon^{2n+1}$. Hence for all $m \in \{1, \ldots, n\}$ we have $\mu \leq k(n,m)\varepsilon^{2n+1}$.

By Lemma 6.2, for every $\ell \in \{1, \ldots, L\}$ there exists $z_\ell \in E_\mu$ with $W(z_\ell) = \|z_\ell - v_\ell\| \leq \varepsilon$.

By Lemma 6.3 applied to

$$\underbrace{z_1, \ldots, z_1}_{d_1}, \underbrace{z_2, \ldots, z_2}_{d_2}, \ldots, \underbrace{z_L, \ldots, z_L}_{d_L}$$

there exists $z'' \in E_\lambda$ such that

$$z'' = \sum_{i=1}^{d} \frac{\lambda(1-\lambda)^{i-1}}{1-(1-\lambda)^d} x_i,$$

where $x_1, \ldots, x_{d_1} = z_1$, $x_{d_1+1}, \ldots, x_{d_1+d_2} = z_2$, $\ldots$, $x_{d_1+\cdots+d_{L-1}+1}, \ldots, x_d = z_L$.

Now

$$
\begin{aligned}
z'' &= \sum_{i=1}^{d} \frac{\lambda(1-\lambda)^{i-1}}{1-(1-\lambda)^d} x_i \\
&= \sum_{\ell=1}^{L} \frac{\lambda}{1-(1-\lambda)^d} z_\ell \left(\sum_{i=d_1+\cdots+d_{\ell-1}+1}^{d_1+\cdots+d_\ell} (1-\lambda)^{i-1} \right) \\
&= \frac{\lambda}{1-(1-\lambda)^d} \sum_{\ell=1}^{L} z_\ell \frac{(1-\lambda)^{d_1+\cdots+d_{\ell-1}} - (1-\lambda)^{d_1+\cdots+d_\ell}}{1-(1-\lambda)} \\
&= \frac{1}{1-(1-\lambda)^d} \sum_{\ell=1}^{L} z_\ell (1-\lambda)^{d_1+\cdots+d_{\ell-1}} (1-(1-\lambda)^{d_\ell}).
\end{aligned}
$$

We have

$$\|z - z''\| \le \|z - z'\| + \|z' - z''\|,$$

and

$$\|z - z'\| \le \varepsilon.$$

Recall that

$$z' = \sum_{\ell=1}^{L} \frac{d_\ell}{d} v_\ell, \quad \text{and} \quad \|z_\ell\| \le n \text{ for each } \ell.$$

Therefore

$$
\begin{aligned}
z' - z'' &= \sum_{\ell=1}^{L} \left(\frac{d_\ell}{d} v_\ell - z_\ell \frac{(1-\lambda)^{d_1+\cdots+d_{\ell-1}}(1-(1-\lambda)^{d_\ell})}{1-(1-\lambda)^d} \right) \\
&= \left(\sum_{\ell=1}^{L} \frac{d_\ell}{d} (v_\ell - z_\ell) \right) + \sum_{\ell=1}^{L} z_\ell \left(\frac{d_\ell}{d} - \frac{(1-\lambda)^{d_1+\cdots+d_{\ell-1}}(1-(1-\lambda)^{d_\ell})}{1-(1-\lambda)^d} \right).
\end{aligned}
$$

Hence

$$
\begin{aligned}
\|z' - z''\| &\le \varepsilon + \left\| \sum_{\ell=1}^{L} z_\ell \left(\frac{d_\ell}{d} - \frac{(1-\lambda)^{d_1+\cdots+d_{\ell-1}}(1-(1-\lambda)^{d_\ell})}{1-(1-\lambda)^d} \right) \right\| \\
&\le \varepsilon + n \sum_{l=1}^{L} \zeta(\lambda, d_\ell, d_1 + \cdots + d_{\ell-1}), \\
&\le \varepsilon + nL\varepsilon,
\end{aligned}
$$

where the last equality follows from (6.23). Therefore

$$\|z - z''\| \le \varepsilon(2 + Ln).$$

□

Section 4. Convergence of E_T

Proof of Lemma 4.2. The proof is similar to the proof of Lemma 3.2. We just need to replace λ with $1/(T+1)$. $\square$

Lemma 6.5. *Fix S with $S \subset N$ and* $\text{card}(S) = s \leq n$. *For all $\varepsilon > 0$ there exists $T \in \mathbb{N}$ and $z \in E_T$ such that $\|z - e_S\| \leq \varepsilon$.*

Proof. If $s = n$, then $e_S \in E_1$. Assume now that $s \leq n-1$, and w.l.o.g. that $e_S = (1, \ldots, 1, 0, \ldots 0)$. Fix $\varepsilon > 0$, and define

$$Y_\varepsilon = \left\{ z = (\underbrace{q, \ldots, q}_{s}, \underbrace{y, \ldots, y}_{n+1-s}, \underbrace{w, \ldots, w}_{n}) \in \mathcal{S}, w \geq y \geq \frac{W(z)}{2n+1} - C\varepsilon \right\},$$

with C as in (6.10) and $W(\cdot)$ as in (6.4). We use $\|\cdot\|_1$, hence $W(z) = \|e_S - z\|$.

The set Y_ε is compact. We now prove that, for T large enough, $Y_\varepsilon \cap E_T \neq \varnothing$.

Denote by $D = \text{card}\{V, V \supset S, \text{card}(V) = n\}$. In the D-repeated game play at each stage a Nash pure equilibrium of the one-shot game corresponding to any V such that $V \supset S$ and $\text{card}(V) = n$. We then obtain some element $z \in E_D$ with

$$z = (\underbrace{1, \ldots, 1}_{s}, \underbrace{y, \ldots, y}_{2n+1-s}),$$

and

$$y = \frac{n-s}{2n+1-s}. \tag{6.24}$$

Take k such that $1/k \leq C\varepsilon$, and fix $T \geq kD$. By Euclidean division

$$T = k'D + r, \tag{6.25}$$

with $k' \geq k$ and $0 \ \leq r < D$.

In the T-repeated game repeat k' times the D-game and for the last r stages play a Nash with payoff

$$(\underbrace{0, \ldots, 0}_{n+1}, \underbrace{1, \ldots, 1}_{n}).$$

Then $z' \in E_T$, with

$$\begin{aligned} z' &= \frac{k'D}{T} z + \frac{r}{T}(\underbrace{0, \ldots, 0}_{n+1}, \underbrace{1, \ldots, 1}_{n}) \\ &= \left(\underbrace{q', \ldots, q'}_{s}, \underbrace{y', \ldots, y'}_{n+1-s}, \underbrace{w', \ldots, w'}_{n}\right), \end{aligned}$$

with

$$
\begin{aligned}
q' &= \frac{k'D}{T} \\
y' &= \frac{k'D}{T} y \\
w' &= \frac{k'D}{T} y + \frac{r}{T}.
\end{aligned}
$$

Therefore, using (6.24) and (6.25), we obtain

$$
\begin{aligned}
W(z') &= s\left(1 - \frac{k'D}{T}\right) + (n+1-s)\frac{k'D}{T} y + n\frac{k'D}{T} y + \frac{nr}{T} \\
&= s\frac{r}{T} + \frac{nr}{T} + (n-s)\frac{k'D}{T},
\end{aligned}
$$

and

$$
\begin{aligned}
y' - \frac{W(z')}{2n+1} + C\varepsilon &= \frac{k'D}{T}\frac{n-s}{2n+1-s} - \frac{1}{2n+1}\left((s+n)\frac{r}{T} + (n-s)\frac{k'D}{T}\right) + C\varepsilon \\
&= \frac{k'D}{T}(n-s)\left(\frac{1}{2n+1-s} - \frac{1}{2n+1}\right) - \frac{(s+n)}{(2n+1)}\frac{r}{T} + C\varepsilon \\
&\geq -\frac{(s+n)r}{(2n+1)T} + C\varepsilon \\
&\geq -\frac{r}{T} + C\varepsilon \\
&\geq -\frac{1}{k} + C\varepsilon \\
&\geq 0.
\end{aligned}
$$

Hence $z' \in E_T \cap Y_\varepsilon$. This proves that $E_T \cap Y_\varepsilon \neq \varnothing$.

For all $T \geq kD =: T_0$, $E_T \cap Y_\varepsilon$ is a non-empty compact subset of $\mathbb{R}^N$, and W is continuous. We define

$$
\beta_T = \min\{W(z), z \in E_T \cap Y_\varepsilon\}.
$$

Fix $T \geq T_0$ and such that

$$
\frac{1}{T+1} \leq k(n,m)\varepsilon^{2m+1}.
$$

Assume that $\beta_T > \varepsilon$. Then take $z \in E_T \cap Y_\varepsilon$ such that $W(z) = \beta_T > \varepsilon$.

We have $z = (q, \ldots, q, y, \ldots, y, w \ldots, w)$, with

$$
w \geq y \geq \frac{W(z)}{2n+1} - C\varepsilon \geq \varepsilon\left(\frac{1}{2n+1} - C\right).
$$

As in Lemma 6.1, we have

$$
\varepsilon\left(\frac{1}{2n+1} - C\right) \geq \left(\frac{1}{T+1}\right)^{1/(m+1)}.
$$

So

$$w \geq y \geq \left(\frac{1}{T+1}\right)^{1/(m+1)},$$

and we can apply Lemma 4.2 to z. With computations similar to the ones used in the proof of Lemma 6.2, we find $z' \in E_{T+1} \cap Y_\varepsilon$ such that

$$W(z') - W(z) < -\frac{\varepsilon}{T+1} + \frac{(n+s)\left(\frac{1}{T+1}\right)^{(m+1)/m}}{\varepsilon^{(m+1)/m}\left(\frac{1}{2n+1} - C\right)^{(m+1)/m}} =: \nu_T,$$

where the inequality comes from an argument similar to (6.19).

We have $\beta_{T+1} - \beta_T \leq W(z') - W(z) < \nu_T$, and

$$\sum_T \nu_T = -\infty.$$

Therefore if we assume that for all $T \geq T_0$, $\beta_T > \varepsilon$, we get that $\beta_T \to -\infty$, which is a contradiction. Therefore there exists T such that $\beta_T \leq \varepsilon$. □

Lemma 6.6. *For all $S \subset N$ with $\mathrm{card}(S) \leq n$, for all $\varepsilon > 0$ there exists T_0 such that for all $T \geq T_0$ there exists $z \in E_T$ with $\|z - e_S\| \leq \varepsilon$.*

Proof. By Lemma 6.5 there exist T_0 and $z \in E_{T_0}$ such that $\|z - e_S\| \leq \varepsilon$.

Take k such that

$$k \geq \frac{2n}{\varepsilon}.$$

Take $T \geq kT_0$. By Euclidean division $T = k'T_0 + r$, with $k' \geq k$ and $r \in \{0, \ldots, T_0 - 1\}$.

In the T-repeated game play the T_0-repeated game k' times and for the last r stages play a Nash equilibrium of the one-shot game with payoff x.

Then $z' \in E_T$ with

$$z' = \frac{k'T_0}{T} z + \frac{r}{T} x.$$

We have

$$\begin{aligned}
\|z' - z\| &= \left\|\frac{r}{T}(-z) + \frac{r}{T}x\right\| \\
&= \frac{r}{T}\|x - z\| \\
&\leq \frac{1}{k} 2n \\
&\leq \varepsilon,
\end{aligned}$$

so:

$$\|z' - e_S\| \leq \|z' - z\| + \varepsilon \leq 2\varepsilon.$$

□

Proof of Theorem 4.1. Recall that $\mathcal{S} = \text{conv}\{e_S, S \subset N \text{ such that } \text{card}(S) \leq n\}$, and fix $\varepsilon > 0$. There exists K such that any element of $\mathcal{S}$ is at distance at most ε from a convex combination of elements e_S with coefficients of the type h/K, with $h \in \{0, ..., K\}$. By Lemma 6.6 , there exists T_0 such that for all $T \geq T_0$, for all $S \subset N$ with $\text{card}(S) \leq n$, there exists z in E_T with $\|z - e_S\| \leq \varepsilon$. Since E_{KT} contains all the convex combinations of points of E_T with coefficients of the type h/K, we have that for all $T \geq T_0$, for all $z \in \mathcal{S}$, $d(z, E_{KT}) \leq \varepsilon + \varepsilon = 2\varepsilon$. Proceeding to chop the remainder as in the proof of Lemma 6.6 gives the conclusion. □

Section 5. Public equilibria

We start with the case of three players ($n = 1$). Consider the stage game, assume that every player $i \in N$ chooses L with probability $\alpha^i \in [0, 1]$, and denote $\alpha = (\alpha^1, \alpha^2, \alpha^3)$. The induced law on the public signal can be represented by the probability that L is the most crowded room, which we denote by

$$\begin{aligned}\ell(\alpha) &= \alpha^1\alpha^2\alpha^3 + \alpha^1\alpha^2(1 - \alpha^3) + \alpha^1(1 - \alpha^2)\alpha^3 + (1 - \alpha^1)\alpha^2\alpha^3, \\ &= \alpha^1\alpha^2 + \alpha^1(1 - \alpha^2)\alpha^3 + (1 - \alpha^1)\alpha^2\alpha^3.\end{aligned}$$

For every $\alpha = (\alpha^1, \alpha^2, \alpha^3) \in [0, 1]^3$ define

$$Z(\alpha) = \{\beta = (\beta^1, \beta^2, \beta^3) \in [0, 1]^3 : \ell(\alpha^{-1}, \beta^1) = \ell(\alpha^{-2}, \beta^2) = \ell(\alpha^{-3}, \beta^3)\}. \tag{6.26}$$

The set $Z(\alpha)$ denotes the set of deviations from α that induce the same public signal. It was first introduced by Tomala (1998, Definition 4.1), and was used afterwards by Renault and Tomala (2004).

The idea is the following: at some stage the players are supposed to play according to α. Consider the following three deviations:

(i) player 1 plays β^1 instead of α^1,

(ii) player 2 plays β^2 instead of α^2,

(iii) player 3 plays β^3 instead of α^3.

By (6.26) the three deviations induce the same law of the public signal, hence they induce the same subsequent play. Even if the players notice that a deviation occurred, they will be unable to determine who deviated, hence, in equilibrium, the subsequent play should punish in an appropriate way all 3 players.

The following lemma is the key to the proofs of Proposition 5.1 and Proposition 5.2.

Lemma 6.7. *For all $\alpha \in [0, 1]^3$ there exists $\beta \in Z(\alpha)$ such that $g^1(\alpha^{-1}, \beta^1) + g^2(\alpha^{-2}, \beta^2) + 2g^3(\alpha^{-3}, \beta^3) \geq 1$.*

Proof. Fix $\alpha = (\alpha^1, \alpha^2, \alpha^3) \in [0,1]^3$. Assume, w.l.o.g. that $g^1(\alpha) + g^2(\alpha) + 2g^3(\alpha) < 1$, and that $\ell(\alpha) \geq 1/2$. We have

$$g^3(\alpha) < 1 - g^1(\alpha) - g^2(\alpha) - g^3(\alpha),$$

hence the probability that player 3 is alone ($= g^3(\alpha)$) is lower than the probability that nobody is alone ($= 1 - g^1(\alpha) - g^2(\alpha) - g^3(\alpha)$). So

$$\alpha^3(1-\alpha^1)(1-\alpha^2) + (1-\alpha^3)\alpha^1\alpha^2 < \alpha^1\alpha^2\alpha^3 + (1-\alpha^1)(1-\alpha^2)(1-\alpha^3),$$

which implies

$$\alpha^1\alpha^2(1-2\alpha^3) < (1-\alpha^1)(1-\alpha^2)(1-2\alpha^3), \tag{6.27}$$

hence $\alpha^3 \neq 1/2$.

If $\alpha^3 < 1/2$, then, by (6.27), $\alpha^1\alpha^2 < (1-\alpha^1)(1-\alpha^2)$, but in this case

$$\begin{aligned}\ell(\alpha) &= \alpha^1\alpha^2 + \alpha^1\alpha^3(1-\alpha^2) + (1-\alpha^1)\alpha^2\alpha^3 \\ &< (1-\alpha^1)(1-\alpha^2) + \alpha^1(1-\alpha^2)(1-\alpha^3) + (1-\alpha^1)\alpha^2(1-\alpha^3) \\ &= 1 - \ell(\alpha),\end{aligned} \tag{6.28}$$

so $2\ell(\alpha) < 1$, which is a contradiction.

Hence necessarily we have $\alpha^3 > 1/2$, and, (6.27) becomes:

$$\alpha^1\alpha^2 > (1-\alpha^1)(1-\alpha^2),$$

which gives

$$\alpha^1 + \alpha^2 > 1.$$

By symmetry assume w.l.o.g. that $\alpha^1 \geq \alpha^2$. We now have two cases.

Case 1. $\alpha^2 \geq \alpha^3$.

In this case we have

$$\alpha^1 \geq \alpha^2 \geq \alpha^3 > 1/2. \tag{6.29}$$

Put $\beta^3 = 0$. Then $\ell(\alpha^{-3}, \beta^3) = \alpha^1\alpha^2$.

Define $\beta^1 \in [0, \alpha^1]$ in such a way that $\ell(\alpha^{-1}, \beta^1) = \alpha^1\alpha^2$, namely, such that

$$\alpha^2\alpha^3 + \beta^1((1-\alpha^2)\alpha^3 + \alpha^2(1-\alpha^3)) = \alpha^1\alpha^2.$$

This is possible since $\alpha^2\alpha^3 + 0 \leq \alpha^1\alpha^2$ and, by (6.28),

$$\alpha^2\alpha^3 + \alpha^1((1-\alpha^2)\alpha^3 + \alpha^2(1-\alpha^3)) = \ell(\alpha) \geq \alpha^1\alpha^2.$$

Similarly define $\beta^2 \in [0, \alpha^2]$ in such a way that $\ell(\alpha^{-2}, \beta^2) = \alpha^1\alpha^2$.

We have

$$\begin{aligned}g^1(\alpha^{-1}, \beta^1) &= \alpha^2\alpha^3 + \beta^1(1-\alpha^2-\alpha^3), \\ g^2(\alpha^{-2}, \beta^2) &= \alpha^1\alpha^3 + \beta^2(1-\alpha^1-\alpha^3), \\ g^3(\alpha^{-3}, \beta^3) &= \alpha^1\alpha^2.\end{aligned}$$

Since, by (6.29), $\alpha^2 + \alpha^3 > 1$ and $\beta^1 \leq \alpha^1$, we have

$$g^1(\alpha^{-1}, \beta^1) \geq \alpha^2\alpha^3 + \alpha^1(1 - \alpha^2 - \alpha^3).$$

Similarly

$$g^2(\alpha^{-2}, \beta^2) \geq \alpha^1\alpha^3 + \alpha^2(1 - \alpha^1 - \alpha^3).$$

So

$$\begin{aligned} & g^1(\alpha^{-1}, \beta^1) + g^2(\alpha^{-2}, \beta^2) + 2g^3(\alpha^{-3}, \beta^3) \\ & \qquad \geq \alpha^2\alpha^3 + \alpha^1(1 - \alpha^2 - \alpha^3) + \alpha^1\alpha^3 + \alpha^2(1 - \alpha^1 - \alpha^3) + 2\alpha^1\alpha^2 \\ & \qquad = \alpha^1 + \alpha^2 \\ & \qquad > 1, \end{aligned}$$

where the last inequality stems from (6.29).

Case 2. $\alpha^2 < \alpha^3$.

We have

$$\alpha^3 > \frac{1}{2}, \quad \alpha^1 + \alpha^2 > 1, \tag{6.30}$$

and

$$((\alpha^1 \geq \alpha^3 > \alpha^2) \text{ or } (\alpha^3 \geq \alpha^1 \geq \alpha^2)).$$

Put $\beta^2 = 0$. So

$$\ell(\alpha^{-2}, \beta^2) = \alpha^1\alpha^3 = g^2(\alpha^{-2}, \beta^2).$$

Put

$$\beta^1 = \frac{(\alpha^1 - \alpha^2)\alpha^3}{\alpha^2(1 - \alpha^3) + \alpha^3(1 - \alpha^2)} \quad \text{and} \quad \beta^3 = \frac{(\alpha^3 - \alpha^2)\alpha^1}{\alpha^1(1 - \alpha^2) + \alpha^2(1 - \alpha^1)}.$$

We have

$$\ell(\alpha^{-1}, \beta^1) = \ell(\alpha^{-2}, \beta^2) = \ell(\alpha^{-3}, \beta^3) = \alpha^1\alpha^3,$$
$$\beta^1 \in [0, \alpha^1], \quad \text{and} \quad \beta^3 \in [0, \alpha^3].$$

Denote by

$$\begin{aligned} A &= 2g^3(\alpha^{-3}, \beta^3) + g^1(\alpha^{-1}, \beta^1) + g^2(\alpha^{-2}, \beta^2) \\ &= 2(\alpha^1\alpha^2 + \beta^3(1 - \alpha^1 - \alpha^2)) + \alpha^2\alpha^3 + \beta^1(1 - \alpha^2 - \alpha^3) + \alpha^1\alpha^3. \end{aligned}$$

We want to prove that $A \geq 1$.

Since $\beta^3 \leq \alpha^3$ and $1 - \alpha^1 - \alpha^2 < 0$, we have

$$\begin{aligned} A &\geq 2\alpha^1\alpha^2 + 2\alpha^3(1 - \alpha^1 - \alpha^2) + \alpha^2\alpha^3 + \beta^1(1 - \alpha^2 - \alpha^3) + \alpha^1\alpha^3 \\ &= 2\alpha^1\alpha^2 - \alpha^2\alpha^3 - \alpha^1\alpha^3 + 2\alpha^3 + \beta^1(1 - \alpha^2 - \alpha^3) \\ &= 2\alpha^1\alpha^2 - \alpha^2\alpha^3 - \alpha^1\alpha^3 + 2\alpha^3 + \frac{(\alpha^1 - \alpha^2)\alpha^3(1 - \alpha^2 - \alpha^3)}{\alpha^2(1 - \alpha^3) + \alpha^3(1 - \alpha^2)} \end{aligned} \tag{6.31}$$

And we denote this last quantity by $B(\alpha^1, \alpha^2, \alpha^3)$. Notice that B is an affine function of α^1, and that, by (6.30), $\alpha^1 \in [1-\alpha^2, 1]$. Denote by

$$\delta = \alpha^2(1-\alpha^3) + \alpha^3(1-\alpha^2) > 0.$$

Then

$$\begin{aligned}
B(1,\alpha^2,\alpha^3) &= 2\alpha^2 - \alpha^2\alpha^3 - \alpha^3 + 2\alpha^3 + \frac{(1-\alpha^2)\alpha^3(1-\alpha^2-\alpha^3)}{\alpha^2(1-\alpha^3)+\alpha^3(1-\alpha^2)} \\
&= 2\alpha^2 + \alpha^3(1-\alpha^2)\left(1 + \frac{1-\alpha^2-\alpha^3}{\alpha^2(1-\alpha^3)+\alpha^3(1-\alpha^2)}\right) \\
&= 2\alpha^2 + \frac{\alpha^3(1-\alpha^2)(1-2\alpha^2\alpha^3)}{\alpha^2(1-\alpha^3)+\alpha^3(1-\alpha^2)} \\
&= \frac{1}{\delta}\left(2(\alpha^2)^2(1-\alpha^3) + 2\alpha^2\alpha^3(1-\alpha^2) + \alpha^3(1-\alpha^2) - 2\alpha^2(\alpha^3)^2(1-\alpha^2)\right) \\
&= \frac{1}{\delta}\left(\alpha^3(1-\alpha^2) + 2\alpha^2\left(\alpha^2(1-\alpha^3) + \alpha^3(1-\alpha^2) - (\alpha^3)^2(1-\alpha^2)\right)\right) \\
&= \frac{1}{\delta}\left(\alpha^3(1-\alpha^2) + 2\alpha^2(1-\alpha^3)\left(\alpha^2 + \alpha^3(1-\alpha^2)\right)\right).
\end{aligned}$$

We have $\alpha^2 + \alpha^3(1-\alpha^2) \geq \alpha^3 > 1/2$, so

$$B(1,\alpha^2,\alpha^3) \geq \frac{1}{\delta}\left(\alpha^2(1-\alpha^3) + \alpha^3(1-\alpha^2)\right) = 1. \tag{6.32}$$

On the other hand

$$\begin{aligned}
B(1-\alpha^2,\alpha^2,\alpha^3) &= 2(1-\alpha^2)\alpha^2 - \alpha^2\alpha^3 - (1-\alpha^2)\alpha^3 + 2\alpha^3 + \frac{(1-2\alpha^2)\alpha^3(1-\alpha^2-\alpha^3)}{\alpha^2(1-\alpha^3)+\alpha^3(1-\alpha^2)} \\
&= \frac{1}{\delta}\Big[\left(2(1-\alpha^2)\alpha^2 + \alpha^3\right)\left(\alpha^2(1-\alpha^3) + \alpha^3(1-\alpha^2)\right) \\
&\quad + (1-2\alpha^2)\alpha^3(1-\alpha^2-\alpha^3)\Big].
\end{aligned}$$

Hence

$$\begin{aligned}
\delta(B(1-\alpha^2,\alpha^2,\alpha^3) - 1) &= \left(2(1-\alpha^2)\alpha^2 + (\alpha^3 - 1)\right)\left(\alpha^2(1-\alpha^3) + \alpha^3(1-\alpha^2)\right) \\
&\quad + (1-2\alpha^2)\alpha^3(1-\alpha^2-\alpha^3) \\
&= \alpha^2\left[2\alpha^2(1-\alpha^2) - 1 + \alpha^3\left(2(\alpha^2)^2 + 2(1-\alpha^2)^2\right)\right].
\end{aligned}$$

Since $\alpha^3 > 1/2$, we have

$$\begin{aligned}
\delta(B(1-\alpha^2,\alpha^2,\alpha^3) - 1) &\geq \alpha^2\left(2(1-\alpha^2)\alpha^2 - 1 + (\alpha^2)^2 + (1-\alpha^2)^2\right) \\
&= 0.
\end{aligned}$$

As a consequence

$$B(1-\alpha^2,\alpha^2,\alpha^3) \geq 1.$$

Since the function B is affine in its first argument, and, by (6.32), $B(1, \alpha^2, \alpha^3) \geq 1$, inequality (6.31) yields

$$A \geq B(\alpha^1, \alpha^2, \alpha^3) \geq 1.$$

□

Corollary 6.8. *If one of the following condition holds*

(i) $z \in E^{\mathrm{p}}_{\lambda}$,

(ii) $z \in E^{\mathrm{p}}_{T}$,

then $z^1 + z^2 + 2z^3 \geq 1$.

Proof. (i): Let σ be a public equilibrium of the λ-discounted game.

For every $t \geq 0$ and for every public history $h_t \in \{L, R\}^t$ denote by $\alpha(h_t) = (\alpha^1(h_t), \alpha^2(h_t), \alpha^3(h_t))$ the mixed action profile that σ requires to play at stage $t+1$, after the public history h_t has occurred. Using Lemma 6.7 it is possible to define $\beta(h_t) \in Z(\alpha(h_t))$ such that

$$g^1(\alpha^{-1}(h_t), \beta^1(h_t)) + g^2(\alpha^{-2}(h_t), \beta^2(h_t)) + 2g^3(\alpha^{-3}(h_t), \beta^3(h_t)) \geq 1.$$

For $i \in N$ define τ^i the strategy of player i that, for every $t \geq 0$ and for every public history $h_t \in \{L, R\}^t$, plays according to $\beta^i(h_t)$ at stage $t+1$ if h_t has previously occurred. By construction (σ^{-1}, τ^1), (σ^{-2}, τ^2), (σ^{-3}, τ^3) induce the same distribution on the sequences of public signals. Moreover for all $t \geq 0$ and for all $h_t \in \{L, R\}^t$ we have

$$\mathbb{E}_{\sigma^{-1},\tau^1}[g^1(a_{t+1})|h_t] + \mathbb{E}_{\sigma^{-2},\tau^2}[g^2(a_{t+1})|h_t] + 2\mathbb{E}_{\sigma^{-3},\tau^3}[g^3(a_{t+1})|h_t] \geq 1,$$

where a_{t+1} denotes the random variable corresponding to the action profile at time $t+1$. Taking expectations, we obtain

$$\mathbb{E}_{\sigma^{-1},\tau^1}[g^1(a_{t+1})] + \mathbb{E}_{\sigma^{-2},\tau^2}[g^2(a_{t+1})] + 2\mathbb{E}_{\sigma^{-3},\tau^3}[g^3(a_{t+1})] \geq 1.$$

So

$$\gamma^1_{\lambda}(\sigma^{-1}, \tau^1) + \gamma^2_{\lambda}(\sigma^{-2}, \tau^2) + 2\gamma^3_{\lambda}(\sigma^{-3}, \tau^3) \geq 1.$$

By the equilibrium property

$$\gamma^1_{\lambda}(\sigma) + \gamma^2_{\lambda}(\sigma) + 2\gamma^3_{\lambda}(\sigma) \geq 1.$$

(ii): The proof is similar. □

Proof of Proposition 5.1. Let $z \in E^{\mathrm{p}}_{\lambda}$ for some $0 < \lambda \leq 1$. By Corollary 6.8 we have $z^1 + z^2 + 2z^3 \geq 1$. By symmetry we have also $z^1 + 2z^2 + z^3 \geq 1$ and $2z^1 + z^2 + z^3 \geq 1$. Since z is feasible, we have $z^1 + z^2 + z^3 \leq 1$, so $z \in \mathcal{C}$. Hence $E^{\mathrm{p}}_{\lambda} \subset \mathcal{C}$.

Assume now that $0 < \lambda \leq 1/4$. Take $z \in \mathcal{C}$. Since $\mathcal{C}$ is just the convex hull of the Nash-payoffs of the one-shot game, it is possible to find non-negative numbers $\mu_1, \mu_2, \mu_3, \mu_4$ such that

$$z = \mu_1(1, 0, 0) + \mu_2(0, 1, 0) + \mu_3(0, 0, 1) + \mu_4\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right),$$

and

$$\sum_{j=1}^{4} \mu_j = 1.$$

We now use the construction described in Mertens et al. (1994, Page 194, proof of Theorem 4.2). One of the μ_j is at least 1/4, and $\lambda \leq 1/4$, so it is possible to write

$$z = \lambda d_1 + (1-\lambda) f_1,$$

with $d_1 \in \left\{(1,0,0),(0,1,0),(0,0,1),\left(\frac{1}{4},\frac{1}{4},\frac{1}{4}\right)\right\}$ and $f_1 \in \operatorname{conv}\left\{(1,0,0),(0,1,0),(0,0,1),\left(\frac{1}{4},\frac{1}{4},\frac{1}{4}\right)\right\}$. By applying the same argument to f_1 we obtain inductively a sequence $(d_t)_{t\geq 1}$ of vectors in $\left\{(1,0,0),(0,1,0),(0,0,1),\left(\frac{1}{4},\frac{1}{4},\frac{1}{4}\right)\right\}$ such that

$$z = \sum_{t\geq 1} \lambda(1-\lambda)^{t-1} d_t.$$

Playing at every stage t according to the one-shot Nash equilibrium with payoff d_t gives a public equilibrium of the λ-discounted game with payoff z. Hence $\mathcal{C} = E^{\mathrm{p}}_{\lambda}$ if $\lambda \leq 1/4$. □

The proof of Proposition 5.2 is easier and is omitted.

Proof of Proposition 5.3. By compactness of the set of public strategies and continuity of the discounted payoff function, we know that the set E^{p}_{λ} is compact.

Consider a public equilibrium $\sigma = (\sigma^i)_{i\in N}$ of the λ-discounted game that minimizes the sum of the players' discounted payoffs in E^{p}_{λ}. Assume, for the sake of contradiction, that

$$\sum_{i\in N} \gamma^i_{\lambda}(\sigma) < \frac{1}{3}.$$

For $i \in N$ denote by $\alpha^i \in [0,1]$ the probability that player i plays L at stage 1. Using the mixed action profile $(\alpha^i)_{i\in N}$, for every player $i \in N$ call:

- a^i the probability that at stage 1 *at most* $(n-1)$ players in $N \setminus \{i\}$ play L (i.e., at least $(n+1)$ players in $N \setminus \{i\}$ play R),
- b^i the probability that at stage 1 *exactly* n players in $N \setminus \{i\}$ play L (i.e., exactly n players in $N \setminus \{i\}$ play R),
- c^i the probability that at stage 1 *at least* $(n+1)$ players in $N \setminus \{i\}$ play L (i.e., at most $(n-1)$ players in $N \setminus \{i\}$ play R).

Of course $a^i + b^i + c^i = 1$, and $g^i(\alpha) = a^i\alpha^i + c^i(1-\alpha^i)$. The probability that L is the most crowded room at stage 1 is $\mathbb{P}_{\alpha}(L) = c^i + b^i\alpha^i$ (for every player i), and similarly the probability that R is the most crowded room at stage 1 is $\mathbb{P}_{\alpha}(R) = a^i + b^i(1-\alpha^i)$.

We now define continuation payoffs. Denote by u^i (resp v^i) the discounted continuation payoff of player i from stage 2 on if L (resp R) was the most crowded room at stage 1. If

$\mathbb{P}_\alpha(L) > 0$ (resp $\mathbb{P}_\alpha(R) > 0$), then the continuation strategy after L (resp after R) is a public equilibrium payoff, so

$$\sum_{i\in N} u^i \geq \sum_{i\in N} \gamma^i_\lambda(\sigma) \quad \text{(resp} \quad \sum_{i\in N} v^i \geq \sum_{i\in N} \gamma^i_\lambda(\sigma)),$$

since σ is the public equilibrium that minimizes the sum of players' payoffs.

If $\mathbb{P}_\alpha(L) = 0$, then at stage 1 at least $(n+1)$ players play R with probability 1. By the equilibrium condition, n players will then play L with probability 1 at the first stage and then

$$\sum_{i\in N} g^i(\alpha) = n,$$

and

$$\begin{aligned}\sum_{i\in N} \gamma^i_\lambda(\sigma) &= \lambda \sum_{i\in N} g^i(\alpha) + (1-\lambda) \sum_{i\in N} v^i \\ &\geq \lambda n + (1-\lambda) \sum_{i\in N} \gamma^i_\lambda(\sigma).\end{aligned}$$

Hence

$$\sum_{i\in N} \gamma^i_\lambda(\sigma) \geq n.$$

This is a contradiction, therefore it must be $\mathbb{P}_\alpha(L) > 0$ (and, by symmetry, $\mathbb{P}_\alpha(R) > 0$).

We have for every $i \in N$

$$\gamma^i_\lambda(\sigma) = \lambda g^i(\alpha) + (1-\lambda)\mathbb{P}_\alpha(L)u^i + (1-\lambda)\mathbb{P}_\alpha(R)v^i.$$

So

$$\begin{aligned}\sum_{i\in N} \gamma^i_\lambda(\sigma) &= \lambda \sum_{i\in N} g^i(\alpha) + (1-\lambda)\mathbb{P}_\alpha(L) \sum_{i\in N} u^i + (1-\lambda)\mathbb{P}_\alpha(R) \sum_{i\in N} v^i \\ &\geq \lambda \sum_{i\in N} g^i(\alpha) + (1-\lambda) \sum_{i\in N} \gamma^i_\lambda(\sigma).\end{aligned}$$

Hence

$$\sum_{i\in N} g^i(\alpha) \leq \sum_{i\in N} \gamma^i_\lambda(\sigma) < \frac{1}{3}. \tag{6.33}$$

With probability greater than 2/3 all players are in the same room at stage 1. W.l.o.g. we assume that with probability greater than 1/3 all players are in room R at stage 1. This implies that for every i

$$a^i(1-\alpha^i) > \frac{1}{3}, \tag{6.34}$$

and in particular every player i plays R with positive probability at stage 1. Consequently $\gamma^i_\lambda(\sigma)$ is the payoff obtained by player i when she chooses R at stage 1.

$$\begin{aligned}\gamma^i_\lambda(\sigma) &= a^i(\lambda 0 + (1-\lambda)v^i) + b^i(\lambda 0 + (1-\lambda)v^i) + c^i(\lambda 1 + (1-\lambda)u^i) \\ &= \lambda c^i + (1-\lambda)c^i u^i + (1-\lambda)(a^i + b^i)v^i.\end{aligned}$$

Denote by w^i the payoff obtained by player i if she plays L at stage 1 and then follows σ^i, whereas all the other players adopt σ. Then

$$\begin{aligned} w^i &= a^i(\lambda 1 + (1-\lambda)v^i) + b^i(\lambda 0 + (1-\lambda)u^i) + c^i(\lambda 0 + (1-\lambda)u^i) \\ &= \lambda a^i + (1-\lambda)a^i v^i + (1-\lambda)(b^i + c^i)u^i. \end{aligned} \tag{6.35}$$

By the equilibrium condition $\gamma^i_\lambda(\sigma) \geq w^i$, so

$$\begin{aligned} \lambda(c^i - a^i) &\geq (1-\lambda)(-b^i v^i + b^i u^i) \\ &= b^i(1-\lambda)(u^i - v^i). \end{aligned} \tag{6.36}$$

We know by (6.34) that $a^i(1-\alpha^i) > 1/3$, and by (6.33) that $g^i(\alpha) = a^i\alpha^i + c^i(1-\alpha^i) < 1/3$. Hence $a^i(1-\alpha^i) > c^i(1-\alpha^i)$ and $a^i > c^i$. Using (6.36) we obtain that $u^i - v^i < 0$, i.e., $u^i < v^i$.

Since $a^i + b^i + c^i = 1$, we have

$$a^i v^i + (b^i + c^i)u^i \geq \min(u^i, v^i) = u^i,$$

and (6.35) gives

$$\gamma^i_\lambda(\sigma) \geq w^i \geq \lambda a^i + (1-\lambda)u^i.$$

Adding up we obtain

$$\begin{aligned} \sum_{i\in N} \gamma^i_\lambda(\sigma) &\geq \lambda \sum_{i\in N} a^i + (1-\lambda)\sum_{i\in N} u^i \\ &\geq \lambda \sum_{i\in N} a^i + (1-\lambda)\sum_{i\in N} \gamma^i_\lambda(\sigma). \end{aligned}$$

Hence

$$\frac{1}{3} > \sum_{i\in N} \gamma^i_\lambda(\sigma) \geq \sum_{i\in N} a^i.$$

From (6.34) we have $a^i > 1/3$, so we obtain

$$\frac{1}{3} > (2n+1)\frac{1}{3},$$

which is clearly false. □

Proof of Proposition 5.4. We prove the result by induction on T. For $T = 1$ the public equilibria are just the Nash equilibria of the stage game. The sum of equilibrium payoffs is minimal when all players use the mixed strategy $\frac{1}{2}L \oplus \frac{1}{2}R$. In this case for each player the probability that she is in the minority room is an increasing function of N. When $N = 3$ this probability is 1/4. Therefore $\sum_{i\in N} \gamma^i_T(\sigma) \geq N/4 \geq 1/3$.

Hence the result is true for $T = 1$. Assume that it is true for $T - 1$, but not for T, where $T \geq 2$ is fixed. Let σ be a public equilibrium of the T-repeated game such that $\sum_{i\in N} \gamma^i_T(\sigma) < 1/3$. A proof similar to the proof of Proposition 5.3 yields a contradiction. □

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December 18, 2007