Majority Voting and the Single Crossing Property when Voters Belong to Separate Groups The Role of the Continuity and Strict Monotonicity Assumptions*

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Abstract

This note extends the classical median voter theorem to the case where voters are exogenously distributed across groups, with preferences satisfying the single-crossing property separately inside each group. We provide conditions under which a Condorcet winner exists. The most important condition is the continuity of the set of most-preferred options in voters' types, and we discuss the importance of both this assumption and those of unicity and strict monotonicity.

Keywords: Spence-Mirrlees condition, unidimensional policy space, median voter, phantom anchors

JEL classification: D71.

1 Introduction

Among the various sufficient conditions for the existence of a majority voting equilibrium on unidimensional choice domains, the single-crossing property (SCP) of preferences is probably the one most often used in the literature: see for instance Gans and Smart (1996) and the 260 papers referring to it.[1] Single-crossing requires, roughly speaking, that if a voter's type prefers a larger option to a smaller one, then so do all voters with a larger type. Moreover, unlike single-peakedness, SCP guarantees that the social preference relationship obtained under majority voting is transitive, and corresponds to the median voter's.

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[1] Source: scholar.google.com, accessed on November 30, 2011

The popularity of this approach is due in part to the large set of environments satisfying Gans and Smart (1996)'s premises. In many applications, voters have to vote over two dimensions linked by some type of budget constraint, such as a proportional tax rate t and the amount of public good G (or lump sum transfer) financed by that tax, given by $G(t)$. The budget constraint $G(t)$ then restricts the set of feasible options to be unidimensional. Gans and Smart (1996) show, under rather weak conditions, the equivalence between the SCP over this unidimensional set and the Spence-Mirrlees condition of monotonicity of the marginal rates of substitution in the (t, G) space. This setting has been used to study majority voting over unidimensional public policies in domains such as income taxation, education provision, environmental policies, retirement, social health insurance, etc.

In this note, we extend the study of the impact of SCP on majority voting equilibrium to a setting where voters belong (exogenously) to different groups and where preferences satisfy SCP inside all those groups taken separately (but not necessarily when voters from all groups are pooled together). Such a setting arises for instance when voters vote over (t, G) but belong to exogenous groups which differ in the budget constraint linking the two choices t and G. We give two such examples drawn from recent papers. In Russo (2011), voters differ both in their preference for driving and their location (city center or suburbs). While the proceeds of a cordon toll are rebated to all individuals, the proceeds from parking charges are rebated only to the city residents. When voting over the cordon toll, city residents then take into account the impact of the toll on the proceeds from the parking charges, while suburban voters do not. In other words, voters who (exogenously) differ in location face a different budget constraint between cordon toll and tax transfers, and Russo (2011) shows that this results in preferences satisfying the SCP inside those two groups, but not necessarily when all agents are pooled together. In Besfamille et al (2011), voters' preferences for the enforcement of indirect taxation depend on an intrinsic characteristic but also on whether they own shares in private firms and are thus entitled to receive a part of these firms' profits. The relationship between enforcement and monetary transfers is thus affected by this ownership share. If consumers are grouped exogenously according to their extent of ownership of private firms, then the setting studied in this note applies.

We show in this note that, under certain assumptions, a Condorcet winner exists when all voters vote simultaneously. The main assumption is that the set of voters' most-preferred options is continuous in type for all types in all groups. Together with some more minor conditions, this assumption guarantees the existence of "anchors" in all groups —i.e., of types with a specific most-preferred option. The existence of these anchors in all groups then allows us to generalize the classical median voter theorem by using a well known separation argument.

2 The Model

There is a continuum of voters distributed among N groups, with μ_i denoting the proportion of voters who belong to group $i = 1, ..., N$. Voters are identified by the group they belong to and by their type, denoted by θ. The distribution of types in group i is denoted by the distribution function F_i, with density f_i over $\Theta_i = [\underline{\theta}_i, \overline{\theta}_i]$.[2] The policy space is unidimensional and is the same across groups. That is, agents in all groups have to choose by majority voting one option (for the whole society) x in the set $X \subset \Re$. The utility function $u_i(x, \theta)$ represents the preferences of voters of type $\theta \in \Theta_i$ belonging to group i for any option $x \in X$.

We assume that these preferences satisfy the Single Crossing Property inside each group i.

Definition 1 (SCP) *The utility functions $u_i(x, \theta)$ satisfy the Single-Crossing Property if*

$$\begin{aligned} \forall i = \ & 1, ..., N, \forall x', x \in X \ \textit{with} \ x' > x, \forall \theta', \theta \in \Theta_i, \ \textit{with} \ \theta' > \theta, \\ & u_i(x', \theta) \geq u_i(x, \theta) \Rightarrow u_i(x', \theta') \geq u_i(x, \theta'). \end{aligned}$$

Our objective is to prove the existence of a Condorcet winner when all voters in all groups vote simultaneously:

[2] Observe that we do not impose that $f_i(\theta_i) > 0$ for all $\theta_i \in \Theta_i$, but only that it is meaningful to talk about any type in Θ_i, even if this type is not represented by actual voters in a particular case.

Definition 2 (CW) *Option x is a Condorcet winner if and only if there is no option $y \in X$ such that a strict majority of voters across groups strictly prefer y to x.*

The set of most-preferred outcomes in X of an individual of type θ belonging to group i is denoted by

$$M_i(\theta, X) = \arg\max_{x \in X} u_i(x, \theta).$$

We can then apply the following result due to Milgrom and Shannon (1994, Theorem 4):

Lemma 1 *$M_i(\theta, X)$ is weakly increasing in (θ, X) if and only if $u_i(.)$ satisfies the Single Crossing Property.*

We make the following assumption.[3]

Assumption 1 *$M_i(\theta)$ is continuous in θ, $\forall i = 1, ..., N$ and $\forall \theta \in \Theta_i$.*

Observe that we do not assume that $M_i(\theta)$ is a function, but that it may be a correspondence: any voter's type may have multiple most-preferred options. We define

$$\underline{x}_i = \min_{\theta \in \Theta_i} \min M_i(\theta), \overline{x}_i = \max_{\theta \in \Theta_i} \max M_i(\theta),$$

as, respectively, the lowest and largest most-preferred option in group i, and

$$\underline{x} = \max_{i=1,...,N} \underline{x}_i, \overline{x} = \min_{i=1,...,N} \overline{x}_i,$$

as, respectively, the maximin and minimax most-preferred option across groups. We assume that preferences are such that the most-preferred options of all groups have some overlap:

Assumption 2 $\underline{x} < \overline{x}$

[3]From now on, and since the set X is constant throughout the analysis, we simplify the notation to $M_i(\theta)$.

The interval $[\underline{x}, \overline{x}]$ is then the largest interval in X which is spanned by some voters' blisspoints in all groups.

Since $M_i(\theta)$ is continuous (Assumption 1) with a range encompassing $[\underline{x}, \overline{x}]$ (Assumption 2), we know from the intermediate value theorem that, for any $x \in [\underline{x}, \overline{x}]$, there exists at least one type θ_i in *all* groups $i = 1, ..., N$ who is such that s/he (weakly) most-prefers x to any other feasible option. Proposition 1 will lean heavily on this property. Since $M_i(\theta)$ may be multi-valued, this type needs not be unique, and we denote

$$\begin{aligned} \underline{\theta}_i^*(x) &= \min_{\theta \in \Theta_i} \text{ such that } x \in M_i(\theta), \\ \overline{\theta}_i^*(x) &= \max_{\theta \in \Theta_i} \text{ such that } x \in M_i(\theta), \end{aligned}$$

so that $\underline{\theta}_i^*(x)$ (resp., $\overline{\theta}_i^*(x)$) represents the lowest (resp., largest) type in group i who weakly most-prefers x to any other option in X. We obviously have that $\overline{\theta}_i^*(x) \geq \underline{\theta}_i^*(x)$, $\forall x \in [\underline{x}, \overline{x}]$ and $\forall i = 1, ..., N$. Figures 1b and 2b give illustrations when $N = 2$.

We make use of the following two conditions.

Condition 1 (C1) $\sum_{i=1,...,N} \mu_i F_i(\underline{\theta}_i^*(\underline{x})) \leq 1/2.$

Condition 2 (C2) $\sum_{i=1,...,N} \mu_i F_i(\overline{\theta}_i^*(\overline{x})) \geq 1/2.$

In the light of Lemma 1, condition C1 (resp., C2) ensures that a majority of voters across groups do not most-prefer options lower (resp., larger) than the lowerbound (resp., upperbound) of the common range of most-preferred options across groups.

We now prove that

Proposition 1 *If Assumptions 1 and 2 together with conditions C1 and C2 hold, then (a) there exists (at least) one option $x \in X$, which we denote x^{CW}, such that*

$$\sum \mu_i F_i(\underline{\theta}_i^*(x^{CW})) \leq 1/2 \text{ and } \sum \mu_i F_i(\overline{\theta}_i^*(x^{CW})) \geq 1/2, \tag{1}$$

and (b) x^{CW} is a Condorcet winner in the set X.

Proof 1 *(a) Observe that if C1 holds with equality, then $x^{CW} = \underline{x}$ while, if C2 holds with equality, then $x^{CW} = \overline{x}$. Assume then that the inequalities in both C1 and C2 hold strictly. Observe that both $\underline{\theta}_i^*(x)$ and $\overline{\theta}_i^*(x)$ are monotonically strictly increasing in x (although not continuous in x when the set $M_i(\theta) \ni x$ is not a singleton). Because of this monotonicity, there exists at least one value of $x \in [\underline{x}, \overline{x}]$, denoted by x°, that is such that*

$$\lim_{\epsilon \to 0} \sum \mu_i F_i(\underline{\theta}_i^*(x^\circ - \epsilon)) \leq 1/2 \text{ and } \lim_{\epsilon \to 0} \sum \mu_i F_i(\underline{\theta}_i^*(x^\circ + \epsilon)) \geq 1/2.$$

Observe also that, by the definition of $\underline{\theta}_i^*(x)$ and $\overline{\theta}_i^*(x)$, we have that

$$\underline{\theta}_i^*(x) = \lim_{\epsilon \to 0} \underline{\theta}_i^*(x - \epsilon) \text{ and } \overline{\theta}_i^*(x) = \lim_{\epsilon \to 0} \underline{\theta}_i^*(x + \epsilon).$$

We then obtain that $x^{CW} = x^\circ$.

(b) We now prove that x^{CW} gathers at least one half of the votes when faced with any other option $y \in X$. Since $x^{CW} \in [\underline{x}, \overline{x}]$, we know that there exists a type θ_i^{CW} in all groups i who is such that $\theta_i^{CW} = \underline{\theta}_i^*(x^{CW})$— i.e., that $u_i(x^{CW}, \theta_i^{CW}) \geq u_i(y, \theta_i^{CW})$, $\forall y \in X$.

Assume that $y < x^{CW}$. This in turns means, using Definition 1, that, inside each group i, we have that

$$\forall \theta \geq \theta_i^{CW}, u_i(x^{CW}, \theta) \geq u_i(y, \theta).$$

This guarantees that at least a fraction $1 - F_i(\theta_i^{CW})$, in each group i, will support x^{CW} when faced against y. By definition of x^{CW} (and more precisely the first inequality in (1)), this support aggregates to at least one half over all groups, so that x^{CW} can not be defeated at the majority by y.

Assume now that $y > x^{CW}$, and take $\theta_i^{CW} = \overline{\theta}_i^*(x^{CW})$. Using the contrapositive of Definition 1, we obtain that, inside each group i, we have that

$$\forall \theta \leq \theta_i^{CW}, u_i(x^{CW}, \theta) \geq u_i(y, \theta).$$

This guarantees that at least a fraction $F_i(\theta_i^{CW})$, in each group i, will support x^{CW} when faced against y. Thanks to the second inequality in (1), this support aggregates to at least one half over all groups, so that x^{CW} can not be defeated at the majority by y.

We now give the intuition for the result. The SCP is satisfied in all groups of voters taken separately. We first identify the minimax and maximin most-preferred options across groups, $\underline{x}$ and $\overline{x}$. Given the continuity of $M_i(\theta)$ postulated in Assumption 1, they together determine the boundaries of the largest interval in X which is spanned by the blisspoints of some voters in all groups. That is, for any option x inside this interval, we know that there exists at least one type of voters in *all* groups that (weakly) most-prefers this option x to any other feasible option in X. This type plays a role of "anchor" in each group, which allows to make use of the SCP. That is, if we want to assess the result of a pairwise vote between x and another option, say $y < x$, we know from the SCP that everyone to the right of this anchor in each and every group also prefers x to y. In other words, we can apply the separation argument that is at the heart of the usual, one-group, median voter theorem, provided that we can identify an anchor in each group who most-prefers this option x. To identify a Condorcet winning outcome, we thus have to find an anchor in each group, with the additional property that, roughly speaking, one half of the global population of voters is located on the same side of these groups' anchors. Observe that these anchors need not be the median θ voters inside each group separately.

Several comments are in order. First, observe that it is important to identify the anchor's type in all groups, but that it is not necessary for this type to be actually represented in *all* groups: that is, we need not impose that $f_i(\theta_i) > 0$ for the anchor θ_i in group i. We only need the total population to be divided, roughly, into two equal halves, each on one side of these anchors. In that sense, we can talk of "phantom anchors" who are not actually represented in all groups.[4] An anchor may actually not be represented in any group (i.e., $f_i(\theta_i) = 0$ for all i), in which case it is easy to see that there exist several Condorcet winners.

Second, we need to differentiate between two anchors in certain groups. This is due to the fact that $M_i(\theta)$ is weakly increasing in θ, so that a subset of voters may share the same most-preferred option x in certain groups: see for instance Figure 1a, where all

[4]See the discussion before footnote 5 and Figures 3c and 3d for an example where an anchor does not exist for certain values of x, as opposed to existing but not being represented in a group.

types in between θ' and θ'' have the same blisspoint x'. We then define the "top anchor" $\overline{\theta}_i^*$ to be the largest-type individual who most prefers option x in group i, and the "lowest anchor" $\underline{\theta}_i^*$ as the lowest type: see for instance Figure 1b, where $\overline{\theta}_i^*(x) = \underline{\theta}_i^*(x)$ for all $x \in [\underline{x}, \overline{x}]$ except for x'. We then have a discontinuity in $\overline{\theta}_i^*(x)$ and $\underline{\theta}_i^*(x)$ at point x', and likewise for $\sum \mu_i F_i(\underline{\theta}_i^*(x))$ and $\sum \mu_i F_i(\overline{\theta}_i^*(x))$ (see Figure 1c). But this discontinuity is easy to deal with, since the identity of both anchors monotonically increases wherever they exist. So, in Figure 1c, we have one of three cases:

(a) if $\sum \mu_i F_i(\underline{\theta}_i^*(x')) > 1/2$, then $x^{CW} < x'$,
(b) if $\sum \mu_i F_i(\overline{\theta}_i^*(x')) < 1/2$, then $x^{CW} > x'$,
(c) if $\sum \mu_i F_i(\underline{\theta}_i^*(x')) < 1/2$ and $\sum \mu_i F_i(\overline{\theta}_i^*(x')) > 1/2$, then $x^{CW} = x'$.

Observe that we have a unique Condorcet winner when $M_i(\theta_i)$ are functions and $f_i(\theta_i) > 0$ for all $\theta_i \in \Theta_i$ in all groups i (i.e., when all voters' types are represented and most-prefer a single option). This is not the case anymore with correspondences, as illustrated in Figure 2a where some voters in group 2 most-prefer a range of options. Figure 2b depicts the identity of the anchors as a function of x, while Figure 2c shows that we may have an interval of Condorcet winners. More precisely:

(a) if $\sum \mu_i F_i(\underline{\theta}_i^*(x')) \geq 1/2$, then $x^{CW} \leq x'$,
(b) if $\sum \mu_i F_i(\overline{\theta}_i^*(x'')) \leq 1/2$, then $x^{CW} \geq x''$,
(c) if $\sum \mu_i F_i(\underline{\theta}_i^*(x')) < 1/2$ and $\sum \mu_i F_i(\overline{\theta}_i^*(x'')) > 1/2$, then $\exists x' < x^0 < x^1 < x''$ such that $\sum \mu_i F_i(\underline{\theta}_i^*(x^1)) = 1/2$ and $\sum \mu_i F_i(\overline{\theta}_i^*(x^0)) = 1/2$. Any $x \in [x^0, x^1]$ is a Condorcet winner.

Before turning to why Proposition 1 cannot be extended to the case of discontinuous $M_i(\theta)$, we first stress what the proposition does *not* say. From Proposition 1, we cannot infer that the ("phantom" or not) anchors are decisive for *all* pairwise majority comparisons —i.e., even those which entail comparing two options other than the Condorcet winning one. The reason is that, when comparing two options x and y which both differ from x^{CW}, some types who most-prefer x^{CW} may prefer x to y while others may prefer y to x. The usual separation argument then shows that, in certain groups, individuals with a type lower than the anchor's prefer x to y, while in other groups types larger than the anchor's prefer x to y. Since anchors need not be the median θ voters of their group, there is no way at this level of generality to assess whether x is majority preferred to y. This line of reasoning seems to open up the possibility of majority voting cycles over X, since no type need be decisive in all pairwise majority comparisons. We show in a companion paper (De Donder (2012)) that this is not the case when $M_i(\theta)$ is a strictly increasing function whose range is $[\underline{x}, \overline{x}]$ for all groups. These assumptions impose sufficient structure on individual preferences to prevent the existence of a majority voting cycle of any length. We leave the exploration of the transitivity of the majority voting social preference under the (weaker) assumptions made in this note to future research.

The reason why Proposition 1 can not be extended to the case where $M_i(\theta)$ is discontinuous for one θ in group i is closely related. Figure 3a illustrates this case, where type θ' in group 1 most-prefers two distinct options. How such preferences may be obtained is illustrated with the help of the setting presented in the Introduction, with the policy space bidimensional with generic element (x, y) and where a government budget constraint (in group i) restricts the choice to the pairs (x, y) where $y = T_i(x)$. In Figure 3d, we plot the indifference curve of θ' in the (x, y) space, together with the government budget constraint $T_i(x)$. Assuming that preferences satisfy the SCP is equivalent to assuming that marginal rates of substitution are monotone in θ. It is clear from the example that voters with steeper indifference curves than θ' (those with $\theta < \theta'$) most prefer a value of x lower than x', while voters to the right of θ', with flatter indifference curves, most prefer larger value of x than x''. Individual θ' most-prefers, and is indifferent between, x' and x''. Figure 3b shows that anchors do not exist in group i for options $x \in]x', x''[$.[5] If only one group were present in society, this would not prevent the existence of a Condorcet winner. As soon as more than one group is present, a Condorcet winner may not exist. This may be the case in Figure 3c if $\sum \mu_i F_i(\underline{\theta}_i^*(x')) < 1/2$ and $\sum \mu_i F_i(\overline{\theta}_i^*(x'')) > 1/2$. In that case, any candidate x for Condorcet winner must be such that $x' < x < x''$, for which there is no anchor in group 1. Figure 3d shows that any option in the range $]x', x^\circ]$ is actually defeated at unanimity in group 1 when faced against x', and thus has lower support against x' than

[5] This means that no type is such that they most prefer such an option x, and not that such a type has zero density in the group.

some option immediately to the left of x' or to the immediate right of x''.

3 Conclusion

In this note, we have extended the classical median voter theorem in a unidimensional policy space to the case where voters are exogenously distributed across groups, with preferences satisfying the single-crossing property separately inside each group. This setting corresponds, among others, to the case where the bidimensional policy space is reduced to one dimension because of the existence of a government budget constraint, but where the budget constraint differs from one group to another, with preferences satisfying the Spence-Mirrlees condition of monotonicity of marginal rates of substitution. We show that a Condorcet winner exists provided that we can find an "anchor" (i.e., a type with this option among its most-preferred ones) in each and every group. These results hold under the assumption that the set of most-preferred options is continuous in types in each and every group.

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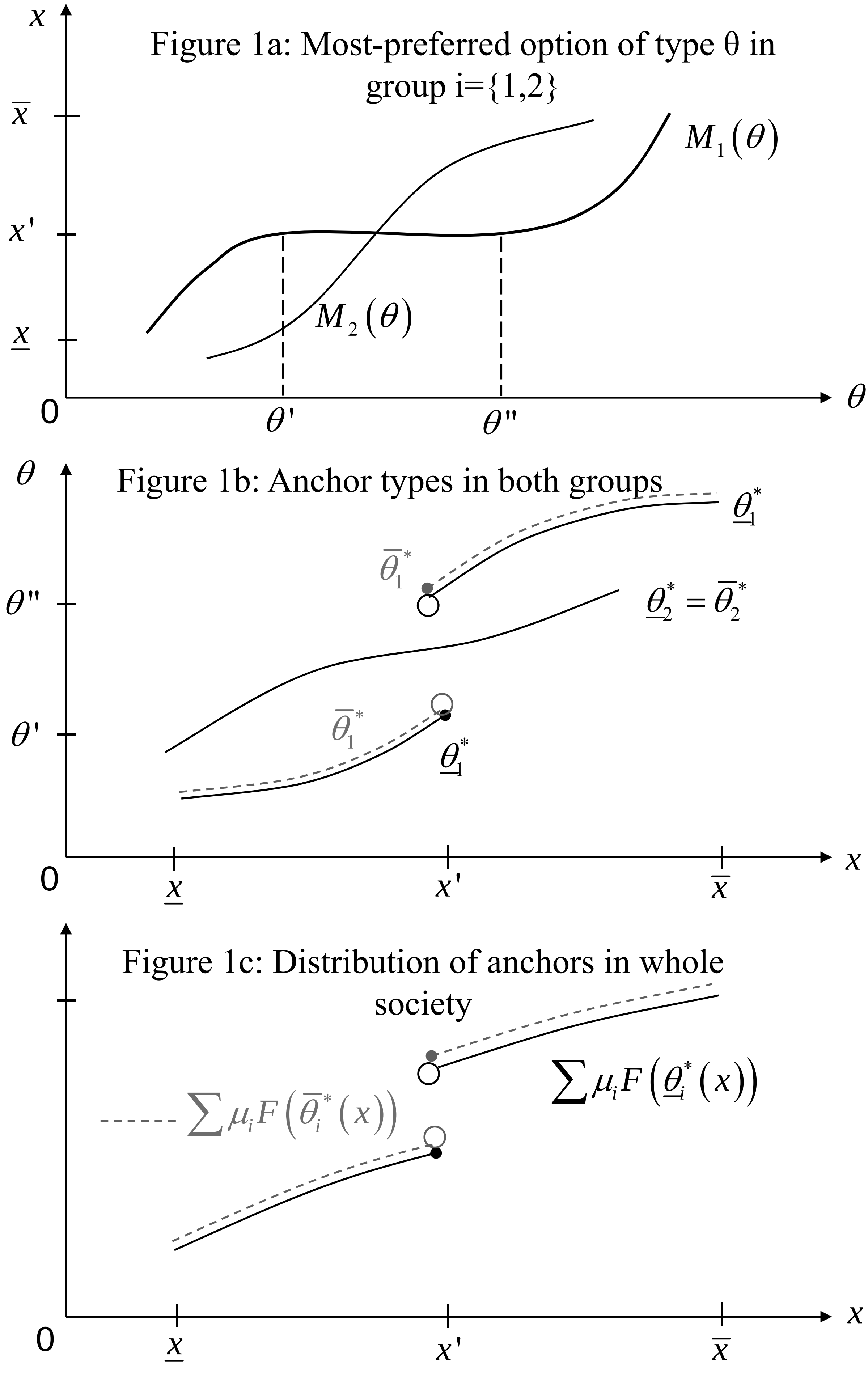


Figure 1a: Most-preferred option of type θ in group i={1,2}
x
$\overline{x}$
x'
$\underline{x}$
0
$M_1(\theta)$
$M_2(\theta)$
θ'
θ''
θ
Figure 1b: Anchor types in both groups
θ
θ''
θ'
$\underline{\theta}_1^*$
$\overline{\theta}_1^*$
$\underline{\theta}_2^* = \overline{\theta}_2^*$
$\overline{\theta}_1^*$
$\underline{\theta}_1^*$
0
$\underline{x}$
x'
$\overline{x}$
x
Figure 1c: Distribution of anchors in whole society
$\sum \mu_i F\left(\underline{\theta}_i^*(x)\right)$
$\sum \mu_i F\left(\overline{\theta}_i^*(x)\right)$
0
$\underline{x}$
x'
$\overline{x}$
x

Figure 2a: Most-preferred options of type θ in group i={1,2}

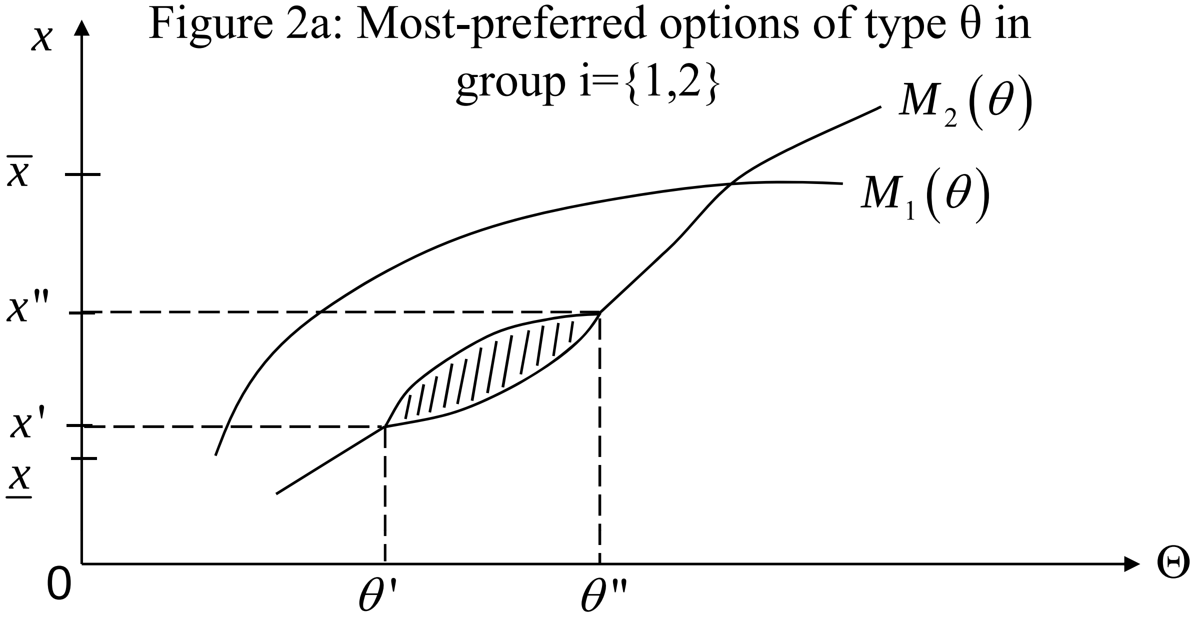


Figure 2b: Anchor types in both groups

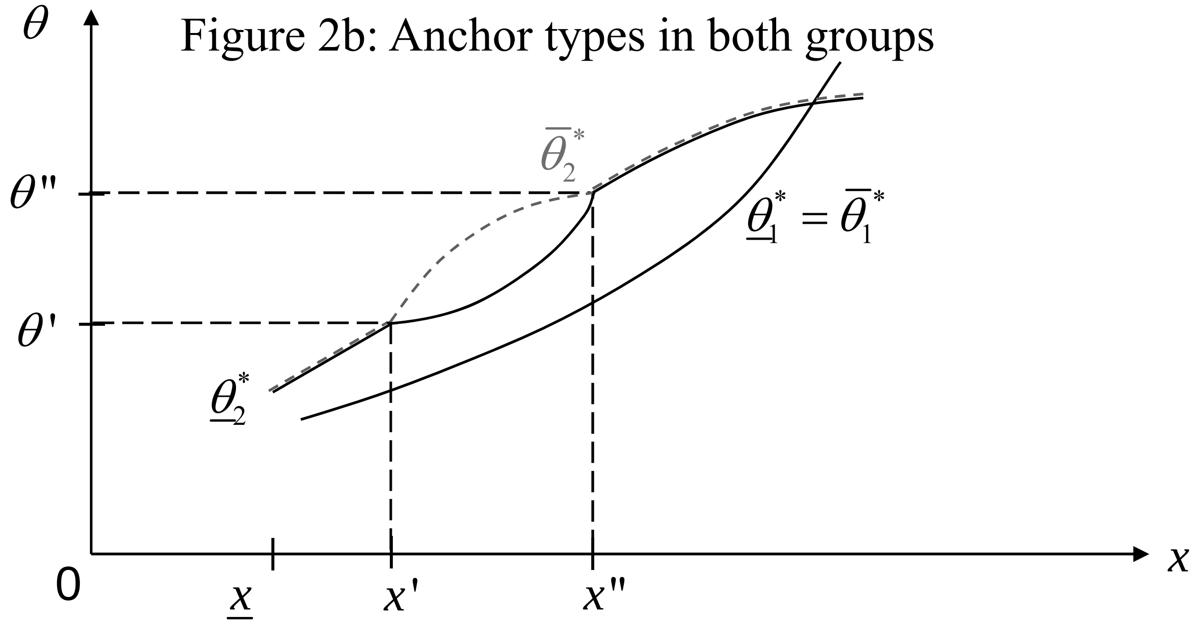
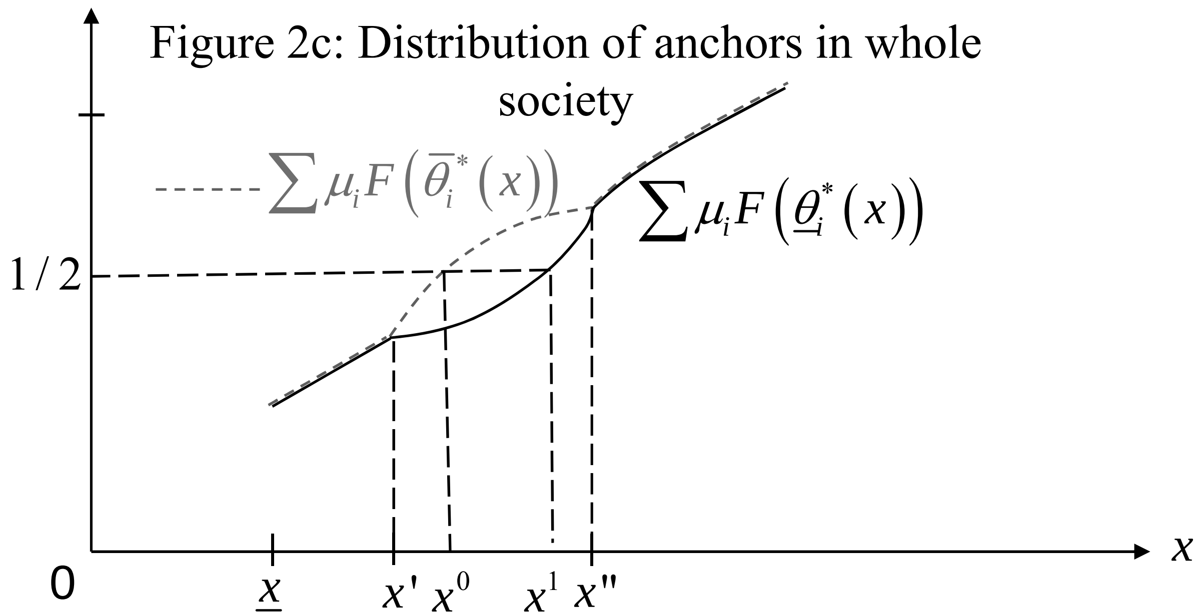


Figure 2c: Distribution of anchors in whole society



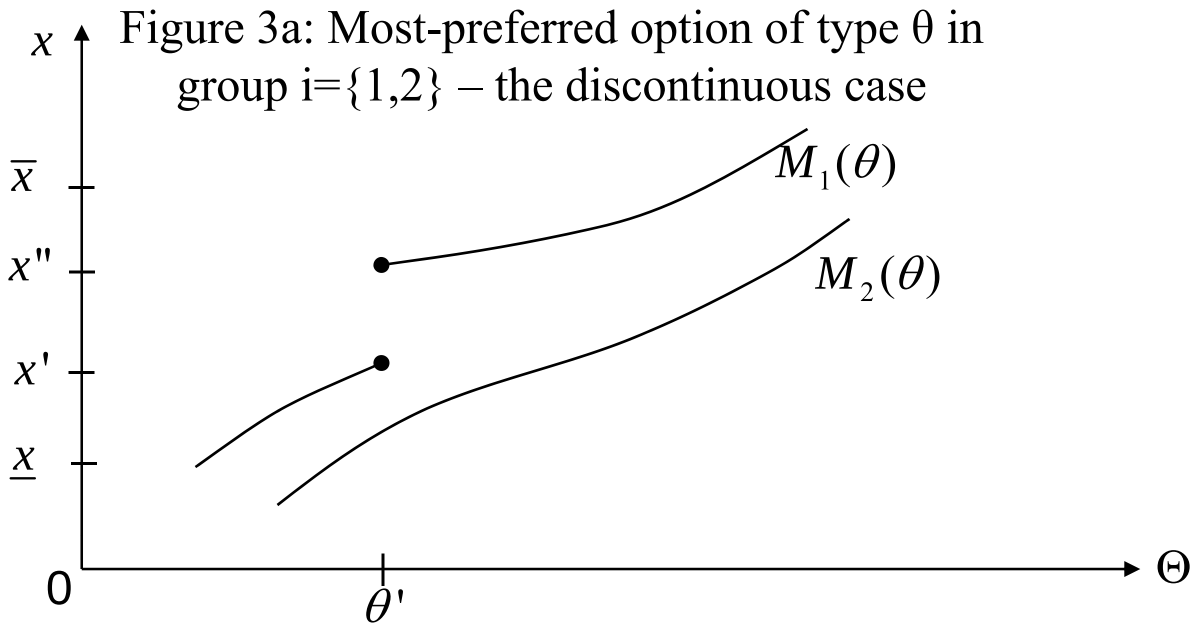


Figure 3a: Most-preferred option of type θ in group i={1,2} – the discontinuous case

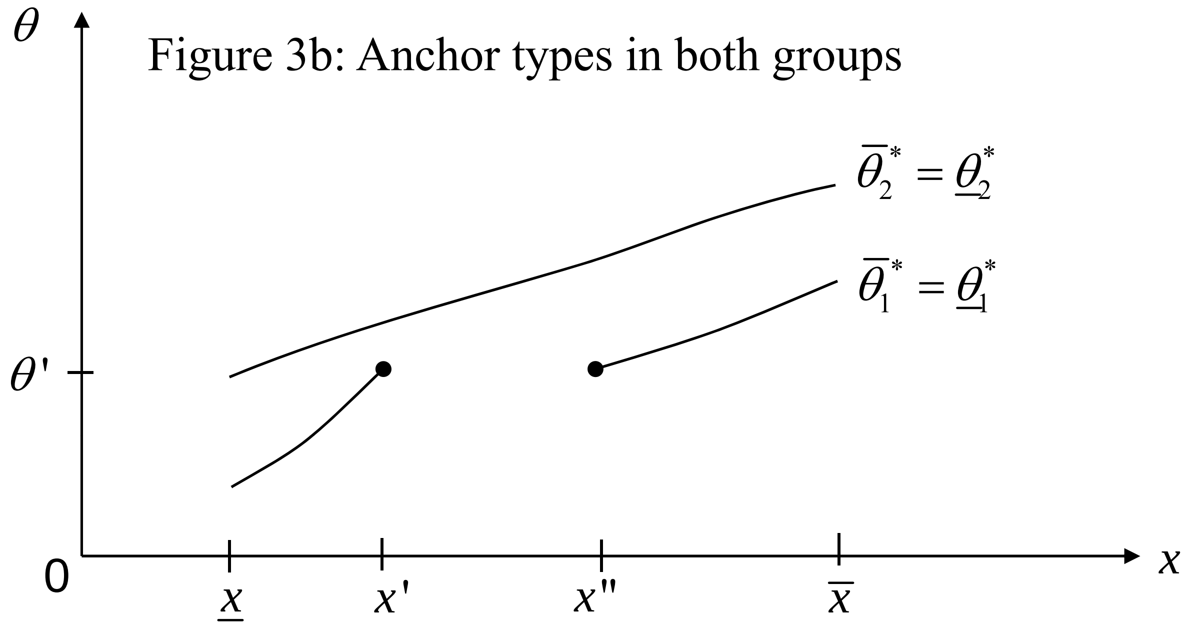


Figure 3b: Anchor types in both groups

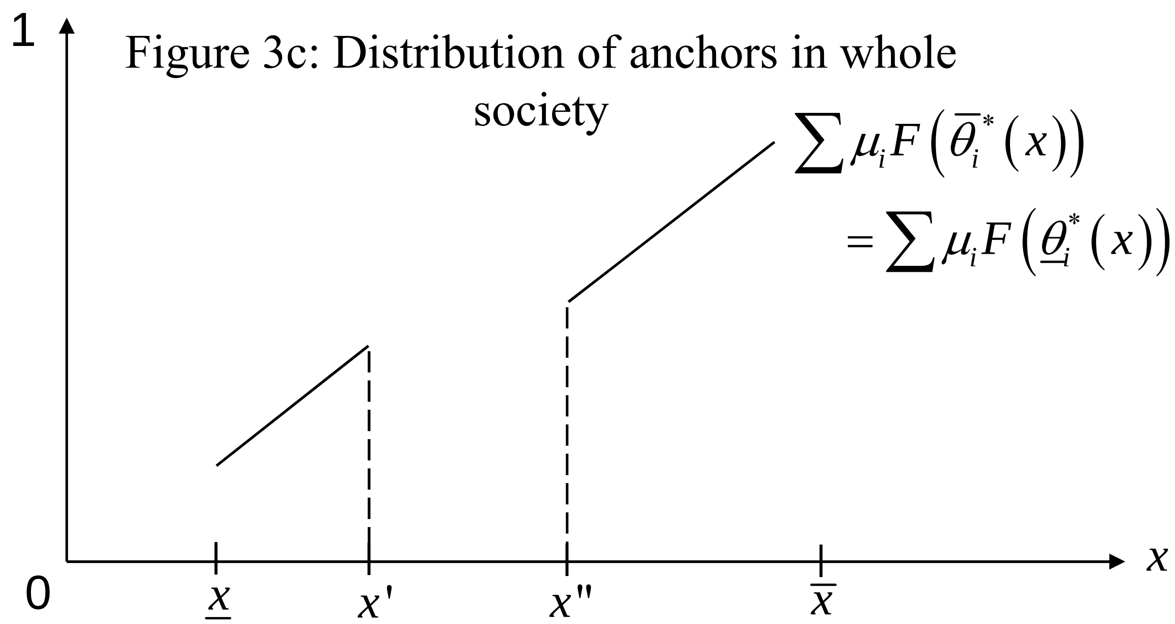


Figure 3c: Distribution of anchors in whole society

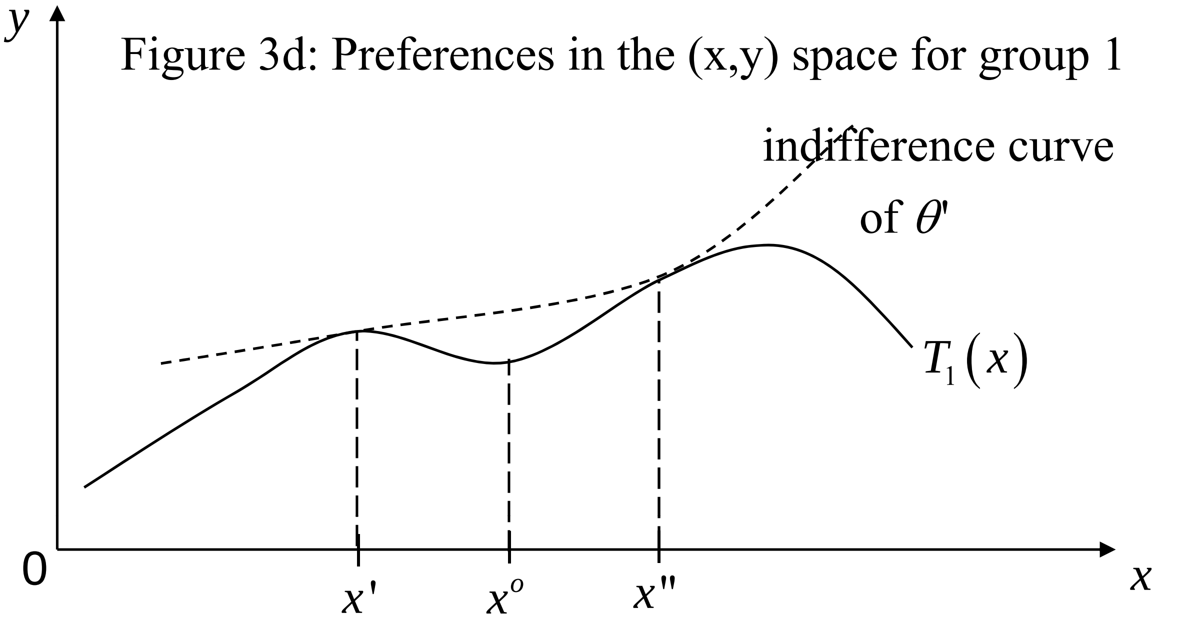


Figure 3d: Preferences in the (x,y) space for group 1