FINITE MASS SELF-SIMILAR BLOWING-UP SOLUTIONS OF A CHEMOTAXIS SYSTEM WITH NON-LINEAR DIFFUSION

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ABSTRACT. For a specific choice of the diffusion, the parabolic-elliptic Patlak-Keller-Segel system with non-linear diffusion (also referred to as the quasi-linear Smoluchowski-Poisson equation) exhibits an interesting threshold phenomenon: there is a critical mass $M_c > 0$ such that all the solutions with initial data of mass smaller or equal to M_c exist globally while the solution blows up in finite time for a large class of initial data with mass greater than M_c. Unlike in space dimension 2, finite mass self-similar blowing-up solutions are shown to exist in space dimension $d \geq 3$.

1. Introduction

In space dimension $d = 2$, the parabolic-elliptic Patlak-Keller-Segel (PKS) system is a simplified model which describes the collective motion of cells in the following situation: cells diffuse in space and emit a chemical signal, the chemo-attractant, which results in the cells attracting each other. If ρ denotes the density of cells and c the concentration of the chemo-attractant, the PKS system reads [13, 19]

$$(1) \qquad \left\{ \begin{array}{l} \partial_t \rho(t,x) = \mathrm{div}\left[\nabla\rho(t,x) - \rho(t,x)\nabla c(t,x)\right] , \\ c(t,x) = (E_2 \star \rho)(t,x) , \quad E_2(x) = -\dfrac{1}{2\pi}\, \ln|x| , \end{array} \right. \qquad (t,x) \in [0,\infty) \times \mathbb{R}^2 .$$

This model may be seen as an elementary brick to understand the aggregation of cells in mathematical biology as it exhibits the following interesting and biologically relevant feature: there is a critical mass above which the density of cells is expected to concentrate near isolated points after a finite time, a property which is related to the formation of fruiting bodies in the slime mold *Dictyostelium discoideum.* Such a phenomenon does not take place if the density of cells is too low. More precisely, given a non-negative integrable initial condition ρ_0 with finite second moment, the system (1) has a unique maximal classical solution (ρ, c) defined on some maximal time interval $[0,T)$, $T \in (0,\infty]$. Its first component ρ is non-negative and the mass of ρ (that is, its L^1-norm) remains constant through time evolution

$$\|\rho(t)\|_1 = M := \|\rho_0\|_1 , \quad t \in [0,T) .$$

It is well-known that, if $M < 8\pi$, the solution to (1) exists globally in time while it blows up in finite time if $M > 8\pi$, see [3, 6, 11, 12] and the references therein.

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More recently, it was shown that there is global existence as well for the critical mass $M = 8\,\pi$, the blowup occurring in infinite time with a profile being a Dirac mass of mass 8π [1]. When the mass M is above 8π, the shape of the finite time blowup is not self-similar according to asymptotic expansions computed in [5, 15] (see also [10] for a related problem in a bounded domain). In addition, there is no integrable and radially symmetric blowing-up self-similar solution to (1) [18, Theorem 8].

In space dimension $d \geq 3$, the system (1) seems to be less relevant from the biological point of view as blowup may occur whatever the value of M [9, 17]. This means that the diffusion is too weak to balance the aggregation resulting from the chemotactic term. It is however well-known that one can enhance the effect of diffusion to prevent crowding by considering a diffusion of porous medium type which increases the diffusion of the cells when their density ρ is large. This is the generalised version of the Patlak-Keller-Segel model considered in, e.g., [2, 4, 22, 23, 24]:

$$\begin{cases} \partial_t \rho(t,x) = \operatorname{div}\left(\nabla\left[\rho^m(t,x)\right] - \rho(t,x)\nabla c(t,x)\right)\,, \\ c(t,x) = (E_d \star \rho)(t,x)\,, \quad E_d(x) = c_d\,|x|^{2-d}\,, \end{cases} \qquad (t,x) \in [0,\infty)\times\mathbb{R}^d\,, \tag{2}$$

where $m > 1$, $c_d := 1/((d-2)\,\sigma_d)$, and $\sigma_d := 2\,\pi^{d/2}/\Gamma(d/2)$ denotes the surface area of the sphere $\mathbb{S}^{d-1}$ of $\mathbb{R}^d$. The system (2) also arises in astrophysics [4] (being then referred to as the generalised Smoluchowski-Poisson equation), and ρ and c denote the density of particles and the gravitational potential, respectively.

For (2), it turns out that there is only one critical exponent of the non-linear diffusion, namely $m_d := 2(d-1)/d$, such that the mass plays a similar role to that in (1). Indeed, if $m > m_d$ the diffusion enhancement is too strong and the solutions always exist globally in time whereas if $m < m_d$ the diffusion is not strong enough to compensate the aggregation term and there are solutions blowing up in finite time whatever the value of the mass [22, 23]. The relevant diffusion is thus achieved in the case when $m = m_d$. In this case, it was proved in [2] that there is a unique threshold mass $M_c > 0$ with the following properties: if the mass $M = \|\rho_0\|_1$ of the initial condition ρ_0 is less or equal to M_c, then the corresponding solution to (2) exists globally in time, whereas given any $M > M_c$ there are initial data ρ_0 with mass M such that the corresponding solution blows up in finite time. Thus, for the peculiar choice $m = m_d$ and $d \geq 3$, the system (2) exhibits the same qualitative behaviour as the PKS system (1) in space dimension 2. Still, there is a fundamental difference as the latter has no fast-decaying stationary solution with mass 8π while the former has a two-parameter family of non-negative, integrable, and compactly supported stationary solutions with mass M_c for each $d \geq 3$ [2, Section 3].

It is then tempting to figure out whether this striking difference extends above the critical mass M_c and this leads us to investigate the existence of blowing-up (or backward) self-similar solutions with finite mass. More precisely, since mass remains unchanged throughout time evolution, we look for solutions (ρ, c) to (2) with $m = m_d$ and $d \geq 3$ of the form

$$\rho(t,x) = \frac{1}{s(t)^d}\,\Phi\left(\frac{x}{s(t)}\right) \quad \text{and} \quad c(t,x) = \frac{1}{s(t)^{d-2}}\,\Psi\left(\frac{x}{s(t)}\right) \tag{3}$$

for $(t,x) \in [0,T) \times \mathbb{R}^d$ with $s(t) := [d(T-t)]^{1/d}$, the time T being an arbitrary positive real number. Note that $s(t)$ converges to zero as t increases to the blowup time T.

Our main result is then the following:

Theorem 1 (Existence of finite mass self-similar blowing-up solutions)**.** *There exists $M_2 \in (M_c, \infty)$ such that, for any M in $(M_c, M_2]$, there exists at least a non-negative self-similar blowing-up solution (ρ_M, c_M) to (2) of the form (3) with a radially symmetric, compactly supported, and non-increasing profile Φ_M satisfying $\|\rho_M(t)\|_1 = \|\Phi_M\|_1 = M$ for $t \in [0,T)$ and $\|\rho_M(t)\|_\infty \to \infty$ as $t \to T$.*

As a consequence of Theorem 1, we realize that non-negative, integrable, and radially symmetric self-similar blowing-up solutions to (2) with a non-increasing profile only exist below a threshold mass. Another by-product of our analysis is the existence of non-negative and non-integrable self-similar blowing-up solutions to (2), see Proposition 8 below.

2. Blowing-up self-similar profiles

From now on,

$$d \geq 3 \quad \text{and} \quad m = m_d = \frac{2(d-1)}{d},$$

and we look for a solution (ρ, c) to (2) of the form

$$\rho(t,x) = \frac{1}{s(t)^d}\, \Phi\left(\frac{x}{s(t)}\right) \quad \text{and} \quad c(t,x) = \frac{1}{s(t)^{d-2}}\, \Psi\left(\frac{x}{s(t)}\right) \tag{4}$$

with $s(t) = [d(T-t)]^{1/d}$ and $(t,x) \in [0,T) \times \mathbb{R}^d$ for some given $T > 0$. We further assume that Φ enjoys the following properties:

$$\begin{cases} \Phi \in \mathcal{C}(\mathbb{R}^d) \cap L^1(\mathbb{R}^d) \ \text{ is radially symmetric and non-negative,} \\ \Phi^{m-1} \in W^{1,\infty}(\mathbb{R}^d) . \end{cases}$$

Inserting the ansatz (4) in (2) gives that (Φ, Ψ) solves

$$\begin{cases} \mathrm{div}\left(\nabla\left[\Phi^m(y)\right] - \Phi(y)\nabla\Psi(y) - \Phi(y)\ y\right) = 0 , \\ \Psi(y) = (E_d \star \Phi)(y) , \end{cases}$$

for $y \in \mathbb{R}^d$. Since $\Psi = E_d \star \Phi$, the radial symmetry of Φ ensures that of Ψ and, introducing the profiles (φ, ψ) of (Φ, Ψ)

$$\Phi(y) = \varphi(|y|) , \qquad \Psi(y) = \psi(|y|) , \qquad y \in \mathbb{R}^d . \tag{5}$$

By [14, Theorem 9.7, Formula (5)], we have

$$\psi(r) = \frac{1}{(d-2)r^{d-2}} \int_0^r \varphi(s)\ s^{d-1}\ \mathrm{d}s + \frac{1}{d-2} \int_r^\infty \varphi(s)\ s\ \mathrm{d}s \tag{6}$$

for $r \geq 0$. We can also write the equation for φ as

$$\partial_r\left(r^{d-1}\ \varphi(r)\ \partial_r J(r)\right) = 0 \quad \text{with} \quad J(r) := \frac{2(d-1)}{d-2}\ \varphi^{(d-2)/d}(r) - \psi(r) - \frac{r^2}{2} \tag{7}$$

for $r \in (0,\infty)$. Since we are looking for an integrable profile, we formally conclude that

$$\partial_r J(r) = 0 \quad \text{for} \quad r \in \mathcal{P}_\varphi := \{s \in (0,\infty) \; : \; \varphi(s) > 0\} \,. \tag{8}$$

In particular, J is constant on any connected component of $\mathcal{P}_\varphi$. But, if $\mathcal{C}$ is a connected component of $\mathcal{P}_\varphi$, we have either

$$\mathcal{C} = (0, R_s) \quad \text{for some} \quad R_s \in (0,\infty] \,, \tag{9}$$

or

$$\mathcal{C} = (R_i, R_s) \quad \text{for some} \quad R_i \in (0,\infty) \quad \text{and} \quad R_s \in (0,\infty] \,.$$

Remark 2. *If we additionally assume that the profile φ is non-increasing then $\mathcal{P}_\varphi$ has only one connected component which is necessarily of the form* (9).

Now, take a connected component $\mathcal{C}$ of $\mathcal{P}_\varphi$. It follows from (8) that there is $\mu \in \mathbb{R}$ such that

$$J(r) = \frac{2(d-1)}{d-2}\, \varphi^{(d-2)/d}(r) - \psi(r) - \frac{r^2}{2} = -\mu \quad \text{for} \quad r \in \mathcal{C} \,. \tag{10}$$

Owing to the assumed integrability of Φ, the function $r \mapsto r^{d-1}\varphi(r)$ belongs to $L^1(0,\infty)$ and it follows from (6) that the function $r \mapsto r^{d-2}\psi(r)$ is bounded in $\mathcal{C}$. Therefore (10) only complies with the integrability of Φ if $R_s < \infty$ which implies the boundedness of $\mathcal{C}$. Introducing

$$\Xi := \varphi^{(d-2)/d}$$

and taking the Laplacian of both sides of (10) yield that Ξ is a positive solution to

$$-\frac{d^2\Xi}{dr^2}(r) - \frac{d-1}{r}\,\frac{d\Xi}{dr}(r) = \frac{d-2}{2(d-1)}\,\left(\Xi(r)^{d/(d-2)} - d\right) \quad \text{in} \quad \mathcal{C} \,, \tag{11}$$

with either

$$\partial_r \Xi(0) = \Xi(R_s) = 0 \quad \text{if} \quad \mathcal{C} = (0, R_s) \tag{12}$$

or

$$\Xi(R_i) = \Xi(R_s) = 0 \quad \text{if} \quad \mathcal{C} = (R_i, R_s) \,. \tag{13}$$

A final change of scale, namely

$$\eta(r) := \frac{1}{\lambda_d}\, \Xi\left(\frac{r}{\mu_d}\right) \,, \quad \lambda_d := d^{(d-2)/d} \,, \quad \mu_d := d^{1/d}\, \left(\frac{d-2}{2(d-1)}\right)^{1/2} \,,$$

leads us to the following boundary-value problem for η: either

$$\begin{cases} \dfrac{d^2\eta}{dr^2}(r) + \dfrac{d-1}{r}\,\dfrac{d\eta}{dr}(r) + \eta(r)^{d/(d-2)} - 1 = 0 \,, \quad r \in (0, \mu_d R_s) \,, \\[2mm] \dfrac{d\eta}{dr}(0) = 0 \,, \quad \eta(\mu_d R_s) = 0 \,, \end{cases} \tag{14}$$

or

$$\begin{cases} \dfrac{d^2\eta}{dr^2}(r) + \dfrac{d-1}{r}\,\dfrac{d\eta}{dr}(r) + \eta(r)^{d/(d-2)} - 1 = 0 \,, \quad r \in (\mu_d R_i, \mu_d R_s) \,, \\[2mm] \eta(\mu_d R_i) = 0 \,, \quad \eta(\mu_d R_s) = 0 \,. \end{cases} \tag{15}$$

We have thus reduced our study to one or several boundary-value problems (depending on the number of connected components of $\mathcal{P}_\varphi$) for a nonlinear second order differential equation. The purpose of the next section is then a precise study of this ordinary differential equation.

However, before going on, let us point out that (11) is not equivalent to (10). Indeed, since

$$\partial_r J(r) = \frac{2(d-1)}{d-2}\ \partial_r \Xi(r) + \frac{1}{r^{d-1}} \int_0^r \Xi(s)^{d/(d-2)}\ s^{d-1}\ \ ds - r\,, \quad r \in \mathcal{C}\,,$$

by (7), the fact that Ξ is a solution to (11) only guarantees that $\partial_r(r^{d-1}\partial_r J(r)) = 0$ for $r \in \mathcal{C}$. Consequently, there are constants C_1 and C_2 such that

$$\partial_r J(r) = -\frac{(d-2)\ C_1}{r^{d-1}}\,, \quad J(r) = \frac{C_1}{r^{d-2}} + C_2\,, \quad r \in \mathcal{C}\,,$$

from which (10) follows only if $C_1 = 0$. On the one hand, if $\mathcal{C} = (R_i, R_s)$ with $0 < R_i < R_s$, it is yet unclear whether the boundary conditions (13) might imply this property. On the other hand, if $\mathcal{C} = (0, R_s)$, the boundary conditions (12) ensure that $\partial_r J(0) = 0$ and thus $C_1 = 0$. We shall only deal with this case in the remaining of this paper and thus focus on the non-increasing profiles φ.

3. An auxiliary ordinary differential equation

For $a \in \mathbb{R}$, let $u(.,a) \in \mathcal{C}^1([0, r_{\max}(a)))$ denote the maximal solution to the Cauchy problem

$$\left\{ \begin{array}{l} u''(r,a) + \dfrac{d-1}{r}\ u'(r,a) + |u(r,a)|^{p-1}\ u(r,a) - 1 = 0\,, \quad r \in [0, r_{\max}(a))\,, \\ \\ u(0,a) = a\,, \quad u'(0,a) = 0\,, \end{array} \right. \tag{16}$$

with $r_{\max}(a) \in (0, \infty]$ and $p = d/(d-2)$.

Clearly, if $a = 1$ then $u(.,1) \equiv 1$ is a stationary solution and $r_{\max}(1) = \infty$. We first show that $u(.,a)$ is global for all $a \in \mathbb{R}$ and oscillates around the value 1 if $a \neq 1$.

Lemma 3. *For each $a \in \mathbb{R} \setminus \{1\}$, $r_{\max}(a) = \infty$, and the solution $u(.,a)$ to (16) is an oscillatory function in $(0,\infty)$. More precisely,*

- *if $a > 1$, there is an increasing sequence $(r_i(a))_{i \geq 0}$ of real numbers such that $r_0(a) = 0$,*

$$\left\{ \begin{array}{l} u'(r_i(a),a) = 0\,, \quad (-1)^i u'(r,a) < 0 \quad \text{for} \quad r \in (r_i(a), r_{i+1}(a))\,, \\ \\ u(r_{2i}(a),a) > u(r_{2i+2}(a),a) > 1 > u(r_{2i+3}(a),a) > u(r_{2i+1}(a),a) \end{array} \right.$$

  for $i \geq 0$,
- *if $a < 1$, there is an increasing sequence $(r_i(a))_{i \geq 1}$ of real numbers such that $r_1(a) = 0$*

$$\left\{ \begin{array}{l} u'(r_i(a),a) = 0\,, \quad (-1)^i u'(r,a) < 0 \quad \text{for} \quad r \in (r_i(a), r_{i+1}(a))\,, \\ \\ u(r_{2i}(a),a) > u(r_{2i+2}(a),a) > 1 > u(r_{2i+1}(a),a) > u(r_{2i-1}(a),a) \end{array} \right.$$

  for $i \geq 1$.

These properties are illustrated in Figure 1. Notice that, for $a = 7$, $u(.,7)$ vanishes at a finite r and thus provides a solution to (14).

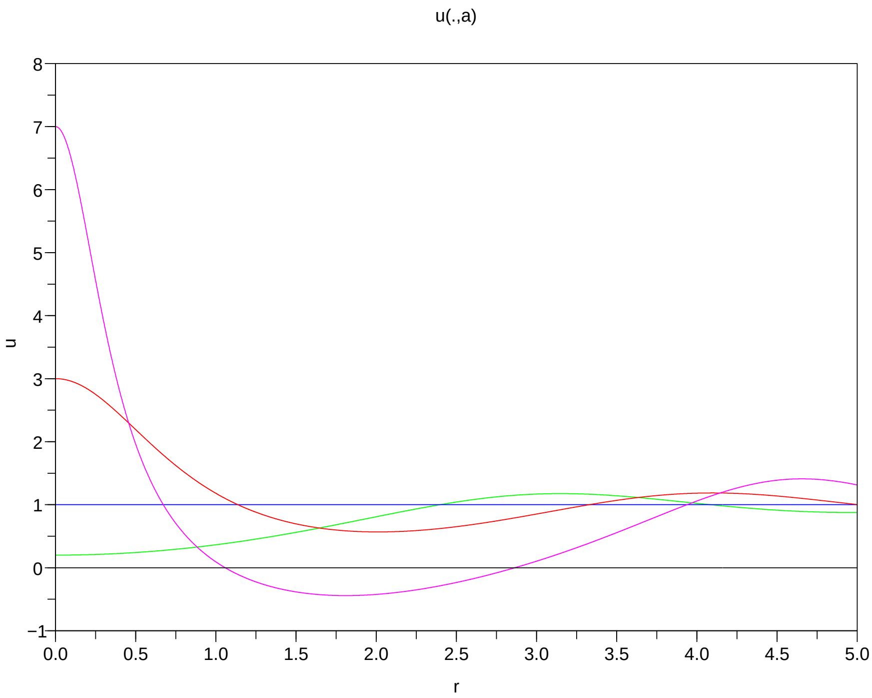

FIGURE 1. Various oscillating behaviours of $u(.,a)$ for $a \in \{0.2, 1, 3, 7\}$.

Proof of Lemma 3. For any $r \in [0, r_{\max}(a))$ consider the functional

$$E(r,a) := \frac{|u'(r,a)|^2}{2} + \frac{|u(r,a)|^{p+1}}{p+1} - u(r,a)\,. \tag{17}$$

By (16), for all $r \in [0, r_{\max}(a))$

$$\frac{dE}{dr}(r,a) = -\frac{d-1}{r}\ |u'(r,a)|^2 \le 0\,, \tag{18}$$

Obviously $E(r,a) \ge -p/(p+1)$. Owing to (18), $E(r,a) \in [-p/(p+1), E(0,a)]$ for $r \in [0, r_{\max}(a))$ which prevents $u(.,a)$ of becoming unbounded at a finite value of r, thereby implying that $r_{\max}(a) = \infty$. We next argue using Sturm's oscillations theorem as in [16, Lemma 9], to establish the oscillatory behaviour of $u(.,a)$ for $a \neq 1$. □

According to (14), we are interested in finding solutions to the initial value problem (16) which are positive and vanish at a finite value of r. We thus focus on the case $a > 0$ and investigate the positivity properties of $u(.,a)$.

Lemma 4. *There is a constant $a_c > 1$ such that*

- *if $a \in (0, a_c)$, then $u(r,a) > 0$ for all $r \ge 0$,*

- *if $a = a_c$, then there is $R(a_c) > 0$ such that*

$$\begin{cases} u(R(a_c), a_c) = 0 \\ u'(R(a_c), a_c) = 0 \\ u(r, a_c) > 0 \quad \textit{for } r \in [0, R(a_c)), \end{cases}$$

- *if $a \in (a_c, \infty)$, then there is $R(a) > 0$ such that*

$$\begin{cases} u(R(a), a) = 0 \\ u'(R(a), a) < 0 \\ u(r, a) > 0 \quad \textit{for } r \in [0, R(a)). \end{cases}$$

These three possibilities are drawn in Figure 2.

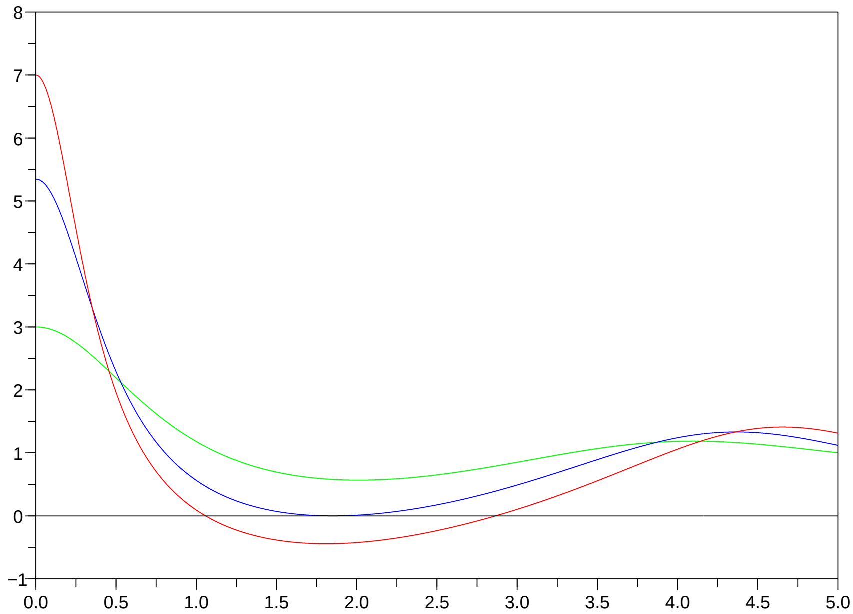

FIGURE 2. Behaviour of $u(., a)$ for $a > a_c$, $a = a_c$ and $a < a_c$.

Proof of Lemma 4. For $a > 0$, we define

$$R(a) := \inf\{R > 0 \ : \ u(r, a) > 0 \quad \text{for} \quad r \in [0, R)\} .$$

Notice that the positivity of a and the continuity of $u(., a)$ guarantee that $R(a) > 0$. We consider the sets

$$\begin{array}{lcl} \mathcal{P} & := & \{a > 0 \ : \ R(a) = \infty\} , \\ \mathcal{N} & := & \{a > 0 \ : \ R(a) < \infty \ \text{ and } \ u'(R(a), a) < 0\} , \\ \mathcal{N}_0 & := & \{a > 0 \ : \ R(a) < \infty \ \text{ and } \ u'(R(a), a) = 0\} . \end{array}$$

Clearly, $\mathcal{P} \cup \mathcal{N} \cup \mathcal{N}_0 = (0,\infty)$ and $1 \in \mathcal{P}$. Actually, if $a \in (0,(p+1)^{1/p})$, then $E(0,a) < 0$ and the monotonicity (18) of E entails that $E(r,a) < 0$ for all $r \ge 0$. But, if $R(a) < \infty$, it readily follows from the definition (17) of the functional E that $E(R(a),a) \ge 0$ whence a contradiction. Therefore, $R(a) = \infty$ for any $a \in (0,(p+1)^{1/p})$ so that

$$\left(0,(p+1)^{1/p}\right) \subset \mathcal{P}\,. \tag{19}$$

Consider now $a \in \mathcal{N}_0$. Then $U(x) := u(|x|,a)$ is a radial positive solution to the homogeneous Dirichlet-Neumann free boundary problem $\Delta U + U^p - 1 = 0$ in $B(0,R(a))$ with $U = \partial_\nu U = 0$ on $\partial B(0,R(a))$. According to [20, Theorem 3 (iii)], there is only one value of a for which this solution has a positive radial solution and it is unique. Consequently, there is a unique $a_c > 0$ such that $\mathcal{N}_0 = \{a_c\}$.

Consider next $a \in \mathcal{N} \cup \mathcal{N}_0$ and recall that $a > 1$ by (19). Following [16, Lemma 11], let us assume for contradiction that there is $\varrho \in (0,R(a))$ such that $u'(\varrho,a) = 0$. Either $\varrho \le 1$ and we infer from the definition, the monotonicity of E, see (17)-(18), and the definition of $R(a)$ that $0 > E(\varrho,a) \ge E(R(a),a) \ge 0$ which is a contradiction. Or $\rho > 1$ and the oscillating behaviour of the solutions implies, using the notation of Lemma 3, that $\varrho \ge r_2(a)$. This implies that $r_1(a) < R(a)$. Then $u(r_1(a),a) \in (0,1)$ and using again (17), (18), and the definition of $R(a)$, we conclude that $0 > E(r_1(a),a) \ge E(R(a),a) \ge 0$, hence a contradiction. Therefore,

$$u'(r,a) < 0 \quad \text{for} \quad r \in (0,R(a)) \quad \text{if} \quad a \in \mathcal{N} \cup \mathcal{N}_0\,. \tag{20}$$

Let us now prove that $\mathcal{P}$ and $\mathcal{N}$ are open subsets of $(0,\infty)$. We first consider $a \in \mathcal{N}$: by (20) there are $\varrho > R(a)$ and $\varepsilon > 0$ such that $u(\varrho,a) < 0$ and $u'(r,a) < -2\varepsilon$ for $r \in (0,\varrho)$. By continuous dependence, there is $\delta \in (0,a)$ such that $u(\varrho,b) < 0$ and $u'(r,b) < -\varepsilon$ for $r \in (0,\varrho)$ and $b \in (a-\delta,a+\delta)$. Since $u(0,b) = b > 0$, we readily deduce that, for each $b \in (a-\delta,a+\delta)$, we have $R(b) \in (0,\varrho)$ with $u'(R(b),b) < -\varepsilon < 0$. Consequently, $(a-\delta,a+\delta)$ and $\mathcal{N}$ is open in $(0,\infty)$. Consider next $a \in \mathcal{P}$, $a > 1$. By Lemma 3 and (17), we have $u(r,a) \ge u(r_1(a),a) \in (0,1)$ for $r \in [0,r_1(a)]$ and $E(r_1(a),a) < 0$. By continuous dependence, there is $\delta > a$ such that $u(r,b) \ge u(r_1(a),a)/2 > 0$ for $r \in [0,r_1(a)]$, $u(r_1(a),b) \in (0,1)$, and $E(r_1(a),b) < 0$ for $b \in (a-\delta,a+\delta)$. Assume now for contradiction that there is $b \in (a-\delta,a+\delta)$ such that $R(b) < \infty$. Owing to (17), (18), and the definition of $R(b)$, we obtain $0 > E(r_1(a),b) > E(R(b),b) \ge 0$ and a contradiction. Consequently, $(a-\delta,a+\delta) \subset \mathcal{P}$ and $\mathcal{P}$ is also open in $(0,\infty)$.

We finally argue as in [16, Lemma 15] to show that there is $A > 0$ such that $(A,\infty) \subset \mathcal{N}$.

Since $\mathcal{P}$ and $\mathcal{N}$ are open subsets of $(0,\infty)$, $\mathcal{N}_0 = \{a_c\}$, $(0,(p+1)^{1/p}) \subset \mathcal{P}$, and $(A,\infty) \subset \mathcal{N}$, we readily conclude that $\mathcal{P} = (0,a_c)$ and $\mathcal{N} = (a_c,\infty)$. □

We next study the properties of the map $a \mapsto R(a)$. An efficient tool for that purpose is the variation of $u(.,a)$ with respect to a defined by

$$\vartheta(r,a) := \frac{\partial u}{\partial a}(r,a)\,, \quad (r,a) \in [0,\infty) \times (0,\infty),$$

which solves the second order linear differential equation

$$\begin{aligned}
&\vartheta''(r,a) + \frac{d-1}{r}\ \vartheta'(r,a) + p\ u(r,a)^{p-1}\ \vartheta(r,a) = 0\,, \quad r\in[0,\infty)\,, \\
&\vartheta(0,a)=1\,, \quad \vartheta'(0,a)=0\,,
\end{aligned} \tag{21}$$

We argue as in [7, 25] to prove the following lemma.

Lemma 5. *If $a > a_c$, there is a unique $z(a)\in(0,R(a))$ such that*

$$\begin{cases} \vartheta(r,a)>0 & \textit{for} \quad r\in[0,z(a))\,, \\ \vartheta(z(a),a)=0 & \\ \vartheta(r,a)<0 & \textit{for} \quad r\in(z(a),R(a)]\,. \end{cases}$$

In addition, $u(z(a),a)>1$ and the ratio $\vartheta(.,a)/u(.,a)$ is a decreasing function of r on $(0,R(a))$.

Proof of Lemma 5. Since the proof follows rather closely that of [25] and [7, Lemma 2.1], we sketch it briefly for the sake of completeness. Fix $a>a_c$ and set $u=u(.,a)$ and $\vartheta=\vartheta(.,a)$ to simplify notations. We first argue as in [16, Lemma 17] to show that ϑ vanishes at least once in the interval $(0,z_1(a))$, where $z_1(a)$ denotes the unique zero in $(0,R(a))$ of $u-1$. Indeed, (16) also reads

$$(u(r)-1)'' + \frac{d-1}{r}\ (u(r)-1)' + \frac{u(r)^p-1}{u(r)-1}\ (u(r)-1) = 0\,, \quad r\in[0,\infty)$$

and $(u(r)^p-1)/(u(r)-1)\le p\ u(r)^{p-1}$ for $r\in[0,z_1(a))$. It then follows from Sturm's comparison theorem that ϑ vanishes at least once in the interval $(0,z_1(a))$. Let $z\in(0,z_1(a))$ denote the first zero of ϑ.

We now aim at showing that ϑ cannot vanish once more in the interval $(z,R(a))$. To this end, we define

$$\xi(r) := r^{d-1}\ [u'(r)\ \vartheta(r) - u(r)\ \vartheta'(r)] = -r^{d-1}\ u(r)^2\ \left(\frac{\vartheta}{u}\right)'(r)\,, \quad r\in[0,R(a))\,,$$

which encodes the monotonicity of ϑ/u. It follows from (16) and (21) that

$$\xi'(r) = r^{d-1}\ \left((p-1)\ u^p(r)+1\right)\ \vartheta(r)\,, \quad r\in[0,R(a))\,. \tag{22}$$

Clearly, $\xi'(r)>0$ for $r\in(0,z)$ and $\xi(0)=0$, so that $\xi(r)>0$ for $r\in(0,z]$. Assume now for contradiction that there is $\varrho\in(z,R(a))$ such that

$$\xi(r)>0 \ \text{ for } \ r\in(0,\varrho) \quad \text{and} \quad \xi(\varrho)=0\,.$$

Observing that $\vartheta'(z)<0$, we realize that, if $\vartheta(\varrho)\ge 0$, there is $\sigma\in(z,\varrho]$ such that $\vartheta(r)<0$ for $r\in(z,\sigma)$ and $\vartheta(\sigma)=0$. In that case, $\vartheta'(\sigma)\ge 0$ and thus $\xi(\sigma)=-\sigma^{d-1}\ u(\sigma)\ \vartheta'(\sigma)\le 0$, leading us to a contradiction. Consequently,

$$\vartheta(\varrho)<0\,. \tag{23}$$

We next introduce the functions

$$
\begin{aligned}
T(r) &:= \frac{2\ (u(r)^p-1)}{(p-1)\ u(r)^p+1}\ \xi(r)-\zeta(r)\,, \\
\zeta(r) &:= r^d\ \left[u'(r)\ \vartheta'(r)+(u(r)^p-1)\ \vartheta(r)\right]+(d-2)\ r^{d-1}\ u'(r)\ \vartheta(r)\,,
\end{aligned}
$$

for $r\in[0,R(a))$ and use (16), (21), and (22) to obtain

$$
\begin{aligned}
\zeta'(r) &= 2\ r^{d-1}\ (u(r)^p-1)\ \vartheta(r)\,, \\
T'(r) &= 2p^2\ \frac{u(r)^{p-1}}{\left[(p-1)\ u(r)^p+1\right]^2}\ u'(r)\ \xi(r)\,,
\end{aligned}
\tag{24}
$$

for $r\in[0,R(a))$. Integrating (24) over $(0,\varrho)$ and using the negativity of u' and the positivity of ξ on this interval give

$$
\zeta(\varrho)=-T(\varrho)>0\,. \tag{25}
$$

Since $\xi(\varrho)=0$, we have $u(\varrho)\ \vartheta'(\varrho)=u'(\varrho)\ \vartheta(\varrho)$ and we have

$$
\zeta(\varrho)=Q(\varrho)\ \frac{\vartheta(\varrho)}{u(\varrho)}\,,
$$

where

$$
Q(r):=r^d\ \left[u'(r)^2+u(r)^{p+1}-u(r)\right]+(d-2)\ r^{d-1}\ u(r)\ u'(r)\,,\quad r\in[0,R(a))\,.
$$

It then follows from (23), (25), and the positivity of u that

$$
Q(\varrho)<0\,. \tag{26}
$$

Finally, define

$$
P(r):=r^d\ \left(u'(r)^2+2\ \frac{u(r)^{p+1}}{p+1}-2\ u(r)\right)+(d-2)\ r^{d-1}\ u(r)\ u'(r)
$$

for $r\in[0,R(a))$. On the one hand, we notice that

$$
P(r)=Q(r)-u(r)-\frac{p-1}{p+1}\ u(r)^{p+1}<Q(r)\,,\quad r\in[0,R(a))\,. \tag{27}
$$

On the other hand, we deduce from (16) and (18) that

$$
P'(r)=r^{d-1}\ u(r)\ \left(\frac{d-2}{d-1}\ u(r)^p-(d+2)\right)\,,\quad r\in[0,R(a))\,.
$$

At this point, we realize that we have necessarily $a>(d+2)(d-1)/(d-2)$ and that there is $s\in(0,R(a))$ such that $P'(r)>0$ if $r\in(0,s)$ and $P'(r)<0$ if $r\in(s,R(a))$. Since $P(0)=0$ and $P(R(a))>0$, we conclude that $P(\varrho)>0$ and then $Q(\varrho)>0$ by (27). But this contradicts (26). We have thus established that ξ is positive in $(0,R(a))$ from which Lemma 5 follows. □

We are now in a position to state and prove some properties of the map $a\mapsto R(a)$.

Proposition 6. *The map $a\mapsto R(a)$ is a decreasing function on (a_c,∞) and there is $z_1>0$ such that*

$$
\lim_{a\searrow a_c} R(a)=R(a_c)\quad \textit{and}\quad \lim_{a\to\infty} a^{(p-1)/2}\ R(a)=z_1\,. \tag{28}
$$

The monotonicity of $a \mapsto R(a)$ is shown in Figure 3. According to numerical simulations, the function $a \mapsto a^{(p-1)/2}\ R(a)$ also seems to be a decreasing function of $a \in [a_c, \infty)$, see Figure 3.

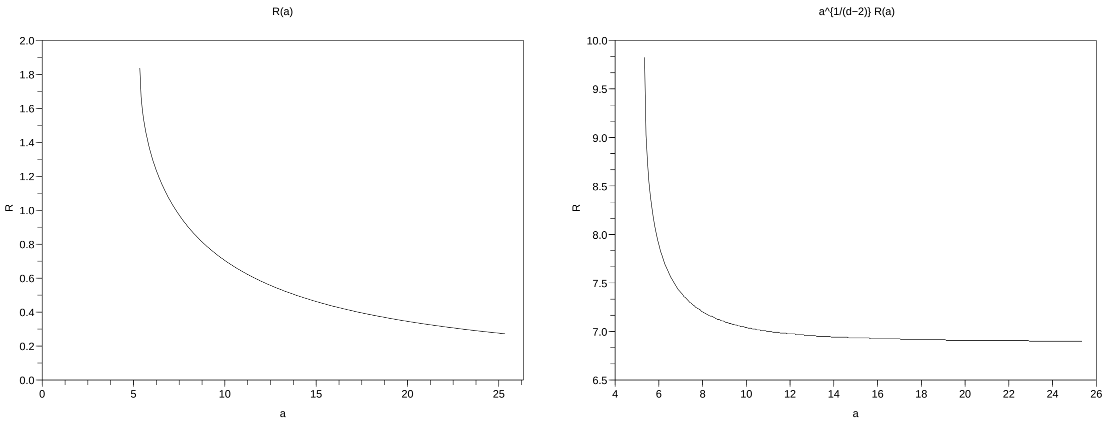

FIGURE 3. Monotonicity of the radius R and $a \mapsto a^{(p-1)/2}\ R(a)$ $(d = 3)$.

Proof of Proposition 6. By Lemma 4, $u'(R(a), a) < 0$ for all $a \in (a_c, \infty)$ and the implicit function theorem warrants that $R \in \mathcal{C}^1((a_c, \infty))$ with

$$\frac{dR}{da}(a) = -\frac{\vartheta(R(a), a)}{u'(R(a), a)} .$$

Since $\vartheta(R(a), a) < 0$ by Lemma 5, the previous formula implies the strict monotonicity of $a \mapsto R(a)$. We next define

$$R_l := \sup_{a \in (a_c, \infty)} R(a) \in (0, \infty] .$$

If $R_l > R(a_c)$, there is $\varrho \in (R(a_c), R_l)$ such that $u(\varrho, a_c) > 0$ by Lemmata 3 and 4. Then, there is $\delta > 0$ such that $R(a) > \varrho$ for $a \in (a_c, a_c + \delta)$. It then follows from the continuous dependence of $u(., a)$ with respect to a and the monotonicity of $u(., a)$ with respect to r that

$$0 = u(R(a_c), a_c) = \lim_{a \searrow a_c} u(R(a_c), a) \ge \lim_{a \searrow a_c} u(\varrho, a) = u(\varrho, a_c) > 0 ,$$

and a contradiction. Therefore, $R_l \le R(a_c)$ is finite and we have

$$u(R_l, a_c) = \lim_{a \searrow a_c} u(R(a), a) = 0 ,$$

from which we conclude that $R_l = R(a_c)$.

Finally, define

$$v(r, a) := \frac{1}{a}\ u\left(\frac{r}{a^{(p-1)/2}}, a\right) , \quad (r, a) \in [0, \infty) \times (0, \infty) . \tag{29}$$

Owing to (16), $v(.,a)$ solves

$$\begin{cases} v''(r,a) + \dfrac{d-1}{r}\ v'(r,a) + |v(r,a)|^{p-1}\ v(r,a) - a^{-p} = 0\,, & r \in [0,\infty)\,, \\ v(0,a) = 1\,, \quad v'(0,a) = 0\,, \end{cases}$$

In addition,

$$v(r,a) > 0 \quad \text{for} \quad r \in \left(0, a^{(p-1)/2}\ R(a)\right) \tag{30}$$

for $a > a_c$ by Lemma 4. Since $a^{-p} \longrightarrow 0$ as $a \to \infty$, we have

$$\lim_{a\to\infty} \sup_{r\in[0,\varrho]} |v(r,a) - w(r)| = 0 \quad \text{for all} \quad \varrho > 0\,, \tag{31}$$

where w denotes the unique solution to

$$\begin{cases} w''(r) + \dfrac{d-1}{r}\ w'(r) + |w(r)|^{p-1}\ w(r) = 0\,, & r \in [0,\infty)\,, \\ w(0) = 1\,, \quad w'(0) = 0\,. \end{cases} \tag{32}$$

By [8], there is $z_1 > 0$ such that

$$w(r) > 0 \ \text{ and } \ w'(r) < 0 \ \text{ for } \ r \in [0,z_1)\,, \quad w(z_1) = 0\,, \quad w'(z_1) < 0\,. \tag{33}$$

Owing to (33), there is $\delta > 0$ such that $w(r) < 0$ for $r \in (z_1, z_1+\delta)$. It then follows from (31) that, given $r \in (z_1, z_1+\delta)$, $v(r,a) < 0$ for a large enough (depending on r), whence $a^{(p-1)/2}\ R(a) \le r$ for a large enough by (30). Letting $r \to z_1$ guarantees that

$$\limsup_{a\to\infty} a^{(p-1)/2}\ R(a) \le z_1\,.$$

Next, if $\varrho \in (0,\gamma)$, we have $w(r) > w(\varrho) > 0$ for $r \in [0,\varrho]$ and we infer from (31) that $v(r,a) > w(\varrho)/2 > 0$ for $r \in [0,\varrho]$ and a large enough. Consequently, $\varrho < a^{(p-1)/2}\ R(a)$ for a large enough, from which we conclude that

$$\liminf_{a\to\infty} a^{(p-1)/2}\ R(a) \ge z_1\,.$$

Combining the above two inequalities completes the proof of Proposition 6. □

The above information allow us to estimate from above and from below a specific integral of $u(.,a)$.

Proposition 7. *For $a \in [a_c,\infty)$, we define*

$$\mathcal{M}(a) := d\ |B(0,1)| \int_0^{R(a)} u(r,a)^p\ r^{d-1}\ \mathrm{d}r\,.$$

Recalling that w is the solution to (32) *and z_1 is its first positive zero, we have*

$$\lim_{a\to\infty} \mathcal{M}(a) \ = \ \mathcal{M}_c := d\ |B(0,1)| \int_0^{z_1} w(r)^p\ r^{d-1}\ \mathrm{d}r\,, \tag{34}$$

$$\mathcal{M}_2 := \sup_{a\in[a_c,\infty)} \mathcal{M}(a) \ < \ \infty\,. \tag{35}$$

Proof of Proposition 7. Let $a \ge a_c$. Since $u(0,a) = a$, it follows from the monotonicity of $u(.,a)$ that

$$\mathcal{M}(a) \le d\ |B(0,1)|\ \int_0^{R(a)} a^p\ r^{d-1}\ \mathrm{d}r = |B(0,1)|\ \left(a^{(p-1)/2}\ R(a)\right)^d .$$

The upper bound (35) is then a straightforward consequence of (28) and the above inequality.

Next, recalling that $v(.,a)$ is defined by (29), we have

$$\mathcal{M}(a) = d\ |B(0,1)|\ \int_0^{a^{(p-1)/2}R(a)} v(r,a)^p\ r^{d-1}\ \mathrm{d}r ,$$

and we infer from (28) and (31) that (34) holds true. □

4. Proof of Theorem 1

Thanks to the analysis done in the previous sections, we are now in a position to construct self-similar blowing-up solutions to (2) having either finite or infinite mass.

Proposition 8. *Given $a > 0$ and $T > 0$, define*

$$\varphi(r) := \lambda_d^{d/(d-2)}\ u(\mu_d r, a)^{d/(d-2)} \quad \textit{for} \quad r \in [0,\infty) \quad \textit{if} \quad a \in (0,a_c) ,$$

and

$$\varphi(r) := \begin{cases} \lambda_d^{d/(d-2)}\ u(\mu_d r, a)^{d/(d-2)} & \textit{for} \quad r \in [0, R(a)/\mu_d] \\ 0 \quad \textit{for} \quad r \ge R(a)/\mu_d , & \end{cases} \qquad \textit{if} \quad a \in [a_c,\infty) .$$

Define next ψ, Φ, and Ψ by (6) and (5), respectively. Then the functions (ρ, c) defined by (4) in $(0,T) \times \mathbb{R}^d$ with $s(t) = [d(T-t)]^{1/d}$ is a non-negative self-similar blowing-up solution to (2) with finite mass if $a \ge a_c$ and infinite mass if $a \in (0,a_c)$.

The proof of Proposition 8 readily follows from the analysis performed in Sections 2 and 3. As for Theorem 1, it is a straightforward consequence of Proposition 8, the threshold values M_c and M_2 being given by

$$M_c := d^{1/d}\ \left(\frac{2(d-1)}{d-2}\right)^{(d-1)/2}\ \mathcal{M}_c \quad \text{and} \quad M_2 := d^{1/d}\ \left(\frac{2(d-1)}{d-2}\right)^{(d-1)/2}\ \mathcal{M}_2 .$$

5. Discussion

We have proved the existence of non-negative, integrable, and radially symmetric self-similar blowing-up solutions for (2). The profile φ of these self-similar solutions is compactly supported and non-increasing, and the mass of the corresponding self-similar solution ranges in the bounded interval $(M_c, M_2]$, the threshold mass M_c corresponding to the onset of blowup found in [2]. Our analysis thus reveals the existence of a second threshold value $M_2 > M_c$ of the mass above which no radially symmetric and non-increasing self-similar blowing-up solution exist. The meaning of this second threshold value for the mass is yet unclear. It is worth mentioning at this point that a related situation was uncovered for the critical unstable thin-film equation

$$\partial_t u = -\partial_x \left(u^n\ \partial_x^3 u + u^{n+2}\ \partial_x u\right) , \quad (t,x) \in [0,\infty) \times \mathbb{R} ,$$

in [21] for $n \in (0, 3/2)$. It is likely that, given $M \in (M_c, M_2]$, there is only a unique radially symmetric and non-increasing self-similar blowing-up solution with mass M and Figure 4 provides some numerical evidence of this fact. Besides this uniqueness question, the question of stability of these blowing-up solutions is also of interest.

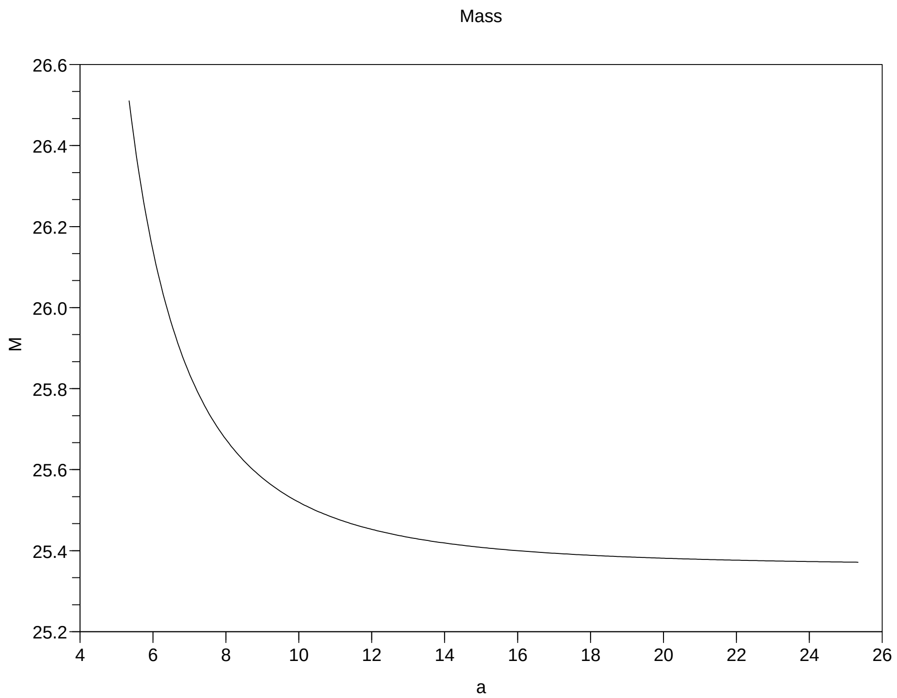

FIGURE 4. Monotonicity of the mass $a \mapsto \mathcal{M}(a)$.

Another challenging question is the existence (or non-existence) of integrable profiles φ with a non-connected positivity set as discussed in Section 2. Figure 5 provides numerical evidence that, if $a > a_c$ is large enough, $u(., a)$ may have several zeroes and each positive "hump" actually corresponds to a solution of (15) for suitable values of R_i and R_s. Whether the additional constraint (10) may be satisfied does not seem to be clear.

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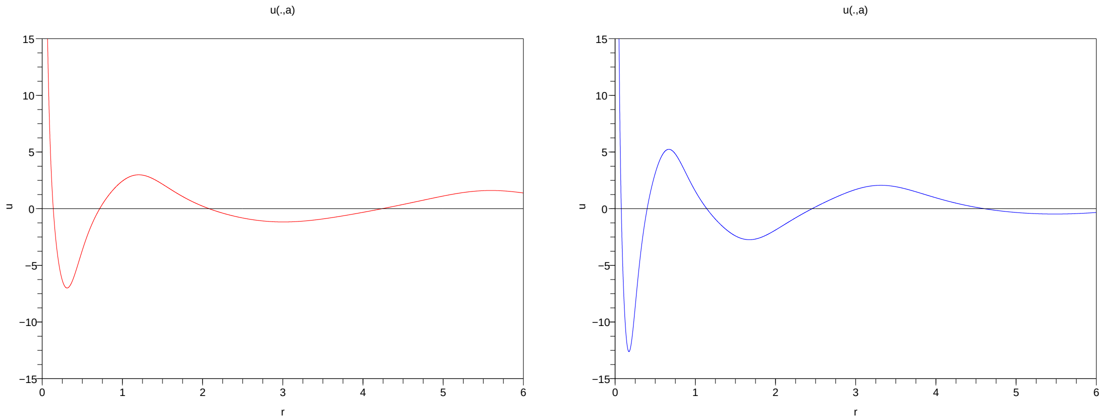

FIGURE 5. Positivity set of $u(.,a)$ with two ($a = 50$, left) and three ($a = 90$, right) connected components ($d = 3$).

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