Realized Variance Disagreement*

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Abstract

We propose realized variance disagreement as a nonparametric measure of segmentation between equity and options markets over a fixed time interval. Defined as the aggregate difference between diffusive variance components of high-frequency asset returns and their risk-neutral conditional short-term expectations recovered from options expiring at the end of the trading day, this statistic captures realized violations of the fundamental no-arbitrage condition that diffusive variance coincides under the physical and risk-neutral probability measures. We derive a central limit theorem that enables feasible inference on integrated variance disagreement between equity and options markets. We further develop market integration tests that can detect spot and integrated variance disagreements. Empirically, we find evidence for episodes of disagreement between equity and options markets. These periods exhibit mild persistence and differ in magnitude across assets.

JEL Classification: C14, C22, C58, G12, G13, G14.

Keywords: empirical characteristic function; high-frequency data; market segmentation; options; risk-neutral probability; rational expectations; stochastic discount factor; stochastic volatility.

1 Introduction

Under the standard full-information rational expectations hypothesis, agents in financial markets are assumed to know the data-generating process of security prices and its input parameters, see e.g., Hansen (2007, 2014) and references therein. In reality, when agents only have imperfect knowledge about models and limited information, asset prices may not always reflect forecasts based on the full information set. This paper investigates to which degree conditional expectations implied by equity and option prices are compatible with the assumption of integrated and arbitrage-free markets.

Options provide important and unique information about the pricing of risks. For example, the conditional risk-neutral return distribution is fully determined in a model-free way by prices

*We would like to thank participants at various conferences and seminars for useful comments and suggestions.

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of a continuum of options written on the underlying asset. Although the risk-neutral and physical return distributions can differ significantly due to compensation for risks, important restrictions apply when investors have access to a continuous record of the true asset price as commonly assumed in prior work. A key restriction concerns diffusive variance, the variance in asset returns due to continuous price movements. Because these moves are frequent and small, their variance can be inferred without error from continuous price observations in a frictionless setting. As a result, when equity and options markets are fully integrated, investors in both markets must agree about diffusive variance.[1] Otherwise, cross-market arbitrage opportunities would arise in the absence of frictions.

In this paper, we conduct inference on the *integrated variance disagreement (IVD)* between equity and options markets, which is defined as the integrated difference over a trading day between the diffusive variance of asset returns under the risk-neutral distribution, embedded in options, and the physical one. In order to measure IVD, we introduce a statistic called *realized variance disagreement (RVD)*, which is constructed from intraday high-frequency asset returns and prices of zero-date options written on the asset expiring at the end of the trading day.[2]

RVD is formed in three steps. We first employ empirical characteristic functions of high-frequency log-returns to estimate the diffusive variance of the physical return distribution. By choosing increasing values of the characteristic exponent, we isolate diffusive return components and remove contributions due to large asset price moves. Next, we construct a corresponding estimate of diffusive variance under the risk-neutral distribution from ratios of conditional characteristic functions recovered from high-frequency options data. In a final step, RVD is obtained as the difference between the option- and the return-based estimates of diffusive variance. The use of high-frequency options data ensures that biases in the risk-neutral and physical variance measures due to nuisance features of the asset price such as jumps and volatility dynamics are asymptotically the same and, therefore, approximately cancel out when forming RVD.

We establish a stable central limit theorem for RVD in a joint asymptotic regime where both the high-frequency sampling interval and the time-to-maturity of the options tend to zero. We exhibit a consistent estimator of the asymptotic variance that enables feasible inference on IVD. The asymptotic distribution of RVD is derived under very weak assumptions on the data-generating process, which cover all continuous-time asset pricing models used in prior work. In particular, we allow for jumps of arbitrary activity and time-varying jump intensities. Similarly, our assumptions on diffusive variance allow for volatility jumps, arbitrary smoothness of volatility paths and time-varying intraday volatility patterns.

We further develop market integration tests based on RVD. We show that a test statistic de-

[1] This does not preclude them, however, from disagreeing about the model driving the asset price dynamics. In this sense, market integration is a weaker assumption than the rational expectations hypothesis.

[2] Trading in zero-date options, which expire on the same day they are traded, has significantly increased in the last few years. In 2024, zero-date options account for more than 50% of the S&P 500 index options volume. Option trading as a whole also surpassed trading of the underlying shares for the first time in 2020. See the *Bloomberg* article "Zero-Day Options Are Most Popular on S&P 500 as Dominance Grows" from January 3, 2025, and the *Quartz* article "Options trading is poised to overtake the stock market" from November 22, 2021.

fined as the ratio of RVD and an estimator of its asymptotic standard error will be asymptotically standard normal if IVD is zero and diverge in absolute value to infinity if IVD is different from zero. The corresponding test will therefore have an asymptotic power of one against alternatives with nonzero integrated variance disagreement. In order to have power against spot variance disagreement, we propose a modified test statistic based on a weighted version of RVD. Under the null hypothesis of no spot variance disagreement, we derive a functional central limit theorem with a mixed Gaussian limit for our weighted test statistic as a function of the weights used in its construction. Under the alternative hypothesis, we show that the modified test statistic diverges to infinity. In order to construct a test of fixed asymptotic size, we develop an easy-to-implement wild bootstrap. The use of high-frequency return data and options close to expiration leads to an asymptotic framework that renders our inference procedures fully nonparametric.

In our empirical application, we use equity and options data for two individual stocks (Amazon and Tesla) and two exchange-traded funds (ETFs) tracking stock market indices (S&P 500 and NASDAQ 100) over the period 2020–2024. Our test detects a nontrivial number of days on which there is significant evidence for market segmentation. This evidence is strongest for Tesla, with the empirical rejection rates for this stock being notably above the nominal size of the test. The average relative size of variance disagreement, quoted in units of volatility, is estimated to be at around 6–7% for the two market indices, and at around 10% and 13% for Amazon and Tesla, respectively. We further detect mild persistence in variance disagreement that can last for several days.

What are the economic implications of these empirical findings? First, equity and options markets do not always operate in tandem and diffusive variance realized in the equity market can be different from the options market's short-term expectations about it. This is true even for the most liquid stocks and market indices. We identify periods of both under- and overestimation of physical volatility by options investors.

Second, even if occasional episodes of market segmentation cannot be exploited due to limits to arbitrage, they have to be accounted for on a theoretical level when trying to explain the joint behavior of equity and options markets within equilibrium or reduced-form models. For example, when markets are segmented, there is no pricing kernel, or stochastic discount factor (SDF), that can rationalize both equity and option prices. This is because rational investors facing no trading frictions can exploit arbitrage opportunities during these periods.

Third, market segmentation restricts how one can use options data for risk management purposes. For example, in the absence of market segmentation, options near expiry provide optimal forecasts of future short-term diffusive variance. These estimates are typically much more efficient than return-based counterparts. The SPOTVOL Index recently launched by the Chicago Board Options Exchange builds on exactly this idea. During periods of market segmentation, however, our results show that option-based estimates of diffusive variance can become biased and noisy, which needs to be accounted for by the econometrician and possibly by the investor as well.

Related Literature

The assumption of rational expectations and the closely related efficient market hypothesis have been subject to intensive empirical examination. When violations are detected, one strand of the literature aims to identify frameworks under which their occurrence can be explained or studied (see e.g., Thaler (2016) for the behavioral approach and Hansen (2014) for the model uncertainty perspective). Another strand investigates why such violations persist. In financial markets, limits to arbitrage (Shleifer and Vishny (1997)) that prevent arbitrageurs from disciplining the market play a key role in this regard. We refer to De Long et al. (1990), Pontiff (1996), Lamont and Thaler (2003), Gârleanu and Pedersen (2011) and Da et al. (2024) for various forms in which limits to arbitrage can appear. Our work differs from this literature in that our focus is on measuring the degree of arbitrage violations in equity and options markets and on developing a nonparametric inference procedure based on realized variance disagreement. Our empirical finding that moderate levels of market segmentation exist occasionally is consistent with Chong and Todorov (2025b)'s result that arbitrage strategies aimed at exploiting them are unprofitable on average due to limits to arbitrage such as transaction costs and information uncertainty.

Tests for the integration between equity and options markets have been proposed in work related to ours. Aït-Sahalia et al. (2001) propose market integration tests by comparing state-price densities inferred from options and returns. They reject a joint hypothesis that the market index follows a one-factor diffusion and that a valid SDF can jointly price the index and options written on it. As Aït-Sahalia et al. (2001) point out, this finding hinges on the strong assumption that the underlying process follows a Markov diffusion, which rules out well-documented features in asset prices such as jumps and stochastic volatility. In more recent work, Augenblick and Lazarus (2025) test rational expectations in the options market by examining violations of variance bounds on investors' belief updates (cf. also Augenblick and Rabin (2021)). They refute a joint hypothesis that index options are rationally priced and that the SDF satisfies a conditional transition independence assumption. As Augenblick and Lazarus (2025) remark, this assumption rules out stochastic volatility and other non-Markovian asset price features.

By contrast, the assumptions underlying our theoretical results are general enough to cover all asset pricing models considered in previous empirical work. Importantly, we allow for specifications in which option prices reflect compensation for both jumps and stochastic volatility, which is well documented in the literature. We achieve this level of generality by studying the pricing of diffusive variance risk. While investors might demand risk premia for disaster risk or the time-variation of diffusive variance, a key insight this paper rests on is that no risk premium can be accorded to the *level* of diffusive variance by investors with full-information rational expectations.

A large literature has studied how to extract information about pricing kernels and SDFs from asset price data. We refer to the foundational work of Hansen and Jagannathan (1991), who derive SDF bounds, and to Alvarez and Jermann (2005) and Hansen and Scheinkman (2009), who analyze the dynamical properties of SDFs. Option prices, more specifically, are known to carry valuable information about risks and the pricing of risks (Bodie and Merton (1995)). Among others, Bakshi and Madan (2000) and Carr and Madan (2001) show that option prices determine the risk-neutral return distribution in a model-free way; Broadie et al. (2007), Bollerslev et al. (2009), Bollerslev and Todorov (2011) and Gabaix (2012) discuss the pricing of variance and jump risks implicit in options; Andersen et al. (2015) and Todorov (2019) estimate physical return dynamics such as diffusive variance using options data.

All the aforementioned papers assume the existence of an SDF that can rationalize the observed security prices. Our paper tests whether such an SDF exists in the first place. We further use variance disagreement to measure the distance between the physical and risk-neutral return distributions in case such an SDF fails to exist. Previous distances based on relative entropy and its generalizations (see e.g., Stutzer (1995), Hansen (2014) and Almeida and Freire (2022)) are infinite in the absence of a valid SDF. Our measure, therefore, provides a notion of distance between physical and risk-neutral distributions precisely when existing measures explode.

Our paper also contributes both theoretically and empirically to previous research documenting puzzling empirical patterns in option prices that are hard to rationalize within standard asset pricing models (Bates (2022)). These include the expensiveness of deep out-of-the-money (OTM) options (see e.g., Coval and Shumway (2001), Bollen and Whaley (2004) and Bondarenko (2014)), the so-called U-shape pricing kernel puzzle (see e.g., Aït-Sahalia and Lo (2000), Jackwerth (2000) and Rosenberg and Engle (2002)) and an inconsistent term structure of option-implied volatility (see e.g., Stein (1989) and Giglio and Kelly (2018)). Theoretically, we develop market integration tests that take the absence of arbitrage (or, equivalently, the existence of an SDF rationalizing both equity and options prices) as the null hypothesis. These tests are fully nonparametric and make no assumptions about investor preferences. Empirically, we reject the hypothesis that equity and options markets are perfectly integrated by documenting sporadic periods in which the two markets are segmented and no SDF can jointly rationalize the observed equity and option prices.

Outline

The rest of the paper is organized as follows. We begin in Section 2 with formally defining the notion of equity and options market integration. Section 3 develops our realized variance disagreement measure and derives its asymptotic distribution. In Section 4 we propose market integration tests building on the realized variance disagreement measure. We conduct a Monte Carlo study to assess the finite-sample behavior of our inference procedures in Section 5. In Section 6, we empirically investigate the integration between equity and options markets. Section 7 concludes. The Appendix contains definitions of option-implied measures used in our inference procedures (Section A), the assumptions behind our main theorems (Section B), explicit formulas for the asymptotic variances of our statistics (Section C), results concerning estimating variance disagreement over several days (Section D) and proofs (Section E).

2 Asset Price Dynamics in Equity and Options Markets

The price X_t of the underlying asset is defined on a standard filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ and evolves as a semimartingale process with dynamics given by

$$dx_t = \alpha_t^{\mathbb{P}} dt + \sigma_t^{\mathbb{P}} dW_t^{\mathbb{P}} + \int_{\mathbb{R}} z[\mu(dt, dz) - \nu_t^{\mathbb{P}}(dz)dt], \tag{2.1}$$

where $x_t = \log(X_t)$ is the log-price, $W^{\mathbb{P}}$ is a Brownian motion and μ is an integer-valued random measure on $\mathbb{R}_+ \times \mathbb{R}$ counting the jumps in X, with compensator $\nu_t^{\mathbb{P}}(dz)dt$. We use the superscript $\mathbb{P}$ to highlight the fact that these quantities are for the dynamics of x under the physical probability measure $\mathbb{P}$. Formal assumptions for the various processes appearing in (2.1) are given in Assumption 1 in the Appendix. The semimartingale assumption is standard in continuous-time econometrics and equivalent to the absence of arbitrage in the equity market by the fundamental theorem of asset pricing.[3]

In this paper, we investigate whether arbitrage opportunities exist *between* the equity and options markets. As we only consider short horizons, without loss of generality, we assume that the risk-free interest rate is zero and that the asset does not pay dividends. In this case, both the asset price and the price of options written the asset are determined as conditional expectations of their terminal payoffs under a risk-neutral pricing measure $\mathbb{Q}$:

$$X_t = \mathbb{E}^{\mathbb{Q}}(X_T \mid \mathcal{F}_t), \quad C_{t,T}(K) = \mathbb{E}^{\mathbb{Q}}\big((X_T - K)^+ \mid \mathcal{F}_t\big), \quad P_{t,T}(K) = \mathbb{E}^{\mathbb{Q}}\big((K - X_T)^+ \mid \mathcal{F}_t\big), \tag{2.2}$$

where $C_{t,T}(K)$ and $P_{t,T}(K)$ are the prices at time t of a European call and put option on the asset, respectively, with strike K and expiration at time T. Allowing for incomplete markets, we do not assume that the risk-neutral measure is unique. Hence, there can be multiple measures $\mathbb{Q}$ that generate the prices of the underlying asset and the observed options according to (2.2). Under $\mathbb{Q}$, the log-price dynamics of the asset are given by

$$dx_t = \alpha_t^{\mathbb{Q}} dt + \sigma_t^{\mathbb{Q}} dW_t^{\mathbb{Q}} + \int_{\mathbb{R}} z[\mu(dt, dz) - \nu_t^{\mathbb{Q}}(dz)dt], \tag{2.3}$$

where $W^{\mathbb{Q}}$ is a $\mathbb{Q}$-Brownian motion and $\nu_t^{\mathbb{Q}}(dz)dt$ is the compensator of μ under $\mathbb{Q}$.[4]

We are interested in the relationship between $\mathbb{P}$ and $\mathbb{Q}$. We say that the equity and options markets are *integrated* on $[0, T]$ if there is a strictly positive aggregate pricing kernel, denoted by M_t, that can rationalize the observed equity and options prices in such a way that for $t \in [0, T]$,

$$X_t = \mathbb{E}_t^{\mathbb{P}}(D_{t,T} X_T), \quad C_{t,T}(K) = \mathbb{E}_t^{\mathbb{P}}\big(D_{t,T}(X_T - K)^+\big), \quad P_{t,T}(K) = \mathbb{E}_t^{\mathbb{P}}\big(D_{t,T}(K - X_T)^+\big), \tag{2.4}$$

[3] Some recent work by Christensen et al. (2022) and Andersen et al. (2025) suggest that stock prices might exhibit explosive drifts over very short time intervals, which might constitute a violation of the semimartingale assumption for the underlying process. We do not consider this situation formally here but we do note that the statistics proposed in this paper are robust to such locally explosive drifts in the asset price.

[4] By (2.2), X is a martingale and therefore has no drift under $\mathbb{Q}$. By Itô's formula, this implies that $\alpha_t^{\mathbb{Q}} = -\frac{1}{2}(\sigma_t^{\mathbb{Q}})^2 - \int_{\mathbb{R}}(e^z - 1 - z)\nu_t^{\mathbb{Q}}(dz)$.

where $D_{t,T} = M_T/M_t$ is the associated SDF and $\mathbb{E}_t^{\mathbb{P}}(\cdot)$ denotes $\mathcal{F}_t$-conditional $\mathbb{P}$-expectation. Otherwise, the two markets are *segmented*. The integration of the equity and options markets is thus equivalent to the existence of a risk-neutral measure $\mathbb{Q}$ that is locally equivalent to $\mathbb{P}$ on $[0, T]$, with M_t being the Radon–Nikodym derivative of $\mathbb{Q}|_{\mathcal{F}_t}$ with respect to $\mathbb{P}|_{\mathcal{F}_t}$. In a sense made precise in Chong and Todorov (2025b), market integration is equivalent to the absence of arbitrage *between* the equity and options markets.

The local equivalence of $\mathbb{P}$ and $\mathbb{Q}$ implies that they agree about events with probability zero, which in turn are related to the small and frequent moves in asset prices. In particular, Girsanov's theorem implies that the diffusion coefficients of x_t under $\mathbb{P}$ and $\mathbb{Q}$ must be equal:

$$\sigma_t^{\mathbb{Q}} = \sigma_t^{\mathbb{P}}, \quad t \in [0, T]. \tag{2.5}$$

Rational investors who participate in both the equity and options markets and know about the diffusive volatility of the underlying asset, $\sigma^{\mathbb{P}}$, will thus determine fair prices of options that are consistent with a risk-neutral diffusive volatility equal to $\sigma^{\mathbb{P}}$. In particular, there will be no risk compensation for variance due to diffusive asset price movements. It is important to note that the absence of arbitrage allows for gaps between the jump compensators $\nu_t^{\mathbb{P}}(dz)$ and $\nu_t^{\mathbb{Q}}(dz)$, reflecting risk premia, or disagreement, due to large or intermediate disaster risk (see e.g., Barro (2006), Chen et al. (2010), Gabaix (2012) and Beason and Schreindorfer (2022)). Similarly, the drift coefficients $\alpha_t^{\mathbb{P}}$ and $\alpha_t^{\mathbb{Q}}$ can differ significantly between $\mathbb{P}$ and $\mathbb{Q}$, with the gap reflecting the instantaneous equity risk premium.[5]

Condition (2.5) allows us to *quantify* the degree of market segmentation between the equity and options markets, or equivalently the extent to which arbitrage violations exist in those markets, by considering the *integrated variance disagreement*

$$\mathrm{IVD}_T = \int_0^T [(\sigma_t^{\mathbb{Q}})^2 - (\sigma_t^{\mathbb{P}})^2] dt. \tag{2.6}$$

IVD can be viewed as a (signed) distance measure between $\mathbb{P}$ and $\mathbb{Q}$. Existing measures in the literature (see e.g., Stutzer (1995), Hansen (2014) and Almeida and Freire (2022)) gauge the discrepancy of $\mathbb{P}$ and $\mathbb{Q}$ through differences between $\alpha_t^{\mathbb{P}}$ and $\alpha_t^{\mathbb{Q}}$ (and between $\nu_t^{\mathbb{P}}$ and $\nu_t^{\mathbb{Q}}$), assuming that IVD is zero. When IVD is nonzero, these measures are typically not defined or equal to infinity. By contrast, IVD is nonzero only if $\mathbb{P}$ and $\mathbb{Q}$ violate the no-arbitrage restriction (2.5). IVD, therefore, measures the degree of segmentation between the equity and options markets.

[5] Mathematically, (2.5) is a necessary condition for the integration of equity and options markets. It is also a sufficient condition in the standard case with finite activity jumps. When jumps are of infinite activity, the local equivalence of $\mathbb{P}$ and $\mathbb{Q}$ also imposes restrictions on the difference between $\nu_t^{\mathbb{P}}(dz)$ and $\nu_t^{\mathbb{Q}}(dz)$ concerning the small jumps in asset prices. These restrictions are of higher-order and we leave their analysis, which is mostly technical in nature, to future work.

3 Inference on Integrated Variance Disagreement

We now construct a realized measure of variance disagreement. The idea is to look at statistics that depend predominantly on the small and frequent moves in asset prices, which $\mathbb{P}$ and $\mathbb{Q}$ should agree about when equity and options markets are fully integrated. One such statistic is the characteristic function of an asset return over a short interval (cf. Todorov and Tauchen (2012) and Jacod and Todorov (2014)). For $\mathbb{S} = \mathbb{P}, \mathbb{Q}$, let us denote

$$\mathcal{L}^{\mathbb{S}}_{t_1,t_2}(u) = \mathbb{E}^{\mathbb{S}}_{t_1}\left(e^{iu(x_{t_2}-x_{t_1})}\right), \quad 0 \leq t_1 < t_2 \leq T. \tag{3.1}$$

If u increases as T shrinks, we can approximate

$$\mathcal{L}^{\mathbb{S}}_{t_1,t_2}(u) \approx \mathbb{E}^{\mathbb{S}}_{t_1}[e^{-\frac{u^2}{2}\int_{t_1}^{t_2}(\sigma^{\mathbb{S}}_s)^2 ds}] \approx e^{-\frac{u^2}{2}(\sigma^{\mathbb{S}}_{t_1})^2(t_2-t_1)}, \tag{3.2}$$

and this statement is made formal in the proofs. Therefore, the characteristic function with asymptotically increasing values of the characteristic exponent separates information about diffusive variance, which $\mathbb{P}$ and $\mathbb{Q}$ must agree about in the absence of arbitrage, from information about jump distributions, which may be different between $\mathbb{P}$ and $\mathbb{Q}$ even in fully integrated equity and options markets. As a consequence, if $\mathbb{P}$ and $\mathbb{Q}$ are related through an SDF, we have $\mathcal{L}^{\mathbb{Q}}_{t_1,t_2}(u) \approx \mathcal{L}^{\mathbb{P}}_{t_1,t_2}(u)$ and $\mathcal{L}^{\mathbb{Q}}_{t_1,t_2}(u)$ is approximately the optimal forecast of $e^{iu(x_{t_2}-x_{t_1})}$.

3.1 Realized Variance Disagreement

In analogy to (2.6), we define the *realized variance disagreement* between the equity and options markets as

$$\mathrm{RVD}^n_T = \widehat{\mathrm{VQ}}^n_T - \widehat{\mathrm{VP}}^n_T, \tag{3.3}$$

where $\widehat{\mathrm{VP}}^n_T$ and $\widehat{\mathrm{VQ}}^n_T$ are estimators of integrated diffusive variance under the physical and risk-neutral probability measures, respectively. These two estimators are constructed from high-frequency price observations of the underlying asset and of options with maturity T at the following time points:

$$\underline{t}^n_i = (i-1)\Delta_n, \quad \bar{t}^n_i = (i-1)\Delta_n + \Delta'_n, \quad i = 1, \ldots, k_n, \tag{3.4}$$

where $0 < \Delta_n < T$, $\Delta'_n = \varphi\Delta_n$ for some $\varphi \in (0,1)$ and k_n is a positive integer such that $k_n\Delta_n = \theta T$ for some $\theta \in (0,1)$. We consider an asymptotic setup where $T \to 0$, $\Delta_n/T \to 0$ and $k_n \to \infty$. Figure 1 illustrates the time points at which equity and option prices are sampled for the construction of RVD.

In our application, we make use of so-called zero-date options, or 0DTEs for short. This means that T will be the end of a trading day, and the interval $[0,T]$ will be one trading day. The reason for using $\Delta'_n < \Delta_n$ is to ensure that the summands defining $\widehat{\mathrm{VQ}}^n_T$ (see (3.8) below) do not use the same options data. Everything that follows will go through with $\Delta'_n = \Delta_n$ but in

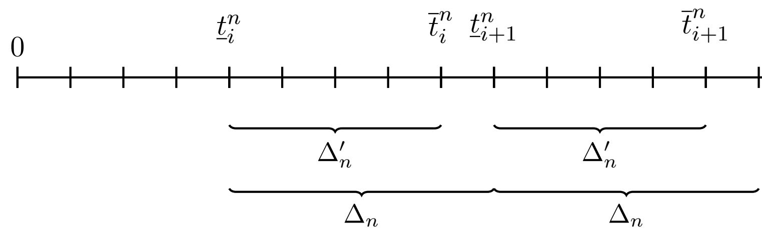

Figure 1: **Illustration of the Sampling Scheme**. RVD is constructed based on high-frequency observations of equity and option prices sampled at a distance of Δ_n' within $[0,T]$, with a gap of $\Delta_n - \Delta_n'$ between consecutive high-frequency intervals. All options expire at time T and sampling stops prior to that, at time θT.

that case our estimate of the asymptotic variance of RVD_T^n will need to be modified slightly. The reason for choosing $\theta < 1$ is to ensure that there is a sufficient strike coverage in the available options data (recall that all available options expire at T).

The choice of Δ_n, Δ_n' and k_n is determined by the econometrician and the availability of data. For example, in one of the settings considered in the Monte Carlo and the empirical sections, Δ_n is six minutes, Δ_n' is five minutes and $k_n = 59$. This corresponds to computing five-minute returns with one-minute gaps in between and stopping half an hour before market close. One can also consider constructing the statistic on part of the trading day only.

As an estimator of $\mathbb{P}$-diffusive variance we choose[6]

$$\widehat{\mathrm{VP}}_T^n = \sum_{i=2}^{k_n} \widehat{V}^{\mathbb{P}}_{\underline{t}_i^n,\overline{t}_i^n}(\widehat{u}_{(i-2)\Delta_n}), \quad \text{where} \quad \widehat{V}^{\mathbb{P}}_{t_1,t_2}(u) = -\frac{2}{u^2}\Re\Big(e^{iu(x_{t_2}-x_{t_1})} - 1\Big). \tag{3.5}$$

The characteristic exponents $\widehat{u}_t$ are chosen in an adaptive way such that

$$\frac{\underline{u}}{\sqrt{T-t}} < \widehat{u}_t < \frac{\overline{u}}{\sqrt{T-t}}, \text{ for some constants } 0 < \underline{u} < \overline{u} < \infty \text{ and } t \in [0,T). \tag{3.6}$$

We detail our choice of $\widehat{u}_t$ in (5.7) in the Monte Carlo section. By assumption, $\widehat{u}_t$ is of the order $1/\sqrt{T}$ and we have $\Delta_n/T \to 0$ and $k_n \to \infty$. Therefore, using (3.2) and a first-order approximation, we see that

$$\begin{aligned} \widehat{\mathrm{VP}}_T^n &\approx \sum_{i=2}^{k_n} \mathbb{E}^{\mathbb{P}}_{\underline{t}_i^n}\Big[\widehat{V}^{\mathbb{P}}_{\underline{t}_i^n,\overline{t}_i^n}(\widehat{u}_{(i-2)\Delta_n})\Big] \approx \sum_{i=2}^{k_n}\Bigg(-\frac{2}{\widehat{u}^2_{(i-2)\Delta_n}}\Bigg)\Big(e^{-\frac{1}{2}\widehat{u}^2_{(i-2)\Delta_n}(\sigma^{\mathbb{P}}_{\underline{t}_i^n})^2\Delta_n'} - 1\Big) \\ &\approx \Delta_n' \sum_{i=2}^{k_n} (\sigma^{\mathbb{P}}_{\underline{t}_i^n})^2. \end{aligned} \tag{3.7}$$

As a result, $\widehat{\mathrm{VP}}_T^n$ consistently estimates integrated diffusive variance under $\mathbb{P}$.

[6] We write $\Re(z)$ and $\Im(z)$ for the real and imaginary part of a complex number z, respectively.

As an estimator of $\mathbb{Q}$-diffusive variance, we choose

$$\widehat{\mathrm{VQ}}_T^n = \sum_{i=2}^{k_n} \widehat{V}^{\mathbb{Q}}_{\underline{t}_i^n, \overline{t}_i^n}(\widehat{u}_{(i-2)\Delta_n}), \quad \text{where} \quad \widehat{V}^{\mathbb{Q}}_{t_1,t_2}(u) = -\frac{2}{u^2}\Re\left(\frac{\widehat{\mathcal{L}}^{\mathbb{Q}}_{t_1,T}(u)}{\widehat{\mathcal{L}}^{\mathbb{Q}}_{t_2,T}(u)} - 1\right). \tag{3.8}$$

This statistic aims to mimic its $\mathbb{P}$-counterpart defined in (3.5) as much as possible. Ideally, we would like to choose

$$\widehat{V}^{\mathbb{Q},\text{infeasible}}_{t_1,t_2}(u) = -\frac{2}{u^2}\Re\left(\widehat{\mathcal{L}}^{\mathbb{Q}}_{t_1,t_2}(u) - 1\right), \tag{3.9}$$

where $\widehat{\mathcal{L}}^{\mathbb{Q}}_{t_1,t_2}(u)$ is an option-based estimate of $\mathcal{L}^{\mathbb{Q}}_{t_1,t_2}(u)$. In practice, however, options have a fixed maturity schedule. In particular, there are no exchange-traded options with intraday expiry, and (3.9) remains infeasible in applications. Instead, we observe at different intraday time points t prices of options expiring at time T, the end of the trading day. Using classical option spanning formulas, which we detail in Appendix A, we can then construct option-based estimates $\widehat{\mathcal{L}}^{\mathbb{Q}}_{t,T}(u)$ of $\mathcal{L}^{\mathbb{Q}}_{t,T}(u)$. As we show in the proofs, we have

$$\mathcal{L}^{\mathbb{Q}}_{t_1,t_2}(u) \approx \frac{\widehat{\mathcal{L}}^{\mathbb{Q}}_{t_1,T}(u)}{\widehat{\mathcal{L}}^{\mathbb{Q}}_{t_2,T}(u)} \quad \text{as} \quad T \to 0. \tag{3.10}$$

The approximation in (3.10) becomes exact in the case where the characteristics of x under $\mathbb{Q}$ (volatility and jump compensator) are $\mathcal{F}_0$-measurable. This covers, in particular, locally deterministic intraday patterns in volatility, which are well-known to be present in real financial data. By (3.2), (3.8) and (3.10), we have

$$\begin{aligned}
\widehat{\mathrm{VQ}}_T^n &\approx \sum_{i=2}^{k_n}\left(-\frac{2}{\widehat{u}^2_{(i-2)\Delta_n}}\right)\Re\left(\mathcal{L}^{\mathbb{Q}}_{\underline{t}_i^n,\overline{t}_i^n}(\widehat{u}_{(i-2)\Delta_n}) - 1\right) \\
&\approx \sum_{i=2}^{k_n}\left(-\frac{2}{\widehat{u}^2_{(i-2)\Delta_n}}\right)\left(e^{-\frac{1}{2}\widehat{u}^2_{(i-2)\Delta_n}(\sigma^{\mathbb{Q}}_{\underline{t}_i^n})^2\Delta'_n} - 1\right) \\
&\approx \Delta'_n \sum_{i=2}^{k_n}(\sigma^{\mathbb{Q}}_{\underline{t}_i^n})^2.
\end{aligned} \tag{3.11}$$

Equations (3.7) and (3.11) imply that RVD_T^n is a realized measure of variance disagreement. In the following, we also consider

$$\mathrm{rRVD}_T^n = \frac{\mathrm{RVD}_T^n}{\widehat{\mathrm{VP}}_T^n} = \frac{\widehat{\mathrm{VQ}}_T^n}{\widehat{\mathrm{VP}}_T^n} - 1 \tag{3.12}$$

as an estimator of the *relative integrated variance disagreement* between the equity and options markets,

$$\mathrm{rIVD}_T = \frac{\int_0^{\theta T}[(\sigma_t^{\mathbb{Q}})^2 - (\sigma_t^{\mathbb{P}})^2]dt}{\int_0^{\theta T}(\sigma_t^{\mathbb{P}})^2 dt} = \frac{\int_0^{\theta T}(\sigma_t^{\mathbb{Q}})^2 dt}{\int_0^{\theta T}(\sigma_t^{\mathbb{P}})^2 dt} - 1. \tag{3.13}$$

Remark 1. Using (3.7), (3.10) and (3.11), we have

$$\mathrm{RVD}_T^n \approx \sum_{i=2}^{k_n}\Big(\mathbb{E}^{\mathbb{Q}}_{\underline{t}_i^n}[\widehat{V}^{\mathbb{P}}_{\underline{t}_i^n,\bar{t}_i^n}] - \widehat{V}^{\mathbb{P}}_{\underline{t}_i^n,\bar{t}_i^n}\Big). \tag{3.14}$$

The full-information rational expectations forecast at time $\underline{t}_i^n$ of $\widehat{V}^{\mathbb{P}}_{\underline{t}_i^n,\bar{t}_i^n}$ is $\mathbb{E}^{\mathbb{P}}_{\underline{t}_i^n}[\widehat{V}^{\mathbb{P}}_{\underline{t}_i^n,\bar{t}_i^n}]$. While $\mathbb{P}$- and $\mathbb{Q}$-forecasts can be very different for general random variables, condition (2.5) implies that we must have

$$\mathbb{E}^{\mathbb{P}}_{\underline{t}_i^n}[\widehat{V}^{\mathbb{P}}_{\underline{t}_i^n,\bar{t}_i^n}] \approx \int_{\underline{t}_i^n}^{\bar{t}_i^n}(\sigma_s^{\mathbb{P}})^2 ds = \int_{\underline{t}_i^n}^{\bar{t}_i^n}(\sigma_s^{\mathbb{Q}})^2 ds \approx \mathbb{E}^{\mathbb{Q}}_{\underline{t}_i^n}[\widehat{V}^{\mathbb{P}}_{\underline{t}_i^n,\bar{t}_i^n}], \tag{3.15}$$

if equity and options markets are integrated. Therefore, RVD_T^n measures the divergence of short-term volatility forecasts between investors in the equity and options markets. When the two markets are integrated, RVD_T^n is approximately a martingale sum and, as we show below, mixed normally distributed around zero. Otherwise, RVD_T^n will be significantly different from zero, and the options market either over- or underestimates the diffusive volatility of the underlying asset.

Remark 2. What if investors (in the options market) have only access to a restricted information set $\mathcal{F}'_t \subset \mathcal{F}_t$ in the spirit of Hansen (2007, 2014)?[7] That is, suppose (2.2) holds with $\mathcal{F}_t$ replaced with $\mathcal{F}'_t$. Since in this case option prices are determined as $\mathcal{F}'_t$-conditional expectations of their terminal payoffs, the option-spanning result from Appendix A shows that $\widehat{\mathcal{L}}^{\mathbb{Q}}_{t,T}(u)$ will be an estimate of $\mathcal{L}'^{\mathbb{Q}}_{t,T}(u) = \mathbb{E}'^{\mathbb{Q}}_t[e^{iu(x_T - x_t)}]$, where $\mathbb{E}'^{\mathbb{Q}}_t$ denotes conditional $\mathbb{Q}$-expectation given $\mathcal{F}'_t$. As a consequence, we have

$$\mathrm{RVD}_T^n \approx \Delta'_n \sum_{i=2}^{k_n}\Big[(\sigma^{\mathbb{Q}}_{\underline{t}_i^n})^2 - (\sigma^{\mathbb{P}}_{\underline{t}_i^n})^2\Big] + \mathrm{FE}_T^n + \mathrm{RP}_T^n, \tag{3.16}$$

where

$$\mathrm{FE}_T^n = \Delta'_n \sum_{i=2}^{k_n}\Big\{\mathbb{E}'^{\mathbb{P}}_{\underline{t}_i^n}[(\sigma^{\mathbb{Q}}_{\underline{t}_i^n})^2] - (\sigma^{\mathbb{Q}}_{\underline{t}_i^n})^2\Big\}, \quad \mathrm{RP}_T^n = \Delta'_n \sum_{i=2}^{k_n}\Big\{\mathbb{E}'^{\mathbb{Q}}_{\underline{t}_i^n}[(\sigma^{\mathbb{Q}}_{\underline{t}_i^n})^2] - \mathbb{E}'^{\mathbb{P}}_{\underline{t}_i^n}[(\sigma^{\mathbb{Q}}_{\underline{t}_i^n})^2]\Big\}. \tag{3.17}$$

The first term, FE_T^n, is the aggregate forecast error in estimating $\mathbb{Q}$-diffusive variance based on the limited information set $\mathcal{F}'_t$. The second term, RP_T^n, is an aggregate risk premium demanded by investors due to uncertainty about $\mathbb{Q}$-diffusive variance.

The magnitude of FE_T^n and RP_T^n depends on how close $\mathcal{F}'_t$ is to $\mathcal{F}_t$. Under the standard assumption that $\mathcal{F}'_t$ contains a discrete price record up to time t of the underlying asset at frequency Δ'_n, investors can employ classical spot volatility estimators to infer $(\sigma^{\mathbb{P}}_{\underline{t}_i^n})^2$ from the price data. As long as investors have access to an estimator $\widehat{\sigma}^{\mathbb{P}}_{\underline{t}_i^n}$ that satisfies $(\widehat{\sigma}^{\mathbb{P}}_{\underline{t}_i^n})^2 = (\sigma^{\mathbb{P}}_{\underline{t}_i^n})^2 + o_p(1/\sqrt{k_n})$ (which will be true for standard spot volatility estimators under the assumptions on

[7] Related to this, Duffie and Lando (2001) study the implications for the term structure of credit spreads of corporate bond holders who have access to imperfect information only.

k_n and Δ'_n we impose in Assumption 5), both FE_T^n and RP_T^n will be negligible for the inference procedures we propose below in the case where condition (2.5) is satisfied. If (2.5) is violated, then RVD_T^n estimates integrated variance disagreement plus a forecast error and a risk premium term, that is,

$$\mathrm{IVD}'_T = \mathrm{IVD}_T + \mathrm{FE}_T + \mathrm{RP}_T, \tag{3.18}$$

where

$$\mathrm{FE}_T = \int_0^T (\mathbb{E}_t'^{\mathbb{P}}[(\sigma_t^{\mathbb{Q}})^2] - (\sigma_t^{\mathbb{Q}})^2)dt, \quad \mathrm{RP}_T = \int_0^T (\mathbb{E}_t'^{\mathbb{Q}}[(\sigma_t^{\mathbb{Q}})^2] - \mathbb{E}_t'^{\mathbb{P}}[(\sigma_t^{\mathbb{Q}})^2])dt. \tag{3.19}$$

Remark 3. The RVD measure defined in (3.3) stands in contrast to a point-in-time (static) estimator of integrated variance disagreement given by

$$\mathrm{RVD}_T^{n,\text{static}} = V_T^{\mathbb{Q}} - V_T^{n,\mathbb{P}}, \tag{3.20}$$

which compares $V_T^{n,\mathbb{P}}$, a nonparametric estimate of the integrated diffusive variance $\int_0^T (\sigma_t^{\mathbb{P}})^2 dt$ constructed from high-frequency returns in the interval $[0,T]$, and $V_T^{\mathbb{Q}}$, a nonparametric estimate of $\int_0^T (\sigma_t^{\mathbb{Q}})^2 dt$ from options observed at time $t=0$ and expiring at time $t=T$. The two nonparametric volatility estimates $V_T^{n,\mathbb{P}}$ and $V_T^{\mathbb{Q}}$ are constructed over different frequencies: the option-based one uses conditional moments of returns over the entire interval $[0,T]$, while the return-based one makes use of high-frequency returns within $[0,T]$ of length Δ_n, with $T/\Delta_n \to \infty$. This mismatch of frequencies leads to biases of different asymptotic orders in $V_T^{n,\mathbb{P}}$ and $V_T^{\mathbb{Q}}$ due to nuisance features in the return dynamics such as jumps, stochastic volatility as well as intraday deterministic patterns in volatility. By contrast, these biases approximately cancel out in the construction of RVD_T^n. This is because the estimates of $\mathbb{P}$- and $\mathbb{Q}$-variance, $\widehat{V}^{\mathbb{Q}}_{\underline{t}_i^n,\overline{t}_i^n}(\widehat{u}_{(i-2)\Delta_n})$ and $\widehat{V}^{\mathbb{P}}_{\underline{t}_i^n,\overline{t}_i^n}(\widehat{u}_{(i-2)\Delta_n})$, are formed over the same high-frequency intervals and use the same characteristic exponents. To achieve this bias reduction and robustness to the above-mentioned nuisance features in the asset dynamics, RVD_T^n requires high-frequency option data, as opposed to $\mathrm{RVD}_T^{n,\text{static}}$ which only uses static option data at one point in time.

3.2 The Asymptotic Distribution of Realized Variance Disagreement

The main theoretical result of this paper, stated in the next theorem, determines the asymptotic distribution of realized variance disagreement. Below, $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}})$ denotes an extension of the original probability space that supports the option observation errors. It is formally introduced in (A.2) in Appendix A.

Theorem 1. *Suppose that Assumptions 1–5 from Appendix B hold and that the characteristic exponent $\widehat{u}_t$ satisfies (3.6) and (B.2). Then*

$$\frac{\mathrm{RVD}_T^n - \varphi \mathrm{IVD}_{\theta T}}{\sqrt{k_n[(\Delta'_n)^2 \mathrm{AVar}_T^{\mathbb{P}} + \delta T^{3/2} \mathrm{AVar}_T^{\mathbb{Q}}]}} \xrightarrow{\mathcal{L}-s} N(0,1), \tag{3.21}$$

where $\mathrm{AVar}_T^{\mathbb{P}} = \frac{2}{\theta T}\int_0^{\theta T}(\sigma_t^{\mathbb{P}})^4 dt$ *and* $\mathrm{AVar}_T^{\mathbb{Q}}$ *is defined in* (C.3) *in Appendix* C*. If* ℓ_n *is an integer sequence satisfying* $\ell_n \to \infty$ *and* $\ell_n/k_n \to 0$ *and*

$$\begin{aligned}\widehat{\mathcal{V}}_T^n &= \sum_{i=2}^{k_n}(\widehat{Z}_{\underline{t}_i^n,\bar{t}_i^n}(\widehat{u}_{(i-2)\Delta_n}))^2,\\ \widehat{\mathcal{M}}_T^n &= \ell_n \sum_{j=1}^{\lfloor k_n/\ell_n \rfloor}(\widehat{M}_{j,n}(\widehat{u}_{(i-2)\Delta_n})^2 \quad \textit{with} \quad \widehat{M}_{j,n}(u) = \frac{1}{\ell_n}\sum_{i=(j-1)\ell_n+1}^{j\ell_n}\widehat{Z}_{\underline{t}_i^n,\bar{t}_i^n}(u),\end{aligned} \tag{3.22}$$

where for $u > 0$ *and* $0 \le t_1 < t_2 < T$,

$$\widehat{Z}_{t_1,t_2}(u) = \widehat{V}_{t_1,t_2}^{\mathbb{Q}}(u) - \widehat{V}_{t_1,t_2}^{\mathbb{P}}(u), \tag{3.23}$$

then $\frac{1}{k_n(\Delta_n')^2}(\widehat{\mathcal{V}}_T^n - \widehat{\mathcal{M}}_T^n)$ *is a consistent estimator of the asymptotic* $\overline{\mathcal{F}}$*-conditional variance and*

$$\frac{\mathrm{RVD}_T^n - \varphi \mathrm{IVD}_{\theta T}}{\sqrt{\widehat{\mathcal{V}}_T^n - \widehat{\mathcal{M}}_T^n}} \xrightarrow{\mathcal{L}-s} N(0,1). \tag{3.24}$$

Several comments are in order. First, the convergence in distribution in (3.21) is stable, see Chapter 2.2.1 of Jacod and Protter (2012), which is important for valid inference when the $\overline{\mathcal{F}}$-conditional asymptotic variance is random. It ensures that studentization by a consistent estimator of the $\overline{\mathcal{F}}$-conditional variance results in convergence to a standard normal limit in (3.24). Moreover, the statistic in (3.24) is of self-normalized type, which implies that it is scale-free.

Second, the statistic is built over a given interval of length θT and uses k_n increments of length Δ_n' sampled at frequency Δ_n. These quantities, as mentioned earlier, are picked by the econometrician. In applications using zero-date options, the maturity T of the options is usually fixed, but θ can be selected based on the econometrician's interval of interest and the availability of strikes in the option data satisfying the conditions of the theorem. For example, in the Monte Carlo and empirical sections, we compute RVD_T^n up to half an hour before market close. Because there are time gaps between consecutive returns in RVD_T^n, the constant φ appears in (3.21). Taking RVD_T^n/φ yields a consistent estimator of $\mathrm{IVD}_{\theta T}$.

Third, the rate of convergence of RVD_T^n is naturally determined by the number of terms in the statistic, k_n. The permissible order of k_n in turn depends on the degree of jump activity and the smoothness of volatility and the jump intensity (see Assumptions 1–3). Our theorem sets k_n in a way that does not assume prior knowledge about these features, enabling robust inference on integrated variance disagreement under very general specifications including infinite variation jumps and rough volatility (Gatheral et al. (2018), Chong and Todorov (2025c)). If, however, the econometrician possesses additional information (e.g., an estimate of jump activity and volatility smoothness from high-frequency return data), then k_n can be chosen adaptively to increase the rate of convergence in the theorem. For simplicity, we do not consider such extensions here.

Fourth, the current choice (3.6) makes the diffusive asset price component more prominent relative to asset price jumps. This allows us to conduct inference on integrated variance disagreement in a jump-robust way without making (semi)parametric assumptions on the jump distribution. If, on the other hand, one is interested in the no-arbitrage restrictions concerning infinite activity price jumps, then one needs a larger rate of growth of the characteristic exponent than the one we are currently using. We leave the exploration of this for future work.

Fifth, there are mainly two sources of error that drive the asymptotic variance of RVD_T^n. On the one hand, the component $\mathrm{AVar}_T^{\mathbb{P}}$ captures the high-frequency variation of the terms $\widehat{V}^{\mathbb{P}}_{\underline{t}_i^n,\bar{t}_i^n}(u) - \mathbb{E}^{\mathbb{P}}_{\underline{t}_i^n}[\widehat{V}^{\mathbb{P}}_{\underline{t}_i^n,\bar{t}_i^n}(u)]$ (i.e., the approximation error in the first step in (3.7)). This component is identical to the asymptotic variance of the classical realized variance estimator (Barndorff-Nielsen and Shephard (2002)). On the other hand, the component $\mathrm{AVar}_T^{\mathbb{Q}}$ captures variance due to option observation errors, which render $\widehat{\mathcal{L}}^{\mathbb{Q}}_{t,T}(u)$ a noisy estimate of $\mathcal{L}^{\mathbb{Q}}_{t,T}(u)$. The relative size of these two errors depends on the length of the high-frequency return intervals, Δ_n', the mesh of the strike grid of the available options, δ, and the maturity of the options, T. The precise definition of δ is given in Assumption 4 in Appendix B. We note that Theorem 1 applies regardless of whether the first, the second, or both errors play an asymptotic role in the limiting distribution.

Finally, the result in Theorem 1 is related to Jacod and Todorov (2014) and Todorov (2019), with important differences between these papers and the current work. Jacod and Todorov (2014) estimate integrated volatility from nonlinear transforms of local empirical characteristic functions of returns over shrinking blocks. By contrast, here we avoid making nonlinear transformations of local statistics (and avoid the associated asymptotic biases resulting from this) by choosing suitable characteristic exponents. Todorov (2019) derives a CLT for an option-based estimator of volatility given by $-2\log|\widehat{\mathcal{L}}^{\mathbb{Q}}_{t,T}(\widehat{u}_t)|/\widehat{u}_t^2$, for fixed t. By contrast, here we use ratios of option-implied characteristic functions integrated over high-frequency intervals, whose asymptotic behavior is quite different from that of the option-based volatility estimator of Todorov (2019). In addition, Theorem 1 is a joint asymptotic result for return- and option-based volatility estimators in which asymptotic biases in the two measures cancel out. Hence, the asymptotic behavior of our statistic is established under significantly weaker conditions on the volatility dynamics and the jumps than those required in Jacod and Todorov (2014) and Todorov (2019). For example, Jacod and Todorov (2014) and Todorov (2019) place much stronger upper bounds on the jump activity in asset prices and lower bounds on the smoothness of volatility than Assumptions 1–3. In contrast to Jacod and Todorov (2014), we do not need to make semiparametric assumptions on the distribution of asset price jumps.

The delta method yields the asymptotic distribution of rRVD_T^n.

Theorem 2. *Under the same assumptions as in Theorem 1, we have*

$$\frac{\mathrm{rRVD}_T^n - \mathrm{rIVD}_{\theta T}}{\sqrt{\mathrm{AVar}(\mathrm{rRVD}_T^n)}} \xrightarrow{\mathcal{L}-s} N(0,1), \tag{3.25}$$

where

$$\text{AVar}(\text{rRVD}_T^n) = k_n(\Delta_n')^2 \frac{(\int_0^{\theta T}(\sigma_t^{\mathbb{Q}})^2 dt)^2}{(\int_0^{\theta T}(\sigma_t^{\mathbb{P}})^2 dt)^4} \text{AVar}_T^{\mathbb{P}} + k_n \delta T^{3/2} \frac{1}{(\int_0^{\theta T}(\sigma_t^{\mathbb{P}})^2 dt)^2} \text{AVar}_T^{\mathbb{Q}}. \tag{3.26}$$

Moreover, if we define

$$\widehat{\text{AVar}}(\text{rRVD}_T^n) = \frac{(\widehat{\text{VQ}}_T^n)^2}{(\widehat{\text{VP}}_T^n)^4} \widehat{\text{AVar}}_T^{\mathbb{P},n} + \frac{1}{(\widehat{\text{VP}}_T^n)^2} \widehat{\text{AVar}}_T^{\mathbb{Q},n}, \tag{3.27}$$

where

$$\widehat{\text{AVar}}_T^{\mathbb{P},n} = \frac{1}{2} \sum_{i=3}^{k_n} \Big(\widehat{V}^{\mathbb{P}}_{\underline{t}_i^n, \bar{t}_i^n}(\widehat{u}_{(i-2)\Delta_n}) - \widehat{V}^{\mathbb{P}}_{\underline{t}_{i-1}^n, \bar{t}_{i-1}^n}(\widehat{u}_{(i-3)\Delta_n}) \Big)^2, \tag{3.28}$$

$$\widehat{\text{AVar}}_T^{\mathbb{Q},n} = \frac{1}{2} \sum_{i=3}^{k_n} \Big(\widehat{V}^{\mathbb{Q}}_{\underline{t}_i^n, \bar{t}_i^n}(\widehat{u}_{(i-2)\Delta_n}) - \widehat{V}^{\mathbb{Q}}_{\underline{t}_{i-1}^n, \bar{t}_{i-1}^n}(\widehat{u}_{(i-3)\Delta_n}) \Big)^2, \tag{3.29}$$

then we have the following feasible central limit theorem:

$$\frac{\text{rRVD}_T^n - \text{rIVD}_{\theta T}}{\sqrt{\widehat{\text{AVar}}(\text{rRVD}_T^n)}} \xrightarrow{\mathcal{L}-s} N(0,1). \tag{3.30}$$

In Appendix D, we present an extension of Theorem 2 that covers inference on relative IVD over multiple days. The main additional challenge consists in identifying bias terms that are of higher order in (3.25) but may no longer be so under time aggregation. On a single day, the results of Appendix D constitute finite-sample improvements over Theorem 2.

4 Testing Market Integration

4.1 A Market Integration Test for Integrated Variance Disagreement

Thanks to the no-arbitrage condition (2.5), Theorem 1 immediately yields a market integration test based on whether integrated variance disagreement is zero or not. The associated market integration test statistic is given by

$$\text{MIT}_T^n = \frac{\text{RVD}_T^n}{\sqrt{\widehat{\mathcal{V}}_T^n - \widehat{\mathcal{M}}_T^n}}, \tag{4.1}$$

where RVD_T^n is the realized variance disagreement measure defined in (3.3) and $\widehat{\mathcal{V}}_T^n$ and $\widehat{\mathcal{M}}_T^n$ are defined in (3.22).

The no-arbitrage condition (2.5) implies that the variance disagreement between integrated equity and options markets is zero. In this case, the terms $\widehat{Z}_{\underline{t}_i^n, \bar{t}_i^n}(\widehat{u}_{(i-2)\Delta_n})$ in (3.3) form an approximate martingale difference sequence and the market integration test statistic is approximately normally distributed by (3.24). By contrast, if the integrated variance disagreement between the two markets is different from zero, (3.24) implies that the absolute value of MIT_T^n diverges to infinity.

The formal description of the null and alternative hypotheses requires care due to the shrinking horizon of the economy. Recognizing that even on short horizons such as one day, volatility can exhibit strong deterministic time-of-day patterns, which may in addition vary from day to day (Andersen et al. (2024)), we model diffusive variance under $\mathbb{P}$ and $\mathbb{Q}$ as

$$(\sigma_t^{\mathbb{P}})^2 = \xi(t/T)\varsigma_t^2, \quad (\sigma_t^{\mathbb{Q}})^2 = (\sigma_t^{\mathbb{P}})^2 + \zeta(t/T)\rho_t, \quad t \in [0, T]. \tag{4.2}$$

Here, ξ is a deterministic function capturing the time-of-day pattern of diffusive variance under $\mathbb{P}$, while ς_t^2 denotes its stochastic component. Similarly, ζ and ρ describe the deterministic and random components in the potential gap between diffusive $\mathbb{P}$- and $\mathbb{Q}$-variance.[8] While ξ and ς_t are nonnegative, there are no restrictions on the sign of ζ and ρ, so $\mathbb{Q}$ can potentially over- or undervalue diffusive variance. Without loss of generality, we assume $\rho_0 \neq 0$ almost surely, so that condition (2.5) corresponds to $\zeta \equiv 0$. Additional regularity assumptions are stated in the Appendix.[9]

Theorem 3. *Consider MIT_T^n from (4.1) as a test statistic to discriminate between the following two hypotheses:*

$$H_0 : \int_0^{\theta} \zeta(u)du = 0, \qquad H_1 : \int_0^{\theta} \zeta(u)du \neq 0. \tag{4.3}$$

If the assumptions of Theorem 1 are satisfied and $(\Delta_n')^2/[(\Delta_n')^2 + \delta T^{3/2}]$ converges to a strictly positive number, then

$$\begin{cases} \mathrm{MIT}_T^n \xrightarrow{\mathcal{L}-s} N(0,1) & \text{under } H_0, \\ |\mathrm{MIT}_T^n| \xrightarrow{\mathbb{P}} \infty \text{ at rate } \sqrt{k_n} & \text{under } H_1. \end{cases} \tag{4.4}$$

The hypotheses in Theorem 3 concern the disagreement between $\mathbb{P}$ and $\mathbb{Q}$, as represented by ζ, about the time-of-day effect of volatility. This deterministic pattern, on average, constitutes the main feature of volatility in intraday asset returns. The market integration test statistic from (4.1) is able to detect differences in the integrated time-of-day effect between $\mathbb{P}$ and $\mathbb{Q}$. No difference, however, does not mean that integrated variance disagreement, $\mathrm{IVD}_{\theta T}$, is exactly zero. We can have a situation where $\int_0^{\theta} \zeta(u)du = 0$ but $\mathrm{IVD}_{\theta T} = -\int_0^{\theta} \zeta(u)\rho_{uT}du \neq 0$ due to the variation generated by ρ over the interval $[0, \theta T]$. Nevertheless, this variation is of higher order, i.e., we have $\mathrm{IVD}_{\theta T} = -\rho_0 \int_0^{\theta} \zeta(u)du + o_p(1) = o_p(1)$ as $T \downarrow 0$, so integrated variance disagreement is approximately zero in this case. The assumption of a positive limit of

[8] The subsequent results clearly extend to the case where ξ and ζ are random but $\mathcal{F}_0$-measurable. This is important if we aggregate the market integration test statistic over multiple days. In this case, ξ and ζ are allowed to vary randomly from one day to another.

[9] Modeling spot variance via (4.2) formally assumes that we consider sequences of physical measures $\mathbb{P}_T$ and risk-neutral measures $\mathbb{Q}_T$ under which we have (2.1)–(2.3) but with $\mathbb{P}$ and $\mathbb{Q}$ replaced by $\mathbb{P}_T$ and $\mathbb{Q}_T$, respectively. To keep notation simple, we omit the subscript T. For instance, we continue to write $\mathbb{P} = \mathbb{P}_T$, $\mathbb{Q} = \mathbb{Q}_T$, $\sigma_t^{\mathbb{P}} = \sigma_t^{\mathbb{P}_T}$ and $\sigma_t^{\mathbb{Q}} = \sigma_t^{\mathbb{Q}_T}$.

$(\Delta_n')^2/[(\Delta_n')^2 + \delta T^{3/2}]$ ensures that option observation errors do not dominate the test statistic, which would lead to a loss of power under the alternative hypothesis.

If the equity and options markets disagree about diffusive variance at specific time points but agree about integrated variance over the whole day, the power of a test based on (4.1) will not exceed the nominal size of the test. Next, we develop a test that holds an asymptotic power of 1 even against such short periods of variance disagreement.

4.2 A Market Integration Test for Spot Variance Disagreement

We design a modified market integration test that can detect disagreement between the equity and options markets about *spot* variance. Because of (2.5), it is also a test for the existence of an SDF relating $\mathbb{P}$ and $\mathbb{Q}$. We introduce the weighted statistic

$$\mathrm{RVD}_T^n(\eta) = \sum_{i=2}^{k_n} w_\eta(\tfrac{i-1}{k_n})\widehat{Z}_{\underline{t}_i^n,\overline{t}_i^n}(\widehat{u}_{(i-2)\Delta_n}), \tag{4.5}$$

where $(w_\eta)_{\eta\in[0,2\pi]}$ is a family of weight functions, and consider the supremum test statistic

$$\mathrm{MIT}_T^{*n} = \frac{1}{\sqrt{k_n}\Delta_n'} \sup_{\eta\in[0,2\pi]} |\mathrm{RVD}_T^n(\eta)|. \tag{4.6}$$

The weight functions w_η are chosen in such a way that any deviation of $\mathbb{P}$- and $\mathbb{Q}$-spot variance results in an abnormally large value of the test statistic. We follow Bierens and Ploberger (1997) and choose

$$w_\eta(t) = \sin(\eta t) + \cos(\eta t).$$

The next theorem shows that the test statistic (4.6) is able to distinguish between

$$H_0^* : \zeta \equiv 0 \text{ on } [0,\theta], \qquad H_1^* : \zeta \not\equiv 0 \text{ on } [0,\theta], \tag{4.7}$$

where ζ is defined in (4.2) and determines whether there is a $\mathbb{P}$–$\mathbb{Q}$ gap in diffusive variance. In order to determine the critical region of the test, we apply a wild bootstrap procedure. We fix an integer B, the number of bootstrap samples, and consider a sequence $(\chi_{i,b})_{i,b\geq 1}$ of independent standard normal random variables defined on a very good extension $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$ of $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}})$. For $b = 1, \dots, B$, we define

$$\mathrm{RVD}_T^{*n,b}(\eta) = \sum_{i=2}^{k_n} w_\eta(\tfrac{i-1}{k_n})\widehat{Z}_{\underline{t}_i^n,\overline{t}_i^n}(\widehat{u}_{(i-2)\Delta_n})\chi_{i,b}$$

and let $\widehat{Q}_{n,T}^B(p)$ be the p-quantile of the sample $\{\frac{1}{\sqrt{k_n}\Delta_n'} \sup_{\eta\in[0,2\pi]}|\mathrm{RVD}_T^{*n,b}(\eta)| : b = 1, \dots, B\}$.

Theorem 4. *Suppose that the assumptions behind Theorem 3 are valid. Under H_0^*,*

$$\mathrm{MIT}_T^{*n} \xrightarrow{\mathcal{L}-s} Z_{\max} = \sup_{\eta\in[0,2\pi]} |Z_\eta|, \tag{4.8}$$

where $Z = (Z_\eta)_{\eta\in[0,2\pi]}$ is an $\overline{\mathcal{F}}$-conditional Gaussian process, defined on a very good extension $(\Omega', \mathcal{F}', \mathbb{P}')$ of $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}})$, with mean 0 and $\overline{\mathcal{F}}$-conditional covariance function given in (C.4). Under H_1^*, we have

$$\mathrm{MIT}_T^{*n} \xrightarrow{\mathbb{P}} \infty. \tag{4.9}$$

In particular, for any $\alpha \in (0,1)$,

$$\lim_{n\to\infty, T\to 0} \lim_{B\to\infty} \widetilde{\mathbb{P}}\left(\mathrm{MIT}_T^{*n} > \widehat{Q}_{n,T}^B(1-\alpha)\right) = \begin{cases} \alpha & \text{under } H_0^*, \\ 1 & \text{under } H_1^*. \end{cases} \tag{4.10}$$

5 Monte Carlo Study

We now evaluate the finite-sample performance of the newly developed inference procedures in empirically realistic settings.

5.1 Setup

We use the following model for the dynamics of X under $\mathbb{S} = \mathbb{P}, \mathbb{Q}$ in the Monte Carlo study:

$$\begin{aligned} \frac{dX_t}{X_{t-}} &= b^{\mathbb{S}}dt + \sigma_t^{\mathbb{S}} dW_t^{\mathbb{S}} + \int_{\mathbb{R}} (e^z - 1)(\mu - \nu^{\mathbb{S}})(dt, dz), \quad V_t^{\mathbb{S}} = (\sigma_t^{\mathbb{S}})^2, \\ dV_t^{\mathbb{S}} &= \kappa(\theta - V_t^{\mathbb{S}})dt + \sigma\sqrt{V_t^{\mathbb{S}}}(\rho dW_t^{\mathbb{S}} + \sqrt{1-\rho^2} d\widetilde{W}_t^{\mathbb{S}}), \\ \nu^{\mathbb{S}}(dt, dz) &= V_t^{\mathbb{P}} \left(c_- \frac{e^{-\lambda_-^{\mathbb{S}}|z|}}{|z|^{1+\alpha}} 1_{\{z<0\}} + c_+ \frac{e^{-\lambda_+^{\mathbb{S}}|z|}}{|z|^{1+\alpha}} 1_{\{z>0\}} \right) dtdz, \end{aligned} \tag{5.1}$$

where $W^{\mathbb{S}}$ and $\widetilde{W}^{\mathbb{S}}$ are two independent Brownian motions and μ is an integer-valued random measure, counting the jumps in the log-price, with compensator $\nu^{\mathbb{S}}$ under $\mathbb{S}$. Parameters that might differ under the two probability measures are indicated by a superscript.

Diffusive volatility follows a Heston model while the jumps in the model follow a time-changed tempered stable process, with the time change being diffusive variance. We calibrate the parameters of the model to roughly match key empirical features of market index dynamics and options written on it. In particular, we set the scale parameters c_- and c_+ as

$$c_- = 0.1 \times \frac{(\lambda_-^{\mathbb{P}})^{2-\alpha}}{\Gamma(2-\alpha)} \quad \text{and} \quad c_+ = 0.1 \times \frac{(\lambda_+^{\mathbb{P}})^{2-\alpha}}{\Gamma(2-\alpha)}. \tag{5.2}$$

This implies that jump variance is 20% of diffusive variance. Setting $\lambda_-^{\mathbb{P}} = \lambda_+^{\mathbb{P}}$, we further have that the jump compensator $\nu^{\mathbb{P}}$ is symmetric around zero. This is roughly consistent with empirical evidence reported in earlier work about the statistical jump distribution. We consider different values for the parameter α, which controls the jump activity. In particular, we consider both finite and infinite activity jumps. The drift under $\mathbb{P}$ is $b^{\mathbb{P}} = 0.05$, while we have $b^{\mathbb{Q}} = 0$ because of the first identity in (2.2).

To keeps things simple, in the case of no arbitrage, the compensation for variance risk, known as the variance risk premium, is generated by a compensation for negative jump risk only and without any compensation for the time variation in diffusive variance. This is why, in the case of no arbitrage, i.e., when $V_t^{\mathbb{Q}} = V_t^{\mathbb{P}}$, these identical processes have the same dynamics (expressed by κ and θ) under $\mathbb{P}$ and $\mathbb{Q}$. The risk premium for negative jumps is generated by setting

$$\left(\frac{\lambda_-^{\mathbb{P}}}{\lambda_-^{\mathbb{Q}}}\right)^{2-\alpha} = \frac{0.1 + 0.2 \times 1.2}{0.1}, \quad \lambda_+^{\mathbb{Q}} = \lambda_+^{\mathbb{P}}. \tag{5.3}$$

In this case, the variance risk premium is 20% of the variance risk in the specifications without arbitrage. This is similar to earlier estimates reported for market index options.

Finally, we set

$$\sigma_t^{\mathbb{Q}} = AW \times \sigma_t^{\mathbb{P}}, \quad \text{for some constant } AW > 0. \tag{5.4}$$

Hence, $AW = 1$ corresponds to integrated equity and options markets satisfying (2.5), $AW > 1$ to the case where the options market overvalues diffusive volatility and $AW < 1$ to the case where the options market undervalues diffusive volatility. The parameters for all specifications are given in Table 1. We consider situations with both small and large disagreement about diffusive volatility.

Table 1: **Parameter Setting for the Monte Carlo**

Case		Variance Parameters				Jump Parameters						
	AW	θ	κ	σ	ρ	α	$\lambda_-^{\mathbb{P}}$	$\lambda_-^{\mathbb{Q}}$	$\lambda_+^{\mathbb{P}}$	$\lambda_+^{\mathbb{Q}}$	c_-	c_+
NA-J1	1.0	0.02	8	0.2	−0.9	−0.5	100	61	100	100	7523	7523
NA-J2	1.0	0.02	8	0.2	−0.9	0.5	100	44	100	100	113	113
AL1-J1	0.9	0.02	8	0.2	−0.9	−0.5	100	100	100	100	7523	7523
AL1-J2	0.9	0.02	8	0.2	−0.9	0.5	100	100	100	100	113	113
AL2-J1	0.8	0.02	8	0.2	−0.9	−0.5	100	100	100	100	7523	7523
AL2-J2	0.8	0.02	8	0.2	−0.9	0.5	100	100	100	100	113	113
AH1-J1	1.1	0.02	8	0.2	−0.9	−0.5	100	100	100	100	7523	7523
AH1-J2	1.1	0.02	8	0.2	−0.9	0.5	100	100	100	100	113	113
AH2-J1	1.2	0.02	8	0.2	−0.9	−0.5	100	100	100	100	7523	7523
AH2-J2	1.2	0.02	8	0.2	−0.9	0.5	100	100	100	100	113	113

5.2 Observation Scheme and Choice of Tuning Parameters

We set $X_0 = 5{,}000$ and draw $V_0^{\mathbb{P}}$ from its stationary distribution. Options are observed on the interval $[0, T]$, with $T = 1/252$ corresponding to one business day. We consider sampling between 5 minutes after market open until 30 minutes prior to market close. We consider two values for Δ_n': $\Delta_n' = (1/252) \times (1/78)$, which corresponds to a 5-minute sampling frequency, and $\Delta_n' = (1/252) \times (1/390)$, which corresponds to a 1-minute sampling frequency in a trading day of 6.5 hours. We create 1-minute gaps between intervals over which the summands in the statistic are computed by setting Δ_n to 6 minutes in the first and to 2 minutes in the second case. In the 5-minute (1-minute) sampling case, this leads to $k_n = 59$ ($k_n = 178$).

At each observation time, out-of-the-money options on the underlying asset are generated on a strike grid given by multiples of 5, resulting in a gap of 5 between strikes. The highest and lowest strikes are the highest and lowest values on the strike grid for which the true option price is above 0.075. This option observation scheme mimics roughly ones available in real data. The option mid-quote is the true option price plus an error that equals 0.025 times a standard normal random variable multiplied by the true option price. The observation errors are $\mathcal{F}$-conditionally independent across time and across strikes.

The characteristic exponent from (3.6) is chosen in an adaptive data-driven way. First, denote

$$\widehat{u}_t^{(1)} = \sqrt{\frac{-2\log(0.2)}{(\widehat{RV}_{t,T}^{\mathbb{Q}} \vee (T-t)\,\underline{v}) \wedge (T-t)\,\overline{v}}}, \tag{5.5}$$

for some finite constants $0 < \underline{v} < \overline{v} < \infty$ and where $\widehat{RV}_{t,T}^{\mathbb{Q}}$ is a measure of risk-neutral conditional return variance over the interval $[t, T]$ that is given in Appendix A. In all applications, we set $\underline{v} = 0.001$ and $\overline{v} = 0.5$, which provide lower and upper bounds on annualized risk-neutral variance. We further let

$$\widehat{u}_t^{(2)} = \inf\{u \geq 0 : |\widehat{\mathcal{L}}_{t,T}^{\mathbb{Q}}(u)| \leq 0.2\} \tag{5.6}$$

and define

$$\widehat{u}_t = \operatorname{argmin}\Big\{|\widehat{\mathcal{L}}_{t,T}^{\mathbb{Q}}(u)| : u \in [1, \widehat{u}_t^{(1)} \wedge \widehat{u}_t^{(2)}]\Big\}. \tag{5.7}$$

This choice of $\widehat{u}_t$ aims at using characteristic exponents for which $|\widehat{\mathcal{L}}_{t,T}^{\mathbb{Q}}(u)|$ is approximately 0.2.

In assessing the market integration test based on spot variance disagreement, we consider aggregation of the test statistic to a single day and five consecutive days of observations. Moreover, we approximate $\sup_{\eta\in[0,2\pi]}|\mathrm{RVD}_T^n(\eta)|$ and $\sup_{\eta\in[0,2\pi]}|\mathrm{RVD}_T^{*n,b}(\eta)|$ by taking the maximum over an equidistant grid of eleven points covering the interval $[0, 2\pi]$.

5.3 Results

We start with evaluating the finite-sample behavior of confidence intervals for rIVD_T, constructed using the result in Theorem 2. The Monte Carlo results are reported in Table 2. They show that the constructed confidence intervals based on the asymptotic limit result exhibit a slight undercoverage. This effect is more pronounced for the coarser sampling frequency of five minutes. One reason is that for moderate sizes of k_n, the left-hand side of (3.24) is closer in distribution to

$$\frac{\sum_{i=1}^{k_n}(Z_i^2-1)}{\sqrt{\sum_{i=1}^{k_n}(Z_i^2-1)^2-(\sum_{i=1}^{k_n}(Z_i^2-1))^2/k_n}}, \quad Z_i \overset{\text{i.i.d.}}{\sim} N(0,1),$$

which has longer tails than the standard normal distribution. A bootstrap approach, similar to the one employed for the test in Section 4.2, can correct for this finite-sample distortion.

Table 2: Monte Carlo Results: Confidence Intervals for rIVD_T

Case	5-minute frequency		1-minute frequency	
	$\gamma = 0.90$	$\gamma = 0.95$	$\gamma = 0.90$	$\gamma = 0.95$
NA-J1	0.8796	0.9328	0.8782	0.9324
NA-J2	0.8618	0.9154	0.8740	0.9332
AL1-J1	0.8786	0.9324	0.8900	0.9450
AL1-J2	0.8730	0.9250	0.8828	0.9322
AL2-J1	0.8762	0.9360	0.8800	0.9366
AL2-J2	0.8750	0.9304	0.8790	0.9386
AH1-J1	0.8542	0.9130	0.8746	0.9312
AH1-J2	0.8358	0.9012	0.8662	0.9212
AH2-J1	0.8656	0.9186	0.8790	0.9308
AH2-J2	0.8322	0.8952	0.8660	0.9204

Note: Reported results are empirical coverage rates of two-sided confidence intervals for rIVD_T with confidence level of γ based on 1,000 replications.

We proceed next with the Monte Carlo results for the market integration test from Section 4.2. They are reported in Table 3. These results show that the test has overall good behavior under the null hypothesis. This holds true for both considered sampling schemes and for both levels of jump activity.

As expected, the power of the test depends on the size of the disagreement between the equity and options markets about diffusive spot volatility and the level of aggregation. In particular, when option-implied spot diffusive volatility is 10% below or above its underlying asset-based counterpart and the level of aggregation is one day, the power is quite low. The power increases significantly when the difference between $\sigma_t^{\mathbb{P}}$ and $\sigma_t^{\mathbb{Q}}$ is 20% and/or when we aggregate over five consecutive days.

Higher jump activity tends to result in a small decrease in the power of the test as it makes the statistic somewhat noisier. This effect is of higher asymptotic order but it is nevertheless noticeable in finite samples. Finally, the increase of the sampling frequency leads to some loss in power of the test. This is most noticeable in the scenarios in which $\sigma_t^{\mathbb{Q}}$ is above $\sigma_t^{\mathbb{P}}$. The reason for this loss of power when the sampling frequency increases is the option observation error. When we increase the sampling frequency, $\widehat{\mathrm{VQ}}_T^n$ gets noisier while the precision of $\widehat{\mathrm{VP}}_T^n$ improves. In our Monte Carlo experiment, the first effect dominates the second one when switching from five-minute to one-minute sampling.[10] Overall, the reported Monte Carlo results indicate good finite-sample behavior of the test in empirically realistic scenarios.

Table 3: Monte Carlo Results: Test for Spot Variance Disagreement

Case	**Aggregation: 1 day**				**Aggregation: 5 days**			
	5-minute frequency		1-minute frequency		5-minute frequency		1-minute frequency	
	$\alpha = 0.10$	$\alpha = 0.05$	$\alpha = 0.10$	$\alpha = 0.05$	$\alpha = 0.10$	$\alpha = 0.05$	$\alpha = 0.10$	$\alpha = 0.05$
NA-J1	0.0972	0.0458	0.0986	0.0502	0.0910	0.0420	0.1080	0.0510
NA-J2	0.0998	0.0452	0.0912	0.0418	0.1040	0.0420	0.1140	0.0520
AL1-J1	0.1384	0.0694	0.1220	0.0580	0.4070	0.2710	0.2380	0.1640
AL1-J2	0.1284	0.0610	0.1174	0.0636	0.4310	0.2870	0.2680	0.1630
AL2-J1	0.3728	0.2312	0.2214	0.1356	0.9420	0.8930	0.6410	0.5340
AL2-J2	0.3154	0.1950	0.2022	0.1200	0.9140	0.8530	0.5810	0.4530
AH1-J1	0.1710	0.0994	0.1052	0.0532	0.3030	0.2120	0.1120	0.0590
AH1-J2	0.1534	0.0898	0.1078	0.0540	0.2200	0.1500	0.1030	0.0550
AH2-J1	0.2892	0.1948	0.1196	0.0594	0.6750	0.5990	0.1910	0.1040
AH2-J2	0.2586	0.1754	0.1114	0.0584	0.5500	0.4600	0.1650	0.0960

Note: Reported results are empirical rejection rates of the test based on 1,000 replications with size α.

6 Empirical Assessment of Market Integration

6.1 Data

We examine the integration of equity and options markets for the following underlying assets: SPY (an ETF tracking the S&P 500 market index), QQQ (an ETF tracking the NASDAQ 100 index), Amazon and Tesla.[11] The data covers the period 2020–2024. Zero-date options for SPY and QQQ for the first part of the sample were available on Mondays, Wednesdays and Fridays, and on every trading day for the rest of the sample. Zero-date options for Amazon and Tesla are available only on Fridays throughout the sample.

[10] In fact, for a given level of option observation error and physical diffusive variance, there is an optimal sampling frequency for realized variance disagreement that achieves the lowest asymptotic variance. To keep things simple, we do not present this result here.

[11] The options considered in the empirical analysis are of American style, which allow their holders to exercise them before the expiration date. While our theory builds on European-style options, it continues to apply when options are of American style. In particular, in absence of dividends (which is always the case for the zero-date options that we work with), it is never optimal to exercise early, see e.g., Merton (1973). For deep in-the-money puts, it can be optimal to exercise them early in order to gain the interest on the strike. But if the interest rate is zero, then it is also never optimal to exercise puts before expiration, see e.g., Schroder (2015). Even if interest rates are different from zero, one can show that the early exercise premium for out-of-the-money options is asymptotically negligible. As existing evidence suggests that the impact of the early exercise premium on volatility estimation is rather small empirically (Carr and Figà-Talamanca (2020)), we do not show this formally. As a further confirmation of the negligible role of the early exercise premium for our results, an analysis of S&P 500 index options, which are European style, yields very similar results to what we find for SPY options.

We apply standard filters to the options data and remove strikes with zero bids for out-of-the-money options. We remove days on which the minimum number of available strikes with nonzero bids for out-of-the-money options at any sampling point during the trading day is below five. Similarly, we discard days for which at any sampling point there are fewer than two out-of-the-money calls or puts. The moneyness of the options is with respect to implied forward extracted via put–call parity from the three available strikes with minimum put–call spread. We take the median of these three implied forward estimates. Finally, we remove from the sample days in which there are more than 20% of zero five-minute returns for the underlying asset. This filter discards mostly days with partial trading around holidays. We also remove days with FOMC announcements.

We sample the equity and options data on each day from 8:35 CST until 14:30 CST for SPY and QQQ and from 8:35 CST until 13:30 CST for Amazon and Tesla. The reason for stopping slightly earlier in the day for the two individual stocks is to guarantee better strike coverage, which is more of an issue towards market close for individual name options than for ETF options.

6.2 Test Results

We implement the market integration test from Section 4.2 on the data at one-minute and five-minute frequencies, aggregated to one and five consecutive days. When forming the statistic, the characteristic exponent is set exactly as in the Monte Carlo. Table 4 reports a summary of the daily rejection frequencies of the test at a nominal size of 5% or 10%.

Table 4: Empirical Test Rejection Rates

Asset	**Aggregation: 1 day**				**Aggregation: 5 days**			
	5-minute frequency		1-minute frequency		5-minute frequency		1-minute frequency	
	$\alpha = 0.10$	$\alpha = 0.05$	$\alpha = 0.10$	$\alpha = 0.05$	$\alpha = 0.10$	$\alpha = 0.05$	$\alpha = 0.10$	$\alpha = 0.05$
Amazon	0.1590	0.0872	0.1587	0.0847	0.2051	0.1282	0.3514	0.2973
QQQ	0.1638	0.1034	0.1376	0.0763	0.2416	0.1544	0.1918	0.1096
SPY	0.1427	0.0796	0.1240	0.0648	0.2056	0.1389	0.1732	0.1117
Tesla	0.2259	0.1213	0.1681	0.0924	0.3191	0.2340	0.4043	0.2979

As seen from the reported results, the empirical rejection rates are above the nominal size of the test across all assets. Exactly as in the Monte Carlo, the rejection frequencies increase as we increase aggregation from one day to five days. This suggests that there is some persistence in market segmentation. We observe the highest rejection rates across the various test configurations for Tesla, with rejection rates for this asset that are significantly above the nominal size of the test. On the other hand, the lowest rejection rates across the various test configurations are for SPY. Nevertheless, even for SPY, the rejection rates of the test when implemented on five consecutive days appear high.

In Figure 2, we plot the time series of the p-values of the test for aggregation to one day. The figure reveals that the violations of the null hypothesis of market integration appear randomly scattered across the sample period. That said, we note some persistence in low p-values, which is consistent with the higher rejection rates of the test when implemented on five consecutive days.

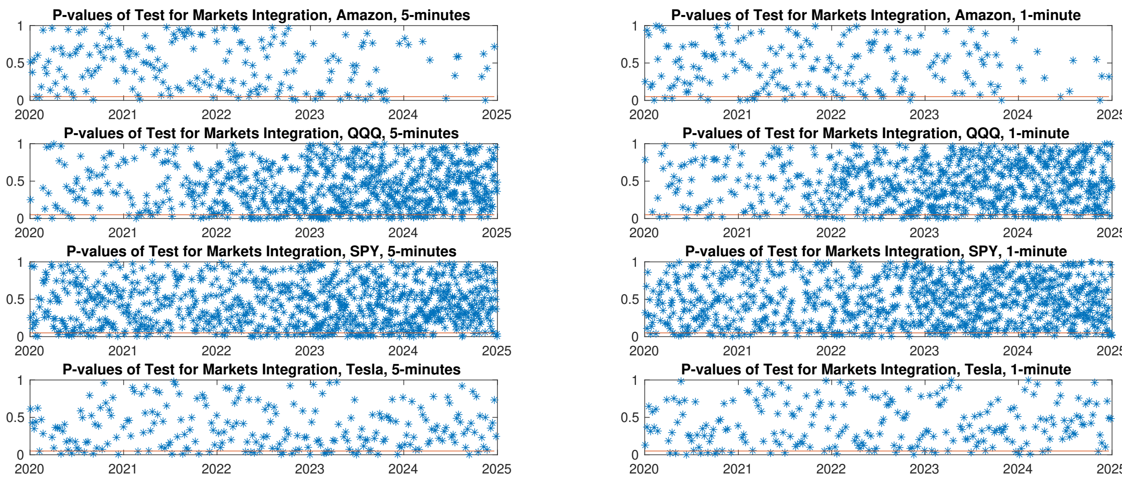

Figure 2: **Empirical Test Results (1-Day Aggregation).**

6.3 Examples of Variance Disagreement

We next illustrate the violations of the market integration null hypothesis with two examples. We use the SPY data for this purpose and pick two days when the test statistic has a low p-value—one day on which the options market overestimates volatility and one day on which it underestimates volatility. In Figure 3, we display the realized integrated diffusive variance implied by the options market against its counterpart constructed from high-frequency returns of the underlying asset. More specifically, recalling the approximations in (3.7) and (3.11), we plot

$$\widehat{\mathrm{VP}}_{i,T}^{n} = \sum_{j=i}^{k_n} \widehat{V}^{\mathbb{P}}_{\underline{t}_j^n,\overline{t}_j^n}(\widehat{u}_{(j-2)\Delta_n}) \quad \text{and} \quad \widehat{\mathrm{VQ}}_{i,T}^{n} = \sum_{j=i}^{k_n} \widehat{V}^{\mathbb{Q}}_{\underline{t}_j^n,\overline{t}_j^n}(\widehat{u}_{(j-2)\Delta_n}), \quad i = 2, \ldots, k_n. \tag{6.1}$$

On the first of the two days (January 31, 2020), the disagreement between the equity and options markets about spot diffusive variance lasted for approximately the first half of the trading day and afterwards the two markets were largely in agreement. The $\mathbb{Q}$-integrated variance estimate on this day appears rather choppy, which suggests a lot of revisions in volatility expectations by the SPY options traders. On this day, the US declared COVID-19 a public health emergency. This triggered moves in the stock markets, with the S&P 500 index falling by around 1.6% during the course of the trading day.

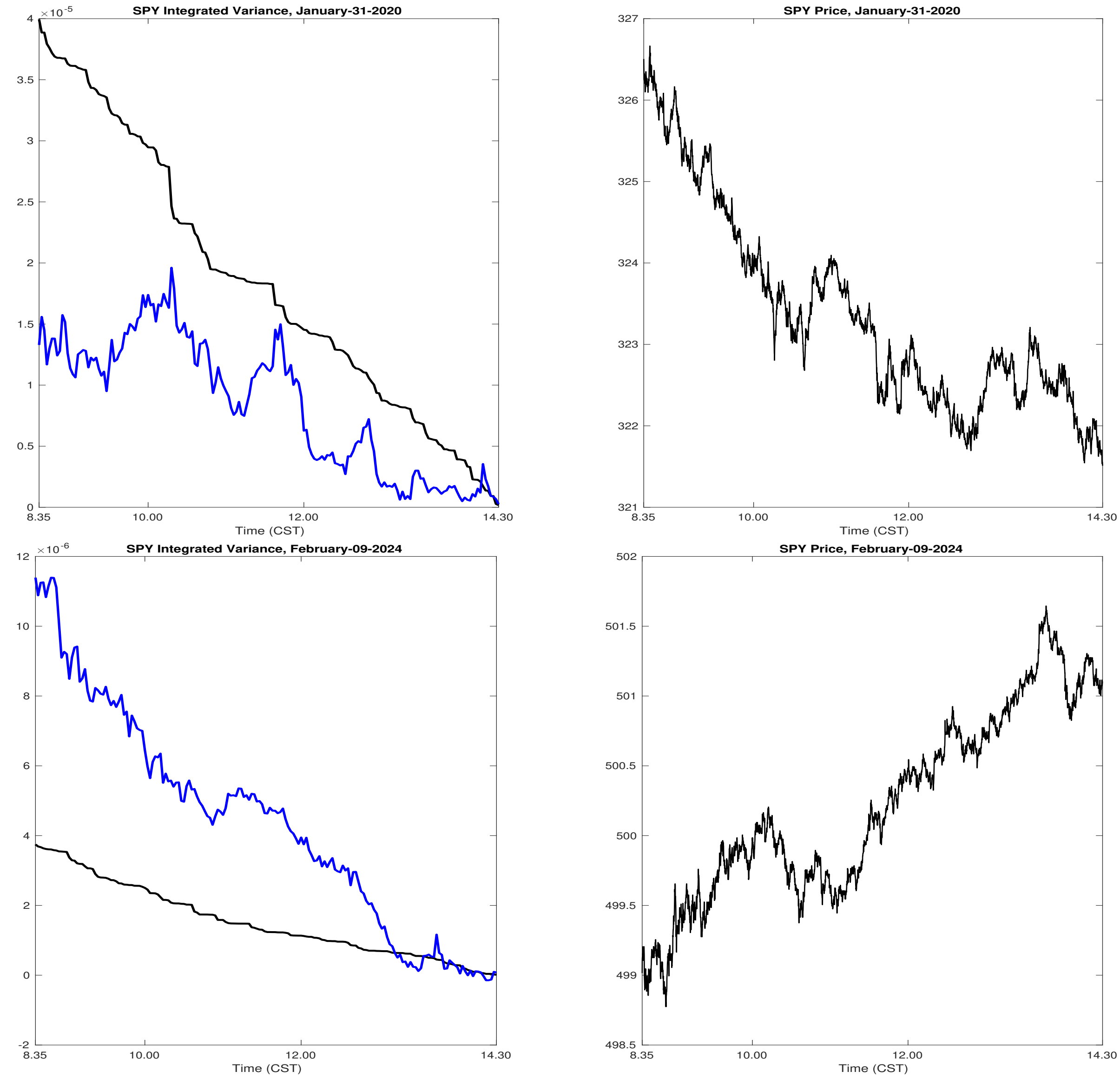

Figure 3: **SPY Integrated Variance on Days with Large Disagreements between Equity and Options Markets**. On each of the two left plots, the black line corresponds to $\widehat{\mathrm{VP}}_{i,T}^{n}$ and the blue line to $\widehat{\mathrm{VQ}}_{i,T}^{n}$ over the interval 8:35–14:30 CST and computed at one-minute frequency.

The second day with large disagreement between equity and options markets about volatility is February 9, 2024. On this day, the equity and options markets were assessing the impact of the highly anticipated seasonal revision to inflation data by the Bureau of Labor Statistics as this can often provide a clue about future FOMC decisions. The news was viewed favorably by investors, with the S&P 500 index gaining around 0.5% during the course of the trading day. Our evidence suggests that the options market was consistently overestimating the volatility throughout this trading day. The relative gap between the $\mathbb{P}$- and $\mathbb{Q}$-volatility estimates shrinks towards market close but remains nontrivial.

The two examples show diverse types of situations in which equity and options markets disagree about volatility. A common observation is that much of the variance disagreement is realized in the morning hours of the trading day, with variance forecasts of the two markets converging in the afternoon. This finding is consistent with DeCanio (1979), who shows that economic agents have the ability to achieve rational expectations through learning.

6.4 The Magnitude of Variance Disagreement

How big is the gap between the estimates of diffusive variance under $\mathbb{P}$ and $\mathbb{Q}$? We can use the value of our measure of relative realized variance disagreement, rRVD_T^n, to answer this question. In Figure 4, we plot 90% confidence intervals for daily rIVD_T using the result in Theorem 2. As seen from the figure, the confidence intervals are in general wide, which reflects the difficulty of the inference problem. For all assets, there are occasional spikes in the length of the confidence bands. This is primarily due to instances in which the quality of the options data is relatively poor. The plot also reveals a degree of commonality between the disagreement measures across different assets. For example, over the period spanning the second half of 2021 and first half of 2022, the options markets were undervaluing volatility while the opposite was true towards the end of 2022.

The estimates of the asymptotic variance $\mathrm{AVar}(\mathrm{rRVD}_T^n)$ defined in (3.26) at the five-minute sampling frequency have median values in the range 0.37–0.43 for the four different assets. This means that only bigger disagreements (in relative terms) can be detected using the daily values of rRVD_T^n while one needs to aggregate across days in order to learn about smaller in magnitude disagreements. For this reason, we make use of the result in Theorem 2* from Appendix D to infer the average value and size of rIVD_T over the sample period. In particular, our goal is to provide estimates of $\mathbb{E}^{\mathbb{P}}(\mathrm{rIVD}_T)$ and $\sqrt{\mathbb{E}^{\mathbb{P}}(\mathrm{rIVD}_T^2)}$.

By Theorem 2, we have

$$\mathrm{rRVD}_d \approx \mathrm{rIVD}_d + Z_d, \tag{6.2}$$

where Z_d is $\overline{\mathcal{F}}$-conditionally normal with mean 0 and variance $\mathrm{AVar}(\mathrm{rRVD}_d)$ and independent for different days. We use the subscript d to indicate quantities computed on day d and we omit n and T from the notation. Taking sample averages allows us to gauge the average magnitude of relative variance disagreement. We define

$$\widehat{\mathrm{rIVD}} = \frac{1}{D}\sum_{d=1}^{D} \mathrm{rRVD}_d \quad \text{and} \quad \widehat{\mathrm{S}}^2(\mathrm{rIVD}) = \frac{1}{D}\sum_{d=1}^{D} \mathrm{rRVD}_d^2. \tag{6.3}$$

Because $\mathrm{AVar}(\mathrm{rRVD}_d)$ shrinks to zero in the asymptotic approximation (6.2), $\widehat{\mathrm{rIVD}}$ and $\widehat{\mathrm{S}}^2(\mathrm{rIVD})$

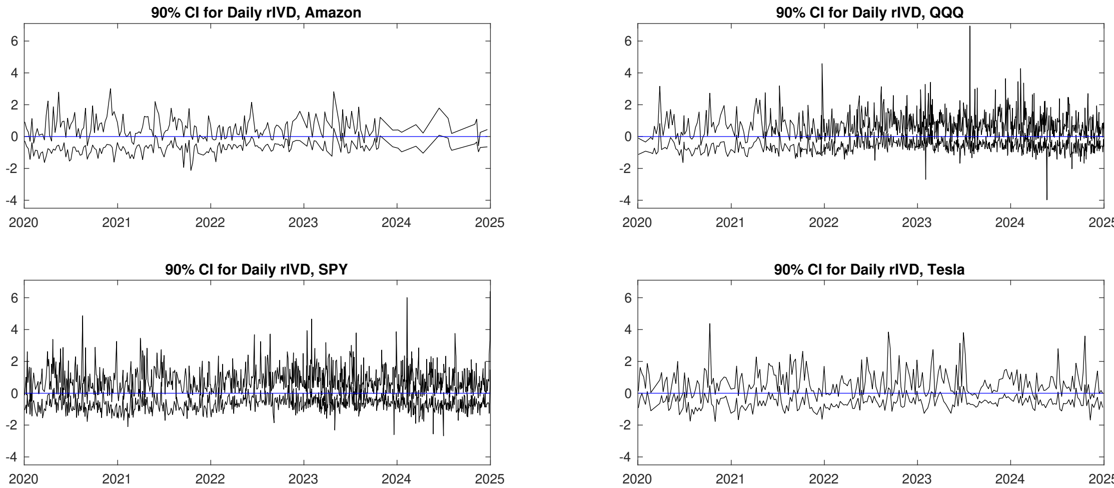

Figure 4: **Confidence Intervals for Daily rIVD.** The statistics are computed using a five-minute sampling frequency.

are estimators of

$$\overline{\mathrm{rIVD}} = \frac{1}{D}\sum_{d=1}^{D} \mathrm{rIVD}_d \quad \text{and} \quad \mathrm{S}^2(\mathrm{rIVD}) = \frac{1}{D}\sum_{d=1}^{D} \mathrm{rIVD}_d^2, \tag{6.4}$$

respectively. Furthermore, because Z_d is independent across days, the autocorrelation of rRVD_d can be used to study the persistence of the variance disagreement between $\mathbb{P}$ and $\mathbb{Q}$.

It is evident, however, that approximation errors in (6.2) lead to biases in the estimators defined in (6.3), especially when aggregated over multiple days. The results below are therefore based on bias-corrected versions of $\widehat{\mathrm{rIVD}}$ and $\widehat{\mathrm{S}}^2(\mathrm{rIVD})$, which are formally defined in (D.5) and (D.6) of Appendix D. Since the bias corrections require a higher frequency for more precise estimation, we use a one-minute sampling frequency for computing $\widehat{\mathrm{rIVD}}$ and $\widehat{\mathrm{S}}^2(\mathrm{rIVD})$.

In Table 5, we report estimates for the size and persistence of relative variance disagreement. We draw several conclusions from the results. First, for all considered assets, there are periods when the options markets over- and underestimate diffusive spot volatility, with the time-series average of the relative variance disagreement, $\widehat{\mathrm{rIVD}}$, being in general small. Second, consistent with our test results reported in Table 4, the size of the $\mathbb{P}$–$\mathbb{Q}$ variance gap appears bigger for the individual stocks and smaller for the ETFs. Indeed, the relative $\mathbb{P}$–$\mathbb{Q}$ variance gap for QQQ and SPY is in the range 13–15% on average, while it is much bigger for Amazon and Tesla, at 22% and 27%, respectively. In volatility terms, these numbers correspond to gaps of around 6–7% for QQQ and SPY and around 10% and 13% for Amazon and Tesla, respectively. One should keep in mind, however, that these are average numbers. They can be realized either through small and frequent deviations or via rare big episodes of disagreements (like the ones illustrated in the previous section). Indeed, the large standard errors for $\widehat{\mathrm{S}}(\mathrm{rIVD})$ for QQQ and SPY suggest that these estimates are driven in part by large outliers. Finally, the gap between estimates of

Table 5: Empirical Estimates of the Relative $\mathbb{P}$–$\mathbb{Q}$ Variance Gap

Asset	$\widehat{\mathbf{rIVD}}$	$\widehat{\mathbf{S}}\mathbf{(rIVD)}$	**P-values of AC Test**	
			Lags 1-5	Lags 6-10
Amazon	-0.1603 (0.0382)	0.2151 (0.0755)	0.25	0.25
QQQ	0.0806 (0.0208)	0.1204 (0.1253)	0.02	0.48
SPY	0.0823 (0.0224)	0.1522 (0.0878)	0.03	0.04
Tesla	-0.0473 (0.0331)	0.2722 (0.0520)	0.03	0.05

Note: The quantities are computed using one-minute sampling frequency and the bias-corrected estimators defined in (D.5) and (D.6) of Appendix D. Numbers in parentheses in the second and third column are long-run standard error estimates. The autocorrelation test is based on sums of squared autocorrelation coefficients multiplied by the number of observations used in computing the autocorrelations. The p-values are computed on the basis of the limiting distribution of these statistics under the null of no serial correlation in the time series, which is $\chi^2(k)$ where k denotes the number of autocorrelations entering the summation. For the two individual stocks Amazon and Tesla, data is available only at weekly frequency and therefore only autocorrelations at lags 5 and 10 are computed.

$\mathbb{P}$- and $\mathbb{Q}$-variance appears to have some mild positive persistence, with the evidence being the strongest for SPY, QQQ and Tesla.

The reported estimates of average gaps between $\mathbb{P}$- and $\mathbb{Q}$-variance can be compared to the transaction costs investors face when trading options. The option portfolio behind $\widehat{\mathcal{L}}_{t,T}^{\mathbb{Q}}(u)$ assigns most weight to around-the-money options. In our sample, the average relative bid–ask spread for at-the-money options equals 13.9% for Amazon, 5.1% for QQQ, 3.9% for SPY and 7.8% for Tesla. With the exception of Amazon, these estimates are small and indicate that the options markets are highly liquid. As shown in Proposition 5 of Chong and Todorov (2025b), one half of the relative bid–ask spread for around-the-money options provides an estimate for the relative transaction costs associated with trading options that exploit discrepancies between $\mathbb{P}$- and $\mathbb{Q}$-spot volatility. For all tickers, the reported average size of the gaps between $\mathbb{P}$- and $\mathbb{Q}$-volatility exceeds the average transaction costs associated with portfolios aimed at exploiting them. This shows that in addition to transaction costs, there are other factors that make eliminating these $\mathbb{P}$–$\mathbb{Q}$ variance gaps difficult. One such factor appears to be of econometric nature: the arbitrageur needs to decide ex ante whether the options market over- or underestimates diffusive volatility. As Figure 4 shows, this is even a difficult econometric task ex post. As a result, arbitrageurs who do not have perfect knowledge about the underlying asset dynamics face significant estimation risk at the time when they need to decide whether to engage in a costly arbitrage strategy or not.

6.5 Economic Implications

Our empirical results challenge the belief that investors in equity and options markets always have full-information rational expectations. The econometric methods we develop to identify and measure market segmentation can be combined with additional data to better understand its economic origin.

One plausible explanation for the existence of market segmentation relates to uncertainty about volatility. In a setting where investors are econometricians and need to learn from past discrete returns about future volatility (see Chen and Epstein (2002), Hansen (2007, 2014) and Epstein and Ji (2013)), there is a nontrivial estimation risk concerning the existence, direction and magnitude of $\mathbb{P}$–$\mathbb{Q}$ variance gaps. If asset prices were observed continuously, investors would be able to know the value of $\mathbb{P}$-diffusive variance without error. In practice, asset prices are only observed discretely and if one samples at very high frequency, then the observed price is contaminated by significant market microstructure noise. There is a large body of work, starting with Barndorff-Nielsen and Shephard (2002, 2004) and Andersen et al. (2003), that studies feasible inference about integrated volatility from high-frequency returns. The estimation problem becomes significantly more complicated when trying to infer *spot* volatility from high-frequency returns due to the presence of a nontrivial intraday volatility pattern that can change between trading days. Thus, even when using the most efficient estimators of volatility, investors are likely to face significant statistical uncertainty about $\mathbb{P}$-diffusive variance. As we noted in Remark 2, if the gap between the full information set $\mathcal{F}_t$ and the information set available to the investors $\mathcal{F}'_t$ is large, this can introduce realized variance disagreement due to uncertainty generated by the reduced knowledge about diffusive variance.

Market segmentation may also be driven by the large percentage of retail investors in the options market (Bryzgalova et al. (2023)), who act as noise traders (De Long et al. (1990)) and create price pressure on option prices (Gârleanu et al. (2009)). When combined with constrained financial intermediaries (as argued for by Gârleanu and Pedersen (2011) and Bates (2022)) and nontrivial transaction costs, temporary arbitrage opportunities may be hard to take advantage of due to the aforementioned limits to arbitrage. This view is consistent with the empirical findings of Chong and Todorov (2025b).

Even if gaps between $\mathbb{P}$- and $\mathbb{Q}$-volatility cannot be arbitraged away due to limits to arbitrage, the existence of episodes with such deviations has to be accounted for when conducting integrated analysis of equity and options market data. For example, Alvarez and Jermann (2005), Hansen and Scheinkman (2009), Christensen (2017) and Qin and Linetsky (2017) decompose the SDF into permanent and transitory components. As we show, such an SDF does not exist at all times for equity and option prices. Thus, our results call for studying the singular component of the risk-neutral measure which cannot be related to the physical measure through a valid pricing kernel.

Another implication of our results pertains to the use of options data for econometric purposes. Most existing studies including Renault (1997), Gagliardini et al. (2011), Andersen et al. (2015), Todorov (2019) and Chong and Todorov (2025a) take integrated equity and options markets as a premise. For example, in the absence of arbitrage, the optimal forecast of volatility on a given day is a weighted average of an option-based and a returns-based volatility estimator, with weights determined by the precision of the two volatility estimators (Todorov (2019)). In practice, the option-based volatility estimate is assigned higher weights, as it is significantly more efficient than its returns-based counterpart. On days with market segmentation, however, variance disagreement introduces a bias in the option-based volatility estimator. Thus, the weighting scheme has to be modified according to a bias–variance analysis. Our measures of disagreement can be used for such analysis.

7 Conclusion

In this paper, we conduct inference on the disagreement of the equity and options markets about diffusive variance. We introduce a novel measure which we call realized variance disagreement and which uses high-frequency data of an underlying asset and zero-date options written on the asset. We prove a stable central limit theorem that enables feasible inference in an asymptotic regime where both the high-frequency sampling interval and the time-to-expiration of the options tend to zero.

We further develop market integration tests based on the insight that integration between equity and options markets is equivalent to no diffusive variance disagreement between the statistical and risk-neutral probability measures. Unlike previous literature, our inference procedures are fully nonparametric and essentially model-free. In particular, we allow for arbitrary activity of jumps, arbitrary volatility dynamics and intraday volatility patterns that may change from day to day, both under the physical and the risk-neutral measure. We also make no assumptions about investors' risk preferences.

Empirically, we document periods of segmentation between equity and options markets, with their frequency and magnitude varying across assets. The degree of variance disagreement also shows mild time-series persistence. Likely explanations for such periods of market segmentation include transactions costs, uncertainty about volatility and active retail trading.

A Option-Implied Measures

Given our interest in zero-date options, we assume that the risk-free rate is zero and that there are no dividend payments during the trading day. These simplifications have no bearing on the asymptotics but somewhat simplify the exposition. Recall from Section 2 that $P_{t,T}(K)$ and $C_{t,T}(K)$ denote the prices at time t of European call and put options, respectively, with strike price K and maturity T. We let $O_{t,T}(K) = \min\{P_{t,T}(K), C_{t,T}(K)\}$ denote the price of the associated OTM option. We assume an observation scheme where OTM option prices are observed at high-frequency time points t on a time-dependent discrete strike grid:

$$0 < K_{t,1} < K_{t,2} < \cdots < K_{t,N_t}, \quad N_t \in \mathbb{N}_+.$$

Options are observed with error. More specifically, the option mid-quote is given by

$$\widehat{O}_{t,T}(K_{t,j}) = O_{t,T}(K_{t,j}) + \epsilon_{t,T}(K_{t,j}), \quad j = 1, \ldots, N_t, \tag{A.1}$$

where $\epsilon_{t,T}(K_{t,j})$ is an observation error. We assume that the observations errors are defined on an extended probability space given by

$$\overline{\Omega} = \Omega \times \Omega^{(1)}, \quad \overline{\mathcal{F}} = \mathcal{F} \otimes \mathcal{F}^{(1)}, \quad \overline{\mathcal{F}}_t = \bigcap_{s>t} \mathcal{F}_s \otimes \mathcal{F}_s^{(1)}, \quad \overline{\mathbb{P}}(d\omega, d\omega^{(1)}) = \mathbb{P}(d\omega)\mathbb{P}^{(1)}(\omega, d\omega^{(1)}), \tag{A.2}$$

where $(\Omega^{(1)}, \mathcal{F}^{(1)}, (\mathcal{F}_t^{(1)})_{t\geq 0})$ is an auxiliary filtered space and $\mathbb{P}^{(1)}(\omega, d\omega^{(1)})$ is a transition probability from Ω, the probability space on which the asset price is defined, to $\Omega^{(1)}$.

A classical option-spanning result by Bakshi and Madan (2000) and Carr and Madan (2001) implies that the risk-neutral conditional characteristic function of price increments as defined in (3.1) can be recovered from OTM option prices via

$$\mathcal{L}_{t,T}^{\mathbb{Q}}(u) = 1 - (u^2 + iu) \int_{\mathbb{R}} e^{iu(k - x_t) - k} O_{t,T}(e^k) dk \tag{A.3}$$

in a model-free way. As a feasible counterpart, we follow Todorov (2019) and use

$$\widehat{\mathcal{L}}_{t,T}^{\mathbb{Q}}(u) = 1 - (u^2 + iu) \sum_{j=2}^{N_t} e^{iu(k_{t,j-1} - x_t) - k_{t,j-1}} \widehat{O}_{t,T}(K_{t,j-1})(k_{t,j} - k_{t,j-1}), \tag{A.4}$$

where $k_{t,j} = \log(K_{t,j})$ is the log-strike. The sum is a Riemann approximation of the integral in (A.3). To improve on the quality of the Riemann sum approximation, we generate option prices on an equidistant strike grid at increments of 0.1 using linear interpolation in Black–Scholes implied volatility space from the available options data[12] and compute $\widehat{\mathcal{L}}_{t,T}^{\mathbb{Q}}(u)$ from the option prices on the finer strike grid.

The selection of the characteristic exponent $\widehat{u}_t$ in (5.5)–(5.7) uses option-implied second moments of log-returns (see e.g., Bakshi and Madan (2000)), defined formally as

$$\widehat{RV}_{t,T}^{\mathbb{Q}} = 2 \sum_{j=2}^{N_t} e^{-k_{t,j-1}} (1 - (k_{t,j-1} - x_t)) \widehat{O}_{t,T}(K_{t,j-1})(k_{t,j} - k_{t,j-1}). \tag{A.5}$$

We note that $\widehat{RV}_{t,T}^{\mathbb{Q}}$ is an estimate of $\mathbb{E}_t^{\mathbb{Q}}[(x_T - x_t)^2]$.

[12] We refer to Gagliardini et al. (2011) for a different interpolation method.

B Assumptions

Because x is an Itô semimartingale, it admits a Grigelionis representation under $\mathbb{S} \in \{\mathbb{P}, \mathbb{Q}\}$ (see Theorem 2.1.2 in Jacod and Protter (2012)):

$$x_t = x_0 + \int_0^t \alpha_s^{\mathbb{S}} ds + \int_0^t \sigma_t^{\mathbb{S}} dW_t^{\mathbb{S}} + \int_0^t \int_{\mathbb{R}} \gamma^{\mathbb{S}}(s,z)(\mathfrak{p}^{\mathbb{S}}(dt,dz) - dtF(dz)), \tag{B.1}$$

where $\mathfrak{p}^{\mathbb{S}}$ is an $\mathbb{S}$-Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}$ with Lévy measure F and $\gamma^{\mathbb{S}} : \Omega \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is a predictable function that determines the jump sizes and jump intensity of x. Note that in the Grigelionis representation of x, the Lévy measure F can be fixed to be the same under $\mathbb{P}$ and $\mathbb{Q}$, while $\mathfrak{p}^{\mathbb{S}}$ and $\gamma^{\mathbb{S}}$ now depend on the probability measure $\mathbb{S}$. Our assumptions on the dynamics of x are formulated in terms of the coefficients of x in the Grigelionis representation.

Assumption 1. *For $\mathbb{S} \in \{\mathbb{P}, \mathbb{Q}\}$, the diffusive spot variance of x is given by (4.2). The processes $\alpha^{\mathbb{S}}$, ς, ς^{-1} and ρ are locally bounded. The function $\xi : [0,1] \to \mathbb{R}$ is càdlàg, positive and bounded away from zero, $\zeta : [0,1] \to \mathbb{R}$ is càdlàg and $\varsigma^2 + \rho_t \sup_{u \in [0,1]} |\zeta(u)|/\xi(u)$ is locally bounded away from zero. One can find a localizing sequence $(\tau_n)_{n \geq 1}$ of stopping times, some constant $\beta \in (1,2)$ and deterministic constants $C_n \in (0,\infty)$ such that for each n, we have $\int_{\mathbb{R}} (|\gamma^{\mathbb{S}}(t,z)|^{\beta} \wedge 1) F(dz) \leq C_n$ for $t \leq \tau_n$. The characteristic exponents $\widehat{u}_t$ are $\overline{\mathcal{F}}_t$-measurable random variables and satisfy (3.6) and*

$$\mathbb{E}[(\widehat{u}_t \sqrt{T-t} - \upsilon_t)^2] = o(1) \tag{B.2}$$

uniformly for all $t \in [0, \theta T]$, where θ is the constant introduced after (3.4) and the process υ is adapted, càdlàg and mean-square continuous.

These assumptions are mild and standard in high-frequency econometrics. The assumption on $\gamma^{\mathbb{S}}$ becomes weaker for larger values of β. Because the maximal value of β is 2 (see e.g., Jacod and Protter (2012)), the restriction on the degree of jump activity in the asset price is very weak. In particular, Assumption 1 allows for jumps of infinite variation in the price and is satisfied by virtually all parametric models used in applied work.

The conditions on ς^{-1}, ξ and $\varsigma^2 + \rho_t \sup_{u \in [0,1]} |\zeta(u)|/\xi(u)$ imply that both $(\sigma^{\mathbb{P}})^2$ and $(\sigma^{\mathbb{Q}})^2$ are locally bounded away from zero. We could have introduced an intraday pattern not only in volatility but also in the drift $\alpha^{\mathbb{S}}$ or the jump function $\gamma^{\mathbb{S}}$. Because we show later that these coefficients do not asymptotically affect the statistics considered in this paper, we refrain from doing so for simplicity. The local boundedness of the drift, however, does exclude explosive drifts as considered by Christensen et al. (2022) and Andersen et al. (2025). Since in these cases the equity market itself allows for arbitrage, and we are interested in market segmentation between the equity and options markets, we do not consider such a setting here.

The assumptions on the characteristic exponents imply that $\widehat{u}_t$ can be chosen in an adaptive data-dependent way Properties (3.6) and (B.2) imply that $\widehat{u}_t$ is of the order $1/\sqrt{T}$ and is locally mean-square continuous. Note that $\widehat{u}_t$, in general, is only $\overline{\mathcal{F}}_t$-measurable because it can depend on the observed option prices which are contaminated by errors defined on the extended probability space given in (A.2).

In the next two assumptions, we denote the spot log-characteristic function of the jump part of x by

$$\psi_t^{\mathbb{S}}(u) = \int_{\mathbb{R}} (e^{iu\gamma^{\mathbb{S}}(t,z)} - 1 - iu\gamma^{\mathbb{S}}(t,z))F(dz), \quad \mathbb{S} \in \{\mathbb{P}, \mathbb{Q}\}. \tag{B.3}$$

Assumption 2. *For any compact subset $U \subseteq (0,\infty)$, we assume that there is a localizing sequence $(\tau_n)_{n\geq 1}$ of stopping times together with nonnegative constants C_n and some $\alpha > 0$ and $\beta \in (1,2)$ such that for all $n \geq 1$, $0 \leq s \leq t \leq T$ and $u \in U$, we have*

$$\mathbb{E}^{\mathbb{P}}\big(|\alpha^{\mathbb{P}}_{t\wedge\tau_n} - \alpha^{\mathbb{P}}_{s\wedge\tau_n}|^2 + |\varsigma^2_{t\wedge\tau_n} - \varsigma^2_{s\wedge\tau_n}|^2 + |\rho_{t\wedge\tau_n} - \rho_{s\wedge\tau_n}|^2 \mid \mathcal{F}_s\big) \leq C_n|t-s|^{2\alpha}, \tag{B.4}$$

$$\mathbb{E}^{\mathbb{P}}\left(\int_{\mathbb{R}} (\gamma^{\mathbb{P}}(t\wedge\tau_n, z) - \gamma^{\mathbb{P}}(s\wedge\tau_n, z))^2 F(dz) \;\middle|\; \mathcal{F}_s\right) \leq C_n|t-s|^{2\alpha}, \tag{B.5}$$

$$\mathbb{E}^{\mathbb{P}}\big(|\psi^{\mathbb{P}}_{t\wedge\tau_n}(u) - \psi^{\mathbb{P}}_{s\wedge\tau_n}(u)|^2 \mid \mathcal{F}_s\big) \leq C_n|u|^{\beta}|t-s|^{2\alpha}. \tag{B.6}$$

Moreover, the function ζ has a finite β-variation, that is,

$$\sup\left\{\sum_{k=1}^{n} |\zeta(u_k) - \zeta(u_{k-1})|^{\beta} : 0 \leq u_0 \leq \cdots \leq u_n \leq 1,\, n \in \mathbb{N}\right\} < \infty. \tag{B.7}$$

Assumption 3. *For any compact subset $U \subseteq (0,\infty)$, we assume that there are $\alpha > 0$, $\beta \in (1,2)$ and a càdlàg adapted process C such that the following holds for all $0 \leq s \leq t \leq T$ and $u \in U$:*

$$\mathbb{E}^{\mathbb{Q}}\big(|\alpha^{\mathbb{Q}}_t - \alpha^{\mathbb{Q}}_s|^2 + |\varsigma^2_t - \varsigma^2_s|^2 + |\rho_t - \rho_s|^2 \mid \mathcal{F}_s\big) \leq C_s|t-s|^{2\alpha}, \tag{B.8}$$

$$\mathbb{E}^{\mathbb{Q}}\left(\int_{\mathbb{R}} (\gamma^{\mathbb{Q}}(t,z) - \gamma^{\mathbb{Q}}(s,z))^2 F(dz)) \;\middle|\; \mathcal{F}_s\right) \leq C_s|t-s|^{2\alpha}, \tag{B.9}$$

$$\mathbb{E}^{\mathbb{Q}}\big(|\psi^{\mathbb{Q}}_t(u) - \psi^{\mathbb{Q}}_s(u)|^2 \mid \mathcal{F}_s\big) \leq C_s|u|^{\beta}|t-s|^{2\alpha}. \tag{B.10}$$

We assume without loss of generality that α, β and τ_n are common to the previous three assumptions. The last two assumptions concern the conditional smoothness in mean square of the coefficients of the asset price process, with α being a lower bound on the regularity. Since these assumptions involve conditional expectations, they do not rule out jumps. When the coefficients of x are themselves Itô semimartingales, which is true in many models used in applied work, then Assumptions 2 and 3 are satisfied with $\alpha = 1/2$. But since α can be arbitrarily low, they also cover rough volatility models as studied by Gatheral et al. (2018) and Chong and Todorov (2025c).

We now detail the observation scheme of the option prices and use the shorthand notation $\underline{k}_t = k_{t,1}$ and $\overline{k}_t = k_{t,N_t}$.

Assumption 4. *The observed option prices $\widehat{O}_{t,T}(K_{t,j})$ from (A.1) are defined on $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}})$ and there is an $\mathbb{F}$-adapted càdlàg process C_t such that the following holds:*

1. *For all* $0 \leq t < u < T$,

$$\mathbb{E}_t^{\mathbb{Q}}\left[|\alpha_u^{\mathbb{Q}}|^4 + |\sigma_u^{\mathbb{Q}}|^6 + e^{4|x_u|} + \left(\int_{\mathbb{R}}(e^{3|\gamma^{\mathbb{Q}}(u,z)|} - 1 - 3|\gamma^{\mathbb{Q}}(u,z)|)F(dz)\right)^4\right] \leq C_t.$$

2. *The number of strikes* N_t *and the log-strike grid* $\{k_{t,j}\}_{j=1}^{N_t}$ *are* $\mathcal{F}_t$*-measurable and*

$$C_t^{-1}\delta \leq k_{t,j} - k_{t,j-1} \leq C_t\delta, \qquad j = 2, \ldots, N_t,$$

for a deterministic sequence $\delta = \delta(T)$. *Furthermore,*

$$\sup_{t\in[0,T]} \sup_{j:|k_{t,j}-x_t|\leq 1} |(k_{t,j} - k_{t,j-1})/\delta - \kappa_t(k_{t,j-1} - x_t)| = o_p(1)$$

where

$$\kappa_t(k) = \overline{\kappa}(t/T; k)\widetilde{\kappa}_t(k) \tag{B.11}$$

and $\overline{\kappa}(t;k)$ *is a family of deterministic measurable time-of-day functions and* $t \mapsto \widetilde{\kappa}_t(k)$, *for each* k, *is* $\mathbb{F}$*-adapted and càdlàg. Moreover,* $\overline{\kappa}(t;k)$ $(\widetilde{\kappa}_t(k))$ *is continuous (mean-square continuous) at* $k = 0$, *uniformly in* t, $\overline{\kappa}(t) = \overline{\kappa}(t;0)$ *has a finite* β*-variation and* $\widetilde{\kappa}_t = \widetilde{\kappa}_t(0)$ *is mean-square continuous.*

3. *We have*

$$\frac{\inf_{i=1,\ldots,k_n}(|\underline{k}_{\underline{t}_i^n}| \wedge |\overline{k}_{\underline{t}_i^n}| \wedge |\underline{k}_{\overline{t}_i^n}| \wedge |\overline{k}_{\overline{t}_i^n}|)}{|\log(\delta/\sqrt{T})|} \to \infty.$$

4. *For* $t > 0$ *and* $j = 1, \ldots, N_t$, *we have*

$$\epsilon_{t,T}(K_{t,j}) = \lambda_t(k_{t,j} - x_t)O_{t,T}(K_{t,j})\overline{\epsilon}_{t,j}, \tag{B.12}$$

where $\overline{\epsilon}_{t,j}$ *is* $\mathcal{F}_t^{(1)}$*-measurable, independent of* $\mathcal{F}$ *under* $\overline{\mathbb{P}}$, *i.i.d. as* j *and* t *vary and satisfies*

$$\mathbb{E}^{\overline{\mathbb{P}}}[\overline{\epsilon}_{t,j} \mid \mathcal{F}] = 0, \quad \mathbb{E}^{\overline{\mathbb{P}}}[(\overline{\epsilon}_{t,j})^2 \mid \mathcal{F}] = 1, \quad \mathbb{E}^{\overline{\mathbb{P}}}[|\overline{\epsilon}_{t,j}|^p \mid \mathcal{F}] < \infty \text{ for all } p > 2. \tag{B.13}$$

The option error intensity process $\lambda_t(k)$ *is of the form*

$$\lambda_t(k) = \overline{\lambda}(t/T; k)\widetilde{\lambda}_t(k), \tag{B.14}$$

where $\overline{\lambda}(t;k)$ *is a family of deterministic measurable time-of-day functions and* $t \mapsto \widetilde{\lambda}_t(k)$, *for each* k, *is* $\mathbb{F}$*-adapted and càdlàg. Moreover,* $\overline{\lambda}(t;k)$ $(\widetilde{\lambda}_t(k))$ *is continuous (mean-square continuous) at* $k = 0$, *uniformly in* t, $\overline{\lambda}(t) = \overline{\lambda}(t;0)$ *has a finite* β*-variation and* $\widetilde{\lambda}_t = \widetilde{\lambda}_t(0)$ *is mean-square continuous.*

Assumption 4 concerns the option prices, the option strike grid and the option observation errors and is adopted from Chong and Todorov (2025a). In particular, for characterizing the quality of the option-based estimator of the characteristic function, we need certain integrability conditions. We note that we require existence of conditional moments over shrinking time intervals, and therefore these moment conditions are relatively weak. In addition to the existence of conditional moments, we also require an asymptotically shrinking mesh of the log-strike grid covering the whole interval $(0, \infty)$. The requirement for the strike range can be further weakened because we are interested in $\widehat{\mathcal{L}}^{\mathbb{Q}}_{t,T}(u)$ for large u, and for such values of the characteristic exponent only options with strikes in the vicinity of the current stock price matter asymptotically. Condition (B.12) implies multiplicative option observation errors, where $\lambda_{t,T}(k)$ are adapted scaling factors and $\bar{\epsilon}_{t,j}$ are centered error variables that are independent of the information set $\mathcal{F}$, which includes the asset price. In contrast to Chong and Todorov (2025a), we allow for time-of-day variation in both the asymptotic mesh size of the strike grid, $\kappa_{t,T}(k)$, and in the intensity process of the option observation errors, $\lambda_{t,T}(k)$.

Assumption 5. *We have $T \downarrow 0$, $\Delta_n \downarrow 0$, $\Delta_n' \downarrow 0$ and $k_n \to \infty$ in such a way that $\Delta_n' = \varphi \Delta_n$ for some $\varphi \in (0,1)$, $k_n \Delta_n = \theta T$ for some $\theta \in (0,1)$, $(\Delta_n')^2/[(\Delta_n')^2 + \delta T^{3/2}] \to q$ for some $q \in [0,1]$, $\delta k_n^{3/2} T^{-1/2} \log(1/T) \to 0$, $k_n T^{\iota} \to 0$ for all $\iota > 0$, $k_n^{-1} \log(1/T) \to 0$.*

Assumption 5 gathers rate conditions involving mainly the observation interval T, the sampling frequency Δ, the number of terms in our statistics, k_n, and the bound on the mesh size of the strike grid, δ. The number q signifies the relative importance of $\mathbb{P}$- and $\mathbb{Q}$-contributions to the asymptotic variance of RVD (see Appendix C for more details). If $q = 0$ ($q = 1$), option observation errors dominate (are negligible) for the asymptotic variance. If $0 < q < 1$, both the high-frequency variation of diffusive asset price moves and option observation errors contribute to the asymptotic variance. Assumption 5 restricts k_n to be slower than any polynomial of T. As discussed after Theorem 1, this can be weakened with a priori information about, or an estimate of, the jump activity in the asset price and the smoothness of volatility.

C Asymptotic Variances

We are now in the position to state the asymptotic variances of the various statistics introduced in the paper. The asymptotic variance of RVD^n_T is given by

$$\mathrm{AVar}_T = q\mathrm{AVar}^{\mathbb{P}}_T + (1-q)\mathrm{AVar}^{\mathbb{Q}}_T \tag{C.1}$$

where

$$\mathrm{AVar}^{\mathbb{P}}_T = \frac{2}{\theta T}\int_0^{\theta T} (\sigma^{\mathbb{P}}_t)^4 dt, \tag{C.2}$$

$$\mathrm{AVar}^{\mathbb{Q}}_T = \frac{1}{\theta T}\int_0^{\theta T} \left[8 e^{\upsilon_t^2 \frac{1}{T-t}\int_t^T (\sigma^{\mathbb{Q}}_s)^2 ds} (1 - \tfrac{t}{T})^{3/2} (\sigma^{\mathbb{P}}_t)^3 \lambda_t^2 \kappa_t \int_{\mathbb{R}} \cos^2(\upsilon_t \sigma^{\mathbb{P}}_t k) \widetilde{\Phi}(k) dk \right] dt. \tag{C.3}$$

In (C.3), $\widetilde{\Phi}(k) = \varphi(k) + |k|\Phi(|k|)$ and φ and Φ are the probability density function and cumulative distribution function of the standard normal distribution, respectively.

The first term, $\mathrm{AVar}_T^{\mathbb{P}}$, is due to high-frequency variation in the underlying asset returns and identical (up to a normalizing factor) to the asymptotic variance of the realized variance estimator (Barndorff-Nielsen and Shephard (2002)). The second term comprises variance due to option observation errors which enter RVD_T^n through the option-based characteristic function estimator in (A.4). The constant q, introduced in Assumption 5, denotes the proportion of the asymptotic variance of RVD_T^n due to $\mathrm{AVar}_T^{\mathbb{P}}$. Importantly, in Theorem 1, q can be any number within $[0,1]$. In the extreme cases, the asymptotic variance of RVD_T^n is asymptotically dominated by return variation ($q = 1$) or option observation errors ($q = 0$).

The asymptotic variance of $\mathrm{RVD}_T^n(\eta)$ from (4.5), viewed as a process in η, is captured by a process $(Z_\eta)_{\eta\in[0,2\pi]}$ which appears first in Theorem 4. It is defined on a very good extension $(\Omega', \mathcal{F}', \mathbb{P}')$ of $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}})$ and an $\overline{\mathcal{F}}$-conditional Gaussian process with mean 0 and $\overline{\mathcal{F}}$-conditional covariance function given by

$$\begin{aligned}
&\mathbb{E}'[Z_{\eta_1} Z_{\eta_2} \mid \overline{\mathcal{F}}] \\
&\quad = \int_0^1 w_{\eta_1}(u) w_{\eta_2}(u) \Bigg[2q\varsigma_0^4 \xi(\theta u)^2 + 8(1-q)\varsigma_0^3 \widetilde{\lambda}_0^2 \widetilde{\kappa}_0 e^{\upsilon_0^2 (1-\theta u)^{-1}(\varsigma_0^2 \int_{\theta u}^1 \xi(v)dv + \rho_0 \int_{\theta u}^1 \zeta(u)du)} \\
&\qquad \times (1-\theta u)^{3/2} \xi(\theta u)^{3/2} \overline{\lambda}(\theta u)^2 \overline{\kappa}(\theta u) \int_{\mathbb{R}} \cos^2(\upsilon_0 \varsigma_0 \xi(\theta u) k) \widetilde{\Phi}(k)^2 dk \Bigg] du
\end{aligned} \tag{C.4}$$

for $\eta_1, \eta_2 \in [0, 2\pi]$.

If $\eta_1 = \eta_2$, the term $\int_0^1 2\varsigma_0^4 \xi(\theta u)^2 du$ is a small-T approximation of $\mathrm{AVar}_T^{\mathbb{P}}$ in the case where $\sigma^{\mathbb{P}}$ satisfies (4.2). Similarly, the second term given by

$$\begin{aligned}
&\int_0^1 8\varsigma_0^3 \widetilde{\lambda}_0^2 \widetilde{\kappa}_0 e^{\upsilon_0^2 (1-\theta u)^{-1}(\varsigma_0^2 \int_{\theta u}^1 \xi(v)dv + \rho_0 \int_{\theta u}^1 \zeta(u)du)} \\
&\qquad \times (1-\theta u)^{3/2} \xi(\theta u)^{3/2} \overline{\lambda}(\theta u)^2 \overline{\kappa}(\theta u) \int_{\mathbb{R}} \cos^2(\upsilon_0 \varsigma_0 \xi(\theta u) k) \widetilde{\Phi}(k)^2 dk du
\end{aligned}$$

is a small-T approximation of $\mathrm{AVar}_T^{\mathbb{Q}}$ in the case where $\sigma^{\mathbb{Q}}$ also satisfies (4.2). Since most information about $\mathbb{Q}$-diffusive variance is contained in around-the-money options as the time-to-maturity of the options shrinks to zero, only the intraday patterns of at-the-money option characteristics, $\overline{\lambda}$ and $\overline{\kappa}$, enter (C.4).

D Variance Disagreement over Multiple Days

We consider estimation of relative IVD based on D days of data, where D can potentially increase to infinity at a slow speed. In this case, higher-order terms in (3.25) may no longer be negligible and require bias correction, which we detail in this section. If D is fixed (e.g., if $D = 1$), the corrected estimators below can be viewed as finite-sample improvements of rRVD_T^n.

We use a subscript d to signify quantities computed on day d. To simplify notation, we omit reference to n and T. For example, rRVD_d signifies rRVD^n_T computed on day d. We also recall from (6.3) and (6.4) the definitions

$$\overline{\mathrm{rIVD}} = \frac{1}{D}\sum_{d=1}^{D} \mathrm{rIVD}_d, \quad \widehat{\mathrm{rIVD}} = \frac{1}{D}\sum_{d=1}^{D} \mathrm{rRVD}_d,$$
$$\mathrm{S}^2(\mathrm{rIVD}) = \frac{1}{D}\sum_{d=1}^{D} \mathrm{rIVD}_d^2, \quad \widehat{\mathrm{S}}^2(\mathrm{rIVD}) = \frac{1}{D}\sum_{d=1}^{D} \mathrm{rRVD}_d^2.$$

Our aim is to conduct inference on $\overline{\mathrm{rIVD}}$, the average relative IVD across D days, and $\mathrm{S}(\mathrm{rIVD})$, the root mean square of relative IVD within the same sample period, using the realized counterparts $\widehat{\mathrm{rIVD}}$ and $\widehat{\mathrm{S}}(\mathrm{rIVD})$. Note that rIVD_d can change signs from one day to another. Therefore, $\mathrm{S}(\mathrm{rIVD})$ measures the typical magnitude (of either sign) of relative IVD, while $\overline{\mathrm{rIVD}}$ is the average size of relative IVD (with possible cancellation between days).

In order to identify bias terms in $\overline{\mathrm{rIVD}}$ and $\mathrm{S}(\mathrm{rIVD})$, we consider a bias–variance decomposition of the estimators of $\mathbb{P}$- and $\mathbb{Q}$-diffusive variance. To this end, we define

$$\mathrm{VP}^n_T = \sum_{i=2}^{k_n} V^{\mathbb{P}}_{i,n}(\widehat{u}_{\underline{t}^n_{i-1}}), \qquad V^{\mathbb{P}}_{i,n}(u) = -\frac{2}{u^2}\mathbb{E}\Big[\Re\big(e^{iu(x_{\overline{t}^n_i}-x_{\underline{t}^n_i})}-1\big) \mid \mathcal{F}_{\underline{t}^n_i}\Big],$$
$$\widetilde{\mathrm{VP}}^n_T = \sum_{i=2}^{k_n}\Big(\widehat{V}^{\mathbb{P}}_{i,n}(\widehat{u}_{\underline{t}^n_{i-1}}) - V^{\mathbb{P}}_{i,n}(\widehat{u}_{\underline{t}^n_{i-1}})\Big),$$
$$\mathrm{VQ}^n_T = \sum_{i=2}^{k_n} V^{\mathbb{Q}}_{i,n}(\widehat{u}_{\underline{t}^n_{i-1}}), \qquad V^{\mathbb{Q}}_{i,n}(u) = -\frac{2}{u^2}\Re\left(\frac{\mathcal{L}^{\mathbb{Q}}_{\underline{t}^n_i,T}(u)}{\mathcal{L}^{\mathbb{Q}}_{\overline{t}^n_i,T}(u)} - 1\right),$$
$$\widetilde{\mathrm{VQ}}^n_T = -\sum_{i=2}^{k_n}\frac{2}{\widehat{u}^2_{\underline{t}^n_{i-1}}}\left\{\frac{\Re(\mathcal{L}^{\mathbb{Q}}_{\overline{t}^n_i,T}(\widehat{u}_{\underline{t}^n_{i-1}}))}{|\mathcal{L}^{\mathbb{Q}}_{\overline{t}^n_i,T}(\widehat{u}_{\underline{t}^n_{i-1}})|^2}\Re\Big(Z^{\mathcal{L}}_{\underline{t}^n_i,T}(\widehat{u}_{\underline{t}^n_{i-1}})\Big)\right.$$
$$\left.-\frac{\Re(\mathcal{L}^{\mathbb{Q}}_{\underline{t}^n_{i-1}}(\widehat{u}_{\underline{t}^n_{i-1}}))\Re(\mathcal{L}^{\mathbb{Q}}_{\overline{t}^n_i,T}(\widehat{u}_{\underline{t}^n_{i-1}}))^2}{|\mathcal{L}^{\mathbb{Q}}_{\overline{t}^n_i,T}(\widehat{u}_{\underline{t}^n_{i-1}})|^4}\Re\Big(Z^{\mathcal{L}}_{\overline{t}^n_i,T}(\widehat{u}_{\underline{t}^n_{i-1}})\Big)\right\},$$
$$\mathrm{VQ}'^n_T = -\sum_{i=2}^{k_n}\frac{2}{\widehat{u}^2_{\underline{t}^n_{i-1}}}\frac{\Re(\mathcal{L}^{\mathbb{Q}}_{\overline{t}^n_i,T}(\widehat{u}_{\underline{t}^n_{i-1}}))^3\Re(\mathcal{L}^{\mathbb{Q}}_{\underline{t}^n_i,T}(\widehat{u}_{\underline{t}^n_{i-1}}))}{|\mathcal{L}^{\mathbb{Q}}_{\overline{t}^n_i,T}(\widehat{u}_{\underline{t}^n_{i-1}})|^6}\Big(\Re(Z^{\mathcal{L}}_{\overline{t}^n_i,T}(\widehat{u}_{\underline{t}^n_{i-1}}))^2 - \Im(Z^{\mathcal{L}}_{\overline{t}^n_i,T}(\widehat{u}_{\underline{t}^n_{i-1}}))^2\Big),$$
$$\mathrm{VQ}''^n_T = -\sum_{i=2}^{k_n}\frac{2}{\widehat{u}^2_{\underline{t}^n_{i-1}}}\left\{\frac{\Re(\mathcal{L}^{\mathbb{Q}}_{\overline{t}^n_i,T}(\widehat{u}_{\underline{t}^n_{i-1}}))}{|\mathcal{L}^{\mathbb{Q}}_{\overline{t}^n_i,T}(\widehat{u}_{\underline{t}^n_{i-1}})|^2}\Re\Big(R^{\mathcal{L}}_{\underline{t}^n_i,T}(\widehat{u}_{\underline{t}^n_{i-1}})\Big)\right.$$
$$\left.-\frac{\Re(\mathcal{L}^{\mathbb{Q}}_{\underline{t}^n_{i-1}}(\widehat{u}_{\underline{t}^n_{i-1}}))\Re(\mathcal{L}^{\mathbb{Q}}_{\overline{t}^n_i,T}(\widehat{u}_{\underline{t}^n_{i-1}}))^2}{|\mathcal{L}^{\mathbb{Q}}_{\overline{t}^n_i,T}(\widehat{u}_{\underline{t}^n_{i-1}})|^4}\Re\Big(R^{\mathcal{L}}_{\overline{t}^n_i,T}(\widehat{u}_{\underline{t}^n_{i-1}})\Big)\right\},$$

where $Z^{\mathcal{L}}_{t,T}(u)$ and $R^{\mathcal{L}}_{t,T}(u)$ are defined in (E.1). Then the estimators of $\mathbb{P}$- and $\mathbb{Q}$-diffusive variance from (3.5) and (3.8) satisfy

$$\widehat{\mathrm{VP}}_T^n = \sum_{i=2}^{k_n} \widehat{V}_{i,n}^{\mathbb{P}}(\widehat{u}_{\underline{t}_{i-1}^n}) = \mathrm{VP}_T^n + \widetilde{\mathrm{VP}}_T^n \tag{D.1}$$

and

$$\widehat{\mathrm{VQ}}_T^n = \sum_{i=2}^{k_n} \widehat{V}_{i,n}^{\mathbb{Q}}(\widehat{u}_{\underline{t}_{i-1}^n}) = \mathrm{VQ}_T^n + \widetilde{\mathrm{VQ}}_T^n + \mathrm{VQ}_T'^n + \mathrm{VQ}_T''^n + \text{higher-order terms.} \tag{D.2}$$

In (D.1) and (D.2), VP_T^n and VQ_T^n are the estimands, $\widetilde{\mathrm{VP}}_T^n$ and $\widetilde{\mathrm{VQ}}_T^n$ are variance terms, and $\mathrm{VQ}_T'^n$ and $\mathrm{VQ}_T''^n$ are bias terms. The following theorem identifies the leading bias terms in the estimation of $\overline{\mathrm{rIVD}}$ and $\mathrm{S(rIVD)}$. We give a proof at the end of Section E.3.

Theorem 2*. *Suppose that the assumptions of Theorem 1 are satisfied on* $[0, DT]$. *Further suppose that* $D = O(\sqrt{k_n})$. *Then*

$$\frac{\widehat{\overline{\mathrm{rIVD}}} - \overline{\mathrm{rIVD}} - \frac{1}{D}\sum_{d=1}^{D}\left(\frac{\mathrm{VQ}_d' + \mathrm{VQ}_d''}{\mathrm{VP}_d} + \frac{\mathrm{VQ}_d \widetilde{\mathrm{VP}}_d^2}{\mathrm{VP}_d^3}\right)}{\sqrt{\frac{1}{D^2}\sum_{d=1}^{D}\mathrm{AVar}(\mathrm{rRVD}_d)}} \xrightarrow{\mathcal{L}-s} N(0,1), \tag{D.3}$$

where $\mathrm{AVar}(\mathrm{rRVD}_d)$ *is the expression in* (3.26) *computed on day* d. *Furthermore,*

$$\begin{aligned} &\frac{1}{\sqrt{\frac{4}{D^2}\sum_{d=1}^{D}\mathrm{rIVD}_d^2\mathrm{AVar}(\mathrm{rRVD}_d)}}\left\{\widehat{\mathrm{S}}^2(\mathrm{rIVD}) - \mathrm{S}^2(\mathrm{rIVD}) - \frac{1}{D}\sum_{d=1}^{D}\left[\frac{\widetilde{\mathrm{VQ}}_d^2 + (\mathrm{VQ}_d' + \mathrm{VQ}_d'')^2}{\mathrm{VP}_d^2}\right.\right. \\ &\left.\left. + \frac{\mathrm{VQ}_d^2\widetilde{\mathrm{VP}}_d^2}{\mathrm{VP}_d^4} + 2\,\mathrm{rIVD}_d\left(\frac{\mathrm{VQ}_d' + \mathrm{VQ}_d''}{\mathrm{VP}_d} + \frac{\mathrm{VQ}_d\widetilde{\mathrm{VP}}_d^2}{\mathrm{VP}_d^3}\right)\right]\right\} \xrightarrow{\mathcal{L}-s} N(0,1). \end{aligned} \tag{D.4}$$

Our feasible implementation of Theorem 2* uses the following bias-corrected estimators of $\overline{\mathrm{rIVD}}$ and $\mathrm{S}^2(\mathrm{rIVD})$:

$$\widehat{\overline{\mathrm{rIVD}}}^{\mathrm{corr}} = \frac{1}{D}\sum_{d=1}^{D}\mathrm{rRVD}_d^{\mathrm{corr}}, \quad \mathrm{rRVD}_d^{\mathrm{corr}} = \mathrm{rRVD}_d - \left(\frac{\widehat{\mathrm{VQ}}_d'}{\widehat{\mathrm{VP}}_d} + \frac{\widehat{\mathrm{VQ}}_d\widehat{\mathrm{AVar}}_d^{\mathbb{P}}}{\widehat{\mathrm{VP}}_d^3}\right), \tag{D.5}$$

$$\begin{aligned} \widehat{\mathrm{S}}^2(\mathrm{rIVD})^{\mathrm{corr}} = \widehat{\mathrm{S}}^2(\mathrm{rIVD}) - \frac{1}{D}\sum_{d=1}^{D}&\left[\frac{\widehat{\mathrm{AVar}}_d^{\mathbb{Q}} + (\widehat{\mathrm{VQ}}_d')^2}{\widehat{\mathrm{VP}}_d^2} + \frac{\widehat{\mathrm{VQ}}_d^2\widehat{\mathrm{AVar}}_d^{\mathbb{P}}}{\widehat{\mathrm{VP}}_d^4}\right. \\ &\left. + 2\,\mathrm{rRVD}_d\left(\frac{\widehat{\mathrm{VQ}}_d'}{\widehat{\mathrm{VP}}_d} + \frac{\widehat{\mathrm{VQ}}_d\widehat{\mathrm{AVar}}_d^{\mathbb{P}}}{\widehat{\mathrm{VP}}_d^3}\right)\right], \end{aligned} \tag{D.6}$$

where

$$\widehat{\mathrm{VQ}}_T'^n = -\sum_{i=2}^{k_n} \frac{\Re(\widehat{\mathcal{L}}^{\mathbb{Q}}_{\overline{t}_i^n,T}(\widehat{u}_{\underline{t}_{i-1}^n}))^3 \Re(\widehat{\mathcal{L}}^{\mathbb{Q}}_{\underline{t}_i^n,T}(\widehat{u}_{\underline{t}_{i-1}^n}))}{|\widehat{\mathcal{L}}^{\mathbb{Q}}_{\overline{t}_i^n,T}(\widehat{u}_{\underline{t}_{i-1}^n})|^6} \times \frac{\Re(\widehat{\mathcal{L}}^{\mathbb{Q}}_{\overline{t}_i^n,T}(\widehat{u}_{\underline{t}_{i-1}^n}) - \widehat{\mathcal{L}}^{\mathbb{Q}}_{\underline{t}_i^n,T}(\widehat{u}_{\underline{t}_{i-1}^n}))^2 - \Im(\widehat{\mathcal{L}}^{\mathbb{Q}}_{\overline{t}_i^n,T}(\widehat{u}_{\underline{t}_{i-1}^n}) - \widehat{\mathcal{L}}^{\mathbb{Q}}_{\underline{t}_i^n,T}(\widehat{u}_{\underline{t}_{i-1}^n}))^2}{\widehat{u}^2_{\underline{t}_{i-1}^n}} \tag{D.7}$$

and $\widehat{\mathrm{AVar}}_d^{\mathbb{P}}$, $\widehat{\mathrm{AVar}}_d^{\mathbb{Q}}$, $\widehat{\mathrm{VP}}_d$, $\widehat{\mathrm{VQ}}_d$ and $\widehat{\mathrm{VQ}}_d'$ denote the quantities in (3.28), (3.29), (D.1), (D.2) and (D.7) computed on day d. We further propose the following estimators of the asymptotic variances in (D.3) and (D.4):

$$\frac{1}{D^2}\sum_{d=1}^{D} \widehat{\mathrm{AVar}}(\mathrm{rRVD}_d) \approx \frac{1}{D^2}\sum_{d=1}^{D} \mathrm{AVar}(\mathrm{rRVD}_d),$$
$$\frac{4}{D^2}\sum_{d=1}^{D} (\mathrm{rRVD}_d^{\mathrm{corr}})^2 \widehat{\mathrm{AVar}}(\mathrm{rRVD}_d) \approx \frac{4}{D^2}\sum_{d=1}^{D} \mathrm{rIVD}_d^2 \mathrm{AVar}(\mathrm{rRVD}_d),$$

where $\widehat{\mathrm{AVar}}(\mathrm{rRVD}_d)$ is the day d version of (3.27). In the Monte Carlo experiment as well as in the empirical application, we account for the bias due to VQ_d'' by linearly interpolating the available strikes in Black–Scholes implied volatility space before computing the Riemann sum approximation in (A.4) (see Appendix A for further details).

E Proofs

E.1 Notation and Decomposition of Terms

For $\mathbb{S} \in \{\mathbb{P}, \mathbb{Q}\}$, $t, t_1, t_2 \in [0, T]$ and $u > 0$, we define $\zeta^{\mathbb{P}} \equiv 0$, $\zeta^{\mathbb{Q}} = \zeta$ and

$$\mathcal{L}^{\mathbb{S}}_{t_1,t_2}(u) = \mathbb{E}^{\mathbb{S}}_{t_1}\left(e^{iu(x_{t_2}-x_{t_1})}\right), \quad \overline{\mathcal{L}}^{\mathbb{S}}_{t_1,t_2}(u) = \exp(\Psi^{\mathbb{S}}_{t_1,t_2}(u)),$$
$$\Psi^{\mathbb{S}}_{t_1,t_2}(u) = iu\alpha^{\mathbb{S}}_{t_1}(t_2-t_1) - \frac{u^2}{2}\left(\varsigma^2_{t_1}\int_{t_1}^{t_2}\xi(t/T)dt + \rho_{t_1}\int_{t_1}^{t_2}\zeta^{\mathbb{S}}(t/T)dt\right) + \psi^{\mathbb{S}}_{t_1}(u)(t_2-t_1),$$

where $\psi^{\mathbb{S}}_t(u)$ was introduced in (B.3). We also make the following bias–variance decomposition of the option-based characteristic function estimator:

$$\widehat{\mathcal{L}}^{\mathbb{Q}}_{t,T}(u) - \mathcal{L}^{\mathbb{Q}}_{t,T}(u) = Z^{\mathcal{L}}_{t,T}(u) + R^{\mathcal{L}}_{t,T}(u), \tag{E.1}$$

where

$$Z^{\mathcal{L}}_{t,T}(u) = -(u^2+iu)\sum_{j=2}^{N_t} e^{iu(k_{t,j-1}-x_t)-k_{t,j-1}}\epsilon_{t,T}(K_{t,j-1})(k_{t,j}-k_{t,j-1}) \tag{E.2}$$

and

$$R^{\mathcal{L}}_{t,T}(u) = (u^2+iu)\sum_{j=2}^{N_t}\int_{k_{t,j-1}}^{k_{t,j}}[e^{iu(k-x_t)-k}O_{t,T}(e^k) - e^{iu(k_{t,j-1}-x_t)-k_{t,j-1}}O_{t,T}(K_{t,j-1})]dk \\ - (u^2+iu)\Big(\int_{-\infty}^{k_{t,1}} e^{iu(k-x_t)-k}O_{t,T}(e^k)dk + \int_{k_{t,N_t}}^{\infty} e^{iu(k-x_t)-k}O_{t,T}(e^k)dk\Big). \tag{E.3}$$

With this notation, we decompose $\widehat{Z}_{t_1,t_2}(u)$ into six terms:

$$\widehat{Z}_{t_1,t_2}(u) = -\Re\Big(Z^{\mathbb{P}}_{t_1,t_2}(u) + B^{\mathbb{P}}_{t_1,t_2}(u) + Z^{\mathbb{Q}}_{t_1,t_2}(u) + R^{\mathbb{Q}}_{t_1,t_2}(u) + B^{\mathbb{Q}}_{t_1,t_2}(u) + D_{t_1,t_2}(u)\Big), \tag{E.4}$$

where

$$Z^{\mathbb{P}}_{t_1,t_2}(u) = -\frac{2}{u^2}\big(e^{iu(x_{t_2}-x_{t_1})} - \mathcal{L}^{\mathbb{P}}_{t_1,t_2}(u)\big), \quad B^{\mathbb{P}}_{t_1,t_2}(u) = -\frac{2}{u^2}\big(\mathcal{L}^{\mathbb{P}}_{t_1,t_2}(u) - \overline{\mathcal{L}}^{\mathbb{P}}_{t_1,t_2}(u)\big), \tag{E.5}$$

$$Z^{\mathbb{Q}}_{t_1,t_2}(u) = -\frac{2}{u^2}\bigg(\frac{\mathcal{L}^{\mathbb{Q}}_{t_1,T}(u)}{\mathcal{L}^{\mathbb{Q}}_{t_2,T}(u)^2}Z^{\mathcal{L}}_{t_2,T}(u) - \frac{Z^{\mathcal{L}}_{t_1,T}(u)}{\mathcal{L}^{\mathbb{Q}}_{t_2,T}(u)}\bigg), \tag{E.6}$$

$$R^{\mathbb{Q}}_{t_1,t_2}(u) = -\frac{2}{u^2}\frac{(\widehat{\mathcal{L}}^{\mathbb{Q}}_{t_1,T}(u) - \mathcal{L}^{\mathbb{Q}}_{t_1,T}(u))(\widehat{\mathcal{L}}^{\mathbb{Q}}_{t_2,T}(u) - \widehat{\mathcal{L}}^{\mathbb{Q}}_{t_2,T}(u))}{\mathcal{L}^{\mathbb{Q}}_{t_2,T}(u)\widehat{\mathcal{L}}^{\mathbb{Q}}_{t_2,T}(u)} \\ + \frac{2}{u^2}\frac{\mathcal{L}^{\mathbb{Q}}_{t_1,T}(u)(\widehat{\mathcal{L}}^{\mathbb{Q}}_{t_2,T}(u) - \mathcal{L}^{\mathbb{Q}}_{t_2,T}(u))^2}{\mathcal{L}^{\mathbb{Q}}_{t_2,T}(u)^2\widehat{\mathcal{L}}^{\mathbb{Q}}_{t_2,T}(u)} - \frac{2}{u^2}\bigg(\frac{\mathcal{L}^{\mathbb{Q}}_{t_1,T}(u)}{\mathcal{L}^{\mathbb{Q}}_{t_2,T}(u)^2}R^{\mathcal{L}}_{t_2,T}(u) - \frac{R^{\mathcal{L}}_{t_1,T}(u)}{\mathcal{L}^{\mathbb{Q}}_{t_2,T}(u)}\bigg), \tag{E.7}$$

$$B^{\mathbb{Q}}_{t_1,t_2}(u) = -\frac{2}{u^2}\bigg(\overline{\mathcal{L}}^{\mathbb{Q}}_{t_1,t_2}(u) - \frac{\mathcal{L}^{\mathbb{Q}}_{t_1,T}(u)}{\mathcal{L}^{\mathbb{Q}}_{t_2,T}(u)}\bigg), \quad D_{t_1,t_2}(u) = -\frac{2}{u^2}\big(\overline{\mathcal{L}}^{\mathbb{P}}_{t_1,t_2}(u) - \overline{\mathcal{L}}^{\mathbb{Q}}_{t_1,t_2}(u)\big). \tag{E.8}$$

In what follows, we use the shorthand notation

$$X_{i,n}(u) = X_{\underline{t}^n_i,\overline{t}^n_i}(u), \quad i = 1,\dots,k_n, \tag{E.9}$$

for $X \in \{Z, Z^{\mathbb{S}}, R^{\mathbb{Q}}, B^{\mathbb{S}}, D\}$ and $\mathbb{S} = \mathbb{P},\mathbb{Q}$.

The signal term is $D_{t_1,t_2}(u)$ and converges, when aggregated over the trading day, to integrated variance disagreement. It captures the $\mathbb{P}$–$\mathbb{Q}$ difference in diffusive variance over the interval $[t_1,t_2]$, extracted from conditional characteristic functions after freezing the coefficients of the asset price at the beginning of the interval, t_1. The terms $B^{\mathbb{P}}_{t_1,t_2}(u)$ and $B^{\mathbb{Q}}_{t_1,t_2}(u)$ contain the biases due to this discretization. Their magnitudes are mainly depend on the smoothness parameter of the coefficient processes, α. The approximation error in (3.10) is also part of $B^{\mathbb{Q}}_{t_1,t_2}(u)$.

There are two variance terms, $Z^{\mathbb{P}}_{t_1,t_2}(u)$ and $Z^{\mathbb{Q}}_{t_1,t_2}(u)$. The former accounts for variation in estimating $\mathbb{P}$-diffusive variance by $\widehat{V}^{\mathbb{P}}$. The terms $Z^{\mathbb{Q}}_{t_1,t_2}(u)$ and $R^{\mathbb{Q}}_{t_1,t_2}(u)$ both arise from estimating $\mathcal{L}^{\mathbb{Q}}_{t_i,T}(u)$ by its feasible counterpart $\widehat{\mathcal{L}}^{\mathbb{Q}}_{t_i,T}(u)$, where $i = 1,2$. While $Z^{\mathbb{Q}}_{t_1,t_2}(u)$ is a centered noise term, $R^{\mathbb{Q}}_{t_1,t_2}(u)$ is a remainder term of higher asymptotic order.

E.2 Preliminary Bounds

A classical localization argument (cf. Jacod and Protter (2012)) shows that in the subsequent results, there is no loss of generality if we assume that all locally bounded processes are uniformly bounded by a fixed constant and that the localizing sequence in the assumptions is $\tau_n = \infty$. The first lemma quantifies the error incurred by freezing coefficients at the beginning of a high-frequency interval.

Lemma 1. *Suppose Assumptions 1–3 hold. For $0 \leq t_1 \leq t_2 \leq T$, $u > 0$ and $\mathbb{S} \in \{\mathbb{P}, \mathbb{Q}\}$,*

$$|\mathcal{L}^{\mathbb{S}}_{t_1,t_2}(u) - \overline{\mathcal{L}}^{\mathbb{S}}_{t_1,t_2}(u)| \leq C|u||t_2 - t_1|^{1/2+\alpha}, \tag{E.10}$$

where C is a constant that does not depend on u, t_1 and t_2.

The next two lemmas contain bounds on $Z^{\mathbb{Q}}_{i,n}(u)$ and $R^{\mathbb{Q}}_{i,n}(u)$, respectively.

Lemma 2. *Suppose Assumptions 1–4 hold. If $k_n\Delta_n \asymp T$, $k_n T^{\iota} \to 0$ for any $\iota > 0$ and $\widehat{u}_t$ satisfies (3.6), then there exists a constant $C > 0$ such that*

$$\sum_{i=2}^{k_n} w_\eta(\tfrac{i-1}{k_n})\Re(Z^{\mathbb{Q}}_{i,n}(\widehat{u}_{(i-2)\Delta_n})) = O_p(\sqrt{\delta k_n T^{3/2}}), \tag{E.11}$$

$$\sum_{i=2}^{k_n} w_\eta(\tfrac{i-1}{k_n})^2\Re(Z^{\mathbb{Q}}_{i,n}(\widehat{u}_{(i-2)\Delta_n}))^2 = O_p(\delta k_n T^{3/2}), \tag{E.12}$$

$$\sum_{i=2}^{k_n} w_\eta(\tfrac{i-1}{k_n})^2\Re(Z^{\mathbb{P}}_{i,n}(\widehat{u}_{(i-2)\Delta_n}))\Re(Z^{\mathbb{Q}}_{i,n}(\widehat{u}_{(i-2)\Delta_n})) = O_p(\sqrt{\delta T^{5/2}\Delta_n}), \tag{E.13}$$

$$\sum_{i=2}^{k_n} w_\eta(\tfrac{i-1}{k_n})^2\Re(D_{i,n}(\widehat{u}_{(i-2)\Delta_n}))\Re(Z^{\mathbb{Q}}_{i,n}(\widehat{u}_{(i-2)\Delta_n})) = O_p(\sqrt{\delta T^{5/2}\Delta_n}), \tag{E.14}$$

uniformly in $\eta \in [0, 2\pi]$.

Lemma 3. *Suppose Assumptions 1–4 hold. If $k_n\Delta_n \asymp T$, $\delta\log(1/T)^2/\sqrt{T} \to 0$, $k_n\delta/T^{1/2} \to 0$, $k_n T^{\iota} \to 0$ for any $\iota > 0$ and $\widehat{u}_t$ satisfies (3.6), then we have*

$$\sum_{i=2}^{k_n} \left|\Re(R^{\mathbb{Q}}_{i,n}(\widehat{u}_{(i-2)\Delta_n}))\right| = O_p(\delta k_n\sqrt{T}\log(1/T)), \tag{E.15}$$

$$\sum_{i=2}^{k_n} \left|\Re(R^{\mathbb{Q}}_{i,n}(\widehat{u}_{(i-2)\Delta_n}))\right|^2 = O_p(\delta^2 k_n T\log^2(1/T)), \tag{E.16}$$

$$\sum_{i=2}^{k_n}\left(\left|\Re(Z^{\mathbb{P}}_{i,n}(\widehat{u}_{(i-2)\Delta_n}))\right| + \left|\Re(D_{i,n}(\widehat{u}_{(i-2)\Delta_n}))\right|\right)\left|\Re(R^{\mathbb{Q}}_{i,n}(\widehat{u}_{(i-2)\Delta_n}))\right| = O_p(\delta T^{3/2}\log(1/T)). \tag{E.17}$$

We continue with bounds on the bias terms $B^{\mathbb{P}}_{i,n}(u)$, $B^{\mathbb{Q}}_{i,n}(u)$ as well as results about the signal term $D_{i,n}(u)$.

Lemma 4. *Suppose Assumptions 1 and 2 hold. If $k_n\Delta_n \asymp T$ and $\widehat{u}_t$ satisfies (3.6), then*

$$\sum_{i=2}^{k_n} \left|\Re(B^{\mathbb{P}}_{i,n}(\widehat{u}_{(i-2)\Delta_n}))\right| = O_p(\sqrt{k_n}T\Delta_n^{\alpha}), \quad \sum_{i=2}^{k_n} \left|\Re(B^{\mathbb{P}}_{i,n}(\widehat{u}_{(i-2)\Delta_n}))\right|^2 = O_p(T^2\Delta_n^{2\alpha}). \tag{E.18}$$

Lemma 5. *Suppose Assumptions 1–3 hold. If $k_n\Delta_n = \theta T$ and $\widehat{u}_t$ satisfies (3.6), then*

$$\sum_{i=2}^{k_n} \left|\Re(B^{\mathbb{Q}}_{i,n}(\widehat{u}_{(i-2)\Delta_n}))\right| = O_p(k_n T^{1+\alpha}), \quad \sum_{i=2}^{k_n} \left|\Re(B^{\mathbb{Q}}_{i,n}(\widehat{u}_{(i-2)\Delta_n}))\right|^2 = O_p(k_n T^{2+2\alpha}). \tag{E.19}$$

Lemma 6. *Suppose that Assumptions 1 and 2 hold. If $k_n\Delta_n = \theta T$, $\Delta_n' = \varphi\Delta_n$ and $\widehat{u}_t$ satisfies (3.6), then*

$$\begin{aligned}&\sum_{i=2}^{k_n} w_\eta(\tfrac{i-1}{k_n}) D_{i,n}(\widehat{u}_{(i-2)\Delta_n}) \\ &\quad = -\theta\varphi T\int_0^1 w_\eta(u)\zeta(\theta u)\rho_{u\theta T}du + O_p(T\Delta_n^{\alpha} + T^{2-\beta/2} + \Delta_n + Tk_n^{-1/\beta}),\end{aligned} \tag{E.20}$$

$$\sum_{i=2}^{k_n} w_\eta(\tfrac{i-1}{k_n})^2 \Re(D_{i,n}(\widehat{u}_{(i-2)\Delta_n}))^2 = \varphi\theta T\Delta_n' \int_0^1 w_\eta(u)^2\zeta(\theta u)^2\rho_{u\theta T}^2 du + o_p(T\Delta_n), \tag{E.21}$$

$$\sum_{i=2}^{k_n} w_\eta(\tfrac{i-1}{k_n})^2 \Re(Z^{\mathbb{P}}_{i,n}(\widehat{u}_{(i-2)\Delta_n}))\Re(D_{i,n}(\widehat{u}_{(i-2)\Delta_n})) = O_p(\sqrt{T}\Delta_n^{3/2} + T^{3/2-\beta/4}\Delta_n), \tag{E.22}$$

uniformly in $\eta \in [0, 2\pi]$.

In the case $\eta = 0$, (4.2) shows that

$$\theta\varphi T\int_0^1 \zeta(\theta u)\rho_{u\theta T}du = \varphi\int_0^{\theta T}\zeta(t/T)\rho_t dt = \varphi\mathrm{IVD}_{\theta T}. \tag{E.23}$$

E.3 Proof of Main Theorems

Recall (4.5) and define, for $\eta \in [0, 2\pi]$,

$$\widehat{\mathcal{V}}^n_T(\eta) = \sum_{i=2}^{k_n} w_\eta(\tfrac{i-1}{k_n})^2(\widehat{Z}_{i,n}(\widehat{u}_{(i-2)\Delta_n}))^2.$$

If (3.6) and Assumptions 1–5 hold, we first use the preliminary bounds above to show that

$$\mathrm{RVD}^n_T(\eta) = -\sum_{i=2}^{k_n} w_\eta(\tfrac{i-1}{k_n})\Re\left[(Z^{\mathbb{P}}_{i,n} + Z^{\mathbb{Q}}_{i,n} + D_{i,n})(\widehat{u}_{(i-2)\Delta_n})\right] + o_p(\sqrt{T\Delta_n}), \tag{E.24}$$

$$\widehat{\mathcal{V}}^n_T(\eta) = \sum_{i=2}^{k_n} w_\eta(\tfrac{i-1}{k_n})^2 \Re\big[(Z^{\mathbb{P}}_{i,n} + Z^{\mathbb{Q}}_{i,n} + D_{i,n})(\widehat{u}_{(i-2)\Delta_n})\big]^2 + o_p(T\Delta_n), \tag{E.25}$$

$$\widehat{\mathcal{M}}^n_T = \theta\varphi T\Delta'_n \int_0^1 \rho^2_{u\theta T}\zeta(\theta u)^2 du + o_p(T\Delta_n). \tag{E.26}$$

The starting point in all three cases is the decomposition (E.4). For (E.24), note that $|w_\eta(u)| \le 2$, so the contributions of $R^{\mathbb{Q}}_{i,n}(u)$, $B^{\mathbb{P}}_{i,n}(u)$ and $B^{\mathbb{Q}}_{i,n}(u)$ (with $u = \widehat{u}_{(i-2)\Delta_n}$) to $\mathrm{RVD}^n_T(\eta)$ are $O_p(\delta k_n\sqrt{T}\log(1/T))$, $O_p(\sqrt{k_n}T\Delta_n^\alpha)$ and $O_p(k_n T^{1+\alpha})$ by Lemmas 3, 4 and 5, respectively. Since $k_n\Delta_n = \theta T$, we have

$$\begin{aligned} O_p(\sqrt{k_n}T\Delta_n^\alpha) &= O_p(\sqrt{T\Delta_n}(k_n T^\iota)^{1-\alpha}) \quad (\text{with } \iota = \alpha/(1-\alpha)),\\ O_p(\delta k_n\sqrt{T}\log(1/T)) &= O_p(\sqrt{T\Delta_n}\delta k_n^{3/2}T^{-1/2}\log(1/T)),\\ O_p(k_n T^{1+\alpha}) &= O_p(\sqrt{T\Delta_n}(k_n T^\iota)^{3/2}) \quad (\text{with } \iota = 2\alpha/3). \end{aligned}$$

Assumption 5 implies that these terms are $o_p(\sqrt{T\Delta_n})$, proving (E.24).

Equation (E.25) is proved in a similar fashion. Viewing $Z^{\mathbb{P}}_{i,n} + D_{i,n}$ as a single term, expanding the square in (E.4) results in 25 expressions, out of which only quadratic terms involving $Z^{\mathbb{P}}_{i,n} + D_{i,n}$ and $Z^{\mathbb{Q}}_{i,n}$ are kept in (E.25). To verify that the remaining products are negligible, we observe that the sums involving squares of $R^{\mathbb{Q}}_{i,n}$, $B^{\mathbb{P}}_{i,n}$ and $B^{\mathbb{Q}}_{i,n}$ are $O_p(\delta^2 k_n T\log^2(1/T) + T^2\Delta_n^{2\alpha} + k_n T^{2+2\alpha}) = o_p(T\Delta_n)$ by (E.16), (E.18), (E.19) and the rate conditions of Assumption 5. The Cauchy–Schwarz inequality implies that also product terms involving only $R^{\mathbb{Q}}_{i,n}$, $B^{\mathbb{P}}_{i,n}$ and $B^{\mathbb{Q}}_{i,n}$ have $o_p(T\Delta_n)$ contributions. Using (E.17), one can also show that the product of $Z^{\mathbb{P}}_{i,n} + D_{i,n}$ and $R^{\mathbb{Q}}_{i,n}$ is $o_p(T\Delta_n)$. By (E.12), (E.16), (E.18), (E.19) and the Cauchy–Schwarz inequality, the same is true for the products of $Z^{\mathbb{Q}}_{i,n}$ with $R^{\mathbb{Q}}_{i,n}$, $B^{\mathbb{P}}_{i,n}$ and $B^{\mathbb{Q}}_{i,n}$. The products of $Z^{\mathbb{P}}_{i,n} + D_{i,n}$ with $B^{\mathbb{P}}_{i,n}$ and $B^{\mathbb{Q}}_{i,n}$ also have $o_p(T\Delta_n)$ contributions, which can be shown by combining the Cauchy–Schwarz inequality with (E.18), (E.19), (E.21) and the fact that

$$\sum_{i=2}^{k_n} w_\eta(\tfrac{i-1}{k_n})^2 \Re\big(Z^{\mathbb{P}}_{i,n}(\widehat{u}_{(i-2)\Delta_n})\big)^2 = O_p(\Delta_n T). \tag{E.27}$$

This estimate in turn follows by combining (E.30) below with (E.21) and (E.22).

To prove (E.26), we observe that $D_{i,n}(\widehat{u}_{(i-2)\Delta_n})$ is the dominating term in $\widehat{M}_{j,n}(\widehat{u}_{(i-2)\Delta_n})$:

$$\begin{aligned} \widehat{M}_{j,n}(\widehat{u}_{(i-2)\Delta_n}) &= -\frac{1}{\ell_n}\sum_{i=(j-1)\ell_n+1}^{j\ell_n} \Re(D_{i,n}(\widehat{u}_{(i-2)\Delta_n})) + o_p(\Delta_n)\\ &= \frac{\theta\varphi T}{\ell_n}\int_{(j-1)\ell_n/k_n}^{j\ell_n/k_n} \rho_{u\theta T}\zeta(\theta u)du + o_p(\Delta_n), \end{aligned} \tag{E.28}$$

where the second line is obtained analogously to (E.20). An analysis similar to the proof of

(E.21) now shows that

$$\widehat{\mathcal{M}}_T^n = \frac{\theta^2\varphi^2T^2}{\ell_n}\sum_{j=1}^{\lfloor k_n/\ell_n\rfloor}\left(\int_{(j-1)\ell_n/k_n}^{j\ell_n/k_n}\rho_{u\theta T}\zeta(\theta u)du\right)^2 + o_p(T\Delta_n)$$
$$= \frac{\theta^2\varphi^2T^2}{k_n}\sum_{j=1}^{\lfloor k_n/\ell_n\rfloor}\int_{(j-1)\ell_n/k_n}^{j\ell_n/k_n}\rho_{u\theta T}^2\zeta(\theta u)^2du + o_p(T\Delta_n)$$
$$= \theta\varphi T\Delta_n'\int_0^1\rho_{u\theta T}^2\zeta(\theta u)^2du + o_p(T\Delta_n),$$

which is (E.26).

Having completed the proof of (E.24)–(E.26), we characterize in the next two lemmas the limiting behavior of the leading terms of $\mathrm{RVD}_T^n(\eta)$ and $\widehat{\mathcal{V}}_T^n$.

Lemma 7. *Suppose Assumptions 1–5 hold and that $k_n\Delta_n = \theta T$ and $\Delta_n' = \varphi\Delta_n$. If $\widehat{u}_t$ satisfies (3.6) and (B.2) and $k_nT^\iota \to 0$ for all $\iota > 0$, then*

$$\left(\frac{1}{\sqrt{k_n[(\Delta_n')^2+\delta T^{3/2}]}}\sum_{i=2}^{k_n} w_\eta(\tfrac{i-1}{k_n})\Re(Z_{i,n}^{\mathbb{P}}(\widehat{u}_{(i-2)\Delta_n}) + Z_{i,n}^{\mathbb{Q}}(\widehat{u}_{(i-2)\Delta_n}))\right)_{\eta\in[0,2\pi]} \overset{\mathcal{L}-s}{\Longrightarrow} Z. \quad \text{(E.29)}$$

Lemma 8. *Suppose Assumptions 1–5 hold and that $k_n\Delta_n = \theta T$ and $\Delta_n' = \varphi\Delta_n$. If $\widehat{u}_t$ satisfies (3.6) and $k_nT^\iota \to 0$ for all $\iota > 0$, we have*

$$\frac{1}{k_n[(\Delta_n')^2+\delta T^{3/2}]}\sum_{i=2}^{k_n} w_\eta(\tfrac{i-1}{k_n})^2[\Re((Z_{i,n}^{\mathbb{P}} + Z_{i,n}^{\mathbb{Q}} + D_{i,n})(\widehat{u}_{(i-2)\Delta_n}))]^2$$
$$= \int_0^1 w_\eta(u)^2\Bigg[2q\varsigma_0^4\xi(\theta u)^2 + q\rho_0^2\zeta(\theta u)^2 + 8(1-q)\varsigma_0^3\widetilde{\lambda}_0^2\widetilde{\kappa}_0 e^{\upsilon_0^2(1-\theta u)^{-1}(\varsigma_0^2\int_{\theta u}^1\xi(v)dv+\rho_0\int_{\theta u}^1\zeta(u)du)}$$
$$\times(1-\theta u)^{3/2}\xi(\theta u)^{3/2}\overline{\lambda}(\theta u)^2\overline{\kappa}(\theta u)\int_{\mathbb{R}}\cos^2(\upsilon_0\varsigma_0\xi(\theta u)k)\widetilde{\Phi}(k)^2dk\Bigg]du + o_p(1). \quad \text{(E.30)}$$

Proof of Theorem 1. By (E.24), (E.20) (with $\eta = 0$) and (E.23),

$$\frac{\mathrm{RVD}_T^n - \varphi\mathrm{IVD}_{\theta T}}{\sqrt{k_n[(\Delta_n')^2+\delta T^{3/2}]}} = -\frac{1}{\sqrt{k_n[(\Delta_n')^2+\delta T^{3/2}]}}\sum_{i=2}^{k_n}\Re(Z_{i,n}^{\mathbb{P}}(\widehat{u}_{(i-2)\Delta_n}) + Z_{i,n}^{\mathbb{Q}}(\widehat{u}_{(i-2)\Delta_n})) + o_p(1).$$

Lemma 7 implies that $(k_n[(\Delta_n')^2+\delta T^{3/2}])^{-1/2}(\mathrm{RVD}_T^n - \varphi\mathrm{IVD}_{\theta T})$ converges stably in law to a centered normal distribution with $\overline{\mathcal{F}}$-conditional variance given by

$$\mathrm{AVar}_0 = \int_0^1\Bigg[2q\varsigma_0^4\xi(\theta u)^2 + 8(1-q)\varsigma_0^3\widetilde{\lambda}_0^2\widetilde{\kappa}_0 e^{\upsilon_0^2(1-\theta u)^{-1}(\varsigma_0^2\int_{\theta u}^1\xi(v)dv+\rho_0\int_{\theta u}^1\zeta(u)du)}$$
$$\times(1-\theta u)^{3/2}\xi(\theta u)^{3/2}\overline{\lambda}(\theta u)^2\overline{\kappa}(\theta u)\int_{\mathbb{R}}\cos^2(\upsilon_0\varsigma_0\xi(\theta u)k)\widetilde{\Phi}(k)^2dk\Bigg]du.$$

The first assertion of the theorem, (3.21), now follows by observing that AVar_T from (C.1) converges in probability to AVar_0 by the continuity properties of the involved coefficients. The second assertion of the theorem, (3.24), is a consequence of the first one and the fact that

$$\frac{1}{k_n[(\Delta_n')^2 + \delta T^{3/2}]}(\widehat{\mathcal{V}}_T^n - \widehat{\mathcal{M}}_T^n) \xrightarrow{\mathbb{P}} \text{AVar}_0 \tag{E.31}$$

by (E.25), (E.26) and (E.30). □

Proof of Theorem 2. The proof of Lemmas 6 and 7 shows that

$$\frac{1}{\sqrt{k_n[(\Delta_n')^2 + \delta T^{3/2}]}}\left(\frac{1}{k_n\Delta_n'}\begin{pmatrix}\widehat{\text{VP}}_T^n \\ \widehat{\text{VQ}}_T^n\end{pmatrix} - \begin{pmatrix}\frac{1}{\theta T}\int_0^{\theta T}(\sigma_t^{\mathbb{P}})^2 dt \\ \frac{1}{\theta T}\int_0^{\theta T}(\sigma_t^{\mathbb{Q}})^2 dt\end{pmatrix}\right)$$

is asymptotically mixed normal with mean 0 and $\overline{\mathcal{F}}$-conditional covariance matrix

$$\begin{pmatrix} q\text{AVar}_T^{\mathbb{P}} & 0 \\ 0 & (1-q)\text{AVar}_T^{\mathbb{Q}}\end{pmatrix}.$$

We derive Theorem 2 from this result and the delta method. □

Proof of Theorem 3. The identity (E.23), property (B.4) and the rate condition $k_n T^\iota \to 0$ for all $\iota > 0$ imply that under H_0,

$$\varphi\text{IVD}_T = \theta\varphi T\rho_0\int_0^1 \zeta(\theta u)du + O_p(T^{1+\alpha}) = O_p(T^{1+\alpha}) = o_p(\sqrt{k_n}\Delta_n').$$

Therefore, by (3.24),

$$\text{MIT}_T^n = \frac{\text{RVD}_T^n - \varphi\text{IVD}_{\theta T}}{\sqrt{\widehat{\mathcal{V}}_T^n - \widehat{\mathcal{M}}_T^n}} + o_p(1) \xrightarrow{\mathcal{L}-s} N(0,1), \tag{E.32}$$

which shows the first part of (4.4). For the second part, note that

$$\begin{aligned}\frac{1}{\sqrt{k_n}}|\text{MIT}_T^n| &= \frac{1}{\sqrt{k_n}}\left|\frac{\text{RVD}_T^n - \varphi\text{IVD}_{\theta T}}{\sqrt{\widehat{\mathcal{V}}_T^n - \widehat{\mathcal{M}}_T^n}} + \frac{\varphi\text{IVD}_{\theta T}}{\sqrt{\widehat{\mathcal{V}}_T^n - \widehat{\mathcal{M}}_T^n}}\right| = \left|\frac{\theta\varphi T\int_0^1\zeta(\theta u)\rho_{u\theta T}du}{k_n\sqrt{(\Delta_n')^2 + \delta T^{3/2}}\sqrt{\text{AVar}_0}}\right| + o_p(1) \\ &\xrightarrow{\mathbb{P}} \frac{\sqrt{q}|\rho_0\int_0^1\zeta(\theta u)du|}{\sqrt{\text{AVar}_0}}.\end{aligned} \tag{E.33}$$

by (E.23), (B.4) and (E.31). By assumption, $\rho_0 \neq 0$ almost surely and $q > 0$, which proves the second part of (4.4). □

Proof of Theorem 4. Similarly to how we obtained (E.32) in the previous proof, we can write the left-hand side of (4.8) under H_0^* as

$$\sup_{\eta\in[0,2\pi]}\left|\frac{1}{\sqrt{k_n}\Delta_n'}\sum_{i=2}^{k_n} w_\eta(\tfrac{i-1}{k_n})\Re(Z_{i,n}^{\mathbb{P}}(\widehat{u}_{(i-2)\Delta_n})+Z_{i,n}^{\mathbb{Q}}(\widehat{u}_{(i-2)\Delta_n}))\right|+o_p(1).$$

Hence, (4.8) follows from Lemma 7, the assumption $q>0$, the continuous mapping theorem and the fact that taking the supremum is a continuous functional on $C([0,2\pi])$ equipped with the uniform topology.

Under H_1^*, similarly to (E.33), we have

$$\sup_{\eta\in[0,2\pi]}\left|\sqrt{k_n}\int_0^1 w_\eta(u)\zeta(\theta u)\rho_{u\theta T}du\right|+o_p(\sqrt{k_n})=|\rho_0|\sqrt{k_n}\sup_{\eta\in[0,2\pi]}\left|\int_0^1 w_\eta(u)\zeta(\theta u)du\right|+o_p(\sqrt{k_n}).$$

Because ζ is not identically zero and càdlàg, it is nonzero on a subset of $[0,\theta]$ with positive Lebesgue measure under the alternative hypothesis. The supremum term on the right-hand side of the previous line is therefore positive by Theorem 1 of Bierens and Ploberger (1997). By assumption, $\rho_0\neq 0$ almost surely, and (4.9) follows.

Conditionally on $\overline{\mathcal{F}}$, the random variables $M_T^{n,b}=\frac{1}{\sqrt{k_n}\Delta_n'}\sup_{\eta\in[0,2\pi]}|\mathrm{RVD}_T^{*n,b}(\eta)|$ are independent and identically distributed as b varies and normal. So if we denote by $F_{n,T}$ the $\overline{\mathcal{F}}$-conditional distribution function of $M_T^{n,1}$, then $F_{n,T}$ is strictly increasing by Theorem 11.11 of Lifshits (1995). A standard result about sample quantiles (see e.g., Theorem 5.9 of Shao (2003)) shows that $\widehat{Q}_{n,T}^B(p)\to F_{n,T}^{-1}(p)$ in probability as $B\to\infty$ for all $p\in(0,1)$. In particular,

$$\lim_{B\to\infty}\widetilde{\mathbb{P}}\Big(\mathrm{MIT}_T^{*n}>\widehat{Q}_{n,T}^B(1-\alpha)\Big)\leq\overline{\mathbb{P}}\Big(\mathrm{MIT}_T^{*n}>F_{n,T}^{-1}(1-\alpha)-\delta\Big)$$

for any $\delta>0$. Note that $F_{n,T}^{-1}(1-\alpha)$ is the $(1-\alpha)$-quantile of $M_T^{n,b}$, which converges in distribution to $Z_{\max}$. Under H_0^*, we know from the first part of the theorem that $\mathrm{MIT}_T^{*n}\xrightarrow{\mathcal{L}-s}Z_{\max}$, which has a density by Theorem 11.11 of Lifshits (1995). Therefore, for any $\epsilon>0$, there exists a choice of $\delta>0$ such that

$$\lim_{n\to\infty,T\to 0}\lim_{B\to\infty}\widetilde{\mathbb{P}}\Big(\mathrm{MIT}_T^{*n}>\widehat{Q}_{n,T}^B(1-\alpha)\Big)<\alpha+\epsilon.$$

A similar argument shows that the left-hand side is larger than $\alpha-\epsilon$. The claim under H_0^* follows by letting $\epsilon\to 0$. Under H_1^*, the same arguments apply except that we use the fact that $\mathrm{MIT}_T^{*n}\xrightarrow{\mathbb{P}}\infty$ in this case. □

Proof of Theorem 2.* We first derive a higher-order expansion of rRVD_T^n as defined in (3.12). By (3.8),

$$\widehat{V}_{i,n}^{\mathbb{Q}}(u)=-\frac{2}{u^2}\left(\frac{\Re\big(\widehat{\mathcal{L}}_{\underline{t}_i^n,T}^{\mathbb{Q}}(u)\big)\Re\big(\widehat{\mathcal{L}}_{\overline{t}_i^n,T}^{\mathbb{Q}}(u)\big)+\Im\big(\widehat{\mathcal{L}}_{\underline{t}_i^n,T}^{\mathbb{Q}}(u)\big)\Im\big(\widehat{\mathcal{L}}_{\overline{t}_i^n,T}^{\mathbb{Q}}(u)\big)}{\Re\big(\widehat{\mathcal{L}}_{\overline{t}_i^n,T}^{\mathbb{Q}}(u)\big)^2+\Im\big(\widehat{\mathcal{L}}_{\overline{t}_i^n,T}^{\mathbb{Q}}(u)\big)^2}-1\right).$$

We now apply a second-order Taylor expansion. If u is of the order $1/\sqrt{T}$, then $\Im(\mathcal{L}^{\mathbb{Q}}_{\underline{t}_i^n,T}(u))$ or $\Im(\mathcal{L}^{\mathbb{Q}}_{\overline{t}_i^n,T}(u))$ are of the order $O_p(T^\iota)$ with $\iota = \alpha \wedge (1-\beta/2)$ by Assumptions 1 and 3. Moreover, $u^{-2}\Re(\widehat{\mathcal{L}}^{\mathbb{Q}}_{\overline{t}_i^n,T}(u) - \mathcal{L}^{\mathbb{Q}}_{\overline{t}_i^n,T}(u))$ and $u^{-2}\Im(\widehat{\mathcal{L}}^{\mathbb{Q}}_{\overline{t}_i^n,T}(u) - \mathcal{L}^{\mathbb{Q}}_{\overline{t}_i^n,T}(u))$ are $O_p(\sqrt{\delta T^{3/2}})$ (see the proof of Lemma 2). Therefore,

$$
\begin{aligned}
\widehat{V}^{\mathbb{Q}}_{i,n}(u) = V^{\mathbb{Q}}_{i,n}(u) - \frac{2}{u^2}\Bigg\{ & \frac{\Re(\mathcal{L}^{\mathbb{Q}}_{\overline{t}_i^n,T}(u))}{|\mathcal{L}^{\mathbb{Q}}_{\overline{t}_i^n,T}(u)|^2}\Re\Big(\widehat{\mathcal{L}}^{\mathbb{Q}}_{\underline{t}_i^n,T}(u) - \mathcal{L}^{\mathbb{Q}}_{\underline{t}_i^n,T}(u)\Big) \\
& - \frac{\Re(\mathcal{L}^{\mathbb{Q}}_{\underline{t}_{i-1}^n}(u))\Re(\mathcal{L}^{\mathbb{Q}}_{\overline{t}_i^n,T}(u))^2}{|\mathcal{L}^{\mathbb{Q}}_{\overline{t}_i^n,T}(u)|^4}\Re\Big(\widehat{\mathcal{L}}^{\mathbb{Q}}_{\overline{t}_i^n,T}(u) - \mathcal{L}^{\mathbb{Q}}_{\overline{t}_i^n,T}(u)\Big) \\
& + \frac{\Re(\mathcal{L}^{\mathbb{Q}}_{\overline{t}_i^n,T}(u))^3\Re(\mathcal{L}^{\mathbb{Q}}_{\underline{t}_i^n,T}(u))}{|\mathcal{L}^{\mathbb{Q}}_{\overline{t}_i^n,T}(u)|^6}\Big(\Re\Big(\widehat{\mathcal{L}}^{\mathbb{Q}}_{\overline{t}_i^n,T}(u) - \mathcal{L}^{\mathbb{Q}}_{\overline{t}_i^n,T}(u)\Big)^2 - \Im\Big(\widehat{\mathcal{L}}^{\mathbb{Q}}_{\overline{t}_i^n,T}(u) - \mathcal{L}^{\mathbb{Q}}_{\overline{t}_i^n,T}(u)\Big)^2\Big) \\
& - \frac{\Re(\mathcal{L}^{\mathbb{Q}}_{\overline{t}_i^n,T}(u))^2}{|\mathcal{L}^{\mathbb{Q}}_{\overline{t}_i^n,T}(u)|^4}\Re\Big(\widehat{\mathcal{L}}^{\mathbb{Q}}_{\underline{t}_i^n,T}(u) - \mathcal{L}^{\mathbb{Q}}_{\underline{t}_i^n,T}(u)\Big)\Re\Big(\widehat{\mathcal{L}}^{\mathbb{Q}}_{\overline{t}_i^n,T}(u) - \mathcal{L}^{\mathbb{Q}}_{\overline{t}_i^n,T}(u)\Big) \\
& + \frac{\Re(\mathcal{L}^{\mathbb{Q}}_{\overline{t}_i^n,T}(u))^2}{|\mathcal{L}^{\mathbb{Q}}_{\overline{t}_i^n,T}(u)|^4}\Im\Big(\widehat{\mathcal{L}}^{\mathbb{Q}}_{\underline{t}_i^n,T}(u) - \mathcal{L}^{\mathbb{Q}}_{\underline{t}_i^n,T}(u)\Big)\Im\Big(\widehat{\mathcal{L}}^{\mathbb{Q}}_{\overline{t}_i^n,T}(u) - \mathcal{L}^{\mathbb{Q}}_{\overline{t}_i^n,T}(u)\Big)\Bigg\} \\
& + O_p(T^\iota\sqrt{\delta T^{3/2}} + \delta^{3/2}T^{1/4}).
\end{aligned}
$$

Next, we sum over i. Using (E.1), the bounds $Z^{\mathcal{L}}_{t,T}(u) = O_p(\delta^{1/2}/T^{1/4})$ and $R^{\mathcal{L}}_{t,T}(u) = O_p(\delta T^{-1/2}\log T^{-1})$ (see the proof of Lemma 3) and the fact that $Z^{\mathcal{L}}_{t,T}(u)$ is $\overline{\mathcal{F}}$-conditionally centered and independent as t varies, we obtain

$$
\widehat{\mathrm{VQ}}^n_T = \mathrm{VQ}^n_T + \widetilde{\mathrm{VQ}}^n_T + \mathrm{VQ}'^n_T + \mathrm{VQ}''^n_T + O_p(k_nT^\iota\sqrt{\delta T^{3/2}} + k_n\delta^{3/2}T^{1/4} + \sqrt{k_nT}\delta). \tag{E.34}
$$

We note that the $O_p(\sqrt{k_nT}\delta)$-term in (E.34) is uncorrelated across different days. Moreover, $\mathrm{VQ}^n_T = O_p(T)$, $\widetilde{\mathrm{VQ}}^n_T = O_p(\sqrt{k_n\delta T^{3/2}})$, $\mathrm{VQ}'^n_T = O_p(k_n\delta\sqrt{T})$ and $\mathrm{VQ}''^n_T = O_p(k_n\delta\sqrt{T}\log T^{-1})$. Similarly, in (D.1), we have $\mathrm{VP}^n_T = O_p(T)$ and $\widetilde{\mathrm{VP}}^n_T = O_p(\sqrt{k_n}\Delta_n)$.

By Taylor's theorem, the previous bounds and the rate conditions $k_nT^\iota = O(1/\sqrt{k_n})$ and $k_n^{3/2}\delta/\sqrt{T} = O(1)$, it follows that

$$
\begin{aligned}
\mathrm{rRVD}^n_T = \frac{\widehat{\mathrm{VQ}}^n_T}{\widehat{\mathrm{VP}}^n_T} - 1 = \frac{\mathrm{VQ}^n_T}{\mathrm{VP}^n_T} - 1 + \frac{\widetilde{\mathrm{VQ}}^n_T + \mathrm{VQ}'^n_T + \mathrm{VQ}''^n_T}{\mathrm{VP}^n_T} - \frac{\mathrm{VQ}^n_T\widetilde{\mathrm{VP}}^n_T}{(\mathrm{VP}^n_T)^2} + \frac{\mathrm{VQ}^n_T(\widetilde{\mathrm{VP}}^n_T)^2}{(\mathrm{VP}^n_T)^3} \\
- \frac{\widetilde{\mathrm{VP}}^n_T(\widetilde{\mathrm{VQ}}^n_T + \mathrm{VQ}'^n_T + \mathrm{VQ}''^n_T)}{(\mathrm{VP}^n_T)^2} + O_p(\sqrt{\delta/(k_n\sqrt{T})} + \sqrt{k_n}\delta/\sqrt{T}).
\end{aligned} \tag{E.35}
$$

Aggregating over multiple days, we obtain

$$
\widehat{\mathrm{rIVD}} = \frac{1}{D}\sum_{d=1}^{D}\mathrm{rRVD}_d = \frac{1}{D}\sum_{d=1}^{D}\Big(\frac{\mathrm{VQ}_d}{\mathrm{VP}_d} - 1\Big) + \frac{1}{D}\sum_{d=1}^{D}\Big(\frac{\mathrm{VQ}'_d + \mathrm{VQ}''_d}{\mathrm{VP}_d} + \frac{\mathrm{VQ}_d\widetilde{\mathrm{VP}}^2_d}{\mathrm{VP}^3_d}\Big)
$$

$$+\frac{1}{D}\sum_{d=1}^{D}\left(\frac{\widetilde{\mathrm{VQ}}_d}{\mathrm{VP}_d}-\frac{\mathrm{VQ}_d\widetilde{\mathrm{VP}}_d}{\mathrm{VP}_d^2}\right)+o_p((k_nD)^{-1/2}(1+\sqrt{k_n^2\delta/\sqrt{T}})),$$

which implies (D.3). For the second step in the previous display, we used the assumption $D = o(k_n^2)$ and the fact that the $O_p(\sqrt{k_n}\delta/\sqrt{T})$-term in (E.35) is centered and uncorrelated on different days.

Taylor expansion of the function $(x/y-1)^2$ yields

$$\begin{aligned}
\mathrm{rRVD}_d^2 &= \left(\frac{\mathrm{VQ}_d}{\mathrm{VP}_d}-1\right)^2+\frac{2}{\mathrm{VP}_d}\left(\frac{\mathrm{VQ}_d}{\mathrm{VP}_d}-1\right)(\widetilde{\mathrm{VQ}}_d+\mathrm{VQ}'_d+\mathrm{VQ}''_d)\\
&\quad-\frac{2\mathrm{VQ}_d}{\mathrm{VP}_d^2}\left(\frac{\mathrm{VQ}_d}{\mathrm{VP}_d}-1\right)\widetilde{\mathrm{VP}}_d+\frac{(\widetilde{\mathrm{VQ}}_d+\mathrm{VQ}'_d+\mathrm{VQ}''_d)^2}{\mathrm{VP}_d^2}\\
&\quad+\frac{2(\mathrm{VP}_d-2\mathrm{VQ}_d)}{\mathrm{VP}_d^3}\widetilde{\mathrm{VP}}_d(\widetilde{\mathrm{VQ}}_d+\mathrm{VQ}'_d+\mathrm{VQ}''_d)\\
&\quad+\frac{\mathrm{VQ}_d(3\mathrm{VQ}_d-2\mathrm{VP}_d)\widetilde{\mathrm{VP}}_d^2}{\mathrm{VP}_d^4}+O_p(k_n^{-3/2}(1+\sqrt{k_n^2\delta/\sqrt{T}})+\sqrt{k_n}\delta/\sqrt{T}).
\end{aligned}$$

Because the $O_p(\sqrt{k_n}\delta/\sqrt{T})$-term is uncorrelated on different days, it follows that

$$\begin{aligned}
\frac{1}{D}\sum_{d=1}^{D}\mathrm{rRVD}_d^2 &= \frac{1}{D}\sum_{d=1}^{D}\mathrm{rIVD}_d^2+\frac{2}{D}\sum_{d=1}^{D}\left(\frac{\mathrm{VQ}_d}{\mathrm{VP}_d}-1\right)\frac{\mathrm{VQ}'_d+\mathrm{VQ}''_d}{\mathrm{VP}_d}\\
&\quad+\frac{1}{D}\sum_{d=1}^{D}\frac{\widetilde{\mathrm{VQ}}_d^2+(\mathrm{VQ}'_d+\mathrm{VQ}''_d)^2}{\mathrm{VP}_d^2}+\frac{1}{D}\sum_{d=1}^{D}\frac{\mathrm{VQ}_d(3\mathrm{VQ}_d-2\mathrm{VP}_d)\widetilde{\mathrm{VP}}_d^2}{\mathrm{VP}_d^4}\\
&\quad+\frac{2}{D}\sum_{d=1}^{D}\mathrm{rIVD}_d\left(\frac{\widetilde{\mathrm{VQ}}_d}{\mathrm{VP}_d}-\frac{\mathrm{VQ}_d\widetilde{\mathrm{VP}}_d}{\mathrm{VP}_d^2}\right)+o_p((k_nD)^{-1/2}(1+\sqrt{k_n^2\delta/\sqrt{T}}))\\
&=\frac{1}{D}\sum_{d=1}^{D}\mathrm{rIVD}_d^2+\frac{1}{D}\sum_{d=1}^{D}\left(\frac{\widetilde{\mathrm{VQ}}_d^2+(\mathrm{VQ}'_d+\mathrm{VQ}''_d)^2}{\mathrm{VP}_d^2}+\frac{\mathrm{VQ}_d^2\widetilde{\mathrm{VP}}_d^2}{\mathrm{VP}_d^4}\right)\\
&\quad+\frac{2}{D}\sum_{d=1}^{D}\mathrm{rIVD}_d\frac{\mathrm{VQ}'_d+\mathrm{VQ}''_d}{\mathrm{VP}_d}+\frac{2}{D}\sum_{d=1}^{D}\mathrm{rIVD}_d\frac{\mathrm{VQ}_d\widetilde{\mathrm{VP}}_d^2}{\mathrm{VP}_d^3}\\
&\quad+\frac{2}{D}\sum_{d=1}^{D}\mathrm{rIVD}_d\left(\frac{\widetilde{\mathrm{VQ}}_d}{\mathrm{VP}_d}-\frac{\mathrm{VQ}_d\widetilde{\mathrm{VP}}_d}{\mathrm{VP}_d^2}\right)+o_p((k_nD)^{-1/2}(1+\sqrt{k_n^2\delta/\sqrt{T}})),
\end{aligned}$$

which implies (D.4). □

E.4 Proof of Preliminary Lemmas

Recall that all locally bounded processes can be assumed to be uniformly bounded thanks to a classical localization argument.

Proof of Lemma 1. By the Lévy–Khintchine formula, $\overline{\mathcal{L}}^{\mathbb{S}}_{t_1,t_2}(u)$ is the characteristic function of

$$\begin{aligned} x^{\mathbb{S}}_{t_1,t_2} = \alpha^{\mathbb{S}}_{t_1}(t_2-t_1) + \int_{t_1}^{t_2} \sqrt{\varsigma^2_{t_1}\xi(s/T) + \rho_{t_1}\zeta^{\mathbb{S}}(s/T)}dW^{\mathbb{S}}_s \\ + \int_{t_1}^{t_2}\int_{\mathbb{R}} \gamma^{\mathbb{S}}(t_1,z)(\mathfrak{p}^{\mathbb{S}}(dt,dz) - dtF(dz)). \end{aligned} \tag{E.36}$$

Using the basic inequality $|e^{ix}-e^{iy}| \leq |x-y|$, valid for $x,y\in\mathbb{R}$, we get

$$\begin{aligned} |\mathcal{L}^{\mathbb{S}}_{t_1,t_2}(u) - \overline{\mathcal{L}}^{\mathbb{S}}_{t_1,t_2}(u)| &\leq |u|\mathbb{E}^{\mathbb{S}}_{t_1}\left(\left|\int_{t_1}^{t_2}(\alpha^{\mathbb{S}}_s - \alpha^{\mathbb{S}}_{t_1})ds\right|\right) \\ &+ |u|\mathbb{E}^{\mathbb{S}}_{t_1}\left(\left|\int_{t_1}^{t_2}\left(\sqrt{\varsigma^2_s\xi(s/T)+\rho_s\zeta^{\mathbb{S}}(s/T)} - \sqrt{\varsigma^2_{t_1}\xi(s/T)+\rho_{t_1}\zeta^{\mathbb{S}}(s/T)}\right)dW^{\mathbb{S}}_s\right|\right) \\ &+ |u|\mathbb{E}^{\mathbb{S}}_{t_1}\left(\left|\int_{t_1}^{t_2}\int_{\mathbb{R}}(\gamma^{\mathbb{S}}(s,z)-\gamma^{\mathbb{S}}(t_1,z))(\mathfrak{p}^{\mathbb{S}}(ds,dz) - dsF(dz))\right|\right). \end{aligned} \tag{E.37}$$

By Assumption 1 and a classical localization argument, we can assume that there is a constant $c>0$ such that the two expressions under the square root sign are larger than c almost surely. Therefore, an application of Jensen's inequality, Itô's isometry, the mean value theorem and Assumptions 2 and 3 yields (E.10). □

Proof of Lemma 2. Because of the first property in (B.13), we have

$$\mathbb{E}^{\overline{\mathbb{P}}}(Z^{\mathbb{Q}}_{i,n}(\widehat{u}_{(i-2)\Delta_n}) \mid \mathcal{F}) = 0. \tag{E.38}$$

Furthermore, by Assumption 2 and (3.6), we have

$$|\mathcal{L}^{\mathbb{Q}}_{t_1,t_2}(\widehat{u}_{(i-2)\Delta_n})| = e^{-\frac{1}{2}\widehat{u}^2_{(i-2)\Delta_n}(\varsigma_0^2\int_{t_1}^{t_2}\xi(t/T)dt + \rho_0\int_{t_1}^{t_2}\zeta(t/T)dt)+o(1)} \begin{cases} \leq 1, \\ \geq e^{-\frac{1}{2}\overline{u}^2(\varsigma_0^2\overline{\xi}+|\rho_0|\overline{\zeta})+o(1)}, \end{cases} \tag{E.39}$$

where $\overline{\xi} = \sup_{t\in[0,1]}\xi(t)$, $\overline{\zeta} = \sup_{t\in[0,1]}|\zeta(t)|$ and the $o(1)$-term and $C\in(0,\infty)$ are uniform in ω, t_1, t_2, i and n. Therefore, by Assumption 4, we have for sufficiently small T that

$$\begin{aligned} \mathbb{E}^{\overline{\mathbb{P}}}(|Z^{\mathbb{Q}}_{i,n}(\widehat{u}_{(i-2)\Delta_n})|^2 \mid \mathcal{F}) &\leq \frac{C}{\widehat{u}^4_{(i-2)\Delta_n}}\left(\mathbb{E}^{\overline{\mathbb{P}}}(|Z^{\mathcal{L}}_{\underline{t}^n_i,T}(\widehat{u}_{(i-2)\Delta_n})|^2 \mid \mathcal{F}) + \mathbb{E}^{\overline{\mathbb{P}}}(|Z^{\mathcal{L}}_{\overline{t}^n_i,T}(\widehat{u}_{(i-2)\Delta_n})|^2 \mid \mathcal{F})\right) \\ &\leq C\delta\sum_{j=2}^{N_{\underline{t}^n_i}} e^{-2k_{\underline{t}^n_i,j-1}}O_{\underline{t}^n_i,T}(K_{\underline{t}^n_i,j-1})^2(k_{\underline{t}^n_i,j} - k_{\underline{t}^n_i,j-1}) \\ &+ C\delta\sum_{j=2}^{N_{\overline{t}^n_i}} e^{-2k_{\overline{t}^n_i,j-1}}O_{\overline{t}^n_i,T}(K_{\overline{t}^n_i,j-1})^2(k_{\overline{t}^n_i,j} - k_{\overline{t}^n_i,j-1}). \end{aligned}$$

By Lemma 8.1 of Chong and Todorov (2025a), we have

$$
\begin{aligned}
O_{t,T}(e^k) \leq C_t(Te^{-(k-x_t)}\mathbf{1}_{\{k-x_t>1\}} + Te^{3(k-x_t)}\mathbf{1}_{\{k-x_t<-1\}} \\
+ \frac{T}{|k-x_t|}\mathbf{1}_{\{\sqrt{T}\leq|k-x_t|\leq 1\}} + \sqrt{T}\mathbf{1}_{\{|k-x_t|\leq\sqrt{T}\}}),
\end{aligned} \tag{E.40}
$$

where C is an adapted process that by localization can be assumed to be bounded. By Riemann approximation, we deduce the bound

$$
\mathbb{E}^{\overline{\mathbb{P}}}(|Z^{\mathbb{Q}}_{i,n}(\widehat{u}_{(i-2)\Delta_n})|^2 \mid \mathcal{F}) \leq C(T-(i-1)\Delta_n)^{3/2}\delta, \tag{E.41}
$$

for some constant C that does not depend on T, i, Δ_n and δ. This implies (E.11) and (E.12).

Next, note that

$$
\Re(\overline{\mathcal{L}}^{\mathbb{S}}_{i,n}(u)) = \cos\big(u\alpha^{\mathbb{S}}_{\underline{t}^n_i}\Delta'_n + \Im(\psi^{\mathbb{S}}_{\underline{t}^n_i}(u))\Delta'_n\big)e^{-\frac{1}{2}u^2\int_{\underline{t}^n_i}^{\overline{t}^n_i}(\varsigma^2_{\underline{t}^n_i}\xi(t/T)+\rho_{\underline{t}^n_i}\zeta^{\mathbb{S}}(t/T))dt+\Re(\psi^{\mathbb{S}}_{\underline{t}^n_i}(u))\Delta'_n}.
$$

We have assumed without loss of generality that the random variables in the previous line are uniformly bounded. Therefore, there exists a constant $C>0$ such that

$$
|\Re(D_{i,n}(u))| \leq \left|\frac{2}{u^2}\Big(\Re(\overline{\mathcal{L}}^{\mathbb{P}}_{i,n}(u))-1\Big)\right| + \left|\frac{2}{u^2}\Big(\Re(\overline{\mathcal{L}}^{\mathbb{Q}}_{i,n}(u))-1\Big)\right| \leq C\Delta'_n.
$$

As a result,

$$
\mathbb{E}^{\overline{\mathbb{P}}}\Big(\big|\Re(D_{i,n}(\widehat{u}_{(i-2)\Delta_n}))\big|^2\Big) \leq C\Delta_n^2, \tag{E.42}
$$

which together with (E.38), (E.41) and the Cauchy–Schwarz inequality yields (E.14).

In order to show (E.13), we write the left-hand side as

$$
\begin{aligned}
&\sum_{i=2}^{k_n} w_\eta(\tfrac{i-1}{k_n})^2\Re(\widetilde{Z}^{\mathbb{P}}_{i,n}(\widehat{u}_{(i-2)\Delta_n}))\Re(Z^{\mathbb{Q}}_{i,n}(\widehat{u}_{(i-2)\Delta_n})) \\
&\quad + \sum_{i=2}^{k_n} w_\eta(\tfrac{i-1}{k_n})^2\big(\Re(Z^{\mathbb{P}}_{i,n}(\widehat{u}_{(i-2)\Delta_n})) - \Re(\widetilde{Z}^{\mathbb{P}}_{i,n}(\widehat{u}_{(i-2)\Delta_n}))\big)\Re(Z^{\mathbb{Q}}_{i,n}(\widehat{u}_{(i-2)\Delta_n})),
\end{aligned} \tag{E.43}
$$

where

$$
\widetilde{Z}^{\mathbb{P}}_{i,n}(u) = -\frac{2}{u^2}\bigg(\exp\Big(iu\int_{\underline{t}^n_i}^{\overline{t}^n_i}\varsigma_s\sqrt{\xi(s/T)}dW^{\mathbb{P}}_s\Big) - \mathbb{E}^{\overline{\mathbb{P}}}_{(i-1)\Delta_n}\Big[\exp\Big(iu\int_{\underline{t}^n_i}^{\overline{t}^n_i}\varsigma_s\sqrt{\xi(s/T)}dW^{\mathbb{P}}_s\Big)\Big]\bigg).
$$

As $|\cos(x)-1| \leq \frac{1}{2}x^2$, it is easy to see that

$$
\mathbb{E}^{\mathbb{P}}_{\underline{t}^n_i}\Big(|\Re(\widetilde{Z}^{\mathbb{P}}_{i,n}(\widehat{u}_{(i-2)\Delta_n}))|^2\Big) \leq C\Delta_n^2, \tag{E.44}
$$

which together with (E.38) and (E.41) shows that the first term on the right-hand side of (E.43) is $O_p(\sqrt{k_n}\Delta_n\sqrt{T^{3/2}\delta}) = O_p(\sqrt{\Delta_n\delta T^{5/2}})$. Regarding the second term, we use the basic inequality

$|\cos(x) - \cos(y)| \leq C(|x-y| \wedge 1)$ as well as Assumptions 1 and 2 and condition (3.6) for the characteristic exponent to derive

$$\begin{aligned}
&\mathbb{E}^{\overline{\mathbb{P}}}_{\underline{t}_i^n}\left(|\Re(Z^{\mathbb{P}}_{i,n}(\widehat{u}_{(i-2)\Delta_n})) - \Re(\widetilde{Z}^{\mathbb{P}}_{i,n}(\widehat{u}_{(i-2)\Delta_n}))|^2\right) \\
&\quad \leq C\widehat{u}^{-2}_{(i-2)\Delta_n}\mathbb{E}^{\overline{\mathbb{P}}}_{\underline{t}_i^n}\left[\left(\int_{\underline{t}_i^n}^{\overline{t}_i^n} \alpha_s^{\mathbb{P}} ds\right)^2\right] + C\widehat{u}^{\beta-4}_{(i-2)\Delta_n}\mathbb{E}^{\overline{\mathbb{P}}}_{\underline{t}_i^n}\left[\left|\int_{\underline{t}_i^n}^{\overline{t}_i^n}\int_{\mathbb{R}} \gamma^{\mathbb{P}}(s,z)(\mathfrak{p}^{\mathbb{P}}(ds,dz) - dsF(dz))\right|^{\beta}\right] \\
&\quad \leq CT\Delta_n^2 + CT^{2-\beta/2}\Delta_n.
\end{aligned} \tag{E.45}$$

Because $k_nT^{\iota} \to 0$ for all $\iota > 0$, it follows that the second term in (E.43) is of the order $O_p(\sqrt{k_n(T\Delta_n^2 + T^{2-\beta/2}\Delta_n)T^{3/2}\delta}) = o_p(\sqrt{\Delta_n\delta T^{5/2}})$, which completes the proof of (E.13). □

Proof of Lemma 3. We start with establishing some bounds for $R^{\mathcal{L}}_{t,T}(u)$ and $Z^{\mathcal{L}}_{t,T}(u)$. By (E.3),

$$\begin{aligned}
|R^{\mathcal{L}}_{t,T}(u)| \leq C(u^2 \vee u)\sum_{j=2}^{N_t}\int_{k_{t,j-1}}^{k_{t,j}}\Big[&u(k-k_{t,j-1})e^{-k}O_{t,T}(e^k) + e^{-k_{t,j-1}}(k-k_{t,j-1})O_{t,T}(e^k) \\
&+ e^{-k_{t,j-1}}|O_{t,T}(e^k) - O_{t,T}(e^{k_{t,j-1}})|\Big]dk \\
&+ (u^2 \vee u)\int_{-\infty}^{k_{t,1}} e^{-k}O_{t,T}(e^k)dk + (u^2 \vee u)\int_{k_{t,N_t}}^{\infty} e^{-k}O_{t,T}(e^k)dk.
\end{aligned}$$

By Lemma 8.1 of Chong and Todorov (2025a), we have

$$|O_{t,T}(e^{k_1}) - O_{t,T}(e^{k_2})| \leq C_t\left[\frac{T}{(k_2-x_t)^4} \wedge \frac{T}{(k_2-x_t)^2} \wedge 1\right]|e^{k_1} - e^{k_2}| \tag{E.46}$$

for all $k_1, k_2 \in \mathbb{R}$, where C has the same properties as in (E.40). In combination with Assumption 4, (3.6) and (E.40), we thus obtain

$$\begin{aligned}
|R^{\mathcal{L}}_{t,T}(u)| &\leq C(u^2 \vee u)\delta u\int_{\mathbb{R}}(Te^{-2|k|}\mathbf{1}_{\{|k|>1\}} + T/|k|\mathbf{1}_{\{\sqrt{T}\leq|k|\leq 1\}} + \sqrt{T}\mathbf{1}_{\{|k|\leq\sqrt{T}\}})dk \\
&\quad + C(u^2 \vee u)\delta\int_{\mathbb{R}}(\mathbf{1}_{\{|k|\leq\sqrt{T}\}} + T/k^2\mathbf{1}_{\{\sqrt{T}\leq|k|\leq 1\}} + T/k^4\mathbf{1}_{\{|k|>1\}})dk \\
&\quad + C(u^2 \vee u)T\left(\int_{-\infty}^{\frac{1}{2}|\log(\delta/\sqrt{T})|} e^{2k}dk + \int_{\frac{1}{2}|\log(\delta/\sqrt{T})|}^{\infty} e^{-2k}dk\right) \\
&\leq C(u^2 \vee u)\delta\sqrt{T}(1 + u\sqrt{T}|\log T|).
\end{aligned} \tag{E.47}$$

Because of (B.13) and (E.2), the same arguments leading to (E.41) show that

$$\begin{aligned}
\mathbb{E}^{\overline{\mathbb{P}}}(|Z^{\mathcal{L}}_{t,T}(u)|^2 \mid \mathcal{F}) &\leq (u^4 \vee u^2)\delta\sum_{j=2}^{N_t} e^{-2k_{t,j-1}}O_{t,T}(K_{t,j-1})^2(k_{t,j} - k_{t,j-1}) \\
&\leq C(u^4 \vee u^2)\delta T^{3/2}.
\end{aligned} \tag{E.48}$$

As to the fourth moment, classical martingale inequalities, the previous bound and (E.40) show that

$$\mathbb{E}^{\overline{\mathbb{P}}}(|Z^{\mathcal{L}}_{t,T}(u)|^4 \mid \mathcal{F}) \leq C(u^8 \vee u^4)\Bigg[\Bigg(\delta \sum_{j=2}^{N_t} e^{-2k_{t,j-1}} O_{t,T}(K_{t,j-1})^2 (k_{t,j} - k_{t,j-1})\Bigg)^2 + \delta^3 \sum_{j=2}^{N_t} e^{-4k_{t,j-1}} O_{t,T}(K_{t,j-1})^4 (k_{t,j} - k_{t,j-1})\Bigg] \leq C(u^8 \vee u^4)[\delta^2 T^3 + \delta^3 T^{5/2}] \leq C(u^8 \vee u^4)\delta^2 T^3, \tag{E.49}$$

where the last step uses the assumption $\delta(\log T)^2/\sqrt{T} \to 0$.

Next, define $\eta = \frac{1}{4} e^{-\frac{1}{2}\overline{u}^2(\varsigma_0^2 \overline{\xi} + |\rho_0|\overline{\zeta})}$. Then by (E.47) and (E.48), we have

$$\begin{aligned}
&\overline{\mathbb{P}}\Bigg(\bigcup_{i=2}^{k_n}\Big\{\big|\widehat{\mathcal{L}}^{\mathbb{Q}}_{\overline{t}_i^n,T}(\widehat{u}_{(i-2)\Delta_n}) - \mathcal{L}^{\mathbb{Q}}_{\overline{t}_i^n,T}(\widehat{u}_{(i-2)\Delta_n})\big| > \eta\Big\}\Bigg)\\
&\quad \leq \eta^{-2} \sum_{i=2}^{k_n} \mathbb{E}^{\overline{\mathbb{P}}}\Big[\big|\widehat{\mathcal{L}}^{\mathbb{Q}}_{\overline{t}_i^n,T}(\widehat{u}_{(i-2)\Delta_n}) - \mathcal{L}^{\mathbb{Q}}_{\overline{t}_i^n,T}(\widehat{u}_{(i-2)\Delta_n})\big|^2\Big]\\
&\quad \leq C k_n \delta T^{-1/2},
\end{aligned}$$

which tends to 0 by assumption. Therefore, we can assume without loss of generality that $|\widehat{\mathcal{L}}^{\mathbb{Q}}_{\overline{t}_i^n,T}(\widehat{u}_{(i-2)\Delta_n}) - \mathcal{L}^{\mathbb{Q}}_{\overline{t}_i^n,T}(\widehat{u}_{(i-2)\Delta_n})| \leq \eta$ and therefore, by (E.39), $|\widehat{\mathcal{L}}^{\mathbb{Q}}_{\overline{t}_i^n,T}(\widehat{u}_{(i-2)\Delta_n})| \geq 2\eta$ for all $i = 2, \ldots, k_n$. In this case, if Tu^2 is bounded, straightforward calculations together with (E.39) yield the following estimate:

$$|R^{\mathbb{Q}}_{i,n}(\widehat{u}_{(i-2)\Delta_n})| \leq \frac{K}{u^2}\big(|R^{\mathcal{L}}_{\underline{t}_i^n,T}(\widehat{u}_{(i-2)\Delta_n})|^2 + |R^{\mathcal{L}}_{\overline{t}_i^n,T}(\widehat{u}_{(i-2)\Delta_n})|^2 + |Z^{\mathcal{L}}_{\underline{t}_i^n,T}(\widehat{u}_{(i-2)\Delta_n})|^2 + |Z^{\mathcal{L}}_{\overline{t}_i^n,T}(\widehat{u}_{(i-2)\Delta_n})|^2 + |R^{\mathcal{L}}_{\underline{t}_i^n,T}(u)| + |R^{\mathcal{L}}_{\overline{t}_i^n,T}(u)|\big). \tag{E.50}$$

Because $\delta(\log T)^2/\sqrt{T} \to 0$ and $\widehat{u}_t$ satisfies (3.6), we obtain from the previous bounds that

$$\begin{aligned}
\mathbb{E}^{\overline{\mathbb{P}}}(|R^{\mathbb{Q}}_{i,n}(\widehat{u}_{(i-2)\Delta_n})| \mid \mathcal{F}) &\leq C\delta\sqrt{T}\big(1 + |\log T|\big) \leq C\delta\sqrt{T}\log(1/T),\\
\mathbb{E}^{\overline{\mathbb{P}}}(|R^{\mathbb{Q}}_{i,n}(\widehat{u}_{(i-2)\Delta_n})|^2 \mid \mathcal{F}) &\leq C\delta^2 T\big(1 + |\log T|^2\big) \leq C\delta^2 T\log^2(1/T).
\end{aligned}$$

From here, the first two results of the lemma follow. For the last result of the lemma, we apply the above bound for $\mathbb{E}^{\overline{\mathbb{P}}}(|R^{\mathbb{Q}}_{i,n}(\widehat{u}_{(i-2)\Delta_n})| \mid \mathcal{F})$ as well as (E.42), (E.44) and (E.45). □

Proof of Lemma 4. The lemma immediately follows from Lemma 1 and (3.6). □

Proof of Lemma 5. We start with the first part of the lemma. By Lemma 1 and (E.39), we have

$$\begin{aligned} B^{\mathbb{Q}}_{t_1,t_2}(u) &= -\frac{2}{u^2}\left(e^{(t_2-t_1)\Psi^{\mathbb{Q}}_{t_1}(u)} - \frac{\overline{\mathcal{L}}^{\mathbb{Q}}_{t_1,T}(u) + (\mathcal{L}^{\mathbb{Q}}_{t_1,T}(u) - \overline{\mathcal{L}}^{\mathbb{Q}}_{t_1,T}(u))}{\overline{\mathcal{L}}^{\mathbb{Q}}_{t_2,T}(u) + (\mathcal{L}^{\mathbb{Q}}_{t_2,T}(u) - \overline{\mathcal{L}}^{\mathbb{Q}}_{t_2,T}(u))}\right) \\ &= -\frac{2}{u^2}e^{(t_2-t_1)\Psi^{\mathbb{Q}}_{t_1}(u)}\left(1 - e^{-(T-t_2)(\Psi^{\mathbb{Q}}_{t_2}(u)-\Psi^{\mathbb{Q}}_{t_1}(u))}\right) + O(|u|^{-1}T^{1/2+\alpha}). \end{aligned} \tag{E.51}$$

where the constant in the O-term does not depend on ω, u, t_1, t_2 and T. Because

$$|\Psi^{\mathbb{Q}}_{t_2}(u) - \Psi^{\mathbb{Q}}_{t_1}(u)| \leq C|u||\alpha^{\mathbb{Q}}_{t_2} - \alpha^{\mathbb{Q}}_{t_1}| + C|\psi^{\mathbb{Q}}_{t_2}(u) - \psi^{\mathbb{Q}}_{t_1}(u)| + \begin{cases} Cu^2|(\sigma^{\mathbb{Q}}_{t_2})^2 - (\sigma^{\mathbb{Q}}_{t_1})^2| \\ Cu^2(|\varsigma^2_{t_2} - \varsigma^2_{t_1}| + |\rho_{t_1} - \rho_{t_2}|) \end{cases}$$

for some constant $C > 0$ that does not depend on u, t_1 and t_2, Assumption 3 implies that if $\widehat{u}_t$ satisfies (3.6), then

$$\mathbb{E}_{\underline{t}^n_i}(|B^{\mathbb{Q}}_{i,n}(\widehat{u}_{(i-2)\Delta_n})|) \leq C(T\Delta^{\alpha}_n + T^{1+\alpha}), \quad \mathbb{E}^{\overline{\mathbb{P}}}_{\underline{t}^n_i}(|B^{\mathbb{Q}}_{i,n}(\widehat{u}_{(i-2)\Delta_n})|^2) \leq C(T^2\Delta^{2\alpha}_n + T^{2+2\alpha}).$$

This proves (E.19). □

Proof of Lemma 6. Isolating the leading-order term in the bias term $D_{i,n}(u)$ defined in (E.8), we can bound

$$\begin{aligned} &\left|D_{i,n}(u) + \rho_{\underline{t}^n_i}\int_{\underline{t}^n_i}^{\overline{t}^n_i}\zeta(t/T)dt\right| \\ &\leq \frac{2}{u^2}\left|\overline{\mathcal{L}}^{\mathbb{P}}_{\underline{t}^n_i,\overline{t}^n_i}(u) - e^{-\frac{1}{2}u^2\varsigma^2_{\underline{t}^n_i}\int_{\underline{t}^n_i}^{\overline{t}^n_i}\xi(t/T)dt}\right| + \frac{2}{u^2}\left|e^{-\frac{1}{2}u^2\int_{\underline{t}^n_i}^{\overline{t}^n_i}(\varsigma^2_{\underline{t}^n_i}\xi(t/T)+\rho_{\underline{t}^n_i}\zeta(t/T))dt} - \overline{\mathcal{L}}^{\mathbb{Q}}_{\underline{t}^n_i,\overline{t}^n_i}(u)\right| \\ &\quad + \frac{2}{u^2}\left|e^{-\frac{1}{2}u^2\varsigma^2_{\underline{t}^n_i}\int_{\underline{t}^n_i}^{\overline{t}^n_i}\xi(t/T)dt} - e^{-\frac{1}{2}u^2\int_{\underline{t}^n_i}^{\overline{t}^n_i}(\varsigma^2_{\underline{t}^n_i}\xi(t/T)+\rho_{\underline{t}^n_i}\zeta(t/T))dt} - \frac{1}{2}u^2\rho_{\underline{t}^n_i}\int_{\underline{t}^n_i}^{\overline{t}^n_i}\zeta(t/T)dt\right|. \end{aligned} \tag{E.52}$$

Because $|e^x - 1 - x| \leq \frac{1}{2}x^2$ and

$$\left|\overline{\mathcal{L}}^{\mathbb{S}}_{\underline{t}^n_i,\overline{t}^n_i}(u) - e^{-\frac{1}{2}u^2\int_{\underline{t}^n_i}^{\overline{t}^n_i}(\varsigma^2_{\underline{t}^n_i}\xi(t/T)+\rho_{\underline{t}^n_i}\varsigma^{\mathbb{S}}(t/T))dt}\right| \leq C(\Delta'_n|u| + \Delta'_n|u|^{\beta}) \leq C\Delta_n(|u| + |u|^{\beta}), \quad \mathbb{S} \in \{\mathbb{P}, \mathbb{Q}\},$$

by Assumption 1, it follows that

$$\left|D_{i,n}(u) + \rho_{\underline{t}^n_i}\int_{\underline{t}^n_i}^{\overline{t}^n_i}\zeta(t/T)dt\right| \leq C(|u|^{-1}\Delta_n + |u|^{\beta-2}\Delta_n + u^2\Delta^2_n) \leq C(|u|^{\beta-2}\Delta_n + u^2\Delta^2_n),$$

for some constant $C > 0$ that does not depend on u and Δ_n. This implies that

$$\sum_{i=2}^{k_n} w_{\eta}(\tfrac{i-1}{k_n})D_{i,n}(\widehat{u}_{(i-2)\Delta_n}) = -\sum_{i=2}^{k_n} w_{\eta}(\tfrac{i-1}{k_n})\rho_{\underline{t}^n_i}\int_{\underline{t}^n_i}^{\overline{t}^n_i}\zeta(t/T)dt + O_p(T^{2-\beta/2} + \Delta_n),$$

$$\sum_{i=2}^{k_n} w_\eta(\tfrac{i-1}{k_n})^2 \Re(D_{i,n}(\widehat{u}_{(i-2)\Delta_n}))^2 = \sum_{i=2}^{k_n} w_\eta(\tfrac{i-1}{k_n})^2 \rho^2_{\underline{t}_i^n} \left(\int_{\underline{t}_i^n}^{\overline{t}_i^n} \zeta(t/T)dt\right)^2 + O_p(T^{2-\beta/2}\Delta_n + \Delta_n^2).$$

Changing variables, using (B.4) and applying (B.7) and Lemma 9 twice, we obtain

$$\begin{aligned}
&\sum_{i=2}^{k_n} w_\eta(\tfrac{i-1}{k_n}) D_{i,n}(\widehat{u}_{(i-2)\Delta_n}) \\
&\quad = -\theta T \sum_{i=2}^{k_n} \int_{\frac{i-1}{k_n}}^{\frac{i-1+\varphi}{k_n}} w_\eta(u)\rho_{u\theta T}\zeta(\theta u)du + O_p(T^{2-\beta/2} + \Delta_n + T\Delta_n^\alpha) \\
&\quad = -\frac{\theta\varphi T}{k_n} \sum_{i=2}^{k_n} w_\eta(\tfrac{i-1}{k_n})\rho_{\frac{i-1}{k_n}\theta T}\zeta(\theta\tfrac{i-1}{k_n}) + O_p(T^{2-\beta/2} + \Delta_n + T\Delta_n^\alpha + Tk_n^{-1/\beta}) \\
&\quad = -\theta\varphi T \int_0^1 w_\eta(u)\rho_{u\theta T}\zeta(\theta u)du + O_p(T^{2-\beta/2} + \Delta_n + T\Delta_n^\alpha + Tk_n^{-1/\beta}).
\end{aligned}$$

This shows (E.20).

To show (E.21), again change variables and use (B.4) to obtain

$$\sum_{i=2}^{k_n} w_\eta(\tfrac{i-1}{k_n})^2 \Re(D_{i,n}(\widehat{u}_{(i-2)\Delta_n}))^2 = \frac{\varphi^2\theta^2T^2}{k_n^2} \sum_{i=2}^{k_n} \left(\frac{k_n}{\varphi} \int_{\frac{i-1}{k_n}}^{\frac{i-1+\varphi}{k_n}} w_\eta(u)\rho_{u\theta T}\zeta(\theta u)du\right)^2 + o_p(T\Delta_n).$$

We claim that

$$\sum_{i=2}^{k_n} w_\eta(\tfrac{i-1}{k_n})^2 \Re(D_{i,n}(\widehat{u}_{(i-2)\Delta_n}))^2 = \frac{\varphi^2\theta^2T^2}{k_n^2} \sum_{i=2}^{k_n} \frac{k_n}{\varphi} \int_{\frac{i-1}{k_n}}^{\frac{i-1+\varphi}{k_n}} w_\eta(u)^2\rho^2_{u\theta T}\zeta(\theta u)^2du + o_p(T\Delta_n).$$

To see this, realize that the difference between the terms on the right-hand side of the previous two lines is

$$\frac{\varphi^2\theta^2T^2}{k_n^2} \sum_{i=2}^{k_n} \widetilde{\mathrm{Var}}\Big(w_\eta(\tfrac{i-1}{k_n} + U_n)\rho_{U_n\theta T}\zeta(\theta U_n)\Big), \tag{E.53}$$

where U_n is uniformly distributed on $[0, \frac{\varphi}{k_n}]$, independent of all other variables and $\widetilde{\mathrm{Var}}$ signifies taking variance only with respect to U_n. Clearly, U_n converges in distribution to 0. Because w_η, ρ and ζ are all uniformly bounded and continuous almost everywhere (recall that ζ is càdlàg and therefore continuous outside a set of measure zero), the continuous mapping theorem together with the dominated convergence theorem implies that $\widetilde{\mathrm{Var}}(w_\eta(\frac{i-1}{k_n} + U_n)\rho_{U_n\theta T}\zeta(\theta U_n)) \to 0$ for every i. Next, write $(k_n-1)^{-1}\sum_{i=2}^{k_n}\widetilde{\mathrm{Var}}(w_\eta(\frac{i-1}{k_n} + U_n)\rho_{U_n\theta T}\zeta(\theta U_n)) = \widetilde{\mathbb{E}}[f_{n,T}(V_n)]$, where V_n is uniformly distributed on $\{\frac{1}{k_n}, \ldots, \frac{k_n-1}{k_n}\}$, independent of all other variables, $\widetilde{\mathbb{E}}$ signifies expectation with respect to V_n only and $f_{n,T}(v) = \widetilde{\mathrm{Var}}(w_\eta(v + U_n)\rho_{U_n\theta T}\zeta(\theta U_n))$. We have just established that $f_{n,T}(v) \to 0$ for all $v \in [0,1]$. Since V_n converges in distribution to a uniform distribution on $[0,1]$, the continuous mapping theorem implies that $\widetilde{\mathbb{E}}[f_{n,T}(V_n)] \to 0$. As a result, we have shown that (E.53) is $o_p(T^2/k_n) = o_p(T\Delta_n)$, which yields the claimed equation and, by Riemann integration and Lemma 9, Equation (E.21).

Lastly, observe that

$$\mathbb{E}^{\overline{\mathbb{P}}}(Z^{\mathbb{P}}_{i,n}(\widehat{u}_{(i-2)\Delta_n}) \mid \overline{\mathcal{F}}_{(i-1)\Delta_n}) = 0.$$

Since $D_{i,n}(\widehat{u}_{(i-2)\Delta_n})$ is $\overline{\mathcal{F}}_{(i-1)\Delta_n}$-measurable, we have

$$\mathbb{E}^{\overline{\mathbb{P}}}[\Re(Z^{\mathbb{P}}_{i,n}(\widehat{u}_{(i-2)\Delta_n}))\Re(D_{i,n}(\widehat{u}_{(i-2)\Delta_n})) \mid \overline{\mathcal{F}}_{(i-1)\Delta_n}] = 0.$$

Moreover, using Assumption 1 and the basic inequality $|\cos(x)-1| \leq C(|x| \wedge 1)^2$, we derive the bound

$$\mathbb{E}^{\overline{\mathbb{P}}}[\Re(Z^{\mathbb{P}}_{i,n}(\widehat{u}_{(i-2)\Delta_n}))^2 \mid \overline{\mathcal{F}}_{(i-1)\Delta_n}] \leq C\Delta_n^2 + C\widehat{u}^{\beta-4}_{(i-2)\Delta_n}\Delta_n$$

for some constant $C > 0$ that does not depend on u and Δ_n. Therefore,

$$\mathbb{E}^{\overline{\mathbb{P}}}[\Re(Z^{\mathbb{P}}_{i,n}(\widehat{u}_{(i-2)\Delta_n}))^2\Re(D_{i,n}(\widehat{u}_{(i-2)\Delta_n}))^2 \mid \overline{\mathcal{F}}_{(i-1)\Delta_n}] \leq C\Delta_n^4 + C\widehat{u}^{\beta-4}_{(i-2)\Delta_n}\Delta_n^3,$$

and we obtain (E.22) using (3.6). □

The following analytical lemma is used in the proof of Lemma 6.

Lemma 9. *If $f : [0,1] \to \mathbb{R}$ is of finite p-variation, then f is Riemann integrable and for any choice of $u_i^n \in [\frac{i-1}{n}, \frac{i}{n}]$, we have*

$$\left|\frac{1}{n}\sum_{i=1}^{n} f(u_i^n) - \int_0^1 f(u)du\right| = O(n^{-1/p}). \tag{E.54}$$

More generally, for any $\varphi \in [0,1]$, we have

$$\left|\frac{1}{n}\sum_{i=1}^{n} \varphi f(u_i^n) - \sum_{i=1}^{n}\int_{\frac{i-1}{n}}^{\frac{i-1+\varphi}{n}} f(u)du\right| = O(n^{-1/p}). \tag{E.55}$$

Proof. Taking $\varphi = 1$ in (E.55) yields (E.54), so it suffices to prove the former. By assumption, we have $V_p(f;[0,1]) < \infty$, where

$$V_p(f;[a,b]) = \sup\left\{\left(\sum_{j=1}^{k}|f(t_j)-f(t_{j-1})|^p\right)^{1/p} : a = t_0 < \cdots < t_k = b,\ k \in \mathbb{N}\right\} \tag{E.56}$$

for $0 \leq a < b \leq 1$. Let $I_i = [\frac{i-1}{n}, \frac{i}{n}]$ and bound

$$\left|\frac{1}{n}\sum_{i=1}^{n} \varphi f(u_i^n) - \sum_{i=1}^{n}\int_{\frac{i-1}{n}}^{\frac{i-1+\varphi}{n}} f(u)du\right| \leq \sum_{i=1}^{n}\int_{[\frac{i-1}{n},\frac{i-1+\varphi}{n}]}|f(u_i^n)-f(u)|du \leq \frac{\varphi}{n}\sum_{i=1}^{n} V_p(f;I_i),$$

where the last step is obtained by choosing the partition $\{\frac{i-1}{n}, u, u_i^n, \frac{i}{n}\}$ and bounding $|f(u_i^n) - f(u)| \leq V_p(f;I_i)$. If $p = 1$, note that $\sum_{i=1}^n V_1(f;I_i) \leq V_1(f;[0,1])$ and we are done. If $p > 1$, use Hölder's inequality to bound

$$\frac{1}{n}\sum_{i=1}^{n} V_p(f;I_i) \le \frac{1}{n}\left(\sum_{i=1}^{n} V_p(f;I_i)^p\right)^{1/p} n^{1-1/p} \le n^{-1/p} V_p(f;[0,1]).$$

The last bound holds because $(\sum_{i=1}^{n} V_p(f;I_i)^p)^{1/p}$ is equal to the supremum in (E.56) restricted to partitions where each $\frac{i}{n}$ is a grid point. □

E.5 Proof of Lemma 7 and Lemma 8

Proof of Lemma 7. We first prove finite-dimensional convergence and compute asymptotic variances and covariances. For the error term under $\mathbb{P}$, we carry out a few preliminary approximations. We have

$$\mathbb{E}^{\overline{\mathbb{P}}}_{\underline{t}_i^n}[Z^{\mathbb{P}}_{i,n}(\widehat{u}_{(i-2)\Delta_n}) - \overline{Z}^{\mathbb{P}}_{i,n}(\widehat{u}_0)] = 0,$$

where

$$\overline{Z}^{\mathbb{P}}_{i,n}(\widehat{u}_0) = -\frac{2}{\widehat{u}_0^2}\left(\exp\left(i\widehat{u}_0\varsigma_0\int_{\underline{t}_i^n}^{\overline{t}_i^n}\sqrt{\xi(t/T)}dW_t^{\mathbb{P}}\right) - \mathbb{E}^{\overline{\mathbb{P}}}_{\underline{t}_i^n}\left[\exp\left(i\widehat{u}_0\varsigma_0\int_{\underline{t}_i^n}^{\overline{t}_i^n}\sqrt{\xi(t/T)}dW_t^{\mathbb{P}}\right)\right]\right).$$

By Assumption 2, (E.45) and (B.2), we have

$$\mathbb{E}^{\overline{\mathbb{P}}}_{\underline{t}_i^n}\left(|\Re(Z^{\mathbb{P}}_{i,n}(\widehat{u}_{(i-2)\Delta_n})) - \Re(\overline{Z}^{\mathbb{P}}_{i,n}(\widehat{u}_{(i-2)\Delta_n}))|^2\right) \le CT^{1+2\alpha}\Delta_n + CT\Delta_n^2 + CT^{2-\beta/2}\Delta_n$$

and

$$\mathbb{E}^{\overline{\mathbb{P}}}_{\underline{t}_i^n}\left(|\Re(\overline{Z}^{\mathbb{P}}_{i,n}(\widehat{u}_{(i-2)\Delta_n})) - \Re(\overline{Z}^{\mathbb{P}}_{i,n}(\widehat{u}_0))|^2\right) = o_p(\Delta_n).$$

Therefore,

$$\sum_{i=2}^{k_n} w_\eta(\tfrac{i-1}{k_n})\Re\Big(Z^{\mathbb{P}}_{i,n}(\widehat{u}_{(i-2)\Delta_n}) - \overline{Z}^{\mathbb{P}}_{i,n}(\widehat{u}_{(i-2)\Delta_n})\Big) = O_p(\sqrt{k_n}(T^{1/2+\alpha}\sqrt{\Delta_n}+\sqrt{T}\Delta_n+T^{1-\beta/4}\sqrt{\Delta_n}))$$

and

$$\sum_{i=2}^{k_n} w_\eta(\tfrac{i-1}{k_n})\Re\Big(\overline{Z}^{\mathbb{P}}_{i,n}(\widehat{u}_{(i-2)\Delta_n}) - \overline{Z}^{\mathbb{P}}_{i,n}(\widehat{u}_0)\Big) = o_p(\sqrt{k_n}\Delta_n).$$

Since $k_n T^{\iota} \to 0$ for all $\iota > 0$ and $\Delta_n' = \varphi\Delta_n$, the previous two lines are $o_p(\sqrt{k_n}\Delta_n')$, which implies that it suffices to show (E.29) with $\overline{Z}^{\mathbb{P}}_{i,n}(\widehat{u}_0)$ instead of $Z^{\mathbb{P}}_{i,n}(\widehat{u}_{(i-2)\Delta_n})$.

Direct calculations show that

$$\mathbb{E}^{\overline{\mathbb{P}}}_{\underline{t}_i^n}(\Re(\overline{Z}^{\mathbb{P}}_{i,n}(\widehat{u}_0))^2) = \frac{2\big(1-\exp\big(-\varsigma_0^2\widehat{u}_0^2\int_{\underline{t}_i^n}^{\overline{t}_i^n}\xi(t/T)dt\big)\big)^2}{\widehat{u}_0^4}, \tag{E.57}$$

so a first-order Taylor expansion and an argument similar to the proof of (E.21) (cf. (E.53)) and applied to ξ imply that for $\eta_1, \eta_2 \in [0, 2\pi]$, we have

$$
\begin{aligned}
&\frac{1}{k_n(\Delta_n')^2}\sum_{i=2}^{k_n} w_{\eta_1}(\tfrac{i-1}{k_n})w_{\eta_2}(\tfrac{i-1}{k_n})\mathbb{E}^{\overline{\mathbb{P}}}_{\underline{t}_i^n}(\Re(\overline{Z}^{\mathbb{P}}_{i,n}(\widehat{u}_0))^2) \\
&= \frac{2\varsigma_0^4}{k_n(\Delta_n')^2}\sum_{i=2}^{k_n} w_{\eta_1}(\tfrac{i-1}{k_n})w_{\eta_2}(\tfrac{i-1}{k_n})\left(\int_{\underline{t}_i^n}^{\overline{t}_i^n}\xi(t/T)dt\right)^2 + O_p(k_n^{-1}) \qquad \text{(E.58)}\\
&\xrightarrow{\mathbb{P}} 2\varsigma_0^4\int_0^1 w_{\eta_1}(u)w_{\eta_2}(u)\xi^2(\theta u)du.
\end{aligned}
$$

Next, we identify the asymptotic variance due to option observation errors. Similarly to (E.39), we have $\mathcal{L}^{\mathbb{Q}}_{\underline{t}_i^n,T}(\widehat{u}_{(i-2)\Delta_n}) = \exp(-\frac{1}{2}\widehat{u}^2_{\underline{t}_{i-1}^n}(\varsigma^2_{\underline{t}_i^n}\int_{(i-1)\Delta_n}^T\xi(t/T)dt + \rho_{\underline{t}_i^n}\int_{(i-1)\Delta_n}^T\zeta(t/T)dt)) + o_p(1)$ by (B.2). Therefore, by (B.2) and (E.6) (cf. also (E.48)),

$$
\begin{aligned}
\mathbb{E}^{\overline{\mathbb{P}}}(\Re(Z^{\mathbb{Q}}_{i,n}(\widehat{u}_{(i-2)\Delta_n}))^2 \mid \mathcal{F}) = {}& \frac{4}{\widehat{u}^4_{\underline{t}_{i-1}^n}}e^{\upsilon^2_{\underline{t}_i^n}(T-\underline{t}_i^n)^{-1}(\varsigma^2_{\underline{t}_i^n}\int_{\underline{t}_i^n}^T\xi(t/T)dt+\rho_{\underline{t}_i^n}\int_{\underline{t}_i^n}^T\zeta(t/T)dt)} \\
&\times \mathbb{E}^{\overline{\mathbb{P}}}(\Re(Z^{\mathcal{L}}_{\underline{t}_i^n,T}(\widehat{u}_{\underline{t}_{i-1}^n}))^2 + \Re(Z^{\mathcal{L}}_{\underline{t}_i^n,T}(\widehat{u}_{\underline{t}_{i-1}^n}))^2 \mid \mathcal{F}) + o_p(\delta T^{3/2}).
\end{aligned}
$$

By (E.2) and Assumption 4, we have

$$
\begin{aligned}
\mathbb{E}^{\overline{\mathbb{P}}}(\Re(Z^{\mathcal{L}}_{\underline{t}_i^n,T}(\widehat{u}_{\underline{t}_{i-1}^n}))^2 \mid \mathcal{F}) = {}& \widehat{u}^4_{\underline{t}_{i-1}^n}e^{-2x_{\underline{t}_i^n}}\sum_{j=2}^{N_{\underline{t}_i^n}}\cos^2(\widehat{u}_{\underline{t}_{i-1}^n}(k_{\underline{t}_i^n,j-1}-x_{\underline{t}_i^n}))e^{-2(k_{\underline{t}_i^n,j-1}-x_{\underline{t}_i^n})} \\
&\times \lambda_{\underline{t}_i^n}(k_{\underline{t}_i^n,j-1}-x_{\underline{t}_i^n})^2 O_{\underline{t}_i^n,T}(e^{k_{\underline{t}_i^n,j-1}})^2(k_{\underline{t}_i^n,j}-k_{\underline{t}_i^n,j-1})^2 + o_p(\delta/\sqrt{T}).
\end{aligned}
$$

As in Lemmas 1 and 2 in Todorov (2019), we have

$$
e^{-(k-x_t)}O_{t,T}(e^k) = \begin{cases} O_p(e^{-|k-x_t|}T) & \text{if } |k-x_t| > \log T^{-1}, \\ O_p(\sqrt{T}/\log T^{-1}) & \text{if } \sqrt{T}\log T^{-1} < |k-x_t| < \log T^{-1} \end{cases} \qquad \text{(E.59)}
$$

and, for $|k-x_t| \le \sqrt{T}\log T^{-1}$,

$$
\left|O_{t,T}(k) - e^{x_t}\sigma_t^{\mathbb{P}}\sqrt{T-t}\,\widetilde{\Phi}\left(\frac{k-x_t}{\sigma_t^{\mathbb{P}}\sqrt{T-t}}\right)\right| = o_p(\sqrt{T}), \qquad \text{(E.60)}
$$

where the O_p- and o_p-terms in (E.59) and (E.60) are uniformly in k and t. As a result, Riemann integration together with Assumption 1 (concerning $\widehat{u}_t$) and Assumption 4 (concerning κ and λ) yields

$$
\begin{aligned}
&\mathbb{E}^{\overline{\mathbb{P}}}(\Re(Z^{\mathcal{L}}_{\underline{t}_i^n,T}(\widehat{u}_{\underline{t}_{i-1}^n}))^2 \mid \mathcal{F}) \\
&= \widehat{u}^4_{\underline{t}_{i-1}^n}(\sigma^{\mathbb{P}}_{\underline{t}_i^n})^2\delta(T-\underline{t}_i^n)\sum_{j:|k_{\underline{t}_i^n,j-1}-x_{\underline{t}_i^n}|\le\sqrt{T}|\log T|}\cos^2(\widehat{u}_{\underline{t}_{i-1}^n}(k_{\underline{t}_i^n,j-1}-x_{\underline{t}_i^n}))e^{-2(k_{\underline{t}_i^n,j-1}-x_{\underline{t}_i^n})}
\end{aligned}
$$

$$\times \lambda_{\underline{t}_i^n}(k_{\underline{t}_i^n,j-1} - x_{\underline{t}_i^n})^2 \widetilde{\Phi}\bigg(\frac{k_{\underline{t}_i^n,j-1} - x_{\underline{t}_i^n}}{\sigma^{\mathbb{P}}_{\underline{t}_i^n}\sqrt{T-\underline{t}_i^n}}\bigg)^2 \kappa_{\underline{t}_i^n}(k_{\underline{t}_i^n,j-1} - x_t)(k_{\underline{t}_i^n,j} - k_{\underline{t}_i^n,j-1}) + o_p(\delta/\sqrt{T})$$

$$= \frac{\delta}{T-\underline{t}_i^n}\upsilon^4_{\underline{t}_{i-1}^n}(\sigma^{\mathbb{P}}_{\underline{t}_i^n})^2 \int_{-\sqrt{T}|\log T|}^{\sqrt{T}|\log T|} \cos^2\bigg(\upsilon_{\underline{t}_{i-1}^n}\frac{k}{\sqrt{T-\underline{t}_i^n}}\bigg)e^{-2k}\lambda_{\underline{t}_i^n}(k)^2\widetilde{\Phi}\bigg(\frac{k}{\sigma^{\mathbb{P}}_{\underline{t}_i^n}\sqrt{T-\underline{t}_i^n}}\bigg)^2\kappa_{\underline{t}_i^n}(k)dk$$

$$+ o_p(\delta/\sqrt{T})$$

$$= \frac{\delta}{\sqrt{T-\underline{t}_i^n}}\upsilon^4_{\underline{t}_{i-1}^n}(\sigma^{\mathbb{P}}_{\underline{t}_i^n})^3\lambda_{\underline{t}_i^n}(0)^2\kappa_{\underline{t}_i^n}(0)\int_{\mathbb{R}}\cos^2(\upsilon_{\underline{t}_i^n}\sigma^{\mathbb{P}}_{\underline{t}_i^n}k)\widetilde{\Phi}(k)^2dk + o_p(\delta/\sqrt{T}).$$

A similar expansion holds for $\mathbb{E}^{\overline{\mathbb{P}}}(\Re(Z^{\mathcal{L}}_{\overline{t}_i^n,T}(\widehat{u}_{(i-2)\Delta_n}))^2 \mid \mathcal{F})$. Therefore,

$$\begin{aligned}\mathbb{E}^{\overline{\mathbb{P}}}(\Re(Z^{\mathbb{Q}}_{i,n}(\widehat{u}_{\underline{t}_{i-1}^n}))^2 \mid \mathcal{F}) = 4\delta(T-\underline{t}_i^n)^{3/2}e^{\upsilon^2_{\underline{t}_i^n}(T-\underline{t}_i^n)^{-1}(\varsigma^2_{\underline{t}_i^n}\int_{\underline{t}_i^n}^T \xi(t/T)dt + \rho_{\underline{t}_i^n}\int_{\underline{t}_i^n}^T\zeta(t/T)dt)} \\ \times\bigg[(\sigma^{\mathbb{P}}_{\underline{t}_i^n})^3\lambda_{\underline{t}_i^n}(0)^2\kappa_{\underline{t}_i^n}(0)\int_{\mathbb{R}}\cos^2(\upsilon_{\underline{t}_i^n}\sigma^{\mathbb{P}}_{\underline{t}_i^n}k)\widetilde{\Phi}(k)^2dk \\ + (\sigma^{\mathbb{P}}_{\overline{t}_i^n})^3\lambda_{\overline{t}_i^n}(0)^2\kappa_{\overline{t}_i^n}(0)\int_{\mathbb{R}}\cos^2(\upsilon_{\overline{t}_i^n}\sigma^{\mathbb{P}}_{\overline{t}_i^n}k)\widetilde{\Phi}(k)^2dk\bigg] + o_p(\delta T^{3/2}).\end{aligned}$$

By (B.2), (B.11), (B.14), the rate conditions $\Delta_n/T = \theta/k_n$, $\Delta_n' = \varphi\Delta_n$ and Lemma 9, we have

$$\begin{aligned}&\frac{1}{k_n\delta T^{3/2}}\sum_{i=2}^{k_n} w_{\eta_1}(\tfrac{i-1}{k_n})w_{\eta_2}(\tfrac{i-1}{k_n})\mathbb{E}^{\overline{\mathbb{P}}}_{\underline{t}_i^n}(\Re(Z^{\mathbb{Q}}_{i,n}(\widehat{u}_{(i-2)\Delta_n}))^2) \\ &= \frac{4}{k_n}\sum_{i=2}^{k_n} w_{\eta_1}(\tfrac{i-1}{k_n})w_{\eta_2}(\tfrac{i-1}{k_n})e^{\upsilon_0^2(1-\theta(i-1)/k_n)^{-1}(\varsigma_0^2\int_{\theta(i-1)/k_n}^1\xi(v)dv + \rho_0\int_{\theta(i-1)/k_n}^1\zeta(v)dv)} \\ &\quad\times(1-\theta\tfrac{i-1}{k_n})^{3/2}\bigg[\varsigma_0^3\widetilde{\lambda}_0^2\widetilde{\kappa}_0\xi(\theta\tfrac{i-1}{k_n})^{3/2}\overline{\lambda}(\theta\tfrac{i-1}{k_n})^2\overline{\kappa}(\theta\tfrac{i-1}{k_n})\int_{\mathbb{R}}\cos^2(\upsilon_0\varsigma_0\xi(\theta\tfrac{i-1}{k_n})k)\widetilde{\Phi}(k)^2dk \\ &\quad+ \varsigma_0^3\widetilde{\lambda}_0^2\widetilde{\kappa}_0\xi(\theta\tfrac{i-1+\varphi}{k_n})^{3/2}\overline{\lambda}(\theta\tfrac{i-1+\varphi}{k_n})^2\overline{\kappa}(\theta\tfrac{i-1+\varphi}{k_n})\int_{\mathbb{R}}\cos^2(\upsilon_0\varsigma_0\xi(\theta\tfrac{i-1+\varphi}{k_n})k)\widetilde{\Phi}(k)^2dk\bigg] + o_p(1) \\ &\xrightarrow{\mathbb{P}} 8\varsigma_0^3\widetilde{\lambda}_0^2\widetilde{\kappa}_0\int_0^1 w_{\eta_1}(u)w_{\eta_2}(u)e^{\upsilon_0^2(1-\theta u)^{-1}(\varsigma_0^2\int_{\theta u}^1\xi(v)dv + \rho_0\int_{\theta u}^1\zeta(u)du)} \\ &\qquad\times(1-\theta u)^{3/2}\xi(\theta u)^{3/2}\overline{\lambda}(\theta u)^2\overline{\kappa}(\theta u)\int_{\mathbb{R}}\cos^2(\upsilon_0\varsigma_0\xi(\theta u)k)\widetilde{\Phi}(k)^2dkdu.\end{aligned} \tag{E.61}$$

Since the option observation errors are $\mathcal{F}$-conditionally centered, the covariance terms are identically zero:

$$\mathbb{E}^{\overline{\mathbb{P}}}_{\underline{t}_i^n}(\Re(Z^{\mathbb{Q}}_{i,n}(\widehat{u}_{(i-2)\Delta_n}))\Re(Z^{\mathbb{P}}_{i,n}(\widehat{u}_{(i-2)\Delta_n}))) = 0. \tag{E.62}$$

As a consequence, we deduce (C.4).

For any $p > 2$, we have

$$\mathbb{E}^{\overline{\mathbb{P}}}_{\underline{t}_i^n}(|\Re(\overline{Z}^{\mathbb{P}}_{i,n}(\widehat{u}_0)|^p) \leq C\mathbb{E}^{\overline{\mathbb{P}}}_{\underline{t}_i^n}(|(\Delta_i^n W^{\mathbb{P}})^2|^p) \leq C\Delta_n^p \tag{E.63}$$

and therefore,

$$\sum_{i=2}^{k_n} |w_\eta(\tfrac{i-1}{k_n})|^p \mathbb{E}^{\overline{\mathbb{P}}}_{\underline{t}_i^n}(|\Re(\overline{Z}^{\mathbb{P}}_{i,n}(\widehat{u}_0))|^p) \leq C k_n \Delta_n^p = o(k_n^{p/2}(\Delta_n')^p). \tag{E.64}$$

Similarly, for $p=4$, conditions (E.6), (E.39) and (E.49) imply that

$$\sum_{i=2}^{k_n} |w_\eta(\tfrac{i-1}{k_n})|^4 \mathbb{E}^{\overline{\mathbb{P}}}_{\underline{t}_i^n}(|\Re(\overline{Z}^{\mathbb{Q}}_{i,n}(\widehat{u}_{(i-2)\Delta_n}))|^4) \leq C k_n \delta^2 T^3 = o(k_n^2 \delta^2 T^3).$$

An argument similar to the proof of Lemma 13.3.15 in Jacod and Protter (2012) shows finite-dimensional stable convergence in law in (E.29).

It remains to verify tightness in the space $C([0,1])$. By the Arzelà–Ascoli theorem, it suffices to show that

$$\sup_{n\in\mathbb{N}} \mathbb{E}\left[\sup_{|\eta_1-\eta_2|\leq\epsilon} \left| \frac{1}{\sqrt{k_n[(\Delta_n')^2+\delta T^{3/2}]}} \sum_{i=2}^{k_n} (w_{\eta_2}(\tfrac{i-1}{k_n}) - w_{\eta_1}(\tfrac{i-1}{k_n})) \Re((Z^{\mathbb{P}}_{i,n}+Z^{\mathbb{Q}}_{i,n})(\widehat{u}_{(i-2)\Delta_n}))\right|\right] \to 0 \tag{E.65}$$

as $\epsilon \to 0$. By definition, $\partial_\eta w_\eta(t) = \eta w_{-\eta}(t)$. Therefore, writing $w_{\eta_2}(\frac{i-1}{k_n}) - w_{\eta_1}(\frac{i-1}{k_n}) = \int_{\eta_1}^{\eta_2} \partial_\eta w_\eta(\frac{i-1}{k_n}) d\eta$, we can bound the expectation in (E.65) by

$$\begin{aligned}
&\mathbb{E}\left[\sup_{|\eta_1-\eta_2|\leq\epsilon} \int_{\eta_1\wedge\eta_2}^{\eta_1\vee\eta_2} \left| \frac{\eta}{\sqrt{k_n[(\Delta_n')^2+\delta T^{3/2}]}} \sum_{i=2}^{k_n} w_{-\eta}(\tfrac{i-1}{k_n}) \Re((Z^{\mathbb{P}}_{i,n}+Z^{\mathbb{Q}}_{i,n})(\widehat{u}_{(i-2)\Delta_n}))\right| d\eta\right] \\
&\leq \frac{1}{\sqrt{k_n[(\Delta_n')^2+\delta T^{3/2}]}} \mathbb{E}\Bigg[\sup_{|\eta_1-\eta_2|\leq\epsilon} \left(\int_{\eta_1\wedge\eta_2}^{\eta_1\vee\eta_2} \eta^2 d\eta\right)^{1/2} \\
&\quad \times \left(\int_0^{2\pi} \left(\sum_{i=2}^{k_n} w_{-\eta}(\tfrac{i-1}{k_n}) \Re((Z^{\mathbb{P}}_{i,n}+Z^{\mathbb{Q}}_{i,n})(\widehat{u}_{(i-2)\Delta_n}))\right)^2 d\eta\right)^{1/2}\Bigg] \\
&\leq \frac{(2\pi)^{3/2}\epsilon^{1/2}}{\sqrt{k_n[(\Delta_n')^2+\delta T^{3/2}]}} \sup_{\eta\in[0,2\pi]} \mathbb{E}\left[\left(\sum_{i=2}^{k_n} w_{-\eta}(\tfrac{i-1}{k_n}) \Re((Z^{\mathbb{P}}_{i,n}+Z^{\mathbb{Q}}_{i,n})(\widehat{u}_{(i-2)\Delta_n}))\right)^2\right]^{1/2}.
\end{aligned}$$

The previous arguments show that this is $O(\epsilon^{1/2})$, uniformly in n, which shows (E.65). □

Proof of Lemma 8. We use the same notation $\overline{Z}^{\mathbb{P}}_{i,n}(u)$ as in the proof of Lemma 7 and decompose

$$\begin{aligned}
&\Re\big(Z^{\mathbb{P}}_{i,n}(\widehat{u}_{(i-2)\Delta_n})\big)^2 - \Re\big(\overline{Z}^{\mathbb{P}}_{i,n}(\widehat{u}_{(i-2)\Delta_n})\big)^2 \\
&= \big(\Re\big(Z^{\mathbb{P}}_{i,n}(\widehat{u}_{(i-2)\Delta_n})\big) - \Re\big(\overline{Z}^{\mathbb{P}}_{i,n}(\widehat{u}_{(i-2)\Delta_n})\big)\big)^2 \\
&\quad + 2\big(\Re\big(Z^{\mathbb{P}}_{i,n}(\widehat{u}_{(i-2)\Delta_n})\big) - \Re\big(\overline{Z}^{\mathbb{P}}_{i,n}(\widehat{u}_{(i-2)\Delta_n})\big)\big)\Re\big(\overline{Z}^{\mathbb{P}}_{i,n}(\widehat{u}_{(i-2)\Delta_n})\big).
\end{aligned}$$

Using (3.6), Assumptions 1 and 2 and a similar estimate to (E.45), we obtain

$$\begin{aligned}
&\mathbb{E}\left[\left(\frac{1}{\Delta_n T}\sum_{i=2}^{k_n} w_\eta(\tfrac{i-1}{k_n})^2\big(\Re\big(Z^{\mathbb{P}}_{i,n}(\widehat{u}_{(i-2)\Delta_n})\big) - \Re\big(\overline{Z}^{\mathbb{P}}_{i,n}(\widehat{u}_{(i-2)\Delta_n})\big)\big)^2\right)^{\beta/2}\right] \\
&\quad\leq \frac{2^\beta}{(\Delta_n T)^{\beta/2}}\sum_{i=2}^{k_n}\mathbb{E}\big[\big|\Re\big(Z^{\mathbb{P}}_{i,n}(\widehat{u}_{(i-2)\Delta_n})\big) - \Re\big(\overline{Z}^{\mathbb{P}}_{i,n}(\widehat{u}_{(i-2)\Delta_n})\big)\big|^\beta\big] \\
&\quad\leq C k_n(\Delta_n T)^{-\beta/2}T^{\beta/2}(T^{\alpha\beta}\Delta_n^{\beta/2} + \Delta_n^\beta + \Delta_n).
\end{aligned} \tag{E.66}$$

If $\widehat{u}_t$ satisfies (3.6) and $k_n T^\iota \to 0$ for all $\iota > 0$, this is $o(1)$. Similarly, if we use Hölder's inequality and the estimate $\mathbb{E}[|\Re(\overline{Z}^{\mathbb{P}}_{i,n}(\widehat{u}_{(i-2)\Delta_n}))|^p]^{1/p} \leq C\Delta_n$, which is valid for all $p > 0$, we obtain

$$\begin{aligned}
&\mathbb{E}\left[\frac{1}{\Delta_n T}\sum_{i=2}^{k_n} w_\eta(\tfrac{i-1}{k_n})^2\big|\Re\big(Z^{\mathbb{P}}_{i,n}(\widehat{u}_{(i-2)\Delta_n})\big) - \Re\big(\overline{Z}^{\mathbb{P}}_{i,n}(\widehat{u}_{(i-2)\Delta_n})\big)\big|\big|\Re\big(\overline{Z}^{\mathbb{P}}_{i,n}(\widehat{u}_{(i-2)\Delta_n})\big)\big|\right] \\
&\quad\leq C k_n(\Delta_n T)^{-1}T^{1/2}(T^\alpha\Delta_n^{1/2} + \Delta_n + \Delta_n^{1/\beta})\Delta_n,
\end{aligned} \tag{E.67}$$

which is $o(1)$ under the same set of assumptions.

Using the inequality $|\cos(x) - 1| \leq |x|^2/2$, we further have

$$\sum_{i=2}^{k_n} w_\eta(\tfrac{i-1}{k_n})^2\Big[\Re\big(\overline{Z}^{\mathbb{P}}_{i,n}(\widehat{u}_{(i-2)\Delta_n})\big)^2 - \mathbb{E}^{\overline{\mathbb{P}}}_{(i-1)\Delta_n}\big(\Re\big(\overline{Z}^{\mathbb{P}}_{i,n}(\widehat{u}_{(i-2)\Delta_n})\big)^2\big)\Big] = O_p(\sqrt{k_n}\Delta_n^2) \tag{E.68}$$

and

$$\sum_{i=2}^{k_n} w_\eta(\tfrac{i-1}{k_n})^2\Big[\Re\big(\overline{Z}^{\mathbb{Q}}_{i,n}(\widehat{u}_{(i-2)\Delta_n})\big)^2 - \mathbb{E}^{\overline{\mathbb{P}}}_{(i-1)\Delta_n}\big(\Re\big(\overline{Z}^{\mathbb{Q}}_{i,n}(\widehat{u}_{(i-2)\Delta_n})\big)^2\big)\Big] = O_p(\sqrt{k_n}\delta T^{3/2}). \tag{E.69}$$

The statement of the lemma now follows by expanding the square on the left-hand side of (E.30) and applying (E.21) to derive the limit of the squared D-term, (E.14) and (E.22) to show that cross-terms involving the D-term are negligible, (E.58) and (E.66)–(E.68) to derive the limit of the squared $Z^{\mathbb{P}}$-term, (E.61) and (E.69) to derive the limit of the squared $Z^{\mathbb{Q}}$-term and (E.62) to show that the product term involving $Z^{\mathbb{P}}$ and $Z^{\mathbb{Q}}$ is negligible. □

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