STOCHASTIC PROCESS FORMS IN VERTICALLY EXTENDED CONTINUOUS TIME

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ABSTRACT. Continuous-time games involving complex uncertainty are a well-known challenge because they can rarely be formulated as well-posed extensive forms. Yet, it is the extensive form that fully specifies the rules of a dynamic game, making it essential for describing information flow and dynamic equilibria ([5]). This fundamental issue concerns large classes of decision problems, including stochastic differential, timing, or Bayesian "games" in continuous time.

The stochastic process form, introduced in this article, provides an abstract game-theoretic model of the extensive form characteristics of such games, formulated in the language of stochastic processes. It is based on outcome processes and information structures on the space of configurations. A strategy is a process on that space describing a complete contingent plan of action compatible with information, and outcomes are induced by strategy profiles. A natural notion of information sets and subgames as well as of dynamic equilibrium obtains.

The problem of instantaneous reaction and information about it is tackled by introducing vertically extended continuous time. The article shows that there is a suitable stochastic analysis on this time axis, generalising the classical one. Via the notion of tilting convergence, action along the vertical half-axis is a limit of accumulating action processes on smaller and smaller grids in classical continuous time. This provides a notion for viewing outcomes in stochastic process form as limits of outcomes in well-posed stochastic extensive forms as introduced in [56].

The theory is finally shown to apply to stochastic differential and timing games. For the latter, we provide a specific model in stochastic process form, construct the symmetric preemption equilibrium described by [58, 29] in a stacked strategic form setting, and explain the equilibrium outcome as a tilting limit of classical continuous-time decision making.

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Introduction

Alice and Bob can try to grab a one-dollar bill once the referee has whistled. The first who grabs gets it. However, in case of simultaneous first grabbing both are fined in the amount of one dollar instead. What can game theory tell about this? If we find it plausible, helpful, or reasonable to model time as a continuum — and there are good points favouring any of these attitudes —, no symmetric "subgame-perfect" equilibrium obtains. This issue has been identified in [29, 58] (see also the references therein), and a solution proposed, yielding a symmetric "equilibrium". However, that construction is based on discrete-time approximations on the level of payoffs and relies heavily on the specific structure of a two-player timing game. Thus, it is relevant to understand

1. whether this solution can be explained by an intrinsic continuous-time model (based on intrinsic continuous-time outcomes and decision points, as well as strategies that are complete contingent plans of action), and
2. whether such a model does arise from abstract decision-theoretic principles, in a way compatible with control-theoretic language used in applied probability theory.

Game-theoretic models in continuous-time formulation are an important object of applied probability theory indeed. One important example for continuous-time models employing game-theoretic terminology is given by (stochastic) differential games. These are usually formulated in terms of a (stochastic) differential equation of the form

$$\mathrm{d}\chi_t = V(\xi_t, \chi_t)\,\mathrm{d}\eta_t, \qquad t \in \mathbb{R}_+, \tag{0.1}$$

where ξ_t is the tuple of all agents' actions at time t, valued in, say, $\mathbb{R}^n$, η is a suitable integrator (for instance, a function of bounded variation, or a continuous $\mathbb{L}^2$-martingale with respect to some probability measure $\mathbb{P}$), valued in, say, $\mathbb{R}^m$, $V\colon \mathbb{R}^{n+d} \to \mathbb{R}^{d\times m}$ is a sufficiently regular map so that this equation admits a unique solution in a reasonable sense, and χ describes the $\mathbb{R}^d$-valued solution, called "state process", which is the payoff-relevant quantity and which agents can partially condition future decisions on. In this context, information available to the agents is modelled via the state process's path χ, filtrations, and the requirement that the processes, particularly ξ and the strategy inducing ξ, are adapted to them. One typical definition of equilibrium would be, given a bounded continuous function u from a suitable path space to $\mathbb{R}$:

$$\mathbb{E}_x u(\chi^s) \geq \mathbb{E}_x u(\chi^{(\tilde{s}^i, s^{-i})}), \qquad \forall x \in \mathbb{R}^d,$$

for all players $i = 1, \ldots, n$ and all unilateral deviations $\tilde{s}^i$. In this example, $s = \xi$ is the strategy profile, directly determining the action process, n is the number of players, and $s^i = \xi^i$ is its i-th component. χ^s describes the solution to Equation 0.1 given $\xi = s$, with initial condition x under the measure $\mathbb{P}_x$, whose expectation is denoted by $\mathbb{E}_x$. This is a stacked strategic form perspective: For any initial value of the state process, an own strategic form is defined.[1]

As another natural example, timing games can be seen as the simplest non-trivial class of dynamic games. In these, at each point in time, agents can choose a number in $\{0, 1\}$, the choice of 0 being irreversible. These games are often formulated in terms of the first times the number 0 is chosen by the agents ("optional" or "stopping" times). An alternative description is based on decreasing $\{0, 1\}$-valued[2] stochastic processes adapted to some information flow modelled by a filtration.[3] Due to their relative simplicity, timing games allow to study conceptual problems in game theory and solutions to these like under a magnifying glass. For example, one classical conundrum in continuous-time preemption games is the difficulty of constructing a symmetric preemption equilibrium, which is readily possible in discrete time.[4] The problem is that complicated chains of randomised action and reaction on smaller and smaller discrete-time grids collapse in the continuous-time limit — or seen from a different perspective, in continuous time, waiting for a tiny, but positive amount of time with positive probability still gives the opportunity to an opponent to preempt you.

However, from a decision-theoretic point of view, a game is specified by a complete set of rules. The concepts of "strategy", "equilibrium", "subgames" (or "information sets") are derived notions, implicitly determined by the primitives. In turn, the primitives of a dynamic game are given in terms of what we may call its *extensive form characteristics*, that is, "the flow of information about past choices and exogenous events, along with a set of adapted choices locally available to decision makers".[5] As argued in [55, Introduction], the "paradigmatic model of these extensive form characteristics is provided by classical *extensive form* theory, as established by von Neumann and Morgenstern in [68] and furthered by Kuhn in [46, 47]". Its two main formulations — using either refined partitions or (decision) trees, illustrated in Figure 1 — can be given a very abstract, general setting. This has first been done in [3, 4, 2, 5], and generalised to a stochastic setting in [57, Chapters 1, 2]. A well-posed extensive form theory obtains under specific order-theoretic conditions on the underlying tree or forest (cf. [4, 2]), which generalises to the stochastic setting in [57]. For (stochastic) extensive forms based on time-indexed paths of action, well-posed is found to be equivalent to the well-orderedness of the time half-axis, by [57, Theorem 2.3.14]. This creates

[1] See [39, 26] for the initial accounts due to Isaacs and Friedman. For a more recent overview on differential games in general, see [22]. Regarding stochastic differential games, see [16, 17] for a recent textbook focusing on stochastic differential games where the "vector field" V depends also on the distribution of (ξ, χ), called "mean field games", introduced independently in [38, 50].

[2] Or $[0, 1]$-valued, which is equivalent, up to taking some conditional expectation of the $\{0, 1\}$-valued process in question, see [14, 67].

[3] See, e.g., [23, 24] for the start of the literature on "Dynkin games", [67, 48, 33] and the references therein for more recent works on this topic. For another, more abstract "game-theoretic" stream of the literature on timing games, see, for instance, [29, 49, 58, 65] and the references therein. For a review of the vast literature on economic applications via real options theory, see [9].

[4] This is the example from the very beginning of this article.

[5] Cf. [57].

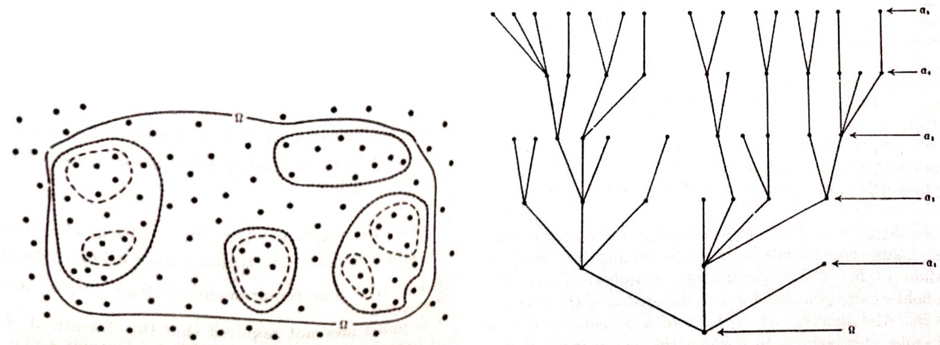

FIGURE 1. Refined partitions on a set of outcomes, and a tree representing outcomes by maximal chains (copies of Figures 5, 9 in [68])

a foundational issue for the huge class of game-theoretic models that are essentially based on continuous time, like those mentioned above.[6]

As a consequence, the interplay of continuous time and complex uncertainty represents both a litmus test and a fundamental challenge for game theory. When time is modelled as a continuum and information is subject to intricate structures of uncertainty, even the most basic notions — such as strategy and outcome, simultaneity and reaction, randomisation and beliefs, subgames and information sets — become elusive. Ultimately, canonical meta-concepts of equilibrium and optimality become difficult to implement rigorously and in a natural way, raising the questions of their general decision-theoretic meaning and of their interpretation in concrete applications.

The reasons for continuous-time modelling are manifold. One is pragmatic. Working in continuous time unlocks the powerful toolbox of (real, stochastic, functional, ...) analysis and considerably helps pushing the frontiers of mathematical tractability. Relatedly, the numerics — calculating optimisers, values, equilibria — may become much more tractable and the convergence of numerical schemes easier to analyse. Another reason is decision-theoretic. Working in discrete time assumes that there is some entity or natural constraint that can enforce agents to act only at the times of the predefined grid — though in many situations it is well conceivable that a real agent could try to act in between two grid points. This is a severe restriction for non-cooperative game theory, where, once the rules have been fixed, agents are otherwise anarchic. A fourth traditional argument is rather "philosophical", based on the statement that time really "is" continuous. If one accepts this point of view, any realistic model must employ continuous time.[7]

For examples, we refer to the discussion of stochastic differential and continuous-time timing games from the beginning. On this occasion we emphasise that in stochastic differential games, the action process ξ and the state process χ do typically not have locally right-constant paths, and hence there is little hope of being able to formulate the problem directly within a well-posed action

[6] We have to mention here the action-reaction framework from [1] which is based on paths of action in continuous time, but still does yield a well-posed extensive form. However, in this model, players cannot *de facto* act continuously because they must determine inertia (or waiting) times after any definitive action, committing to inaction during the corresponding period of time. This structure restores well-orderedness of all plays alias decision paths; see [57, Section 2.3.2].

[7] We express no philosophical opinion regarding the validity of this argument; we just mention its existence. It has a prominent tradition in the debate about the interpretation of physics. Another example for its use may be volatility modelling in finance. At least when assuming the concept of volatility to represent reality in one way or the other as opposed to a mainly instrumental view on it, the claim that "volatility is rough" (cf. [32], see also [31] and there references therein, not least their titles) would make little sense without the implicit claim that "time is continuous".

path stochastic extensive form, by the results of [57, Chapter 2].[8] We also note that in timing games instantaneous reaction to new exogenous information, i.e. a Brownian motion hitting some boundary, may be a relevant optimiser or best response candidate which is also incompatible with the well-posed action path stochastic extensive form models constructed in [57, Chapter 2].

The sheer size, breadth, and relevance of the literature on continuous-time games and control, which involve uncertainty (stochastic noise, randomisation, "incomplete information", ...) to varying, but often high degrees, makes it necessary to develop an abstract decision-theoretic understanding. Existing standard approaches typically fall into two extremes: either a stacked strategic form, as is common in stochastic differential games and control (see [26, 58, 19]), without any canonical way of implementing extensive form notions like subgame-perfect equilibrium (cf. [60]), or a fully developed well-posed extensive form (cf. [68, 47, 5]), based on action paths indexed over time (cf. [3, 4] for the basic deterministic example, and [57, Chapters 1 and 2] for a general stochastic theory). One must further mention the product form (cf. [69, 70, 37]) which leaves the extensive form terrain in a decent way in order to center on measurability, also with respect to information on past choices. Yet, to best of the author's knowledge, measurability along time (or more generally, instances of decision making, e.g. subgames) is not focal in this formal model, while it is critical for continuous-time theory and the application of (stochastic) analysis.

All these perspectives aim to capture the essence of dynamic strategic interaction, and all seek to be analysable through dynamic refinements of the Nash equilibrium concept. This text argues that these approaches, despite their differences, share underlying extensive form characteristics and can be unified through an abstract and general formal model. We show that this can be done in a way incorporating the necessary structure to implement dynamic equilibrium (like perfect Bayesian or subgame-perfect) and to be compatible with complex forms of probabilistic uncertainty in continuous time and, in particular, the general theory of stochastic processes (cf. [21, 25, 43, 54]).

In a temporal setup involving probabilistic uncertainty, focusing on extensive form characteristics leads to the basic insight that the fundamental object describing outcomes of possible choices are not decision trees or forests (as in extensive form theory), nor mere collections of random variables over decision points or "agents" (as in the product form), but rather stochastic processes satisfying particular measurability properties with respect to a given filtration. Yet, strategies must be understandable as complete contingent plans of action, alias local choices, given these informational alias measurability constraints; it must be explained which outcome processes they induce; it must be explained what are the "decision points", "subgames", "information sets" (including counterfactuals) conditional on that outcome processes are induced and, ultimately, preferences are formulated. As in extensive form theory, the stacked strategic form ultimately used in equilibrium analysis must derive from these data.

Thus, we develop a framework that speaks the language of stochastic processes while systematically preserving the structural hallmarks of the underlying extensive form characteristics. This abstract model is deliberately general: it subsumes a broad class of games and control problems formulated in terms of stochastic processes and offers a conceptual bridge to extensive form-based reasoning. It enables rigorous comparisons with stochastic extensive forms (in the sense of [57, Chapter 2], a generalisation of [5, 47, 68]) and captures key strategic properties without relying on case-by-case ad-hoc adaptations of extensive form theory.

The relevance of such a model lies not only in its capacity to represent strategic dynamics with greater clarity, but also in its ability to enable a general theory of a) reformulating classical

[8] In [3], game trees based on action paths are actually linked to "differential games". This neglects the fact that usually differential games are not formulated via trees, but via differential equations and filtrations. Be it as it may, the corresponding decision tree, or decision forest in the stochastic framework we propose in this text, is not such as to give rise to extensive forms or, even if so, to ensure their well-posedness, as we conclude in [57, Chapter 2], thereby generalising findings from [4, 2].

continuous-time models like stochastic differential games and then approximating them using rigorous action path extensive form models, and of b) conversely describing decision-theoretic limits of action path extensive form models as reaction lags shrink to zero. Indeed, as mentioned above, a very general class of action path stochastic extensive forms — that is stochastic extensive forms wherein outcomes are given by pairs (ω, f) of an exogenous scenario and a time-indexed path f of action — can be constructed, as demonstrated in the thesis [57, Chapter 2]. It is shown in [57, Theorem 2.3.14] of that thesis that, if restricting to a well-ordered time grid in $\mathbb{R}_+$, these stochastic extensive forms are well-posed, i.e. strategy profiles induced unique outcomes. Under this hypothesis, action paths can be seen as locally right-constant paths in continuous time whose jump times lie in a fixed, well-ordered grid.

With this approximation intuition in mind, we make a fundamental observation concerning the limit behaviour of outcomes: as intervals of interaction shrink to zero, information about potential reactions vanishes. Precisely, if on the grid G_n, $n \in \mathbb{Z}$, given by $G_n(k) = k2^{-n}$ for all $k = 0, 1, \dots,$ Alice acts according to the process $\xi^n = 1[0, 2^{-n})_{\mathbb{R}_+}$ and Bob according to ξ^{n-1}, then Alice switches to zero strictly before Bob, and there is no difficulty in modelling Bob observing this until time 2^{1-n}. Letting go $n \to \infty$, both ξ^n and ξ^{n-1} converge pointwise to the action process $1\{0\}$. The order of action and also Bob's observation of Alice's action is lost in the limit. Moreover, there is no instant of time at that the stopping really occurs: at time zero, the value is still 1, while at any time $\varepsilon > 0$, the stopping must have already happened before — a paradox.

This problem essentially underlies the challenge described and analysed in [29, 58] in the context of continuous-time preemption games. Yet, the analysis of these articles does not formally describe the "limit" outcome processes, which yet are a crucial part of the extensive form characteristics of the problem. If we wish to do so in the language of stochastic processes, the described phenomenon suggests allowing for well-ordered chains of "reaction nodes" at a single real point in time. Interpreting these as instances of instantaneous reaction, it becomes natural to glue well-orders above every real time point — leading to a vertically extended continuous-time structure. That is, we consider the vertically extended set of continuous time $\tilde{\mathbb{T}} = \mathbb{R}_+ \times \alpha$ for some well-order α, whose smallest three elements we call $0, 1, 2$, and we equip $\tilde{\mathbb{T}}$ with lexicographic order. Then, the intuitive limit as $n \to \infty$ of Alice's behaviour ξ^n is $1\{(0, 0)\}$, whereas that of Bob's ξ^{n-1} is $1\{(0, 0), (0, 1)\}$. Then, there is no loss of information in the limit. The set $\tilde{\mathbb{T}}$ and the notion of "tilting" convergence just described are illustrated in Figure 2.

This extension raises several foundational questions: What size and structure should such well-orders have in order to be consistent with the above-mentioned limit procedure? How are stochastic processes to be defined on such an extended time scale? What are the appropriate notions of order, topology, and measurability on the vertically extended time half-axis? How can we formalise "points" that agents can consider their options and revise their plans at, and what are counterfactuals? What concept to use in order to describe the corresponding instances of time, optional times, and, based on that, optional processes generated by these times and corresponding actions, adapted to the information flow? Finally, how can we formally describe the "tilting" limit procedure motivated above, and establish a link between grid-dependent decision making in continuous time and decision making in vertically extended time? These questions, rooted in game, decision, and control theory, lead to an extended theory of continuous time and stochastic analysis on that time half-axis, which is the subject of the first two sections of this article. In the third section, a rigorous abstract game-theoretic model of stochastic extensive form characteristics based on stochastic processes in vertically extended time is introduced, which we call *stochastic process form*, and we argue that it responds to the above challenges. Within this model, we derive notions of strategy and outcomes, information sets and subgames, randomisation and beliefs, and of equilibrium. We examine the contrasts with classical extensive, product, and strategic form approaches and explore

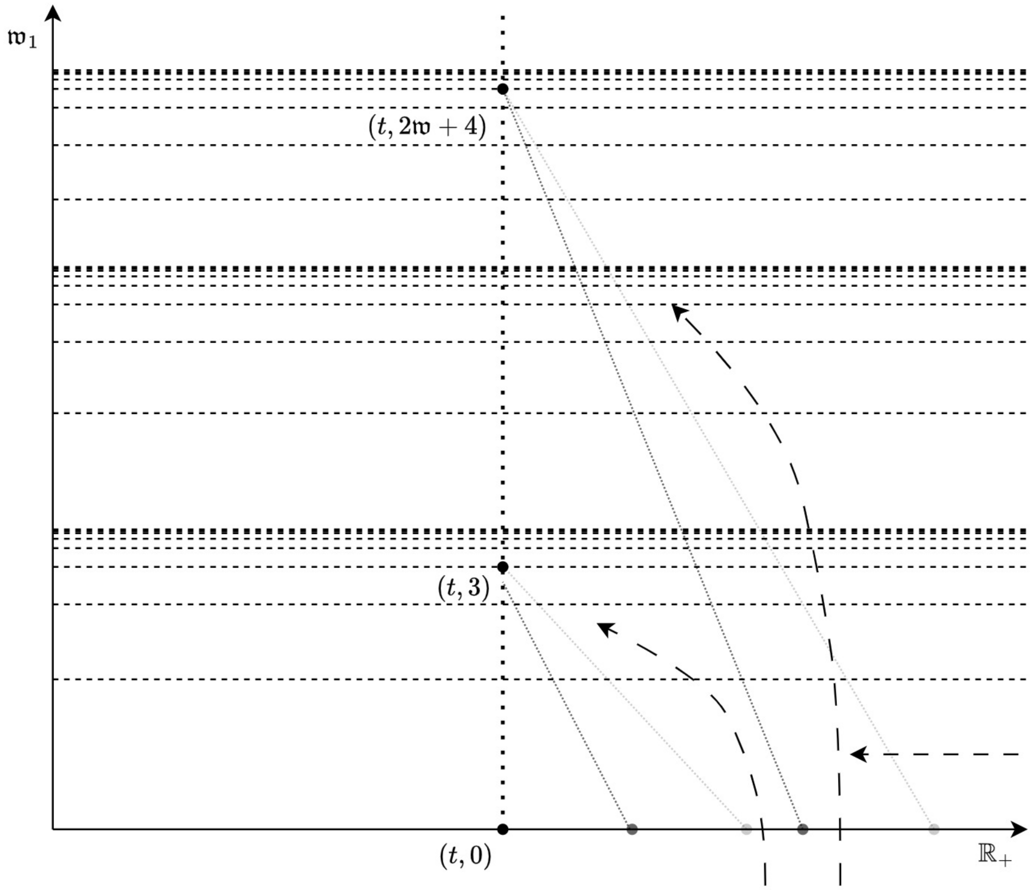

FIGURE 2. An illustration of vertically extended continuous time and tilting convergence: above any real time $t \in \mathbb{R}_+$ we attach a sufficiently large well-order. The action process consisting of three actions at times $(t, 0)$, $(t, 3)$, and $(t, 2\mathfrak{w}+4)$ can be obtained by a limit of sequences of classical continuous-time decision making on refining, convergent grids in $\mathbb{R}_+$, on which these actions occur at the grid points with indices 0, 3, and $2\mathfrak{w}+4$, respectively. The limit procedure is illustrated by the dashed arrow pushing the classical continuous-time decision making to the left. They eventually accumulate at the real time t; but, via the special notion of convergence proposed here, they are tilted by 90° counterclockwise and end up on the vertical axis above t. — Notation: Roughly speaking, up to unique isomorphism, $\mathfrak{w}$ is the "smallest infinite well-order", and $2\mathfrak{w}+4$ is the twofold concatenation of this well-order concatenated with the four-element well-order. $\mathfrak{w}_1$ can be seen as the set containing, up to unique isomorphisms, all well-orders embeddable into $\mathbb{R}_+$. For rigorous definitions and more details, see Section 1.

the consequences of our framework for dynamic equilibrium analysis. To demonstrate both the breadth and specificity of the theory, we apply it to stochastic differential games in general and provide a detailed treatment of a prototypical class: timing games.

One the one hand, we discuss on a rather abstract level how and why stochastic differential games fall within the framework of stochastic process forms, which provides a stronger decision-theoretic footing than the traditional stacked strategic form framework. A detailed analysis is performed in the case of timing games. Concerning the latter, we recall that [29, 58, 65] give a solution to the continuous-time preemption problem in terms of a stacked strategic form approach, based on extended mixed strategies. There, payoffs are defined as direct functions of strategy profiles, using a "discrete time with an infinitesimally fine grid" limit consideration. This approach does not formally describe outcomes, nor does it describe how payoffs arise from outcomes. One side effect of this is that payoffs are rather hard to formulate, and it probably would be even harder for more than two players. Moreover, in the cited approach strategies are actually "stacked strategies" alias large families of stochastic processes, one for each subgame. Thus, strategies formally depend on subgames — though this dependency is *ex post* weakened by a dynamic consistency requirement. However, this consistency condition is not further justified and appears as a solution to a difficulty arising from the formalism rather than from the formalised problem itself. After all, a strategy is a complete contingent plan of action alias local choices. Further, a "subgame" is a "point" where agents can revise these local choices. Both notions are no primitives in any strong sense, but arise from the description of information flow and local choices, i.e., the extensive form characteristics. Provided well-posedness of the game-theoretic model, a strategy profile induces a unique outcome in any "subgame". In the stacked strategic form model from the cited literature, these decision-theoretically important steps are skipped. This raises the question whether the "stacked strategies" can be integrated into one strategy process, defining a complete contingent plan of action. Similarly, we ask how "subgames", "decision points", "information sets" arise from information flow and local choices. This would clearly strengthen the interpretation of subgame-perfect equilibrium, because an equilibrium must be a strategy profile in its own right, understandable in terms of the basic extensive form characteristics.

Responding to this, we propose a well-posed stochastic process form model for timing games. Aside from being more general (general finite number of players, asymmetric information, full closed-loop setting, including a larger class of subgames), it gives a systematic explanation of outcomes and strategies, information, subgames, and equilibrium, deriving from the basic principles of the stochastic process form model. We show that the expected symmetric preemption equilibrium obtains via one global strategy profile which induces the expected outcome of randomisation on the vertical half-axis above the preemption boundary. As a corollary of the theory of "tilting convergence", transforming reaction on smaller and smaller time lags into chains of infinitesimal reaction, we obtain a representation of this outcome in terms of a limit of discrete-time approximations. This extends a similar result on the deterministic two-player timing game from [64].

Organisation of the text. The paper is organised as follows. In Section 1, vertically extended continuous time $\overline{\mathbb{T}}$ is introduced as the smallest complete total order containing all countable accumulations of well-orders embedded into $\mathbb{R}_+$. Topology and measurable structures on $\overline{\mathbb{T}}$ are studied. In Section 2, stochastic processes and random times are investigated, and suitable notions of progressive measurability, optional and predictable times and processes are introduced and their basic properties analysed. The notion of tilting convergence is introduced and so the fundamental link to outcomes of well-posed action path stochastic extensive forms is establishes. In Section 3, stochastic process forms are introduced in full generality and their information sets analysed. The section is concluded with a case study of continuous-time stochastic timing games and a short discussion of stochastic differential games.

All proofs can be found in the appendix. A pedagogic, self-contained treatment of the Dedekind-MacNeille completion of partially ordered sets, which we use in Section 1, is found in the appendix as well.

Related texts. This text is essentially equal to parts of the doctoral thesis [57] of E.R. The main body and the appendices are more or less exactly taken from [57, Chapter 3 and Appendix]. The introduction is a combination of parts of [57, Introduction]; the conclusion is a combination of parts of [57, Conclusion]; — in both cases, some changes have been made.

Further declarations. In selected places and to a very limited extent, the software *ChatGPT-4* in its current version was used to check spelling and improve the English formulation of the authors' thoughts. Any suggestion has been critically reviewed by the authors and the authors take full responsibility for resulting modifications of the text. LaTeX has been used to generate this manuscript technically. Figure 2 has been generated by E.R. using the graphics software *draw.io*, version v27.0.9.

Notations. We give a list of some notations or conventions used throughout the text some of which are not completely standard.

- $\mathbb{N} = \mathbb{Z}_+ =$ the positive integers including zero, $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$, $\mathbb{Q} =$ the rational numbers, $\mathbb{R}$ = the real numbers, $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x \geq 0\}$, all of them understood to be equipped with the standard order and algebraic structure;
- $\bigcup M =$ the union of a set $M =$ the set of all x that are the element of some $S \in M$, also written $\bigcup_{i \in I} S_i$ in case M is the image of some function $I \ni i \mapsto S_i$, for some set I;
- $f \times g =$ the function $D_1 \times D_2 \to V_1 \times V_2$, $(x, y) \mapsto (f(x), g(y))$ for functions $f\colon D_1 \to V_1$, $g\colon D_2 \to V_2$;
- $\mathfrak{w} =$ the smallest infinite ordinal, $\mathfrak{w}_1 =$ the smallest uncountable ordinal;
- $|M| =$ the cardinality of a set M;
- $x < y$ means "$x \leq y$ and $x \neq y$", given a partial order $\leq$ an a set M and $x, y \in M$; similarly, $>$ denotes the strict partial order associated to a partial order $\geq$;
- $[x, y)_T = \{z \in T \mid x \leq z < y\}$ for any poset (partially ordered set) T and all $x, y \in T$; intervals $[x, y]_T$, $(x, y]_T$, $(x, y)_T$ are defined similarly according to usual conventions;
- $\mathscr{B}_T =$ the Borel σ-algebra of a given topological space T;
- $\mathscr{E}|_D = \{E \cap D \mid E \in \mathscr{E}\}$, for any measurable space $(\Omega, \mathscr{E})$ and any subset $D \subseteq \Omega$;
- $[\![\sigma, \tau)\!) = \{(t, \omega) \in \overline{\mathbb{T}} \times \Omega \mid \sigma(\omega) \leq t < \tau(\omega)\}$ for the poset $\overline{\mathbb{T}}$ introduced in Section 1, any set Ω, and all maps $\sigma, \tau\colon \Omega \to \overline{\mathbb{T}}$, known under the name *stochastic interval* in probability theory; the intervals $[\![\sigma, \tau]\!]$, $(\!(\sigma, \tau]\!]$, and $(\!(\tau, \sigma)\!)$ are defined similarly according to usual conventions; moreover, $[\![\tau]\!] = [\![\tau, \tau]\!]$;
- $f_*\mu$ is the push-forward of a measure μ on a given measurable space $(\Omega, \mathscr{E})$ by a map $f\colon \Omega \to Y$ into some set Y, defined on the σ-algebra $f_*\mathscr{E} = \{B \subseteq Y \mid f^{-1}(B) \in \mathscr{E}\}$ and given by $f_*\mu(B) = \mu(f^{-1}(B))$, $B \in f_*\mathscr{E}$;
- $f^*\mathscr{Y} = \{f^{-1}(B) \mid B \in \mathscr{Y}\}$ is the pull-back of $\mathscr{Y}$ by f, for any map $f\colon \Omega \to Y$ and any σ-algebra $\mathscr{Y}$ on Y;
- $\mathfrak{P}_{\mathscr{E}}$ is the set of probability measures on a measurable space $(\Omega, \mathscr{E})$;
- $\mathscr{E}^{\mathrm{u}}$ denotes the universal completion of a σ-algebra $\mathscr{E}$, that is, the intersection of the completions of $\mathscr{E}$ with respect to all elements of $\mathfrak{P}_{\mathscr{E}}$.

1. Vertically extended continuous time

In [57, Chapter 2], we have shown the well-posedness of action path stochastic extensive forms on well-ordered time grids. We have also seen that, essentially, one cannot go beyond this within a rigorous action path stochastic extensive form setting. Thus, as described in the introduction to this text, we use the outcomes of these action path stochastic extensive forms defined on well-ordered time grids and let the grids become finer and finer to obtain asymptotic action processes, as illustrated in Figure 2 printed in the introduction. The corresponding notion of convergence is discussed later in Section 2. First, we need to vertically extend the continuous-time half-axis in order to faithfully represent relevant patterns of reaction on these smaller and smaller time grids in the limit.

The aim of this subsection is to introduce this vertically extended continuous time, that is, the smallest complete total order containing $\mathbb{R}_+$ and any tilted well-order embeddable into $\mathbb{R}_+$. We equip it with a suitable topology and with suitable σ-algebras. The order topology being too large, the interesting σ-algebra is not given by the Borel sets. Hence, we are led to studying the problems of the measurability of continuous functions and of measurable projection and section.

1.1. **Preliminaries on order theory.** We start with recalling some basic, well-known facts from order theory and thereby fix notation and conventions.

Completions. Basic notions from order theory are recalled in the appendix, see Section B. Here, we only recall the following special notions which are introduced in that appendix. Let P be a poset and $\varphi\colon P \hookrightarrow L$ be a completion. Then,

1. we call φ *dense* iff $\operatorname{im}\varphi$ is both join- and meet-dense in L;
2. we call φ *small* iff for any completion $\psi\colon P \hookrightarrow M$, there is an embedding $f\colon L \to M$ with $\psi = f \circ \varphi$.

It can be shown that a completion is dense iff it is small, for any poset P there is a small completion (φ, L), for any small completion (ψ, M) there is a unique isomorphism $f\colon L \to M$ such that $\psi = f \circ \varphi$, and the small completion of P can be represented by the Dedekind-MacNeille completion. For details, see Section B.

Topology on total orders. Without further mention, we equip any total order T with the order topology $\mathscr{O}_T$. Namely, this is the topology generated by all sets of the form $\uparrow t \setminus \{t\} = \{u \in T \mid t < u\}$ and $\downarrow t \setminus \{t\} = \{u \in T \mid u < t\}$, $t \in T$. In the following definitions, let T be a total order.

Recall that in a topological space Y, a neighbourhood of a point $y \in Y$ is a set $V \subseteq Y$ containing an open set V' with $y \in V' \subseteq V$. A point $t \in T$ is a *left-limit point* iff for every neighbourhood U of t we have $U \cap (\downarrow t \setminus \{t\}) \neq \emptyset$. A point $t \in T$ is a *right-limit point* iff for every neighbourhood U of t we have $U \cap (\uparrow t \setminus \{t\}) \neq \emptyset$. Denote the sets of left-limit (right-limit) points by $T_{\nearrow}$ ($T_{\swarrow}$), respectively.

If $f\colon T \to S$ is a function on T into some set S and $t \in T$ is a left-limit (right-limit) point, then we call f *left-constant at* t (*right-constant at* t) iff there is a neighbourhood U of t such that $f|_{U \cap \downarrow t}$ ($f|_{U \cap \uparrow t}$) is constant, respectively.[9]

If $f\colon T \to Y$ is a function on T into some topological space Y and $t \in T$ is a left-limit (right-limit) point, then $y \in Y$ is said *left-limit (right-limit) of* f *at* t iff for every neighbourhood $V \subseteq Y$ of y, there is a neighbourhood $U \subseteq T$ of t such that $U \cap (\downarrow t \setminus \{t\}) \subseteq f^{-1}(V)$ ($U \cap (\uparrow t \setminus \{t\}) \subseteq f^{-1}(V)$, respectively). Provided Y is Hausdorff, if existent such y is unique, and denoted by $y = \lim_{u \nearrow t} f(u)$ ($y = \lim_{u \searrow t} f(u)$, respectively). We call f *left-continuous at* t (*right-continuous at* t iff, provided t is a left-limit (right-limit) point, $f(t)$ is a left-limit (right-limit) of f at t, respectively. With these definitions, f is continuous at t iff it is both left- and right-continuous at t; and f is left-continuous (right-continuous) at t if it is right-constant (left-constant) at t, respectively. If Y is a total order as well, and $t \in T$ is a right-limit point, f is said *lower semicontinuous from the right at* t iff for any $y \in Y$ with $y < f(t)$, there is $v \in T$ with $t < v$ such that for all $u \in (t, v)_T$, we have $y < f(u)$.

A function $f\colon T \to S$ into some set S is said *locally right-constant* (*locally left-constant*) iff it is right-constant (left-constant) at any left-limit (right-limit) point $t \in T$, respectively. In direct generalisation of the standard framework, we call a function $f\colon T \to Y$ into a topological space Y

[9] We choose this convention; a weaker alternative would have been to ask for $f|_{U \cap (\downarrow t \setminus \{t\})}$ ($f|_{U \cap (\uparrow t \setminus \{t\})}$, respectively) to be constant.

làg (*làd*) iff it has left-limits (right-limits) at all left-limit (right-limit) points, respectively. We call a function $f\colon T \to Y$ into a topological space Y *left-continuous*, or *càg* (*right-continuous*, or *càd*) iff it is left-continuous (right-continuous) at any $t \in T$, respectively. Agglutinations, like *càdlàg*, stand for the conjunction, like "càd and làg". So, for example, f is continuous iff it is càdcàg.[10] A function $f\colon T \to Y$ from a totally ordered set T into a further totally ordered set Y is said *lower semicontinuous from the right* iff it lowersemicontinuous from the right at all right-limit points in T.

Well-orders and ordinals. We recall basic notions from the theory of well-orders within ZFC, see [71, Chapter 2]. A *well-order* is a poset S such that any non-empty subset $M \subseteq S$ has a minimum. An *ordinal (number)* is a set α such that all $y \in \alpha$ satisfy $y \subseteq \alpha$ and the element relation $\in$ defines a strict partial order on α. The class of ordinals is denoted by On. Then (see [71, Section 9]),

1. any set of ordinals is strictly totally ordered by the element relation $\in$,
2. any non-empty set of ordinals has a minimum with respect to that partial order,
3. for every $\alpha \in \mathrm{On}$, $\alpha = \{\beta \in \mathrm{On} \mid \beta < \alpha\}$, and
4. On is not a set.

Ordinals are important to us because any well-order is isomorphic to a unique ordinal via a unique isomorphism (cf. [71, Section 9]).

The empty set is an ordinal, i.e. $\emptyset \in \mathrm{On}$, which is called *zero* and denoted by $0 = \emptyset$, any $\alpha \in \mathrm{On}$ has a unique successor, given by $\alpha + 1 = \alpha \cup \{\alpha\}$, and any downward closed set of ordinals is itself an ordinal. The successor of zero is *one*, denoted by $1 = 0 + 1$ — and so on. A *successor ordinal* is an ordinal β such that there is $\alpha \in \mathrm{On}$ with $\beta = \alpha + 1$. A *limit ordinal* is a non-zero, non-successor ordinal. For any set $S \subseteq \mathrm{On}$ of ordinals, there is a smallest ordinal β such that all $\alpha \in S$ satisfy $\alpha \leq \beta$, given by $\bigcup S$. Hence, if $S \subseteq \alpha + 1$ for some $\alpha \in \mathrm{On}$, then $\sup S = \bigcup S$ in the poset $\alpha + 1$. Based on the zero ordinal, the successor operation $+1$, and the supremum operation sup, one can recursively define ordinal arithmetic in terms of an associative addition, a left-distributive multiplication, and a multiplicative exponentiation on ordinals. Care must be taken, however, since the algebraic structure of these operations is relatively limited in general, see [71, p. 80] for details.

As a direct consequence of the infinity axiom, there exists a smallest infinite ordinal, denoted by $\mathfrak{w}$. Equipped with ordinal addition and multiplication, and the respective neutral elements 0 and 1, $\mathfrak{w}$ is identical to the algebraic structure $\mathbb{N} = \mathbb{Z}_+$ of natural numbers. One can show the existence of a smallest (alias first) uncountable ordinal, denoted by $\mathfrak{w}_1$.[11] Its successor is $\mathfrak{w}_1 + 1 = \mathfrak{w}_1 \cup \{\mathfrak{w}_1\}$. By the results cited above, $\mathfrak{w}_1$ is the set of all countable ordinal numbers. An ordinal can be embedded into $\mathbb{R}_+$ iff it is countable. For any countable set $S \subseteq \mathfrak{w}_1$ of countable ordinals, $\bigcup S$ is countable, hence it admits a supremum in $\mathfrak{w}_1$. Recall that, without further mention, we equip any totally ordered set with the order topology. $\mathfrak{w}_1$ is sequentially compact, but not compact. As a consequence, any uncountable ordinal is not metrisable. Hence, an ordinal is Polish iff it is countable.

1.2. The complete total order $\overline{\overline{\mathbb{T}}}$.

In view of the theory developed in [57, Chapters 1 and 2], and by [57, Theorem 2.3.14] in particular, one can construct a broad class well-posed stochastic extensive forms based on paths of action indexed over well-ordered subsets of $\mathbb{R}_+$. The theory of this text is based on the intuition of eventually accumulating these time well-orders from the "right". Here, we graphically represent continuous, or "real", time $\mathbb{R}_+$ as a *horizontal* half-axis

[10] In principle, to all these definitions concerning the entire function f one must add the qualifier "locally". As continuity is a local property, this makes no difference however. By contrast, "constant" is not a local property: for example, the identity on $\{0, 1\}$ is locally right- and left-constant, but not constant.

[11] $\mathfrak{w}$ is typically denoted by ω, and $\mathfrak{w}_1$ by ω_1, but in view of the dominant role of probability in this text, and the typical notation "ω" for scenarios, we choose this unusual notation.

oriented towards the right. As discussed later on in Subsections 2.5, in the limit, such well-orders can collapse form a purely horizontal point of view. As we wish to keep track of this order structure of decisions in the limit, we extend real time on a well-ordered *vertical* half-axis.

Recalling that well-orders embedded in $\mathbb{R}_+$ are always countable, the time half-axis we require is

$$\mathbb{T} = \mathbb{R}_+ \times \mathfrak{w}_1. \tag{1.1}$$

We equip $\mathbb{T}$ with lexicographic order, that is, $(t,\beta),(u,\gamma) \in \mathbb{T}$ satisfy

$$(t,\beta) \leq (u,\gamma) \qquad \Longleftrightarrow \qquad t < u, \text{ or } [t = u \text{ and } \beta \leq \gamma]. \tag{1.2}$$

A stochastic analysis based on this time half-axis requires understanding two things: first, the small completion of $\mathbb{T}$ (for taking infima and suprema); second, suitable topologies and σ-algebras on that completion (for convergence, probability, and integration).

In this subsection, we deal with the first question, but also prepare our later treatment of the second one. For that, the approach will consist in exhausting $\mathbb{T}$ by sufficiently "small" extensions. Namely, fix some $\alpha \in \mathfrak{w}_1 + 1$, and let

$$\mathbb{T}_\alpha = \mathbb{R}_+ \times \alpha = \{t \in \mathbb{T} \mid \pi(t) < \alpha\}. \tag{1.3}$$

Clearly, we have $\mathbb{T}_\alpha = \mathbb{T}$ if $\alpha = \mathfrak{w}_1$. More importantly, $\mathbb{T} = \bigcup_{\alpha\in\mathfrak{w}_1} \mathbb{T}_\alpha$.

Let us note that, via set inclusion, $\mathbb{T}_\alpha$ is embedded into $\overline{\mathbb{R}_+} \times (\mathfrak{w}_1 + 1)$ endowed with lexicographic order, i.e. all $(t,\beta),(u,\gamma) \in \overline{\mathbb{R}_+} \times (\mathfrak{w}_1 + 1)$ satisfy Statement 1.2. Here, $\overline{\mathbb{R}_+}$ is the small completion of the poset $\mathbb{R}_+$, given by $\overline{\mathbb{R}_+} = \mathbb{R}_+ \cup \{\infty\}$, with $\infty = \sup \mathbb{R}_+$. To obtain a (candidate for a) small completion of $\mathbb{T}_\alpha$, let

$$\overline{\mathbb{T}_\alpha} = \begin{cases} \mathbb{T}_{\alpha+1} \cup \{\infty\}, & \text{if } \alpha \text{ is a limit ordinal,} \\ \mathbb{T}_\alpha \cup \{\infty\}, & \text{else,} \end{cases} \tag{1.4}$$

equipped with induced order. If $\alpha = \mathfrak{w}_1$, we simply write $\overline{\mathbb{T}} = \overline{\mathbb{T}_\alpha}$, that is, we let

$$\overline{\mathbb{T}} = [\mathbb{R}_+ \times (\mathfrak{w}_1 + 1)] \cup \{\infty\}. \tag{1.5}$$

Note that $\overline{\mathbb{T}_0} = \{\infty\}$ and, if $\alpha > 0$,

$$\overline{\mathbb{T}_\alpha} = \mathbb{T}_\alpha \cup \{(t, \sup \alpha) \mid t \in \mathbb{R}_+\} \cup \{\infty\}, \tag{1.6}$$

where $\sup \alpha$ is the supremum of the set α in $\mathfrak{w}_1 + 1$. This union is disjoint iff α is a limit ordinal.

There are embeddings $\overline{\mathbb{R}_+} \to \overline{\mathbb{T}} \to \overline{\mathbb{R}_+} \times (\mathfrak{w}_1 + 1)$ mapping $t \mapsto (t, 0)$ and $(t,\alpha) \mapsto (t,\alpha)$, by means of which we treat $\overline{\mathbb{R}_+}$ as a subset of $\overline{\mathbb{T}}$, and $\overline{\mathbb{T}}$ as a subset of $\overline{\mathbb{R}_+} \times (\mathfrak{w}_1 + 1)$. Moreover, let $p\colon \overline{\mathbb{R}_+} \times (\mathfrak{w}_1 + 1) \to \overline{\mathbb{R}_+}$ and $\pi\colon \overline{\mathbb{R}_+} \times (\mathfrak{w}_1 + 1) \to (\mathfrak{w}_1 + 1)$ be the canonical projections of the set-theoretic product. Clearly, p is monotone and π is not.

We now answer the first question above.

Proposition 1.1. *Via set inclusion, $\overline{\mathbb{T}_\alpha}$ is a small completion of $\mathbb{T}_\alpha$. In particular, $\overline{\mathbb{T}}$ is a small completion of $\mathbb{T}$.*

The proof is based on the following lemma which is of independent interest.

Lemma 1.2. *Let $\alpha \in \mathfrak{w}_1 + 1$ and $S \subseteq \overline{\mathbb{T}_\alpha}$ be a subset. Furthermore, let*

$$a = \inf \mathcal{P}p(S), \qquad b = \sup \mathcal{P}p(S), \qquad \text{in } \overline{\mathbb{R}_+}.$$

Then, S has both an infimum and a supremum in $\overline{\mathbb{T}_\alpha}$, given by

$$\inf S = \begin{cases} (a,\gamma), & \text{if } a \in \mathcal{P}p(S) \text{ and } \gamma = \inf \mathcal{P}\pi(S \cap [\{a\} \times (\sup\alpha + 1)]) \text{ in } \sup\alpha + 1, \\ (a, \sup\alpha), & \text{if } a \in \mathbb{R}_+ \setminus \mathcal{P}p(S), \\ \infty, & \text{else,} \end{cases}$$

and

$$\sup S = \begin{cases} (b,\gamma), & \text{if } b \in \mathcal{P}p(S) \text{ and } \gamma = \sup \mathcal{P}\pi(S \cap [\{b\} \times (\sup\alpha + 1)]) \text{ in } \sup\alpha+1, \\ b, & \text{if } b \notin \mathcal{P}p(S). \end{cases}$$

Definition 1.3. The complete and totally ordered lattice $\overline{\mathbb{T}}$ is said *vertically extended continuous, or real, time*, or shorter *vertically extended time*, or even shorter, if the context permits, *time*.

As detailed above, the elements of $\mathbb{T}$ have a direct interpretation in terms of "accumulated" well-orders embedded into $\mathbb{R}_+$; this point is further detailed by the notion of tilting convergence, introduced in Subsection 2.5. The other elements of $\overline{\mathbb{T}}$ arise by taking suprema and infima of elements of $\mathbb{T}$, as described in Proposition 1.1 and, more explicitly, in Lemma 1.2. In most decision-theoretic contexts, the element ∞ can be interpreted as "never" or as a terminal time. A natural interpretation of $(t,\mathfrak{w}_1)$, for $t \in \mathbb{R}_+$, is "never at real time t" – a contradiction, of course. Yet, this contradiction is later resolved by the fact that the relevant objects describing dynamic decision making, introduced in Section 2, — optional times and optional processes — overlook these vertical endpoints.

1.3. **Topology and σ-algebras on $\overline{\mathbb{T}}$.** In the remainder of this and the following section, we study suitable σ-algebras on the vertically extended continuous time half-axis $\overline{\mathbb{T}}$, the small completion of $\mathbb{T}$. First, we note that $\overline{\mathbb{T}}$ is equipped with the order topology $\mathscr{O}_{\overline{\mathbb{T}}}$. For any subset $S \subseteq \overline{\mathbb{T}}$, let $\mathscr{G}_{\overline{\mathbb{T}}}(S)$ be the set of all down- and up-sets of the form $[0,t)_{\overline{\mathbb{T}}}$ and $(t,\infty]_{\overline{\mathbb{T}}}$, $t \in S$. Then, by definition, $\mathscr{G}_{\overline{\mathbb{T}}}(\overline{\mathbb{T}})$ is a subbase of the topology $\mathscr{O}_{\overline{\mathbb{T}}}$. This is slightly strengthened by the following simple result.

Lemma 1.4. *The set*

$$\mathscr{U}_{\overline{\mathbb{T}}}(\mathbb{T}) = \{[0,u)_{\overline{\mathbb{T}}} \mid u \in \mathbb{T}\} \cup \{(t,\infty]_{\overline{\mathbb{T}}} \mid t \in \mathbb{T}\} \cup \{(t,u)_{\overline{\mathbb{T}}} \mid t,u \in \mathbb{T}\} \tag{1.7}$$

is a base and $\mathscr{G}_{\overline{\mathbb{T}}}(\mathbb{T})$ is a subbase of the topology $\mathscr{O}_{\overline{\mathbb{T}}}$ on $\overline{\mathbb{T}}$.

Let $\mathscr{B}_{\overline{\mathbb{T}}} = \sigma(\mathscr{O}_{\overline{\mathbb{T}}})$ be the Borel σ-algebra on $(\overline{\mathbb{T}}, \mathscr{O}_{\overline{\mathbb{T}}})$. By definition, basically, $\mathscr{B}_{\overline{\mathbb{T}}}$

1. is the smallest σ-algebra $\mathscr{B}'$ on $\overline{\mathbb{T}}$ such that for all topological spaces Y, all continuous maps $\overline{\mathbb{T}} \to Y$ on are $\mathscr{B}'$-$\mathscr{B}_Y$-measurable,
2. and is the largest σ-algebra $\mathscr{B}'$ on $\overline{\mathbb{T}}$ such that for all topological spaces X, all continuous maps $X \to \overline{\mathbb{T}}$ are $\mathscr{B}_X$-$\mathscr{B}'$-measurable.

Deleting sets from $\mathscr{B}_{\overline{\mathbb{T}}}$ removes some continuous maps $\overline{\mathbb{T}} \to Y$ – in probabilistic terms, *a fortiori*, also some stochastic processes with continuous paths – from our reach. Adding sets to $\mathscr{B}_{\overline{\mathbb{T}}}$ removes some continuous maps $X \to \overline{\mathbb{T}}$ – in probabilistic terms, some random times – from our reach. In that sense, $\mathscr{B}_{\overline{\mathbb{T}}}$ is quite natural. However, despite being Hausdorff the topology $\mathscr{O}_{\overline{\mathbb{T}}}$ on $\overline{\mathbb{T}}$ is non-metrisable, because $\mathfrak{w}_1$ is not metrisable. This makes standard methods from stochastic analysis hard to apply. Moreover, it is shown further below that $\mathscr{B}_{\overline{\mathbb{T}}}$ is neither generated by the (compact) class of (closed) intervals, respectively, nor by that of products of compacts in $\overline{\mathbb{R}_+} \times \alpha$, $\alpha \in \mathfrak{w}_1$, and basic results from stochastic analysis, such as measurable projection, do not generalise to that σ-algebra.

Therefore, we consider the σ-algebras generated by intervals on the one hand, and by the products of compacts $\overline{\mathbb{R}_+} \times \alpha$, $\alpha \in \mathfrak{w}_1$ on the other. This defines the following programme which we are concerned with in most of Sections 1 and 2:

1. Introduce these σ-algebras formally, precisely describe their relevant generators, and establish their relationship;
2. Determine a relevant class of topological spaces Y such that continuous maps $\overline{\mathbb{T}} \to Y$ are measurable;

3. Study whether the theorem of measurable projection and section holds true for these σ-algebras;
4. Study stochastic processes and random times with time $\overline{\mathbb{T}}$ regarding relevant measurability properties.

We start with Step 1. Let $\mathscr{I}_{\overline{\mathbb{T}}}(\mathbb{T}) = \sigma(\mathscr{G}_{\overline{\mathbb{T}}}(\mathbb{T}))$ be the σ-algebra generated by $\mathscr{G}_{\overline{\mathbb{T}}}(\mathbb{T})$, the complements of principal up- and down-sets of elements of $\mathbb{T}$ in $\overline{\mathbb{T}}$. Next, for any $\alpha \in \mathfrak{w}_1$, equip $\overline{\mathbb{R}_+}$, $\alpha+1$, and their product $\overline{\mathbb{R}_+} \times (\alpha+1)$ with their Polish topologies $\mathscr{O}_{\overline{\mathbb{R}_+}}$, $\mathscr{O}_{\alpha+1}$, and $\mathscr{O}_{\overline{\mathbb{R}_+}\times(\alpha+1)} = \mathscr{O}_{\overline{\mathbb{R}_+}} \otimes \mathscr{O}_{\alpha+1}$, respectively, let

$$\rho^\alpha \colon \overline{\mathbb{T}} \to \overline{\mathbb{R}_+} \times (\alpha+1),\ t \mapsto \big(p(t), \sup[(\pi(t)+1) \cap (\alpha+1)]\big), \tag{1.8}$$

and let $\mathscr{P}^\alpha_{\overline{\mathbb{T}}}$ be the σ-algebra on $\overline{\mathbb{T}}$ generated by ρ^α and $\mathscr{B}_{\overline{\mathbb{R}_+}\times(\alpha+1)} = \sigma(\mathscr{O}_{\overline{\mathbb{R}_+}\times(\alpha+1)})$, called the *projection σ-algebra of rank α*. Let $\mathscr{P}_{\overline{\mathbb{T}}}$ be the σ-algebra on $\overline{\mathbb{T}}$ generated by the set of pairs ρ^α, $\mathscr{B}_{\overline{\mathbb{R}_+}\times(\alpha+1)} = \sigma(\mathscr{O}_{\overline{\mathbb{R}_+}\times(\alpha+1)})$, ranging over $\alpha \in \mathfrak{w}_1$, called the *projection σ-algebra*.[12] In formulae,

$$\mathscr{P}^\alpha_{\overline{\mathbb{T}}} = \big\{(\rho^\alpha)^{-1}(B) \mid B \in \mathscr{B}_{\overline{\mathbb{R}_+}\times(\alpha+1)}\big\}, \qquad \alpha \in \mathfrak{w}_1, \tag{1.9}$$

and $\mathscr{P}_{\overline{\mathbb{T}}} = \bigvee_{\alpha\in\mathfrak{w}_1} \mathscr{P}^\alpha_{\overline{\mathbb{T}}}$.

We now describe several generators of $\mathscr{I}_{\overline{\mathbb{T}}}(\mathbb{T})$, making precise the statement that "$\mathscr{I}_{\overline{\mathbb{T}}}(\mathbb{T})$ is the σ-algebra generated by the intervals". For this, let

$$\mathscr{K}_{\overline{\mathbb{T}}}(\mathbb{T}) = \{[t,u]_{\overline{\mathbb{T}}} \mid t,u \in \mathbb{T}\}.$$

Proposition 1.5. *For any $t,u \in \overline{\mathbb{T}}$, $[t,u]_{\overline{\mathbb{T}}}$ is compact. $\mathscr{K}_{\overline{\mathbb{T}}}(\mathbb{T})$ is an intersection-stable compact class satisfying*

$$\mathscr{I}_{\overline{\mathbb{T}}}(\mathbb{T}) = \sigma(\mathscr{G}_{\overline{\mathbb{T}}}(\mathbb{T})) = \sigma(\mathscr{K}_{\overline{\mathbb{T}}}(\mathbb{T})).$$

Corollary 1.6. *The following sets of intervals are intersection-stable and generate the σ-algebra $\mathscr{I}_{\overline{\mathbb{T}}}(\mathbb{T})$:*

$$\{[0,u]_{\overline{\mathbb{T}}} \mid u \in \mathbb{T}\}, \quad \{[0,u)_{\overline{\mathbb{T}}} \mid u \in \mathbb{T}\}, \quad \{[t,\infty]_{\overline{\mathbb{T}}} \mid t \in \mathbb{T}\}, \quad \{(t,\infty]_{\overline{\mathbb{T}}} \mid t \in \mathbb{T}\}.$$

Hence, we call $\mathscr{I}_{\overline{\mathbb{T}}}(\mathbb{T})$ the $\mathbb{T}$-*interval σ-algebra on* $\overline{\mathbb{T}}$, and abbreviate it by $\mathscr{I}_{\overline{\mathbb{T}}}$. We continue with discussing the projection σ-algebras. Generators of $\mathscr{P}^\alpha_{\overline{\mathbb{T}}}$, for $\alpha \in \mathfrak{w}_1$, and $\mathscr{P}_{\overline{\mathbb{T}}}$ are spelled out in the following lemma.

Lemma 1.7. *For any $Y \in \{\overline{\mathbb{R}_+}\} \cup \mathfrak{w}_1$, let $\mathscr{G}_Y$ be an intersection-stable generator of the σ-algebra $\mathscr{B}_Y$ such that Y is the countable union of elements of $\mathscr{G}_Y$. For $\alpha \in \mathfrak{w}_1$, let*

$$\mathscr{G}^\alpha_{\overline{\mathbb{T}},\times} = \big\{(\rho^\alpha)^{-1}(B \times C) \mid B \in \mathscr{G}_{\overline{\mathbb{R}_+}},\, C \in \mathscr{G}_{\alpha+1}\big\},$$

and let

$$\mathscr{G}_{\overline{\mathbb{T}},\times} = \bigcup_{\alpha\in\mathfrak{w}_1} \mathscr{G}^\alpha_{\overline{\mathbb{T}},\times}. \tag{1.10}$$

Then,

1. *for all $\alpha \in \mathfrak{w}_1$, $\mathscr{G}^\alpha_{\overline{\mathbb{T}},\times}$ is an intersection-stable generator of the σ-algebra $\mathscr{P}^\alpha_{\overline{\mathbb{T}}}$ on $\overline{\mathbb{T}}$;*
2. *$\mathscr{G}_{\overline{\mathbb{T}},\times}$ is a generator of the σ-algebra $\mathscr{P}_{\overline{\mathbb{T}}}$ on $\overline{\mathbb{T}}$; moreover, if, for all $\alpha,\beta \in \mathfrak{w}_1$ with $\alpha < \beta$ and all $C \in \mathscr{G}_{\alpha+1}$, we have either $\alpha \in C$ and $C \cup (\alpha,\beta]_{\mathfrak{w}_1} \in \mathscr{G}_{\beta+1}$, or $\alpha \notin C$ and $C \in \mathscr{G}_{\beta+1}$, then the union in Equation 1.10 is increasing in $\alpha \in \mathfrak{w}_1$ and $\mathscr{G}_{\overline{\mathbb{T}},\times}$ is intersection-stable;*
3. *if $\mathscr{G}_Y$ consists of compact sets in Y for all $Y \in \{\overline{\mathbb{R}_+}\} \cup \mathfrak{w}_1$ and the only element B of $\mathscr{G}_{\overline{\mathbb{R}_+}}$ with $\infty \in B$ is $B = \{\infty\}$, then the preceding generators are compact classes, respectively.*

[12] One may interpret the notation "$\mathscr{P}$" also by the words "product", "Polish", "preimage".

Example 1.8. A typical example for the generators $\mathscr{G}_{\alpha+1}$, $\alpha \in \mathfrak{w}_1$, is given by $\mathscr{G}_{\alpha+1} = \alpha + 2 = \{\beta \in \mathfrak{w}_1 \mid \beta \subseteq \alpha + 1\}$. This provides an intersection-stable generator of $\alpha + 1$, $\alpha \in \mathfrak{w}_1$. Moreover, the hypothesis in the second sentence of Part 2 is satisfied, i.e. for all $\alpha, \beta \in \mathfrak{w}_1$ with $\alpha < \beta$ and all $C \in \mathscr{G}_{\alpha+1}$, we have either $\alpha \in C$ and $C \cup (\alpha, \beta]_{\mathfrak{w}_1} \in \mathscr{G}_{\beta+1}$, or $\alpha \notin C$ and $C \in \mathscr{G}_{\beta+1}$. In order to obtain compact classes as in Part 3, we could restrict to those β that are not limit ordinals, leading to $\{0\} \cup \{\gamma + 1 \mid \gamma \in \alpha\} \subseteq \alpha + 1$.

The σ-algebras $\mathscr{P}^{\alpha}_{\overline{\mathbb{T}}}$, for $\alpha \in \mathfrak{w}_1$, are defined by projecting down on the vertical levels α and below. As the following lemma indicates, they are insensitive to "new information" above these levels, and — a result which is important in the context of stochastic processes — this result is compatible with taking the product with a measurable space.

Lemma 1.9. *Let $\alpha \in \mathfrak{w}_1$ be a countable ordinal and $(\Omega, \mathscr{E})$ be a measurable space. Then, for any $M \in \mathscr{P}^{\alpha}_{\overline{\mathbb{T}}} \otimes \mathscr{E}$, any $t \in \mathbb{R}_+$, and any $\omega \in \Omega$, we have*

$$(u, \omega) \in M$$

for all or no $u \in p^{-1}(t) \cap (\overline{\mathbb{T}} \setminus \mathbb{T}_\alpha)$.

Next, we ask the following: Is there an easy representation of the "vertical limit" σ-algebra $\mathscr{P}_{\overline{\mathbb{T}}} = \bigvee_{\alpha \in \mathfrak{w}_1} \mathscr{P}^{\alpha}_{\overline{\mathbb{T}}}$? And, if so, is it compatible with taking the product with a measurable space? The next result gives a simple answer to these questions. It illustrates the interplay of the uncountable vertical half-axis and the countable additivity of σ-algebras.

Proposition 1.10. *Let $(\Omega, \mathscr{E})$ be a measurable space. Then,*

$$\mathscr{P}_{\overline{\mathbb{T}}} \otimes \mathscr{E} = \bigcup_{\alpha \in \mathfrak{w}_1} \mathscr{P}^{\alpha}_{\overline{\mathbb{T}}} \otimes \mathscr{E} = \left\{ (\rho^\alpha \times \mathrm{id}_\Omega)^{-1}(S) \mid \alpha \in \mathfrak{w}_1,\, S \in \mathscr{B}_{\overline{\mathbb{R}_+} \times (\alpha+1)} \otimes \mathscr{E} \right\}. \tag{1.11}$$

The union is increasing in $\alpha \in \mathfrak{w}_1$.

In particular, taking singleton Ω, and recalling Equation 1.9, we get the representation

$$\mathscr{P}_{\overline{\mathbb{T}}} = \left\{ (\rho^\alpha)^{-1}(B) \mid \alpha \in \mathfrak{w}_1,\, B \in \mathscr{B}_{\overline{\mathbb{R}_+} \times (\alpha+1)} \right\}. \tag{1.12}$$

Furthermore, we obtain the following corollary.

Corollary 1.11. *We have $\mathscr{P}_{\overline{\mathbb{T}}} \subseteq \mathscr{B}_{\overline{\mathbb{T}}} \vee \sigma(B \times \{0\} \mid B \in \mathscr{B}_{\overline{\mathbb{R}_+}})$ and $\mathscr{P}_{\overline{\mathbb{T}}} \neq \mathscr{B}_{\overline{\mathbb{T}}}$.*

We continue with a result linking $\mathscr{I}_{\overline{\mathbb{T}}}$ and $\mathscr{P}_{\overline{\mathbb{T}}}$.

Lemma 1.12. *We have $\mathscr{I}_{\overline{\mathbb{T}}} \subseteq \mathscr{P}_{\overline{\mathbb{T}}}$.*

We close this subsections with confirming that the relevant structural maps are measurable with respect to the σ-algebras under scrutiny. Let us denote, for any $\alpha \in \mathfrak{w}_1$, the set-theoretic inclusion map $\overline{\mathbb{T}}_{\alpha+1} \to \overline{\mathbb{T}}$ by ι_α. Note that, as follows from Equation 1.4, we have $\overline{\mathbb{T}}_{\alpha+1} \in \mathscr{B}_{\overline{\mathbb{R}_+} \times (\alpha+1)}$.

Lemma 1.13. *Let $\alpha \in \mathfrak{w}_1$. Then, ι_α is $\mathscr{B}_{\overline{\mathbb{R}_+} \times (\alpha+1)}|_{\overline{\mathbb{T}}_{\alpha+1}}$-$\mathscr{P}_{\overline{\mathbb{T}}}$-measurable and p is both $\mathscr{I}_{\overline{\mathbb{T}}}$-$\mathscr{B}_{\overline{\mathbb{R}_+}}$- and $\mathscr{P}^{\alpha}_{\overline{\mathbb{T}}}$-$\mathscr{B}_{\overline{\mathbb{R}_+}}$-measurable.*

Note that, by Lemma 1.12, ι_α is also $\mathscr{B}_{\overline{\mathbb{R}_+} \times (\alpha+1)}|_{\overline{\mathbb{T}}_{\alpha+1}}$-$\mathscr{I}_{\overline{\mathbb{T}}}$-measurable, and p is also $\mathscr{P}_{\overline{\mathbb{T}}}$-$\mathscr{B}_{\overline{\mathbb{R}_+}}$-measurable.

1.4. **Continuous functions on $\overline{\mathbb{T}}$.** We continue with Step 2. Actually, a more granular answer is of interest, because it is natural in stochastic analysis to consider processes whose paths satisfy the weaker requirement of having left- and right-limits. For this, recall the definitions of limit points and continuity in Subsection 1.1, which require some additional care when dealing with $\overline{\mathbb{T}}$ instead of $\overline{\mathbb{R}_+}$. To start, we determine the left- and right-limit points in $\overline{\mathbb{T}}$.

Lemma 1.14. *The set $\overline{\mathbb{T}}_{\nearrow}$ of left-limit points in $\overline{\mathbb{T}}$ is given by all $t \in \overline{\mathbb{T}} \setminus \{0\}$ such that $\pi(t)$ is not a successor ordinal. The set $\overline{\mathbb{T}}_{\swarrow}$ of right-limit points in $\overline{\mathbb{T}}$ is given by all $(t, \mathfrak{w}_1)$, $t \in \mathbb{R}_+$.*

We now answer the main question of this subsection. In a first step, we note the special role of the points $t \in \overline{\mathbb{T}}$ with $\pi(t) = \mathfrak{w}_1$.

Lemma 1.15. *Let Y be a metrisable topological space and $f : \overline{\mathbb{T}} \to Y$ be $\mathscr{P}_{\overline{\mathbb{T}}}$-$\mathscr{B}_Y$-measurable. Then, for all $t \in \mathbb{R}_+$, f is left-constant and, in particular, left-continuous at $(t, \mathfrak{w}_1)$.*

This left-continuity requirement at the "never at t"-instants is thus necessary for $\mathscr{P}_{\overline{\mathbb{T}}}$-$\mathscr{B}_Y$-measurability (and, in particular, for $\mathscr{I}_{\overline{\mathbb{T}}}$-$\mathscr{B}_Y$-measurability), but it is not a restriction as it may seem at first sight. First, for all topological spaces Y, any function $g : \overline{\mathbb{R}_+} \to Y$ induces a function $f : \overline{\mathbb{T}} \to Y$ that is left-continuous at $(t, \mathfrak{w}_1)$, for all $t \in \mathbb{R}_+$, namely $f = g \circ p$. f inherits relevant properties from g, like làg, continuity, measurability.[13] Moreover, note that a làg function $\overline{\mathbb{T}} \to \mathbb{R}$ that is left-continuous at $(t, \mathfrak{w}_1)$, for all $t \in \mathbb{R}_+$, is essentially a làg function $\mathbb{T} \cup \{\infty\} \to \mathbb{R}$ that can be continuously extended from below, or the left, to the point at infinity of any vertical half-axis. That is, asymptotic behaviour of the function along the vertical half-axis can be explained by countably many values along each vertical half-axis. Speaking game-theoretically, the formally uncountable chains of vertical, infinitesimal (randomised or not) reaction are actually described by countably many ones.

Provided this unproblematic regularity assumption in $t \in \overline{\mathbb{T}}$ with $\pi(t) = \mathfrak{w}_1$, measurability can be assured under mild regularity conditions, as the following proposition clarifies.

Proposition 1.16. *Let Y be a metrisable topological space. Any làg function $\overline{\mathbb{T}} \to Y$ that is left-continuous at $(t, \mathfrak{w}_1)$, for all $t \in \mathbb{R}_+$, is $\mathscr{I}_{\overline{\mathbb{T}}}$-$\mathscr{B}_Y$-measurable. In particular, it is $\mathscr{P}_{\overline{\mathbb{T}}}$-$\mathscr{B}_Y$-measurable.*

We emphasise some of the special cases covered by the preceding proposition.

Corollary 1.17. *Let Y be a metrisable topological space. The following functions $\overline{\mathbb{T}} \to Y$ are $\mathscr{I}_{\overline{\mathbb{T}}}$-$\mathscr{B}_Y$-measurable, and, in particular, $\mathscr{P}_{\overline{\mathbb{T}}}$-$\mathscr{B}_Y$-measurable:*

1. *provided $Y = \mathbb{R}$, any monotone function that is left-continuous at $(t, \mathfrak{w}_1)$, for all $t \in \mathbb{R}_+$,*
2. *any làdlàg function that is left-continuous at $(t, \mathfrak{w}_1)$, for all $t \in \mathbb{R}_+$,*
3. *any càdlàg function that is left-continuous at $(t, \mathfrak{w}_1)$, for all $t \in \mathbb{R}_+$,*
4. *any continuous function.* □

1.5. **Measurable projection and section.** Next, we discuss Step 3. The basic idea of the proof is already expressed in Proposition 1.10 and Lemma 1.9. As a consequence, the following proposition obtains.

Proposition 1.18. *Let $(\Omega, \mathscr{E})$ be a measurable space, $\mathrm{prj}_\Omega : \overline{\mathbb{T}} \times \Omega \to \Omega$ denote the canonical projection onto Ω, and $M \in \mathscr{P}_{\overline{\mathbb{T}}} \otimes \mathscr{E}$. Then, there is $\alpha \in \mathfrak{w}_1$ such that*

$$M_\alpha = M \cap (\overline{\mathbb{T}}_{\alpha+1} \times \Omega) \in \mathscr{B}_{\overline{\mathbb{R}_+} \times (\alpha+1)} \otimes \mathscr{E}, \qquad \textit{and} \qquad \mathcal{P}\mathrm{prj}_\Omega(M) = \mathcal{P}\mathrm{prj}_\Omega(M_\alpha).$$

We now state the theorem of measurable projection and section for vertically extended real time. For the setup, let us introduce some notation. For a measurable space $(\Omega, \mathscr{E})$, let $\mathscr{E}^{\mathrm{u}}$ be the universal

[13] The latter in view of Lemma 1.13.

completion of $\mathscr{E}$, that is, the intersection of the completions of $\mathscr{E}$ with respect to all probability measures on $(\Omega, \mathscr{E})$. Moreover, given a measurable space $(\Omega, \mathscr{E})$, for any map $\tau\colon \Omega \to \overline{\mathbb{T}}$, we let

$$\llbracket \tau \rrbracket = \{(\tau(\omega), \omega) \mid \omega \in \Omega\} \tag{1.13}$$

be its converse graph. Then, we have the following result.

Theorem 1.19 (Measurable Projection and Section)**.** *Let $(\Omega, \mathscr{E})$ be a measurable space, $M \in \mathscr{P}_{\overline{\mathbb{T}}} \otimes \mathscr{E}$ and $\mathrm{prj}_\Omega\colon \overline{\mathbb{T}} \times \Omega \to \Omega$ be the projection onto Ω. Then,*

$$\mathcal{P}\mathrm{prj}_\Omega(M) = \{\omega \in \Omega \mid \exists t \in \overline{\mathbb{T}}\colon (t, \omega) \in M\} \in \mathscr{E}^{\mathrm{u}}.$$

Moreover, there is an $\mathscr{E}^{\mathrm{u}}|_{\mathcal{P}\mathrm{prj}_\Omega(M)}$-$\mathscr{P}_{\overline{\mathbb{T}}}$-measurable map $\sigma\colon \mathcal{P}\mathrm{prj}_\Omega(M) \to \overline{\mathbb{T}}$ such that $\llbracket \sigma \rrbracket \subseteq M$.

2. Stochastic processes in vertically extended continuous time

In this section, we tackle Step 4. For this, and for the entire Section 2, we fix a measurable space $(\Omega, \mathscr{E})$ and a filtration $\mathscr{F} = (\mathscr{F}_t)_{t \in \overline{\mathbb{T}}}$ on it with time index set $\overline{\mathbb{T}}$, and a measurable space $(Y, \mathscr{Y})$. Moreover, we will introduce a model of instants of vertically extended continuous time that options can be evaluated or decisions be made at, called optional times. Based on this, we are led to a theory of optional and predictable processes, yielding a natural model for decision making in vertically extended continuous time. We will see that these notions non-trivially extend the classical notions from stochastic calculus on $\mathbb{R}_+$. Finally, we show that these optional processes arise naturally by accumulating discrete-time decision making in classical continuous time $\mathbb{R}_+$, using the new concept of tilting convergence.

2.1. Augmentation and right-limits of information flow.

As the theory of stochastic processes and stopping times involves both approximations from the right and projections, we have to discuss right-continuity and completeness assumption on $\mathscr{F}$.

Regarding right-continuity, approximation from the right in the Polish spaces $\overline{\mathbb{R}_+} \times (\alpha + 1)$, $\alpha \in \mathfrak{w}_1$, may require the *strong right-continuous extension* $\mathscr{F}_{\boldsymbol{+}} = (\mathscr{F}_{t\boldsymbol{+}})_{t \in \overline{\mathbb{T}}}$ which we define as follows:

$$\mathscr{F}_{\infty\boldsymbol{+}} = \mathscr{F}_\infty, \qquad \mathscr{F}_{(t,\beta)\boldsymbol{+}} = \bigcap_{u \in (t,\infty)_{\overline{\mathbb{R}_+}}} \mathscr{F}_{(u,\beta)}, \quad t \in \mathbb{R}_+,\, \beta \in \mathfrak{w}_1 + 1.$$

As $\mathscr{F}$ is a filtration, this implies that for all $t \in \overline{\mathbb{T}}$, we have $\mathscr{F}_{t\boldsymbol{+}} = \mathscr{F}_{p(t)\boldsymbol{+}}$ — a strong property. Namely, this is equivalent to saying that there is a filtration $\mathscr{G} = (\mathscr{G}_t)_{t \in \overline{\mathbb{R}_+}}$ with time index set $\overline{\mathbb{R}_+}$ such that $\mathscr{F}_{\boldsymbol{+}} = \mathscr{G} \circ p$. We note that $(\mathscr{F}_{\boldsymbol{+}})_{\boldsymbol{+}} = \mathscr{F}_{\boldsymbol{+}}$ and we call $\mathscr{F}$, idem $(\Omega, \mathscr{E}, \mathscr{F})$, *strongly right-continuous* iff $\mathscr{F} = \mathscr{F}_{\boldsymbol{+}}$.

Note that the right-continuous extension defined above is much larger than the one with respect to the order topology on $\overline{\mathbb{T}}$. With respect to the order topology, the *right-continuous extension* is $\mathscr{F}_+ = (\mathscr{F}_{t+})_{t \in \overline{\mathbb{T}}}$, given by $\mathscr{F}_{t+} = \bigcap_{u \in (t,\infty]_{\overline{\mathbb{T}}}} \mathscr{F}_u = \mathscr{F}_{t\boldsymbol{+}}$ if $t \in \overline{\mathbb{T}}_{\swarrow} = \pi^{-1}(\{\mathfrak{w}_1\})$,[14] and $\mathscr{F}_{t+} = \mathscr{F}_t$ else. $\mathscr{F}$, idem $(\Omega, \mathscr{E}, \mathscr{F})$, is said *right-continuous* iff $\mathscr{F} = \mathscr{F}_+$. In the strong right-continuous extension information about events collapses along the vertical axes — for any $x \in \mathbb{R}_+$, all information about the realised scenario included in $\mathscr{F}_x$ suffices to describe randomisation along the vertical half-axis $p = x$.

The second issue is completion. As beliefs are treated as part of agents' preferences, we do not fix probability measures at this stage. Hence, we make use of universal completions. We fix our conventions for this and recall basic properties.[15]

[14] See Lemma 1.14.

[15] For the essence of these notions and their properties, the methods of proof from the classical case can be directly adapted to the present setting. For the classical case, see, for instance, [62, Chapter 1].

Let $\mathfrak{P}_{\mathscr{E}}$ denote the set of probability measures on $(\Omega, \mathscr{E})$ and, for any $\mathbb{P} \in \mathfrak{P}_{\mathscr{E}}$, $\mathscr{N}_{\mathbb{P}} = \{M \subseteq \Omega \mid \exists N \in \mathscr{E} \colon M \subseteq N, \mathbb{P}(N) = 0\}$. Then, for any sub-$\sigma$-algebra $\mathscr{A} \subseteq \mathscr{E}$, call *universal augmentation of $\mathscr{A}$ in $\mathscr{E}$* the σ-algebra

$$\overline{\mathscr{A}} = \bigcap_{\mathbb{P} \in \mathfrak{P}_{\mathscr{E}}} (\mathscr{A} \vee \mathscr{N}_{\mathbb{P}}). \tag{2.1}$$

Moreover, we have

$$\mathscr{A}^{\mathrm{u}} \subseteq \overline{\mathscr{A}} \subseteq \overline{\mathscr{E}} = \mathscr{E}^{\mathrm{u}}, \qquad \overline{\overline{\mathscr{A}}} = \overline{\mathscr{A}},$$

and the inequalities can be strict. $\mathscr{A}$ is said *universally augmented in $\mathscr{E}$* iff $\overline{\mathscr{A}} = \mathscr{A}$. We recall that, with this convention, $\mathscr{E}$ is universally complete iff it is universally augmented in itself.

The *universal augmentation of $\mathscr{F}$ in $\mathscr{E}$* is the filtration $\overline{\mathscr{F}} = (\overline{\mathscr{F}_t})_{t \in \overline{\mathbb{T}}}$. The filtration $\mathscr{F}$ is said *universally augmented in $\mathscr{E}$* iff $\mathscr{F} = \overline{\mathscr{F}}$. We call the filtered measurable space $(\Omega, \mathscr{E}, \mathscr{F})$ *universally complete* iff $\mathscr{E} = \overline{\mathscr{E}}$ and $\mathscr{F} = \overline{\mathscr{F}}$. We finally note that universal augmentation and right-continuous extension commute, i.e.[16]

$$\overline{\mathscr{F}_{\boldsymbol{+}}} = \overline{\mathscr{F}}_{\boldsymbol{+}}. \tag{2.2}$$

As a direct consequence, the same equation holds true with "$\boldsymbol{+}$" replaced by "$+$".

2.2. **Progressively measurable processes.** We continue with recalling some basic notions and thereby fixing notation. A *stochastic process*, with time $\overline{\mathbb{T}}$ and valued in $(Y, \mathscr{Y})$, is a map $\xi \colon \overline{\mathbb{T}} \times \Omega \to Y$ such that the maps $\xi_t = \xi(t, .)$, $t \in \overline{\mathbb{T}}$, are $\mathscr{E}$-$\mathscr{Y}$-measurable. This is equivalent to the map $\Omega \to Y^{\overline{\mathbb{T}}}$, $\omega \mapsto (t \mapsto \xi(t, \omega))$, also denoted by ξ, being $\mathscr{E}$-$\mathscr{Y}^{\otimes \overline{\mathbb{T}}}$-measurable. A stochastic process $\xi \colon \overline{\mathbb{T}} \times \Omega \to Y$ is said *(strictly) $\mathscr{F}$-adapted* iff, for all $t \in \overline{\mathbb{T}}$, ξ_t is even $\mathscr{F}_t$-$\mathscr{Y}$-measurable. We emphasise the following generalisation, which is less evident, because it depends on the choice of the σ-algebra on $\overline{\mathbb{T}}$.

Definition 2.1. Let $\mathscr{T}_{\overline{\mathbb{T}}}$ be a σ-algebra on $\overline{\mathbb{T}}$ containing the $\mathbb{T}$-intervals, i.e. with $\mathscr{I}_{\overline{\mathbb{T}}}(\mathbb{T}) \subseteq \mathscr{T}_{\overline{\mathbb{T}}}$.

1. A subset $M \subseteq \overline{\mathbb{T}} \times \Omega$ is said *$\mathscr{F}$-progressively measurable with respect to $\mathscr{T}_{\overline{\mathbb{T}}}$* iff, for all $t \in \overline{\mathbb{T}}$,
$$M \cap ([0, t]_{\overline{\mathbb{T}}} \times \Omega) \in \mathscr{T}_{\overline{\mathbb{T}}} \otimes \mathscr{F}_t. \tag{2.3}$$
The set of $\mathscr{F}$-progressively measurable subsets of $\overline{\mathbb{T}} \times \Omega$ with respect to $\mathscr{T}_{\overline{\mathbb{T}}}$ is denoted by $\mathrm{Prg}(\mathscr{T}_{\overline{\mathbb{T}}}, \mathscr{F})$.
2. A stochastic process $\xi \colon \overline{\mathbb{T}} \times \Omega \to Y$ is said *$\mathscr{F}$-progressively measurable* with respect to $\mathscr{T}_{\overline{\mathbb{T}}}$ iff $\xi^{-1}(B)$ is $\mathscr{F}$-progressively measurable with respect to $\mathscr{T}_{\overline{\mathbb{T}}}$ for any $B \in \mathscr{Y}$.
3. If the qualifier "with respect to $\mathscr{T}_{\overline{\mathbb{T}}}$" is omitted, then $\mathscr{T}_{\overline{\mathbb{T}}} = \mathscr{P}_{\overline{\mathbb{T}}}$. Moreover, let $\mathrm{Prg}(\mathscr{F}) = \mathrm{Prg}(\mathscr{P}_{\overline{\mathbb{T}}}, \mathscr{F})$.

Remark 2.2. The following facts are easily shown. Let the objects and notation be given as in the definition. Then:

1. Standard real time notions of adapted and progressively measurable processes are essentially retrieved by considering stochastic processes $\xi \colon \overline{\mathbb{T}} \times \Omega \to Y$ and filtrations $\mathscr{F}$ such that $\xi_t = \xi_{p(t)}$ and $\mathscr{F}_t = \mathscr{F}_{p(t)}$ for all $t \in \overline{\mathbb{T}}$.
2. $\mathrm{Prg}(\mathscr{T}_{\overline{\mathbb{T}}}, \mathscr{F})$ defines a σ-algebra, and a stochastic process $\xi \colon \overline{\mathbb{T}} \times \Omega \to Y$ is $\mathscr{F}$-progressively measurable with respect to $\mathscr{T}_{\overline{\mathbb{T}}}$ iff it is $\mathrm{Prg}(\mathscr{T}_{\overline{\mathbb{T}}}, \mathscr{F})$-$\mathscr{Y}$-measurable.
3. If $\mathscr{T}_{\overline{\mathbb{T}}}, \mathscr{T}'_{\overline{\mathbb{T}}}$ are two σ-algebras on $\overline{\mathbb{T}}$ containing the $\mathbb{T}$-intervals and such that $\mathscr{T}_{\overline{\mathbb{T}}} \subseteq \mathscr{T}'_{\overline{\mathbb{T}}}$, then $\mathrm{Prg}(\mathscr{T}_{\overline{\mathbb{T}}}, \mathscr{G}) \subseteq \mathrm{Prg}(\mathscr{T}'_{\overline{\mathbb{T}}}, \mathscr{G})$, i.e. $\mathscr{G}$-progressively measurability with respect to $\mathscr{T}_{\overline{\mathbb{T}}}$ implies that with respect to $\mathscr{T}'_{\overline{\mathbb{T}}}$.

[16]This follows from the analogous classical result. For the reader's convenience, the appendix contains a proof nevertheless.

4. We have $\mathscr{T}_{\overline{\mathbb{T}}} \otimes \mathscr{F}_0 \subseteq \mathrm{Prg}(\mathscr{T}_{\overline{\mathbb{T}}}, \mathscr{F}) \subseteq \mathscr{T}_{\overline{\mathbb{T}}} \otimes \mathscr{F}_\infty$. Hence, with respect to $\mathscr{T}_{\overline{\mathbb{T}}}$, any set of the form $T \times E$, with $T \in \mathscr{T}_{\overline{\mathbb{T}}}$, $A \in \mathscr{F}_0$, is $\mathscr{F}$-progressively measurable, and any progressively measurable set is $\mathscr{T}_{\overline{\mathbb{T}}} \otimes \mathscr{F}_\infty$-measurable.
5. If $\mathscr{T}_{\overline{\mathbb{T}}} \subseteq \mathscr{P}_{\overline{\mathbb{T}}}$ and Y is a metrisable topological space, then for any, with respect to $\mathscr{T}_{\overline{\mathbb{T}}}$, $\mathscr{F}$-progressively measurable $\xi \colon \overline{\mathbb{T}} \times \Omega \to Y$, all $\omega \in \Omega$ and all $t \in \mathbb{R}_+$, $\xi(.,\omega)$ is left-constant at $(t, \mathfrak{w}_1)$. This follows from the fact that $\overline{\mathbb{T}} \to \overline{\mathbb{T}} \times \Omega$, $t \mapsto (t,\omega)$ is $\mathscr{P}_{\overline{\mathbb{T}}}$-$\mathscr{T}_{\overline{\mathbb{T}}} \otimes \mathscr{E}$-measurable, and Lemma 1.15.[17]
6. Any, with respect to $\mathscr{T}_{\overline{\mathbb{T}}}$, $\mathscr{F}$-progressively measurable stochastic process $\xi \colon \overline{\mathbb{T}} \times \Omega \to Y$ as in Definition 2.1 is adapted in a relaxed sense. Namely, ξ_t is $\mathscr{F}_t$-$\mathscr{Y}$-measurable (at least) for any $t \in \overline{\mathbb{T}}$ with $\pi(t) < \mathfrak{w}_1$, alias $t \in \mathbb{T} \cup \{\infty\}$. In view of the discussion in Subsection 1.2, these can be interpreted as (deterministic) optional times. The result follows from the previous item with $T = \{t\}$ and $E = \Omega$.

The simplest non-trivial progressively measurable processes are those generated by *certain* random times – making the emphasised qualifier precise, is our concern for the remainder of this and the following subsection. A *random time* with respect to a σ-algebra $\mathscr{T}_{\overline{\mathbb{T}}}$ on $\overline{\mathbb{T}}$ containing the intervals is an $\mathscr{E}$-$\mathscr{T}_{\overline{\mathbb{T}}}$-measurable map $\tau \colon \Omega \to \overline{\mathbb{T}}$. These are in one-to-one correspondence to certain subsets of $\overline{\mathbb{T}} \times \Omega$, by considering its converse graph $[\![\tau]\!]$ as defined in Equation 1.13, its converse epi- or its converse hypograph. These sets can be defined in terms of stochastic intervals as well, which in turn are defined in complete analogy to the real-time case.[18] By assigning to any $t \in \overline{\mathbb{T}}$ the constant map $\Omega \to \overline{\mathbb{T}}$ with value t, we obtain an injection of $\overline{\mathbb{T}}$ into the set of random times, and we identify $\overline{\mathbb{T}}$ with its image under this injection. Note that, hence, any set $M \subseteq \overline{\mathbb{T}} \times \Omega$ is $\mathscr{F}$-progressively measurable with respect to $\mathscr{T}_{\overline{\mathbb{T}}}$ as in the definition iff, for any $t \in \overline{\mathbb{T}}$, $M \cap [\![0,t]\!] \in \mathscr{T}_{\overline{\mathbb{T}}} \otimes \mathscr{F}_t$.

As in the classical theory, the notion of adaptedness for stochastic processes alias compatibility with the filtration $\mathscr{F}$ motivates defining the following subclass of random times (with respect to $\mathscr{I}_{\overline{\mathbb{T}}}(\mathbb{T})$). An *$\mathscr{F}$-stopping time* is a map $\tau \colon \Omega \to \overline{\mathbb{T}}$ with $\{\tau \leq t\} \in \mathscr{F}_t$ for all $t \in \overline{\mathbb{T}}$. In other words, $\tau \colon \Omega \to \overline{\mathbb{T}}$ is an $\mathscr{F}$-stopping time iff the process $1[\![0,\tau)\!)$ is $\mathscr{F}$-adapted. That is, by Corollary 1.6, τ is a random time with respect to $\mathscr{I}_{\overline{\mathbb{T}}}(\mathbb{T})$ such that at any (order-completed) time $t \in \overline{\mathbb{T}}$, the given information flow $\mathscr{F}$ can tell whether τ lies not in the future. It describes information about whether a fixed event in $\mathscr{E}$ has already happened — including the present —, or not. To any $\overline{\mathbb{T}}$-valued stopping time τ, we can associate the set

$$\mathscr{F}_\tau = \{E \in \mathscr{E} \mid \forall t \in \overline{\mathbb{T}} \colon E \cap \{\tau \leq t\} \in \mathscr{F}_t\}. \tag{2.4}$$

Remark 2.3. Standard arguments, combined with Corollary 1.6, show that, for any sequence $(\tau_n)_{n\in\mathbb{N}}$ of $\mathscr{F}$-stopping times and $\tau = \tau_0$ we have:

1. Scenariowise supremum $\sup_{n\in\mathbb{N}} \tau_n$ is an $\mathscr{F}$-stopping time;
2. Scenariowise infimum $\sigma = \inf_{n\in\mathbb{N}} \tau_n$ is an $\mathscr{F}_+$-stopping time, and even an $\mathscr{F}$-stopping time in case $\bigcup_{n\in\mathbb{N}}\{\sigma = \tau_n\} = \Omega$;
3. For all $t \in \overline{\mathbb{T}}$ with $\pi(t) < \mathfrak{w}_1$, we have $\{\tau < t\}$, $\{\tau = t\} \in \mathscr{F}_t$;
4. $\mathscr{F}_\tau$ is a sub-σ-algebra of $\mathscr{E}$ and moreover, if $(\Omega, \mathscr{E}, \mathscr{F})$ is universally complete, then it is universally augmented in $\mathscr{E}$;
5. If $\tau = t$ holds true, for some $t \in \overline{\mathbb{T}}$, then $\mathscr{F}_\tau = \mathscr{F}_t$;
6. If $\tau_0 \leq \tau_1$ holds true, then $\mathscr{F}_{\tau_0} \subseteq \mathscr{F}_{\tau_1}$;

[17] As $\mathfrak{w}_1$ is uncountable, this does not have clear implications on the random variable $\xi_{(t,\mathfrak{w}_1)}$ in general.

[18] For instance, given two maps $\sigma, \tau \colon \Omega \to \overline{\mathbb{T}}$, we have

$$[\![\sigma,\tau)\!) = \{(t,\omega) \in \overline{\mathbb{T}} \times \Omega \mid \sigma(\omega) \leq t < \tau(\omega)\}.$$

That way, the converse graph of τ is given by $[\![\tau]\!] = [\![\tau,\tau]\!]$, the converse weak and strict epigraphs of τ are given by $[\![\tau,\infty]\!]$ and $(\!(\tau,\infty]\!]$, and the converse weak and strict hypographs of τ are given by $[\![0,\tau]\!]$ and $[\![0,\tau)\!)$.

7. τ is $\mathscr{F}_\tau$-$\mathscr{I}_{\overline{\mathbb{T}}}(\mathbb{T})$-measurable.

The following counterexample shows that other classical results about stopping times do not generalise. Detailed verifications can be found in the appendix.

Example 2.4. Suppose that $\Omega = \mathbb{R}_+$, equipped with Lebesgue σ-algebra $\mathscr{E}$ and the exponential distribution $\mathbb{P}$ with parameter 1, i.e. the identity σ on $\mathbb{R}_+$ satisfies $\mathbb{P}(\sigma > t) = \mathrm{e}^{-t}$, $t \in \mathbb{R}_+$. We can see σ as the set-theoretic inclusion map $\mathbb{R}_+ \to \overline{\mathbb{T}}$. Let $\mathscr{F} = (\mathscr{F}_t)_{t\in\overline{\mathbb{T}}}$ be the $\mathbb{P}$-augmented filtration generated by σ, i.e. the collections $\mathscr{F}_t$ of sets $\{\sigma \le s\}$, $s \le t$, ranging over $t \in \overline{\mathbb{T}}$. Let $V \subseteq \mathbb{R}_+$. Define $\tau\colon \Omega \to \overline{\mathbb{T}}$ by letting $\tau(\omega) = \sigma(\omega)$ if $\omega \in V$, and $\tau(\omega) = (\sigma(\omega), 1)$ else. Then the following statements hold true:

1. τ is an $\mathscr{F}$-stopping time;
2. $\mathbb{P}_\tau = \mathbb{P}_\sigma$ on $\mathscr{I}_{\overline{\mathbb{T}}}(\mathbb{T})$;
3. $\sigma \le \tau$ and $\{\sigma = \tau\} = V$;
4. If $V \notin \mathscr{E}$, then:
   (a) τ is not $\mathscr{F}_\tau$-$\mathscr{P}_{\overline{\mathbb{T}}}$-measurable;
   (b) $[\![\tau]\!]$, $[\![0,\tau)\!)$, and $(\!(\tau,\infty]\!]$ are not $\mathscr{F}$-progressively measurable (i.e. not even with respect to $\mathscr{P}_{\overline{\mathbb{T}}}$).

Note that Part 4 is not void, since $\mathscr{E} \subsetneq \mathcal{P}\mathbb{R}_+$ (Vitali).

We conclude that there are $\mathscr{F}$-adapted processes with càg paths that are not $\mathscr{F}$-progressively measurable (i.e. not even with respect to $\mathscr{P}_{\overline{\mathbb{T}}}$), for instance, $1(\!(\tau,\infty]\!]$ from Example 2.4, Part 4. Moreover, the example demonstrates that two stopping times can have the same distribution, be ordered and still be different in any scenario (case $V = \emptyset$). We can also infer from Item 4(a), combined with Remark 2.3, Item 7, that the inclusion in Lemma 1.12 is strict:

Corollary 2.5. *We have* $\mathscr{I}_{\overline{\mathbb{T}}}(\mathbb{T}) \subsetneq \mathscr{P}_{\overline{\mathbb{T}}}$. □

2.3. **Optional times.** The preceding discussion raises the question whether there is a natural subclass of stopping times exhibiting stronger measurability properties. This can mean different things: the measurability of the stopping time itself and the progressive measurability of its converse graph, epi- and hypograph. Making the σ-algebra on $\overline{\mathbb{T}}$ larger works against the first, but in favour of the second requirement. Thus, we may rephrase the questions as whether there is a σ-algebra that solves this trade-off. Indeed, there is one, namely the projection σ-algebra, as the following theorem affirms.

Theorem 2.6. *For a map $\tau\colon \Omega \to \overline{\mathbb{T}}$ consider the following seven statements.*

1. *The converse graph $[\![\tau]\!]$ is $\mathscr{F}$-progressively measurable.*
2. *The converse strict hypograph $[\![0,\tau)\!)$ is $\mathscr{F}$-progressively measurable.*
3. *The converse weak epigraph $[\![\tau,\infty]\!]$ is $\mathscr{F}$-progressively measurable.*
4. *The converse strict epigraph $(\!(\tau,\infty]\!]$ is $\mathscr{F}$-progressively measurable and $\pi\circ\tau < \mathfrak{w}_1$.*
5. *The converse weak hypograph $[\![0,\tau]\!]$ is $\mathscr{F}$-progressively measurable and $\pi\circ\tau < \mathfrak{w}_1$.*
6. *There is $\alpha \in \mathfrak{w}_1$ such that $\pi\circ\tau \le \alpha$ and, for all $\beta \in \alpha+1$ and $t \in \overline{\mathbb{T}}$:*

$$\{\pi\circ\tau = \beta,\, \tau \le t\} \in \mathscr{F}_t.$$

7. *τ is an $\mathscr{F}$-stopping time, there is $\alpha \in \mathfrak{w}_1$ such that $\pi\circ\tau \le \alpha$, and τ is $\mathscr{F}_\tau$-$\mathscr{P}_{\overline{\mathbb{T}}}$-measurable.*

The following holds true:

– *Statements 2 and 3 are equivalent, and statements 4 and 5 are equivalent.*
– *Statements 6 and 7 are equivalent, and they imply all the others.*
– *If $(\Omega, \mathscr{E}, \mathscr{F})$ is universally complete, then all statements are equivalent.*

Definition 2.7. An *$\mathscr{F}$-optional time* is an $\mathscr{F}$-stopping time satisfying $\pi \circ \tau \leq \alpha$ for some $\alpha \in \mathfrak{w}_1$ and being $\mathscr{F}_\tau$-$\mathscr{P}_{\overline{\mathbb{T}}}$-measurable.

Remark 2.8. A map $\tau\colon \Omega \to \overline{\mathbb{R}_+}$ is an $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$-optional time, or equivalently, an $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$-stopping time, both in the classical sense (see e.g. Dellacherie–Meyer [21], Kallenberg [44]), iff it is an $\mathscr{F}$-stopping time in the extended sense, iff it is an $\mathscr{F}$-optional time. In the extended setting, the notion of optional times is stronger than that of stopping times.

Example 2.4 together with Theorem 2.6 already illustrates the difference. On a more abstract level, a stopping time is a random time such that one can tell whether it is already over or not; this time can indeed be thought of as the time some given process on $(\Omega, \mathscr{E})$ stops.

In addition, an optional time lies in $\mathbb{T}$, unless it takes the value ∞, meaning "never"; more precisely, even on $\mathbb{T}_{\alpha+1}$ for some countable $\alpha \in \mathfrak{w}_1$. Recalling that the vertical axis stands for accumulating embedding of well-orders into $\mathbb{R}_+$ and that any well-order embedded into $\mathbb{R}_+$ must be countable, we see that a fixed optional time must be thinkable in terms of one such an embedding (or at most countably many) — the discussion in Subsection 2.5 and, in particular, Proposition 2.33 will make this precise. Moreover, the time itself must be measurable at the time of its realisation, with respect to the σ-algebra $\mathscr{P}_{\overline{\mathbb{T}}}$, which takes into account information about the vertical time coordinate as well. In total, optional times describe times where one can revise options and make decisions in a way compatible with the given information flow. This is applied and further elaborated on in Section 3 to formulate a general notion of "information sets" and "subgames" in a stochastic process-based game-theoretic model.

Corollary 2.9. *If $(\Omega, \mathscr{E}, \mathscr{F})$ is universally complete, $\xi\colon \overline{\mathbb{T}} \times \Omega \to Y$ is $\mathscr{F}$-progressively measurable, and τ is an $\mathscr{F}$-optional time, then*
$$\xi_\tau\colon \Omega \to Y,\ \omega \mapsto \xi_{\tau(\omega)}(\omega)$$
is $\mathscr{F}_\tau$-$\mathscr{Y}$-measurable.

the following propositions underline the importance of optional times, illustrate the utility of the previous theorem, and moreover prove useful in some of the upcoming proofs.

Proposition 2.10. *Let $(\tau_n)_{n\in\mathbb{N}}$ be a sequence of $\mathscr{F}$-optional times and $\tau\colon \Omega \to \overline{\mathbb{T}}$ be a map. The following statements hold true:*

1. *If τ is an $\mathscr{F}$-stopping time and $\varepsilon \geq 0$ is real, then $p \circ \tau + \varepsilon$ is an $\mathscr{F}_{+}$-optional time, and provided $\varepsilon > 0$, it is even an $\mathscr{F}$-optional time.*
2. *If $\alpha \in \mathfrak{w}_1$ is such that, for all $n \in \mathbb{N}$, $\pi \circ \tau_n \leq \alpha$, and*
$$\tau(\omega) = \begin{cases} \lim_{n\to\infty} \tau_n(\omega), & \text{if this limit exists,} \\ \infty, & \text{else,} \end{cases} \qquad \omega \in \Omega,$$
the limit being taken in the Polish space $\overline{\mathbb{R}_+} \times (\alpha + 1)$, then τ is an $\mathscr{F}_{+}$-optional time.
3. *We have $\{\tau_0 \leq \tau_1\} \in \mathscr{F}_{\tau_0} \cap \mathscr{F}_{\tau_1}$.*
4. *$\tau_0 \wedge \tau_1$ and $\tau_0 \vee \tau_1$ are $\mathscr{F}$-optional times.*

In general, the proofs are a bit tedious. In the universally complete setting, they can be substantially simplified. This is discussed in the appendix, together with the proofs.

As an illustration, we present the following result about approximating optional times with respect to the augmented filtration by optional times with respect to the right-limit of the original one — the case of $\overline{\mathbb{R}_+}$-valued $\overline{\tau}$ is well-known (see, e.g., [62]).

Proposition 2.11. *Suppose that $\mathscr{E}$ is universally complete and let $\overline{\mathscr{F}} = (\overline{\mathscr{F}_t})_{t\in\overline{\mathbb{T}}}$. Then, for any $\mathbb{P} \in \mathfrak{P}_{\mathscr{E}}$ and any $\overline{\mathscr{F}}$-optional time $\overline{\tau}$ there is an $\mathscr{F}_{+}$-optional time τ with $\mathbb{P}(\tau = \overline{\tau}) = 1$.*

A large class of stopping and optional times can be constructed as follows. For a set $M \subseteq \overline{\mathbb{T}} \times \Omega$, let

$$D_M : \Omega \to \overline{\mathbb{T}},\ \omega \mapsto \inf\{t \in \overline{\mathbb{T}} \mid (t,\omega) \in M\}$$

be its *début* alias entry time. For example, $M = \{\xi \in B\}$ for a Y-valued stochastic process ξ and a measurable set $B \in \mathscr{Y}$. Recall that ξ is $\mathscr{F}$-progressively measurable iff M is for any $B \in \mathscr{Y}$.

Theorem 2.12 (Début)**.** *Suppose that $\mathscr{E}$ is universally complete. Let $M \subseteq \overline{\mathbb{T}} \times \Omega$ be $\mathscr{F}$-progressively measurable. Then, D_M is an $\overline{\mathscr{F}}_+$-stopping time. Moreover, the following conditions are equivalent:*

1. *D_M is an $\overline{\mathscr{F}}$-optional time.*
2. *$\pi \circ D_M < \mathfrak{w}_1$.*
3. *$[\![D_M]\!] \cap [\![0,\infty)\!) \subseteq M$.*

Remark 2.13. Note that, by Theorem 2.6, conversely, any optional time τ is of the form D_M with progressively measurable M, namely, with $M = [\![\tau,\infty]\!]$. This sheds further light on the interpretation of optional times. Namely, D_M being an $\overline{\mathscr{F}}$-optional time means that the option-revising or decision-making agent really does so *at* time τ, and not only in the infinitesimal future after τ. In classical continuous time $\mathbb{R}_+$, an analogous statement is not true for optional times. For instance, the début of Brownian motion ξ into an open set U is the smallest time τ such that infinitesimally after this the Brownian motion enters U — but, typically,[19] $\xi_\tau \notin U$! Hence, the notion of the first entry time is problematic: it actually does not exist in this situation, there is only a greatest lower bound on all times t with $\xi_t \in U$.

Not so for optional times in vertically extended continuous time! Here, Property 3 in Theorem 2.12 essentially expresses the fact that if in scenario $\omega \in \Omega$ the set M is reached at some (finite or infinite) time, then it is reached at the infimal time D_M, i.e. $(D_M(\omega),\omega) \in M$.

2.4. **Optional processes.** So far we have focused on the basic case of stopping times and corresponding processes. We continue with discussing a classes of progressively measurable processes describing decision making at optional times. By Theorem 2.6, the σ-algebras on $\overline{\mathbb{T}} \times \Omega$

$$\mathrm{Prd}(\mathscr{F}) = \big\{\{0\} \times E \mid E \in \mathscr{F}_0\big\} \vee \sigma\Big([\![0,\tau]\!] \mid \tau\ \mathscr{F}\text{-optional time}\Big),$$

$$\mathrm{Opt}(\mathscr{F}) = \big\{\{\infty\} \times E \mid E \in \mathscr{F}_\infty\big\} \vee \sigma\Big([\![0,\tau)\!) \mid \tau\ \mathscr{F}\text{-optional time}\Big),$$

are contained in $\mathrm{Prg}(\mathscr{F})$. In analogy with the classical case, call a subset $M \subseteq \overline{\mathbb{T}} \times \Omega$ *$\mathscr{F}$-predictable* (*$\mathscr{F}$-optional*) iff $M \in \mathrm{Prd}(\mathscr{F})$ ($M \in \mathrm{Opt}(\mathscr{F})$, respectively); and idem for a stochastic process $\xi : \overline{\mathbb{T}} \times \Omega \to Y$ iff this holds true for any M of the form $M = \xi^{-1}(B)$, $B \in \mathscr{Y}$. As in the classical case, predictability implies optionality:

Lemma 2.14. *We have* $\mathrm{Prd}(\mathscr{F}) \subseteq \mathrm{Opt}(\mathscr{F}) \subseteq \mathrm{Prg}(\mathscr{F})$.

An important and illustrative example of optional processes is the following. Let us call *real-valued simple $\mathscr{F}$-optional process* any map $\xi : \overline{\mathbb{T}} \times \Omega \to \mathbb{R}$ of the form

$$\xi = \xi^\alpha \circ \mathrm{prj}_\Omega\, 1[\![\tau_\alpha]\!] + \sum_{\beta \in \alpha} \xi^\beta \circ \mathrm{prj}_\Omega\, 1[\![\tau_\beta, \tau_{\beta+1})\!)\,, \tag{2.5}$$

for a countable ordinal $\alpha \in \mathfrak{w}_1$, a family $(\tau_\beta)_{\beta\in\alpha+1}$ of $\mathscr{F}$-optional times with $\tau_0 = 0$, $\tau_\alpha = \infty$, $\tau_\beta \leq \tau_\gamma$ for all $\beta, \gamma \in \alpha+1$ with $\beta \leq \gamma$, and $\tau_\gamma = \sup_{\beta\in\gamma} \tau_\beta$ for all limit ordinals $\gamma \in \alpha + 1$, and a family $(\xi^\beta)_{\beta\in\alpha+1}$ of real-valued $\mathscr{F}_{\tau_\beta}$-measurable ξ^β. Given a Polish space Y, a *simple $\mathscr{F}$-optional process* is a map $\xi : \overline{\mathbb{T}} \times \Omega \to Y$ such that for any measurable map $\varphi : Y \to \mathbb{R}$, $\varphi \circ \xi$ is a real-valued simple $\mathscr{F}$-optional process.

[19] That is, if ξ does not already start in U, i.e. $\xi_0 \notin U$, ...

Let us briefly note that, as a special case, real-valued simple $\mathscr{F}$-optional processes contain what, in view of the classical stochastic analysis literature (see, e.g., [21, 44]), we might call *real-valued simple $\mathscr{F}$-predictable processes*. That is the case where $\tau_1 = (0,1)$, $\alpha = \gamma + 1$ for some $\gamma \in \mathfrak{w}_1$, $\xi^\gamma = \xi^\alpha$, and for all $\beta \in \alpha \setminus \{0\}$, there is an $\mathscr{F}$-optional time σ_β with $p \circ \tau_\beta = p \circ \sigma_\beta$ and $\pi \circ \tau_\beta = \pi \circ \sigma_\beta + 1$, implying $[\![\tau_0, \tau_1)\!) = [\![0]\!]$, $[\![\tau_\beta, \tau_{\beta+1})\!) = (\!(\sigma_\beta, \sigma_{\beta+1}]\!]$ for $\beta \in (0,\gamma)_{\mathfrak{w}_1}$, and $\xi^\alpha \circ \mathrm{prj}_\Omega 1[\![\tau_\alpha]\!] + \xi^\gamma \circ \mathrm{prj}_\Omega 1[\![\tau_\gamma, \tau_{\gamma+1})\!) = \xi^\gamma \circ \mathrm{prj}_\Omega 1(\!(\sigma_\gamma, \infty]\!]$.

Lemma 2.15. *$\mathrm{Opt}(\mathscr{F})$ is the smallest σ-algebra $\mathscr{M}$ on $\overline{\mathbb{T}} \times \Omega$ such that all real-valued simple $\mathscr{F}$-optional processes are $\mathscr{M}$-measurable.*

Lemma 2.16. *The set of real-valued $\mathscr{F}$-optional processes is the smallest set of maps $\overline{\mathbb{T}} \times \Omega \to \mathbb{R}$ a) containing $1[\![0,\tau)\!)$ for all $\mathscr{F}$-optional times and $1(\{\infty\} \times E)$ for all $E \in \mathscr{F}_\infty$, b) closed under pointwise addition and real scalar multiplication, and c) closed under pointwise convergence.*

Similar results may be obtained for predictable processes. By definition, optional (and predictable) processes can be seen as a limit object of locally right-constant sequential decision making, progressively measurable with respect to the information flow. Note that we further elaborate on this in the following Subsection 2.5. Here, for predictable processes, the approximators are such that any action can be predicted (see the discussion above about simple predictable processes). Generalising these concepts to vertically extended real time allows to fully and consistently model an agent's capacity of sequential instantaneous re- or proaction with respect to information flow, including information progressively revealed only during this "instantaneous" process (of course, it is only instantaneous in the $\overline{\mathbb{R}_+}$-coordinate).

Remark 2.17.
1. In view of the preceding discussion and Lemma 2.14, one can further develop the subtleness of re- or proaction with respect to information flow in the framework of vertically extended time, by considering generalised *Meyer σ-algebras*, i.e. σ-algebras $\mathscr{M}$ on $\overline{\mathbb{T}} \times \Omega$ satisfying
$$\mathrm{Prd}(\mathscr{F}) \subseteq \mathscr{M} \subseteq \mathrm{Opt}(\mathscr{F}).$$
These objects have been introduced in classical continuous times by [51], further developed in both theory and applications in [10, 11, 13, 12]. A detailed mathematical development of the related stochastic analysis in vertically extended continuous time is beyond the scope of the present work. It seems worth further inquiry, because it naturally arises in control- and game-theoretic models in order to mix predictable and optional decision making with respect to different sources of information (see Subsection 3.1).
2. The notions of predictability and optionality correspond to the classical notions when supposing $\mathscr{F}$ and ξ such that $\mathscr{F}_t = \mathscr{F}_{p(t)}$ and $\xi_t = \xi_{p(t)}$ for all $t \in \overline{\mathbb{T}}$.

The following proposition provides a hierarchical description of optional processes in the spirit of descriptive set theory (see [45]). This allows to better understand the measurability of optional processes at all $t \in \overline{\mathbb{T}}$ with $\pi(t) = \mathfrak{w}_1$. For this, let us introduce the following notation. Let $\mathcal{V}_0$ denote the real vector space, with pointwise addition and scalar multiplication, generated by all maps of the form

$$1[\![0,\tau)\!), \quad 1\{\infty\} \times E, \qquad \tau\ \mathscr{F}\text{-optional time}, E \in \mathscr{F}_\infty. \tag{2.6}$$

For any ordinal $\alpha \in \mathfrak{w}_1$, let $\mathcal{V}_{\alpha+1}$ be the set of pointwise limits of $\mathcal{V}_\alpha$-valued sequences. For any limit ordinal $\alpha \in \mathfrak{w}_1$, let $\mathcal{V}_\alpha = \bigcup_{\beta \in \alpha} \mathcal{V}_\beta$.

By transfinite induction, one shows that for all $\alpha, \beta \in \mathfrak{w}_1 + 1$ with $\alpha \leq \beta$, we have $\mathcal{V}_\alpha \subseteq \mathcal{V}_\beta$. Moreover, for all $\alpha \in \mathfrak{w}_1$, $\mathcal{V}_\alpha$ is an $\mathbb{R}$-vector space — for if not, there would be a smallest $\alpha \in$

$\mathfrak{w}_1 + 1$ without that property, which is impossible.[20] We can now state and prove the announced proposition.

Proposition 2.18. *The set of real-valued $\mathscr{F}$-optional processes equals $\mathcal{V}_{\mathfrak{w}_1}$.*

Corollary 2.19. *For any $\mathscr{F}$-optional process ξ valued in a Polish space Y, there is $\alpha \in \mathfrak{w}_1$ such that for all $t \in \overline{\mathbb{T}}$ with $\pi(t) \geq \alpha$ and all $\omega \in \Omega$, $\xi(t,\omega) = \xi((p(t),\alpha),\omega)$ holds true.*

Note that, in view of Remark 2.2, Part 5, we already knew that for any progressively measurable process ξ valued in a metrisable topological space, any $u \in \mathbb{R}_+$ and any $\omega \in \Omega$, there is $\alpha \in \mathfrak{w}_1$ such that all $\beta \in \mathfrak{w}_1 + 1$ with $\alpha \leq \beta$ satisfy $\xi((u,\beta),\omega) = \xi((u,\alpha),\omega)$. By the preceding corollary, this can be considerably strengthened for optional processes: α can be chosen independent of both u and ω.

Definition 2.20. For any $\mathscr{F}$-optional process ξ valued in a Polish space Y, its *vertical level* is the smallest $\alpha \in \mathfrak{w}_1$ such that for all $t \in \overline{\mathbb{T}}$ with $\pi(t) \geq \alpha$ and all $\omega \in \Omega$, $\xi(t,\omega) = \xi((p(t),\alpha),\omega)$ holds true. The *upper vertical level* of ξ is the smallest $\beta \in \mathfrak{w}_1$ satisfying the following property: For all $x \in \mathbb{R}_+$ and all $\omega \in \Omega$, there is $\alpha \in \beta$ such that for all $t \in \overline{\mathbb{T}}$ with $p(t) = x$ and $\pi(t) \geq \alpha$, we have $\xi(t,\omega) = \xi((x,\alpha),\omega)$.

Remark 2.21. Note that the vertical level is inferior to the upper vertical level. Let $\alpha \in \mathfrak{w}_1$. If α is the vertical level of ξ, then its upper vertical level is either α or $\alpha + 1$. If $\alpha + 1$ is the upper vertical level of ξ, then α is the vertical level of ξ. So, the upper vertical level is a more general and flexible concept than that of the vertical level, while the latter is a bit more accessible.

2.5. **Tilting convergence.** In this subsection, we show that optional processes in vertically extended time arise by closing the set of classical, very simple optional processes with respect to binary continuous operations and *natural* limit procedures. Given a Polish space Y, a *very simple $\mathscr{F}$-optional process* is a map $\xi \colon \overline{\mathbb{T}} \times \Omega \to Y$ such that for any measurable map $\varphi \colon Y \to \mathbb{R}$, $\varphi \circ \xi$ is a real-valued simple $\mathscr{F}$-optional process admitting a representation according to Equation 2.5 with deterministic τ_β, for all $\beta \in \alpha + 1$. We recall that a process $\xi \colon \overline{\mathbb{T}} \times \Omega \to Y$ is said *classical* iff $\xi = \xi \circ (p \times \mathrm{id}_\Omega)$. A classical, very simple $\mathscr{F}$-optional process can be represented as just mentioned with deterministic, $\overline{\mathbb{R}_+}$-valued τ_β.

The subtle point is to clarify what "natural" limit procedure means in our context and to formulate a suitable notion of convergence. Pointwise convergence has already been considered in the preceding section — by Lemma 2.16, optional processes are essentially generated by simple processes via pointwise limits. Stability under pointwise convergence is a requirement that emanates from basic measure and integration theory: Any σ-additive measure on the sample space $\overline{\mathbb{T}} \times \Omega$ that can integrate (bounded measurable) functions of all simple $\mathscr{F}$-optional processes, can also integrate (bounded measurable) functions of general $\mathscr{F}$-optional processes.

In order to obtain general simple optional processes out of classical very simple ones — and therefore give an interpretation of (simple, then general) optional processes in vertically extended time —, one needs an additional notion of convergence, namely one capturing infinitesimally accumulating information. With the game-theoretic background of this text, mostly developed in Section 3, this is in particular information on decisions which, under pointwise convergence, may collapse in the limit. To give a visual description of the process, imagine an infinitely long sentence written in one horizontal half-axis (think of $\mathbb{R}_+$). A schematic representation of this is given in Figure 2, printed in the introduction. Words are indicated by points, and the position of the words at the beginning of the convergence procedure are indicated by light gray.

Now, there is a person sitting at $+\infty$, requesting an executive summary. The person therefore pushes the sentence with infinite strength in the direction of the start of the sentence (that is, zero)

[20] The arguments are similar to those used in the context of the Borel hierarchy, see, e.g. [45].

— this is the dashed arrow pointing to the left in Figure 2, moving the words (indicated by points) towards the left. The movement is indicated by the points being printed in darker gray. Now, at any point $t \in \mathbb{R}_+$ on the half-axis, there are parts of the sentence that accumulate in its right-hand neighbourhood. Things turn out such that these parts are *tilted* by 90 degrees counterclockwise as to build a vertical strip just above t, to be read starting from below. In Figure 2, the tilting is indicated by the curved dashed arrows. The accumulation process is most easily imagined at zero; but, given a sufficient kind of periodicity of the sentence, the asymptotics work out at any $t \in \mathbb{R}_+$ as well.

Note that the sentence, although written on a continuous paper roll, is well-ordered and so are the vertical strips. On the paper roll, severely reshaped by the person, a continuous structure may yet arise. This should remind us of various pointwise approximation procedure in analysis, e.g. the approximation of measurable functions valued in $\mathbb{R}_+$ by simple ones, or even by step functions in certain cases. What we see in addition here, is the preservation of information on order about "words" (or more generally objects) that collapse at a single position t on the paper roll within the limit. The game-theoretic importance of this construction seems rather obvious then: the order of actions, whose execution times collapse to one identical "real" time $t \in \mathbb{R}_+$ in the limit, should be preserved.

Before getting to a formal description of this concept, let us note by means of an example that: a) pointwise convergence is far too restrictive, and b) convergence of the sequence $(\xi^n)_{n\in\mathbb{N}}$ alone does not lead to a satisfactory notion, i.e. the "convergence" depends on the sequence of grids.

Example 2.22. For this, consider the real-valued processes indexed over $n \in \mathbb{Z}$ given by $\xi^n = 1[\![0, 2^{-n})\!)$. Imagine that the process describes the actions of some agent called Alice on the dyadic grid G_n given by $k2^{-n}$, $k \in \mathbb{N}$. What is a limit of $(\xi^n)_{n\in\mathbb{N}}$? The pointwise limit, as $n \to +\infty$, is $1[\![0, (0, \mathfrak{w}_1)]\!]$ — but this gives no interesting information about the (vertically extended) time at that the agent switches to value zero (in short: stops). On the other hand, for any optional time τ with $p \circ \tau = 0$ and $\pi \circ \tau > 0$, $\xi = 1[\![0, \tau)\!)$ is a limit horizontally, i.e. $\xi^n(t) \to \xi(t)$ at any $t \in \overline{\mathbb{R}_+}$. As only Alice is involved, the exact choice of τ does not seem to reflect much essential information.

Now, add another agent called Bob acting according to ξ^{n-1} in grid G_n. Horizontally, the same limit obtains, and for Bob alone the choice of the vertical stopping time seems irrelevant. However, in any grid G_n, Alice switches to zero strictly before Bob: Alice uses the second, Bob the third opportunity to stop. In that context, the adequate limit outcome for Alice would be $\xi^A = 1[\![0, \tau^A)\!)$ with $\tau^A = (0, 1)$ and, similarly, for Bob $\xi^B = 1[\![0, \tau^B)\!)$ with $\tau^B = (0, 2)$.[21] Hence, the game-theoretically plausible limit depends on the chosen grid sequence and not only on the sequence $(\xi^n)_{n\in\mathbb{N}}$. It also depends on the fact that both agents use the same grid sequence. Next, suppose that, in any grid G_n, $n \in \mathbb{N}$, a third agent called Carol acts according to ξ^{n-2} if n is even and according to ξ^n if n is odd. Then, comparing with Bob, in any grid both agents do not stop simultaneously, but any susceptible limits ξ^B for Bob and ξ^C for Carol, in any scenario $\omega \in \Omega$, must either let them stop — also vertically — instantaneously or gives one of them the priority — which both would remain unexplained by the approximating sequences. A similar comparison with Alice is equally inconclusive. Thus, on the vertical half-axis, there may be no plausible limit at all. This holds true despite the facts that a) all sequences do converge in the strongest possible sense (pointwise on the full domain $\overline{\mathbb{T}} \times \Omega$, and they are uniformly bounded) and b) the horizontal limit is plausible. Hence, for understanding "limit behaviour along the vertical half-axis", looking at convergence of the sequence $(\xi^n)_{n\in\mathbb{N}}$ alone is insufficient.

Based on the theoretical motivation outlined in the beginning, underlined by the preceding example, we give the following definitions.

[21]Note that, in the von Neumann hierarchy, the second ordinal is 1, and the third ordinal is 2.

Definition 2.23. An $\mathscr{F}$*-adapted grid* is a map $G\colon (\alpha+1)\times\Omega \to \overline{\mathbb{T}}$ for some $\alpha \in \mathfrak{w}_1$ such that:

1. for each $\beta \in \alpha$, $\tau^G_\beta\colon \Omega \to \overline{\mathbb{T}}$, $\omega \mapsto G(\beta,\omega)$ is an $\mathscr{F}$-optional time;
2. $\tau^G_0 = 0$, $\tau^G_\alpha = \infty$;
3. for all $\beta,\gamma \in \alpha$ with $\beta < \gamma$, we have $\tau^G_\beta(\omega) < \tau^G_\gamma(\omega)$ for all $\omega \in \{\tau^G_\beta < \infty\}$;
4. for all limit ordinals $\gamma \in \alpha+1$ and all $\omega \in \Omega$, we have $G(\gamma,\omega) = \sup_{\beta\in\gamma} G(\beta,\omega)$.

An $\mathscr{F}$-adapted grid is said *classical* iff it is $\overline{\mathbb{R}_+}$-valued. A *(deterministic) grid* is an $\mathscr{F}$-random grid as above such that, for all $\beta \in \alpha$, τ^G_β is deterministic (i.e. constant).

Given two $\mathscr{F}$-adapted grids $G\colon (\alpha+1)\times\Omega \to \overline{\mathbb{T}}$, $G'\colon (\alpha'+1)\times\Omega \to \overline{\mathbb{T}}$, G' is said to *refine* or to be a *refinement of* G iff there is an order-embedding $j\colon \alpha+1 \hookrightarrow \alpha'+1$ such that $G = G' \circ (j \times \mathrm{id}_\Omega)$. For any $\mathscr{F}$-adapted grid $G\colon (\alpha+1)\times\Omega \to \overline{\mathbb{T}}$, let the *grid size* at $\omega \in \Omega$ be given by

$$\Delta(G,\omega) = \begin{cases} \infty, & \text{if } \sup_{\beta\in\alpha} G(\beta,\omega) < \infty, \\ \sup_{\substack{\beta\in\alpha: \\ \tau^G_\beta(\omega)<\infty}} \big(p\circ G(\beta+1,\omega) - p\circ G(\beta,\omega)\big), & \text{else.} \end{cases}$$

Let $(G_n)_{n\in\mathbb{N}}$ be a sequence of $\mathscr{F}$-adapted grids. It is said *refining* iff for all $n \in \mathbb{N}$, G_{n+1} refines G_n. It is said *(pointwise uniformly) convergent* if, for every $\omega \in \Omega$, $\Delta(G_n,\omega) \to 0$ as $n \to \infty$.

Remark 2.24. Any $\mathscr{F}$-adapted grid $G\colon (\alpha+1)\times\Omega \to \overline{\mathbb{T}}$ can be seen as a map $\mathfrak{w}_1 \times \Omega \to \overline{\mathbb{T}}$, by letting $G(\beta,\omega) = \infty$ for all arguments $(\beta,\omega) \in [\mathfrak{w}_1 \setminus (\alpha+1)] \times \Omega$. We use this convention in the following.

Simple / very simple (classical) $\mathscr{F}$-optional processes are defined via $\mathscr{F}$-adapted / deterministic (classical) grids, respectively. Let us fix a name for that.

Definition 2.25. Let Y be a Polish space and ξ' be a simple $\mathscr{F}$-optional process valued in Y. An $\mathscr{F}$-adapted grid $G\colon (\alpha+1)\times\Omega \to \overline{\mathbb{T}}$ is said *compatible with* ξ' iff there are a real-valued simple $\mathscr{F}$-optional process ξ, given by Equation 2.5, with $\tau_\beta = G(\beta,.)$ for all $\beta \in \alpha+1$, and measurable $\varphi\colon \mathbb{R} \to Y$ such that $\xi' = \varphi \circ \xi$.

In other words: Compatibility essentially means that all jump times of ξ are part of grid. By definition, for any simple (very simple) $\mathscr{F}$-optional process, there is a compatible $\mathscr{F}$-adapted (deterministic) grid. For simple $\mathscr{F}$-optional processes, there is even a smallest such grid, provided measure-theoretic completeness:

Lemma 2.26. *Suppose that $(\Omega,\mathscr{E},\mathscr{F})$ is universally complete. Let Y be a Polish space and ξ be a simple $\mathscr{F}$-optional process valued in Y. Then, there is an $\mathscr{F}$-adapted grid $G\colon (\alpha+1)\times\Omega \to \overline{\mathbb{T}}$ satisfying, for all $\beta \in \alpha$ and $\omega \in \Omega$:*

$$G(\beta+1,\omega) = \inf\{t \in [G(\beta,\omega),\infty]_{\overline{\mathbb{T}}} \mid \xi_t(\omega) \neq \xi_{G(\beta,\omega)}(\omega)\}. \tag{2.7}$$

Moreover, ξ can be represented as in Equation 2.5 with $\tau_\beta = G(\beta,.)$ and $\xi^\beta = \xi_\beta$ for all $\beta \in \alpha+1$. If ξ is classical, then so is G.

It is easily shown using transfinite induction that if there is an order embedding $\alpha \hookrightarrow \alpha'$ of one ordinal α into another α', then $\alpha \le \alpha'$. As an immediate consequence of this, if G, G' are as in the definition such that G' refines G, then $\alpha \le \alpha'$. As another consequence, we obtain the following lemma.

Lemma 2.27. *Let $(G_n)_{n\in\mathbb{N}}$ be a refining, convergent sequence of $\mathscr{F}$-adapted grids $G_n\colon (\alpha_n+1)\times\Omega \to \overline{\mathbb{T}}$ and let $(t,\omega) \in \overline{\mathbb{R}_+} \times \Omega$. Then,*

1. *for any $n \in \mathbb{N}$, there is a unique ordinal $\delta^n(t,\omega)$ admitting an order isomorphism*

$$\psi^n(t,\omega)\colon \delta^n(t,\omega)+1 \to \{\beta \in \alpha_n+1 \mid G_n(\beta,\omega) \ge t\},$$

and, moreover, this order isomorphism is unique and given by

$$\beta' \mapsto \inf\{\beta \in \alpha_n + 1 \mid G_n(\beta,\omega) \geq t\} + \beta';$$

2. *for all* $n \in \mathbb{N}$, $\delta^n(t,\omega) \leq \delta^{n+1}(t,\omega)$, *and* $\delta(t,\omega) = \sup_{n\in\mathbb{N}}(\delta^n(t,\omega)+1)$ *is non-zero and countable.*

Hence, to any refining, convergent sequence of $\mathscr{F}$-adapted grids $G_n\colon (\alpha_n+1)\times\Omega \to \overline{\mathbb{T}}$, $n \in \mathbb{N}$, we can assign maps with domain $\overline{\mathbb{R}_+}\times\Omega$ denoted by ψ^n, δ^n, δ, and γ and given as in the lemma and by the following formula[22]

$$\gamma(t,\omega) = \inf\{\beta \in \delta(t,\omega) \mid \limsup_{n\to\infty} p\circ G_n(\psi^n(t,\omega)(\beta),\omega) > t\}, \quad (t,\omega) \in \overline{\mathbb{R}_+}\times\Omega. \tag{2.8}$$

As $(G_n)_{n\in\mathbb{N}}$ is convergent, $\gamma(t,\omega)$ is a limit ordinal for all $(t,\omega) \in \mathbb{R}_+\times\Omega$.

Definition 2.28. Let Y be a Polish space, ξ and ξ^n, $n \in \mathbb{N}$, be stochastic processes $\overline{\mathbb{T}}\times\Omega \to Y$, and $(G_n)_{n\in\mathbb{N}}$ be a refining, convergent sequence of $\mathscr{F}$-adapted grids $G_n\colon (\alpha_n+1)\times\Omega\to\overline{\mathbb{T}}$.

$(\xi^n \mid G_n)_{n\in\mathbb{N}}$ *converges tiltingly to* ξ, or $(\xi^n)_{n\in\mathbb{N}}$ *converges tiltingly along* $(G_n)_{n\in\mathbb{N}}$ or $(\xi^n \mid G_n) \xrightarrow{\mathsf{T}} \xi$ as $n\to\infty$, iff, for all $(t,\beta,\omega) \in \overline{\mathbb{T}}\times\Omega$, we have the following convergence in Y:

$$\xi(t,\beta,\omega) = \begin{cases} \lim_{n\to\infty} \xi^n\Big(G_n(\psi^n(t,\omega)(\beta),\omega),\omega\Big), & \text{if } \beta \in \gamma(t,\omega),\\ \lim_{\beta'\nearrow\gamma(t,\omega)} \xi(t,\beta',\omega), & \text{else.}\end{cases} \tag{2.9}$$

Note that, for a fixed refining, convergent sequence of $\mathscr{F}$-adapted grids G_n, $n\in\mathbb{N}$, the tilting convergence $(\xi^n \mid G_n) \xrightarrow{\mathsf{T}} \xi$ as $n\to\infty$ determines ξ uniquely at all arguments of the form $(t,\beta,\omega) \in \overline{\mathbb{T}}\times\Omega$ with $\beta\in\gamma(t,\omega)$, by the first case in Equation 2.9. This includes $\overline{\mathbb{R}_+}$, but not all of $\overline{\mathbb{T}}$. The values on the (uncountable) remainder the vertical half-axis above t are determined by extending $\xi(.,\omega)$ left-continuously at $(t,\gamma(t,\omega))$ and then constantly until $(t,\mathfrak{w}_1)$. Indeed, using the metaphor from the subsections' beginning, these arguments are not attained by the "infinitely strong push" initiated by the person at $+\infty$. In other words, they do not contain relevant information about the asymptotics of $(\xi^n)_{n\in\mathbb{N}}$ along $(G_n)_{n\in\mathbb{N}}$. Note that, for this to work, an asymptotic limit must exist at the right-hand end of the information that accumulates near t – formally, left-continuity of $\xi(.,\omega)$ at $(t,\gamma(t,\omega))$ is necessary.

Remark 2.29. Let Y be Polish spaces, ξ and ξ^n, $n\in\mathbb{N}$, be stochastic processes $\overline{\mathbb{T}}\times\Omega\to Y$, and $(G_n)_{n\in\mathbb{N}}$ be a refining, convergent sequence of $\mathscr{F}$-adapted grids $G_n\colon(\alpha_n+1)\times\Omega\to\overline{\mathbb{T}}$ such that $(\xi^n \mid G_n) \xrightarrow{\mathsf{T}} \xi$.

1. If $\hat{\xi}\colon \overline{\mathbb{T}}\times\Omega\to Y$ is another stochastic process such that $(\xi^n \mid G_n)\xrightarrow{\mathsf{T}}\hat{\xi}$, then $\hat{\xi}=\xi$.
2. Let Z be another Polish space and $f\colon Y\to Z$ continuous. Then, $(f\circ\xi^n \mid G_n)\xrightarrow{\mathsf{T}} f\circ\xi$.
3. Let Y' be another Polish space, and ξ' and ξ'^n, $n\in\mathbb{N}$, be further stochastic processes $\overline{\mathbb{T}}\times\Omega\to Y'$ such that $(\xi'^n \mid G_n)\xrightarrow{\mathsf{T}}\xi'$. Then, $((\xi^n,\xi'^n)\mid G_n)\xrightarrow{\mathsf{T}}(\xi,\xi')$, with respect to the topological product $Y\times Y'$.
4. As a consequence, if Y', ξ', $(\xi'^n)_{n\in\mathbb{N}}$ are given as in the preceding item, and if $Y=Y'$ is also a topological vector space on $\mathbb{R}$ and $(a_n)_{n\in\mathbb{N}}$ a sequence of scalars converging to $a\in\mathbb{R}$, then $(\xi^n + a_n\xi'^n \mid G_n)\xrightarrow{\mathsf{T}}\xi + a\xi'$ as $n\to\infty$.

Note that we refrain from embedding tilting convergence into the language of general topology; though an interesting question, this is clearly beyond the scope of this text.

[22]The following infimum is computed in the complete lattice $\delta(t,\omega)+1$.

To the best of the author's knowledge, the notion of "tilting convergence" is a new contribution to both the literature on stochastic analysis and control and that on limits in continuous-time games. Stochastic analysis in vertically extended time needs a notion of convergence that is adapted to both information flow given by the filtration $\mathscr{F}$ and to the vertical extension of time. Direct extensions of classical notions (such as pointwise / almost sure, measure-, or $\mathbb{L}^p$-convergence) seem inappropriate for this. Regarding game theory, in [28], Fudenberg and Levine study approximations of outcomes in continuous-time games in terms of outcomes generated by embedded refining, convergent sequences of grids, but they do not consider instantaneous reaction. The "discrete time with an infinitesimally fine grid" approximation by Simon and Stinchcombe (cf. [63]) does so, but it restricts to a deterministic setting, to approximators ξ^n with a finite number of jumps and with stationary actions,[23] and with piecewise constant ξ. Having said that, formally, tilting convergence can be seen a (though broad) generalisation of the convergence implied by the metric in [63, Section 4, p. 1185].

Example 2.30. Reconsider Example 2.22. Let $\xi^n = 1[\![0, 2^{-n})\!)$, and let $G_n \colon (\mathfrak{w}+1) \times \Omega \to \overline{\mathbb{T}},\ (k,\omega) \mapsto k2^{-n}$, for any $n \in \mathbb{Z}$, with the understanding $\mathfrak{w}2^{-n} = \infty$. Then, Alice's and Bob's behaviour on $(G_n)_{n\in\mathbb{N}}$ converge tiltingly: $(\xi^n \mid G_n) \xrightarrow{\mathsf{T}} 1[\![0]\!] = 1[\![0,(0,1))\!)$ and $(\xi^{n-1} \mid G_n) \xrightarrow{\mathsf{T}} 1[\![0,(0,1)]\!] = 1[\![0,(0,2))\!)$. Carol's, however, does not. Indeed, let, for $n \in \mathbb{Z}$,

$$\tilde{\xi}^n = \begin{cases} \xi^{n-2}, & \text{if } 2 \mid n, \\ \xi^n, & \text{else.} \end{cases}$$

For all $n \in \mathbb{N}$ and $\omega \in \Omega$, $\psi^n(0,\omega) = \mathrm{id}_{\mathfrak{w}+1}$; whence

$$\tilde{\xi}^n\Big(G_n(\psi^n(t,\omega)(1),\omega),\omega\Big) = \tilde{\xi}^n(2^{-n},\omega) = \begin{cases} 1, & \text{if } 2 \mid n, \\ 0 & \text{else,} \end{cases}$$

which does not converge in $\mathbb{R}$ as $n \to +\infty$.

Example 2.31. We again consider Example 2.22 in order to illustrate the grid-dependence. Let $\xi^n = 1[\![0, 2^{-n})\!)$, and let $G_n \colon (\mathfrak{w}^2+1) \times \Omega \to \overline{\mathbb{T}},\ (k\mathfrak{w}+m,\omega) \mapsto (k+1-2^{-m})2^{-n}$, for any $n \in \mathbb{Z}$, with the understanding $\mathfrak{w}2^{-n} = \infty$ (case $k\mathfrak{w}+m = \mathfrak{w}^2$ alias $(k,m) = (\mathfrak{w},0)$). Then, Alice's and Bob's behaviour on $(G_n)_{n\in\mathbb{N}}$ converge tiltingly, but to other limits: $(\xi^n \mid G_n) \xrightarrow{\mathsf{T}} 1[\![0]\!] = 1[\![0,(0,\mathfrak{w}))\!)$ and $(\xi^{n-1} \mid G_n) \xrightarrow{\mathsf{T}} 1[\![0,(0,2\mathfrak{w}))\!)$. Alice only switches at the $\mathfrak{w}$th moment, Bob only at the $2\mathfrak{w}$th moment. Carol's behaviour does not converge on $(G_n)_{n\in\mathbb{N}}$ for similar reason as those from the previous example.

To some extent, it is possible to represent tilting convergence in terms of pointwise convergence:

Lemma 2.32. *Let ξ^n, $n \in \mathbb{N}$, be real-valued stochastic processes $\overline{\mathbb{T}} \times \Omega \to \mathbb{R}$, and $(G_n)_{n\in\mathbb{N}}$ be a refining, convergent sequence of $\mathscr{F}$-adapted grids $G_n \colon (\alpha_n+1) \times \Omega \to \overline{\mathbb{T}}$. Let $\mathrm{L}(\mathrm{On})$ denote the class of limit ordinals. Then, for all $(t,\beta,\omega) \in \overline{\mathbb{T}} \times \Omega$ with $\beta \in \delta^n(t,\omega)+1$, we have:*

$$\begin{aligned} \xi^n\Big(G_n(\psi^n(t,\omega)(\beta),\omega),\omega\Big) &= \xi^n_{\tau^{G_n}_\beta}(\omega)\, 1[\![0]\!](t,0,\omega) \\ &\quad + \sum_{\beta_0 \in \alpha_n+1} \xi^n_{\tau^{G_n}_{\beta_0+1+\beta}}(\omega)\, 1(\!(\tau^{G_n}_{\beta_0}, \tau^{G_n}_{\beta_0+1}]\!](t,0,\omega) \\ &\quad + \sum_{\beta_0 \in (\alpha_n+1)\cap \mathrm{L}(\mathrm{On})} \xi^n_{\tau^{G_n}_{\beta_0+\beta}}(\omega)\, 1[\![\tau^{G_n}_{\beta_0}]\!](t,0,\omega). \end{aligned} \tag{2.10}$$

[23] That is, for large n, the action at the βth point of grid G_n does not depend on n, for a fixed grid index $\beta \in \alpha_n$.

With this representation, we see directly that tilting convergence "accumulates information from the future". In particular, it is – in general – not correct that any sequence of $\mathscr{F}$-optional processes converging tiltingly along a sequence of refining, convergent $\mathscr{F}$-adapted grids has an $\mathscr{F}$-optional (tilting) limit. Provided the grid converges sufficiently strongly, it appears natural to hope for an $\mathscr{F}_{+}$-optional tilting limit. Making this precise is beyond the scope of the present work.

Here, the more relevant question for us is the following: What processes can be generated out of classical, very simple $\mathscr{F}$-optional processes via tilting and pointwise convergence?

Proposition 2.33. *Let τ be an $\mathscr{F}$-optional time. Then, there are a sequence $(\xi^n)_{n\in\mathbb{N}}$ of classical, very simple $\mathscr{F}$-optional processes and a refining, convergent sequence $(G_n)_{n\in\mathbb{N}}$ of classical, deterministic grids G_n compatible with ξ^n, for all $n \in \mathbb{N}$, such that $(\xi^n \mid G_n) \xrightarrow{\mathsf{T}} 1[\![0,\tau)\!)$ as $n \to \infty$.*

For the following result, let us adopt the following conventions. A set S of maps $\overline{\mathbb{T}} \times \Omega \to Y$ is said *optionally closed under tilting convergence* iff, for all S-valued sequences $(\xi^n)_{n\in\mathbb{N}}$ of $\mathscr{F}$-optional processes, all maps $\xi \colon \overline{\mathbb{T}} \times \Omega \to Y$, and all refining, convergent sequences of $\mathscr{F}$-adapted grids $(G_n)_{n\in\mathbb{N}}$ such that a) $(\xi^n \mid G_n) \xrightarrow{\mathsf{T}} \xi$ as $n \to \infty$ and b) ξ is $\mathscr{F}$-optional, it necessarily holds true that $\xi \in S$. Further, a set S of maps $\overline{\mathbb{T}} \times \Omega \to Y$ is said *closed under continuous binary operations* iff for all $\xi, \xi' \in S$ and all continuous $f \colon Y \times Y \to Y$, the map $\xi'' \colon \overline{\mathbb{T}} \times \Omega \to Y$, $(t,\omega) \mapsto f(\xi_t(\omega), \xi'_t(\omega))$ satisfies $\xi'' \in S$.

Theorem 2.34. *Let Y be a Polish space. The set of Y-valued $\mathscr{F}$-optional processes equals the smallest set of maps $\overline{\mathbb{T}} \times \Omega \to Y$ a) containing all Y-valued classical, very simple $\mathscr{F}$-optional processes, b) closed under continuous binary operation, c) closed under pointwise convergence, and d) optionally closed under tilting convergence.*

We conclude that — in the sense made precise in this subsection — $\mathscr{F}$-optional processes are the processes generated by all classical simple $\mathscr{F}$-optional processes defined on deterministic grids (= very simple), by means of "continuous completion" (continuous binary operations), "measurable completion" (pointwise convergence), and "decision-theoretic completion" (tilting convergence).

3. Stochastic process forms

In this section, we introduce the abstract game- and decision-theoretic model of stochastic process forms. These implement extensive form characteristics using the language of stochastic processes, giving rise to a model that encompasses much of the continuous-time stochastic control literature, including stochastic differential games and timing games, but comes as close as arguably possible to an extensive form. The stochastic process form comes with a subtle model of information flow and information sets, or "subgames", using techniques from stochastic analysis. In stochastic process forms, strategies are complete contingent plans of action given by one stochastic process. A minimal requirement is well-posedness, i.e. any strategy profile induces a unique outcome compatible with it. This gives rise to a canonical way of implementing abstract concepts of dynamic equilibrium, including perfect Bayesian and subgame-perfect equilibrium. These point are illustrated concretely by a case study of the stochastic timing game and further discussed in the context of stochastic differential games.

3.1. **Introduction of stochastic process forms.**

This text studies game-, decision-, control-theoretic models in that action is described by stochastic processes evolving in (possibly) continuous time. As shown in [57, Chapters 1 and 2] — and in particular in [57, Theorem 2.3.14] — a large class of well-posed stochastic extensive forms based on paths of action indexed over well-ordered subsets of $\mathbb{R}_+$ can be constructed. The construction of action path stochastic extensive forms in [57, Chapters 1 and 2] moreover reveals that the induced outcomes generate adapted processes with respect to the exogenous information flow. These adapted processes on a well-ordered grid in $\mathbb{R}_+$ can be equivalently seen as locally right-constant adapted processes with time index set $\mathbb{R}_+$.[24] Thus, we see that there is an extensive form footing to stochastic games with locally right-constant continuous-time paths of action, given a fixed grid of admissible action times.

By the classical results due to [63, 66] and [4, 2], going beyond such a locally right-constant setting while remaining strictly within extensive form theory is doomed to failure. However, when starting out of well-posed stochastic extensive forms with adapted (and thus optional) locally right-constant action process with time index set $\mathbb{R}_+$, continuous, measurable and decision-theoretic completion yields exactly the class of general optional processes in vertically extended continuous time, by Theorem 2.34. This implies two things. First, in that limit sense, defining a game-theoretic form on the basis of action processes with these properties has a footing on well-posed action path stochastic extensive forms. Second, decision-theoretic generality requires to work in vertically extended continuous time, based on the stochastic analysis developed so far in this text.

Therefore, for abstract game-theoretic reasons, the question arises what game-theoretic structures obtain when we describe action by stochastic (and in particular optional) processes in vertically extended time. This is moreover motivated by the existence of a huge literature on games, decision and control problems in continuous time using stochastic processes, including timing games and differential games, in various formulations.[25] A third reason, linked to the two previously mentioned ones, is that explaining these games in terms of their extensive form characteristics also suggests a limit theory using action path stochastic extensive forms as approximators. In a first step, however, a susceptible limit must be identified, if we are interested in more than mere existence of it. Indeed, we wish to provide an abstract and general model *a priori* of the extensive form characteristics of games based on stochastic processes. As this formulation is not an extensive form and the basic structure of it are not decision trees, but stochastic processes, it receives the name *stochastic process form.*

We motivate main parts of the following definition of the stochastic process form beforehand, and continue the detailed discussion afterwards. How do we model the "extensive form characteristics" in a stochastic process form, defined as "the flow of information about past choices and exogenous events, along with a set of adapted choices locally available to decision makers" in the introduction? The flow of an agent i's information about past choices and exogenous events is given by a stochastic process χ on the one hand and on the other a pair consisting of a filtration $\mathscr{H}^i$ and a σ-algebra $\mathscr{M}^i$, respectively. This stochastic process is called state process, valued in some state space $\mathbb{B}$, as in the control-theoretic literature. The filtration is defined on the configuration space $W = \Omega \times \mathbb{B}^{\overline{\mathbb{T}}}$, which is the product of the set of exogenous scenarios Ω and the path space for the state process $\mathbb{B}^{\overline{\mathbb{T}}}$, as in the Witsenhausen product form (cf. [69, 70, 37]). Departing from a product form setting, $\mathscr{M}^i$ is a σ-algebra $\overline{\mathbb{T}} \times W$ with $\mathrm{Prd}(\mathscr{H}^i) \subseteq \mathscr{M}^i \subseteq \mathrm{Opt}(\mathscr{H}^i)$, describing in flexible way what pieces of information revealed at time τ^i the agent i can condition her action on (roughly speaking). These measurability conditions are also a clear departure from the extensive form setting because they use properties of entire processes and not of their evaluations at fixed deterministic times.

The condition defining $\mathscr{M}^i$ reveals it as what it well-known in stochastic analysis as *Meyer σ-algebra* (a.k.a. σ-field) with respect to $\mathscr{H}^i$, introduced and developed in [51, 25, 10] in the classical setting; for recent work on applications to stochastic control, see, e.g. [12, 13, 11]. Following this literature, $\mathscr{M}^i$ allows to express the amount of information revealed "at time τ^i" agent i can use for

[24] More precisely, in the language and notation of [57], given a well-ordered subset $\tilde{\mathbb{T}} \subseteq \mathbb{R}_+$ with $0 \in \tilde{\mathbb{T}}$, the collection of induced outcomes $(\omega, f') \in W \subseteq \Omega \times \mathbb{A}^{\tilde{\mathbb{T}}}$ of a strategy profile s, given a random move $\mathsf{x} = \mathsf{x}_t(f)$ with domain $D_{\mathsf{x}} = D_{t,f}$, $(t, f) \in \tilde{\mathbb{T}} \times \mathbb{A}^{\mathbb{T}}$, and given scenario $\omega \in \Omega$, can be seen as map $\mathbb{R}_+ \times D_{\mathsf{x}} \to \mathbb{A}$ with locally right-constant paths jumping only at times $\tilde{\mathbb{T}}$.

[25] For a recent textbook with many examples focusing on "mean field games", see [16].

action at time τ^i, as opposed to action at times succeeding τ^i. The presented framework of Meyer σ-algebras in vertically extended time permits to use the power of classical Meyer σ-algebras in describing action with respect to information at a given time in settings where longer well-ordered chains of instantaneous pro- and reaction are relevant, including games.

The set of choices locally available to decision makers is described by stochastic processes $s^i\colon \overline{\mathbb{T}}\times W \to \mathbb{A}^i$, valued in some personal action space $\mathbb{A}^i$, and called strategies. These are thus formal primitives, but they are required to be $\mathscr{H}^i$-progressively measurable and "locally" $\mathscr{M}^i$-measurable processes. The meaning of "locally" is subtle: what are these loci alias decision points? When equipped with the σ-algebra $\mathscr{M}^i$, agent i can check her options at any $\mathscr{H}^i$-optional time τ^i such that $[\![0,\tau^i)\!) \in \mathscr{M}^i$; for exactly in that case, she can really follow the abstract strategy $1[\![0,\tau^i)\!)$ of opting for value zero at time τ^i. At time τ^i, agent i can observe the state process $\tilde{\chi}$ up to time τ^i, that is everything she can see of $\mathscr{M}^i$-measurable functions (thus, possible strategies) of it up to time τ^i. However, already optional times equipped with pointwise order do clearly not define a tree or forest; this is a clear departure from the extensive form. Still, we obtain a notion modelling instances $(\tau^i,\tilde{\chi})$ at that choices alias options are available to agents, and strategies can be seen as complete contingent plans of action at all these instances, compatible with the information structure $\mathscr{M}^i$. We nevertheless insist on the difference to extensive forms where strategies are all complete contingent plans of locally available choices, without any condition on measurability along the time axis, or more precisely, over option-revision instances $(\tau^i,\tilde{\chi})$. In the stochastic process form setting, it is the $\mathscr{H}^i$-progressive measurability and "local" $\mathscr{M}^i$-measurability that imply such a condition.

Based on this, one can also develop a notion of outcome, or actually, state processes induced by strategy profiles, given a starting point $(\tau^i,\tilde{\chi})$ as above. Then, a minimal requirement for a stochastic process form in order to give rise to a proper game-theoretic model is well-posedness: that any strategy profile induces a unique state process, given any starting point.

With these preparations, we introduce the formal definition. In what follows, $\overline{\mathbb{T}}$ denotes vertically extended time as introduced in Subsection 1.2. In addition, we fix a measurable space $(\Omega,\mathscr{E})$ with $\Omega\neq\emptyset$. The elements of Ω represent *exogenous scenarios*, those of $\mathscr{E}$ *events*.[26] Moreover, fix a σ-ideal $\mathscr{N}$ on $\mathscr{E}$, that is a non-empty and strict subset of $\mathscr{E}$, stable under both intersection with elements of $\mathscr{E}$ and countable union. The relevant example for this is $\mathscr{N}=\mathscr{E}\cap\bigcap_{\mathbb{P}\in\mathfrak{P}}\mathscr{N}_{\mathbb{P}}$ for a non-empty set $\mathfrak{P}\subseteq\mathfrak{P}_{\mathscr{E}}$ of prior beliefs alias probability measures on $\mathscr{E}$. This also includes the case $\mathscr{N}=\{\emptyset\}$. Given this σ-ideal $\mathscr{N}$, we say that a property holds for *$\mathscr{N}$-almost all* $\omega\in\Omega$ or *$\mathscr{N}$-almost surely* iff there is $N\in\mathscr{N}$ such that the property holds for all $\omega\in N^{\complement}$. For any set $S\subseteq\overline{\mathbb{T}}\times\Omega$, two maps $\chi,\chi'\colon S\to Y$ are said *$\mathscr{N}$-indistinguishable*, denoted by $\chi\cong_{\mathscr{N}}\chi'$ if $\mathscr{N}$-almost all $\omega\in\Omega$ satisfy the following property: for all $t\in\overline{\mathbb{T}}$ with $(t,\omega)\in S$ we have $\chi_t(\omega)=\chi'_t(\omega)$. If $\mathscr{N}=\mathscr{E}\cap\mathscr{N}_{\mathbb{P}}$ for some $\mathbb{P}\in\mathfrak{P}_{\mathscr{E}}$, then "$\mathscr{N}$" is replaced by "$\mathbb{P}$" in these phrases, as usual.

Definition 3.1. (Part A): For fixed $(\Omega,\mathscr{E},\mathscr{N})$, consider the data

$$\mathbf{F}=(I,\mathbb{A},\mathbb{B},W,\mathcal{W},\mathscr{H},\mathscr{M},\mathcal{S}),$$

where:

- I is a non-empty, finite set — its elements are called *agents*;
- $\mathbb{A}=\prod_{i\in I}\mathbb{A}^i$ is the topological product of Polish spaces $\mathbb{A}^i$, $i\in I$ — the elements of $\mathbb{A}^i$ are called *i's actions* and the elements of $\mathbb{A}$ *action profiles*;
- $\mathbb{B}$ is a Polish space — it is called *state space*, its elements are called *states*;
- $W\subseteq\Omega\times\mathbb{B}^{\overline{\mathbb{T}}}$ is a subset — its elements are called *configurations*;
- $\mathcal{W}$ is a set of pairs $\zeta=(\xi,\chi)$ of maps $\xi\colon\overline{\mathbb{T}}\times\Omega\to\mathbb{A}$ and $\chi\colon\overline{\mathbb{T}}\times\Omega\to\mathbb{B}$ such that, for all $\omega\in\Omega$, $(\omega,\chi(\omega))\in W$ — an *action process* is a ξ such that there is χ with $(\xi,\chi)\in\mathcal{W}$, a

[26]In the language of [57], $(\Omega,\mathscr{E})$ is an exogenous scenario space.

state process is a χ such that there is ξ with $(\xi,\chi)\in\mathcal{W}$, an *outcome process* is an element of $\mathcal{W}$, seen as a map $\overline{\mathbb{T}}\times\Omega\to\mathbb{A}\times\mathbb{B}$;
- $\mathscr{H}=(\mathscr{H}^i)_{i\in I}$ is a family of filtrations $\mathscr{H}^i=(\mathscr{H}^i_t)_{t\in\overline{\mathbb{T}}}$ on the sample space W, $i\in I$ — for any $i\in I$, $\mathscr{H}^i$ is called *basic information structure for* i;
- $\mathscr{M}=(\mathscr{M}^i)_{i\in I}$ is a family of σ-algebras $\mathscr{M}^i$ on $\overline{\mathbb{T}}\times W$ satisfying $\mathrm{Prd}(\mathscr{H}^i)\subseteq\mathscr{M}^i\subseteq\mathrm{Opt}(\mathscr{H}^i)$, $i\in I$ — for any $i\in I$, $\mathscr{M}^i$ is called *Meyer information structure for* i;
- $\mathcal{S}=\bigtimes_{i\in I}\mathcal{S}^i$ is the set-theoretic product of sets $\mathcal{S}^i$ of $\mathscr{H}^i$-progressively measurable maps $s^i\colon\overline{\mathbb{T}}\times W\to\mathbb{A}^i$ — for any $i\in I$, a *strategy process for* i is an element of $\mathcal{S}^i$.

(Part B): Let $i\in I$. An *optional time for* i is an $\mathscr{H}^i$-optional time such that $[\![0,\tau^i)\!)\in\mathscr{M}^i$.[27] A *history for* i is a pair (τ^i,χ) consisting of an optional time τ^i for i and a state process χ. Let χ,χ' be state processes and τ^i be an optional time for i. Then, we say that χ' *cannot be distinguished from* χ *until* τ^i *by* i, or that (τ^i,χ') *cannot be distinguished from* (τ^i,χ), in symbols $\chi'\approx_{i,\tau^i}\chi$, iff we have, for $\mathscr{N}$-almost all $\omega\in\Omega$ and all $t\in[0,\tau^i(\omega,\chi(\omega))]_{\overline{\mathbb{T}}}$, and for all real-valued $\mathscr{M}^i$-measurable maps $f\colon\overline{\mathbb{T}}\times W\to\mathbb{R}$:

$$f(t,\omega,\chi(\omega))=f(t,\omega,\chi'(\omega)). \tag{3.1}$$

An *(endogenous) information set for* i is a pair $\mathfrak{p}=(\tau^i,\mathfrak{x})$ for an optional time τ^i for i and an equivalence class $\mathfrak{x}$ with respect to $\approx_{i,\tau^i}$ on the set of state processes.[28] τ^i is said the *time* of the information set. The set of information sets for i is denoted by $\mathfrak{P}^i$, and its subset of information sets with time τ^i is denoted by $\mathfrak{P}^i(\tau^i)$, for any optional time τ^i for i.

Let, for any stochastic process $s\colon\overline{\mathbb{T}}\times W\to\mathbb{A}$ and any stochastic process $\chi\colon\overline{\mathbb{T}}\times\Omega\to\mathbb{B}$, the stochastic process $s\llcorner v$ be given by

$$s\llcorner\chi\colon\overline{\mathbb{T}}\times\Omega\to\mathbb{A},\ (t,\omega)\mapsto(s^i(t,\omega,\chi(\omega)))_{i\in I}.$$

We call s *admissible* iff for all $i\in I$, all optional times τ^i for i, all state processes $\tilde{\chi}$, there is an, up to $\mathscr{N}$-indistinguishability, unique state process χ extending $\tilde{\chi}$, i.e. satisfying $\chi|_{[\![0,\tau^i\circ(\mathrm{id}_\Omega\star\chi))\!)}\cong_{\mathscr{N}}\tilde{\chi}|_{[\![0,\tau^i\circ(\mathrm{id}_\Omega\star\chi))\!)}$, and indistinguishable from it until τ^i, i.e. satisfying $\chi\approx_{i,\tau^i}\tilde{\chi}$,[29] that admits an action process ξ with $(\xi,\chi)\in\mathcal{W}$ satisfying

$$(s\llcorner\chi)|_{[\![\tau^i\circ(\mathrm{id}_\Omega\star\chi),\infty]\!]}\cong_{\mathscr{N}}\xi|_{[\![\tau^i\circ(\mathrm{id}_\Omega\star\chi),\infty]\!]}. \tag{3.2}$$

We call the — up to $\mathscr{N}$-indistinguishability uniquely determined — processs χ the *state process induced by* s *given* $(\tau^i,\tilde{\chi})$, respectively, and use the notation $\chi=\mathrm{Out}^\star(s\mid\tau^i,\tilde{\chi})$.

(Part C): A *stochastic process form* is given by data $\mathbf{F}$ as above such that

1. for all $i\in I$, $\mathscr{H}^i$ is non-anticipative, that is, there is a family of σ-algebras $\tilde{\mathscr{H}}^i_t$ on $\Omega\times\mathbb{B}^{[0,t]_{\overline{\mathbb{T}}}}$, ranging over $t\in\overline{\mathbb{T}}$, such that, with $\mathrm{proj}_{[0,t]_{\overline{\mathbb{T}}}}\colon\mathbb{B}^{\overline{\mathbb{T}}}\to\mathbb{B}^{[0,t]_{\overline{\mathbb{T}}}},\ f\mapsto f|_{[0,t]_{\overline{\mathbb{T}}}}$, we have
$$\mathscr{H}^i_t=\{(\mathrm{id}_\Omega\times\mathrm{proj}_{[0,t]_{\overline{\mathbb{T}}}})^{-1}(H)\cap W\mid H\in\tilde{\mathscr{H}}^i_t\};$$
2. for all $\zeta=(\xi,\chi)\in\mathcal{W}$, all $i\in I$, the map
$$\mathrm{id}_\Omega\star\chi\colon\Omega\to W,\ \omega\mapsto(\omega,\chi(\omega))$$
is $\mathscr{E}$-$\mathscr{H}^i_\infty$-measurable;

[27] By Theorem 2.6, $[\![0,\tau^i)\!)\in\mathrm{Opt}(\mathscr{H}^i)$ for any $\mathscr{H}^i$-optional time τ^i. If $\mathscr{H}^i$ is augmented, the converse is true as well, by the same theorem. Note that this is a generalisation of stopping times with respect to Meyer-σ-algebras, going back to [51, 25], see also [10, Subsection 2.1].

[28] It follows from the definition that $\approx_{i,\tau^i}$ is an equivalence relation. For the proof, take $f=1[\![0,\tau^i)\!)$. Inserting $t=\tau^i(\omega,\chi(\omega))$ yields $\tau^i(\omega,\chi'(\omega))\leq\tau^i(\omega,\chi(\omega))$. Hence, we can insert, in a second step, $t=\tau^i(\omega,\chi'(\omega))$ which yields $\tau^i(\omega,\chi(\omega))\leq\tau^i(\omega,\chi'(\omega))$. See Proposition 3.4 for further discussion.

[29] Both properties are not necessarily equivalent. See Proposition 3.4 for a discussion of this.

3. for all outcome processes $\zeta = (\xi, \chi)$, $\zeta' = (\xi', \chi') \in \mathcal{W}$, for all $i \in I$, all optional times τ^i for i, such that, with $\hat{\tau}^i = \tau^i \circ (\mathrm{id}_\Omega \star \chi)$, $\xi|_{[\![0,\hat{\tau}^i]\!]} \cong_{\mathscr{N}} \xi'|_{[\![0,\hat{\tau}^i]\!]}$ holds true, we have $\chi|_{[\![0,\hat{\tau}^i]\!]} \cong_{\mathscr{N}} \chi'|_{[\![0,\hat{\tau}^i]\!]}$;
4. for any $i \in I$, any optional time τ^i for i, any $\beta \in \mathfrak{w}_1$, any $s^i \in \mathcal{S}^i$, there is an $\mathscr{M}^i$-measurable process $\tilde{s}^i \colon \overline{\mathbb{T}} \times W \to \mathbb{A}^i$ such that $\tilde{s}^i \in \mathcal{S}^i$ and $s^i_{\tau^i} = \tilde{s}^i_{\tau^i}$ on $\{\pi \circ \tau^i = \beta\}$.[30]

(Part D): An SPF $\mathbf{F}$ is said *well-posed* iff all $s \in \mathcal{S}$ are admissible.

We make some additional remarks. Let us first note that, since everything is encoded as processes in time, information is non-anticipative (Axiom 1) and the state is a non-anticipative function of action (Axiom 3), we obtain a basic structure for "causality" from the beginning. In that sense, the stochastic process form is more similar to the stochastic extensive form than to the product form. The stochastic process form therefore merges different concepts of information in order to provide a general and tractable setting for problems with uncountably many decision situations ("agents" in the language of Witsenhausen; information sets in the setting of this text), as discussed in the sequel.

Furthermore, outcome processes are pairs of action and state processes, where the former determine the latter in a non-anticipative way compatible with optional times (Axiom 3). However, this mapping need not be scenariowise; it can be purely "statistical". This is a further departure from the strict stochastic extensive form setting, but is common in many contexts. For instance, the state may arise through stochastic integration of a function of the action process with respect to a semimartingale. "Endogenous" information (that is, information about agents' behaviour) is only transmitted via the state process, and we may assume that only the state process is payoff-relevant — both without loss of generality, because the state could include a copy of action.

Moreover, let us insist on the fact that the term "information set" does, of course, not have the same *formal* meaning as in extensive forms. An information set $\mathfrak{p} = (\tau^i, \mathfrak{x})$ describes the time τ^i at that an agent $i \in I$ currently considers her options and which set $\mathfrak{x}$ of state processes is still possible to be realised. This explains in particular the information the agent has about the behaviour of all agents up to time τ^i (whence the qualifier "endogenous"). Agent i's information about W, including realisations of exogenous scenario $\omega \in \Omega$ and state processes $\chi(\omega)$, and including information "at" time τ^i, can then be derived from $\mathscr{M}^i$. τ^i itself is compatible with this information because $[\![0, \tau^i)\!) \in \mathscr{M}^i$. In a considerable generalisation of [60], another name for information sets would be *subgame* because information sets are the instances that agents consider options or revise decisions at. We refrain from this usage in general because the present setting goes beyond the situation of perfect information.

Finally, let us close the bracket opened in the beginning of this subsection by noting that in a well-posed stochastic extensive form, the action process ξ with $\xi = s_\llcorner\chi$, for the state process χ induced by s given some history, is progressively measurable with respect to the filtration induced by all $\mathscr{H}^i$, $i \in I$, and the map $\mathrm{id}_\Omega \star \chi$. This follows from Axiom 2 and the optionality of s^i, for any $i \in I$. The proof is elementary; it is given in the special case of timing games later in the text. Moreover, by Axiom 4, at any joint optional time τ for all $i \in I$ (for example, elements of $\mathbb{T}$), for any $\beta \in \mathfrak{w}_1$, there is $\tilde{s} \in \mathcal{S}$ with $\tilde{s}_\tau = s_\tau$ on $\{\pi \circ \tau = \beta\}$ whose i-th component is $\mathscr{M}^i$-measurable and therefore $\mathscr{H}^i$-optional, for all $i \in I$. Then, the action process $\tilde{\xi}$ with $\tilde{\xi} = \tilde{s}_\llcorner\tilde{\chi}$, for the state process $\tilde{\chi}$ induced by $\tilde{s}$ given some history, is even optional with respect to the filtration induced by all $\mathscr{H}^i$, $i \in I$, and the map $\mathrm{id}_\Omega \star \tilde{\chi}$. We conclude that "induced" action processes are progressively measurable and "locally" optional, and if strategies for $i \in I$ are $\mathscr{M}^i$-measurable, even optional.

[30] Following the setting of [51, 25], this means nothing else than the measurability of $s^i_{\tau^i}$ with respect to the filtration associated with $\mathscr{M}^i$ evaluated at τ^i, where all this is considered on the measurable space given by $\{\pi \circ \tau^i = \beta\}$ with induced σ-algebra.

3.2. **Information sets, counterfactuals, and equilibrium.** In this subsection, we further discuss the "problem of information" [47] in stochastic process forms. We treat the role of $\mathscr{H}$ and $\mathscr{M}$, and we analyse information sets. We first give some examples for classical choices of $\mathscr{H}$.

Example 3.2. Fix a stochastic process form $\mathbf{F}$ and an agent $i \in I$. We consider the separable, symmetric case where $\mathscr{H}^i_t$ equals $\mathscr{F}^i_t \otimes \mathscr{B}'^i_t|_W$, $i \in I$, $t \in \overline{\mathbb{T}}$, for some filtration $\mathscr{F}^i$ on $(\Omega, \mathscr{E})$ and some sub-σ-algebra $\mathscr{B}'^i_t$ of $(\mathscr{B}_{\mathbb{B}})^{\otimes \overline{\mathbb{T}}}$. One can take $(\Omega, \mathscr{E}, \mathscr{F}^i)$ to be universally complete. The following cases, classical in control theory,[31] obtain:

1. The case of *state-independent exogenous information* obtains iff every state process χ is $\mathscr{F}^i$-progressively measurable.
2. The case of *dynamic learning of exogenous information* is the opposite. It can be formulated by fixing some filtration $\mathscr{G}^i$ with $\mathscr{F}^i_t \subseteq \mathscr{G}^i_t$ for all $t \in \overline{\mathbb{T}}$ and considering state processes χ that are $\mathscr{G}^i$-adapted but not necessarily $\mathscr{F}^i$-adapted. Given two outcome processes $\zeta = (\xi, \chi)$ and $\zeta' = (\xi', \chi')$, the filtrations on Ω describing i's information flow for either outcome, which are those induced by $\mathscr{H}^i$ and $\mathrm{id}_\Omega \star \chi$ and $\mathrm{id}_\Omega \star \chi'$, respectively, may therefore be strictly larger than $\mathscr{F}^i$ and differ.
3. *Open-loop* strategies (or controls) obtain iff $\mathscr{B}'^i_t = \{\emptyset, \mathbb{B}^{\overline{\mathbb{T}}}\}$.
4. *Closed-loop* strategies (or controls) obtain iff, for all $t \in \overline{\mathbb{T}}$, $(\mathscr{B}_{\mathbb{B}})^{\otimes [0,t)_{\overline{\mathbb{T}}}} \otimes \{\emptyset, \mathbb{B}^{\otimes [t,\infty]_{\overline{\mathbb{T}}}}\} \subseteq \mathscr{B}'^i_t \subseteq (\mathscr{B}_{\mathbb{B}})^{\otimes [0,t]_{\overline{\mathbb{T}}}} \otimes \{\emptyset, \mathbb{B}^{\otimes (t,\infty]_{\overline{\mathbb{T}}}}\}$ (for $t = 0$, this reads: $\mathscr{B}'^i_0 \subseteq (\mathscr{B}_{\mathbb{B}})^{\otimes \{0\}} \otimes \{\emptyset, \mathbb{B}^{\otimes (0,\infty]_{\overline{\mathbb{T}}}}\}$).[32]

These are stylised special cases, of course, and mixed regimes obtain quite easily. For instance, in a game, some dynamic learning of exogenous information may already come in by other players' using private randomisation devices.

Out of the separable case one can construct more complicated information structures, intertwining information on exogenous and endogenous events. For example, $\mathscr{H}^i_t$ ($i \in I$, $t \in \overline{\mathbb{T}}$) could also be given by $(\mathscr{F}^i_t \otimes \mathscr{B}'^i_t) \vee (\mathscr{G}^i_t \otimes \mathscr{B}''^i_t)|_W$ for some filtration $\mathscr{G}^i$ on $(\Omega, \mathscr{E})$ and some suitable sub-σ-algebra $\mathscr{B}''^i_t$ of $(\mathscr{B}_{\mathbb{B}})^{\otimes \overline{\mathbb{T}}}$. We also note that $\mathbb{B}$ could be a product of Polish spaces, and the information structures could depend in different ways on the different factors of that product. For example, one factor could describe a partially observable signal the agent cannot condition on (open-loop) and a second factor could describe an observation process the agent can condition on (closed-loop) and whose underlying exogenous randomness the agent try to learn. Moreover, information can be asymmetric in that not all agents can observe all factors in the same way. For example, any agent could have its own observation process, described by a corresponding component of χ.

Next, we discuss an example for the choice of $\mathscr{M}$ and elaborate on the combination of Meyer σ-algebras and vertically extended time.

Example 3.3. Meyer σ-algebra have been used in the financial literature recently in order to model information about imminent trade signals (see, for instance, [13]). So far the focus has been set mainly on the single-agent setting however.[33] Reinterpreting this modelling approach in the language of the present setting, and adding vertically extended time in particular, yields a stochastic process form where $\mathscr{M}$ is given by

$$\mathscr{M}^i = \mathrm{Prd}(\mathscr{H}^i) \vee \sigma(Z^i), \qquad i \in I,$$

for a fixed $\mathscr{H}^i$-optional process Z^i, for any $i \in I$, describing the signal observed by agent i. In [13], only a single agent is considered and this agent can essentially act twice per instance of time.

[31] See, e.g., the textbooks [19, 16].

[32] We could, without loss of generality, ask for $\mathscr{B}'^i_t = (\mathscr{B}_{\mathbb{B}})^{\otimes [0,t]_{\overline{\mathbb{T}}}} \otimes \{\emptyset, \mathbb{B}^{\otimes (t,\infty]_{\overline{\mathbb{T}}}}\}$ if $t > 0$ since this does not necessarily imply that strategies can condition at time t on the state at time t. Indeed, in that case, we could still have $\mathscr{M}^i = \mathrm{Opt}(\hat{\mathscr{H}}^i)$, for $\hat{\mathscr{H}}^i_t = \mathscr{F}^i_t \otimes (\mathscr{B}_{\mathbb{B}})^{\otimes [0,t)_{\overline{\mathbb{T}}}} \otimes \{\emptyset, \mathbb{B}^{\otimes [t,\infty]_{\overline{\mathbb{T}}}}\}$, $t > 0$, $\mathscr{B}'^i_0 = \{\emptyset, \mathbb{B}^{\otimes [0,\infty]_{\overline{\mathbb{T}}}}\}$, and $\hat{\mathscr{H}}^i_0 = \mathscr{H}^i_0$.

[33] Yet, see [12] for a multi-agent model.

In the interactive setting or settings involving non-trivial consecutive, infinitesimal randomisation, e.g. in the framework of preemption as in [58], one might wish to extend this. This requires adding additional virtual instants of time along the vertical half-axis, in order to keep track of the chain of reactions. Moreover, the signals Z^i would no more be sufficient for describing the information agent i has at the different vertical versions of the given real instant τ^i of time: if Z^i contains sufficient information about agents' behaviour at τ^i, there may be more than one ("I do what you do") or no ("I do not do what I do") outcome process $\zeta = (\xi, \chi)$ compatible with a given strategy profile (in the sense of satisfying Equation 3.2), therefore destroying well-posedness. In the well-posed setting, Z^i will mainly be useful for describing *exogenous* signal observations and pro- and reaction with respect to these, while vertically extended time accounts for *endogenous* signal observations and pro- and reaction with respect to the latter. The formulation in terms of stochastic analysis in vertically extended time assures that both sorts of observation and action are consistent.

In stochastic dynamic games, it is important to analyse strategy profiles given counterfactual "histories" in a way compatible with the agents' information. By construction, the extensive form offers natural concepts for this, based on moves and, more generally, information sets. These concepts extend naturally to stochastic extensive forms, as introduced and studied in [57, Chapter 2]. In stochastic process forms these concepts are no more available in a strict sense; we have mimicked them in the definition above.

In a stochastic process form, an information set for an agent $i \in I$ is given by an optional time τ^i with respect to i's information structure and an equivalence class of state processes χ, where χ is identified with all other state processes χ' that i cannot distinguish from it given the information $\mathscr{M}^i$ and time τ^i. Here, χ and χ' cannot be distinguished given $\mathscr{M}^i$ and time τ^i iff any strategy based on information $\mathscr{M}^i$ ($= \mathscr{M}^i$-measurable) yields the same result up to time τ^i inclusively. It is clear that if the amount of information is too large — implying too much future knowledge, and in particular times τ^i that depend on agents' action at τ^i, i.e. arbitrary $\mathscr{H}^i$-optional times — too many state processes can be distinguished which may prevent well-posedness. At the same time it may seem natural to permit "subgames" starting at the first jump of an exogenous Poisson process, for example. Hence, restricting to predictable $\mathscr{H}^i$-optional times is not a convincing solution either. Here, Meyer information structures provide a subtle device for managing this trade-off. This point is illustrated by the following proposition.

Before stating it, we note that the mathematical tractability of this analysis is ensured by the fact that in stochastic process forms information is modelled on a "universal" configuration space $W \subseteq \Omega \times \mathbb{B}^{\overline{\mathbb{T}}}$, not only on Ω. This namely allows for a description of information that is independent of the choice of a concrete state process, and from which the state-dependent exogenous information — that is, given a state process χ, the filtration on $(\Omega, \mathscr{E})$ generated by $\mathscr{H}^i$ and $\mathrm{id}_\Omega \star \chi$ — can be derived. However, our approach is not primarily chosen for mathematical convenience, but because we esteem it natural in a stochastic process-based game-theoretic setting.[34]

Proposition 3.4. *Let* $\mathbf{F}$ *be a stochastic process form on* $(\Omega, \mathscr{E}, \mathscr{N})$ *and* $i \in I$. *Consider the following additional assumptions:*

(A) *We have* $\mathscr{H}^i_0 \subseteq \mathscr{E} \otimes \{\emptyset, \mathbb{B}^{\overline{\mathbb{T}}}\}$;
(B) *For all* $t \in \overline{\mathbb{T}} \setminus \{0\}$, $\{\emptyset, \Omega\} \otimes \mathbb{B}^{[0,t]_{\overline{\mathbb{T}}}} \otimes \{\emptyset, \mathbb{B}^{(t,\infty]_{\overline{\mathbb{T}}}}\} \subseteq \mathscr{H}^i_t$.
(C) *We have* $\mathscr{M}^i \subseteq \mathrm{Opt}(\mathscr{E} \otimes \{\emptyset, \mathbb{B}^{\overline{\mathbb{T}}}\}) \vee \mathrm{Prd}(\mathscr{H}^i)$, *where* $\mathscr{E} \otimes \{\emptyset, \mathbb{B}^{\overline{\mathbb{T}}}\}$ *denotes the filtration equal to that* σ*-algebra at any time* $t \in \overline{\mathbb{T}}$.

Suppose that Assumptions (A) and (C) are satisfied. Then, we have:

[34] Compare the paper [42] which treats a general adaptive stochastic control framework in a classical continuous-time setting.

1. *For any $\mathscr{M}^i$-measurable $f\colon \overline{\mathbb{T}} \times W \to \mathbb{R}$ and all $t \in \overline{\mathbb{T}}$, $\omega \in \Omega$, and $h, h' \in \mathbb{B}^{\overline{\mathbb{T}}}$ with $h|_{[0,t)_{\overline{\mathbb{T}}}} = h'|_{[0,t)_{\overline{\mathbb{T}}}}$:*
$$f(t,\omega,h) = f(t,\omega,h').$$
2. *For any optional time τ^i for i and all state processes χ, χ' with*
$$\chi|_{[\![0,\tau^i\circ(\mathrm{id}_\Omega\star\chi))\!)} \cong_{\mathscr{N}} \chi'|_{[\![0,\tau^i\circ(\mathrm{id}_\Omega\star\chi))\!)},$$
we have $\chi \approx_{i,\tau^i} \chi'$.
3. *If in addition Assumption (B) is satisfied, then, for any optional time τ^i for i and all state processes χ, χ' that are left-continuous*[35] *at all $u \in \overline{\mathbb{T}}$ with $\pi(u) = \mathfrak{w}_1$ and satisfy $\chi \approx_{i,\tau^i} \chi'$, we have $\chi|_{[\![0,\tau^i\circ(\mathrm{id}_\Omega\star\chi))\!)} \cong_{\mathscr{N}} \chi'|_{[\![0,\tau^i\circ(\mathrm{id}_\Omega\star\chi))\!)}$.*

In particular, if all three assumptions above are satisfied, then, for all optional times τ^i for i and all state processes χ, χ' that are left-continuous at all $u \in \overline{\mathbb{T}}$ with $\pi(u) = \mathfrak{w}_1$, we have the equivalence:
$$\chi \approx_{i,\tau^i} \chi' \qquad \Longleftrightarrow \qquad \chi|_{[\![0,\tau^i\circ(\mathrm{id}_\Omega\star\chi))\!)} \cong_{\mathscr{N}} \chi'|_{[\![0,\tau^i\circ(\mathrm{id}_\Omega\star\chi))\!)}.$$

Remark 3.5 ("Nodes")**.** As a consequence of this proposition, if Assumptions (A) and (C) are satisfied, then, for any $i \in I$, one could call *nodes for i* all sets of the form
$$x_{\tau^i}(\tilde{\chi}) = \{\chi \text{ state processes} \mid \chi|_{[\![0,\tau^i\circ(\mathrm{id}_\Omega\star\chi))\!)} \cong_{\mathscr{N}} \chi'|_{[\![0,\tau^i\circ(\mathrm{id}_\Omega\star\chi))\!)}\},$$
ranging over all optional times τ^i for $i \in I$ and all state processes $\tilde{\chi}$. This is analogous to the definition of nodes in action path stochastic decision forests in [57, Section 1.2] (similarly, in a deterministic and very special setting, under the name "differential games", in [3]). However, it is clear that the set of all these $x_{\tau^i}(\tilde{\chi})$ does not at all define a tree or forest. This yields no extensive form in any strict sense.

In this SPF language for describing extensive form characteristics based on stochastic processes, including a model of information sets alias "subgames" alias instances of decision revision, we can conclude this subsection with a definition of equilibrium. It is a refinement of the classical Nash equilibrium concept in two ways: 1) the best-response condition must also be satisfied given counterfactual histories ("off the equilibrium path"), in the spirit of subgame-perfect equilibria (cf. [60, 61]); 2) the beliefs agents form about exogenous information and, in the case of imperfect information, the current "move" must be consistent, in the spirit of perfect Bayesian equilibria. As the stochastic process form is still very similar to stochastic extensive forms, it is unsurprising that the following definition is an adaptation of Definitions 2.3.16, 2.3.17, and 2.3.18 to the setting of stochastic process forms.

Yet, there is an important difference here since in SPF time provides a "uniformising structure" among information sets (making possible a refined meaning of conditional probability). Moreover, calculating expectations with respect to the posterior at a given endogenous information set requires to determine an adequate σ-algebra to condition on. The natural concepts for this, once again, derive from the general theory of Meyer-σ-algebras because at an information set $\mathfrak{p} = (\tau^i, \mathfrak{x})$ for agent i, the agent can condition on all $\mathscr{M}^i$-measurable maps evaluated at τ^i, at the realised scenario ω, and at the belief $\chi_{i,\mathfrak{p}} \in \mathfrak{x}$ about the actual endogenous "history". We propose a relaxed version of dynamic rationality and equilibrium — allowing for a selection of information sets (and thus counterfactuals) to be checked[36] — because this a restriction typically made in the literature, for reasons that will become clearer in the case study of the timing game. For instance, one may focus

[35] Note that the pathwise left-continuity at times $u \in \overline{\mathbb{T}}$ with $\pi(u) = \mathfrak{w}_1$ is not at all a strong requirement; it suffices for χ to be progressively measurable with respect to some filtration (see Remark 2.2, Part 5).

[36] In the static Nash equilibrium case, one would only check the best response condition given the information set at time zero, provided it is unique. In a rigorous dynamic setting, on the contrary, one would wish to check given all information sets, of course.

on information sets $\mathfrak{p} = (\tau^i, \mathfrak{x})$ such that $\mathfrak{x}$ contains a deterministic (if not even constant) state process.

Definition 3.6. Let $\mathbf{F}$ be a well-posed SPF on $(\Omega, \mathscr{E}, \mathscr{N})$.

1. A *belief system on* $\mathbf{F}$ is a family $\Pi = (\mathscr{P}^{i,\tau^i}, \kappa^{i,\tau^i}, p_{i,\mathfrak{p}}, \mathscr{P}_{i,\mathfrak{p}})_{\mathfrak{p}=(\tau^i,\mathfrak{x})\in\mathfrak{P}^i,\, i\in I}$ such that, for any $i \in I$ and any optional time τ^i for i, $\mathscr{P}^{i,\tau^i}$ is a σ-algebra on $\mathfrak{P}^i(\tau^i)$, κ^{i,τ^i} is a Markov kernel from $(\mathfrak{P}^i(\tau^i), \mathscr{P}^{i,\tau^i})$ to $(\Omega, \mathscr{E})$ with $\mathscr{N} \subseteq \mathscr{N}_{\kappa^{i,\tau^i}(.,\mathfrak{p})}$ for all $\mathfrak{p} \in \mathfrak{P}^i(\tau^i)$, and, moreover, for any information set $\mathfrak{p} = (\tau^i, \mathfrak{x}) \in \mathfrak{P}^i(\tau^i)$ for i with time τ^i, $\mathscr{P}_{i,\mathfrak{p}}$ is a σ-algebra on $\mathfrak{x}$,[37] and $p_{i,\mathfrak{p}} \colon \Omega \to \mathfrak{x}$ is an $\mathscr{E}$-$\mathscr{P}_{i,\mathfrak{p}}$-measurable map.
2. A *taste system on* $\mathbf{F}$ is a family $U = (u_{i,\mathfrak{p}})_{\mathfrak{p}\in\mathfrak{P}^i,\, i\in I}$ of maps $u_{i,\mathfrak{p}} \colon W \to \mathbb{R}$.
3. An *expected utility (EU) preference structure on* $\mathbf{F}$ is a tuple $\mathrm{Pr} = (\Pi, U, \mathscr{W})$ where
   – Π is a belief system on $\mathbf{F}$,
   – U is a taste system on $\mathbf{F}$, and
   – $\mathscr{W}$ is a σ-algebra on W,

   such that, we have, for all $i \in I$, all optional times τ^i for i and information sets $\mathfrak{p} = (\tau^i, \mathfrak{x}) \in \mathfrak{P}^i(\tau^i)$:
   (a) $u_{i,\mathfrak{p}}$ is bounded and $\mathscr{W}$-Borel-measurable;
   (b) $\mathrm{Out}^s_{i,\mathfrak{p}} \colon \Omega \to W$, $\omega \mapsto (\omega, \mathrm{Out}^\star(s \mid \tau^i, p_{i,\mathfrak{p}}(\omega))(\omega))$ is $\mathscr{E}$-$\mathscr{W}$-measurable for all $s \in \mathcal{S}$;
   (c) for any $s \in \mathcal{S}$ and any optional time σ^i for i with $\tau^i \leq \sigma^i$, the map
   $$\varphi^s_{i,\mathfrak{p},\sigma^i} \colon \mathfrak{x} \to \mathfrak{P}^i(\sigma^i),$$
   assigning to any $\chi \in \mathfrak{x}$ the unique $\mathfrak{p}' = (\sigma^i, \mathfrak{x}') \in \mathfrak{P}^i(\sigma^i)$ with $\mathrm{Out}^\star(s \mid \tau^i, \chi) \in \mathfrak{x}'$, is $\mathfrak{P}_{i,\mathfrak{p}}$-$\mathfrak{P}^{i,\sigma^i}$-measurable;
   (d) the map
   $$\chi_{i,\mathfrak{p}} \colon \Omega \to \mathbb{B}^{\overline{\mathbb{T}}},\ \omega \mapsto p_{i,\mathfrak{p}}(\omega)(\omega)$$
   is a state process.
4. Let $\mathrm{Pr} = (\Pi, U, \mathscr{W})$ be an EU preference structure on $\mathbf{F}$ and $s \in \mathcal{S}$ a strategy profile. (Pr, s) is said *dynamically consistent* iff:
   (a) for all $i \in I$, there is u_i with $u_{i,\mathfrak{p}} = u_i$ for all $\mathfrak{p} \in \mathfrak{P}^i$;
   (b) for all $i \in I$, all optional times τ^i, σ^i for i with $\tau^i \leq \sigma^i$, all $\mathfrak{p} \in \mathfrak{P}^i(\tau^i)$, all $\omega \in \Omega$ satisfy[38]
   $$p_{i,\varphi^s_{i,\mathfrak{p},\sigma^i}(p_{i,\mathfrak{p}}(\omega))}(\omega) = \mathrm{Out}^\star(s \mid \tau^i, p_{i,\mathfrak{p}}(\omega));$$
   (c) for all $i \in I$, all optional times τ^i, σ^i for i with $\tau^i \leq \sigma^i$, all $\mathfrak{p} \in \mathfrak{P}^i(\tau^i)$ and the measure $\mathbb{P}_{i,\mathfrak{p}} = \kappa^{i,\tau^i}(.,\mathfrak{p})$, and all $E \in \mathscr{E}$, we have, $\mathbb{P}_{i,\mathfrak{p}}$-almost surely:[39]

$$\kappa^{i,\sigma^i}(E, \varphi^s_{i,\mathfrak{p},\sigma^i} \circ p_{i,\mathfrak{p}}) = \mathbb{P}_{i,\mathfrak{p}}\Big(E \mid \varphi^s_{i,\mathfrak{p},\sigma^i} \circ p_{i,\mathfrak{p}}\Big). \tag{3.3}$$

[37] According to a classical choice of $\mathscr{P}_{i,\mathfrak{p}}$, any $\chi, \chi' \in \mathfrak{x}$, that coincide $\mathscr{N}$-almost surely on $[\![0, \tau^i]\!]$, would have to be inseparable by $\mathscr{P}_{i,\mathfrak{p}}$, that is, for all $P \in \mathscr{P}_{i,\mathfrak{p}}$, $\chi \in P$ iff $\chi' \in P$.

[38] Note that for $\tau^i = \sigma^i$ this implies the equality $p_{i,\mathfrak{p}}(\omega) = \mathrm{Out}^\star(s \mid p_{i,\mathfrak{p}}(\omega))$. This reflects the fact that in SPF, formally, information sets actually partition all possible state processes, rather than "moves". In SEF already, moves contained in an information sets form a partition of attainable outcomes, and information sets partition moves, see [57, Section 2.1]. However, note that there is no rigorous meaning to "moves" in SPF.

[39] We recall that Equation 3.3 is, by definition of conditional expectation (and "probability"), equivalent to the following statement: for all bounded, $\mathscr{P}^{i,\sigma^i}$-Borel-measurable maps $f \colon \mathfrak{P}^i(\sigma^i) \to \mathbb{R}$, we have
$$\mathbb{E}^{\mathbb{P}_{i,\mathfrak{p}}}\left(f(\varphi^s_{i,\mathfrak{p},\sigma^i} \circ p_{i,\mathfrak{p}}) \cdot 1_E\right) = \mathbb{E}^{\mathbb{P}_{i,\mathfrak{p}}}\left(f(\varphi^s_{i,\mathfrak{p},\sigma^i} \circ p_{i,\mathfrak{p}}) \cdot \kappa^{i,\tau^i}(E, \varphi^s_{i,\mathfrak{p},\sigma^i} \circ p_{i,\mathfrak{p}})\right).$$

5. Let $\mathrm{Pr} = (\Pi, U, \mathscr{W})$ be an EU preference structure on $\mathbf{F}$. For any $i \in I$, any optional time τ^i for i, and any information set $\mathfrak{p} \in \mathfrak{P}^i(\tau^i)$, let $\mathbb{P}_{i,\mathfrak{p}} = \kappa^{i,\tau^i}(.,\mathfrak{p})$ and

$$\mathscr{F}_{i,\mathfrak{p}} = \Big\{\{\omega \in \Omega \mid f_{\tau^i(\omega,\chi_{i,\mathfrak{p}}(\omega))}(\omega, \chi_{i,\mathfrak{p}}(\omega)) \in B\} \mid f \colon \overline{\mathbb{T}} \times W \to \mathbb{R}\ \mathscr{M}^i\text{-measurable},\ B \in \mathscr{B}_{\mathbb{R}}\Big\}.^{40}$$

Further, for any strategy profile $s \in \mathcal{S}$, let $\pi_{i,\mathfrak{p}}(s)$ denote the conditional expectation of $u_{i,\mathfrak{p}} \circ \mathrm{Out}^s_{i,\mathfrak{p}}$ with respect to $\mathbb{P}_{i,\mathfrak{p}}$ given $\mathscr{F}_{i,\mathfrak{p}}$, that is,[41]

$$\pi_{i,\mathfrak{p}}(s) = \mathbb{E}^{\mathbb{P}_{i,\mathfrak{p}}}\Big(u_{i,\mathfrak{p}} \circ \mathrm{Out}^s_{i,\mathfrak{p}} \mid \mathscr{F}_{i,\mathfrak{p}}\Big).$$

For any $i \in I$, fix a set of information sets $\tilde{\mathfrak{P}}^i \subseteq \mathfrak{P}^i$, and let $\tilde{\mathfrak{P}} = (\tilde{\mathfrak{P}}^i)_{i\in I}$. A strategy profile $s \in \mathcal{S}$ is said *dynamically rational on* $\tilde{\mathfrak{P}}$ *given* Pr iff for all $i \in I$, all $\mathfrak{p} \in \tilde{\mathfrak{P}}^i$, and all $\tilde{s} \in \mathcal{S}$ with $\tilde{s}^{-i} = s^{-i}$, we have, $\mathbb{P}_{i,\mathfrak{p}}$-almost surely,

$$\pi_{i,\mathfrak{p}}(s) \geq \pi_{i,\mathfrak{p}}(\tilde{s}).$$

Let $s \in \mathcal{S}$ be a strategy profile. Then, (s, Pr) is said *in equilibrium on* $\tilde{\mathfrak{P}}$ iff it is dynamically rational on $\tilde{\mathfrak{P}}$ given Pr and (Pr, s) is dynamically consistent. The qualifier "on $\tilde{\mathfrak{P}}$" can be dropped iff $\tilde{\mathfrak{P}}^i = \mathfrak{P}^i$ for all $i \in I$.

Remark 3.7. Consider a stochastic process form $\mathbf{F}$ satisfying Assumption (A) in Proposition 3.4 (i.e. at time zero, agents have no information about the state process), and an EU preference structure Pr as in the definition. Then, it follows directly from the definition that, for any $i \in I$, there is a unique (endogenous!) information set at time zero. Denote this information set by $\mathfrak{p}_0 = (0, \mathfrak{x}_0)$. It clearly does not depend on the agent i.

Then, for any $s \in \mathcal{S}$ such that (Pr, s) is dynamically consistent, $p_{i,\mathfrak{p}_0}$ is constant with value $\chi = \mathrm{Out}^\star(s \mid 0, \tilde{\chi})$ (where $\tilde{\chi}$ can be any state process). More generally, for any information set $\mathfrak{p} = (\tau^i, \mathfrak{x})$ for i with $\chi \in \mathfrak{x}$, $p_{i,\mathfrak{p}}$ is constant with value χ, and, in particular, $\mathbb{P}_{i,\mathfrak{p}} = \mathbb{P}_{i,\mathfrak{p}_0}$. Off the equilibrium path (or without Assumption (A)), this need not hold true.

Moreover, in Bayesian language, for any $i \in I$, $\mathbb{P}_{i,\mathfrak{p}_0}$ is the *prior* of agent i. The *common prior assumption*, alias *Harsanyi doctrine*, says that $\mathbb{P}_{i,\mathfrak{p}_0} = \mathbb{P}_{j,\mathfrak{p}_0}$ for all $i, j \in I$.[42] The *posterior for* i *at information set* $\mathfrak{p} \in \mathfrak{P}^i$ is the "conditional probability" $\mathbb{P}_{i,\mathfrak{p}}(. \mid \mathscr{F}_{i,\mathfrak{p}})$, though this need not admit a representation via a Markov kernel in the spirit of regular conditional probabilities.

Example 3.8. Consider, for example, the special case of "quasi"-perfect information as expressed by the Assumptions (A), (C), and (B). Then, with κ^{i,τ^i} being constant in the second component and independent of i and τ^i, the equilibrium definition above implements a generalised stochastic version of the concept of subgame-perfect equilibrium (cf. [60]). More generally, the definition above implements the concept of perfect Bayesian equilibrium (cf. [36, 30]) in stochastic process forms. In Subsections 3.3 and 3.4, we discuss this definition in the context of stochastic timing and differential games, with a concrete detailed example for the former.

3.3. **Timing games.** We apply the developed theory to the simplest non-trivial dynamic game-theoretic model. That is, we introduce the general continuous-time stochastic timing game in stochastic process form, and illustrate it by proving the existence of the symmetric preemption equilibrium in the grab-the-dollar game. While natural in discrete time, it is hard to justify in

[40] This is nothing else than the σ-algebra of the τ^i-past associated with the Meyer-σ-algebra $\mathscr{M}^i$, pulled back onto Ω by $\mathrm{id}_\Omega \star \chi_{i,\mathfrak{p}}$; introduced in [51], see also [25], and [10, Subsection 2.1]. Using the notation from the latter survey, and of the pullback of σ-algebras, we thus have: $\mathscr{F}_{i,\mathfrak{p}} = (\mathrm{id}_\Omega \star \chi_{i,\mathfrak{p}})^* \mathscr{F}^{\mathscr{M}^i}_{\tau^i} = \{(\mathrm{id}_\Omega \star \chi_{i,\mathfrak{p}})^{-1}(M) \mid M \in \mathscr{F}^{\mathscr{M}^i}_{\tau^i}\}$.

[41] Note that the following expression is well-defined in view of Equation 3.3, because $\mathrm{Out}^s_{i,\mathfrak{p}} \colon \Omega \to W$ is uniquely defined up to $\mathscr{N}$-indistinguishability, and $\mathscr{N} \subseteq \mathscr{N}_{\mathbb{P}_{i,\mathfrak{p}}}$ by assumption.

[42] See [34, 35, 36, 8, 6].

continuous time. Yet, preemption is both an interesting theoretical example of subgame-perfection and a crucial phenomenon in many economic applications. In seminal papers, [29] and, in a general stochastic setting, [58, 65] provide a theory explaining the symmetric preemption equilibrium in continuous time using a stacked strategic form.

However, in such a formulation, strategies necessarily depend on subgames. Although consistency across subgames may be assumed *ex post*, the question remains on what grounds this happens. Moreover, the cited literature formally explains neither outcomes, that is, track records of the players' action, nor how payoffs derive from outcomes. Instead, payoffs are a direct function of strategy profiles, without factorising over (induced) outcomes. Hence, these models do not explain what happens, but what abstract strategies are chosen in each strategic form and what the payoffs are. This is reflected by the fact that payoffs are derived via a discrete-time approximation, see [64]. As discussed in the introduction of [58], and shown in the already-mentioned literature [66, 4, 2], a well-posed extensive form formulation based solely on paths with time index set $\mathbb{R}_+$ requires locally right-constant paths, so that reaction can only occur with a positive time lag of delay — such a model is not very conclusive regarding preemption. The action-reaction model in [1] provides a well-posed extensive form model making instantaneous reaction, and even $\mathfrak{w}$ many of them, possible;[43] however, as this model leaves the question of randomised strategies open, this theory is not sufficiently applicable to the preemption problem as well. This explains why the stacked strategic form approach has been chosen in [29] and *idem* in [58].

One problem about passing from discrete-time preemption to continuous-time preemption lies in the collapse of complicated patterns action and reaction near the preemption boundary to action at the preemption boundary with probability one. In classical continuous time, this implies a dramatic loss of information on the action process containing information about the action-reaction behaviour. However, this is a result of looking at action process convergence in terms of pointwise convergence. By contrast, tilting convergence preserves the detailed information by writing it on the vertical axis above the preemption boundary. Hence, stochastic process forms in vertically extended continuous time can bridge the gap between a) the need of a faithful model of the extensive form characteristics, based on an explicit description of outcomes, choices and information flow and including randomisation, and b) the desire to formally describe action and reaction behaviour, e.g. in preemption games, arising in continuous time via game-theoretic equilibrium analysis.

The mentioned kind of preemption is a very interesting example from the literature on timing games, but only one among many. Many other of these timing game models also use stacked strategic forms and ad-hoc variants of Nash equilibrium, which makes it difficult to understand the dynamic aspect of strategy and equilibrium. This includes the rich economic literature on real option games as well as the mathematical theory of Dynkin games — we refer to the introduction of [58] for an overview on the literature. Another example would be timing games of asymmetric information, e.g. about price signals. Here, one agent can trade proactively thereby making profit and another one can only react (instantaneously in real time, one level higher on the vertical half-axis). But this other person's reaction may impact the price, and other players may pro- or react with respect to this.[44] The stochastic process form and the included abstract, dynamic equilibrium concept allows formulating a very general stochastic timing game model, which can shed light on this problem in general and is therefore of general interest.

Let us start with introducing our formal model of timing problems. For convenience, we focus on the case of "full" endogenous information alias closed-loop controls (more precisely: at any time, all players know what the other players have done up to, exclusively, that time). This is the

[43]The model in [1] is formulated with one reaction node per action node (i.e. roughly per instant of real time); the extension to $\mathfrak{w}$ many reaction nodes is immediate.

[44]For related models, see, e.g. [12, 13].

comparatively complicated case, weaker informational settings can be analysed similarly without essential additional effort. For this subsection's purpose, let $(\Omega, \mathscr{E})$ be a universally complete measurable space. Suppose that $(\Omega, \mathscr{E})$ is large enough to support a probability measure $\mathbb{P}$ and a, with respect to $\mathbb{P}$, $[0,1]$-uniformly distributed random variable. Let $\mathscr{N} = \{\emptyset\}$.

Let us fix an important technical convention. To any decreasing map $h\colon \overline{\mathbb{T}} \to \{0,1\}$ we assign the map $h_- = h(.-)\colon \overline{\mathbb{T}} \to \{0,1\}$, defined in any $t \in \overline{\mathbb{T}}$ as follows. We let $h_-(0) = h(0-) = 1$ if $t = 0$; $h(t-) = \lim_{u \nearrow t} h(u)$ if t is a left-limit point; and, else, that is, if $t = (p(t), \beta + 1)$ for some $\beta \in \mathfrak{w}_1$, $h(t-) = h(p(t), \beta)$. We extend this convention to componentwise decreasing maps valued in $\{0,1\}^I$ in a componentwise manner, and to stochastic processes with decreasing paths valued in $\{0,1\}^I$.

Let the data $\mathbf{F} = (I, \mathbb{A}, \mathbb{B}, W, \mathcal{W}, \mathscr{H}, \mathscr{M}, \mathcal{S})$ and $(\alpha, \upsilon, \mathscr{G}, \mathscr{F}, \mathscr{F}^\vee, (\tau_{\boldsymbol{b}}, \tau_{\boldsymbol{b}}^-)_{\boldsymbol{b} \in \mathbb{B}}, z)$ be given as follows:

- I is a non-empty, finite set;
- $\mathbb{A}^i = \{0,1\}$ for any $i \in I$, $\mathbb{A} = \prod_{i \in I} \mathbb{A}^i$, equipped with discrete topology and the product order;
- $\mathbb{B} = \mathbb{A}$, thus also equipped with discrete topology and the product order, let $\mathbf{1} = (1, \ldots, 1) \in \mathbb{B}$ and $\mathbf{0} = (0, \ldots, 0) \in \mathbb{B}$ denote the constant functions $I \to \{0,1\}$ with value 1 and 0, respectively;
- $\alpha \in \mathfrak{w}_1 \setminus \{0\}$ is a countable non-zero ordinal;
- W is the set of pairs $(\omega, h) \in \Omega \times \mathbb{B}^{\overline{\mathbb{T}}}$ such that h is right-continuous, decreasing, and has upper vertical level smaller than or equal to α, and satisfies $h(\infty) = \mathbf{0}$;
- $\mathscr{G} = (\mathscr{G}^i)_{i \in I}$ is a family of universally augmented filtrations on $(\Omega, \mathscr{E})$ with time index set $\overline{\mathbb{T}}$;
- $\upsilon \cong (\upsilon^i)_{i \in I}\colon \Omega \to [0,1]^I$ is $\mathscr{E}$-Borel-measurable, such that there is probability measure $\mathbb{P}$ on $(\Omega, \mathscr{E})$ making υ uniformly distributed and independent from $\mathscr{G}$;
- $\mathscr{F} = (\mathscr{F}^i)_{i \in I}$, and for any $i \in I$, $\mathscr{F}^i = (\mathscr{F}^i_t)_{t \in \overline{\mathbb{T}}}$ is the filtration given by $\mathscr{F}^i_t = \overline{\mathscr{G}^i_t \vee \sigma(\upsilon^i)}$, the universal augmentation being taken in $\mathscr{E}$;
- $\mathscr{F}^\vee = (\mathscr{F}^\vee_t)_{t \in \overline{\mathbb{T}}}$ is the in $\mathscr{E}$ universally augmented filtration generated by this family, i.e. $\mathscr{F}^\vee_t = \overline{\bigvee_{i \in I} \mathscr{F}^i_t}$;
- $\mathcal{W}$ is the set of pairs $\zeta = (\xi, \chi)$ where $\xi\colon \overline{\mathbb{T}} \times \Omega \to \mathbb{A}$ is an $\mathscr{F}^\vee$-optional, right-continuous, componentwise decreasing process with upper vertical level smaller than or equal to α satisfying $\xi_\infty = 0$, and $\chi = \xi$;[45]
- let

$$z\colon \overline{\mathbb{T}} \times W \to \mathbb{B},\ (t, \omega, h) \mapsto \begin{cases} \mathbf{1}, & \text{if } t = 0, \\ h(t), & \text{if } t > 0, \end{cases} \tag{3.4}$$

  and, for any $\boldsymbol{b} \in \mathbb{B}$,

$$\tau_{\boldsymbol{b}} = \inf\{u \in \overline{\mathbb{T}} \mid z_u \leq \boldsymbol{b}\}, \qquad \tau_{\boldsymbol{b}}^- = \inf\{u \in \overline{\mathbb{T}} \mid z_{u-} \leq \boldsymbol{b}\};$$

- for any $i \in I$ and $t \in \overline{\mathbb{T}}$, let $\mathscr{H}^{i,0}_t$ be the smallest σ-algebra on W containing $\mathscr{F}^i_t \otimes \mathscr{B}_{\mathbb{B}}^{[0,t]_{\overline{\mathbb{T}}}} \otimes \{\emptyset, \mathbb{B}^{(t,\infty]_{\overline{\mathbb{T}}}}\}|_W$ and such that, for all $u \in [0,t]_{\overline{\mathbb{T}}}$ and $\beta \in \mathfrak{w}_1$,

$$\{\tau_{\boldsymbol{b}} \leq u,\ \pi \circ \tau_{\boldsymbol{b}} = \beta\} \in \mathscr{H}^{i,0}_t,$$

  let $\mathscr{H}^{i,1}_t$ be the universal augmentation of $\mathscr{H}^{i,0}_t$ in $[\mathscr{H}^{i,0}_\infty]^{\mathrm{u}}$; then, let, if $t > 0$,

$$\mathscr{H}^i_t = \{H \in \mathscr{H}^{i,1}_t \mid \exists \tilde{H} \subseteq \Omega \times \mathbb{B}^{[0,t]_{\overline{\mathbb{T}}}}\colon H = \mathrm{proj}^{-1}_{[0,t]_{\overline{\mathbb{T}}}}(\tilde{H}) \cap W\},$$

[45] The condition $\chi = \xi$, for instance, could be relaxed; χ could be the solution to some (stochastic) differential equation depending non-anticipatively on ξ.

and, else,

$$\mathscr{H}_0^i = \{(E \times \mathbb{B}^{\overline{\mathbb{T}}}) \cap W \mid E \in \mathscr{F}_0^i\};$$

– for any $i \in I$, $\mathscr{M}^i = \mathrm{Opt}(\mathscr{F}^i \otimes \{\emptyset, \mathbb{B}^{\overline{\mathbb{T}}}\}) \vee \mathrm{Prd}(\mathscr{H}^i)$, where $\mathscr{F}^i \otimes \{\emptyset, \mathbb{B}^{\overline{\mathbb{T}}}\}$ denotes the filtration given by $\mathscr{F}_t^i \otimes \{\emptyset, \mathbb{B}^{\overline{\mathbb{T}}}\}$, $t \in \overline{\mathbb{T}}$;
– for any $i \in I$, let $\mathcal{S}_0^i$ be the set of $\mathscr{H}^i$-progressively measurable, $\mathbb{A}^i$-valued processes $s^i \colon \overline{\mathbb{T}} \times W \to \mathbb{A}^i$, lower semicontinuous from the right, with upper vertical level smaller than or equal to α, and satisfying both $s_t^i(\omega, h) \leq h^i(t-)$, for all $(t, \omega, h) \in \overline{\mathbb{T}} \times W$, and $s_\infty^i = 0$; then, let $\mathcal{S}^i$ be the set of $s^i \in \mathcal{S}_0^i$ such that any optional time τ^i for i admits $\mathscr{M}^i$-measurable $\tilde{s}^i \colon \overline{\mathbb{T}} \times W \to \mathbb{A}^i$ with $\tilde{s}^i \in \mathcal{S}_0^i$ and $s_{\tau^i}^i = \tilde{s}_{\tau^i}^i$.

The basis of this model are the outcomes, like in extensive form models, and in contrast to the stacked strategic form model. Outcomes are exactly the outcomes of a timing game which, by definition, is a game with two actions, one of them being irreversible: that is, collections of decreasing $\{0, 1\}$-valued paths for any player. There are no grounds for considering additional stopping intensities as in [29, 58], nor for acting on the whole unit interval (as in almost all of the timing game literature with randomisation). The assumption of optionality of action processes is not even strictly required, since in the proof of well-posedness it can be seen that only optional action processes can be generated by strategic decision making according to $\mathcal{S}$. We have a countable uniform bound on activity; in the perspective of approximation of action processes via tilting convergence this corresponds to an upper bound on the well-order type of approximating grids. The assumption that the state process induced by a given action process equals the action process can be relaxed; it is made for simplicity here. Regarding the information structure $\mathscr{H}^i$ and $\mathscr{M}^i$ of player $i \in I$, we are in the setting of Proposition 3.4. Here, $\mathscr{H}^i$ has been sufficiently enlarged to make the stochastic analysis of relevant débuts on W possible, without violating the non-anticipativity axiom, Axiom 1, in the definition of SPF.

Let us discuss strategies in a bit more detail. In the SPF setting, strategies are complete contingent plans of action – contrasting stacked strategic form frameworks. They are globally defined objects, assembling local decisions based on available information (though this also requires measurability along "nodes", see Remark 3.5). They must be globally progressively measurable, and, moreover, at any optional time for i, representable by an $\mathscr{M}^i$-measurable strategy. Lower semicontinuity from the right implies that the choice to remain at 1 at the upper end of the vertical half-axis above some real time $t \in \mathbb{R}_+$ implies the agent to choose 1 as well on some positive interval $((t, \mathfrak{w}_1), t + \varepsilon)_{\overline{\mathbb{T}}}$, for some $\varepsilon > 0$, depending on (ω, h). On a scenariowise level, this reflects the "identifiability" axiom in [66] and the inertia time lags in [1]; yet, it is only a weak restriction because the player has the whole of $\{u \in \overline{\mathbb{T}} \mid p(u) = t,\, \pi(u) < \alpha\}$ to react infinitesimally. However, as a difference to (the natural stochastic generalisation of) [1], the inertia time lag can depend on information revealed only at time $t + \varepsilon$. The inequality "$s_t^i(\omega, h) \leq h^i(t-)$" simply expresses that at time t and conditional on a history h according to that i has already chosen 0, there is no other choice than 0 left. Note, however, that it is absolutely possible that $s_t^i(\omega, h) = 1$ while $s_u^i(\omega, h) = 0$ for $t, u \in \overline{\mathbb{T}}$ with $u < t$: $s^i(\omega, h)$ need not be decreasing! For instance, consider the strategy choosing 0 whenever some real-valued Markov process η stays in a closed set $C \subseteq \mathbb{R}$, and 1 otherwise. Starting in a subgame $(\tau^i, \tilde{\chi})$ with $\eta_{\tau^i \circ (\mathrm{id}_\Omega \star \tilde{\chi})} \notin C$, the agent will stay at 1 for a positive amount of classical real time. Yet, η may have hit C strictly before $\tau^i \circ (\mathrm{id}_\Omega \star \tilde{\chi})$. Dropping this monotonicity assumption is crucial in the endeavour of formulating strategies as complete contingent plans of action, and independently of subgames. At the same time, it requires a consistent and careful application of stochastic analysis.

Our first aim is to show well-posedness of $\mathbf{F}$. For this, we start with the stochastic analysis of the data introduced before. The first step is about the processes z and z_- and its débuts $\tau_{\boldsymbol{b}}$

and $\tau_{\boldsymbol{b}}^-$. They are crucial in constructing the induced outcome map; hence, we must verify their measurability properties beforehand.

Lemma 3.9. *For any $\boldsymbol{b} \in \mathbb{B}$, $\tau_{\boldsymbol{b}}$ and $\tau_{\boldsymbol{b}}^-$ are $\mathscr{H}^i$-optional times satisfying $\tau_{\mathbf{1}} = \tau_{\mathbf{1}}^- = 0$, and, if $\boldsymbol{b} \neq \mathbf{1}$, $\pi \circ \tau_{\boldsymbol{b}} < \alpha$, $\pi \circ \tau_{\boldsymbol{b}}^- = \pi \circ \tau_{\boldsymbol{b}} + 1$, and $[\![0, \tau_{\boldsymbol{b}}^-)\!) = [\![0, \tau_{\boldsymbol{b}}]\!]$. z is $\mathscr{H}^i$-optional and has the upper vertical level α. z_- is $\mathscr{H}^i$-predictable and has the vertical level α, and even upper vertical level α if α is a limit ordinal.*

In what follows, let $\mathscr{H}^\vee$ denote the augmented filtration generated by the family of $\mathscr{H}^j$, $j \in I$. That is, with $\mathscr{H}_t^{\vee,0} = \bigvee_{j \in I} \mathscr{H}_t^j$ for all $t \in \overline{\mathbb{T}}$, $\mathscr{H}_t^\vee = \overline{\mathscr{H}_t^{\vee,0}}$ is the augmentation of $\mathscr{H}_t^{\vee,0}$ in $[\mathscr{H}_\infty^{\vee,0}]^{\mathrm{u}}$.

The next two lemmata are concerned with the questions whether optionality and progressive measurability are preserved under natural operations on the path space of the state process.

Lemma 3.10. *Let $f \colon \mathbb{T} \times W \to \mathbb{B}$ be $\mathscr{H}^\vee$-optional with right-continuous, decreasing paths, and upper vertical level smaller than or equal to α satisfying $f_\infty = \mathbf{0}$. Then, with f seen as a map $W \to \mathbb{B}^{\overline{\mathbb{T}}}$, there is a map*

$$f^\# \colon \overline{\mathbb{T}} \times W \to \overline{\mathbb{T}} \times W,\ (t, \omega, h) \mapsto (t, \omega, f(\omega, h)) \tag{3.5}$$

which is both $\mathrm{Opt}(\mathscr{H}^\vee)$*-*$\mathrm{Opt}(\mathscr{H}^\vee)$*- and* $\mathrm{Prg}(\mathscr{H}^\vee)$*-*$\mathrm{Prg}(\mathscr{H}^\vee)$*-measurable.*

Lemma 3.11. *Let $\eta \colon \mathbb{T} \times W \to \mathbb{B}$ be $\mathscr{H}^\vee$-optional and χ be a state process. Then, $\eta \circ [\mathrm{id}_{\overline{\mathbb{T}}} \times (\mathrm{id}_\Omega \star \chi)]$ is $\mathscr{F}^\vee$-optional.*

Now, we can state the well-posedness theorem.

Theorem 3.12. $\mathbf{F}$ *is a well-posed stochastic process form on $(\Omega, \mathscr{E}, \{\emptyset\})$.*

We call $\mathbf{F}$ the *timing SPF of upper vertical level α*. Combined with the equilibrium concept in Definition 3.6, it provides a general continuous-time timing game model for finitely many players $i \in I$ with possibly asymmetric exogenous information $\mathscr{G}^i$, and augmented with private randomisation devices υ^i. Players can react instantaneously to new information α times vertically above any real time $t \in \mathbb{R}_+$. By the choice of $\mathscr{M}^i$, at any optional time τ^i for i, a decision can be based on information $\mathscr{H}^i$ in a predictable way in general, but in addition on exogenous information $\mathscr{F}^i$ in a fully optional way. Loosely speaking, the player $i \in I$ can base a decision at time τ^i on exogenous information $\mathscr{F}^i$ until τ^i inclusively, but only on endogenous information that can be explained by endogenous information accumulated over previous instants of time.[46] Note that "indistinguishability up to an optional time τ^i for i" and information sets can be easily characterised in this setting, see Proposition 3.4.

Theorem 3.12 is remarkable because already in the relatively simple case of timing games, counterexamples to well-posedness are well-known (see, in particular, [63, 66]). The analysis of a similar, but deterministic setting in [66] concludes that well-posedness can only hold true for a specific subset of strategies, including a) a restriction of the number of simultaneous action, and b) an "identifiability" requirement regarding accumulating action from the right.[47] An analogue to a) is given by the assumption of a uniform upper bound on the upper vertical level of outcomes processes and strategies. Note, however, that — in contrast to the assumptions and conclusions in [66] — infinitely many jumps at a given real time are possible without risking well-posedness, as long as there is a uniform upper bound α.[48] On the other hand, b) is addressed by the regularity

[46] A formally precise description of this can be given in terms of the σ-algebras $\mathscr{F}_{\tau^i}^{\mathscr{M}^i}$, in the sense of [10, Subsection 2.1], following [51, 25].

[47] As already noted, the inertia nodes in [1] play a similar role. However, this latter model is a rigorous extensive form model, in contrast to Stinchcombe's *ex post* strategy set restriction.

[48] A similar, but finite structural requirement is contained in [63, Assumption F.1], also in a deterministic setting.

requirement — namely lower semicontinuity from the right — of strategies.[49] By the inclusion of vertically extended real time, this requirement does not preclude instantaneous reaction.

Remark 3.13. Theorem 3.12 also obtains if we do not ask for a uniform upper bound on upper vertical levels. Formally, such an SPF obtains by formally taking $\alpha = \mathfrak{w}_1$ in the definition of the SPF's data. One can indeed show that the corresponding SPF is well-posed, but we omit this result and the proof here because in many relevant applications, it appears, there is a uniform upper bound on the vertical level.

We next illustrate **F** by constructing an equilibrium in a specific toy example. This particular example is motivated by the literature on — deterministic and stochastic — games of preemption in continuous time, see [65, 64, 58, 29]. As already said before, the existence and nature of symmetric equilibria crucially depends on the way instantaneous pro- and reaction is modelled and combined with randomisation. While the just-cited works take a "discrete time with an infinitesimally fine grid" perspective (see, e.g. [58, 63]), we propose a formulation using abstract stochastic process forms. This is for the following reasons. First, this theory directly addresses the extensive form characteristics of the problem and provides canonical concepts of equilibrium in a general dynamic and Bayesian setting, including the here-relevant version of subgame-perfect equilibrium. Second, we insist on the fact that exogenous randomness, a Bayesian uncertainty domain, or randomisation can all be taken care of using the generalised stochastic calculus for vertically extended time proposed in Section 2. Third, the concrete definition of the game, including payoffs, in stochastic process form given below is "intrinsic" and does — even not implicitly — rely on approximation arguments, though approximation is an important interpretation device. As a consequence, the definition of the game in general, and of the payoffs in particular, can seem a little simpler while being more general than that in [29, Subsection 4.B] or [58, Definition 2.9] — provided the abstract theory of stochastic process forms in vertically extended continuous time has been accepted. Fourth, by reformulating the stochastic timing game in the language of stochastic process forms we indicate how much more general games in continuous time, which critically involve instantaneous pro- and reaction, can be analysed in an abstract and tractable framework.

We consider a very stylised example illustrating key structures of the theory, and appearing quite similarly in the cited literature. It is a stochastic version of the "grab-the-dollar game" as described in the deterministic setting in [29]. Let us recall the basic facts about this example, following, e.g., [58] and the references therein. The story behind is about two players sitting in front of a one dollar bill. At any time, they can decide to (try to) grab it. The player grabbing first gets the dollar. If both grab at the same time, however, they both have to pay a fine. Clearly, this is a toy model for the modelling of preemption. The discrete-time version of the game admits a symmetric equilibrium, given by the behaviour strategy of grabbing with probability 1/2 at any feasible time. In a standard continuous-time version such an equilibrium does not exist, as is well-known. Indeed, simultaneous grabbing at time zero with probability one is not in equilibrium. Hence, in any symmetric equilibrium in classical continuous time, both players have not stopped with probability one at some time $\varepsilon > 0$. Given this behaviour of one player, the other player can do strictly better by stopping before ε with probability one, in a way that reduces simultaneous grabbing.[50] Therefore, there is no evident symmetric continuous-time equilibrium.

As a solution, Fudenberg and Tirole [29], as well as Riedel and Steg [58], argue within a "discrete time with an infinitesimally fine grid"-approach, and introduce "extended mixed strategies" (cf. [58, Definition 2.7]) consisting not only of (mixed) stopping decisions along the time axis, but also of

[49] A similar, but finite structural requirement is contained in [63, Assumption F.3]. An analogue to [63, Assumption F.2] (piecewise continuity) is automatically satisfied for the timing game.

[50] The latter can be achieved by a distribution of stopping that is absolutely continuous with respect to Lebesgue measure on $(0, \varepsilon]_{\mathbb{R}_+}$.

processes describing the "conditional stopping probabilities" of reaction on the infinitesimally fine grid (cf. [58]). At least in the deterministic case, these have a precise interpretation as limits of grabbing probabilities in behaviour strategies on refining, convergent sequences of discrete-time grids (cf. [64]). They are motivated as a device to effectively control the order of stopping of the two agents if they happen to stop simultaneously, and this becomes clear in the definition of the payoffs in [58, Definitions 2.9 and 2.11]. Hence, that approach contains an idea of different instances of time attached to one real point in time — however, in the cited texts, this notion is not formally spelled out. As a consequence, the definition of payoffs is relatively hard to state, and the definition of subgame-perfect equilibria (cf. [58, Definition 2.14]) is not tightly linked to the abstract game-theoretic concept of subgame-perfect equilibrium, which is based on extensive forms — or, at least, extensive form characteristics. This is, of course, linked to the mentioned general difficulties of formulating continuous-time and stochastic games and to the decision of these articles' authors to work with a stacked family of strategic forms rather than with an "approximate" extensive form. The latter did simply not exist and so it could not be applied.

In the present text, we have developed such a theory out of abstract principles underlying extensive forms and decision making under probabilistic uncertainty. The notion of "approximate extensive form" we suggest is the stochastic process form. We note that it combines extensive form and stochastic aspects — without being an extensive form, but based on outcomes that derive from extensive forms. Below, we see that the use of stochastic calculus in vertically extended time can give a meaning to a) non-simultaneous and ordered action at the same time that is b) measurable with respect to information given by σ-algebras. Moreover, c) it provides further fundamental insights regarding the definition of "subgames" in the stochastic setting, supporting and extending one key innovation of [58].

We now discuss the model formally. As in the cited literature, we focus on the two-player case, $I = \{1, 2\}$, with symmetric exogenous information, $\mathscr{G}^1 = \mathscr{G}^2$. Fix the level $\alpha = \mathfrak{w} + 1$ and a probability measure $\mathbb{P}$ on $(\Omega, \mathscr{E})$ making υ uniformly distributed and independent from $\mathscr{G}^i$, $i \in I$. Slightly developing [58, Example 2.4], the stochastic component of the "grab-the-dollar game" now consists in two things. First, player 2 is not American, and at time $t \in \mathbb{R}_+$, one dollar is worth η_t in 2's local currency, where $\eta = (\eta_t)_{t\in\mathbb{R}_+}$ is $\mathscr{G}^1$-adapted, takes values in $(0,\infty)_{\mathbb{R}}$, and has continuous paths. Second, the dollar is only released at some exogenously given classical optional (alias stopping) time, for example, the time a neutral referee whistles. If a player acts before that time, both players are fined. All fines are payed in local currency. Let τ be an $\overline{\mathbb{R}_+}$-valued $\mathscr{G}$-stopping time modelling the "whistle". For $i, j \in I$ with $i \neq j$, we consider the tastes $u_i, u_j : W \to \mathbb{R}$, given as follows. Let $(\omega, h) \in W$, and, for all $k \in I$, define $\sigma_k(h) = \inf\{t \in \overline{\mathbb{T}} \mid h^k(t) = 0\}$. Further, let η^1 denote the constant process with value 1, and $\eta^2 = \eta$. Let $a_i(h) = 1\{\pi \circ \sigma_i(h) < \mathfrak{w}\}$ and

$$
\begin{array}{llll}
u_i(\omega,h) = u_j(\omega,h) & = -1, & & \text{if } \sigma_i(h) < \tau(\omega); \\
u_i(\omega,h) = \eta^i_{p(\sigma_i(h))} a_i(h), & u_j(\omega,h) = 0, & & \text{if } \tau(\omega) \le \sigma_i(h) < \sigma_j(h); \\
u_i(\omega,h) = u_j(\omega,h) & = -a_i(h), & & \text{if } \tau(\omega) \le \sigma_i(h) = \sigma_j(h) < \infty; \\
u_i(\omega,h) = u_j(\omega,h) & = 0, & & \text{if } \sigma_i(h) = \sigma_j(h) = \infty.
\end{array}
\tag{3.6}
$$

Now, we define the strategy profile $s = (s^i)_{i\in I}$. Let $\phi \cong (\phi^n)_{n\in\mathfrak{w}} : [0,1] \to [0,1]^{\mathfrak{w}}$ be measurable such that, for each $i \in I$, $\phi(\upsilon^i) = (\phi^n(\upsilon^i))_{n\in\mathfrak{w}} = (\upsilon^{i,n})_{n\in\mathfrak{w}}$ is an i.i.d. sequence of $[0,1]$-uniformly distributed random variables, according to $\mathbb{P}$. Let $i \in I$, $j = 3 - i$, and $(t, \omega, h) \in \overline{\mathbb{T}} \times W$. If $h^i(t-) = 0$, or if $t = \infty$, let $s^i_t(\omega, h) = 0$. Else, let

$$
s^i_t(\omega,h) = \begin{cases} 1, & \text{if } t < \tau(\omega), \\ & \quad \text{or } \Big(\tau(\omega) \le t,\ \pi(t) < \mathfrak{w},\ \upsilon^{i,\pi(t)}(\omega) \ge \frac{\eta^j_{p(t)}}{1+\eta^j_{p(t)}}(\omega), \\ 0, & \text{else.} \end{cases}
$$

That is, an agent i pursuing strategy s^i, starting at an optional time τ^i of her's does the following, provided $\chi = \mathrm{Out}^\star(s \mid \tau^i, \tilde{\chi})$ denotes the state process induced by s given the process $\tilde{\chi}$ constant to $\mathbf{1}$ (all players are still active exclusively until τ^i). If τ is not yet reached, i.e. on $\{\tau^i \circ (\mathrm{id}_\Omega \star \chi) < \tau\}$, player i waits until τ, and then "stops" (i.e. switches to zero) with probability $\frac{\eta^j_\tau}{1+\eta^j_\tau}(\omega)$ at any $t \in \overline{\mathbb{T}}$ with $\tau = p(t)$ and $\pi(t) < \mathfrak{w}$, independently along the vertical axis below level $\mathfrak{w}$, until player i manages to actually stop. This implies that i reaches state zero with probability one before time $(p \circ \tau, \mathfrak{w})$.

If τ has already been reached, two cases arise. Let $\hat{\tau}^i = \tau^i \circ (\mathrm{id}_\Omega \star \chi)$. On $\{\hat{\tau}^i \geq \tau,\, \pi \circ \hat{\tau}^i < \mathfrak{w}\}$, analogously at any instance of time on the vertical $\mathfrak{w}$-axis above $\hat{\tau}^i$ the player stops with probability $\frac{\eta^j_{\hat{\tau}^i}}{1+\eta^j_{\hat{\tau}^i}}(\omega)$, until stopping occurs, reaching state zero almost surely before level $\mathfrak{w}$. In the remaining event, on $\{\hat{\tau}^i \geq \tau,\, \pi \circ \hat{\tau}^i \geq \mathfrak{w}\}$, i stops immediately (and so does the opponent); given the payoffs, they are indifferent about different options at this time, and if i wishes to stop "in the near future", then by lower semicontinuity from the right and the fixed bound $\alpha = \mathfrak{w} + 1$ on the upper vertical level, then she even has to stop immediately. Anyway, this latter case is reached with probability zero from the "classical" part of time, the real time axis $\mathbb{R}_+$.

The outcome induced by s can be obtained directly by tilting convergence, as states the following theorem. For simplicity, we restrict the statement to the unique information set at time 0.[51] The theorem can be seen as a stochastic variant of a similar result in [64], and as a representation of it on the level of outcomes. Conversely, this shows that tilting convergence allows for a substantial generalisation of the "discrete time with an infinitesimally fine grid" approach (see Subsection 2.5 for further discussion). The theorem follows directly from Proposition 2.33. By the very definition of tilting convergence, the approximating sequence ξ^n locally equals the outcome of the classical discrete-time symmetric equilibrium, written onto the grid G_n, for any $n \in \mathbb{N}$.

Theorem 3.14. *Let χ be the state process induced by s given the unique information set at time 0, and $\xi = \chi$ be the corresponding action process. Then, there is a sequence $(\xi^n)_{n\in\mathbb{N}}$ of classical, very simple $\mathscr{F}^\vee$-optional processes and a refining, convergent sequence $(G_n)_{n\in\mathbb{N}}$ of classical, deterministic grids G_n compatible with ξ^n, for all $n \in \mathbb{N}$, such that $(\xi^n \mid G_n) \xrightarrow{\mathsf{T}} \xi$ as $n \to \infty$.*

It is shown in the following that s defines an equilibrium on some relatively large $\tilde{\mathfrak{P}}$ and with respect to the payoffs u_i, $i \in I$, and the EU preference structure arising from $\mathbb{P}$ in the most consistent way possible. Precisely, we let $\tilde{\mathfrak{P}} = (\tilde{\mathfrak{P}}^i)_{i\in I}$, where for each $i \in I$, $\tilde{\mathfrak{P}}^i$ is the set of information sets $\mathfrak{p} = (\tau^i, \mathfrak{x})$ such that $\mathfrak{x}$ contains an $\mathscr{F}^i$-progressively measurable state process. That is, counterfactual histories must be independent of the opponent's randomisation device. This restriction appears necessary to the author because the agents cannot use "new" independent randomisation devices at each real time t — it is well known from stochastic analysis that this is incompatible with measurability in the time variable, and in particular, it obstructs progressive measurability and $\mathscr{M}^i$-measurability.[52] If we wish to obtain i.i.d. randomisation over all instants of real time such as to obtain a random distribution in the Schwartz sense, we obtain "white noise" which, in that sense, cannot be represented as a function, or process, valued in the original state space. Hence, in order to go beyond $\tilde{\mathfrak{P}}$ as above, one would have to model strategies by random distributions rather than processes on action spaces. This is deliberately left for future research.

Nevertheless, the setting with $\tilde{\mathfrak{P}}$ as above is already more general than the setting in [29] and that, more general one, in [58]. In the latter, dependence on endogenous histories is encoded by the dependence of each strategic form on *modes* describing which players have already stopped and

[51] See Remark 3.7.

[52] We also refer to the proof of the theorem which would not work for general information sets because, in general, the history of the state process could correlate with the future randomisation of the opponent.

plans are only revised at real instants of time. In our setting, this would correspond to information sets $\mathfrak{p} = (\tau^i, \mathfrak{x})$ such that $\mathfrak{x}$ contains a deterministic path and τ^i is real-valued; and to strategies that can only condition on the left-limit of this path at the current instant of time.

By contrast, in the model of the present text, s^i is a best response to s^j, for both $i, j \in I$, $i \neq j$, with respect to a larger class of (generalised) "subgames", given by a general class of optional times $\sigma = \sigma^i$ possibly taking values on higher levels of the vertical half-axis, by our definition of equilibrium and information sets. Note that the model of subgame-perfection in [58, 29] does not contain that feature: Once the "atomic" randomisation procedure for "extended mixed strategies" has been started, players can no more revise their plan. By contrast, the stochastic process form in vertically extended time model allows for this. Agents can revise plans at all those cross-sections — given by information sets — through "two-dimensional" time measurable with respect to information and below level $\alpha = \mathfrak{w} + 1$. Agents can even perform different actions, with different probabilities, on different levels within the same vertical strip.

Furthermore, we do not restrict to strategies that are Markovian in the "mode", and find that the strategy profile s (which is "horizontally Markovian") is an equilibrium even within the larger and more natural strategy space defined here. We also note that most timing game formulations, including the just-cited ones, integrate out the randomisation of both players, including the opponent, by considering the action space $[0, 1]$ rather than $\{0, 1\}$. In such an approach, the above-mentioned problem becomes invisible by construction yet remains unsolved.

Due to the specific information structure, especially the closed-loop information on all players' states and the symmetric exogenous information, we may interpret the equilibrium property as a stochastic version of subgame-perfect equilibrium (see Remark 3.8). This is indeed the term used for this equilibrium in the stochastic stacked strategic form setting by Riedel and Steg in [58], and by Fudenberg and Tirole in [29].

Theorem 3.15. *There is an* EU *preference structure* $\mathrm{Pr} = (\Pi, U, \mathscr{W})$ *such that:*

- (s, Pr) *is in equilibrium on* $\tilde{\mathfrak{P}}$*,*
- U *is given by* $(u_i)_{i \in I}$ *defined above,*
- $\mathscr{W} = \mathscr{H}_\infty^\vee$.

In this equilibrium, no stopping occurs strictly before τ. At all $\mathbb{R}_+$-valued $\mathscr{G}$-stopping times σ not earlier than τ, both players stop with $\mathbb{P}$-probability one on the vertical half-axis above σ. More precisely, they do so on the initial leg $\{\pi \in \mathfrak{w},\, p = \sigma\}$. Simultaneous stopping, sole stopping by player i, sole stopping by player j (these terms referring to the extended time half-axis $\overline{\overline{\mathbb{T}}}$) — all these three events have the probabilities known from [58]. If $\eta = 1$ is constant equal to one, then all these probabilities are $1/3$ under $\mathbb{P}$. This confirms the findings from [58, 29].

3.4. **Stochastic differential games and control.** We conclude our study with discussing how stochastic differential games and control problems based on differential equations can be formulated using the language of stochastic process forms. The basic idea is as follows. On a complete probability space $(\Omega, \mathscr{E}, \mathbb{P})$, with $\mathscr{N} = \mathscr{N}_{\mathbb{P}}$, we consider the system of abstract stochastic differential equations

$$\mathrm{d}\chi_t^\beta = V_{(t,\beta)}(\chi_{[0,(t,\beta))_{\overline{\overline{\mathbb{T}}}}}, \xi_{[0,(t,\beta))_{\overline{\overline{\mathbb{T}}}}}) \cdot \mathrm{d}\eta_t, \qquad \beta \in \mathfrak{w}_1 + 1, \tag{3.7}$$

on the stochastic intervals consisting of $(t, \omega) \in \overline{\mathbb{R}_+} \times \Omega$ with $\hat{\tau}(\omega) \leq (t, \beta) \leq \infty$,[53] with initial condition $\chi|_{[\![0,\hat{\tau}]\!]} = \hat{\chi}$, where:

- I is a non-empty, finite set of *agents*, or *players*;
- $\xi\colon \overline{\mathbb{T}} \times \Omega \to \mathbb{A}$ is a stochastic processes describing the agents' *action*, $\mathbb{A} = \prod_{i \in I} \mathbb{A}^i$ and $\mathbb{A}^i = \mathbb{R}^{a_i}$ for $a_i \in \mathbb{N}^*$ and $i \in I$;
- $\chi\colon \overline{\mathbb{T}} \times \Omega \to \mathbb{R}^d$ is a stochastic process describing a *state* or *signal*, not fully observable by the agents, for $d \in \mathbb{N}^*$, for any $(t, \beta) \in \overline{\mathbb{T}}$, $\chi^\beta_t = \chi_{(t,\beta)}$, and $\hat{\chi}\colon [\![0, \hat{\tau}]\!] \to \mathbb{R}^d$ is (the restriction of) a stochastic process describing a given initial state;
- $\hat{\tau}\colon \Omega \to \overline{\mathbb{T}}$ is a random time with respect to $\mathscr{P}_{\overline{\mathbb{T}}}$ such that $\pi \circ \hat{\tau} \leq \alpha$ for some $\alpha \in \mathfrak{w}_1$, describing the initial time;
- $\eta\colon \mathbb{R}_+ \times \Omega \to \mathbb{R}^m$ describes an exogenous "random" perturbation of the state, where $m \in \mathbb{N}^*$;
- the maps $V_t\colon (\mathbb{R}^d)^{[0,t)_{\overline{\mathbb{T}}}} \times (\mathbb{R}^a)^{[0,t)_{\overline{\mathbb{T}}}} \to \mathbb{R}^{d \times m}$, where $t \in \overline{\mathbb{T}}$, describes the infinitesimal linear effect of these perturbations on the state process χ at time t, where $m \in \mathbb{N}^*$ and $a = \sum_{i \in I} a_i$.

Stochastic analysis treats the meaning and further properties of such equations, understood as integral equations with respect to the measure $\mathbb{P}$.[54] The dependence on the measure may be crucial, of course. For example, η could be a continuous $\mathbb{R}^m$-valued $\mathbb{L}^2$-semimartingale with respect to $\mathbb{P}$ and the integral be understood in the sense of $\mathbb{L}^2(\mathbb{P})$-convergence, following Ito. We only formulate one abstract non-anticipativity assumption on System 3.7, which is satisfied by the usual integration concepts:

- **Assumption SDG.** Suppose that, with the notation just introduced, χ solves System 3.7 for ξ and initial condition $\chi|_{[\![0,\hat{\tau}]\!]} = \hat{\chi}$ and initial time $\hat{\tau}$. Further, let $\xi'\colon \overline{\mathbb{T}} \times \Omega \to \mathbb{A}$ and $\chi'\colon \overline{\mathbb{T}} \times \Omega \to \mathbb{R}^d$ constitute another pair of stochastic processes satisfying $\xi'|_{[\![0,\hat{\tau}]\!]} = \xi|_{[\![0,\hat{\tau}]\!]}$ and $\chi'|_{[\![0,\hat{\tau}]\!]} = \hat{\chi}$ and such that χ' solves System 3.7 for ξ' and the initial condition $\chi'|_{[\![0]\!]} = \hat{\chi}|_{[\![0]\!]}$ and initial time 0. Then, χ solves System 3.7 for ξ and the initial condition $\chi|_{[\![0]\!]} = \hat{\chi}|_{[\![0]\!]}$ and initial time 0.

Imitating the stochastic differential games and control literature, System 3.7 can be used to construct the set of outcomes $\mathcal{W}$. For this, we select a subset $W \subseteq \Omega \times \mathbb{B}^{\overline{\mathbb{T}}}$ and fix the relevant information structures. Let, for any $i \in I$, $\mathscr{H}^i$ be a filtration on W satisfying Axiom 1 in Definition 3.1 above, and $\mathscr{M}^i$ be a σ-algebra on $\overline{\mathbb{T}} \times W$ satisfying $\mathrm{Prd}(\mathscr{H}^i) \subseteq \mathscr{M}^i \subseteq \mathrm{Opt}(\mathscr{H}^i)$. Let $\mathscr{H} = (\mathscr{H}^i)_{i \in I}$, $\mathscr{M} = (\mathscr{M}^i)_{i \in I}$. Then, let $\mathcal{W}$ be *a* non-empty set of pairs $\zeta = (\xi, \chi)$ satisfying the following properties:

1. for all $\zeta = (\xi, \chi)$, $\zeta' = (\xi', \chi') \in \mathcal{W}$, $\chi_0 = \chi'_0$;
2. for all $\zeta = (\xi, \chi)$ and all $\omega \in \Omega$, $(\omega, \chi(\omega)) \in W$;
3. for all $\zeta = (\xi, \chi) \in \mathcal{W}$, χ is the, up to $\mathscr{N}$-indistinguishability, unique stochastic process χ satisfying System 3.7 for ξ and initial data $(0, \chi|_{[\![0]\!]})$;
4. for all $i \in I$, all $\mathscr{H}^i$-optional times τ^i with $[\![0, \tau^i)\!) \in \mathscr{M}^i$, all $\tilde{\zeta} = (\tilde{\xi}, \tilde{\chi})$, $\zeta' = (\xi', \chi') \in \mathcal{W}$, for $\xi = \tilde{\xi}$ and the initial data $(\hat{\tau}, \hat{\chi}) = (\tau^i \circ (\mathrm{id}_\Omega \star \chi'), \chi'|_{[\![0,\hat{\tau}]\!]})$, there is an, up to $\mathbb{P}$-indistinguishability, unique stochastic process χ satisfying System 3.7, and (for at least one representative thereof with respect to $\mathbb{P}$-indistinguishability) we have $(\xi, \chi) \in \mathcal{W}$.

Let $\hat{\chi}^0 = \chi|_{[\![0]\!]}$ for some (and all) $(\xi, \chi) \in \mathcal{W}$. Moreover, let $\mathcal{I}$ denote the set of pairs $(\hat{\tau}, \hat{\chi}) = (\tau^i \circ (\mathrm{id}_\Omega \star \chi), \chi|_{[\![0,\hat{\tau}]\!]})$, where $i \in I$, τ^i is an $\mathscr{H}^i$-optional time with $[\![0, \tau^i)\!) \in \mathscr{M}^i$, χ is such that

[53] In an alternative "localised" setting, we would only consider the stochastic interval consisting of $(t, \omega) \in \mathbb{R}_+ \times \Omega$ with $\hat{\tau}(\omega) \leq (t, \beta)$ and fix a value of χ at infinity, thereby restricting our attention to $\overline{\mathbb{T}} \setminus \{\infty\}$. The difference between both settings is smaller than one might suppose at first sight, because of the stopping property of stochastic integrals and because $[\![0, 1]\!] \subseteq [\![0, \infty)\!) \subseteq [\![0, \infty]\!]$, and $[\![0, 1]\!] \to [\![0, \infty]\!]$, $(t, \omega) \mapsto (-\log(1 - t), \omega)$ is a map preserving many relevant structures.

[54] Going back to [40, 41], see, for example, the textbooks [21, 43, 54, 27].

there is ξ with $(\xi, \chi) \in \mathcal{W}$. For any $i \in I$, fix a set $\mathcal{S}^i$ of $\mathscr{M}^i$-measurable maps $s^i \colon \overline{\mathbb{T}} \times W \to \mathbb{A}^i$, such that all elements of the product $\mathcal{S} = \bigtimes_{i\in I} \mathcal{S}^i$ are admissible.

Proposition 3.16. *The data* $\mathbf{F} = (I, \mathbb{A}, \mathbb{B}, W, \mathcal{W}, \mathscr{H}, \mathscr{M}, \mathcal{S})$ *defines a well-posed stochastic process form.*

Remark 3.17. Note that, by well-posedness, for every $s \in \mathcal{S}$, the ansatz $\xi = s_{\llcorner}\chi$ plugged into System 3.7 yields the system of abstract stochastic differential equations

$$\mathrm{d}\chi_t^\beta = V_{(t,\beta)}(\chi_{[0,(t,\beta))_{\overline{\mathbb{T}}}}, (s_{\llcorner}\chi)_{[0,(t,\beta))_{\overline{\mathbb{T}}}}) \cdot \mathrm{d}\eta_t, \qquad \beta \in \mathfrak{w}_1 + 1, \tag{3.8}$$

on the stochastic intervals[55] consisting of $(t, \omega) \in \overline{\mathbb{R}_+} \times \Omega$ satisfying $(t, \beta) \leq \infty$ and with initial condition $\chi|_{[\![0]\!]} = \hat{\chi}^0$. If $\chi = \mathrm{Out}^\star(s \mid 0, \tilde{\chi})$ and $\xi = s_{\llcorner}\chi$, where $\tilde{\chi}$ is a state process with $\tilde{\chi}|_{[\![0]\!]} = \hat{\chi}^0$, then χ solves System 3.8. If this system is, up to $\mathbb{P}$-indistinguishability, uniquely solvable, then the induced outcome processes of all strategy profiles, given the information set at time 0, can be characterised by it. Counterfactual induced outcome processes are less handily characterised because they involve conditioning on counterfactual information sets.

Remark 3.18. We note that we have covered stochastic differential games and control problems in the so-called "strong formulation" here. What about the so-called "weak formulation"? As a side remark, we note that often there is a way to translate a weakly formulated problem into the language of the strong formulation, e.g. by change-of-measure techniques (then, the agents' action consists in determining a density process).[56] Relatedly, by the Yamada-Watanabe theorem, under certain conditions, weak existence of solutions for stochastic differential equations already implies strong existence.[57] So, in these cases, there is not much to worry about. However, what about an untranslated, and possibly untranslatable, weakly formulated problem?

It can be seen as a relaxation of the above-discussed formalism. Indeed, this is by accepting that the probability space $(\Omega, \mathscr{E}, \mathbb{P})$ and the information structure $\mathscr{H}$ need no more be fixed, but may vary depending on the outcome and strategy process. Moreover, the random perturbation and the initial states are only fixed in distribution, not (almost surely) pathwise. This situation can be interpreted in the sense of an outcome-dependent extension of the exogenous scenario space, adding sufficient randomisation devices in order make sense of the state dynamics in distribution.

The fundamental decision-theoretic problem with this model lies in that the mere existence of certain scenarios then depends on agents' strategies (creating "unknown unknowns"). This can be compensated for by directly working on the path space for outcome and strategy processes, thereby fixing $(\Omega, \mathscr{E})$ and a canonical outcome and strategy process on it, an approach often adopted in the stochastic analysis and control literature, already for reasons of mathematical convenience. Then, the decision making of any agent no more consists in choosing a strategy, but in selecting a "non-anticipative" probability measure on this path space $(\Omega, \mathscr{E})$. In our approach,[58] that is, essentially, a belief on the realised own strategy — a true paradox which highlights that this weak approach does really relax the basic modelling principles of the present text.

Remark 3.19. We note that, without loss of generality, we have restricted our attention to a particular type of stochastic differential equations in this subsection. Instead of controlling the "velocity"

[55] Idem.

[56] See, for instance, the [19, Chapter 21] for a textbook account on that.

[57] See, for instance, the [44, Theorem 32.14] for a textbook account on that.

[58] Note that, in the present text, a "mixed strategy" in the von Neumann, Morgenstern and Nash sense is modelled simply by a strategy (perhaps only conditionally) independent from the exogenous information of other agents and inducing the same distributions under all agent's beliefs ("secret" and "objective" in the sense of [7]). The randomising nature of a strategy is entirely based on its dependence on $(\Omega, \mathscr{E})$. This is discussed in more detail in [57, Subsections 2.1.4 and 2.3.5].

(like drift rate or volatility) with respect to some exogenous random driver as in Equation 3.7, one might also control the driver itself, as, for instance, done in stochastic singular control or optimal stopping (or timing), and related multiple-agent (games) models. One could also make the field V dependent on the distribution of χ, leading to so-called McKean-Vlasov dynamics with applications in non-atomic stochastic or mean-field games (introduced in [50, 38], see, e.g. [16, 17, 15]). The discussion of this subsection can be adapted to these cases as well.

Remark 3.20. We briefly comment — on a very informal level, without going into technical details — on the equilibrium concept for stochastic process forms in the context of this subsection. Stochastic differential games and control problems are often formulated in stacked strategic form. For instance, the optimality criterion in a strongly Markovian setting with a single agent i may be of the form

$$(*)\qquad \text{Maximise } J(x;s) = \mathbb{E}_x[u_i(\chi^s_\infty)] \text{ over } s = s^i, \qquad \forall x \in \mathbb{R}^d,$$

supposing χ^s to be a strong Markov process, given by $\chi^s = \mathrm{Out}^\star(s \mid 0, \tilde{\chi})$ for some fixed state process $\tilde{\chi}$ corresponding to an initial condition, with continuous $u_i : \mathbb{R}^m \to [-4, 2]$. Very roughly speaking, one may (try to) apply the strong Markov property in order to rewrite this as a maximisation problem of conditional expectations given a suitable filtration evaluated at relevant optional times. This in turn lies within the framework of the equilibrium (in the single-agent case, say, optimiser) concept for stochastic process forms in Definition 3.6. If all this works out, then the stacked strategic forms thus turn out as a special case of a stochastic process form, also from the perspective of equilibria and optimisers.

This perspective on dynamic optimisers (and equilibria) is also compatible with and explains standard methods from control theory. The representation in $(*)$ is used because it gives rise to a function in x, the value function $V(x) = \sup_s J(x;s)$. The dynamic programming (alias Bellman) principle exploits the optimality of s at different "information sets" or "subgames", given sufficient regularity of V. It can be used to study local properties of V. This can be a powerful method to characterise, verify, or compute optimisers or equilibria using partial differential equations (called Hamilton-Jacobi-Bellman equations), thereby further justifying the approach using stochastic process forms in (possibly vertically extended) continuous time.[59]

Conclusion

An abstract and general language of continuous-time games based on stochastic processes has been introduced. Taking limits of the outcomes from well-posed action path stochastic extensive forms, in a way that preserves accumulating reaction behaviour, leads to the notions of tilting convergence and of *vertically extended continuous time*. On this extended time half-axis, a suitable stochastic analysis — with consistent notions of progressive measurability, optional and decision times and processes, a Début Theorem etc. — can be defined. The resulting game-theoretic model based on the *stochastic process form*, which avoids a) well-posedness problems by reducing the set of strategies and b) measurability problems by supposing strategies to be progressively measurable (a strengthening, or, depending on the perspective, weakening, of the product form approach [69, 70, 37]), can be justified on the grounds of tilting approximation of outcomes, but, at the same time, encompasses a vast class of applications: stochastic differential games, continuous-time timing games, continuous-time Bayesian games (e.g. principal-agent problems). In a case study of the timing game, we see that the symmetric, randomised preemption equilibrium predicted by [29, 58] obtains also in this setting, conditional on a vast class of subgames.

The language proposed here is developed out of first principles of game and probability theory, which thus encompass different disciplines. Despite their strong conceptual and historical links

[59] For the purely control-theoretic aspects, we refer, for instance, to the textbooks [19, 53].

these disciplines sometimes speak quite different languages, which blurs the view on those principles. [57, Chapters 1 and 2] and the present text have pointed to some of the key difficulties, which makes it understandable that one often recurs to specific ad-hoc formulations. Yet, those works also demonstrate that it is possible to make the link in a certain sense, given sufficient mathematical effort, and that this effort may improve the conceptual understanding. For example, the symmetric preemption equilibrium arises rather naturally from the general theory; no specific theory of "stopping intensities" alias conditional stopping probabilities as in [29, 58], which heavily uses the two-player and timing games structure, is needed. Using mathematics, a convincing but seemingly ad-hoc solution in "economic" game theory can be represented by a canonical "economic" game-theoretic solution concept.

The present text focuses on abstract and general theory, illustrated via simple examples, and on the general link to stochastic process-based game and control theory. Moreover, we have seen that this theory yields a well-posed model of timing games, compatible with and providing further footing to the existing theory on preemption games. Other things remain to be addressed. For example, we argue that stochastic differential games are an important class of problems covered by stochastic process forms. It is beyond the scope of this text to provide another detailed case study of a differential game in stochastic process form within the necessary detail. We think that future research on stochastic differential games with preemption features, or asymmetric or partial information, with applications in economics and finance (as in [13, 12]), for example, could benefit from and draw upon the game-theoretic formalism introduced in [57] and this text.

Furthermore, using tilting convergence we have provided a general approximation mechanism on the level of outcomes. A game-theoretically very relevant question would be how this can be lifted to equilibria (see, e.g., [63, 28, 64] for related literature). However, an approximation on the level of equilibria is more dependent on specific assumptions on the concrete problems at hand, e.g. regularity assumptions on tastes alias payoffs function. This is beyond the scope of this text, but could be analysed in more specific situations, under specific regularity assumptions, using the language from the present text.

This text also provides a contribution to stochastic analysis, which is formally independent from the decision-theoretic motivation underlying the present text. It might be useful at all places where accumulation creates discontinuities that become invisible in the usual pointwise limit. In that sense, this relates to the literature on stochastic integration and stochastic differential equations, which is a theory about limits of simple integrals as the time grids become arbitrarily fine. The question of finding an adequate notion for this in situations involving jumps arises in applications (see, e.g., [13] and the references therein) and has motivated abstract theory (see, e.g., [18, 52] and the references therein). Stochastic analysis in vertically extended continuous time provides an alternative candidate for this. A next step would necessarily involve attempts to formulate stochastic integration intrinsically in this setting, based on the notions of optional and predictable times and processes etc. introduced and studied in the present text.[60]

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Appendix A. Proofs

A.1. **Section 1.**

The complete total order $\overline{\mathbb{T}}$.

Proof of Lemma 1.2. Let $\alpha \in \mathfrak{w}_1 + 1$ and $S \subseteq \overline{\mathbb{T}_\alpha}$ be some subset, $a = \inf \mathcal{P}p(S)$, $b = \sup \mathcal{P}p(S)$ in $\overline{\mathbb{R}_+}$.

If $a \in \mathcal{P}p(S)$, then $\mathcal{P}\pi(S \cap [\{a\} \times (\sup \alpha + 1)]) \neq \emptyset$. Hence, this set has a minimum γ in $\sup \alpha + 1$. Then, (a, γ) defines a minimum of S. If $a \in \mathbb{R}_+ \setminus \mathcal{P}p(S)$, then $(a, \sup \alpha)$ is an infimum of S in $\overline{\mathbb{T}_\alpha}$. Else, $a = \infty$ and $S = \emptyset$. Then, ∞ is an infimum of S in $\overline{\mathbb{T}}$.

If $b \in \mathcal{P}p(S)$, then $\mathcal{P}\pi(S \cap [\{b\} \times (\sup \alpha + 1)]) \neq \emptyset$. Thus, this set has a supremum γ in $\sup \alpha + 1$. Then, (b, γ) defines a supremum of S. If $b \notin \mathcal{P}p(S)$, then $(b, 0)$ is a supremum of S. □

Proof of Proposition 1.1. $\overline{\mathbb{T}_\alpha}$ is a complete lattice by Lemma 1.2. It then suffices to show that, via set inclusion, $\overline{\mathbb{T}_\alpha}$ defines a dense completion of $\mathbb{T}_\alpha$ (see Corollary B.6).

For this, note that, in case $\alpha > 0$, for any $t \in \mathbb{R}_+$, we have $\sup A_t = (t, \sup \alpha) = \inf B_t$, where

$$A_t = p^{-1}([0, t]_{\mathbb{R}_+}) \cap \mathbb{T}_\alpha, \qquad B_t = p^{-1}((t, \infty)_{\mathbb{R}_+}) \cap \mathbb{T}_\alpha,$$

by Lemma 1.2. Moreover, the same lemma implies $\inf \emptyset = \infty = \sup \mathbb{T}$. In view of Equation 1.6, $\overline{\mathbb{T}_\alpha}$ is a dense completion of $\mathbb{T}_\alpha$. □

Topology and σ-algebras on $\overline{\mathbb{T}}$.

Proof of Lemma 1.4. (Ad "$\mathscr{G}_{\overline{\mathbb{T}}}(\mathbb{T})$ is a subbase of $\mathscr{O}_{\overline{\mathbb{T}}}$"): It suffices to show that elements of $\mathscr{G}_{\overline{\mathbb{T}}}(\overline{\mathbb{T}})$ are unions of subsets of $\mathscr{G}_{\overline{\mathbb{T}}}(\mathbb{T})$. Let $t \in \overline{\mathbb{T}}$. Then, there are $A_t, B_t \subseteq \mathbb{T}$ with $\sup A_t = t = \inf B_t$. We infer that

$$[0, t)_{\overline{\mathbb{T}}} = \bigcup_{u \in A_t} [0, u)_{\overline{\mathbb{T}}}, \qquad (t, \infty]_{\overline{\mathbb{T}}} = \bigcup_{u \in B_t} (u, \infty]_{\overline{\mathbb{T}}}.$$

(Ad "$\mathscr{U}_{\overline{\mathbb{T}}}$ is a base of $\mathscr{O}_{\overline{\mathbb{T}}}$"): We have just seen that $\mathscr{G}_{\overline{\mathbb{T}}}(\mathbb{T})$ is a subbase of $\mathscr{O}_{\overline{\mathbb{T}}}$. It is evident that $\mathscr{U}_{\overline{\mathbb{T}}} \cup \{\overline{\mathbb{T}}\}$ is the set of intersections of finite subsets of $\mathscr{G}_{\overline{\mathbb{T}}}(\mathbb{T})$,[61] and, by basic topology, a base of $\mathscr{O}_{\overline{\mathbb{T}}}$. Moreover, $\overline{\mathbb{T}} = [0, 1)_{\overline{\mathbb{T}}} \cup (0, \infty]_{\overline{\mathbb{T}}}$. Hence, $\mathscr{U}_{\overline{\mathbb{T}}}$ is a base, too. □

Proof of Proposition 1.5. (Ad compactness of $[t, u]_{\overline{\mathbb{T}}}$): It is well-known that complete totally ordered lattices are compact. We nevertheless give a proof for the reader's convenience.

Let $t, u \in \overline{\mathbb{T}}$ and $\mathscr{C}$ be an open covering of $[t, u]_{\overline{\mathbb{T}}}$ in $\overline{\mathbb{T}}$, i.e. $\mathscr{C} \subseteq \mathscr{O}_{\overline{\mathbb{T}}}$ and $[t, u]_{\overline{\mathbb{T}}} \subseteq \bigcup \mathscr{C}$. We have to show that $\mathscr{C}$ admits a finite subcovering. By Lemma 1.4 and basic topology (Alexander subbase theorem), it suffices to consider the case where $\mathscr{C} \subseteq \mathscr{G}_{\overline{\mathbb{T}}}$. Let, in the complete lattice $\overline{\mathbb{T}}$,

$$a = \inf\{t' \in \mathbb{T} \mid (t', \infty]_{\overline{\mathbb{T}}} \in \mathscr{C}\}, \qquad b = \sup\{u' \in \mathbb{T} \mid [0, u')_{\overline{\mathbb{T}}} \in \mathscr{C}\}.$$

If $a < t$, then there is $t' \in \mathbb{T}$ with $t' < t$ such that $(t', \infty]_{\overline{\mathbb{T}}} \in \mathscr{C}$, whence the finite subcovering $[t, u]_{\overline{\mathbb{T}}} \subseteq (t', \infty]_{\overline{\mathbb{T}}}$. Similarly, if $u < b$, we get the a finite subcovering $[t, u]_{\overline{\mathbb{T}}} \subseteq [0, u')_{\overline{\mathbb{T}}} \in \mathscr{C}$ for some $u' \in \mathbb{T}$.

It remains to consider the case $t \leq a$ and $b \leq u$. We claim that $a < b$. As $b \leq u$ and $\mathscr{C}$ covers $[t, u]_{\overline{\mathbb{T}}}$, there is $t' \in \mathbb{T}$ such that $u \in (t', \infty]_{\overline{\mathbb{T}}} \in \mathscr{C}$, whence $a < u$. Thus $a \in [t, u]_{\overline{\mathbb{T}}}$, whence we infer – using the definition of a and the covering property of $\mathscr{C}$ – the existence of $u' \in \mathbb{T}$ such that $a \in [0, u')_{\overline{\mathbb{T}}} \in \mathscr{C}$. Hence, $a < b$. As a consequence, there are $t', u' \in \mathbb{T}$ with $a \leq t' < u' \leq b$ such that $[0, u')_{\overline{\mathbb{T}}}, (t', \infty]_{\overline{\mathbb{T}}} \in \mathscr{C}$. We then have $[t, u]_{\overline{\mathbb{T}}} \subseteq \overline{\mathbb{T}} = [0, u')_{\overline{\mathbb{T}}} \cup (t', \infty]_{\overline{\mathbb{T}}}$.

[61]... the empty intersection being equal to $\overline{\mathbb{T}}$.

(Ad "$\mathscr{K}_{\overline{\mathbb{T}}}(\mathbb{T})$ is an intersection-stable compact class"): As a total order, $\mathbb{T}$ is a lattice. Hence, $\mathscr{K}_{\overline{\mathbb{T}}}(\mathbb{T})$ is intersection-stable. It is a compact class, by basic topology, because its elements are compact with respect to the fixed topology $\mathscr{O}_{\overline{\mathbb{T}}}$.

(Ad $\mathscr{I}_{\overline{\mathbb{T}}}(\mathbb{T}) = \sigma(\mathscr{G}_{\overline{\mathbb{T}}}(\mathbb{T})) = \sigma(\mathscr{K}_{\overline{\mathbb{T}}}(\mathbb{T}))$): The first equality is the definition. For the second one, let $u, t \in \mathbb{T}$. Then, $\pi(t)$ is countable. Thus,

$$(u, \infty]_{\overline{\mathbb{T}}} = \overline{\mathbb{T}} \setminus [0, u]_{\overline{\mathbb{T}}}, \qquad [0, t)_{\overline{\mathbb{T}}} = \bigcup_{\gamma \in \pi(t)} \big[0, (p(t), \gamma)\big]_{\overline{\mathbb{T}}},$$

are elements of $\sigma(\mathscr{K}_{\overline{\mathbb{T}}}(\mathbb{T}))$. Conversely,

$$[t, u]_{\overline{\mathbb{T}}} = \overline{\mathbb{T}} \setminus \big([0, t)_{\overline{\mathbb{T}}} \cup (u, \infty]_{\overline{\mathbb{T}}}\big),$$

which is an element of $\sigma(\mathscr{G}_{\overline{\mathbb{T}}}(\mathbb{T}))$. We conclude that $\sigma(\mathscr{G}_{\overline{\mathbb{T}}}(\mathbb{T})) = \sigma(\mathscr{K}_{\overline{\mathbb{T}}}(\mathbb{T}))$. □

Proof of Corollary 1.6. We denote the four sets in the claim, ordered from left to right, by $\mathcal{M}_i$, $i = 1, \ldots, 4$. Clearly, all these four sets are intersection-stable and, by the preceding Proposition 1.5 and complement-stability, contained in $\mathscr{I}_{\overline{\mathbb{T}}}(\mathbb{T})$. It is also clear that $\mathscr{G}_{\overline{\mathbb{T}}}(\mathbb{T}) = \mathcal{M}_2 \cup \mathcal{M}_4$ and $\mathscr{K}_{\overline{\mathbb{T}}}(\mathbb{T}) \subseteq \sigma(\mathcal{M}_1 \cup \mathcal{M}_3)$. We conclude that

$$\mathscr{I}_{\overline{\mathbb{T}}}(\mathbb{T}) = \sigma(\mathcal{M}_1 \cup \mathcal{M}_3) = \sigma(\mathcal{M}_2 \cup \mathcal{M}_4).$$

It remains to show that $\sigma(\mathcal{M}_i) = \sigma(\mathcal{M}_{i+2})$, for both $i = 1, 2$. As $\sigma(\mathcal{M}_1) = \sigma(\mathcal{M}_4)$ and $\sigma(\mathcal{M}_2) = \sigma(\mathcal{M}_3)$, this is equivalent to proving $\sigma(\mathcal{M}_1) = \sigma(\mathcal{M}_2)$.

Let $u \in \mathbb{T}$. Note that

$$[0, u]_{\overline{\mathbb{T}}} = \big[0, (p(u), \pi(u) + 1)\big)_{\overline{\mathbb{T}}}.$$

This is an element of $\mathcal{M}_2$ since $\pi(u) + 1 < \mathfrak{w}_1$. Hence, $\mathcal{M}_1 \subseteq \mathcal{M}_2$, and thus $\sigma(\mathcal{M}_1) \subseteq \sigma(\mathcal{M}_2)$.

Regarding the converse inclusion, if $u = 0$, then $[0, u)_{\overline{\mathbb{T}}} = \emptyset$. If $u \in \mathbb{R}_+ \setminus \{0\}$, then there is a sequence $(u_n)_{n \in \mathbb{N}}$ valued in $[0, u)_{\mathbb{R}_+}$ converging to u, and we have

$$[0, u)_{\overline{\mathbb{T}}} = \bigcup_{n \in \mathbb{N}} [0, u_n]_{\overline{\mathbb{T}}}.$$

Else, $u \in \mathbb{T} \setminus \mathbb{R}_+$. Then, $\pi(u) > 0$ and

$$[0, u)_{\overline{\mathbb{T}}} = \bigcup_{\gamma \in \pi(u)} \big[0, (p(u), \gamma)\big]_{\overline{\mathbb{T}}},$$

a countable union. In all three cases, $[0, u)_{\overline{\mathbb{T}}} \in \sigma(\mathcal{M}_1)$. We conclude that $\sigma(\mathcal{M}_1) = \sigma(\mathcal{M}_2)$, completing the proof. □

Proof of Lemma 1.7. (Ad Part 1): This property is the subject of a classical exercise on product σ-algebras; its proof is sketched for the reader's convenience. Let $\alpha \in \mathfrak{w}_1$ and $\mathscr{A}^\alpha$ denote the σ-algebra on $\overline{\mathbb{T}}$ generated by $\mathscr{G}^\alpha_{\overline{\mathbb{T}},\times}$.

First, for fixed $C \in \mathscr{G}_{\alpha+1}$, consider the set $\mathscr{D}^\alpha_{\overline{\mathbb{R}_+}}$ of $D \in \mathscr{B}_{\overline{\mathbb{R}_+}}$ such that $(\rho^\alpha)^{-1}(D \times C) \in \mathscr{A}^\alpha$. By construction, $\mathscr{D}_{\overline{\mathbb{R}_+}}$ contains the intersection-stable generator $\mathscr{G}_{\overline{\mathbb{R}_+}}$ of $\mathscr{B}_{\overline{\mathbb{R}_+}}$. Moreover, it is a Dynkin system on $\overline{\mathbb{R}_+}$. Indeed, there is a countable subset $\mathscr{C}_{\overline{\mathbb{R}_+}} \subseteq \mathscr{G}_{\overline{\mathbb{R}_+}}$ with $\bigcup \mathscr{C}_{\overline{\mathbb{R}_+}} = \overline{\mathbb{R}_+}$, whence

$$(\rho^\alpha)^{-1}\big(\overline{\mathbb{R}_+} \times C\big) = \bigcup_{B \in \mathscr{C}_{\overline{\mathbb{R}_+}}} (\rho^\alpha)^{-1}\big(B \times C\big) \ \in \mathscr{A}^\alpha.$$

Stability under complements and countable (disjoint) unions is easily verified. Hence, by Dynkin's π-λ-theorem, $\mathscr{D}^\alpha_{\overline{\mathbb{R}_+}} = \mathscr{B}_{\overline{\mathbb{R}_+}}$.

Second, for fixed $B \in \mathscr{B}_{\overline{\mathbb{R}_+}}$ consider the set $\mathscr{D}_{\alpha+1}$ of $D \in \mathscr{B}_{\alpha+1}$ such that $(\rho^\alpha)^{-1}(B \times D) \in \mathscr{A}^\alpha$. By the first step, $\mathscr{D}_{\alpha+1}$ contains the intersection-stable generator $\mathscr{G}_{\alpha+1}$ of $\mathscr{B}_{\alpha+1}$. Moreover, it is a

Dynkin system on $\alpha+1$. Indeed, there is a countable subset $\mathscr{C}_{\alpha+1} \subseteq \mathscr{G}_{\alpha+1}$ with $\bigcup \mathscr{C}_{\alpha+1} = \alpha+1$, whence

$$(\rho^\alpha)^{-1}\big(B \times (\alpha+1)\big) = \bigcup_{C \in \mathscr{C}_{\alpha+1}} (\rho^\alpha)^{-1}\big(B \times C\big) \ \in \mathscr{A}^\alpha.$$

Stability under complements and countable (disjoint) unions is also easily verified. Hence, by Dynkin's π-λ-theorem, $\mathscr{D}_{\alpha+1} = \mathscr{B}_{\alpha+1}$.

As the set of products $B \times C$ ranging over $B \in \mathscr{B}_{\overline{\mathbb{R}_+}}$ and $C \in \mathscr{B}_{\alpha+1}$ generates $\mathscr{B}_{\overline{\mathbb{R}_+}\times(\alpha+1)}$, we infer that ρ^α is $\mathscr{A}^\alpha$-$\mathscr{B}_{\overline{\mathbb{R}_+}\times(\alpha+1)}$-measurable. By definition of $\mathscr{P}^\alpha_{\overline{\mathbb{T}}}$, we infer that $\mathscr{P}^\alpha_{\overline{\mathbb{T}}} \subseteq \mathscr{A}^\alpha$. On the other hand, we have $\mathscr{G}^\alpha_{\overline{\mathbb{T}},\times} \subseteq \mathscr{P}^\alpha_{\overline{\mathbb{T}}}$, whence $\mathscr{A}^\alpha \subseteq \mathscr{P}^\alpha_{\overline{\mathbb{T}}}$. As $\mathscr{G}^\alpha_{\overline{\mathbb{T}},\times}$ inherits intersection-stability from $\mathscr{G}_{\overline{\mathbb{R}_+}}$ and $\mathscr{G}_{\alpha+1}$, Claim 1 obtains.

(Ad Part 2): Let $\mathscr{A}$ denote the σ-algebra on $\overline{\mathbb{T}}$ generated by $\mathscr{G}_{\overline{\mathbb{T}},\times}$. By construction and Part 1, we have $\mathscr{P}^\alpha_{\overline{\mathbb{T}}} \subseteq \mathscr{A}$ for all $\alpha \in \mathfrak{w}_1$. Hence, by Part 1, for any $\alpha \in \mathfrak{w}_1$, ρ^α is $\mathscr{A}$-$\mathscr{B}_{\overline{\mathbb{R}_+}\times(\alpha+1)}$-measurable. By definition of $\mathscr{P}_{\overline{\mathbb{T}}}$, therefore, $\mathscr{P}_{\overline{\mathbb{T}}} \subseteq \mathscr{A}$. On the other hand, $\mathscr{G}_{\overline{\mathbb{T}},\times} \subseteq \mathscr{P}_{\overline{\mathbb{T}}}$, whence the converse inclusion $\mathscr{A} \subseteq \mathscr{P}_{\overline{\mathbb{T}}}$.

Regarding the second sentence, suppose that for all $\alpha, \beta \in \mathfrak{w}_1$ with $\alpha < \beta$ and all $C \in \mathscr{G}_{\alpha+1}$, we have either $\alpha \in C$ and $C \cup (\alpha,\beta]_{\mathfrak{w}_1} \in \mathscr{G}_{\beta+1}$, or $\alpha \notin C$ and $C \in \mathscr{G}_{\beta+1}$. Then, we claim that for all $\alpha,\beta \in \mathfrak{w}_1$ with $\alpha<\beta$, $\mathscr{G}^\alpha_{\overline{\mathbb{T}},\times} \subseteq \mathscr{G}^\beta_{\overline{\mathbb{T}},\times}$ which then implies the claim, because $\mathscr{G}^\beta_{\overline{\mathbb{T}},\times}$ is intersection-stable by Part 1. For proving the claim just made, let $\alpha,\beta \in \mathfrak{w}_1$ be such that $\alpha<\beta$ and let $B \in \mathscr{G}_{\overline{\mathbb{R}_+}}$ and $C \in \mathscr{G}_{\alpha+1}$. If $\alpha \notin C$, then $C \in \mathscr{G}_{\beta+1}$ and, thus $(\rho^\alpha)^{-1}(B\times C) = (\rho^\beta)^{-1}(B \times C) \in \mathscr{G}^\beta_{\overline{\mathbb{T}},\times}$. If $\alpha \in C$, then $C \cup (\alpha,\beta]_{\mathfrak{w}_1} \in \mathscr{G}_{\beta+1}$, and, thus, $(\rho^\alpha)^{-1}(B\times C) = (\rho^\beta)^{-1}(B \times (C \cup (\alpha,\beta]_{\mathfrak{w}_1})) \in \mathscr{G}^\beta_{\overline{\mathbb{T}},\times}$. This proves the claim.

(Ad Part 3): Suppose that $\mathscr{G}_Y$ consists of compact sets in Y for all $Y \in \{\overline{\mathbb{R}_+}\} \cup \mathfrak{w}_1$ and the only element B of $\mathscr{G}_{\overline{\mathbb{R}_+}}$ with $\infty \in B$ is $B = \{\infty\}$. To start, we recall that the product of two compact topological spaces is compact.[62] Hence, if $B \subseteq \overline{\mathbb{R}_+}$ and $C \subseteq \alpha+1$ are compact, for some $\alpha \in \mathfrak{w}_1$, then $B \times C$, equipped with product topology, is compact as well.

Let $B \in \mathscr{G}_{\overline{\mathbb{R}_+}}$, $\alpha \in \mathfrak{w}_1$, and $C \in \mathscr{G}_{\alpha+1}$. If $\infty \notin B$, then

$$(\rho^\alpha)^{-1}(B\times C) = \begin{cases} B \times C, & \text{if } \alpha \notin C, \\ B \times (C \cup [\alpha+1, \mathfrak{w}_1]_{\mathfrak{w}_1+1}), & \text{else,} \end{cases}$$

which is compact with respect to the product topology. If $\infty \in B$, then $B = \{\infty\}$, whence

$$(\rho^\alpha)^{-1}(B\times C) = \begin{cases} B \times \{0\}, & \text{if } 0 \in C, \\ \emptyset, & \text{else,} \end{cases}$$

which is equally compact. Hence, under the hypotheses made, all elements of $\mathscr{G}_{\overline{\mathbb{T}},\times}$ can be seen as compact subsets of $\overline{\mathbb{R}_+} \times (\mathfrak{w}_1+1)$ equipped with product topology. Therefore, $\mathscr{G}_{\overline{\mathbb{T}},\times}$, and also its subsets $\mathscr{G}^\alpha_{\overline{\mathbb{T}},\times}$, $\alpha \in \mathfrak{w}_1$, all yield compact classes. □

Proof of Lemma 1.9. Let

$$\mathscr{M}_\alpha = \Big\{ M \in \mathscr{P}_{\overline{\mathbb{T}}} \otimes \mathscr{E} \mid \forall x,y \in \overline{\mathbb{T}} \setminus \mathbb{T}_\alpha \forall \omega \in \Omega\colon \big[(x,\omega) \in M \text{ and } p(x)=p(y) \ \Rightarrow\ (y,\omega) \in M\big] \Big\}.$$

We have to show that $\mathscr{P}^\alpha_{\overline{\mathbb{T}}} \otimes \mathscr{E} \subseteq \mathscr{M}_\alpha$. In view of the definition of $\mathscr{P}^\alpha_{\overline{\mathbb{T}}}$ and basic measure theory, it suffices to show that:

1. for all $B \in \mathscr{B}_{\overline{\mathbb{R}_+}\times(\alpha+1)}$ and $E \in \mathscr{E}$, $(\rho^\alpha)^{-1}(B) \times E \in \mathscr{M}_\alpha$,

[62] See, e.g., [59, Appendix A1].

2. $\mathscr{M}_\alpha$ is a σ-algebra on $\overline{\mathbb{T}} \times \Omega$.

(Ad Statement 1): Let $B \in \mathscr{B}_{\overline{\mathbb{R}_+} \times (\alpha+1)}$ and $E \in \mathscr{E}$. Let $x, y \in \overline{\mathbb{T}} \setminus \mathbb{T}_\alpha$ and $\omega \in \Omega$ such that $(x, \omega) \in (\rho^\alpha)^{-1}(B) \times E$ and $p(x) = p(y)$. As $\pi(x), \pi(y) \geq \alpha$,

$$\rho^\alpha(x) = (p(x), \alpha) = (p(y), \alpha) = \rho^\alpha(y).$$

Hence $(y, \omega) \in (\rho^\alpha)^{-1}(B) \times E$.

(Ad Statement 2): Clearly, $\emptyset \in \mathscr{M}_\alpha$. Next, let $M \in \mathscr{M}_\alpha$. Then, $M^\complement \in \mathscr{P}_{\overline{\mathbb{T}}} \otimes \mathscr{E}$. For the proof, let $x, y \in \overline{\mathbb{T}} \setminus \mathbb{T}_\alpha$ and $\omega \in \Omega$ such that $(y, \omega) \notin M^\complement$. Thus, $(y, \omega) \in M$. If $p(x) = p(y)$, then $(x, \omega) \in M$, whence $(x, \omega) \notin M^\complement$. By contraposition, $M^\complement \in \mathscr{M}_\alpha$. Finally, let $(M_n)_{n\in\mathbb{N}}$ be an $\mathscr{M}_\alpha$-valued sequence and $M = \bigcup_{n\in\mathbb{N}} M_n$. Let $x, y \in \overline{\mathbb{T}} \setminus \mathbb{T}_\alpha$ and $\omega \in \Omega$ such that $(x, \omega) \in M$ and $p(x) = p(y)$. Then, there is $n \in \mathbb{N}$ such that $(x, \omega) \in M_n$. Hence, $(y, \omega) \in M_n$. Therefore, $(y, \omega) \in M$. The proof is complete. □

Proof of Proposition 1.10. The second equality a direct consequence of the definition of $\mathscr{P}^\alpha_{\overline{\mathbb{T}}}$, $\alpha \in \mathfrak{w}_1$, and the product σ-algebra. We therefore focus on the proof of the first equality.

(Ad "$\supseteq$" and monotonicity of the union): Let $\alpha, \beta \in \mathfrak{w}_1$ be such that $\alpha \leq \beta$. Then, by Lemma 1.7 and Example 1.8, we find generators $\mathscr{G}^\gamma_{\overline{\mathbb{T}}}$ of $\mathscr{P}^\gamma_{\overline{\mathbb{T}}}$, $\gamma \in \mathfrak{w}_1$, such that $\mathscr{G}^\alpha_{\overline{\mathbb{T}}} \subseteq \mathscr{G}^\beta_{\overline{\mathbb{T}}}$. Hence, $\mathscr{P}^\alpha_{\overline{\mathbb{T}}} \subseteq \mathscr{P}^\beta_{\overline{\mathbb{T}}}$. By construction, $\mathscr{P}^\beta_{\overline{\mathbb{T}}} \subseteq \mathscr{P}_{\overline{\mathbb{T}}}$. Then, it follows directly from the definition of the product σ-algebra – i.e. the smallest one making the set-theoretic projections measurable – that

$$\mathscr{P}_{\overline{\mathbb{T}}} \otimes \mathscr{E} \supseteq \mathscr{P}^\beta_{\overline{\mathbb{T}}} \otimes \mathscr{E} \supseteq \mathscr{P}^\alpha_{\overline{\mathbb{T}}} \otimes \mathscr{E}.$$

(Ad "$\subseteq$"): Let $\mathscr{A} = \bigcup_{\alpha\in\mathfrak{w}_1} \mathscr{P}^\alpha_{\overline{\mathbb{T}}} \otimes \mathscr{E}$. In view of Equation 1.9, and by basic measure theory, $\mathscr{A}$ contains a generator of $\mathscr{P}_{\overline{\mathbb{T}}} \otimes \mathscr{E}$, namely

$$\{(\rho^\alpha)^{-1}(B) \times E \mid \alpha \in \mathfrak{w}_1,\, B \in \mathscr{B}_{\overline{\mathbb{R}_+} \times (\alpha+1)},\, E \in \mathscr{E}\}.$$

It therefore suffices to show that $\mathscr{A}$ is a σ-algebra on $\overline{\mathbb{T}} \times \Omega$. As $\mathscr{P}^0_{\overline{\mathbb{T}}} \otimes \mathscr{E}$ is a σ-algebra on $\overline{\mathbb{T}} \times \Omega$, $\emptyset \in \mathscr{A}$. If $A \in \mathscr{A}$, then there is $\alpha \in \mathfrak{w}_1$ such that $A \in \mathscr{P}^\alpha_{\overline{\mathbb{T}}} \otimes \mathscr{E}$. Hence, $A^\complement \in \mathscr{P}^\alpha_{\overline{\mathbb{T}}} \otimes \mathscr{E} \subseteq \mathscr{A}$. Finally, let $(A_n)_{n\in\mathbb{N}}$ be an $\mathscr{A}$-valued sequence. For any $n \in \mathbb{N}$, there is minimal $\alpha_n \in \mathfrak{w}_1$ such that $A_n \in \mathscr{P}^{\alpha_n}_{\overline{\mathbb{T}}} \otimes \mathscr{E}$. Let $\alpha = \sup_{n\in\mathbb{N}} \alpha_n$. As the supremum of a countable set of countable ordinals, α is a countable ordinal as well, i.e. $\alpha \in \mathfrak{w}_1$. Using monotonicity of the union, proven in the first step above, we infer that, for all $n \in \mathbb{N}$, $A_n \in \mathscr{P}^\alpha_{\overline{\mathbb{T}}} \otimes \mathscr{E}$. Hence, $\bigcup_{n\in\mathbb{N}} A_n \in \mathscr{P}^\alpha_{\overline{\mathbb{T}}} \otimes \mathscr{E} \subseteq \mathscr{A}$. We conclude that $\mathscr{A}$ is a σ-algebra, thereby completing the proof. □

Proof of Corollary 1.11. (Ad inclusion): Let $\alpha \in \mathfrak{w}_1$. Let, in addition, $B = [0, t)_{\overline{\mathbb{R}_+}}$ and $C = \beta \setminus \{0\}$ for $t \in \mathbb{R}_+$ and $\beta \in \alpha + 1$. Then

$$(\rho^\alpha)^{-1}(B \times C) = B \times C = \bigcup_{x\in B} ((x, 0), (x, \beta))_{\overline{\mathbb{T}}} \in \mathscr{O}_{\overline{\mathbb{T}}} \subseteq \mathscr{B}_{\overline{\mathbb{T}}}.$$

Furthermore, for open real intervals B as above, we have

$$(\rho^\alpha)^{-1}(B \times (\alpha + 1)) = B \times (\mathfrak{w}_1 + 1) = \bigcap_{n\in\mathbb{N}} [0, t + 2^{-n})_{\overline{\mathbb{T}}} \in \mathscr{B}_{\overline{\mathbb{T}}}.$$

Moreover,

$$(\rho^\alpha)^{-1}(B \times \{0\}) = \begin{cases} (\rho^\alpha)^{-1}(B \times (\alpha + 1)), & \text{if } \alpha = 0, \\ B \times \{0\}, & \text{else.} \end{cases}$$

Hence, for B as above and any $\beta \in \alpha + 2$, we get

$$(\rho^\alpha)^{-1}(B \times \beta) \in \mathscr{B}_{\overline{\mathbb{T}}} \vee \sigma(B' \times \{0\} \mid B' \in \mathscr{B}_{\overline{\mathbb{R}_+}}).$$

Moreover, for $B = \{\infty\}$ and $C \in \alpha + 2$, we have

$$(\rho^\alpha)^{-1}(B \times C) = \begin{cases} \emptyset, & \text{if } C \cap \{0\} = \emptyset, \\ \{\infty\}, & \text{else,} \end{cases} \quad \in \mathscr{B}_{\overline{\mathbb{T}}}.$$

Let $\mathscr{G}_{\overline{\mathbb{R}_+}}$ be the set consisting of a) all intervals $[0,t)_{\overline{\mathbb{R}_+}}$ running over $t \in \mathbb{R}_+$ and b) the set $\{\infty\}$, and $\mathscr{G}_{\alpha+1} = \alpha + 2$, for any $\alpha \in \mathfrak{w}_1$. With the notations from Lemma 1.7, we have $\mathscr{G}_{\overline{\mathbb{T}},\times} \subseteq \mathscr{B}_{\overline{\mathbb{T}}} \vee \sigma(B' \times \{0\} \mid B' \in \mathscr{B}_{\overline{\mathbb{R}_+}})$. Hence, by Lemma 1.7, the claimed inclusion obtains.

(Ad inequality): Let $V \subseteq \mathbb{R}_+$ a non-Lebesgue-measurable set, and $U = V \times \{1\} \subseteq \overline{\mathbb{T}}$. Then,

$$U = \bigcup_{t \in V} (t,(t,2))_{\overline{\mathbb{T}}} \in \mathscr{O}_{\overline{\mathbb{T}}} \subseteq \mathscr{B}_{\overline{\mathbb{T}}}.$$

If we had $U \in \mathscr{P}_{\overline{\mathbb{T}}}$, then there would be $\alpha \in \mathfrak{w}_1$ with $U \in \mathscr{P}^\alpha_{\overline{\mathbb{T}}}$, by Equation 1.12 following Proposition 1.10. More precisely, $U = (\rho^\alpha)^{-1}(B)$ for some $B \in \mathscr{B}_{\overline{\mathbb{R}_+} \times (\alpha+1)}$. Then, $\alpha \geq 2$, and $B = U = V \times \{1\}$. As $\alpha+1$ is Polish, measurable projection onto $\mathbb{R}_+$ would imply that $V \in (\mathscr{B}_{\overline{\mathbb{R}_+}})^{\mathrm{u}}$ — which is false. We conclude that $U \notin \mathcal{P}_{\overline{\mathbb{T}}}$. □

Proof of Lemma 1.12. It suffices to show that $\mathscr{G}_{\overline{\mathbb{T}}}(\mathbb{T}) \subseteq \mathscr{P}_{\overline{\mathbb{T}}}$. Let $t \in \mathbb{T}$ and $\alpha = \pi(t)$. Then,

$$[0,t)_{\overline{\mathbb{T}}} = (\rho^0)^{-1}([0,p(t))_{\overline{\mathbb{R}_+}} \times 1) \cup (\rho^\alpha)^{-1}(\{p(t)\} \times \alpha),$$

which is an element of $\mathscr{P}^\alpha_{\overline{\mathbb{T}}} \subseteq \mathscr{P}_{\overline{\mathbb{T}}}$. Similarly,

$$(t,\infty]_{\overline{\mathbb{T}}} = (\rho^0)^{-1}((p(t),\infty]_{\overline{\mathbb{R}_+}} \times 1) \cup (\rho^{\alpha+1})^{-1}(\{p(t)\} \times \{\alpha+1\}),$$

which is an element of $\mathscr{P}^{\alpha+1}_{\overline{\mathbb{T}}} \subseteq \mathscr{P}_{\overline{\mathbb{T}}}$. □

Proof of Lemma 1.13. (Ad ι_α): Let $\alpha, \beta \in \mathfrak{w}_1$ and $B \in \mathscr{B}_{\overline{\mathbb{R}_+}}$, $C \subseteq \beta + 1$. Then,

$$\iota_\alpha^{-1}((\rho^\beta)^{-1}(B \times C)) = (\rho^\beta \circ \iota_\alpha)^{-1}(B \times C) = \begin{cases} B \times (C \cap (\alpha+1)), & \text{if } \alpha \leq \beta, \\ B \times C, & \text{if } \alpha > \beta,\ \beta \notin C, \\ B \times (C \cup (\beta,\alpha]_{\mathfrak{w}_1}), & \text{if } \alpha > \beta,\ \beta \in C. \end{cases}$$

In view of the definition of $\mathscr{P}_{\overline{\mathbb{T}}}$, we infer that ι_α is $\mathscr{B}_{\overline{\mathbb{R}_+} \times (\alpha+1)}|_{\overline{\mathbb{T}}_{\alpha+1}}$-$\mathscr{P}_{\overline{\mathbb{T}}}$-measurable.

(Ad p): Let $c \in \mathbb{R}_+$ with $c > 0$. Then, for all $\alpha \in \mathfrak{w}_1$,

$$p^{-1}([0,c)_{\overline{\mathbb{R}_+}}) = \bigcup_{n \in \mathbb{N}} [0, c(1-2^{-n})]_{\overline{\mathbb{T}}} = (\rho^\alpha)^{-1}([0,c)_{\overline{\mathbb{R}_+}} \times (\alpha+1)),$$

which is thus an element of both $\mathscr{I}_{\overline{\mathbb{T}}}(\mathbb{T})$ and $\mathscr{P}^\alpha_{\overline{\mathbb{T}}}$. Hence, p is measurable with respect to both σ-algebras. □

Continuous functions on $\overline{\mathbb{T}}$.

Proof of Lemma 1.14. If $t \in \overline{\mathbb{T}}$ is such that $\pi(t) = \beta + 1$ for some $\beta \in \mathrm{On}$, then for $u = (p(t), \beta)$, we have $(u,\infty]_{\overline{\mathbb{T}}} \cap [0,t)_{\overline{\mathbb{T}}} = \emptyset$. Thus, t is not a left-limit point. Moreover, $\overline{\mathbb{T}} \cap [0,0)_{\overline{\mathbb{T}}} = \emptyset$, hence 0 is not a left-limit point either.

Let $t \in \overline{\mathbb{T}} \setminus \{0\}$. Then, every neighbourhood U of t contains an open interval $(t_0,u_0)_{\overline{\mathbb{T}}}$, with $t_0 < t < u_0$, $t_0, u_0 \in \mathbb{T}$, by Lemma 1.4. If $\pi(t) = 0$, then $p(t_0) < p(t)$. Hence, there is $x \in \mathbb{R}_+$ with $p(t_0) < x < p(t)$, whence $x \in U \cap [0,t)_{\overline{\mathbb{T}}}$. Thus, t is a left-limit point. If $\pi(t)$ is a limit ordinal, then $u = (p(t_0), \pi(t_0) + 1) \in \mathbb{T}$ satisfies $t_0 < u < t$. Thus, t is a left-limit point.

∞ is clearly not a right-limit point, because $\overline{\mathbb{T}} \cap (\infty,\infty]_{\overline{\mathbb{T}}} = \emptyset$. Let $t \in \overline{\mathbb{T}} \setminus \{\infty\}$. If $\pi(t) < \mathfrak{w}_1$, then $u = (p(t), \pi(t)+1) \in \mathbb{T}$. Thus, $[0,u)_{\overline{\mathbb{T}}} \cap (t,\infty]_{\overline{\mathbb{T}}} = \emptyset$, and t is therefore not a right-limit point.

If $\pi(t) = \mathfrak{w}_1$, then, again, every neighbourhood U of t contains an open interval $(t_0, u_0)_{\overline{\mathbb{T}}}$, with $t_0 < t < u_0$, $t_0, u_0 \in \mathbb{T}$, by Lemma 1.4. Hence, $p(t) < p(u_0)$. Therefore, there is $x \in \mathbb{R}$ with $p(t) < x < p(u_0)$. Hence, $x \in U \cap (t, \infty]_{\overline{\mathbb{T}}}$. Therefore, t is a right-limit point. □

Proof of Lemma 1.15. Let $t \in \mathbb{R}_+$ and $y = f(t, \mathfrak{w}_1)$. As Y is metrisable, $\{y\} \in \mathscr{B}_Y$ and $f^{-1}(\{y\}) \in \mathscr{P}_{\overline{\mathbb{T}}}$, by hypothesis. By Proposition 1.10 applied to singleton Ω, there is $\alpha \in \mathfrak{w}_1$ with $f^{-1}(\{y\}) \in \mathscr{P}^{\alpha}_{\overline{\mathbb{T}}}$. By Lemma 1.9 applied to singleton Ω, $f(t, \beta) \in \{y\}$ for all $\beta \in [\alpha, \mathfrak{w}_1]_{\mathfrak{w}_1+1}$, and, in particular, $f(u) = y$ for all $u \in ((t, \alpha), (t, \mathfrak{w}_1)]_{\overline{\mathbb{T}}}$. Thus, f is left-constant at $(t, \mathfrak{w}_1)$.

In particular, f is left-continuous at $(t, \mathfrak{w}_1)$. □

Our proof of Proposition 1.16 is based on several lemmata. As explained in the proof below, it suffices to perform the main analysis under the assumption that $Y = \mathbb{R}$. The first lemma is about the cardinality of the set of left-jumps of làg functions, which generalises its well-known counterpart in classical theory. For this, let us note that for any làg function $f \colon \overline{\mathbb{T}} \to \mathbb{R}$, there is a function $\Delta_{\nearrow} f \colon \overline{\mathbb{T}} \to \mathbb{R}$ uniquely defined by $\Delta_{\nearrow} f(0) = 0$ and for $t \in \overline{\mathbb{T}} \setminus \{0\}$:

$$\Delta_{\nearrow} f(t) = \begin{cases} f(t) - f(p(t), \beta), & \text{if } \pi(t) = \beta + 1 \text{ for some } \beta \in \mathfrak{w}_1, \\ f(t) - \lim_{u \nearrow t} f(u), & \text{else.} \end{cases}$$

Lemma A.1. *Let $f \colon \overline{\mathbb{T}} \to \mathbb{R}$ be làg. Then, the set*

$$N = \{t \in \overline{\mathbb{T}} \mid \Delta_{\nearrow} f(t) \neq 0\} \tag{A.1}$$

is countable. Moreover, for all $t, u \in \overline{\mathbb{T}}$ with $t \leq u$ and $(t, u]_{\overline{\mathbb{T}}} \subseteq \overline{\mathbb{T}} \setminus N$, we have

$$p(t) = p(u) \qquad \Longrightarrow \qquad f(t) = f(u).$$

Proof of Lemma A.1. (Ad "N is countable"): Let $\varepsilon > 0$ and

$$N_\varepsilon = \{t \in \overline{\mathbb{T}} \mid |\Delta_{\nearrow} f(t)| \geq 2\varepsilon\}.$$

If N_ε were infinite, the axiom of choice would yield a strictly increasing sequence $(t_n)_{n \in \mathbb{N}}$ valued in N_ε and such that $(t_n, t_{n+1})_{\overline{\mathbb{T}}} \neq \emptyset$ for all $n \in \mathbb{N}$. Then, the axiom of choice would yield a sequence $(u_n)_{n \in \mathbb{N}^*}$ with $t_n < u_{n+1} < t_{n+1}$ for all $n \in \mathbb{N}$ such that $|f(t_n) - f(u_n)| \geq \varepsilon$ for all $n \in \mathbb{N}^*$, because $(t_n)_{n \in \mathbb{N}}$ would be N_ε-valued.

As both sequences would be increasing and $\overline{\mathbb{T}}$ is a complete lattice, they would converge in $\overline{\mathbb{T}}$, and, by choice of $(u_n)_{n \in \mathbb{N}}$, their limits coincide. Let $t = \lim_{n \to \infty} t_n = \lim_{n \to \infty} u_n$. $\pi(t)$ could not be a successor ordinal of the form $\beta + 1$, $\beta \in \mathrm{On}$, since then, for all but finitely many $n \in \mathbb{N}$, we would have $t_n \in ((p(t), \beta), \infty]_{\overline{\mathbb{T}}}$, that is, $t \leq t_n$, which contradicts the strict monotonicity of $(t_n)_{n \in \mathbb{N}}$. Moreover, by strict monotonicity of the considered sequences, $t > 0$.

Hence, by Lemma 1.14, t would be a left-limit point. By hypothesis, f would then have left-limits in t. In particular, for all but finitely many $n \in \mathbb{N}$, $|f(t_n) - f(u_n)| < \varepsilon$ – in contradiction to the choice of $(u_n)_{n \in \mathbb{N}}$. Thus, N_ε must be finite.

We conclude that

$$N = \bigcup_{n \in \mathbb{N}} N_{2^{-n}}$$

is countable.

(Ad second claim): Let $t, u \in \overline{\mathbb{T}}$ satisfy $t \leq u$ and $p(t) = p(u)$. We show that if $f(t) \neq f(u)$, then $(t, u]_{\overline{\mathbb{T}}} \cap N \neq \emptyset$.

If $f(t) \neq f(u)$, then the set

$$S = \{\alpha \in (\pi(t), \pi(u)]_{\mathfrak{w}_1+1} \mid f(t) \neq f(p(t), \alpha)\}$$

contains $\pi(u)$. Therefore, it has a minimum, which we denote by α. Clearly, $\pi(t) < \alpha$, and, for all $\beta \in [\pi(t), \alpha)_{\mathfrak{w}_1+1}$, we have $f(p(t), \beta) = f(t)$. We infer that $\Delta_{\nearrow} f(p(t), \alpha) = f(p(t), \alpha) - f(t) \neq 0$. Therefore, $(t, u]_{\overline{\mathbb{T}}} \cap N \neq \emptyset$. $\square$

For the next lemma, let us note that for any function $f\colon \overline{\mathbb{T}} \to \mathbb{R}$, N as in the preceding Lemma A.1, and $M = \mathcal{P}p(N)$, there is a map $\gamma\colon M \times (\mathfrak{w}_1 + 1) \to (\mathfrak{w}_1 + 1)$ satisfying, for any $t \in M$:

- $\gamma(t, 0) = 0$,
- for any ordinal $\alpha \in \mathfrak{w}_1$, $\gamma(t, \alpha + 1) = \inf N(t, \alpha)$ is the infimum of
$$N(t, \alpha) = \{\beta \in (\gamma(t, \alpha), \mathfrak{w}_1]_{\mathfrak{w}_1+1} \mid (t, \beta) \in N\}$$
in $\mathfrak{w}_1 + 1$,
- for any limit ordinal $\alpha \in \mathfrak{w}_1$:
$$\gamma(t, \alpha) = \sup_{\beta \in \alpha} \gamma(t, \beta).$$

For $t \in M$, let $D_t = \{\alpha \in \mathfrak{w}_1 + 1 \mid \gamma(t, \alpha) < \mathfrak{w}_1\}$. The aim of γ is to count the jumps of f.

Lemma A.2. *Let $f\colon \overline{\mathbb{T}} \to \mathbb{R}$ be làg, and left-continuous at $(t, \mathfrak{w}_1)$, for any $t \in \mathbb{R}_+$. Let N, M, γ, and $(D_t)_{t\in M}$ be defined as just before. Then, for all $t \in M$:*

1. *for all $\alpha, \beta \in \mathfrak{w}_1 + 1$ with $\alpha < \beta$ we have $\gamma(t, \alpha) \leq \gamma(t, \beta)$ with strict inequality iff $\alpha \in D_t$,*
2. *D_t is countable,*
3. *there is $\delta_t \in \mathfrak{w}_1$ with $D_t = \delta_t + 1$.*

Proof. Let $t \in M$.

(Ad Part 1): For the proof of monotonicity, we use transfinite induction. Let $S = \{\beta \in \mathfrak{w}_1 + 1 \mid \forall \alpha \in \beta\colon \gamma(t, \alpha) \leq \gamma(t, \beta)\}$. Clearly, $0 \in S$. If $\beta \in S$, then $\beta + 1 \in S$, because $\gamma(t, \beta) \leq \gamma(t, \beta + 1)$ by definition. If $\beta \in \mathfrak{w}_1 + 1$ is a limit ordinal such that $\beta \subseteq S$, and $\alpha \in \beta$, then $\gamma(t, \alpha) \leq \gamma(t, \alpha + 1) \leq \gamma(t, \beta)$, because $\alpha + 1 < \beta$ and by definition of γ. Hence, $S = \mathfrak{w}_1 + 1$.[63]

For the claim about strict inequality, let $\alpha, \beta \in \mathfrak{w}_1 + 1$ with $\alpha < \beta$. If $\gamma(t, \alpha) < \mathfrak{w}_1$, then, by definition of γ, $\gamma(t, \alpha) < \gamma(t, \alpha + 1) \leq \gamma(t, \beta)$. If $\gamma(t, \alpha) = \mathfrak{w}_1$, then, by monotonicity, $\gamma(t, \alpha) = \mathfrak{w}_1 = \gamma(t, \beta)$.

(Ad Part 2): Let $\gamma_t^* = \sup \mathcal{P}\pi(N \cap p^{-1}(t))$, which is countable. Let $\alpha \in D_t$. If α is not a limit ordinal, then clearly $\gamma(t, \alpha) \in \gamma_t^* + 1$. If α is a limit ordinal, then $\gamma(t, \alpha) = \sup\{\gamma(t, \beta + 1) \mid \beta \in \alpha\}$ – because, by definition of a limit ordinal, $\beta \in \alpha$ implies $\beta + 1 \in \alpha$ and $\gamma(t, .)$ is monotone. Hence, $\gamma(t, \alpha) \leq \gamma_t^*$, i.e. $\gamma(t, \alpha) \in \gamma_t^* + 1$. Therefore, and by Part 1, $\gamma(t, .)|_{D_t}$ defines an injection of D_t into the countable set $\gamma_t^* + 1$. We conclude that D_t is countable.

(Ad Part 3): Note that, by monotonicity of $\gamma(t, .)$, D_t is a downward closed subset of the ordinal $\mathfrak{w}_1$, hence an ordinal itself. As D_t is countable, therefore, $D_t \in \mathfrak{w}_1$. If D_t were a limit ordinal, then $\gamma(t, D_t) = \sup_{\beta \in D_t} \gamma(t, \beta)$ would be countable as the supremum of a countable set of countable ordinals, whence the contradiction $D_t \in D_t$.[64] Therefore, as $0 \in D_t$, D_t is a successor ordinal. Because D_t is countable, its predecessor δ_t is countable as well. $\square$

Given a function $f\colon \overline{\mathbb{T}} \to \mathbb{R}$, N, M, γ, $(D_t)_{t\in M}$, and $(\delta_t)_{t\in M}$ as defined before Lemma A.2, let $D = \bigcup_{t\in M} \{t\} \times D_t$, let $g\colon D \to \mathbb{T}$ be given by $g(t, \alpha) = (t, \gamma(t, \alpha))$, and define the set

$$\mathcal{N} = \big\{[g(t, \alpha), g(t, \alpha + 1))_{\overline{\mathbb{T}}} \mid t \in M,\ \alpha \in \delta_t\big\} \cup \big\{[g(t, \delta_t), (t, \mathfrak{w}_1)]_{\overline{\mathbb{T}}} \mid t \in M\big\}. \tag{A.2}$$

[63] Indeed, the principle of transfinite induction could be reformulated here by saying that: if this were not the case, then $(\mathfrak{w}_1 + 1) \setminus S$ would have a minimum, a contradiction to the afore-said.

[64] Recall that $D_t \in D_t$ is equivalent to $D_t < D_t$, and that in ZFC, a set cannot contain itself as an element.

So, g simply represents the jump counter γ in terms of the time half-axis $\overline{\mathbb{T}}$ (or more precisely, its subset $\mathbb{T}$).

Lemma A.3. *Let $f\colon \overline{\mathbb{T}} \to \mathbb{R}$ be làg, and left-continuous at $(t, \mathfrak{w}_1)$, for any $t \in \mathbb{R}_+$. Let N, M, and $\mathcal{N}$ be defined as just before. Then, the following statements hold true:*

1. *$\mathcal{N} \subseteq \mathscr{I}_{\overline{\mathbb{T}}}(\mathbb{T})$,*
2. *$\mathcal{N}$ is countable,*
3. *$\bigcup \mathcal{N} \in \mathscr{I}_{\overline{\mathbb{T}}}(\mathbb{T})$,*
4. *$\mathcal{N}$ is a partition of $\bigcup_{t\in M} p^{-1}(t)$, i.e. $\emptyset \notin \mathcal{N}$, $\mathcal{N}$ is disjoint and $\bigcup \mathcal{N} = \bigcup_{t\in M} p^{-1}(t)$,*
5. *for all $t \in \overline{\mathbb{T}} \setminus \bigcup \mathcal{N}$, $f(t) = f(p(t))$,*
6. *for any $S \in \mathcal{N}$ and all $t, u \in S$, $f(t) = f(u)$.*

Proof. For the proof, let $f\colon \overline{\mathbb{T}} \to \mathbb{R}$ be làg, and left-continuous at $(t, \mathfrak{w}_1)$, for any $t \in \mathbb{R}_+$, and let N, M, γ, $(D_t)_{t\in M}$, D, g, and $\mathcal{N}$ be defined as described above Lemmata A.2 and A.3. We use Lemma A.2 throughout the whole proof.

(Ad Part 1): It suffices to note that, by definition, g maps to $\mathbb{T}$ and that for any $t \in \mathbb{T}$ and $u \in \overline{\mathbb{T}}$ with $\pi(u) = \mathfrak{w}_1$, we have

$$[t, u]_{\overline{\mathbb{T}}} = \bigcap_{n\in\mathbb{N}} [t, p(u) + 2^{-n})_{\overline{\mathbb{T}}} \quad \in \mathscr{I}_{\overline{\mathbb{T}}}.$$

(Ad Part 2): N and D_t, for all $t \in M$, are countable. Thus, $M = \mathcal{P}p(N)$ is countable as well, and so is D. In particular, $\mathcal{N}$ is countable.

(Ad Part 3): This statement follows from the preceding Parts 1 and 2.

(Ad Part 4): For any $t \in M$, $g(t, .)$ is strictly increasing and maps into $\mathbb{T}$. Hence, $\emptyset \notin \mathcal{N}$. It is moreover clear that $\bigcup \mathcal{N} \subseteq \bigcup_{t\in M} p^{-1}(t)$. On the other hand, let $t \in M$ and $u \in p^{-1}(t)$. Let $\alpha = \{\beta \in D_t \mid 0 \leq \gamma(t, \beta) \leq \pi(u)\}$. By monotonicity of $\gamma(t, .)$, α is a downward closed subset of the ordinal D_t and thus an ordinal itself. We have $0 \in \alpha$ alias $\alpha > 0$. α cannot be a limit ordinal, because if it were, then $\gamma(t, \alpha) = \sup_{\beta\in\alpha} \gamma(t, \beta) \leq \pi(u)$, whence the contradiction $\alpha \in \alpha$. Hence, α is a successor ordinal. In particular, α has a maximum β and $\gamma(t, \beta) \leq \pi(u) < \gamma(t, \beta + 1)$. Thus, $u \in [g(t, \beta), g(t, \beta + 1))_{\overline{\mathbb{T}}}$ if $\beta < \delta_t$, and $u \in [g(t, \delta_t), (t, \mathfrak{w}_1)]_{\overline{\mathbb{T}}}$ if $\beta = \delta_t$. $\mathcal{N}$ is disjoint because $g(t, .) = (t, \gamma(t, .))$ is strictly monotone on D_t, and the fibres of p are disjoint.

(Ad Parts 5 and 6): These parts follow from the previous one by a direct application of Lemma A.1. Regarding Part 5, let $t \in \overline{\mathbb{T}} \setminus \bigcup \mathcal{N}$. For all $u \in (p(t), t]_{\overline{\mathbb{T}}}$, we have $p(u) = p(t) \notin M$, by Part 4, and, in particular, $u \notin N$. Thus, $(p(t), t]_{\overline{\mathbb{T}}} \cap N = \emptyset$, whence $f(u) = f(t)$ for all $u \in (p(t), t]_{\overline{\mathbb{T}}}$, by the mentioned lemma. Regarding Part 6, let $S \in \mathcal{N}$ and $t, u \in S$ with $t \leq u$, without loss of generality. By construction of S (based on the definition of g and γ), we have $p(t) = p(u)$ and there is no $x \in N$ with $t < x \leq u$, that is, $(t, u]_{\overline{\mathbb{T}}} \cap N = \emptyset$. Hence, by the mentioned lemma, $f(t) = f(u)$. ☐

Proof of Proposition 1.16. The main case $Y = \mathbb{R}$: We first assume that $Y = \mathbb{R}$. Let $f\colon \overline{\mathbb{T}} \to \mathbb{R}$ be làg, and left-continuous at $(t, \mathfrak{w}_1)$ for any $t \in \mathbb{R}_+$. Furthermore, let N, M, γ, $(D_t)_{t\in M}$, D, g, and $\mathcal{N}$ be defined as described above Lemmata A.2 and A.3.

(First argument): Let $h\colon \overline{\mathbb{R}_+} \to \mathbb{R}$ be defined by $h(0) = f(0)$ and, for all $t \in \overline{\mathbb{R}_+} \setminus \{0\}$, $h(t) = \lim_{u\nearrow t} f(u)$. Then, by adapting classical methods, it follows that h is left-continuous. Indeed, let $\varepsilon > 0$. Then, for any $t \in \overline{\mathbb{R}_+} \setminus \{0\}$, there is $a_t \in \overline{\mathbb{T}}$ with $a_t < t$ such that, for all $x \in (a_t, t)_{\overline{\mathbb{T}}}$, we have $|f(x) - h(t)| < \varepsilon/2$. As $t \in \overline{\mathbb{R}_+}$, $p(a_t) < p(t)$, and hence we can, upon making a_t larger, assume $a_t \in \overline{\mathbb{R}_+}$. Let us fix some $t \in \overline{\mathbb{R}_+} \setminus \{0\}$ and take arbitrary $x \in (a_t, t)_{\overline{\mathbb{R}_+}}$. If $x \notin N$, we have $h(x) = f(x)$ and thus $|h(x) - h(t)| < \varepsilon/2 < \varepsilon$. If $x \in N$, then there is $y \in [(a_x, x)_{\overline{\mathbb{R}_+}} \cap (a_t, t)_{\overline{\mathbb{R}_+}}] \setminus N$, because N is countable and $a_x, a_t \in \mathbb{R}_+$. Therefore, $|h(x) - h(t)| \leq |f(y) - h(x)| + |f(y) - h(t)| < \varepsilon/2 + \varepsilon/2 = \varepsilon$. We conclude that h is left-continuous.

(Second argument): As a left-continuous function $\overline{\mathbb{R}_+} \to \mathbb{R}$, h is $\mathscr{B}_{\overline{\mathbb{R}_+}}$-$\mathscr{B}_{\mathbb{R}}$-measurable. Hence, by Lemma 1.13, $h \circ p$ is $\mathscr{I}_{\overline{\mathbb{T}}}(\mathbb{T})$-$\mathscr{B}_{\mathbb{R}}$-measurable. By Lemma A.3, Parts 3, 4, and 5, $h \circ p(t) = f \circ p(t) = f(t)$ for all $t \in \overline{\mathbb{T}} \setminus \bigcup \mathcal{N}$, and $\overline{\mathbb{T}} \setminus \bigcup \mathcal{N} \in \mathscr{I}_{\overline{\mathbb{T}}}(\mathbb{T})$. Hence,

$$f\, 1_{\overline{\mathbb{T}} \setminus \bigcup \mathcal{N}} = h \circ p\, 1_{\overline{\mathbb{T}} \setminus \bigcup \mathcal{N}}$$

is $\mathscr{I}_{\overline{\mathbb{T}}}(\mathbb{T})$-$\mathscr{B}_{\mathbb{R}}$-measurable.

(Third argument): f is constant on any $S \in \mathcal{N}$, by Lemma A.3, Part 6. Hence, by Part 1 of the same lemma, $f 1_S$ is $\mathscr{I}_{\overline{\mathbb{T}}}(\mathbb{T})$-$\mathscr{B}_{\mathbb{R}}$-measurable for any $S \in \mathcal{N}$. Hence, using that $\mathcal{N}$ is a countable partition (Parts 2 and 4 of the aforementioned lemma), we infer that

$$f\, 1_{\bigcup \mathcal{N}} = \sum_{S \in \mathcal{N}} f\, 1_S$$

is $\mathscr{I}_{\overline{\mathbb{T}}}(\mathbb{T})$-$\mathscr{B}_{\mathbb{R}}$-measurable.

Combining the results of the second and third arguments, we infer that

$$f = f\, 1_{\overline{\mathbb{T}} \setminus \bigcup \mathcal{N}} + f\, 1_{\bigcup \mathcal{N}}$$

is $\mathscr{I}_{\overline{\mathbb{T}}}(\mathbb{T})$-$\mathscr{B}_{\mathbb{R}}$-measurable.

The case of general metrisable Y: Let Y be a metrisable topological space and $f\colon \overline{\mathbb{T}} \to Y$ be làg, and left-continuous at $(t, \mathfrak{w}_1)$ for any $t \in \mathbb{R}_+$. Let d_Y be a metric generating the topology on Y, let $\varepsilon > 0$ and $y \in Y$. It suffices to show that $f^{-1}(B_\varepsilon(y)) \in \mathscr{I}_{\overline{\mathbb{T}}}(\mathbb{T})$. For this, define

$$\tilde{f}\colon \overline{\mathbb{T}} \to \mathbb{R},\ t \mapsto d_Y(f(t), y).$$

By the triangular inequality, any $t \in \overline{\mathbb{T}}$ and $z \in Y$ satisfy

$$d_Y(f(t), y) \leq d_Y(f(t), z) + d_Y(z, y),$$

whence $|\tilde{f}(t) - d_Y(y, z)| \leq d_Y(f(t), z)$. We infer that $\tilde{f}$ is làg, and left-continuous at $(t, \mathfrak{w}_1)$ for any $t \in \mathbb{R}_+$. Thus, by the main case studied just above, $\tilde{f}$ is $\mathscr{I}_{\overline{\mathbb{T}}}(\mathbb{T})$-$\mathscr{B}_{\mathbb{R}}$-measurable. Therefore,

$$f^{-1}(B_\varepsilon(y)) = \tilde{f}^{-1}(B_\varepsilon(0)) \in \mathscr{I}_{\overline{\mathbb{T}}}(\mathbb{T}).$$

This completes the proof. □

Measurable projection and section.

Proof of Proposition 1.18. Let $M \in \mathscr{P}_{\overline{\mathbb{T}}} \otimes \mathscr{E}$. By Proposition 1.10, there is $\alpha \in \mathfrak{w}_1$ and $M_\alpha \in \mathscr{B}_{\overline{\mathbb{R}_+} \times (\alpha+1)} \otimes \mathscr{E}$ with $M = (\rho^\alpha \times \mathrm{id}_\Omega)^{-1}(M_\alpha)$. As $\overline{\mathbb{T}}_{\alpha+1} = \mathbb{T}_{\alpha+1} \cup \{\infty\} \in \mathscr{B}_{\overline{\mathbb{R}_+} \times (\alpha+1)}$ and $\operatorname{im} \rho^\alpha \subseteq \overline{\mathbb{T}}_{\alpha+1}$, we can assume without loss of generality that $M_\alpha \subseteq \overline{\mathbb{T}}_{\alpha+1} \times \Omega$. As $\rho^\alpha(t) = t$ for all $t \in \overline{\mathbb{T}}_{\alpha+1}$, we obtain $M_\alpha = M \cap (\overline{\mathbb{T}}_{\alpha+1} \times \Omega)$.

As $M_\alpha \subseteq M$, it is clear that $\mathcal{P}\mathrm{prj}_\Omega(M_\alpha) \subseteq \mathcal{P}\mathrm{prj}_\Omega(M)$. For the converse inclusion, let $\omega \in \mathcal{P}\mathrm{prj}_\Omega(M)$. Then, there is $t \in \overline{\mathbb{T}}$ such that $(t, \omega) \in M$. If $\pi(t) \leq \alpha$, then $(t, \omega) \in M_\alpha$, whence $\omega \in \mathcal{P}\mathrm{prj}_\Omega(M_\alpha)$. Else $\pi(t) > \alpha$. Then $t \in \overline{\mathbb{T}} \setminus \mathbb{T}_\alpha$ and $p(t) < \infty$. By Lemma 1.9, we infer that $u = (p(t), \alpha)$ satisfies $(u, \omega) \in M$. As $p(u) < \infty$ and $\pi(u) \leq \alpha$, we get $(u, \omega) \in M_\alpha$. Hence, $\omega \in \mathcal{P}\mathrm{prj}_\Omega(M_\alpha)$. We conclude that $\mathcal{P}\mathrm{prj}_\Omega(M) = \mathcal{P}\mathrm{prj}_\Omega(M_\alpha)$. □

Proof of Theorem 1.19. Let $M \in \mathscr{P}_{\overline{\mathbb{T}}} \otimes \mathscr{E}$. By Proposition 1.18, there is $\alpha \in \mathfrak{w}_1$ with

$$(*) \qquad M_\alpha = M \cap (\overline{\mathbb{T}}_{\alpha+1} \times \Omega) \ \in \mathscr{B}_{\overline{\mathbb{R}_+} \times (\alpha+1)} \otimes \mathscr{E}, \quad \text{and} \quad \mathcal{P}\mathrm{prj}_\Omega(M) = \mathcal{P}\mathrm{prj}_\Omega(M_\alpha).$$

As $\overline{\mathbb{R}_+} \times (\alpha+1)$ is a Polish space, the classical theorems on measurable projection and section apply to M_α and the canonical projection $(\overline{\mathbb{R}_+} \times (\alpha+1)) \times \Omega \to \Omega$, which is the restriction of $\mathrm{prj}_\Omega \colon \overline{\mathbb{T}} \times \Omega \to \Omega$. Thus, $\mathcal{P}\mathrm{prj}_\Omega(M) = \mathcal{P}\mathrm{prj}_\Omega(M_\alpha) \in \mathscr{E}^{\mathrm{u}}$, and there is $\mathscr{E}^{\mathrm{u}}|_{\mathcal{P}\mathrm{prj}_\Omega(M_\alpha)}$-$\mathscr{B}_{\overline{\mathbb{R}_+}\times(\alpha+1)}$-measurable $\sigma_\alpha \colon \mathcal{P}\mathrm{prj}_\Omega(M_\alpha) \to \overline{\mathbb{R}_+} \times (\alpha+1)$ such that $[\![\sigma_\alpha]\!] \subseteq M_\alpha$. In particular, σ_α is $\overline{\mathbb{T}}_{\alpha+1}$-valued.

Let $\sigma = \iota_\alpha \circ \sigma_\alpha$. Then, by Lemma 1.13, and another application of $(*)$, σ is $\mathscr{E}^{\mathrm{u}}|_{\mathcal{P}\mathrm{prj}_\Omega(M)}$-$\mathscr{P}_{\overline{\mathbb{T}}}$-measurable and $[\![\sigma]\!] \subseteq M$. This completes the proof. $\square$

A.2. **Section 2.**

Augmentation and right-limits of exogenous information flow.

Proof of Equation 2.2. This is a classical argument in stochastic analysis, added here for the reader's convenience only. By definition, $\overline{\mathscr{F}_{\infty+}} = \overline{\mathscr{F}_\infty} = \overline{\mathscr{F}}_\infty = \overline{\mathscr{F}}_{\infty+}$. It remains to show that, for all $t \in \overline{\mathbb{T}} \setminus \{\infty\}$, we have

$$\bigcap_{\mathbb{P}\in\mathfrak{P}_{\mathscr{E}}} \Big[\Big(\bigcap_{\mathbb{R}\ni u>t} \mathscr{F}_u\Big) \vee \mathscr{N}_{\mathbb{P}}\Big] = \bigcap_{\mathbb{R}\ni u>t}\bigcap_{\mathbb{P}\in\mathfrak{P}_{\mathscr{E}}} \Big(\mathscr{F}_u \vee \mathscr{N}_{\mathbb{P}}\Big).$$

To start, note that $(\bigcap_{\mathbb{R}\ni u>t} \mathscr{F}_u) \vee \mathscr{N}_{\mathbb{P}} \subseteq \mathscr{F}_v \vee \mathscr{N}_{\mathbb{P}}$ for all real $v > t$ and all $\mathbb{P} \in \mathfrak{P}_{\mathscr{E}}$. Whence the inclusion "$\subseteq$". For the other inclusion's proof, let $E \in \bigcap_{\mathbb{R}\ni u>t}\bigcap_{\mathbb{P}\in\mathfrak{P}_{\mathscr{E}}}(\mathscr{F}_u \vee \mathscr{N}_{\mathbb{P}})$ and $\mathbb{P} \in \mathfrak{P}_{\mathscr{E}}$. For $n \in \mathbb{N}$, let $u_n = p(t) + 2^{-n}$. Then, for any $n \in \mathbb{N}$, there is $E^{\mathbb{P}}_n \in \mathscr{F}_{u_n}$ such that $\mathbb{P}(E\Delta E^{\mathbb{P}}_n) = 0$. Let $E^{\mathbb{P}} = \limsup_{n\to\infty} E^{\mathbb{P}}_{u_n}$. Then, $E^{\mathbb{P}} \in \bigcap_{\mathbb{R}\ni u>t} \mathscr{F}_u$. As

$$\bigcap_{n\in\mathbb{N}} E^{\mathbb{P}}_{u_n} \subseteq E^{\mathbb{P}} \subseteq \bigcup_{n\in\mathbb{N}} E^{\mathbb{P}}_{u_n},$$

σ-subadditivity yields

$$\mathbb{P}(E\Delta E^{\mathbb{P}}) \le \sum_{n\in\mathbb{N}} \mathbb{P}(E\Delta E^{\mathbb{P}}_{u_n}) = 0.$$

Hence, $E \in (\bigcap_{\mathbb{R}\ni u>t} \mathscr{F}_u) \vee \mathscr{N}_{\mathbb{P}}$. As this holds true for any $\mathbb{P} \in \mathfrak{P}_{\mathscr{E}}$, we infer that

$$E \in \bigcap_{\mathbb{P}\in\mathfrak{P}_{\mathscr{E}}} \Big[\Big(\bigcap_{\mathbb{R}\ni u>t} \mathscr{F}_u\Big) \vee \mathscr{N}_{\mathbb{P}}\Big].$$

$\square$

Progressively measurable processes.

Sketch of a proof of Remark 2.3. Most claims can be proven by just copying the standard arguments from the classical theory of stochastic processes (as exposed, for example, in [44]). So we limit ourselves to explaining those points that require some additional thought.

Regarding Properties 1 and 2, the standard argument can directly by applied. Concerning Property 3, note that for any $t \in \overline{\mathbb{T}}$ with $\pi(t) < \mathfrak{w}_1$, $\pi(t)$ is countable. Hence, with $Q_t = [0,t)_{\overline{\mathbb{T}}} \cap \mathbb{Q}$, we have

$$\{\tau < t\} = \bigcup_{q\in Q_t} \{\tau \le q\} \cup \bigcup_{\beta\in\pi(t)} \{\tau \le (p(t),\beta)\} \in \mathscr{F}_t.$$

As a consequence, $\{\tau = t\} = \{\tau \le t\} \setminus \{\tau < t\} \in \mathscr{F}_t$.

Next, consider Claim 4. The fact that $\mathscr{F}_\tau$ is a sub-σ-algebra of $\mathscr{E}$ follows exactly as in the classical case. The claim on augmentedness has also nothing special in the vertically extended case, but it seems less standard; hence, we give the proof for the reader's convenience. It suffices to show that $\overline{\mathscr{F}_\tau} \subseteq \mathscr{F}_\tau$. Indeed, we have — using universal completeness of $\mathscr{E}$ and universal augmentedness of $\mathscr{F}$ in $\mathscr{E}$ —

$$\begin{aligned}
\overline{\mathscr{F}_\tau} &= \bigcap_{\mathbb{P}\in\mathfrak{P}_{\mathscr{E}}} \left\{F \in \mathscr{E} \mid \forall t \in \overline{\mathbb{T}}\colon F \cap \{\tau \le t\} \in \mathscr{F}_t\right\} \vee \mathscr{N}_{\mathbb{P}} \\
&= \bigcap_{\mathbb{P}\in\mathfrak{P}_{\mathscr{E}}} \Big\{E \in \mathscr{E} \vee \mathscr{N}_{\mathbb{P}} \mid \exists F \in \mathscr{E}\colon \big[\mathbb{P}(F\Delta E) = 0,\ \forall t \in \overline{\mathbb{T}}\colon F \cap \{\tau \le t\} \in \mathscr{F}_t\big]\Big\} \\
&= \Big\{E \in \underbrace{\mathscr{E}^{\mathrm{u}}}_{=\mathscr{E}} \mid \forall \mathbb{P} \in \mathfrak{P}_{\mathscr{E}} \exists F \in \mathscr{E}\colon \big[\mathbb{P}(F\Delta E) = 0,\ \forall t \in \overline{\mathbb{T}}\colon F \cap \{\tau \le t\} \in \mathscr{F}_t\big]\Big\} \\
&\subseteq \left\{E \in \mathscr{E} \mid \forall \mathbb{P} \in \mathfrak{P}_{\mathscr{E}} \forall t \in \overline{\mathbb{T}}\colon E \cap \{\tau \le t\} \in \mathscr{F}_t \vee \mathscr{N}_{\mathbb{P}}\right\} \\
&= \Big\{E \in \mathscr{E} \mid \forall t \in \overline{\mathbb{T}}\colon E \cap \{\tau \le t\} \in \underbrace{\bigcap_{\mathbb{P}\in\mathfrak{P}_{\mathscr{E}}} (\mathscr{F}_t \vee \mathscr{N}_{\mathbb{P}})}_{=\mathscr{F}_t} \Big\} \\
&= \mathscr{F}_\tau.
\end{aligned}$$

Properties 5 and 6 are shown as in the classical case. Property 7 follows as in the classical case, using the fact that $\mathscr{I}_{\overline{\mathbb{T}}}(\mathbb{T})$ is generated by all sets of the form $[0,t]_{\overline{\mathbb{T}}}$, $t \in \mathbb{T}$, according to Corollary 1.6. □

Proof of Example 2.4. (Ad Part 1): Let $t \in \mathbb{R}_+$. Then,

$$\{\tau \le t\} = \{\sigma < t\} \cup (\{\sigma = t\} \cap V) \in \mathscr{F}_t,$$

because $\{\sigma = t\} \cap V \subseteq \{\sigma = t\}$ and $\mathscr{F}$ and $\mathscr{E}$ are $\mathbb{P}$-complete. Moreover, for any $\alpha \in \mathfrak{w}_1 \setminus \{0\}$:

$$\{\tau \le (t,\alpha)\} = \{\sigma \le t\} \in \mathscr{F}_t = \mathscr{F}_{(t,\alpha)}.$$

Hence, τ is a $\mathscr{F}$-stopping time in $\overline{\mathbb{T}}$.

(Ad Part 2): From the preceding equations, we infer that $\mathbb{P}(\tau \le t) = \mathbb{P}(\sigma \le t)$ for all $t \in \mathbb{T}$. Hence, $\mathbb{P}_\tau = \mathbb{P}_\sigma$, by Corollary 1.6.

(Ad Part 3): This third claim is evident from the definition of τ.

(Ad Part 4): Suppose that $V \notin \mathscr{E}$. Then

$$\{\pi \circ \tau = 0\} = V \notin \mathscr{E},$$

and, in particular, $\{\pi \circ \tau = 0\} \notin \mathscr{F}_\tau$. As $\overline{\mathbb{R}_+} \times 1 \in \mathscr{P}_{\overline{\mathbb{T}}}$, τ is not $\mathscr{F}_\tau$-$\mathscr{P}_{\overline{\mathbb{T}}}$-measurable. This proves Claim 4(a).

Regarding Claim 4(b), note that

$$V = \mathrm{prj}_\Omega(\llbracket\tau\rrbracket \cap (\rho^1)^{-1}(\overline{\mathbb{R}_+} \times \{0\})).$$

As $V \notin \mathscr{E}$ and $\mathscr{E}$ is $\mathbb{P}$-complete, by the Measurable Projection Theorem 1.19, we obtain $\llbracket\tau\rrbracket \cap (\rho^1)^{-1}(\overline{\mathbb{R}_+} \times \{0\}) \notin \mathscr{P}_{\overline{\mathbb{T}}} \otimes \mathscr{E}$. Hence, $\llbracket\tau\rrbracket \notin \mathscr{P}_{\overline{\mathbb{T}}} \otimes \mathscr{E}$, and, in particular, $\llbracket\tau\rrbracket$ is not $\mathscr{F}$-progressively measurable.

Moreover, by classical theory, it follows that $\llbracket\sigma\rrbracket$ is $\mathscr{F}$-progressively measurable. Indeed, for every $x \in \overline{\mathbb{R}_+}$ and every $\alpha \in \mathfrak{w}_1$,

$$\begin{aligned}
(*) \qquad &\{(\sigma(\omega), \alpha, \omega) \mid \omega \in \Omega\colon \sigma(\omega) \le x\} \\
&= ([0,x]_{\overline{\mathbb{R}_+}} \times \{\alpha\} \times \Omega) \setminus \Big(\bigcup_{\mathbb{Q}\ni q<x} \big[([0,q)_{\overline{\mathbb{R}_+}} \times \{\alpha\} \times \{q < \sigma\}) \cup ((q,x]_{\overline{\mathbb{R}_+}} \times \{\alpha\} \times \{\sigma \le q\})\big]\Big) \\
&\in \mathscr{B}_{\overline{\mathbb{R}_+} \times (\alpha+2)} \times \mathscr{F}_x.
\end{aligned}$$

Taking the preimage of this under $\rho^2 \times \mathrm{id}_\Omega$, for every $t \in \overline{\mathbb{T}}$, with $x = p(t)$ and $\alpha = 0$, yields:

$$\llbracket\sigma\rrbracket \cap \llbracket 0, t\rrbracket = (\rho^2 \times \mathrm{id}_\Omega)^{-1}\Big(\{(\sigma(\omega), 0, \omega) \mid \omega \in \Omega\colon \sigma(\omega) \le x\}\Big) \in \mathscr{P}_{\overline{\mathbb{T}}} \times \mathscr{F}_x \subseteq \mathscr{P}_{\overline{\mathbb{T}}} \otimes \mathscr{F}_t.$$

Furthermore,

$$V^{\complement} = \mathrm{prj}_\Omega(\llbracket 0, \tau)\!\rangle \cap \llbracket\sigma\rrbracket).$$

As $V^{\complement} \notin \mathscr{E}$ and $\mathscr{E}$ is $\mathbb{P}$-complete, by the Measurable Projection Theorem 1.19, we infer $\llbracket 0, \tau)\!\rangle \cap \llbracket\sigma\rrbracket \notin \mathscr{P}_{\overline{\mathbb{T}}} \otimes \mathscr{E}$. Hence, $\llbracket 0, \tau)\!\rangle \notin \mathscr{P}_{\overline{\mathbb{T}}} \otimes \mathscr{E}$, and, in particular, $\llbracket 0, \tau)\!\rangle$ is not $\mathscr{F}$-progressively measurable.

Regarding $\langle\!(\tau, \infty\rrbracket$ a similar argument can be used. Let $\sigma'\colon \Omega \to \overline{\mathbb{T}}$, $\omega \mapsto (\sigma(\omega), 1)$. Taking the preimage of expression $(*)$ under $\rho^2 \times \mathrm{id}_\Omega$ for any $t \in \overline{\mathbb{T}}$, $x = p(t)$, and $\alpha = 1$, we get, if $\pi(t) > 0$:

$$\llbracket\sigma'\rrbracket \cap \llbracket 0, t\rrbracket = (\rho^3 \times \mathrm{id}_\Omega)^{-1}\Big(\{(\sigma(\omega), 1, \omega) \mid \omega \in \Omega\colon \sigma(\omega) \le x\}\Big) \in \mathscr{P}_{\overline{\mathbb{T}}} \times \mathscr{F}_x \subseteq \mathscr{P}_{\overline{\mathbb{T}}} \otimes \mathscr{F}_t.$$

If $\pi(t) = 0$, then

$$\llbracket\sigma'\rrbracket \cap \llbracket 0, t\rrbracket = \bigcup_{\mathbb{Q} \ni q < t} \llbracket\sigma'\rrbracket \cap \llbracket 0, (q, 1)\rrbracket \in \mathscr{P}_{\overline{\mathbb{T}}} \otimes \mathscr{F}_t.$$

Then, note that

$$V = \mathrm{prj}_\Omega(\langle\!(\tau, \infty\rrbracket \cap \llbracket\sigma'\rrbracket).$$

We conclude as above, using the Measurable Projection Theorem 1.19, that $\langle\!(\tau, \infty\rrbracket$ is not $\mathscr{F}$-progressively measurable. □

Optional times. We prepare the proof of Theorem 2.6 with a basic lemma. It is based on the classical argument showing progressive measurability of the converse graph of stopping times in standard real time. Let us introduce the following notation. If $\tau\colon \Omega \to \overline{\mathbb{T}}$ is a map and $\alpha \in \mathfrak{w}_1 + 1$, then let τ_α be the map $\Omega \to \overline{\mathbb{T}}$ satisfying $p \circ \tau = p \circ \tau_\alpha$ and $\pi(\tau_\alpha(\omega)) = \alpha$ for all $\omega \in \{\tau < \infty\}$.

Lemma A.4. *Let $\tau\colon \Omega \to \overline{\mathbb{T}}$ be a map satisfying $\{\tau < t\} \in \mathscr{F}_t$ for any $t \in \overline{\mathbb{R}_+}$.*[65] *The sets $\llbracket 0, \tau_{\mathfrak{w}_1}\rrbracket$ and $\langle\!(\tau_{\mathfrak{w}_1}, \infty\rrbracket$ are $\mathscr{F}$-progressively measurable with respect to the interval σ-algebra $\mathscr{I}_{\overline{\mathbb{T}}}(\mathbb{T})$.*

Proof. Let $t \in \overline{\mathbb{T}}$, and $Q_t = [0, p(t))_{\overline{\mathbb{R}_+}} \cap \mathbb{Q}$. Then, using the fact that Q_t is countable, and by Lemma 1.6:

$$\langle\!(\tau_{\mathfrak{w}_1}, \infty\rrbracket \cap ([0, t]_{\overline{\mathbb{T}}} \times \Omega) = \bigcup_{q \in Q_t} (q, t]_{\overline{\mathbb{T}}} \times \{\tau < q\} \in \mathscr{I}_{\overline{\mathbb{T}}}(\mathbb{T}) \otimes \mathscr{F}_t.$$

Hence, $\langle\!(\tau_{\mathfrak{w}_1}, \infty\rrbracket \in \mathrm{Prg}(\mathscr{I}_{\overline{\mathbb{T}}}(\mathbb{T}), \mathscr{F})$. As a consequence, by Remark 2.2, Item 2,

$$\llbracket 0, \tau_{\mathfrak{w}_1}\rrbracket = (\overline{\mathbb{T}} \times \Omega) \setminus \langle\!(\tau_{\mathfrak{w}_1}, \infty\rrbracket \in \mathrm{Prg}(\mathscr{I}_{\overline{\mathbb{T}}}(\mathbb{T}), \mathscr{F})$$

as well. □

Proof of Theorem 2.6. For a plan of the proof, see Figure 3. The completeness assumption is only made in the proof of implications "$k \Rightarrow 6$" for $k \in \{1, 2, 4\}$.

(Ad "2 ⇔ 3" and "4 ⇔ 5"): This follows directly from the fact that $\mathrm{Prg}(\mathscr{P}_{\overline{\mathbb{T}}}, \mathscr{F})$ is stable under complements (see Remark 2.2, Item 2).

(Ad 1 ⇒ 6): Suppose that $(\Omega, \mathscr{E}, \mathscr{F})$ is universally complete and that $\llbracket\tau\rrbracket$ is $\mathscr{F}$-progressively measurable. Then, as $\mathscr{F}_\infty \subseteq \mathscr{E}$, applying Relation 2.3 with $t = \infty$, basic measure theory yields $\llbracket\tau\rrbracket \in \mathscr{P}_{\overline{\mathbb{T}}} \otimes \mathscr{E}$. Hence, by Proposition 1.10, $\llbracket\tau\rrbracket \in \mathscr{P}^{\alpha}_{\overline{\mathbb{T}}} \otimes \mathscr{E}$ for some $\alpha \in \mathfrak{w}_1$. Then, $\pi \circ \tau \le \alpha$. Indeed, if there existed $\omega \in \Omega$ with $\pi \circ \tau(\omega) > \alpha$, Lemma 1.9 would imply that $(u, \omega) \in \llbracket\tau\rrbracket$ for all $u \in \overline{\mathbb{T}} \setminus \mathbb{T}_\alpha$ with $p(u) = p(\tau(\omega))$. $(\overline{\mathbb{T}} \setminus \mathbb{T}_\alpha) \cap p^{-1}(p(\tau(\omega))$ is infinite, but, as τ is a set-theoretic function, $\llbracket\tau\rrbracket \cap (\overline{\mathbb{T}} \times \{\omega\})$ is a singleton – whence a contradiction.

[65] By Remark 2.3, Item 3, any $\mathscr{F}$-stopping time satisfies this condition.

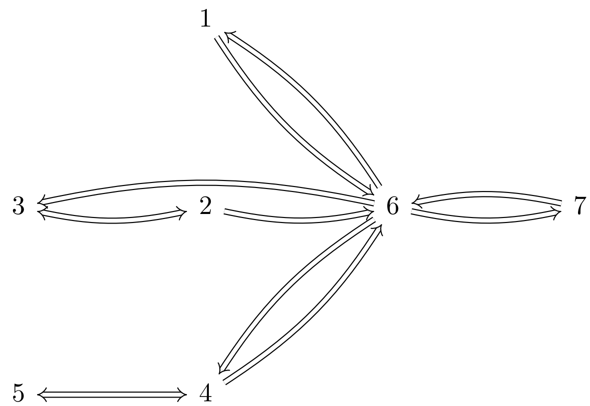

FIGURE 3. Plan of the proof of Theorem 2.6

For the remaining part of the claim, let $\beta \in \alpha+1$ and $t \in \overline{\mathbb{T}}$. Let $M_t = [\![\tau]\!] \cap ([0,t]_{\overline{\mathbb{T}}} \times \Omega)$. Then, by progressive measurability, $M_t \in \mathscr{P}_{\overline{\mathbb{T}}} \otimes \mathscr{F}_t$. As a consequence,

$$M_t^\beta = M_t \cap [(\rho^{\beta+1})^{-1}(\overline{\mathbb{R}_+} \times \{\beta\}) \times \Omega] \in \mathscr{P}_{\overline{\mathbb{T}}} \otimes \mathscr{F}_t.$$

By Theorem 1.19, and universal completeness of $\mathscr{F}_t$,

$$\{\pi \circ \tau = \beta,\, \tau \leq t\} = \mathcal{P}\mathrm{prj}_\Omega(M_t^\beta) \in \mathscr{F}_t.$$

(Ad 2 $\Rightarrow$ 6): Suppose that $(\Omega, \mathscr{E}, \mathscr{F})$ is universally complete and that $[\![0,\tau)\!)$ is $\mathscr{F}$-progressively measurable. As in the proof of the implication (1 $\Rightarrow$ 6) we get that $[\![0,\tau)\!) \in \mathscr{P}^\alpha_{\overline{\mathbb{T}}} \otimes \mathscr{E}$ for some $\alpha \in \mathfrak{w}_1$. We infer that $\pi \circ \tau \leq \alpha$. Indeed, if there existed $\omega \in \Omega$ with $\pi \circ \tau(\omega) > \alpha$, then, with $t = (p(\tau(\omega)), \alpha)$, we would have $(t,\omega) \in [\![0,\tau)\!)$. Hence, Lemma 1.9 would imply that, for $u = (p(\tau(\omega)), \mathfrak{w}_1)$, $(u,\omega) \in [\![0,\tau)\!)$, which is absurd.

Note that as a direct consequence of the hypothesis, $[\![\tau,\infty]\!]$ is also $\mathscr{F}$-progressively measurable. By Theorem 1.19, we infer that, for any $t \in \overline{\mathbb{T}}$:

$$\{\tau \leq t\} = \mathcal{P}\mathrm{prj}_\Omega\big([\![\tau,\infty]\!] \cap ([0,t]_{\overline{\mathbb{T}}} \times \Omega)\big) \in \mathscr{F}_t.$$

Thus, τ is an $\mathscr{F}$-stopping time, and satisfies the hypothesis of Lemma A.4, by Remark 2.3, Item 3.

For the remaining part of the claim, let $\beta \in \alpha+1$ and $t \in \overline{\mathbb{T}}$. Let $M_t = [\![\tau, \tau_{\mathfrak{w}_1}]\!] \cap ([0,t]_{\overline{\mathbb{T}}} \times \Omega)$. By hypothesis, and Lemma A.4, $M_t \in \mathscr{P}_{\overline{\mathbb{T}}} \otimes \mathscr{F}_t$. In particular,

$$M_t^\beta = M_t \cap [(\rho^{\beta+1})^{-1}(\overline{\mathbb{R}_+} \times \{\beta\}) \times \Omega] \in \mathscr{P}_{\overline{\mathbb{T}}} \otimes \mathscr{F}_t.$$

Hence, by Theorem 1.19, and universal completeness of $\mathscr{F}_t$,

$$\{\pi \circ \tau \leq \beta,\, \tau_\beta \leq t\} = \mathcal{P}\mathrm{prj}_\Omega(M_t^\beta) \in \mathscr{F}_t.$$

If $\beta \leq \pi(t)$, then

$$\{\pi \circ \tau \leq \beta,\, \tau \leq t\} = \{\pi \circ \tau \leq \beta,\, \tau_\beta \leq t\} \in \mathscr{F}_t.$$

Else, $\pi(t) < \beta$, so that $\pi(t)+1$ is countable. Let $Q_t = [0, p(t))_{\overline{\mathbb{R}_+}} \cap \mathbb{Q}$, a countable set as well. As $\beta \leq \mathfrak{w}_1$, Remark 2.3, Item 3, and the previous case yield:

$$\{\pi \circ \tau \leq \beta,\, \tau \leq t\} = \bigcup_{\gamma \in \pi(t)+1} \{\tau = (p(t),\gamma)\} \cup \bigcup_{q \in Q_t} \{\pi \circ \tau \leq \beta,\, \tau \leq (q, \mathfrak{w}_1)\} \in \mathscr{F}_t.$$

We conclude that

$$\{\pi\circ\tau=\beta,\,\tau\le t\}=\{\pi\circ\tau\le\beta,\,\tau\le t\}\setminus\bigcup_{\gamma\in\beta}\{\pi\circ\tau\le\gamma,\,\tau\le t\}\in\mathscr{F}_t,$$

because β is countable.

(Ad 4 $\Rightarrow$ 6): Suppose that $(\Omega,\mathscr{E},\mathscr{F})$ is universally complete and that $(\!(\tau,\infty]\!]$, or equivalently $[\![0,\tau]\!]$, is $\mathscr{F}$-progressively measurable and $\pi\circ\tau<\mathfrak{w}_1$. As in the proof of the implication (1 $\Rightarrow$ 6) we infer that $[\![0,\tau]\!]\in\mathscr{P}^{\alpha}_{\overline{\mathbb{T}}}\otimes\mathscr{E}$ for some $\alpha\in\mathfrak{w}_1$. We infer that $\pi\circ\tau\le\alpha$. Indeed, if there existed $\omega\in\Omega$ with $\pi\circ\tau(\omega)>\alpha$, then, with $t=(p(\tau(\omega)),\alpha)$, we would have $(t,\omega)\in[\![0,\tau]\!]$. Hence, Lemma 1.9 would imply that, for $u=(p(\tau(\omega)),\mathfrak{w}_1)$, $(u,\omega)\in[\![0,\tau]\!]$, whence $\pi\circ\tau(\omega)=\mathfrak{w}_1$ – in contradiction to the second part of the hypothesis.

By Theorem 1.19, we infer from the hypothesis that, for any $t\in\overline{\mathbb{R}_+}$:

$$\{\tau<t\}=\mathcal{P}\mathrm{prj}_\Omega\big((\!(\tau,\infty]\!]\cap([0,t]_{\overline{\mathbb{T}}}\times\Omega)\big)\in\mathscr{F}_t.$$

Thus, τ satisfies the hypothesis of Lemma A.4.

For the remaining part of the claim, let $\beta\in\mathfrak{w}_1$, $t\in\overline{\mathbb{T}}$, and $M_t=(\!(\tau,\tau_{\mathfrak{w}_1}]\!]\cap([0,t]_{\overline{\mathbb{T}}}\times\Omega)$. By hypothesis, and Lemma A.4, $M_t\in\mathscr{P}_{\overline{\mathbb{T}}}\otimes\mathscr{F}_t$. In particular,

$$M_t^\beta=M_t\cap[(\rho^{\beta+1})^{-1}(\overline{\mathbb{R}_+}\times\{\beta\})\times\Omega]\in\mathscr{P}_{\overline{\mathbb{T}}}\otimes\mathscr{F}_t.$$

Hence, by Theorem 1.19, and universal completeness of $\mathscr{F}_t$,

$$\{\pi\circ\tau<\beta,\,\tau_\beta\le t\}=\mathcal{P}\mathrm{prj}_\Omega(M_t^\beta)\in\mathscr{F}_t.$$

If $\beta\le\pi(t)$, then

$$\{\pi\circ\tau<\beta,\,\tau\le t\}=\{\pi\circ\tau<\beta,\,\tau_\beta\le t\}\in\mathscr{F}_t.$$

Else, $\pi(t)<\beta$, so that $\pi(t)+1$ is countable. Let $Q_t=[0,p(t))_{\overline{\mathbb{R}_+}}\cap\mathbb{Q}$, a countable set as well. As $\beta\le\mathfrak{w}_1$, Remark 2.3, Item 3, and the previous case yield:

$$\{\pi\circ\tau<\beta,\,\tau\le t\}=\bigcup_{\gamma\in\pi(t)+1}\{\tau=(p(t),\gamma)\}\cup\bigcup_{q\in Q_t}\{\pi\circ\tau<\beta,\,\tau\le(q,\mathfrak{w}_1)\}\in\mathscr{F}_t.$$

We conclude that

$$\{\pi\circ\tau=\beta,\,\tau\le t\}=\{\pi\circ\tau<\beta+1,\,\tau\le t\}\setminus\bigcup_{\gamma\in\beta+1}\{\pi\circ\tau<\gamma,\,\tau\le t\}\in\mathscr{F}_t,$$

because $\beta+1\in\mathfrak{w}_1$. In particular, this relation holds true if $\beta\in\alpha+1$.

(Ad 7 $\Rightarrow$ 6): Suppose Condition 7 to hold true. Let $\alpha\in\mathfrak{w}_1$ such that $\pi\circ\tau\le\alpha$. Hence, for all $\beta\in\alpha+1$ and $t\in\overline{\mathbb{T}}$, we have, by definition of $\mathscr{F}_\tau$ and $\mathscr{P}_{\overline{\mathbb{T}}}$,

$$\{\pi\circ\tau=\beta,\,\tau\le t\}=\big\{\rho^{\alpha+1}\circ\tau\in\overline{\mathbb{R}_+}\times\{\beta\}\big\}\cap\{\tau\le t\}\in\mathscr{F}_t.$$

(Ad 6 $\Rightarrow$ 7): Suppose Condition 6 to hold true. Let $\alpha\in\mathfrak{w}_1$ such that $\pi\circ\tau\le\alpha$. Then, $\alpha+1$ is countable, hence, for any $t\in\overline{\mathbb{T}}$, we have:

$$\{\tau\le t\}=\bigcup_{\beta\in\alpha+1}\{\pi\circ\tau=\beta,\,\tau\le t\}\in\mathscr{F}_t.$$

Hence, τ is an $\mathscr{F}$-stopping time. Note that, for all $\gamma\in\mathfrak{w}_1$, all $\beta\in\gamma+1$, and all $s\in\overline{\mathbb{R}_+}$, we have

$$\begin{aligned}\big\{\rho^\gamma\circ\tau\in[0,s]_{\overline{\mathbb{R}_+}}\times\{\beta\}\big\}&=\begin{cases}\{\pi\circ\tau=\beta,\,p\circ\tau\le s\}, & \text{if }\beta<\gamma,\\ \{\beta\le\pi\circ\tau\le\alpha,\,p\circ\tau\le s\}, & \text{else}\end{cases}\\ &=\begin{cases}\{\pi\circ\tau=\beta,\,\tau\le(s,\mathfrak{w}_1)\}, & \text{if }\beta<\gamma,\\ \{\beta\le\pi\circ\tau\le\alpha,\,\tau\le(s,\mathfrak{w}_1)\}, & \text{else.}\end{cases}\end{aligned}\tag{A.3}$$

Hence, for all $t \in \overline{\mathbb{T}}$, using that α is countable, we obtain:

$$\{\rho^\gamma \circ \tau \in [0,s]_{\overline{\mathbb{T}}} \times \{\beta\}\} \cap \{\tau \le t\} = \begin{cases} \{\pi \circ \tau = \beta,\, \tau \le (s,\mathfrak{w}_1) \wedge t\}, & \text{if } \beta < \gamma, \\ \{\beta \le \pi \circ \tau \le \alpha,\, \tau \le (s,\mathfrak{w}_1) \wedge t\}, & \text{else} \end{cases} \in \mathscr{F}_t.$$

By Lemma 1.7, and the hypothesis, it follows easily that τ is $\mathscr{F}_\tau$-$\mathscr{P}_{\overline{\mathbb{T}}}$-measurable.

(Ad 6 $\Rightarrow$ [1, 3, and 4]): Suppose that Property 6 is satisfied. By what has already been proven, Statement 7 then also holds. Let $\alpha \in \mathfrak{w}_1$ be such that $\pi \circ \tau \le \alpha$. Then, in particular, $\pi \circ \tau < \mathfrak{w}_1$.

In a *first step*, we show the following intermediate result.

Lemma A.5. *Let $\beta \in \alpha + 1$, $\Omega^\beta = \{\pi \circ \tau = \beta\}$, $\mathscr{F}^\beta = (\mathscr{F}^\beta_t)_{t \in \overline{\mathbb{R}_+}}$ be given by $\mathscr{F}^\beta_t = \mathscr{F}_{(t,\beta)}|_{\Omega^\beta}$ for $t \in \mathbb{R}_+$ and $\mathscr{F}^\beta_\infty = \mathscr{F}_\infty|_{\Omega^\beta}$, and $\tau^\beta = p \circ \tau|_{\Omega^\beta}$. Then, $\Omega^\beta \in \mathscr{E}$, $\mathscr{F}^\beta$ defines a filtration on $(\Omega^\beta, \mathscr{E}|_{\Omega^\beta})$ with classical real time half-axis $\overline{\mathbb{R}_+}$, and τ^β is an $\mathscr{F}^\beta$-stopping time in the classical sense, that is, for all $t \in \mathbb{R}_+$, we have:*

$$\{\omega \in \Omega^\beta \mid \tau^\beta(\omega) \le t\} \in \mathscr{F}^\beta_t.$$

Proof of Lemma A.5. First, note that, by Property 6:

$$\Omega^\beta = \{\pi \circ \tau = \beta,\, \tau \le \infty\} \in \mathscr{F}_\infty \subseteq \mathscr{E}.$$

As $\mathscr{F}^\beta$ clearly inherits the monotonicity from $\mathscr{F}$, we infer that $\mathscr{F}^\beta$ defines a filtration on $(\Omega^\beta, \mathscr{E}|_{\Omega^\beta})$ with classical real time half-axis $\overline{\mathbb{R}_+}$. Concerning the proof of the stopping time property, let $t \in \mathbb{R}_+$. Then,

$$\{\omega \in \Omega^\beta \mid \tau^\beta(\omega) \le t\} = \{\pi \circ \tau = \beta,\, \tau \le (t,\beta)\} \in \mathscr{F}_{(t,\beta)}|_{\Omega^\beta} = \mathscr{F}^\beta_t,$$

by Property 6. □

For the *second step* of Part "6 $\Rightarrow$ [1, 3, and 4]" in the proof of Theorem 2.6, let $\beta \in \alpha + 1$, and $t \in \overline{\mathbb{T}}$. Let $x = p(t)$, $Q_x = [0,x)_{\overline{\mathbb{R}_+}} \cap \mathbb{Q}$, and extend $\mathscr{F}$ so that $\mathscr{F}_{(x,\beta)} = \mathscr{F}_\infty$ in case $x = \infty$. Let us define the following sets, describing the graph, epigraph, strict epigraph of τ^β below x:

$$\begin{aligned} G^\beta_x &= \{(u,\omega) \in \overline{\mathbb{R}_+} \times \Omega^\beta \mid u = \tau^\beta(\omega)\} \cap ([0,x]_{\overline{\mathbb{R}_+}} \times \Omega^\beta), \\ E^\beta_x &= \{(u,\omega) \in \overline{\mathbb{R}_+} \times \Omega^\beta \mid \tau^\beta(\omega) \le u\} \cap ([0,x]_{\overline{\mathbb{R}_+}} \times \Omega^\beta), \\ \mathrm{s}E^\beta_x &= \{(u,\omega) \in \overline{\mathbb{R}_+} \times \Omega^\beta \mid \tau^\beta(\omega) < u\} \cap ([0,x]_{\overline{\mathbb{R}_+}} \times \Omega^\beta). \end{aligned}$$

In this step, we wish to analyse measurability properties of τ^β and of these sets in particular. By Lemma A.5, τ^β is an $\mathscr{F}^\beta$-stopping time in the classical sense. In particular,

$$\{\omega \in \Omega^\beta \mid \tau^\beta(\omega) < x\} = \bigcup_{q \in Q_x} \{\omega \in \Omega^\beta \mid \tau^\beta(\omega) \le q\} \in \bigvee_{q \in Q_x} \mathscr{F}^\beta_q \subseteq \mathscr{F}^\beta_x. \tag{A.4}$$

Moreover, classical theory (or Lemma A.4) implies that

$$G^\beta_x,\, E^\beta_x,\, \mathrm{s}E^\beta_x \in \mathscr{B}_{\overline{\mathbb{R}_+}} \otimes \mathscr{F}^\beta_x.$$

By basic measure theory, we have:

$$\mathscr{B}_{\overline{\mathbb{R}_+}} \otimes \mathscr{F}^\beta_x = \mathscr{B}_{\overline{\mathbb{R}_+}} \otimes \mathscr{F}_{(x,\beta)}|_{\Omega^\beta} \subseteq \big(\mathscr{B}_{\overline{\mathbb{R}_+}} \otimes \mathscr{F}_{(x,\beta)}\big)|_{\overline{\mathbb{R}_+} \times \Omega^\beta}.$$

Hence, there are $\tilde{G}^\beta_x, \tilde{E}^\beta_x, \widetilde{\mathrm{s}E}^\beta_x \in \mathscr{B}_{\overline{\mathbb{R}_+}} \otimes \mathscr{F}_{(x,\beta)}$ with

$$G^\beta_x = \tilde{G}^\beta_x \cap (\overline{\mathbb{R}_+} \times \Omega^\beta), \qquad E^\beta_x = \tilde{E}^\beta_x \cap (\overline{\mathbb{R}_+} \times \Omega^\beta), \qquad \mathrm{s}E^\beta_x = \widetilde{\mathrm{s}E}^\beta_x \cap (\overline{\mathbb{R}_+} \times \Omega^\beta).$$

Then, by hypothesis,

$$G^\beta_x = G^\beta_x \cap [\overline{\mathbb{R}_+} \times \{\tau \le (x,\beta)\}] = \tilde{G}^\beta_x \cap [\overline{\mathbb{R}_+} \times \{\pi \circ \tau = \beta,\, \tau \le (x,\beta)\}] \in \mathscr{B}_{\overline{\mathbb{R}_+}} \otimes \mathscr{F}_{(x,\beta)},$$

and similarly,

$$E_x^\beta = E_x^\beta \cap [\overline{\mathbb{R}_+} \times \{\tau \leq (x,\beta)\}] = \tilde{E}_x^\beta \cap [\overline{\mathbb{R}_+} \times \{\pi \circ \tau = \beta,\, \tau \leq (x,\beta)\}] \ \in \mathscr{B}_{\overline{\mathbb{R}_+}} \otimes \mathscr{F}_{(x,\beta)},$$

$$\mathrm{s}E_x^\beta = \mathrm{s}E_x^\beta \cap [\overline{\mathbb{R}_+} \times \{\tau \leq (x,\beta)\}] = \widetilde{\mathrm{s}E}_x^\beta \cap [\overline{\mathbb{R}_+} \times \{\pi \circ \tau = \beta,\, \tau \leq (x,\beta)\}] \ \in \mathscr{B}_{\overline{\mathbb{R}_+}} \otimes \mathscr{F}_{(x,\beta)}.$$

In particular, for any $\gamma \in \mathfrak{w}_1$ and any $S \in \mathscr{B}_{\gamma+1}$, we get that

$$S \times G_x^\beta,\ S \times E_x^\beta,\ S \times \mathrm{s}E_x^\beta \in \mathscr{B}_{\gamma+1} \otimes \mathscr{B}_{\overline{\mathbb{R}_+}} \otimes \mathscr{F}_{(x,\beta)}.$$

For any $\gamma \in \mathfrak{w}_1$, the map $f_\gamma \colon \overline{\mathbb{T}} \times \Omega \to (\gamma+1) \times \overline{\mathbb{R}_+} \times \Omega$, $(u,\omega) \mapsto (\pi \circ \rho^\gamma(u), p \circ \rho^\gamma(u), \omega)$ is a composition of suitably measurable transformations, and is therefore $\mathscr{P}_{\overline{\mathbb{T}}} \otimes \mathscr{F}_{(x,\beta)}$-$\mathscr{B}_{\gamma+1} \otimes \mathscr{B}_{\overline{\mathbb{R}_+}} \otimes \mathscr{F}_{(x,\beta)}$-measurable. Hence, for any $\gamma \in \mathfrak{w}_1$ and any $S \in \mathscr{B}_{\gamma+1}$,

$$f_\gamma^{-1}(S \times G_x^\beta),\ f_\gamma^{-1}(S \times E_x^\beta),\ f_\gamma^{-1}(S \times \mathrm{s}E_x^\beta) \ \in \mathscr{P}_{\overline{\mathbb{T}}} \otimes \mathscr{F}_{(x,\beta)}. \tag{A.5}$$

In a *third step*, we show that, for all $t \in \overline{\mathbb{T}}$, $x = p(t)$, and $Q_x = [0,x)_{\overline{\mathbb{R}_+}} \cap \mathbb{Q}$, we have the following two decompositions:

$$[\![\tau]\!] \cap ([0,t]_{\overline{\mathbb{T}}} \times \Omega) = \bigcup_{\substack{\beta \in \alpha+1: \\ \beta > \pi(t)}} \bigcup_{q \in Q_x} f_{\alpha+1}^{-1}(\{\beta\} \times G_q^\beta) \cup \bigcup_{\substack{\beta \in \alpha+1: \\ \beta \leq \pi(t)}} f_{\alpha+1}^{-1}(\{\beta\} \times G_x^\beta), \tag{A.6}$$

and

$$\begin{aligned} &[\![\tau,\infty]\!] \cap ([0,t]_{\overline{\mathbb{T}}} \times \Omega) \\ &= \bigcup_{\substack{\beta \in \alpha+1: \\ \beta > \pi(t)}} \Bigg[\bigcup_{q \in Q_x} \Big[f_\alpha^{-1}([(\alpha+1) \setminus \beta] \times E_q^\beta) \cup f_\alpha^{-1}(\beta \times \mathrm{s}E_q^\beta) \Big] \\ &\qquad \cup \Big[[x,t]_{\overline{\mathbb{T}}} \times \{\omega \in \Omega^\beta \mid \tau^\beta(\omega) < x\} \Big] \Bigg] \\ &\quad \cup \bigcup_{\substack{\beta \in \alpha+1: \\ \beta \leq \pi(t)}} \Bigg[\Big[f_\alpha^{-1}([(\alpha+1) \setminus \beta] \times E_x^\beta) \cap ([0,t]_{\overline{\mathbb{T}}} \times \Omega) \Big] \cup f_\alpha^{-1}(\beta \times \mathrm{s}E_x^\beta) \Bigg] \end{aligned} \tag{A.7}$$

For the proof of these two decompositions, let $t \in \overline{\mathbb{T}}$ and $(u,\omega) \in \overline{\mathbb{T}} \times \Omega$. We first prove Decomposition A.6. By definition, $(u,\omega) \in [\![\tau]\!] \cap ([0,t]_{\overline{\mathbb{T}}} \times \Omega)$ iff $u = \tau(\omega) \leq t$. This latter statement splits into two disjunct cases, generated by the alternative "$\pi(\tau(\omega)) > \pi(t)$" or "$\pi(\tau(\omega)) \leq \pi(t)$".

1. On the one hand, $u = \tau(\omega) \leq t$ and $\pi(\tau(\omega)) > \pi(t)$ both hold true iff there is $q \in \mathbb{Q}$ with $u = \tau(\omega) \leq q < p(t)$ and $\pi(u) > \pi(t)$. This is equivalent to the relation $\pi(t) < \pi(u) \leq \alpha$ and the existence of $q \in Q_x$ with $\omega \in \Omega^{\pi(u)}$ and $p(u) = \tau^{\pi(u)}(\omega) \leq q$, i.e. $(p(u),\omega) \in G_q^{\pi(u)}$. This in turn is equivalent to the existence of $q \in Q_x$ and $\beta \in \alpha+1$ with $\beta > \pi(t)$ such that $(u,\omega) \in f_{\alpha+1}^{-1}(\{\beta\} \times G_q^\beta)$.
2. On the other hand, $u = \tau(\omega) \leq t$ and $\pi(\tau(\omega)) \leq \pi(t)$ both hold true iff $p(u) = p \circ \tau(\omega) \leq p(t)$ and $\pi(u) = \pi \circ \tau(\omega) \leq \pi(t)$. This is equivalent to the conjunction of the relation $\pi(u) \leq \alpha \wedge \pi(t)$ and the statement that $\omega \in \Omega^{\pi(u)}$ and $p(u) = \tau^{\pi(u)}(\omega) \leq x$ hold true, i.e. $(p(u),\omega) \in G_x^{\pi(u)}$. This in turn is equivalent to the existence of $\beta \in \alpha+1$ with $\beta \leq \pi(t)$ such that $(u,\omega) \in f_{\alpha+1}^{-1}(\{\beta\} \times G_x^\beta)$.

The first decomposition, Equation A.6, is proven.

We continue with proving the second decomposition, Equation A.7. By definition, $(u,\omega) \in [\![\tau,\infty]\!] \cap ([0,t]_{\overline{\mathbb{T}}} \times \Omega)$ iff $\tau(\omega) \leq u \leq t$. This latter statement splits into four disjunct cases, generated by the following two alternatives: “$\pi(\tau(\omega)) > \pi(t)$” or “$\pi(\tau(\omega)) \leq \pi(t)$”; “$\pi(\tau(\omega)) \leq \pi(u)$” or “$\pi(\tau(\omega)) > \pi(u)$”.

1. First, $\tau(\omega) \leq u \leq t$, $\pi(\tau(\omega)) > \pi(t)$, and $\pi(\tau(\omega)) \leq \pi(u)$ all hold true iff $p \circ \tau(\omega) \leq p(u) < p(t)$ and $\pi(u) \geq \pi(\tau(\omega)) > \pi(t)$; a condition which is satisfied iff there is $q \in \mathbb{Q}$ with $p \circ \tau(\omega) \leq p(u) \leq q < x$ and $\pi(u) \geq \pi \circ \tau(\omega) > \pi(t)$. This is equivalent to the existence of $q \in Q_x$ and $\beta \in \alpha + 1$ with $\beta > \pi(t)$ such that $(p(u), \omega) \in E_q^\beta$ and $\pi(u) \geq \beta$, i.e. the fact that $(u, \omega) \in f_\alpha^{-1}([(\alpha+1) \setminus \beta] \times E_q^\beta)$.
2. Second, $\tau(\omega) \leq u \leq t$, $\pi(\tau(\omega)) > \pi(t)$, and $\pi(\tau(\omega)) > \pi(u)$ all hold true iff a) $p \circ \tau(\omega) < p(u) < p(t)$ and $\pi(u) \vee \pi(t) < \pi(\tau(\omega))$, or b) $p \circ \tau(\omega) < p(u) = p(t)$ and $\pi(u) \leq \pi(t) < \pi(\tau(\omega))$. Condition a) is equivalent to the existence of $q \in Q_x$ and $\beta \in \alpha+1$ with $\beta > \pi(t)$ such that $(p(u), \omega) \in \mathrm{s}E_q^\beta$ and $\pi(u) < \beta$, i.e. $f_\alpha(u, \omega) \in \beta \times \mathrm{s}E_q^\beta$, while condition b) is equivalent to $x \leq u \leq t$ and the existence of $\beta \in \alpha+1$ with $\beta > \pi(t)$ such that $\omega \in \Omega^\beta$ and $\tau^\beta(\omega) < x$.
3. Third, $\tau(\omega) \leq u \leq t$, $\pi(\tau(\omega)) \leq \pi(t)$, and $\pi(\tau(\omega)) \leq \pi(u)$ all hold true iff a) $p \circ \tau(\omega) \leq p(u) < p(t)$ and $\pi \circ \tau(\omega) \leq \pi(t) \wedge \pi(u)$, or b) $p \circ \tau(\omega) \leq p(u) = p(t)$ and $\pi \circ \tau(\omega) \leq \pi(u) \leq \pi(t)$. a) is equivalent to the existence of $q \in Q_x$ and $\beta \in \alpha+1$ with $\beta \leq \pi(t)$ such that $(p(u), \omega) \in E_q^\beta$ and $\pi(u) \geq \beta$, while b) is equivalent to the existence of $\beta \in \alpha+1$ with $\beta \leq \pi(t)$ such that $(p(u), \omega) \in E_x^\beta$, $\pi(u) \geq \beta$, and $u \in [x, t]_{\overline{\mathbb{T}}}$. Hence, the conjunction of a) and b) is equivalent to the existence of $\beta \in \alpha+1$ with $\beta \leq \pi(t)$ such that $f_\alpha(u, \omega) \in [(\alpha+1) \setminus \beta] \times E_x^\beta$ and $u \leq t$.
4. Fourth, $\tau(\omega) \leq u \leq t$, $\pi(\tau(\omega)) \leq \pi(t)$, and $\pi(\tau(\omega)) > \pi(u)$ all hold true iff $p \circ \tau(\omega) < p(u) \leq p(t)$, $\pi(u) < \pi \circ \tau(\omega) \leq \pi(t)$. This is equivalent to the existence of $\beta \in \alpha+1$ with $\beta \leq \pi(t)$ such that $(p(u), \omega) \in \mathrm{s}E_x^\beta$ and $\pi(u) < \beta$, i.e. $f_\alpha(u, \omega) \in \beta \times \mathrm{s}E_x^\beta$.

In a *fourth step*, we combine the second and third steps, namely the Statements A.6, A.7, A.5, and A.4, recall that, with the notation from the third step,

$$\{\omega \in \Omega^\beta \mid \tau^\beta(\omega) < x\} = \{\pi \circ \tau = \beta,\, \tau < x\} = \bigcup_{\mathbb{Q} \ni q < x} \{\pi \circ \tau = \beta,\, \tau \leq q\} \in \mathscr{F}_x,$$

and infer that $[\![\tau]\!]$ and $[\![\tau, \infty]\!]$ are $\mathscr{F}$-progressively measurable. By complement-stability of $\mathrm{Prg}(\mathscr{P}_{\overline{\mathbb{T}}}, \mathscr{F})$, then also

$$(\!(\tau, \infty]\!] = [\![\tau, \infty]\!] \setminus [\![\tau]\!]$$

is $\mathscr{F}$-progressively measurable. The proof is complete. □

Proof of Corollary 2.9. Under the completeness hypothesis, the map $f\colon \Omega \to \overline{\mathbb{T}} \times \Omega$, $\omega \mapsto (\tau(\omega), \omega)$ is $\mathscr{F}_\tau$-$\mathrm{Prg}(\mathscr{F})$-measurable. Indeed, for any $t \in \overline{\mathbb{T}}$ and $M \in \mathrm{Prg}(\mathscr{F})$, we then have $[\![\tau]\!] \cap M \cap [\![0, t]\!] \in \mathscr{P}_{\overline{\mathbb{T}}} \otimes \mathscr{F}_t$, by Theorem 2.6, and, by Theorem 1.19 — now using completeness —

$$f^{-1}(M) \cap \{\tau \leq t\} = \mathrm{prj}_\Omega([\![\tau]\!] \cap M \cap [\![0, t]\!]) \in \mathscr{F}_t.$$

Hence, $\xi_\tau = \xi \circ f$ is $\mathscr{F}_\tau$-$\mathscr{Y}$-measurable. □

Proof of Proposition 2.10. (Ad 1): Let $\sigma = p \circ \tau + \varepsilon$. Then, $\pi \circ \sigma = 0$. Moreover, we have

$$\{\pi \circ \sigma = 0,\, \sigma \leq \infty\} = \Omega \in \mathscr{F}_\infty = \mathscr{F}_{\infty+},$$

and, for all $t \in \overline{\mathbb{T}} \setminus \{\infty\}$:

$$\begin{aligned}\{\pi \circ \sigma = 0,\, \sigma \leq t\} &= \{p \circ \tau + \varepsilon \leq t\} \\ &= \{p \circ \tau + \varepsilon \leq p(t)\} \\ &= \{p \circ \tau \leq p(t) - \varepsilon\} \\ &= \{\tau \leq (p(t) - \varepsilon, \mathfrak{w}_1)\} \in \mathscr{F}_{(p(t)-\varepsilon, \mathfrak{w}_1)}.\end{aligned}$$

Note that $\mathscr{F}_{(p(t),\mathfrak{w}_1)} \subseteq \mathscr{F}_u$ for all real $u > t$. Hence, if $\varepsilon = 0$, $\{\pi\circ\sigma = 0,\, \sigma \le t\} \in \mathscr{F}_{t+}$. If $\varepsilon > 0$, then $\mathscr{F}_{(p(t)-\varepsilon,\mathfrak{w}_1)} \subseteq \mathscr{F}_t$.

(Ad 2): Let C be the set of $\omega \in \Omega$ such that $(\tau_n(\omega))_{n\in\mathbb{N}}$ converges in $\overline{\mathbb{R}_+}\times(\alpha+1)$. As $\pi|_{\overline{\mathbb{R}_+}\times(\alpha+1)}$ is continuous as a map $\overline{\mathbb{R}_+}\times(\alpha+1) \to (\alpha+1)$, we have $\pi\circ\tau(\omega) = \lim_{n\to\infty}\pi\circ\tau_n(\omega) \le \alpha$ for all $\omega \in C$. For all $\omega \in C^{\complement}$, we have $\pi\circ\tau(\omega) = \pi(\infty) = 0 \le \alpha$.

In view of Theorem 2.6, it remains to consider arbitrary $\beta \in \alpha+1$ and $t \in \overline{\mathbb{T}}$ and check the $\mathscr{F}_{t+}$-measurability of $\{\pi\circ\tau = \beta,\, \tau \le t\}$. In doing so, we use the following facts: a) convergence of a sequence in $\overline{\mathbb{R}_+}\times(\alpha+1)$ is equivalent to the convergence of the component sequences; b) convergence of a sequence $(x_n)_{n\in\mathbb{N}}$ in $\overline{\mathbb{R}_+}$ is equivalent to $(x_n)_{n\in\mathbb{N}}$ being Cauchy or the set $\{n \in \mathbb{N} \mid x_n \le \ell\}$ being finite for all $\ell \in \mathbb{N}$; c) for all $n \in \mathbb{N}$, $p\circ\tau_n$ is an $\mathscr{F}_{+}$-optional time (by Part 1 just proven before), and thus, by classical theory, $\{p\circ\tau_n \le p\circ\tau_m + \kappa\} \in (\mathscr{F}_{+})_{p\circ\tau_n}$, for all $m, n \in \mathbb{N}$, $\kappa \in \mathbb{R}_+$.

i) We first focus on the case $\beta \le \pi(t)$ and $t < \infty$. In this case, using Properties a), b), and c), we infer that all $m \in \mathbb{N}$ satisfy:

$$
\begin{aligned}
&\{\pi\circ\tau = \beta,\, \tau \le t\} = \{\pi\circ\tau = \beta,\, p\circ\tau \le p(t)\}\\
&= \begin{cases}
\begin{aligned}\bigcap_{\gamma\in\beta}\bigcap_{\ell=m}^{\infty}\bigcup_{k\in\mathbb{N}}\bigcap_{n,m=k}^{\infty}\Big(&\{\pi\circ\tau_n \in (\gamma,\beta],\, \tau_n \le p(t)+2^{-\ell}\}\\
&\cap\{p\circ\tau_n \le p\circ\tau_m + 2^{-\ell},\, p\circ\tau_n \le p(t)+2^{-\ell}\}\\
&\cap\{p\circ\tau_m \le p\circ\tau_n + 2^{-\ell},\, p\circ\tau_m \le p(t)+2^{-\ell}\}\Big),\end{aligned} & \text{if } \beta \text{ is a limit ordinal},\\
\begin{aligned}\bigcap_{\ell=m}^{\infty}\bigcup_{k\in\mathbb{N}}\bigcap_{n,m=k}^{\infty}\Big(&\{\pi\circ\tau_n = \beta,\, \tau_n \le p(t)+2^{-\ell}\}\\
&\cap\{p\circ\tau_n \le p\circ\tau_m + 2^{-\ell},\, p\circ\tau_n \le p(t)+2^{-\ell}\}\\
&\cap\{p\circ\tau_m \le p\circ\tau_n + 2^{-\ell},\, p\circ\tau_m \le p(t)+2^{-\ell}\}\Big),\end{aligned} & \text{else}
\end{cases}\\
&\in \mathscr{F}_{(p(t)+2^{-m})+}.
\end{aligned}
$$

As this holds true for all $m \in \mathbb{N}$, we obtain $\{\pi\circ\tau = \beta,\, \tau \le t\} \in \mathscr{F}_{t+}$.

ii) Second, we consider the case $\pi(t) < \beta$ and $t < \infty$. Then, by case i) just studied, we get

$$
\begin{aligned}
&\{\pi\circ\tau = \beta,\, \tau \le t\} = \{\pi\circ\tau = \beta,\, \tau < p(t)\}\\
&= \bigcup_{\substack{q\in\mathbb{Q}:\\ q<p(t)}} \{\pi\circ\tau = \beta,\, \tau \le (q,\beta)\} \in \bigvee_{\substack{q\in\mathbb{Q}:\\ q<p(t)}} \mathscr{F}_{(q,\beta)+} \subseteq \mathscr{F}_{p(t)} \subseteq \mathscr{F}_{t+}.
\end{aligned}
$$

iii) Finally, we consider the case $t = \infty$. Then, if $\beta > 0$, using results i) and ii) yield

$$
\{\pi\circ\tau = \beta,\, \tau \le t\}\cap C = \{\pi\circ\tau = \beta,\, \tau < \infty\}\cap C = \bigcup_{n\in\mathbb{N}}\{\pi\circ\tau = \beta,\, \tau \le n\} \in \mathscr{F}_\infty;
$$

and

$$
\{\pi\circ\tau = \beta,\, \tau \le t\}\setminus C = \{\pi\circ\tau = \beta,\, \tau = \infty\}\setminus C = \emptyset \in \mathscr{F}_\infty,
$$

whence $\{\pi\circ\tau = \beta,\, \tau \le t\} \in \mathscr{F}_\infty = \mathscr{F}_{\infty+}$. If $\beta = 0$, then, again by i),

$$
\begin{aligned}
&\{\pi\circ\tau = \beta,\, \tau \le t\}\cap C\\
&= \bigcup_{n\in\mathbb{N}}\{\pi\circ\tau = 0,\, \tau \le n\} \cup \Big(\liminf_{n\to\infty}\{\pi\circ\tau_n = 0\} \cap \bigcap_{k\in\mathbb{N}}\liminf_{n\to\infty}\{\tau_n > k\}\Big) \in \mathscr{F}_\infty;
\end{aligned}
$$

and

$$\begin{aligned}
&\{\pi\circ\tau=\beta,\,\tau\le t\}\setminus C\\
&=C^{\complement}\\
&=\Big[\bigcap_{\gamma\in(\alpha+1)\setminus\{0\}}\bigcap_{\delta\in\gamma}\big(\liminf_{n\to\infty}\{\delta<\pi\circ\tau_n\le\gamma\}\big)^{\complement}\cap\big(\liminf_{n\to\infty}\{\pi\circ\tau_n=0\}\big)^{\complement}\Big]\\
&\quad\cup\Bigg[\Big(\bigcup_{\ell\in\mathbb{N}}\bigcap_{k\in\mathbb{N}}\bigcup_{n,m=k}^{\infty}\{p\circ\tau_n\le p\circ\tau_m+2^{-\ell}\}^{\complement}\cup\{p\circ\tau_m\le p\circ\tau_n+2^{-\ell}\}^{\complement}\Big)\\
&\qquad\cap\Big(\bigcup_{\ell\in\mathbb{N}}\limsup_{n\to\infty}\{\tau_n\le\ell\}\Big)\Bigg]\quad\in\mathscr{F}_{\infty^{+}},
\end{aligned}$$

whence again $\{\pi\circ\tau=\beta,\,\tau\le t\}\in\mathscr{F}_{\infty^{+}}$.

(Ad 3): Let $t\in\overline{\mathbb{T}}$ and $Q_t=[0,t)_{\overline{\mathbb{T}}}\cap\mathbb{Q}$. Note that

$$\{p\circ\tau_0<p\circ\tau_1,\ \tau_1\le t\}=\bigcup_{q\in Q_t}\{\tau_0<q<\tau_1\le t\}\ \in\mathscr{F}_t,$$

and that

$$\{p\circ\tau_0=p\circ\tau_1,\,\tau_0\le t,\,\tau_1\le t\}=\Big[\bigcap_{q\in Q_t}\{\tau_0<q<\tau_1\}^{\complement}\cap\{\tau_1<q<\tau_0\}^{\complement}\Big]\cap\{\tau_0\le t,\,\tau_1\le t\}\ \in\mathscr{F}_t.$$

As both τ_0 and τ_1 are optional times, there is $\alpha\in\mathfrak{w}_1$ such that $\pi\circ\tau_k\le\alpha$ for both $k\in\{0,1\}$. Hence,

$$\begin{aligned}
&\{\tau_0\le\tau_1\}\cap\{\tau_1\le t\}\\
&=\{p\circ\tau_0<p\circ\tau_1,\,\tau_1\le t\}\cup\Big(\{p\circ\tau_0=p\circ\tau_1,\,\tau_0\le t,\,\tau_1\le t\}\cap\{\pi\circ\tau_0\le\pi\circ\tau_1,\,\tau_0\le t,\,\tau_1\le t\}\Big)\\
&=\{p\circ\tau_0<p\circ\tau_1,\,\tau_1\le t\}\cup\Big(\{p\circ\tau_0=p\circ\tau_1,\,\tau_0\le t,\,\tau_1\le t\}\\
&\qquad\cap\bigcup_{\beta\in\alpha+1}\bigcup_{\gamma\in\beta+1}\{\pi\circ\tau_0=\gamma,\,\tau_0\le t\}\cap\{\pi\circ\tau_1=\beta,\,\tau_1\le t\}\Big)\ \in\mathscr{F}_t.
\end{aligned}$$

Thus, $\{\tau_0\le\tau_1\}\in\mathscr{F}_{\tau_1}$. Similarly, we get

$$\begin{aligned}
&\{\tau_0<\tau_1\}\cap\{\tau_1\le t\}\\
&=\{p\circ\tau_0<p\circ\tau_1,\,\tau_1\le t\}\cup\Big(\{p\circ\tau_0=p\circ\tau_1,\,\tau_0\le t,\,\tau_1\le t\}\cap\{\pi\circ\tau_0<\pi\circ\tau_1,\,\tau_0\le t,\,\tau_1\le t\}\Big)\\
&=\{p\circ\tau_0<p\circ\tau_1,\,\tau_1\le t\}\cup\Big(\{p\circ\tau_0=p\circ\tau_1,\,\tau_0\le t,\,\tau_1\le t\}\\
&\qquad\cap\bigcup_{\beta\in\alpha+1}\bigcup_{\gamma\in\beta}\{\pi\circ\tau_0=\gamma,\,\tau_0\le t\}\cap\{\pi\circ\tau_1=\beta,\,\tau_1\le t\}\Big)\ \in\mathscr{F}_t.
\end{aligned}$$

Hence, $\{\tau_0<\tau_1\}\in\mathscr{F}_{\tau_1}$. By complement-stability, $\{\tau_1\le\tau_0\}\in\mathscr{F}_{\tau_1}$. Upon switching the roles of τ_0 and τ_1, we infer $\{\tau_0\le\tau_1\}\in\mathscr{F}_{\tau_0}$.

(Ad 4): For $k\in\{0,1\}$, let $\alpha_k\in\mathfrak{w}_1$ be such that $\pi\circ\tau_k\le\alpha_k$. Let $\alpha=\alpha_0\vee\alpha_1$. Hence, $\pi\circ(\tau_0\wedge\tau_1)\le\alpha$ and $\pi\circ(\tau_0\vee\tau_1)\le\alpha$.

Furthermore, let $\beta\in\alpha+1$ and $t\in\overline{\mathbb{T}}$. Then, using Part 3,

$$\begin{aligned}
&\{\pi\circ(\tau_0\wedge\tau_1)=\beta,\,\tau_0\wedge\tau_1\le t\}\\
&=\big(\{\tau_0\le\tau_1,\,\tau_0\le t\}\cap\{\pi\circ\tau_0=\beta,\,\tau_0\le t\}\big)\cup\big(\{\tau_1\le\tau_0,\,\tau_1\le t\}\cap\{\pi\circ\tau_1=\beta,\,\tau_1\le t\}\big)\ \in\mathscr{F}_t.
\end{aligned}$$

Similarly, using Part 3 again, we show that

$$\begin{aligned} &\{\pi\circ(\tau_0\vee\tau_1)=\beta,\,\tau_0\vee\tau_1\le t\}\\ =&\big(\{\tau_0\le\tau_1,\,\tau_1\le t\}\cap\{\pi\circ\tau_1=\beta,\,\tau_1\le t\}\big)\cup\big(\{\tau_1\le\tau_0,\,\tau_0\le t\}\cap\{\pi\circ\tau_0=\beta,\,\tau_0\le t\}\big)\in\mathscr{F}_t. \end{aligned}$$

□

Remark A.6. If $(\Omega,\mathscr{E},\mathscr{F})$ is universally complete, some of the proofs above can be simplified, by using Theorem 2.6. For example, Property 3 can be proven as follows.

Proof of Property 3 under universal completeness. Note that, for any $t\in\overline{\mathbb{T}}$:

$$\begin{aligned} \{\tau_0\le\tau_1\}\cap\{\tau_1\le t\}&=\mathrm{prj}_\Omega(\llbracket\tau_1\rrbracket\cap\llbracket\tau_0,t\rrbracket),\\ \{\tau_1<\tau_0\}\cap\{\tau_0\le t\}&=\mathrm{prj}_\Omega(\llbracket\tau_0\rrbracket\cap(\!(\tau_1,t\rrbracket). \end{aligned}$$

As τ_0 and τ_1 are $\mathscr{F}$-optional times, $\llbracket\tau_1\rrbracket\cap\llbracket\tau_0,\infty\rrbracket$ and $\llbracket\tau_0\rrbracket\cap(\!(\tau_1,t\rrbracket$ are $\mathscr{F}$-progressively measurable, by Theorem 2.6. Hence, their respective intersections with $\llbracket 0,t\rrbracket$ are elements of $\mathscr{P}_{\overline{\mathbb{T}}}\otimes\mathscr{F}_t$. Applying the Measurable Projection Theorem 1.19 and using the universal completeness assumption, we obtain that the left-hand side of the two equations above are elements of $\mathscr{F}_t$.

As this holds true for any $t\in\overline{\mathbb{T}}$, we infer $\{\tau_0\le\tau_1\}\in\mathscr{F}_{\tau_1}$ and $\{\tau_0\le\tau_1\}=\{\tau_1<\tau_0\}^{\complement}\in\mathscr{F}_{\tau_0}$. □

In addition, the following claims can be easily shown using universal completeness.

Proposition A.7. *Suppose that $(\Omega,\mathscr{E},\mathscr{F})$ is a universally complete and let $(\tau_n)_{n\in\mathbb{N}}$ be a sequence of $\mathscr{F}$-optional times. Then, the following statements holds true:*

5. *The scenariowise supremum $\sup_{n\in\mathbb{N}}\tau_n$ is an $\mathscr{F}$-optional time.*
6. *The scenariowise infimum $\sigma=\inf_{n\in\mathbb{N}}\tau_n$ is an $\mathscr{F}$-optional time iff $\bigcup_{n\in\mathbb{N}}\{\sigma=\tau_n\}=\Omega$.*

Proof. (Ad 5): We assume that $\tau=\sup_{n\in\mathbb{N}}\tau_n$. By Theorem 2.6, $\llbracket 0,\tau_n)\!)$ is $\mathscr{F}$-progressively measurable for any $n\in\mathbb{N}$. As

$$\llbracket 0,\tau)\!)=\bigcup_{n\in\mathbb{N}}\llbracket 0,\tau_n)\!),$$

$\llbracket 0,\tau)\!)$ is so, too. Using the completeness assumption and applying Theorem 2.6 again, we infer that τ is an $\mathscr{F}$-optional time.

(Ad 6): By Theorem 2.6, $(\!(\tau_n,\infty\rrbracket$ is $\mathscr{F}$-progressively measurable for any $n\in\mathbb{N}$. As

$$(\!(\sigma,\infty\rrbracket=\bigcup_{n\in\mathbb{N}}(\!(\tau_n,\infty\rrbracket,$$

$(\!(\sigma,\infty\rrbracket$ is so, too. If $\pi\circ\sigma<\mathfrak{w}_1$, then, using the completeness assumption and applying Theorem 2.6 again, we infer that σ is an $\mathscr{F}$-optional time. Conversely, if σ is an optional time, then $\pi\circ\sigma<\mathfrak{w}_1$, by the same theorem.

It remains to prove that

$$\bigcup_{n\in\mathbb{N}}\{\sigma=\tau_n\}=\{\pi\circ\sigma<\mathfrak{w}_1\}.$$

As $\pi\circ\tau_n<\mathfrak{w}_1$ for any $n\in\mathbb{N}$, the inclusion "$\subseteq$" obtains. For the proof of the inclusion "$\supseteq$", let $\omega\in\Omega$ satisfy $\pi\circ\sigma<\mathfrak{w}_1$. Then, $\{p\circ\tau_n(\omega)\mid n\in\mathbb{N}\}$ has a minimum x in $\overline{\mathbb{R}_+}$. Hence, $\{\pi\circ\tau_n(\omega)\mid n\in\mathbb{N}\colon p\circ\tau_n(\omega)=x\}$ has a minimum because $\mathfrak{w}_1$ is well-ordered. If n denotes an element of $\mathbb{N}$ that this minimum is attained at, then $\sigma(\omega)=\tau_n(\omega)$. □

Proof of Proposition 2.11. Let $\mathbb{P}\in\mathfrak{P}_{\mathscr{E}}$ and $\overline{\tau}$ be an $\overline{\mathscr{F}}$-optional time. Then, there is $\alpha\in\mathfrak{w}_1$ such that $\pi\circ\overline{\tau}\le\alpha$. For $n\in\mathbb{N}$, define

$$\overline{\tau}_n=\big(\inf\{k\in\mathbb{N}\mid\overline{\tau}\le k2^{-n}\}\cdot 2^{-n},\pi\circ\overline{\tau}\big).$$

Then, $\pi \circ \overline{\tau}_n \leq \alpha$. Moreover, for any $\beta \in \alpha + 1$ and $t \in \overline{\mathbb{T}}$, we have

$$\{\pi \circ \overline{\tau}_n = \beta,\, \overline{\tau}_n \leq t\} = \bigcup_{\substack{k \in \mathbb{N}:\\ (k2^{-n},\beta) \leq t}} \{\pi \circ \overline{\tau} = \beta,\, \overline{\tau} \leq k2^{-n}\} \in \overline{\mathscr{F}}_t.$$

Hence, $(\overline{\tau}_n)_{n\in\mathbb{N}}$ is a sequence of $\mathscr{F}$-optional times.

For $k, n \in \mathbb{N}$ and $\beta \in \alpha + 1$, let

$$\overline{E}^n_{k,\beta} = \{\overline{\tau}_n = (k2^{-n}, \beta)\}.$$

$\overline{E}^n_{k,\beta} \in \overline{\mathscr{F}}_{(k2^{-n},\beta)}$, because $\overline{\tau}_n$ is an $\overline{\mathscr{F}}$-optional time. Hence, there is a family $(E^n_{k,\beta})_{k,n\in\mathbb{N},\,\beta\in\alpha+1}$ of events such that, for all $k, n \in \mathbb{N}$ and $\beta \in \alpha + 1$:

$$E^n_{k,\beta} \in \mathscr{F}_{(k2^{-n},\beta)}, \qquad \mathbb{P}(E^n_{k,\beta} \Delta \overline{E}^n_{k,\beta}) = 0.$$

For each $n \in \mathbb{N}$, define

$$M^n = \bigcup_{k\in\mathbb{N}} \bigcup_{\beta\in\alpha+1} \{(k2^{-n}, \beta)\} \times E^n_{k,\beta}, \qquad \tau_n = D_{M^n}.$$

We clearly have $\pi \circ \tau_n \leq \alpha$. Moreover, for any $k \in \mathbb{N}$ and $\beta \in \alpha + 1$, we have

$$\{\tau_n = (k2^{-n}, \beta)\} = E^n_{k,\beta} \setminus \Big(\bigcup_{\substack{\ell\in\mathbb{N},\,\gamma\in\alpha+1:\\ (\ell 2^{-n},\gamma)<(k2^{-n},\beta)}} E^n_{\ell,\gamma} \Big),$$

which is an element of $\mathscr{F}_{(k2^{-n},\beta)}$. As a consequence,

$$\{\tau_n = \infty\} = \bigcap_{k\in\mathbb{N},\,\beta\in\alpha+1} \{\tau_n = (k2^{-n}, \beta)\}^{\complement} \in \mathscr{F}_\infty.$$

Hence, for any $\beta \in \alpha + 1$ and all $t \in \overline{\mathbb{T}}$, we have — with the understanding that $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ and $\infty \cdot 2^{-n} = \infty$ —

$$\{\pi \circ \tau_n = \beta,\, \tau_n \leq t\} = \bigcup_{\substack{k\in\overline{\mathbb{N}}:\\ (k2^{-n},\beta)\leq t}} \{\tau_n = (k2^{-n}, \beta)\} \in \mathscr{F}_t.$$

Hence, by Theorem 2.6 (the claims without completeness assumption), τ_n is an $\mathscr{F}$-optional time. Moreover,

$$\begin{aligned}\mathbb{P}(\tau_n \neq \overline{\tau}_n,\, \tau_n < \infty) &\leq \sum_{k\in\mathbb{N}} \sum_{\beta\in\alpha+1} \mathbb{P}(\{\tau_n = (k2^{-n}, \beta)\} \setminus \overline{E}^n_{k,\beta}) \\ &\leq \sum_{k\in\mathbb{N}} \sum_{\beta\in\alpha+1} \mathbb{P}(E^n_{k,\beta} \setminus \overline{E}^n_{k,\beta}).\end{aligned}$$

Noting that

$$\{\tau_n < \infty\} = \bigcup_{k\in\mathbb{N},\,\beta\in\alpha+1} E^n_{k,\beta}, \qquad \{\overline{\tau}_n < \infty\} = \bigcup_{k\in\mathbb{N},\,\beta\in\alpha+1} \overline{E}^n_{k,\beta},$$

we obtain

$$\begin{aligned}\mathbb{P}(\tau_n \neq \overline{\tau}_n,\, \tau_n = \infty) &= \mathbb{P}(\overline{\tau}_n < \infty,\, \tau_n = \infty) \\ &\leq \sum_{k\in\mathbb{N}} \sum_{\beta\in\alpha+1} \mathbb{P}(\overline{E}^n_{k,\beta} \cap \{\tau_n = \infty\}) \\ &\leq \sum_{k\in\mathbb{N}} \sum_{\beta\in\alpha+1} \mathbb{P}(\overline{E}^n_{k,\beta} \setminus E^n_{k,\beta}).\end{aligned}$$

Hence,

$$\mathbb{P}(\tau_n \neq \overline{\tau}_n) \leq \sum_{k\in\mathbb{N}} \sum_{\beta\in\alpha+1} \mathbb{P}(\overline{E}^n_{k,\beta} \Delta E^n_{k,\beta}) = 0.$$

Thus, $\mathbb{P}(\tau_n = \overline{\tau}_n) = 1$. Let τ be the map satisfying $\tau(\omega) = \lim_{n\to\infty} \tau_n(\omega)$ for all $\omega \in \Omega$ that this limit exists for, and $\tau(\omega) = \infty$ otherwise. By Proposition 2.10, τ is an $\mathscr{F}_{\boldsymbol{+}}$-optional time. Since $\overline{\tau}_n \to \overline{\tau}$ as $n \to \infty$ pointwise in $\overline{\mathbb{R}_+} \times (\alpha+1)$, we have:

$$\{\tau \neq \overline{\tau}\} \subseteq \bigcup_{n\in\mathbb{N}} \{\tau_n \neq \overline{\tau}_n\}.$$

This leads to

$$\mathbb{P}(\tau \neq \overline{\tau}) \leq \sum_{n\in\mathbb{N}} \mathbb{P}(\tau_n \neq \overline{\tau}_n) = 0,$$

whence $\mathbb{P}(\tau = \overline{\tau}) = 1$. □

Proof of Theorem 2.12. (Ad first statement): Let $t \in \overline{\mathbb{T}}$. Then, by progressive measurability, $M \cap [\![0,t]\!] \in \mathscr{P}_{\overline{\mathbb{T}}} \otimes \mathscr{F}_t$. Hence, if t is not a right-limit point, or equivalently, $\pi(t) < \mathfrak{w}_1$ (Lemma 1.14), measurable projection (Theorem 1.19) yields

$$\{D_M \leq t\} = \mathcal{P}\mathrm{prj}_\Omega\Big(M \cap [\![0,t]\!]\Big) \in (\mathscr{F}_t)^{\mathrm{u}} \subseteq \overline{\mathscr{F}}_t = \overline{\mathscr{F}}_{t+}, \tag{A.8}$$

because the π-fibres of $\overline{\mathbb{T}}$ are well-ordered and $\mathscr{F}_t$ is universally complete. As a consequence, if t is a right-limit point, or equivalently, $\pi(t) = \mathfrak{w}_1$, i.e. $t = (p(t), \mathfrak{w}_1)$, then

$$\{D_M \leq t\} = \bigcap_{k\in\mathbb{N}} \bigcap_{\ell=k}^{\infty} \{D_M \leq p(t) + 2^{-\ell}\} \in \bigcap_{k\in\mathbb{N}} \overline{\mathscr{F}}_{p(t)+2^{-k}} = \overline{\mathscr{F}}_{t+},$$

because $p(t) + 2^{-\ell} \in \mathbb{R}_+$ for all $\ell \in \mathbb{N}$, which is therefore not a right-limit point (Lemma 1.14), so that Equation A.8 applies.

(Ad second statement): As M is $\mathscr{F}$-progressively measurable, there is $\alpha \in \mathfrak{w}_1$ such that $M \in \mathscr{P}^{\alpha}_{\overline{\mathbb{T}}} \otimes \mathscr{E}$. Therefore, for all $t \in \overline{\mathbb{T}}$ with $\pi(t) > \alpha$ we have:

$$(*) \qquad (t,\omega) \in M \implies (p(t),\alpha,\omega) \in M,$$

by Lemma 1.9.

Regarding the implication "1 ⇒ 2" we simply note that, if D_M is an optional time, then $\pi \circ D_M < \mathfrak{w}_1$ (see Theorem 2.6).

For the implication "2 ⇒ 3", suppose that $\pi \circ D_M < \mathfrak{w}_1$ and let $\omega \in \Omega$ such that $D_M(\omega) < \infty$. Let $x = p \circ D_M(\omega)$ and $S = \{\gamma \in \mathfrak{w}_1 + 1 \mid (x,\gamma,\omega) \in M\}$. If S were empty, then $D_M(\omega) = (x, \mathfrak{w}_1)$ which is excluded by hypothesis. Hence, S has a minimum γ_* and, in particular, $D_M(\omega) = (x, \gamma_*)$. Therefore, $(D_M(\omega), \omega) \in M$.

For the proof of the remaining implication "3 ⇒ 1", suppose that $[\![D_M]\!] \cap [\![0,\infty)\!) \subseteq M$. First, we get that $\pi \circ D_M \leq \alpha$. Indeed, if we had $\pi \circ D_M(\omega) > \alpha$, then $t = D_M(\omega)$ and $u = (p(t), \alpha)$ would satisfy $(t,\omega) \in M$ and $u < t$. By $(*)$, we would obtain $(u,\omega) \in M$. Hence, $D_M(\omega) = t \leq u$ — a contradiction. So we infer that $\pi \circ D_M \leq \alpha$.

For the remainder of the proof, define, for any $\beta \in \alpha + 1$,

$$M^\beta = M \cap (\rho^{\alpha+1} \times \mathrm{id}_\Omega)^{-1}(\overline{\mathbb{R}_+} \times \{\beta\} \times \Omega).$$

This set is $\mathscr{F}$-progressively measurable. Hence, by the first statement proven above, D_{M^β} is an $\overline{\mathscr{F}}_+$-stopping time for any $\beta \in \alpha+1$. Thus, for any $t \in \overline{\mathbb{R}_+}$, we have, with $Q_t = [0,t)_{\overline{\mathbb{T}}} \cap \mathbb{Q}$,

$$\{D_{M^\beta} < t\} = \bigcup_{q\in Q_t} \{D_{M^\beta} \leq q\} \in \bigvee_{q\in Q_t} \overline{\mathscr{F}}_{q+} \subseteq \overline{\mathscr{F}}_t.$$

By Lemma A.4, then, for any $\beta \in \alpha+1$, the set

$$N^\beta = [\![0, (D_{M^\beta})_{\mathfrak{w}_1}]\!]$$

is $\overline{\mathscr{F}}$-progressively measurable.[66]

First intermediate claim: Then, we claim that for any $\beta \in \alpha+1$ and $t \in \overline{\mathbb{T}}$, we have

$$\begin{aligned} \operatorname{prj}_\Omega(N^\beta \cap M^\beta \cap [\![0,t]\!]) &= \{\omega \in \Omega \mid (D_{M^\beta}(\omega),\omega) \in M^\beta \cap [\![0,t]\!]\} \\ &= \{\pi \circ D_{M^\beta} = \beta,\ D_{M^\beta} \leq t\}. \end{aligned} \tag{A.9}$$

For the *proof*, let $\beta \in \alpha+1$ and $t \in \overline{\mathbb{T}}$. Regarding the first equality, the inclusion "$\supseteq$" is evident. Concerning the proof of inclusion "$\subseteq$", let $(u,\omega) \in N^\beta \cap M^\beta \cap [\![0,t]\!]$. First, we infer directly that, by the definition of the début, we have $D_{M^\beta}(\omega) \leq u \leq t$. Second, as $N^\beta \cap M^\beta$ is $\overline{\mathscr{F}}$-progressively measurable, there is $\gamma \in \mathfrak{w}_1$ such that $N^\beta \cap M^\beta \in \mathscr{P}^{\gamma}_{\overline{\mathbb{T}}} \otimes \mathscr{E}^{\mathrm{u}}$. If $\pi(u) \geq \gamma$, then, by Lemma 1.9, all $v \in \overline{\mathbb{T}}$ with $p(v) = p(u)$ and $\pi(v) \geq \gamma$ satisfy $(v,\omega) \in N^\beta \cap M^\beta$ as well. Hence, $D_{M^\beta}(\omega) \leq (p(u),\gamma)$. Moreover, as $(u,\omega) \in N^\beta$, we have $p(u) \leq p \circ D_{M^\beta}(\omega)$. These both relations imply $\pi \circ D_{M^\beta}(\omega) \leq \gamma$. As $\mathfrak{w}_1$ is well-ordered, we infer that $(D_{M^\beta}(\omega),\omega) \in M^\beta$. Indeed, otherwise the definition of the début would imply the impossible statement $D_{M^\beta}(\omega) \geq (p \circ D_{M^\beta}(\omega), \pi \circ D_{M^\beta}(\omega)+1) > D_{M^\beta}(\omega)$.

Regarding the second equality, the inclusion "$\subseteq$" is clear that time. Inclusion "$\supseteq$" follows again from the well-ordering on $\mathfrak{w}_1$. Indeed, let $\omega \in \{\pi \circ D_{M^\beta} = \beta\}$. If we had $(D_{M^\beta}(\omega),\omega) \notin M^\beta$, then, by definition of the début, we would again obtain the impossible statement $D_{M^\beta}(\omega) \geq (p \circ D_{M^\beta}(\omega), \beta+1) > D_{M^\beta}(\omega)$. This completes the proof of the first intermediate claim.

Second intermediate claim: For any $\beta \in \alpha+1$ and $t \in \overline{\mathbb{T}}$, we have

$$\{\pi \circ D_{M^\beta} = \beta,\ D_{M^\beta} \leq t\} \in \overline{\mathscr{F}}_t. \tag{A.10}$$

For the *proof*, note that the set under scrutiny equals $\operatorname{prj}_\Omega(N^\beta \cap M^\beta \cap [\![0,t]\!])$, by the first intermediate claim, see Equation A.9. The $\in$-relation follows from the fact that $N^\beta \cap M^\beta$ is $\overline{\mathscr{F}}$-progressively measurable, the fact that

$$(\overline{\mathscr{F}}_t)^{\mathrm{u}} \subseteq \overline{\overline{\mathscr{F}}_t} = \overline{\mathscr{F}_t} = \overline{\mathscr{F}}_t$$

(when augmenting in $\mathscr{E}$), and the Measurable Projection Theorem 1.19.

Third intermediate claim: For any $\beta \in \alpha+1$, the map

$$\sigma_\beta \colon \Omega \to \overline{\mathbb{T}},\ \omega \mapsto \begin{cases} D_{M^\beta}(\omega), & \text{if } \pi \circ D_{M^\beta}(\omega) = \beta, \\ \infty, & \text{else,} \end{cases} \tag{A.11}$$

is an $\overline{\mathscr{F}}$-optional time.

For the *proof*, let $\beta \in \alpha+1$. We clearly have $\pi \circ \sigma_\beta \leq \beta$. Next, we study the measurability of $\{\pi \circ \sigma_\beta,\ \sigma_\beta \leq t\}$ for all $\gamma \in \beta+1$ and $t \in \overline{\mathbb{T}}$. If $\gamma = \beta > 0$, we obtain

$$\{\pi \circ \sigma_\beta = \beta,\ \sigma_\beta \leq t\} = \{\pi \circ D_{M^\beta} = \beta,\ D_{M^\beta} \leq t\} \in \overline{\mathscr{F}}_t,$$

by the second intermediate claim, see Equation A.10. If $\beta > 0 = \gamma$, then we obtain

$$\{\pi \circ \sigma_\beta = 0,\ \sigma_\beta \leq t\} = \begin{cases} \emptyset, & \text{if } t < \infty, \\ \{\pi \circ D_{M^\beta}\}^{\complement}, & \text{else,} \end{cases} \quad \in \overline{\mathscr{F}}_t,$$

again, by the second intermediate claim applied to $t = \infty$, see Equation A.10. If $\gamma \in (\beta+1) \setminus \{0,\beta\}$, then

$$\{\pi \circ \sigma_\beta = 0,\ \sigma_\beta \leq t\} = \emptyset \in \overline{\mathscr{F}}_t.$$

Finally, if $\beta = \gamma = 0$, then, by looking separately at the cases $t < \infty$ and $t = \infty$ we infer that

$$\{\pi \circ \sigma_\beta = 0,\ \sigma_\beta \leq t\} = \{D_{M^0} \leq p(t)\} \in \overline{\mathscr{F}}_{p(t)+} = \overline{\mathscr{F}}_{p(t)} \subseteq \overline{\mathscr{F}}_t,$$

since D_{M^0} is an $\overline{\mathscr{F}}_+$-stopping time. This completes the proof of the third intermediate claim.

[66] We recall that $(D_{M^\beta})_{\mathfrak{w}_1}$ is the map $\Omega \to \overline{\mathbb{T}}$ with $p \circ (D_{M^\beta})_{\mathfrak{w}_1} = p \circ D_{M^\beta}$ and, for all $\omega \in \{D_{M^\beta} < \infty\}$, $\pi \circ (D_{M^\beta})_{\mathfrak{w}_1} = \mathfrak{w}_1$.

Fourth intermediate claim: We have

$$D_M = \inf_{\beta\in\alpha+1} \sigma_\beta, \qquad \Omega = \bigcup_{\beta\in\alpha+1} \{D_M = \sigma_\beta\}. \tag{A.12}$$

For the *proof*, note that, for any $\beta \in \alpha + 1$, the definitions of M^β and σ_β directly imply that

$$(\dagger) \qquad D_M \leq D_{M^\beta} \leq \sigma_\beta,$$

whence $D_M \leq \inf_{\beta\in\alpha+1} \sigma_\beta$. For the remainder of the proof of this intermediate claim, let $\omega \in \Omega$. Then, $D_M(\omega) = \infty$ or $D_M(\omega) < \infty$. In the first case, by (†), $D_M(\omega) = \sigma_\beta(\Omega)$ for all $\beta \in \alpha + 1$. In the second case, we have $(D_M(\omega), \omega) \in M$ by assumption (Property 3). As $\pi \circ D_M(\omega) \leq \alpha$, there is $\beta \in \alpha + 1$ such that $(D_M(\omega), \omega) \in M^\beta$. Hence, $\pi \circ D_{M^\beta}(\omega) = \beta$ and $D_{M^\beta}(\omega) \leq D_M(\omega)$. In view of the definition of σ_β, we infer that equality holds true in (†). In total, we conclude that $D_M(\omega) = \inf_{\beta\in\alpha+1} \sigma_\beta(\omega)$ and $\Omega = \bigcup_{\beta\in\alpha+1}\{D_M = \sigma_\beta\}$.

Conclusion: By the third intermediate claim (cf. Equation A.11), σ_β is an $\overline{\mathscr{F}}$-optional time for any $\beta \in \alpha + 1$. Hence, by the fourth intermediate claim (cf. Equation A.12) and Proposition A.7, Part 6, D_M is an $\overline{\mathscr{F}}$-optional time — because $\alpha + 1$ is countable and $(\Omega, \mathscr{E}^{\mathrm{u}}, \overline{\mathscr{F}})$ is universally complete. This completes the proof of the theorem. □

Optional processes.

Proof of Lemma 2.14. The second inclusion follows directly from Definition 2.1, Remark 2.2, and Theorem 2.6. Regarding the first inclusion, let $E \in \mathscr{F}_0$. Then, $M = (\{0\} \times E) \cup (\{(0, 1)\} \times E^{\complement})$ is clearly $\mathscr{F}$-progressively measurable since $\mathscr{F}_0 \subseteq \mathscr{F}_{(0,1)}$. Moreover, one easily sees (or otherwise applies Theorem 2.12 to see) that D_M is an $\mathscr{F}$-optional time. Hence,

$$\{0\} \times E = [\![0, D_M)\!) \in \mathrm{Opt}(\mathscr{F}).$$

Furthermore, let τ be an $\mathscr{F}$-optional time. Then, the $E = \{\tau = \infty\}$ satisfies $E \in \mathscr{F}_\infty$, by Remark 2.3, Part Remark 3. Moreover, the map $\sigma \colon \Omega \to \overline{\mathbb{T}}$ such that, first, $p \circ \sigma = p \circ \tau$ and, second, for all $\omega \in E^{\complement}$, $\pi \circ \sigma(\omega) = \pi \circ \tau(\omega) + 1$, is an $\mathscr{F}$-optional time.

Indeed, there is $\alpha \in \mathfrak{w}_1$ with $\pi \circ \tau \leq \alpha$. Thus, $\pi \circ \sigma \leq \pi \circ \tau + 1 \leq \alpha + 1$. On $E^{\complement}$, $\pi \circ \sigma$ is valued in the set of successor ordinals in $\mathfrak{w}_1$, and equal to zero on E. Moreover, for all $\beta \in \mathfrak{w}_1$ and $t \in \overline{\mathbb{T}}$, we have

$$\{\pi \circ \sigma = \beta + 1,\, \sigma \leq t\} = \bigcup_{\mathbb{Q}\ni q<t} \{\pi \circ \tau = \beta,\, \tau \leq q\} \cup \bigcup_{\gamma\in\pi(t)\cap(\alpha+1)} \{\pi \circ \tau = \beta,\, \tau \leq (p(t), \gamma)\} \in \mathscr{F}_t,$$

and

$$\{\pi \circ \sigma = 0,\, \sigma \leq t\} = \begin{cases} \emptyset, & \text{if } t < \infty, \\ E, & \text{else,} \end{cases} \in \mathscr{F}_\infty.$$

This shows that σ is an $\mathscr{F}$-optional time.

Hence,

$$[\![0, \tau]\!] = [\![0, \sigma)\!) \cup (\{\infty\} \times E) \in \mathrm{Opt}(\mathscr{F}).$$

□

Proof of Lemma 2.15. Let $\mathscr{M}$ be a σ-algebra as in the lemma. Then, $\mathscr{M}$ must contain $[\![0, \tau)\!)$ for any $\mathscr{F}$-optional time τ, since we can take $\alpha = 2$, $\xi_0 = 1$, $\xi_1 = \xi_2 = 0$, $\tau_1 = \tau$ in Equation 2.5. Moreover, let $E \in \mathscr{F}_\infty$. Then, taking $\alpha = 1$, $\xi_0 = 0$ and $\xi_1 = 1_E$ in Equation 2.5, we infer that $\{\infty\} \times E \in \mathscr{M}$. Hence, $\mathrm{Opt}(\mathscr{F}) \subseteq \mathscr{M}$.

On the other hand, all processes as in Equation 2.5 are $\mathrm{Opt}(\mathscr{F})$-measurable. Indeed, let $\alpha \in \mathfrak{w}_1$, $(\tau_\beta)_{\beta\in\alpha+1}$ and $(\xi^\beta)_{\beta\in\alpha+1}$ be given as in the statement of the lemma. Then, $[\![\tau_\alpha]\!] = \{\infty\}\times\Omega \in \mathrm{Opt}(\mathscr{F})$ and, for $\beta\in\alpha$,

$$[\![\tau_\beta, \tau_{\beta+1})\!) = [\![0, \tau_{\beta+1})\!) \setminus [\![0, \tau_\beta)\!) \ \in \mathrm{Opt}(\mathscr{F}).$$

In a first step, suppose that, for all $\beta \in \alpha+1$,

$$(*) \qquad \xi^\beta = 1(E_\beta)$$

for some $E_\beta \in \mathscr{F}_{\tau_\beta}$. We have

$$(\overline{\mathbb{T}} \times E_\alpha) \cap [\![\tau_\alpha]\!] = \{\infty\} \times E_\alpha \ \in \mathrm{Opt}(\mathscr{F}).$$

In addition, for any $\beta \in \alpha$, we see, using Theorem 2.6, Property 6, that (with the usual measure-theoretic convention $\infty \cdot 0 = 0$)

$$\sigma_\beta = \tau_\beta\, 1_{E_\beta} + \infty\, 1_{E_\beta^\complement}$$

is an $\mathscr{F}$-optional time. Hence,

$$(\overline{\mathbb{T}} \times E_\beta) \cap [\![\tau_\beta, \tau_{\beta+1})\!) = [\![\sigma_\beta, \tau_{\beta+1})\!) = [\![0, \tau_{\beta+1})\!) \setminus [\![0, \sigma_\beta)\!) \ \in \mathrm{Opt}(\mathscr{F}).$$

As α is countable, this implies that under Assumption $(*)$, the process in Equation 2.5 is $\mathrm{Opt}(\mathscr{F})$-measurable. In a second step, we directly infer that the same result obtains under the weaker hypothesis that for all $\beta \in \alpha+1$,

$$\xi^\beta = \sum_{\ell=1}^{N_\beta} x_{\ell,\beta} 1_{E_{\ell,\beta}},$$

for some integer N_β, some reals $x_{1,\beta}, \dots, x_{N_\beta,\beta} \in \mathbb{R}$, and some $E_{1,\beta}, \dots, E_{N_\beta,\beta} \in \mathscr{F}_{\tau_\beta}$, i.e. ξ^β is a simple function with respect to $\mathscr{F}_{\tau_\beta}$. In the third and final step, we consider the general case. Then, for any $\beta \in \alpha+1$, there is a sequence $(\xi^{\beta,m})_{m\in\mathbb{N}}$ of simple functions with respect to $\mathscr{F}_{\tau_\beta}$ converging pointwise to ξ^β. As measurability is stable under pointwise convergence of real-valued functions, we infer that

$$\xi^\alpha \circ \mathrm{prj}_\Omega\, 1[\![\tau_\alpha]\!] + \sum_{\beta\in\alpha} \xi^\beta \circ \mathrm{prj}_\Omega\, 1[\![\tau_\beta, \tau_{\beta+1})\!) = \lim_{m\to\infty} \Big(\xi^{\alpha,m} \circ \mathrm{prj}_\Omega\, 1[\![\tau_\alpha]\!] + \sum_{\beta\in\alpha} \xi^{\beta,m} \circ \mathrm{prj}_\Omega\, 1[\![\tau_\beta, \tau_{\beta+1})\!)\Big)$$

is $\mathrm{Opt}(\mathscr{F})$-measurable. □

Proof of Lemma 2.16. Let $\mathcal{S}$ be a set of maps $\overline{\mathbb{T}} \times \Omega \to \mathbb{R}$ satisfying conditions a), b), and c). In a *first step*, let $\mathscr{D}$ be the set of $M \in \mathrm{Opt}(\mathscr{F})$ such that $1_M \in \mathcal{S}$.

We first show that $\mathscr{D}$ is a Dynkin system, and then infer that $\mathscr{D} = \mathrm{Opt}(\mathscr{F})$. Taking $\tau = \infty$ and $E = \Omega$ in a), we get that $1_\Omega \in \mathcal{S}$, whence $\Omega \in \mathscr{D}$. Moreover, for all $M \in \mathscr{D}$, $1_{M^\complement} = 1_\Omega - 1_M \in \mathcal{S}$ by b), whence $M^\complement \in \mathscr{D}$. Moreover, if $(D_n)_{n\in\mathbb{N}}$ is a pairwise disjoint $\mathscr{D}$-valued sequence, then $D = \bigcup_{n\in\mathbb{N}} D_n$ satisfies

$$1_D = \sum_{k=0}^{\infty} 1_{D_k} = \lim_{n\to\infty} \sum_{k=0}^{n} 1_{D_k},$$

whence $1_D \in \mathcal{S}$ by b) and c), thus $D \in \mathscr{D}$.

Next, we note that $\mathscr{D}$ contains an intersection-stable generator of $\mathrm{Opt}(\mathscr{F})$. Namely, a) implies that, for any $\mathscr{F}$-optional time, $[\![0,\tau)\!) \in \mathscr{D}$, and, for any $E \in \mathscr{F}_\infty$, $\{\infty\} \times E \in \mathscr{D}$. Hence,

$$\mathscr{G} = \Big\{\{\infty\} \times E \mid E \in \mathscr{F}_\infty\Big\} \cup \Big\{[\![0,\tau)\!) \mid \tau\ \mathscr{F}\text{-optional time}\Big\} \tag{A.13}$$

satisfies $\mathscr{G} \subseteq \mathscr{D}$. By definition, $\mathscr{G}$ generates the σ-algebra $\mathrm{Opt}(\mathscr{F})$. Proposition 2.10, Part 4, implies that the pointwise minimum $\sigma \wedge \tau$ of two $\mathscr{F}$-optional times σ, τ is again an $\mathscr{F}$-optional time. As $[\![0,\sigma)\!) \cap [\![0,\tau)\!) = [\![0, \sigma\wedge\tau)\!)$, we readily infer that $\mathscr{G}$ intersection-stable.

Hence, by Dynkin's π-λ-theorem, $\mathscr{D} = \mathrm{Opt}(\mathscr{F})$. In a *second step*, we show that all $\mathscr{F}$-optional processes are elements of $\mathcal{S}$. Let ξ be an $\mathscr{F}$-optional process. Then, there is a sequence $(\xi_n)_{n\in\mathbb{N}}$ of simple functions with respect to $\mathrm{Opt}(\mathscr{F})$ that converges pointwise to ξ. By the first step and Property b), $\xi_n \in \mathcal{S}$ for any $n \in \mathbb{N}$. Therefore, by Property c), $\xi = \lim_{n\to\infty} \xi_n \in \mathcal{S}$ as well.

It remains to show that the set of $\mathscr{F}$-optional processes satisfies properties a), b), and c). Property a) is satisfied by construction. Properties b) and c) follow from the fact that the real-valued $\mathscr{F}$-optional processes are exactly the $\mathrm{Opt}(\mathscr{F})$-$\mathscr{B}_{\mathbb{R}}$-measurable functions, combined with basic measure theory. $\square$

Proof of Proposition 2.18. Denote the set of real-valued $\mathscr{F}$-optional processes by $L^0(\mathrm{Opt}(\mathscr{F});\mathbb{R})$. We have to show that

$$L^0(\mathrm{Opt}(\mathscr{F});\mathbb{R}) = \mathcal{V}_{\mathfrak{w}_1}.$$

(Step 1): We first show that for all $\alpha \in \mathfrak{w}_1 + 1$, we have $\mathcal{V}_\alpha \subseteq L^0(\mathrm{Opt}(\mathscr{F});\mathbb{R})$. For $\alpha = \mathfrak{w}_1$, this yields the inclusion "$\supseteq$".

If the claim did not hold true, then it would fail for at least one $\alpha \in \mathfrak{w}_1 + 1$. Hence, by the well-order property of ordinals, there would be a smallest one that fails. Let us call it α^*. By definition of $\mathcal{V}_0$ and since $\mathbb{R}$-linear combinations of real-valued measurable functions are measurable, we have $\alpha^* > 0$. If α^* were a limit ordinal, we would get

$$\mathcal{V}_{\alpha^*} = \bigcup_{\beta\in\alpha^*} \mathcal{V}_\beta \subseteq \bigcup_{\beta\in\alpha^*} L^0(\mathrm{Opt}(\mathscr{F});\mathbb{R}) = L^0(\mathrm{Opt}(\mathscr{F});\mathbb{R}),$$

which contradicts the definition of α^*. If α^* were a successor ordinal, there would be $\gamma \in \mathfrak{w}_1$ with $\alpha^* = \gamma + 1$. Hence, if $\xi \in \mathcal{V}_{\alpha^*}$, then there is a $\mathcal{V}_\gamma$-valued sequence $(\xi_n)_{n\in\mathbb{N}}$ with $\xi = \lim_{n\to\infty} \xi_n$ pointwise as $n \to \infty$. By definition of α^* and γ, $\xi_n \in L^0(\mathrm{Opt}(\mathscr{F});\mathbb{R})$ for all $n \in \mathbb{N}$. As pointwise limits of real-valued measurable functions are measurable, ξ would be measurable again. This would prove that $\mathcal{V}_{\alpha^*} \subseteq L^0(\mathrm{Opt}(\mathscr{F});\mathbb{R})$, contradicting the definition of α^*. As non-zero ordinals are either successors or limits, we conclude that the claim must be correct.

(Step 2): Let

$$\mathscr{D} = \{M \in \mathrm{Opt}(\mathscr{F}) \mid 1_M \in \mathcal{V}_{\mathfrak{w}_1}\}.$$

Using Dynkin's π-λ-theorem, we show that $\mathscr{D} = \mathrm{Opt}(\mathscr{F})$.

First, we show that $\mathscr{D}$ is a Dynkin system. Note that

$$1(\overline{\mathbb{T}} \times \Omega) = 1[\![0,\infty)\!) + 1(\{\infty\} \times \Omega) \ \in \mathcal{V}_0 \subseteq \mathcal{V}_{\mathfrak{w}_1},$$

whence $\overline{\mathbb{T}} \times \Omega \in \mathscr{D}$. Moreover, if $M_0, M_1 \in \mathscr{D}$ with $M_0 \subseteq M_1$, then there are $\alpha_0, \alpha_1 \in \mathfrak{w}_1$ such that $1(M_k) \in \mathcal{V}_{\alpha_k}$ for $k = 0, 1$. Without loss of generality, $\alpha_0 \leq \alpha_1$. Hence, $M_0, M_1 \in \mathcal{V}_{\alpha_1}$. As $\mathcal{V}_{\alpha_1}$ is an $\mathbb{R}$-vector space, we infer

$$1(M_1 \setminus M_0) = 1(M_1) - 1(M_0) \ \in \mathcal{V}_{\alpha_1} \subseteq \mathcal{V}_{\mathfrak{w}_1},$$

whence $M_1 \setminus M_0 \in \mathscr{D}$. Next, let $(M_n)_{n\in\mathbb{N}}$ be an increasing sequence valued in $\mathscr{D}$ and $M = \bigcup_{n\in\mathbb{N}} M_n$. Then, there is a sequence $(\alpha_n)_{n\in\mathbb{N}}$ valued in $\mathfrak{w}_1$ such that $1(M_n) \in \mathcal{V}_{\alpha_n}$, for all $n \in \mathbb{N}$. Let $\alpha = \sup_{n\in\mathbb{N}} \alpha_n$, an element of $\mathfrak{w}_1$ again. Then, $1(M_n) \in \mathcal{V}_\alpha$ for all $n \in \mathbb{N}$. Hence,

$$1(M) = \lim_{n\to\infty} 1(M_n) \ \in \mathcal{V}_{\alpha+1} \subseteq \mathcal{V}_{\mathfrak{w}_1},$$

whence $M \in \mathscr{D}$. We have proven that $\mathscr{D}$ is a Dynkin system.

Second, we show that $\mathscr{D}$ contains an intersection-stable generator of $\mathrm{Opt}(\mathscr{F})$. By definition, $\mathcal{V}_0$ contains 1_M, for any $M \in \mathscr{G}$, where $\mathscr{G}$ is the intersection-stable generator of $\mathrm{Opt}(\mathscr{F})$ from

Equation A.13.[67] We infer that $\mathscr{G} \subseteq \mathscr{D}$. Thus, $\mathscr{D}$ contains an intersection-stable generator of $\mathrm{Opt}(\mathscr{F})$.

Third, combining these two intermediate results implies that $\mathscr{D} = \mathrm{Opt}(\mathscr{F})$, by Dynkin's π-λ-theorem.

(Step 3): We now show the inclusion "$\subseteq$". Let $\xi \in L^0(\mathrm{Opt}(\mathscr{F}); \mathbb{R})$. Then, by basic measure theory, there is a $\mathrm{Opt}(\mathscr{F})$-valued family $(M_{n,k})_{n\in\mathbb{N},\, k\in\{0,\dots,n\}}$ and a real-valued family $(x_{n,k})_{n\in\mathbb{N},\, k\in\{0,\dots,n\}}$ such that with

$$\xi_n = \sum_{k=0}^{n} x_{n,k}\, 1(M_{n,k}), \qquad n \in \mathbb{N},$$

we have $\xi_n \to \xi$ pointwise as $n \to \infty$. For each pair $(n, k) \in \mathbb{N}^2$ with $k \leq n$, there is $\alpha_{n,k} \in \mathfrak{w}_1$ such that $1(M_{n,k}) \in \mathcal{V}_{\alpha_{n,k}}$, by Step 2. Let $\alpha = \sup_{(n,k)\in\mathbb{N}^2 :\, k\leq n} \alpha_{n,k}$, which is an element of $\mathfrak{w}_1$. Hence, for all $n \in \mathbb{N}$, $\xi_n \in V_\alpha$. Therefore, $\xi \in \mathcal{V}_{\alpha+1} \subseteq \mathcal{V}_{\mathfrak{w}_1}$. □

Proof of Corollary 2.19. Without loss of generality, we may assume that $Y = \mathbb{R}$. We say that a map $\xi \colon \overline{\mathbb{T}} \times \Omega \to \mathbb{R}$ has Property P iff there is $\alpha \in \mathfrak{w}_1$ such that for all $t \in \overline{\mathbb{T}}$ with $\pi(t) \geq \alpha$ and all $\omega \in \Omega$, $\xi(t, \omega) = \xi((p(t), \alpha), \omega)$ holds true. Now, for any $\gamma \in \mathfrak{w}_1 + 1$, we make the following claim $C(\gamma)$: any $\xi \in \mathcal{V}_\gamma$ has Property P. Using Proposition 2.18, the corollary is just the special case of $C(\mathfrak{w}_1)$.

If $C(\gamma)$ were not correct for all $\gamma \in \mathfrak{w}_1 + 1$, then there would be a minimal $\gamma \in \mathfrak{w}_1 + 1$ such that $C(\gamma)$ is incorrect. Denote this hypothetical minimum by γ^*.

(Step 1): We claim that if $\xi^1, \xi^2 \colon \overline{\mathbb{T}} \times \Omega \to \mathbb{R}$ are maps having Property P, respectively, then for any $x_1, x_2 \in \mathbb{R}$, the linear combination $\xi = x_1\xi^1 + x_2\xi^2$ does so, too. Indeed, let $\alpha_1, \alpha_2 \in \mathfrak{w}_1$ be such that for both $k = 0, 1$, all $t \in \overline{\mathbb{T}}$ with $\pi(t) \geq \alpha_k$ and all $\omega \in \Omega$, $\xi^k(t, \omega) = \xi^k((p(t), \alpha_k), \omega)$. Then, let $\alpha = \alpha_1 \vee \alpha_2$, which is an element of $\mathfrak{w}_1$. Then, for all $t \in \overline{\mathbb{T}}$ with $\pi(t) \geq \alpha$ and all $\omega \in \Omega$, we have

$$\xi(t, \omega) = x_1\xi^1(t, \omega) + x_2\xi^2(t, \omega) = x_1\xi^1((p(t), \alpha), \omega) + x_2\xi^2((p(t), \alpha), \omega) = \xi((p(t), \alpha_k), \omega).$$

Hence, ξ has Property P.

(Step 2): We claim that if $(\xi^n)_{n\in\mathbb{N}}$ is a family of maps $\overline{\mathbb{T}} \times \Omega \to \mathbb{R}$ having Property P and converging pointwise to a map ξ, then ξ has Property P as well. Indeed, let $(\alpha_n)_{n\in\mathbb{N}}$ be a $\mathfrak{w}_1$-valued sequence such that for all $n \in \mathbb{N}$, all $t \in \overline{\mathbb{T}}$ with $\pi(t) \geq \alpha_n$ and all $\omega \in \Omega$, $\xi^n(t, \omega) = \xi^n((p(t), \alpha_n), \omega)$ holds true. Let $\alpha = \sup_{n\in\mathbb{N}} \alpha_n$, which is an element of $\mathfrak{w}_1$ again. Then, for all $t \in \overline{\mathbb{T}}$ with $\pi(t) \geq \alpha$ and all $\omega \in \Omega$, we get:

$$\xi(t, \omega) = \lim_{n\to\infty} \xi^n(t, \omega) = \lim_{n\to\infty} \xi^n((p(t), \alpha), \omega) = \xi((p(t), \alpha), \omega).$$

Thus, ξ has Property P.

(Step 3): $C(0)$ is correct, so that $\gamma^* > 0$. Indeed, if $\xi = 1[\![0, \tau)\!)$ for an $\mathscr{F}$-optional time τ, then $\pi \circ \tau \leq \alpha$ for some $\alpha \in \mathfrak{w}_1$. Now, for $(t, \omega) \in \overline{\mathbb{T}} \times \Omega$ we have

$$t < \tau(\omega) \qquad \Longleftrightarrow \qquad \Big[p(t) < p \circ \tau(\omega) \text{ or } \Big(p(t) = p \circ \tau(\omega) \text{ and } \pi(t) < \pi \circ \tau(\omega)\Big)\Big].$$

If $\pi(t) \geq \alpha$, then the right-hand side is equivalent to $p(t) < p \circ \tau(\omega)$. Hence, if $\pi(t) \geq \alpha$, $t < \tau(\omega)$ iff $(p(t), \alpha) < \tau(\omega)$; in other words:

$$\xi(t, \omega) = \xi((p(t), \alpha), \omega).$$

[67] Recall that the only non-trivial assertion to verify in order to prove intersection-stability of $\mathscr{G}$ is that the pointwise minimum of two $\mathscr{F}$-optional times σ, τ is an $\mathscr{F}$-optional time again, because $[\![0, \sigma)\!) \cap [\![0, \tau)\!) = [\![0, \sigma \wedge \tau)\!)$. But this follows from Proposition 2.10, Part 4.

If $\xi = 1(\{\infty\} \times E)$ for $E \in \mathscr{F}_\infty$, then $\xi(t,\omega) = \xi(\underline{p}(t),\omega)$ for all $t \in \overline{\mathbb{T}}$; here, $\alpha = 0$ does the job already. Moreover, by Step 1, the set S of maps $\overline{\mathbb{T}} \times \Omega \to \mathbb{R}$ having Property P is an $\mathbb{R}$-vector space. As S contains all maps of the form given in Equation 2.6, S contains the $\mathbb{R}$-vector space generated by them, namely $\mathcal{V}_0$. Hence, $C(0)$ is correct and $\gamma^* > 0$.

(Step 4): If γ^* were a limit ordinal, then for any $\xi \in \mathcal{V}_{\gamma^*}$, there would be $\beta \in \gamma^*$ with $\xi \in V_\beta$. As $C(\beta)$ is assumed to hold true, ξ would have Property P. Thus, any element of V_{γ^*} would have Property P, i.e. $C(\gamma^*)$ would be correct — a contradiction to the definition of γ^*. If γ^* were a successor ordinal, there would be $\beta \in \gamma^*$ with $\gamma^* = \beta + 1$. Then, for any $\xi \in \mathcal{V}_{\gamma^*}$, there would be a $\mathcal{V}_\beta$-valued sequence $(\xi^n)_{n\in\mathbb{N}}$ converging pointwise to ξ, by construction of the hierarchy. As $\beta < \gamma^*$, all members of that sequence would have Property P. Hence, by Step 2, ξ would also have Property P. Thus, all elements of V_{γ^*} would have Property P, i.e. $C(\gamma^*)$ would be correct — contradicting again the definition of γ^*. As a non-zero ordinal is either a limit or a successor, we conclude that Claim $C(\gamma)$ is correct for all $\gamma \in \mathfrak{w}_1 + 1$. □

Tilting convergence.

Proof of Lemma 2.26. (Ad grid property, last sentence): Represent ξ as in Equation 2.5, for suitable $\alpha \in \mathfrak{w}_1$, τ_β and ξ^β, $\beta \in \alpha + 1$. Define $G\colon (\alpha+1) \times \Omega \to \overline{\mathbb{T}}$ by $G(0,.) = 0$, Equation 2.7 for all $\beta \in \alpha$ and $\omega \in \Omega$, and $G(\gamma,\omega) = \sup_{\beta\in\gamma} G(\beta,\omega)$ for all limit ordinals $\gamma \in \alpha+1$ and $\omega \in \Omega$.

Then, by transfinite induction, we infer using completeness, the Début Theorem 2.12, and Proposition 2.10 (Part 5), that $G(\beta,.)$ is an $\mathscr{F}$-optional time, for any $\beta \in \alpha+1$. By construction, $G(\beta,\omega) < G(\gamma,\omega)$ holds true for all $\beta,\gamma \in \alpha+1$ and $\omega \in \Omega$ with $G(\beta,\omega) < \infty$ because ξ has locally right-constant paths. Moreover, using transfinite induction, again, we see that, for all $\beta \in \alpha+1$, we have $\tau_\beta \leq G(\beta,.)$. In particular, $\infty = \tau_\alpha \leq G(\alpha,.)$. Thus, $\tau^G_\alpha = G(\alpha,.) = \infty$. Hence, G is an $\mathscr{F}$-adapted grid. It follows directly from the definition, using transfinite induction, that G is classical if ξ is so.

(Ad representation): Let

$$\xi' = \xi_{\tau^G_\alpha} \circ \mathrm{prj}_\Omega\, 1[\![\tau^G_\alpha]\!] + \sum_{\beta\in\alpha} \xi_{\tau^G_\beta} \circ \mathrm{prj}_\Omega\, 1[\![\tau^G_\beta, \tau^G_{\beta+1})\!),$$

and let $M = \{\xi \neq \xi'\}$, an $\mathscr{F}$-optional set, by Corollary 2.9 and completeness. As both ξ and ξ' have locally right-constant paths, the Début Theorem 2.12 implies, together with completeness, that D_M is an $\mathscr{F}$-optional time. We show that $D_M = \infty$. We do so by showing the claim (C1) $D_M \geq \tau^G_\beta$ for all $\beta \in \alpha+1$, using transfinite induction. Inserting $\beta = \alpha$ into (C1) then yields $D_M \geq \infty$, whence $D_M = \infty$. (C1) holds true for $\beta = 0$ because $\tau^G_0 = 0$.

Suppose that (C1) holds true for $\beta \in \alpha$, and let $\omega \in \Omega$. If we had $D_M(\omega) < \tau^G_{\beta+1}(\omega)$, then, using local right-constancy of paths, the induction hypothesis, and the definition of ξ', $\xi_{D_M}(\omega) \neq {\xi'}_{D_M}(\omega) = \xi_{\tau^G_\beta}(\omega)$. Thus, $\tau^G_\beta(\omega) < D_M(\omega) < \tau^G_{\beta+1}(\omega)$. As $\xi'(\omega)$ is constant on $[\tau^G_\beta(\omega), \tau^G_{\beta+1}(\omega))_{\overline{\mathbb{T}}}$, this would imply that $\xi_{D_M}(\omega) \neq {\xi'}_{D_M}(\omega) = {\xi'}_{\tau^G_\beta}(\omega) = \xi_{\tau^G_\beta}(\omega)$, whence the contradiction $\tau^G_{\beta+1}(\omega) = G(\beta+1,\omega) \leq D_M(\omega) < \tau_{\beta+1}(\omega)$. Therefore, $D_M(\omega) \geq \tau^G_{\beta+1}(\omega)$.

For all limit ordinals $\gamma \in \alpha+1$ such that (C1) holds true for all $\beta \in \gamma$, we have $D_M \geq \tau^G_\beta$ for all $\beta \in \gamma$, and thus $D_M \geq \sup_{\beta\in\gamma} \tau^G_\beta = \tau^G_\gamma$, because G is an $\mathscr{F}$-adapted grid. The proof of claim (C1) for all $\beta \in \alpha+1$ is complete, and we infer $D_M = \infty$.

By definition of ξ', we have ${\xi'}_\infty = \xi_{\tau^G_\alpha} = \xi_\infty$. Thus, $D_M = \infty$ implies that $M = \emptyset$ which is equivalent to $\xi = \xi'$. □

Proof of Lemma 2.27. (Ad Part 1): As a subset of a well-order, $\{\beta \in \alpha_n \mid G_n(\beta,\omega) \geq t\}$ is itself well-ordered, and therefore, by basic well-order theory (see, e.g., [71, Lemma 10.1]), there is a unique ordinal $\delta^n(t,\omega)$ admitting an order isomorphism

$$\tilde{\psi}^n(t,\omega)\colon \delta^n(t,\omega) \to \{\beta \in \alpha_n \mid G_n(\beta,\omega) \geq t\}$$

and $\tilde{\psi}^n(t,\omega)$ is uniquely determined by the requirement that, for all $\beta' \in \delta^n(t,\omega)$:

$$\tilde{\psi}^n(t,\omega)(\beta') = \min\{\beta \in \alpha_n \mid G_n(\beta,\omega) \geq t\} \setminus [\mathcal{P}\tilde{\psi}^n(t,\omega)](\beta')\Big);$$

that is, $\tilde{\psi}^n(t,\omega)(0) = \min\{\beta \in \alpha_n \mid G_n(\beta,\omega) \geq t\}$, $\tilde{\psi}^n(t,\omega)(\beta'+1) = \tilde{\psi}^n(t,\omega)(\beta') + 1$ for all $\beta' \in \delta^n(t,\omega)$ with $\beta' \in \delta^n(t,\omega)$, and $\tilde{\psi}^n(t,\omega)(\gamma') = \sup_{\beta' \in \gamma'} \tilde{\psi}^n(t,\omega)(\beta')$ for all limit ordinals $\gamma' \in \delta^n(t,\omega)$. $\tilde{\psi}^n(t,\omega)$ can be extended to $\delta^n(t,\omega)+1$ by letting $\psi^n(t,\omega)(\delta^n(t,\omega)) = \alpha_n$. As $G_n(\alpha_n,\omega) = \infty \geq t$, this yields an order isomorphism $\psi^n(t,\omega)$ as claimed. From the recursion above, we infer that $\psi^n(t,\omega)(\beta') = \psi^n(t,\omega)(0) + \beta'$ for all $\beta' \in \delta^n(t,\omega)+1$.

Uniqueness of the isomorphism follows directly from what has been shown before, by basic well-order theory. We give an argument here for the reader's convenience only. For this, consider an arbitrary order isomorphism $f\colon \delta^n(t,\omega)+1 \to \{\beta \in \alpha_n + 1 \mid G_n(\beta,\omega) \geq t\}$, and let $S = \{\beta' \in \delta^n(t,\omega)+1 \mid f(\beta') \neq \psi^n(t,\omega)(0) + \beta'\}$. If S were non-empty, it would have a minimum β_0. There would be $\beta_1, \beta_2 \in \delta^n(t,\omega)+1$ with $f(\beta_0) = \psi^n(t,\omega)(0) + \beta_1$ and $\psi^n(t,\omega)(0) + \beta_0 = f(\beta_2)$. In particular, $\beta_1, \beta_2 > \beta_0$, and

$$f(\beta_0) = \psi^n(t,\omega)(0) + \beta_1 > \psi^n(t,\omega)(0) + \beta_0 = f(\beta_2),$$

implying the contradiction $\beta_0 > \beta_2$.

(Ad Part 2): Let $n \in \mathbb{N}$. There is an order-embedding $j\colon \alpha_n + 1 \hookrightarrow \alpha_{n+1}+1$ such that $G_n = G_{n+1} \circ (j \times \mathrm{id}_\Omega)$. Hence, if $(\beta,\omega) \in (\alpha_n+1)\times\Omega$ is such that $G_n(\beta,\omega) \geq t$, then $G_{n+1}(j(\beta),\omega) \geq t$. Therefore, for any $\omega \in \Omega$, $\{\beta \in \alpha_n + 1 \mid G_n(\beta,\omega) \geq t\}$ can be order-embedded into $\{\beta \in \alpha_{n+1}+1 \mid G_{n+1}(\beta,\omega) \geq t\}$. Via the order isomorphism from Part 1, $\delta^n(t,\omega)+1$ can be embedded in to $\delta^{n+1}(t,\omega)+1$, whence $\delta^n(t,\omega)+1 \leq \delta^{n+1}(t,\omega)+1$ which implies $\delta^n(t,\omega) \leq \delta^{n+1}(t,\omega)$.

Regarding the second claim, as a supremum of countably many countable, non-zero ordinals, $\delta(t,\omega)$ is countable and non-zero. □

Proof of Lemma 2.32. Let $n \in \mathbb{N}$, $(t,\beta,\omega) \in \overline{\mathbb{T}} \times \Omega$ with $\beta \in \delta^n(t,\omega)+1$. For all $\beta_0 \in \alpha_n + 1$ we infer, using that G_n is an $\mathscr{F}$-adapted grid:

$$\begin{aligned}
&\psi^n(t,\omega)(0) = \beta_0\\
\Leftrightarrow\ &\Big(\tau^{G_n}_{\beta_0}(\omega) \geq t,\ \forall \beta' \in \beta_0\colon \tau^{G_n}_{\beta'}(\omega) < t\Big)\\
\Leftrightarrow\ &\begin{cases} t = 0, & \text{if } \beta_0 = 0,\\ \tau^{G_n}_{\beta_0}(\omega) \geq t > \tau^{G_n}_{\beta_0'}(\omega), & \text{if } \beta_0 = \beta_0' + 1 \text{ for some } \beta_0' \in \mathfrak{w}_1,\\ \tau^{G_n}_{\beta_0}(\omega) = t, & \text{if } \beta_0 \in \mathrm{L}(\mathrm{On}).\end{cases}
\end{aligned}$$

Hence, in view of Lemma 2.27, we get:

$$\begin{aligned}
&\xi^n\Big(G_n(\psi^n(t,\omega)(\beta),\omega),\omega\Big)\\
&= \xi^n\Big(G_n(\psi^n(t,\omega)(0)+\beta,\omega),\omega\Big)\\
&= \sum_{\beta_0 \in \alpha_n+1} \xi^n_{\tau^{G_n}_{\beta_0+\beta}}(\omega)\, 1\{\psi^n(t,\omega)(0) = \beta_0\}\\
&= \xi^n_{\tau^{G_n}_{\beta}}(\omega)\, 1[\![0]\!](t,0,\omega)\\
&\quad + \sum_{\beta_0 \in \alpha_n+1} \xi^n_{\tau^{G_n}_{\beta_0+1+\beta}}(\omega)\, 1(\!(\tau^{G_n}_{\beta_0}, \tau^{G_n}_{\beta_0+1}]\!](t,0,\omega)
\end{aligned}$$

$$+ \sum_{\beta_0 \in (\alpha_n+1)\cap \mathrm{L(On)}} \xi^n_{\tau^{G_n}_{\beta_0+\beta}}(\omega)\, 1[\![\tau^{G_n}_{\beta_0}]\!](t,0,\omega).$$

□

Proof of Proposition 2.33. Let τ be an $\mathscr{F}$-optional time. By definition, there is $\alpha \in \mathfrak{w}_1$ with $\pi \circ \tau < \alpha$. We choose α to be a limit ordinal. This is possible because, if necessary, we could replace α with the ordinal $\alpha' = \sup\{\alpha + \beta \mid \beta \in \mathfrak{w}\}$. Clearly, the updated bound then satisfies $\alpha' \in \mathfrak{w}_1$ and $\pi \circ \tau < \alpha'$.

(Construction of $(G_n)_{n\in\mathbb{N}}$): For any $a, b \in \overline{\mathbb{R}_+}$ with $a < b$, fix a continuous order embedding $h_{a,b} \colon \alpha \hookrightarrow [a,b)_{\overline{\mathbb{R}_+}}$ such that:

- $h_{a,b}(0) = a$;
- $\sup_{\beta\in\alpha} h_{a,b}(\beta) = b$;
- $\sup_{\beta\in\alpha} \big(h_{a,b}(\beta+1) - h_{a,b}(\beta)\big) \leq \frac{b-a}{2} \wedge 1$.

For any $n \in \mathbb{N}$, define $\alpha_n \in \mathfrak{w}_1$ and an order embedding $g_n \colon (\alpha_n+1) \to \overline{\mathbb{R}_+}$ recursively as follows. Let $\alpha_0 = \alpha$, and g_0 be given by $g_0(\beta) = h_{0,\infty}(\beta)$ for $\beta \in \alpha$ and $g_0(\alpha) = \infty$. Further, let $n \in \mathbb{N}$ and suppose that $\alpha_n \in \mathfrak{w}_1$ and $g_n \colon (\alpha_n + 1) \to \overline{\mathbb{R}_+}$ is an order embedding. Define g_{n+1} as follows. Let $\tilde{g}_{n+1} \colon \alpha_n \times \alpha \to \mathbb{R}_+$ be given by

$$\tilde{g}_{n+1}(\beta_0, \beta) = h_{g_n(\beta_0), g_n(\beta_0+1)}(\beta).$$

Equip $\alpha_n \times \alpha$ with lexicographic order. Then, $\tilde{g}_{n+1}$ defines a continuous order embedding and $\alpha_n \times \alpha$ defines a well-order. Therefore, the latter is order-isomorphic to a unique ordinal α_{n+1}, and it is countable, thus $\alpha_{n+1} \in \mathfrak{w}_1$. As a consequence, $\tilde{g}_{n+1}$ induces a continuous order embedding $\alpha_{n+1} \to \mathbb{R}_+$ which we extend to an order embedding $g_{n+1} \colon (\alpha_{n+1}+1) \to \overline{\mathbb{R}_+}$ by letting $g_{n+1}(\alpha_{n+1}) = \infty$. It is clear from this construction that the sequence $(G_n)_{n\in\mathbb{N}}$ given by

$$G_n \colon (\alpha_n + 1) \times \Omega \to \overline{\mathbb{T}},\ (\beta, \omega) \mapsto g_n(\beta), \qquad n \in \mathbb{N},$$

defines a refining sequence of classical, deterministic grids. Moreover, it is convergent, since — by choice of the family $(h_{a,b})_{a,b}$ — we have $\Delta(G_n, .) \leq 2^{-n}$ for all $n \in \mathbb{N}^*$.

(Construction of $(\xi^n)_{n\in\mathbb{N}}$): Let $n \in \mathbb{N}$ and, for any $\omega \in \Omega$,

$$\sigma_n(\omega) = \inf\{\beta \in \alpha_n + 1 \mid p \circ \tau(\omega) \leq g_n(\beta)\} + \pi \circ \tau(\omega).$$

Upon extending g_n to $\mathfrak{w}_1$ by $g_n(\beta) = \infty$ for $\beta \in (\alpha_n, \mathfrak{w}_1)_{\mathfrak{w}_1}$, let $\tau_n = g_n \circ \sigma_n$, and $\xi^n = 1[\![0, \tau_n)\!)$.

We show that τ_n is an $\mathscr{F}$-optional time. As $\pi \circ \tau_n = 0$, $\operatorname{im} \tau_n \subseteq \operatorname{im} g_n$, and g_n has countable image, it suffices to show that $\{\tau_n = t\} \in \mathscr{F}_t$ for all $t \in \operatorname{im} g_n \setminus \{\infty\}$. This follows once we show that $\{\sigma_n = \beta^\star\} \in \mathscr{F}_{g_n(\beta^\star)}$ for all $\beta^\star \in \alpha_n$.

Let $\beta_1, \beta_2 \in \mathfrak{w}_1$. Then, we have the following equivalences:

$$\begin{aligned} &\inf\{\beta \in \alpha_n + 1 \mid p \circ \tau(\omega) \leq g_n(\beta)\} = \beta_1,\ \pi \circ \tau(\omega) = \beta_2 \\ \Leftrightarrow\quad &\forall \beta \in \beta_1 \colon \Big((g_n(\beta), \beta_2) < \tau(\omega) \leq (g_n(\beta_1), \beta_2)\Big),\ \pi \circ \tau(\omega) = \beta_2. \end{aligned}$$

Hence, for $\beta^\star \in \alpha_n$,

$$\begin{aligned} &\{\sigma_n = \beta^\star\} \\ = &\bigcup_{\substack{\beta_1,\beta_2 \in \mathfrak{w}_1 \colon \\ \beta_1+\beta_2=\beta^\star}} \bigcap_{\beta\in\beta_1} \{(g_n(\beta), \beta_2) < \tau(\omega) \leq (g_n(\beta_1), \beta_2)\} \cap \{\pi \circ \tau = \beta_2\} \quad \in \mathscr{F}_{g_n(\beta^\star)}, \end{aligned}$$

because $(g_n(\beta_1), \beta_2) \leq g_n(\beta_1 + \beta_2)$ in $\overline{\mathbb{T}}$ for all $\beta_1, \beta_2 \in \mathfrak{w}_1$, and τ is an $\mathscr{F}$-optional time. Thus, τ_n is an $\mathscr{F}$-optional time.

As τ_n is an $\mathscr{F}$-optional time, ξ^n is $\mathscr{F}$-optional. As τ_n is $\overline{\mathbb{R}_+}$-valued, ξ^n is classical and very simple, and, by construction, the grid G_n is compatible with ξ^n. It remains to show that $(\xi^n \mid G_n) \xrightarrow{\mathsf{T}} 1[\![0,\tau)\!)$ as $n \to \infty$.

(Ad Convergence): Let $(t,\beta,\omega) \in \overline{\mathbb{T}} \times \Omega$ with $\beta \in \gamma(t,\omega)$, and let $n \in \mathbb{N}$ be large enough such that $\beta \in \delta^n(t,\omega)+1$. Then, $\xi^n\Big(G_n(\psi^n(t,\omega)(\beta),\omega),\omega\Big) = 1$ iff $G_n(\psi^n(t,\omega)(\beta),\omega) < \tau^n(\omega)$. By definition of G_n and τ^n, the latter is equivalent to $g_n(\psi^n(t,\omega)(\beta)) < g_n(\sigma_n(\omega))$. By definition of g_n, and because $\psi^n(t,\omega)$ maps into α_n+1, this is equivalent to $\psi^n(t,\omega)(\beta) < \sigma_n(\omega)$. By Lemma 2.27, and by definition of σ_n, this is equivalent to

$$\inf\{\beta' \in \alpha_n+1 \mid g_n(\beta') \geq t\} + \beta < \inf\{\beta' \in \alpha_n+1 \mid g_n(\beta') \geq p \circ \tau(\omega)\} + \pi \circ \tau(\omega). \tag{A.14}$$

If $(t,\beta) \geq \tau(\omega)$, then, by construction of $(G_k)_{k\in\mathbb{N}}$ and the fact that $\pi \circ \tau < \alpha$, there is $N \in \mathbb{N}$ such that for all integers $n \geq N$ Inequality A.14 is not satisfied. Thus, in that case,

$$\xi^n\Big(G_n(\psi^n(t,\omega)(\beta),\omega),\omega\Big) \to 0 = 1[\![0,\tau)\!)(t,\beta,\omega), \qquad \text{as } n \to \infty.$$

If, conversely, $(t,\beta) < \tau(\omega)$, then, again by construction of $(G_k)_{k\in\mathbb{N}}$ and the inequality $\pi \circ \tau < \alpha$, there is $N \in \mathbb{N}$ such that for all integers $n \geq N$ Inequality A.14 is satisfied. Thus, in that case,

$$\xi^n\Big(G_n(\psi^n(t,\omega)(\beta),\omega),\omega\Big) \to 1 = 1[\![0,\tau)\!)(t,\beta,\omega), \qquad \text{as } n \to \infty.$$

Furthermore, note that $\gamma(t,\omega) \geq \alpha$ for all $(t,\omega) \in \mathbb{R}_+ \times \Omega$, by construction of $(G_k)_{k\in\mathbb{N}}$, and $\gamma(\infty,\omega) > 0$ for all $\omega \in \Omega$ by the general construction of γ. Hence, for $(t,\beta,\omega) \in \overline{\mathbb{T}} \times \Omega$ with $\beta \notin \gamma(t,\omega)$, we have $\beta \geq \alpha$. As a consequence, $1[\![0,\tau)\!)(t,\beta,\omega) = \lim_{\beta' \nearrow \gamma(t,\omega)} 1[\![0,\tau)\!)(t,\beta',\omega)$, since $\gamma(t,\omega)$ is a limit ordinal and $\gamma(t,\omega) \geq \alpha > \pi \circ \tau(\omega)$.

We conclude that $(\xi^n \mid G_n) \xrightarrow{\mathsf{T}} 1[\![0,\tau)\!)$ as $n \to \infty$. □

Proof of Theorem 2.34. As any Polish space Y can be measure-theoretically embedded into $\mathbb{R}$,[68] we can suppose without loss of generality that $Y = \mathbb{R}$. For any two $\mathscr{F}$-optional processes ξ, ξ' valued in $\mathbb{R}$, the process $\tilde{\xi} \colon \overline{\mathbb{T}} \times \Omega \to \mathbb{R}^2$, $(t,\omega) \mapsto (\xi_t(\omega), \xi'_t(\omega))$ is again $\mathscr{F}$-optional because for all $B, B' \in \mathscr{B}_{\mathbb{R}}$, we have

$$\tilde{\xi}^{-1}(B \times B') = \xi^{-1}(B) \cap {\xi'}^{-1}(B') \in \operatorname{Opt}(\mathscr{F}).$$

Thus, for any continuous — and *a fortiori* Borel-measurable — $f \colon \mathbb{R}^2 \to \mathbb{R}$, $\xi'' = f \circ \tilde{\xi}$ is $\mathscr{F}$-optional.

Furthermore, pointwise addition and scalar multiplication with $\lambda \in \mathbb{R}$ can be described by the continuous map $\mathbb{R}^2 \to \mathbb{R}$, $(x,y) \mapsto \lambda x + y$. With this, the theorem is a direct consequence of Lemma 2.16 and Proposition 2.33, together with the fact that, for any $\mathscr{F}_\infty$-measurable, real-valued random variable ξ^∞, the process $\xi^\infty \circ \operatorname{prj}_\Omega 1[\![\infty]\!]$ is a classical, very simple $\mathscr{F}$-optional process. □

A.3. Section 3.

Information sets, counterfactuals, and equilibrium.

Proof of Proposition 3.4. (Ad Part 1): Let V be the set of $(\mathscr{P}_{\overline{\mathbb{T}}} \otimes \mathscr{E} \otimes \{\emptyset, \mathbb{B}^{\overline{\mathbb{T}}}\}) \vee \operatorname{Prd}(\mathscr{H}^i)$-measurable $f \colon \overline{\mathbb{T}} \times W \to \mathbb{R}$ such that for $t \in \overline{\mathbb{T}}$, $\omega \in \Omega$, and $h, h' \in \mathbb{B}^{\overline{\mathbb{T}}}$ with $h|_{[0,t)_{\overline{\mathbb{T}}}} = h'|_{[0,t)_{\overline{\mathbb{T}}}}$, we have $f(t,\omega,h) = f(t,\omega,h')$. Let $V_b \subseteq V$ be the subset of bounded $f \in V$.

V_b is clearly stable under pointwise addition, multiplication, and real scalar multiplication and contains all constant functions. It remains to show the Claim **(CL1)** that V_b contains the functions 1_M for all $M \in \mathcal{G}_k$, $k = 1,2$, for an intersection-stable generator $\mathcal{G}_1$ of $\mathscr{P}_{\overline{\mathbb{T}}} \otimes \mathscr{E} \otimes \{\emptyset, \mathbb{B}^{\overline{\mathbb{T}}}\}$ and an intersection-stable generator $\mathcal{G}_2$ of $\operatorname{Prd}(\mathscr{H}^i)$. Indeed, if such $\mathcal{G}_k$, $k = 1,2$, exist, then we may assume that they contain $\overline{\mathbb{T}} \times \Omega \times \mathbb{B}^{\overline{\mathbb{T}}}$. Under this assumption, $\mathcal{G} = \{M \cap N \mid M \in \mathcal{G}_1,\, N \in \mathcal{G}_2\}$

[68] That is, there is a measurable injection $\varphi \colon Y \hookrightarrow \mathbb{R}$ with measurable image, and a measurable inverse $\operatorname{im}\varphi \to Y$.

defines an intersection-stable generator of $(\mathscr{P}_{\overline{\mathbb{T}}}\otimes\mathscr{E}\otimes\{\emptyset,\mathbb{B}^{\overline{\mathbb{T}}}\})\vee\mathrm{Prd}(\mathscr{H}^i)$. Then, as V_b is stable under multiplication, $1_G \in V_b$ for all $G \in \mathcal{G}$, and products of such indicators are again in V_b. Moreover, it is clear that V is closed under pointwise convergence and the limit of a V_b-valued pointwise converging sequence $(f_n)_{n\in\mathbb{N}}$ such that $0 \le f_n \le C$ for some real constant $C > 0$ is again bounded. Hence, using the functional monotone class theorem and Claim (CL1), we infer that V equals the set of all $(\mathscr{P}_{\overline{\mathbb{T}}}\otimes\mathscr{E}\otimes\{\emptyset,\mathbb{B}^{\overline{\mathbb{T}}}\})\vee\mathrm{Prd}(\mathscr{H}^i)$-measurable $f\colon \overline{\mathbb{T}}\times W \to \mathbb{R}$. As $\mathrm{Opt}(\mathscr{E}\otimes\{\emptyset,\mathbb{B}^{\overline{\mathbb{T}}}\}) \subseteq \mathscr{P}_{\overline{\mathbb{T}}}\otimes\mathscr{E}\otimes\{\emptyset,\mathbb{B}^{\overline{\mathbb{T}}}\}$, and as we make Assumption (C), the claim of the proposition's first part follows.

We prove Claim (CL1). For this, let $t \in \overline{\mathbb{T}}$, $\omega \in \Omega$, and $h, h' \in \mathbb{B}^{\overline{\mathbb{T}}}$. First, let $T \in \mathscr{P}_{\overline{\mathbb{T}}}$, $E \in \mathscr{E}$, and $f = 1(T \times E \times \mathbb{B}^{\overline{\mathbb{T}}})$. Then, for all t, ω, h, h' as above, we have $f(t, \omega, h) = f(t, \omega, h')$. Thus, $f \in V_b$. Next, let $H \in \mathscr{H}^i_0$ and $f = 1(\{0\} \times H)$. By Assumption (A), there is $E \in \mathscr{E}$ such that $H = E \times \mathbb{B}^{\overline{\mathbb{T}}}$. Then, again, for all t, ω, h, h' as above, we have $f(t, \omega, h) = f(t, \omega, h')$. Hence, $f \in V_b$. Next, take an $\mathscr{H}^i$-optional time σ and let $f = 1[\![0, \sigma]\!]$. If $t = 0$, then $f(t, \omega, h) = 1 = f(t, \omega, h')$. Suppose that $t > 0$. We have $\{f_t = 0\} = \{\sigma < t\}$. As $\pi \circ \sigma$ is bounded above by some fixed countable ordinal, there is a countable subset $Q \subseteq [0, t)_{\overline{\mathbb{T}}}$ such that

$$\{f_t = 0\} = \{\sigma < t\} = \bigcup_{u\in Q} \{\sigma \le u\}.$$

Let $u \in Q$. Then, we have $\{\sigma \le u\} \in \mathscr{H}^i_u$. Hence, by SPF Axiom 1, using the notation from Definition 3.1, there is $H_u \in \tilde{\mathscr{H}}^i_u$ with

$$1\{\sigma \le u\}(\omega, h) = 1(H_u)(\omega, \mathrm{proj}_{[0,u]_{\overline{\mathbb{T}}}}(h)) = 1(H_u)(\omega, \mathrm{proj}_{[0,u]_{\overline{\mathbb{T}}}}(h')) = 1\{\sigma \le u\}(\omega, h'),$$

because $u < t$. Thus, $f_t(\omega, h) = 1-\sup_{u\in Q} 1\{\sigma \le u\}(\omega, h) = 1-\sup_{u\in Q} 1\{\sigma \le u\}(\omega, h') = f_t(\omega, h')$. Thus, $f \in V_b$. Furthermore, if τ is another $\mathscr{H}^i$-optional time, the minimum $\tau \wedge \sigma$ is as well by Proposition 2.10, and as V_b is stable under differences, we also get that $f = 1(\!(\sigma, \tau]\!] \in V_b$. This proves Claim (CL1).

(Ad Part 2): Let τ^i be an optional time for i and let χ, χ' be state processes such that there is $N \in \mathscr{N}$ satisfying, for all $\omega \in \Omega \setminus N$,

$$\chi|_{[0,\tau^i(\omega,\chi(\omega))_{\overline{\mathbb{T}}}} = \chi'|_{[0,\tau^i(\omega,\chi(\omega))_{\overline{\mathbb{T}}}}.$$

Hence, by Part 1, any $\omega \in \Omega \setminus N$, any $\mathscr{M}^i$-measurable $f\colon \overline{\mathbb{T}} \times W \to \mathbb{R}$ and any $t \in [0, \tau^i(\omega, \chi(\omega))]_{\overline{\mathbb{T}}}$ satisfy

$$(*) \qquad f(t, \omega, \chi(\omega)) = f(t, \omega, \chi'(\omega)).$$

Whence, $\chi \approx_{i,\tau^i} \chi'$.

(Ad Part 3): Let τ^i be an optional time for i and let χ, χ' be state processes that are left-continuous at all $u \in \overline{\mathbb{T}}$ with $\pi(u) = \mathfrak{w}_1$ such that $\chi \approx_{i,\tau^i} \chi'$. We make Assumption (B). There is $N \in \mathscr{N}$ such that for all $\omega \in \Omega\setminus N$, all $t \in [0, \tau^i(\omega, \chi(\omega))]_{\overline{\mathbb{T}}}$, and all $\mathscr{M}^i$-measurable $f\colon \mathbb{T}\times W \to \mathbb{R}$, Equation $(*)$ is satisfied.

As $\mathrm{Prd}(\mathscr{H}^i) \subseteq \mathscr{M}^i$, it suffices to select some embedding of measurable spaces $\varphi\colon \mathbb{B} \hookrightarrow [0, 1]$,[69] and to show the intermediate claim that for all $u \in \overline{\mathbb{T}}$ with $\pi(u) < \mathfrak{w}_1$, the map

$$f^u\colon \mathbb{T} \times W \to \mathbb{R},\ (t, \omega, h) \mapsto \begin{cases} \varphi \circ h(u), & \text{if } u < t, \\ -1, & \text{else,} \end{cases}$$

is $\mathscr{H}^i$-predictable. Indeed, then we infer that, for all $\omega \in \Omega \setminus N$ and $u \in [0, \tau^i(\omega, \chi(\omega)))_{\overline{\mathbb{T}}}$ with $\pi(u) < \mathfrak{w}_1$, it holds true that $\chi(u, \omega) = \chi'(u, \omega)$. By the left-continuity assumption, this extends to all $u \in [0, \tau^i(\omega, \chi(\omega)))_{\overline{\mathbb{T}}}$.

[69] That is, φ is Borel measurable, injective, has Borel-measurable image, and Borel measurable inverse $\mathrm{im}\,\varphi \to \mathbb{B}$.

For the proof of the intermediate claim above, fix $u \in \overline{\mathbb{T}}$ with $\pi(u) < \mathfrak{w}_1$. Then, $\{f^u = -1\} = [\![0, u]\!] \in \operatorname{Prd}(\mathscr{H}^i)$. Moreover, for any Borel set $B \in \mathscr{B}_{\mathbb{R}}$ with $-1 \notin B$,

$$\sigma^u \colon W \to \overline{\mathbb{T}},\, (\omega, h) \mapsto \begin{cases} u, & \text{if } \varphi \circ h(u) \in B, \\ \infty, & \text{else,} \end{cases}$$

is clearly an $\mathscr{H}^i$-optional time, in view of Assumption (B). As

$$\{f^u \in B\} = (\!(\sigma^u, \infty]\!] \in \operatorname{Prd}(\mathscr{H}^i),$$

f^u is $\mathscr{H}^i$-predictable, and the proof of this part of the proposition is complete.

The final statement of the proposition follows directly from the last two parts. □

Timing games.

Proof of Lemma 3.9. (Ad $\tau_{\boldsymbol{b}}$): Let $\boldsymbol{b} \in \mathbb{B}$. As z has locally right-constant paths, we have, for all $(t, \omega, h) \in [\![0, \tau_{\boldsymbol{b}})\!)$, $h(\tau_{\boldsymbol{b}}(\omega, h)) = z_{\tau_{\boldsymbol{b}}}(\omega, h) \leq \boldsymbol{b}$ and $h(t) = z_t(\omega, h) > \boldsymbol{b}$. Hence, by definition of W and the upper vertical level, $\pi \circ \tau_{\boldsymbol{b}}(\omega, h) < \alpha$. Moreover, for all $t, u \in \overline{\mathbb{T}}$ with $u \leq t$, and $\beta \in \mathfrak{w}_1$, we have

$$\begin{aligned} &\{\tau_{\boldsymbol{b}} \leq u,\, \pi \circ \tau_{\boldsymbol{b}} = \beta\} \\ &= \Big(\Omega \times \{h \in \mathbb{B}^{\overline{\mathbb{T}}} \mid \exists v \in [0, u]_{\overline{\mathbb{T}}} \colon (h(v) \leq \boldsymbol{b}, \forall v' \in [0, v)_{\overline{\mathbb{T}}} \colon h(v') > \boldsymbol{b}, \pi(v) = \beta)\}\Big) \cap W \\ &= (\mathrm{id}_\Omega \times \mathrm{proj}_{[0,t]_{\overline{\mathbb{T}}}})^{-1}(\tilde{H}) \cap W, \end{aligned}$$

for some subset $\tilde{H} \subseteq \Omega \times \mathbb{B}^{[0,t]_{\overline{\mathbb{T}}}}$. Hence, by definition of $\mathscr{H}^i$, $\tau_{\boldsymbol{b}}$ is an $\mathscr{H}^i$-optional time.

(Ad $\tau_{\boldsymbol{b}}^-$): We clearly have $\tau_{\boldsymbol{1}} = \tau_{\boldsymbol{1}}^-$. In the following, we suppose that $\boldsymbol{b} \neq \boldsymbol{1}$. Let $w \in W$. Then, $z_{\tau_{\boldsymbol{b}}}(w) \leq \boldsymbol{b} < z_{\tau_{\boldsymbol{b}}^-}(w)$, because z has locally right-constant paths. Hence, if $\tau_{\boldsymbol{b}}(w) = (t, \beta)$ for some $t \in \overline{\mathbb{R}_+}$ and $\beta \in \mathfrak{w}_1$, then $\tau_{\boldsymbol{b}}^-(w) = (t, \beta + 1)$. As $\pi \circ \tau_{\boldsymbol{b}}(w) < \alpha$, we infer that $\tau_{\boldsymbol{b}}^-(w) = (p \circ \tau_{\boldsymbol{b}}(w), \pi \circ \tau_{\boldsymbol{b}}(w) + 1)$. Hence, $[\![0, \tau_{\boldsymbol{b}}^-)\!) = [\![0, \tau_{\boldsymbol{b}}]\!]$, and $\pi \circ \tau_{\boldsymbol{b}}^- = \pi \circ \tau_{\boldsymbol{b}} + 1 \leq \alpha$. Let $t \in \overline{\mathbb{T}}$ and $\beta \in \mathfrak{w}_1$. We have $\{\tau_{\boldsymbol{b}}^- \leq t,\, \pi \circ \tau_{\boldsymbol{b}}^- = \beta\} = \emptyset$, if β is not a successor ordinal or if $t = 0$. Else, there is $\gamma \in \mathfrak{w}_1$ with $\beta = \gamma + 1$ and $t > 0$. If $\pi(t)$ is a successor ordinal as well, there is $\delta \in \mathfrak{w}_1$ with $\pi(t) = \delta + 1$, whence

$$\{\tau_{\boldsymbol{b}}^- \leq t,\, \pi \circ \tau_{\boldsymbol{b}}^- = \beta\} = \{\tau_{\boldsymbol{b}} \leq (p(t), \delta),\, \pi \circ \tau_{\boldsymbol{b}} = \gamma\} \in \mathscr{H}^i_{(p(t),\delta)} \subseteq \mathscr{H}^i_t.$$

If $\pi(t)$ is not a successor ordinal and $t > 0$, then there is a strictly increasing sequence $(u_n)_{n \in \mathbb{N}} \in \overline{\mathbb{T}}^{\mathbb{N}}$ with $u_n \nearrow t$. Hence,

$$\{\tau_{\boldsymbol{b}}^- \leq t,\, \pi \circ \tau_{\boldsymbol{b}}^- = \beta\} = \bigcup_{n \in \mathbb{N}} \{\tau_{\boldsymbol{b}} \leq u_n,\, \pi \circ \tau_{\boldsymbol{b}} = \gamma\} \in \bigvee_{n \in \mathbb{N}} \mathscr{H}^i_{u_n} \subseteq \mathscr{H}^i_t.$$

Thus, $\tau_{\boldsymbol{b}}^-$ is an $\mathscr{H}^i$-optional time with the claimed properties.

(Intermediate result): Let $\boldsymbol{b} \in \mathbb{B}$ and

$$\tau_{<\boldsymbol{b}} = \inf\{t \in \overline{\mathbb{T}} \mid z_u < \boldsymbol{b}\}.$$

As z is decreasing, we have, with $\downarrow \boldsymbol{b} = \{\boldsymbol{b}' \in \mathbb{B} \mid \boldsymbol{b}' \leq \boldsymbol{b}\}$,

$$\tau_{<\boldsymbol{b}} = \inf_{\boldsymbol{b}' \in \downarrow \boldsymbol{b} \setminus \{\boldsymbol{b}\}} \tau_{\boldsymbol{b}'}.$$

As an infimum of a finite number of $\mathscr{H}^i$-optional times, this is again an $\mathscr{H}^i$-optional time by an application of Proposition 2.10, Part 2.10.

(Ad z): Then, as z has locally right-constant and componentwise decreasing paths, for any $\boldsymbol{b}\in\mathbb{B}$, we obtain the representations

$$(*)\qquad \{z=\boldsymbol{b}\}=\begin{cases}[\![\tau_{\boldsymbol{b}},\infty]\!], & \text{if } \boldsymbol{b}=\boldsymbol{0},\\ [\![\tau_{\boldsymbol{b}},\tau_{<\boldsymbol{b}})\!), & \text{else,}\end{cases}\qquad \{z_-=\boldsymbol{b}\}=\begin{cases}[\![0,\tau_{<\boldsymbol{b}}]\!], & \text{if } \boldsymbol{b}=\boldsymbol{1},\\ (\!(\tau_{\boldsymbol{b}},\tau_{<\boldsymbol{b}}]\!], & \text{else.}\end{cases}$$

Hence, z is $\mathscr{H}^i$-optional and has upper vertical level smaller than or equal to α, and z_- is $\mathscr{H}^i$-predictable and has (upper) vertical level smaller than or equal to α (if α is a limit ordinal, respectively). Let $\beta\in\alpha$ and $h\colon\overline{\mathbb{T}}\to\mathbb{B}$ be given by $h(t)=\boldsymbol{1}$ for $t\in[0,(0,\beta))_{\overline{\mathbb{T}}}$, and $h(t)=\boldsymbol{0}$ for $t\in[(0,\beta),\infty]_{\overline{\mathbb{T}}}$. Then, for any $\omega\in\Omega$, we have $(\omega,h)\in W$. Plugging in (ω,h) into z and z_- for all $\beta\in\alpha$ shows that z has upper vertical level α and that z_- has (upper) vertical level α (if α is a limit ordinal). ☐

Proof of Lemma 3.10. The map $f^\#$ is well-defined because for any $(\omega,h)\in W$, $(\omega,f(\omega,h))\in W$. Using Theorem 2.6, Corollary 2.9, Theorem 2.12, and the completeness assumption on the data $\mathscr{H}^\vee$, we infer that f is a simple $\mathscr{H}^\vee$-optional process of the form

$$f_t(w)=\begin{cases}f_{\tau_k}(w), & \text{if } (t,w)\in[\![\tau_k,\tau_{k+1})\!) \text{ for } k=0,\dots,|I|,\\ \boldsymbol{0}, & \text{if } t=\infty,\end{cases}$$

for $\mathscr{H}^\vee$-optional times $\tau_0,\dots,\tau_{|I|}$ with $0=\tau_0\le\dots\le\tau_{|I|+1}=\infty$.

(First step): Let $\mathrm{id}_\Omega\star f$ denote the map $W\to W$, $(\omega,h)\mapsto(\omega,f(\omega,h))$. We show that, for any $t\in\overline{\mathbb{T}}$, $\mathrm{id}_\Omega\star f$ is $\mathscr{H}^\vee_t$-$\mathscr{H}^\vee_t$-measurable. By basic measure theory, using universal completeness of $(W,\mathscr{H}^\vee_\infty,\mathscr{H}^\vee)$, it suffices to show that it is $\mathscr{H}^\vee_t$-$\mathscr{H}^{i,0}_t$-measurable for all $i\in I$, with the notation from the definition of the data $\mathbf{F}$.

Let $i\in I$, $E\in\mathscr{F}^i_0$. Then, $(\mathrm{id}_\Omega\star f)^{-1}(E\times\mathbb{B}^{\overline{\mathbb{T}}})=E\times\mathbb{B}^{\overline{\mathbb{T}}}\in\mathscr{H}^\vee_0$. Next, let $t\in\overline{\mathbb{T}}\setminus\{0\}$, $i\in I$, $E\in\mathscr{F}^i_t$, $n\in\mathbb{N}^*$, for $\ell=1,\dots,n$, $(u_\ell,\boldsymbol{b}_\ell)\in[0,t]_{\overline{\mathbb{T}}}\times\mathbb{B}$, and $B=\{h\in\mathbb{B}^{\overline{\mathbb{T}}}\mid\forall\ell=1,\dots,n\colon h(u_\ell)=\boldsymbol{b}_\ell\}$. Then, for $H_t=E\times B$,

$$\begin{aligned}(\mathrm{id}_\Omega\star f)^{-1}(H_t)=(E\times\mathbb{B}^{\overline{\mathbb{T}}})\cap\bigcap_{\ell=1}^n\Big(&\bigcup_{k=0}^{|I|}\{w\in W\mid\tau_k(w)\le u_\ell<\tau_{k+1}(w),\ f_{\tau_k}(w)=\boldsymbol{b}_\ell\}\\ &\cup\{w\in W\mid u_\ell=\infty,\ \boldsymbol{b}_\ell=\boldsymbol{0}\}\Big)\end{aligned}$$

is an element of $\mathscr{H}^\vee_t$. Finally, let again $t\in\overline{\mathbb{T}}\setminus\{0\}$, $\beta\in\mathfrak{w}_1$, and $\boldsymbol{b}\in\mathbb{B}$. Let

$$\tilde f\colon\overline{\mathbb{T}}\times W\to\mathbb{B},\ (t,w)\mapsto\begin{cases}\boldsymbol{1}, & \text{if } t=0,\\ f_t(w), & \text{else,}\end{cases}$$

and $\sigma_{\boldsymbol{b}}=\inf\{u\in\overline{\mathbb{T}}\mid\tilde f_u\le\boldsymbol{b}\}$. Then, Theorem 2.12, local right-constancy and optionality of $\tilde f$ as well as the completeness assumption on $\mathscr{H}^\vee$ imply that $\sigma_{\boldsymbol{b}}$ is an $\mathscr{H}^\vee$-optional time. For $H_t=\{\tau_{\boldsymbol{b}}\le t,\ \pi\circ\tau_{\boldsymbol{b}}=\beta\}$, we infer that

$$(\mathrm{id}_\Omega\star f)^{-1}(H_t)=\{\sigma_{\boldsymbol{b}}\le t,\ \pi\circ\sigma_{\boldsymbol{b}}=\beta\}\in\mathscr{H}^\vee_t.$$

We conclude that $\mathrm{id}_\Omega\star f$ is $\mathscr{H}^\vee_t$-$\mathscr{H}^\vee_t$-measurable.

($\mathrm{Opt}(\mathscr{H}^\vee)$-$\mathrm{Opt}(\mathscr{H}^\vee)$-measurability): Now let τ be an $\mathscr{H}^\vee$-optional time and let $\tau^f=\tau\circ(\mathrm{id}_\Omega\star f)$. There is $\alpha\in\mathfrak{w}_1$ with $\pi\circ\tau\le\alpha$, whence $\pi\circ\tau^f\le\alpha$. Moreover, for any $t\in\overline{\mathbb{T}}$ and $\beta\in\mathfrak{w}_1$, we have, using the first step's result,

$$\{\tau^f\le t,\ \pi\circ\tau^f=\beta\}=(\mathrm{id}_\Omega\star f)^{-1}(\{\tau\le t,\ \pi\circ\tau=\beta\})\in\mathscr{H}^\vee_t.$$

Hence,

$$(f^\#)^{-1}([\![0,\tau)\!))=[\![0,\tau^f)\!)\in\mathrm{Opt}(\mathscr{H}^\vee).$$

Moreover, for any $E \in \mathscr{H}^\vee_\infty$, we have $(\mathrm{id}_\Omega \star)^{-1}(E) \in \mathscr{H}^\vee_\infty$ by the first step, whence

$$(f^\#)^{-1}(\{\infty\} \times E) = \{\infty\} \times (\mathrm{id}_\Omega \star f)^{-1}(E) \in \mathrm{Opt}(\mathscr{H}^\vee).$$

This completes the proof of $\mathrm{Opt}(\mathscr{H}^\vee)$-$\mathrm{Opt}(\mathscr{H}^\vee)$-measurability.

($\mathrm{Prg}(\mathscr{H}^\vee)$-$\mathrm{Prg}(\mathscr{H}^\vee)$-measurability): Let $M \in \mathrm{Prg}(\mathscr{H}^\vee)$ and $t \in \overline{\mathbb{T}}$. Then, $M \cap [\![0,t]\!] \in \mathscr{P}_{\overline{\mathbb{T}}} \otimes \mathscr{H}^\vee_t$, whence

$$\begin{aligned}(f^\#)^{-1}(M) \cap [\![0,t]\!] &= [\mathrm{id}_{\overline{\mathbb{T}}} \times (\mathrm{id}_\Omega \star f)]^{-1}(M) \cap [\![0,t]\!] \\ &= [\mathrm{id}_{\overline{\mathbb{T}}} \times (\mathrm{id}_\Omega \star f)]^{-1}(M \cap [\![0,t]\!]) \in \mathscr{P}_{\overline{\mathbb{T}}} \otimes \mathscr{H}^\vee_t,\end{aligned}$$

because $\mathrm{id}_{\overline{\mathbb{T}}} \times (\mathrm{id}_\Omega \star f)$ is $\mathscr{P}_{\overline{\mathbb{T}}} \otimes \mathscr{H}^\vee_t$-$\mathscr{P}_{\overline{\mathbb{T}}} \otimes \mathscr{H}^\vee_t$-measurable by the first step and basic measure theory. This proves $\mathrm{Prg}(\mathscr{H}^\vee)$-$\mathrm{Prg}(\mathscr{H}^\vee)$-measurability. □

Proof of Lemma 3.11. (First step): As χ is $\mathscr{F}^\vee$-optional, it is $\mathscr{F}^\vee$-adapted (this follows from Corollary 2.19). Using this and the fact that $\mathscr{F}^\vee$ is universally augmented in $\mathscr{E}$, applying methods similar to those employed in the proof of Lemma 3.10, one proves that $\mathrm{id}_\Omega \star \chi$ is $\mathscr{F}^\vee_t$-$\mathscr{H}^{i,0}_t$-measurable for any $i \in I$ and $t \in \overline{\mathbb{T}}$. Hence, it is $\mathscr{F}^\vee_t$-$\mathscr{H}^\vee_t$-measurable for any $t \in \overline{\mathbb{T}}$, by basic measure theory and the universal completeness of $(\Omega, \mathscr{E}, \mathscr{F}^\vee)$.

(Second step): Let τ be an $\mathscr{H}^\vee$-optional time, and $\tau^\chi = \tau \circ (\mathrm{id}_\Omega \star \chi)$. Then, it follows from the first step, in a way completely analogous to the proof of Lemma 3.10, second step, that τ^χ is an $\mathscr{F}^\vee$-optional time. Hence,

$$[\mathrm{id}_{\overline{\mathbb{T}}} \times (\mathrm{id}_\Omega \star \chi)]^{-1}([\![0,\tau[\![) = [\![0,\tau^\chi[\![\in \mathrm{Opt}(\mathscr{F}^\vee).$$

Moreover, for any $E \in \mathscr{H}^\vee_\infty$, we have $(\mathrm{id}_\Omega \star \chi)^{-1}(E) \in \mathscr{F}^\vee_\infty$, by the first step, whence

$$[\mathrm{id}_{\overline{\mathbb{T}}} \times (\mathrm{id}_\Omega \star \chi)]^{-1}(\{\infty\} \times E) = \{\infty\} \times (\mathrm{id}_\Omega \star \chi)^{-1}(E) \in \mathrm{Opt}(\mathscr{F}^\vee).$$

Thus, $[\mathrm{id}_{\overline{\mathbb{T}}} \times (\mathrm{id}_\Omega \star \chi)]$ is $\mathrm{Opt}(\mathscr{F}^\vee)$-$\mathrm{Opt}(\mathscr{H}^\vee)$-measurable. As a consequence, the composition $\eta \circ [\mathrm{id}_{\overline{\mathbb{T}}} \times (\mathrm{id}_\Omega \star \chi)]$ is $\mathrm{Opt}(\mathscr{F}^\vee)$-$\mathscr{B}_{\mathbb{B}}$-measurable, i.e. $\mathscr{F}^\vee$-optional. □

Proof of Theorem 3.12. (Ad "**F** is SPF": basic properties of the data): Any $(\xi, \chi) \in \mathcal{W}$ satisfies $(\omega, \chi(\omega)) \in W$ for all $\omega \in \Omega$. Moreover, for all $i \in I$ and $t, u \in \overline{\mathbb{T}}$ with $t \leq u$, it follows readily from the definition that $\mathscr{H}^i_t$ is a σ-algebra on W and that we have $\mathscr{H}^i_t \subseteq \mathscr{H}^i_u$, which proves that $\mathscr{H}^i$ is a filtration. Moreover, $\mathscr{F}^i_t \otimes \{\emptyset, \mathbb{B}^{\overline{\mathbb{T}}}\} \subseteq \mathscr{H}^i_t$, for all $t \in \overline{\mathbb{T}}$; hence, $\mathrm{Prd}(\mathscr{H}^i) \subseteq \mathscr{M}^i \subseteq \mathrm{Opt}(\mathscr{H}^i)$. It remains to show the Axioms in Definition 3.1.

(Ad Axiom 1): This axiom is satisfied by construction.

(Ad Axiom 2): Let χ be such that there is ξ with $(\xi, \chi) \in \mathcal{W}$ and $i \in I$. χ is $\mathscr{F}^\vee$-optional, hence -adapted. Thus, for any $i \in I$, all

$$H_t \in \mathscr{F}^i_\infty \otimes \mathscr{B}_{\mathbb{B}}^{[0,t]_{\overline{\mathbb{T}}}} \otimes \{\emptyset, \mathbb{B}^{(t,\infty]_{\overline{\mathbb{T}}}}\}|_W$$

satisfy $(\mathrm{id}_\Omega \star \chi)^{-1}(H_t) \in \mathscr{F}^\vee_\infty \subseteq \mathscr{E}$.

Let $i \in I$, $t \in \overline{\mathbb{T}}$, $\beta \in \mathfrak{w}_1$. For any state process χ, let

$$\tilde{\chi} \colon \overline{\mathbb{T}} \times \Omega \to \mathbb{B},\ (t,\omega) \mapsto \begin{cases} \mathbf{1}, & \text{if } t = 0, \\ \chi_t(\omega), & \text{else,} \end{cases}$$

and $M^{\boldsymbol{b}}_\chi = \{(u,\omega) \in \overline{\mathbb{T}} \times \Omega \mid \tilde{\chi}(u) \leq \boldsymbol{b}\}$ and $D_{M^{\boldsymbol{b}}_\chi}$ be the début of the set $M^{\boldsymbol{b}}_\chi$. By construction, $\tilde{\chi}$ is left-constant at all $u \in \overline{\mathbb{T}}$ with $\pi(u) = \mathfrak{w}_1$. Moreover, $\tilde{\chi}$ is also right-constant at all those u. Thus, as $\mathscr{F}^\vee$ is universally augmented in the universally complete σ-algebra $\mathscr{E}$, $D_{M^{\boldsymbol{b}}_\chi}$ is an $\mathscr{F}^\vee$-optional time, by Theorem 2.12. Hence,

$$(\mathrm{id}_\Omega \star \chi)^{-1}(\{\tau_{\boldsymbol{b}} \leq t,\ \pi \circ \tau_{\boldsymbol{b}} = \beta\}) = \{D_{M^{\boldsymbol{b}}_\chi} \leq t,\ \pi \circ D_{M^{\boldsymbol{b}}_\chi} = \beta\} \in \mathscr{F}^\vee_t \subseteq \mathscr{E}.$$

We conclude that for $i \in I$, $\mathrm{id}_\Omega \star \chi$ is $\mathscr{E}$-$\mathscr{H}^{i,0}_\infty$-measurable, with $\mathscr{H}^{i,0}_\infty$ as in the definition of the data $\mathbf{F}$. As $\mathscr{E}$ is universally complete, basic measure theory implies that $\mathrm{id}_\Omega \star \chi$ is also $\mathscr{E}$-$\mathscr{H}^{i,1}_\infty$-measurable, because $\mathscr{H}^{i,1}_\infty = [\mathscr{H}^{i,0}_\infty]^{\mathrm{u}}$, as in the definition of the data $\mathbf{F}$. As $\mathscr{H}^{i}_\infty \subseteq \mathscr{H}^{i,1}_\infty$, we infer that, in particular, $\mathrm{id}_\Omega \star \chi$ is $\mathscr{E}$-$\mathscr{H}^{i}_\infty$-measurable.

(Ad Axiom 3): This is evident because $\xi = \chi$ for all $(\xi, \chi) \in \mathcal{W}$. We conclude that $\mathbf{F}$ is an SPF.

(Ad Axiom 4): This axiom is satisfied by construction.

(Ad well-posedness): Let $s \in \mathcal{S}$, $i \in I$, τ^i be an optional time for i, and $\tilde{\chi}$ be a state process.

We now define two sequences $(\sigma_n)_{n\in\mathbb{N}}$ of $\mathscr{H}^\vee$-optional times $\sigma_n \colon W \to \overline{\mathbb{T}}$ with $\pi \circ \sigma_n < \alpha$ for all $n \in \mathbb{N}$ and $(\eta^n)_{n\in\mathbb{N}}$ of right-continuous, decreasing $\mathscr{H}^\vee$-optional processes $\eta^n \colon \overline{\mathbb{T}} \times W \to \mathbb{B}$ with upper vertical level smaller than or equal to α and $\eta^n_\infty = \mathbf{0}$, satisfying the following "extra properties", for all $n \in \mathbb{N}$:

1. If $n > 0$, then $\sigma_{n-1} \leq \sigma_n$ and, for all $w \in \{\sigma_{n-1} < \infty\}$, we have $\sigma_{n-1}(w) < \sigma_n(w)$.
2. If $n > 0$, then $s_{\sigma_n(\omega,h)}(\omega, \eta^n(\omega,h)) < \eta^n_{\sigma_n(\omega,h)}(\omega,h)$ for all $(\omega,h) \in W$ with $\sigma_n(\omega,h) < \infty$.
3. If $n > 0$, then for all $t, u \in \overline{\mathbb{T}}$ and $w \in W$ with $(t,w), (u,w) \in [\![\sigma_{n-1}, \sigma_n)\!)$, we have $\eta^n_t(w) = \eta^n_u(w)$ and $|\eta^n_t(w)|_1 \leq |I| - n + 1$.[70]
4. If $n > 0$, then we have $\eta^{n-1}|_{[\![0,\sigma_{n-1})\!)} = \eta^n|_{[\![0,\sigma_{n-1})\!)}$.
5. $\eta^n|_{[\![0,\sigma_0)\!)} = \eta^0|_{[\![0,\sigma_0)\!)}$.
6. For all $(t,\omega,h) \in [\![\sigma_0, \sigma_n)\!)$ we have
$$\eta^n_t(\omega,h) = s_t(\omega, \eta^n(\omega,h)).$$

Let $\sigma_0 = \tau^i$ and $\eta^0 = z$. By construction, σ_0 is an $\mathscr{H}^\vee$-optional time, and η^0 is right-continuous, decreasing and $\mathscr{H}^\vee$-optional. Moreover, $\eta^0_\infty = \mathbf{0}$ by assumption. The extra properties above are clearly satisfied for $n = 0$.

Now let $k \in \mathbb{N}$ and σ_n and η^n with the claimed properties be given for all $n = 0, \dots, k$. Let, for any $t \in \overline{\mathbb{T}}$ and $(\omega,h) \in W$:

$$\eta^{k+1}_t(\omega,h) = \begin{cases} \eta^k_t(\omega,h), & \text{if } (t,\omega,h) \in [\![0,\sigma_k)\!), \\ s_{\sigma_k(\omega,h)}(\omega, \eta^k(\omega,h)), & \text{if } (t,\omega,h) \in [\![\sigma_k,\infty)\!), \\ \mathbf{0}, & \text{else}, \end{cases}$$
$$\sigma_{k+1}(\omega,h) = \inf\{t \geq \sigma_k(\omega,h) \mid s_t(\omega, \eta^{k+1}(\omega,h)) \neq \eta^{k+1}_t(\omega,h)\}.$$

With the notation from Lemma 3.10, $\overline{\mathbb{T}} \times W \to \mathbb{B}$, $(t,\omega,h) \mapsto s_t(\omega, \eta^k(\omega,h))$ equals the composition $s \circ (\eta^k)^\#$, and, by that lemma and the induction hypothesis, it is $\mathscr{H}^\vee$-progressively measurable. Therefore, using again the induction hypothesis, completeness, and Corollary 2.9, we obtain that $[s \circ (\eta^k)^\#]_{\sigma_k}$ is $\mathscr{H}^\vee_{\sigma_k}$-measurable and η^{k+1} is a (simple) $\mathscr{H}^\vee$-optional process. Moreover, η^{k+1} clearly has right-continuous and decreasing paths, because $s_t(\omega,h) \leq h(t-)$ for all $(t,\omega,h) \in \overline{\mathbb{T}} \times W$. Moreover, $\eta^{k+1}_\infty = \mathbf{0}$ by definition. As $\pi \circ \sigma_k < \alpha$ and as s has upper vertical level smaller than or equal to α, η^{k+1} has upper vertical level smaller than or equal to α.

Now we show that σ_{k+1} is an $\mathscr{H}^\vee$-optional time with $\pi \circ \sigma_{k+1} < \alpha$. As η^{k+1} is $\mathscr{H}^\vee$-optional, $s \circ (\eta^{k+1})^\#$ is $\mathscr{H}^\vee$-progressively measurable, by Lemma 3.10. In particular, both $s \circ (\eta^{k+1})^\#$ and η^{k+1} are $\mathscr{H}^\vee$-progressively measurable. η^{k+1} is left- and right-constant and s is left-constant at all $u \in \overline{\mathbb{T}}$ with $\pi(u) = \mathfrak{w}_1$. If we had $\pi \circ \sigma_{k+1}(\omega,h) = \mathfrak{w}_1$ for some $(\omega,h) \in W$, then $\sigma_{k+1}(\omega,h) < \infty$ and $(s \circ (\eta^{k+1})^\#)_{\sigma_{k+1}}(\omega,h) = \eta^{k+1}_{\sigma_{k+1}}(\omega,h)$. Using local right-constanty of η^{k+1} and the fact that the paths of s are lower semicontinuous from the right, we see that there would be $i \in I$ such that, for any $t \in \overline{\mathbb{T}}$ with $\sigma_{k+1}(\omega,h) < t$ and infinitely many $u \in (\sigma_{k+1}(\omega,h), t)_{\overline{\mathbb{T}}}$, we have $(s^i \circ (\eta^{k+1})^\#)_{\sigma_{k+1}}(\omega,h) = 0$

[70]Here, $|.|_1$ denotes the $\mathbb{L}^1$-norm. That is, for $\boldsymbol{b} = (b^i)_{i\in I} \in \mathbb{B}$, we have $|\boldsymbol{b}|_1 = \sum_{i\in I} |b^i| = \sum_{i\in I} b^i$ which is the number of components with value one, or game-theoretically speaking, the number of active agents.

and $(s^i \circ (\eta^{k+1})^\#)_u(\omega, h) = 1$. Then, choosing t sufficiently small so that η^{k+1} is constant on $[\sigma_{k+1}(\omega, h), t)_{\overline{\mathbb{T}}}$, there would be such u with

$$1 = (s^i \circ (\eta^{k+1})^\#)_u(\omega, h) \leq \eta^{k+1}_{u-}(\omega, h) = \eta^{k+1}_{\sigma_{k+1}}(\omega, h) = (s^i \circ (\eta^{k+1})^\#)_{\sigma_{k+1}}(\omega, h) = 0$$

— a contradiction. Hence, $\pi \circ \sigma_{k+1} < \mathfrak{w}_1$. Thus, by augmentedness of $\mathscr{H}^\vee$ and Theorem 2.12, σ_{k+1} is an $\mathscr{H}^\vee$-optional time. Moreover, as s and η^{k+1} have upper vertical level inferior or equal to α, the preceding inequality even implies $\pi \circ \sigma_{k+1} < \alpha$.

It remains to show the extra properties for $n = k + 1$. Property 1 follows directly from the definitions of η^{k+1} and σ_{k+1}; the fact that $\pi \circ \sigma_k < \mathfrak{w}_1$ (σ_k is an optional time by induction hypothesis); and the equality

$$(\dagger) \qquad s_{\sigma_k(\omega,h)}(\omega, \eta^k(\omega, h)) = s_{\sigma_k(\omega,h)}(\omega, \eta^{k+1}(\omega, h)), \qquad (\omega, h) \in W.$$

We briefly prove (†). For this, fix $(\omega, h) \in W$ and let $t = \sigma_k(\omega, h)$, a deterministic optional time. By Axiom 4 applied to t and $\beta = \pi(t)$, for all $i \in I$, there is $\mathscr{M}^i$-measurable $\tilde{s}^i$ with $\tilde{s}^i_t = s^i_t$. By definition, we have $\eta^k(\omega, h)|_{[0,t)_{\overline{\mathbb{T}}}} = \eta^{k+1}(\omega, h)|_{[0,t)_{\overline{\mathbb{T}}}}$. Hence, by Proposition 3.4, Part 1, all $i \in I$ satisfy

$$s^i_{\sigma_k(\omega,h)}(\omega, \eta^k(\omega, h)) = \tilde{s}^i_t(\omega, \eta^k(\omega, h)) = \tilde{s}^i_t(\omega, \eta^{k+1}(\omega, h)) = s^i_{\sigma_k(\omega,h)}(\omega, \eta^{k+1}(\omega, h)),$$

which proves (†).

Regarding Property 2, let $(\omega, h) \in W$ such that $\sigma_{k+1}(\omega, h) < \infty$. By construction, $\eta^{k+1}(\omega, h)$ is constant on $[\sigma_k(\omega, h), \infty)_{\overline{\mathbb{T}}}$. By Property 1, just shown above, $[\sigma_k(\omega, h), \sigma_{k+1}(\omega, h))_{\overline{\mathbb{T}}}$ is non-empty. Hence,

$$s_{\sigma_{k+1}(\omega,h)}(\omega, \eta^{k+1}(\omega, h)) \leq \eta^{k+1}_{\sigma_{k+1}(\omega,h)-}(\omega, h) = \eta^{k+1}_{\sigma_{k+1}(\omega,h)}(\omega, h).$$

Note that

$$(*) \qquad s_{\sigma_{k+1}(\omega,h)}(\omega, \eta^{k+1}(\omega, h)) \neq \eta^{k+1}_{\sigma_{k+1}(\omega,h)}(\omega, h),$$

because $\pi \circ \sigma_{k+1} < \mathfrak{w}_1$ as shown above. Thus, we get $s_{\sigma_{k+1}(\omega,h)}(\omega, \eta^{k+1}(\omega, h)) < \eta^{k+1}_{\sigma_{k+1}(\omega,h)}(\omega, h)$ as claimed.

Regarding Property 3, by construction η^{k+1} is scenariowise constant on $[\![\sigma_k, \sigma_{k+1})\!)$. Let $(t, w) \in [\![\sigma_k, \sigma_{k+1})\!)$. If $k = 0$, then $|\eta^{k+1}_t(w)| \leq |I| = |I| - (k + 1) + 1$. If $k > 0$, then, by the induction hypothesis on Property 2 (by the choice of (t, w), $\sigma_k(w) < \infty$), and by monotonicity of η^k, we have

$$|\eta^{k+1}_t(w)|_1 = |s_{\sigma_k(\omega,h)}(\omega, \eta^k(\omega, h))|_1 < |\eta^k_{\sigma_k(\omega,h)}(\omega, h)|_1 \leq |\eta^k_{\sigma_{k-1}(\omega,h)}(\omega, h)|_1 \leq |I| - k + 1.$$

Hence, $|\eta^{k+1}_t(w)|_1 \leq |I| - k + 1 - 1 = |I| - (k + 1) + 1$.

Property 4 holds true for $n = k + 1$ by definition. By Property 4 and the induction hypothesis for Properties 1 and 5, we infer Property 5 for $n = k + 1$.

Regarding Property 6, first let $(t, \omega, h) \in [\![\sigma_0, \sigma_k)\!)$. Then, if $\pi(t) < \mathfrak{w}_1$, using the induction hypothesis, Proposition 3.4 (Part 1) combined with Axiom 4 applied to the optional time t and $\beta = \pi(t)$, and Property 4 for $n = k + 1$ which we have proven above, we get

$$\eta^{k+1}_t(\omega, h) = \eta^k_t(\omega, h) = s_t(\omega, \eta^k(\omega, h)) = s_t(\omega, \eta^{k+1}(\omega, h)).$$

If $\pi(t) = \mathfrak{w}_1$, both $\eta^{k+1}(\omega, h)$ and $s(\omega, \eta^{k+1}(\omega, h))$ are left-constant at t by progressive measurability of η^{k+1} and s (Remark 2.2, Part 5). Whence, using the fact that $\pi \circ \sigma_0 < \mathfrak{w}_1$ and what has been shown just before, we infer $\eta^{k+1}_t(\omega, h) = s_t(\omega, \eta^{k+1}(\omega, h))$. Second, let $(t, \omega, h) \in [\![\sigma_k, \sigma_{k+1})\!)$. Then, by definition of σ_{k+1}, we obtain

$$\eta^{k+1}_t(\omega, h) = s_t(\omega, \eta^{k+1}(\omega, h)).$$

The construction is complete.

Next, let $n_* = |I| + 1$ and $\eta = \eta^{n_*}$. We claim that:

7. $\sigma_{n_*} = \infty$.
8. $\eta|_{[\![0,\tau^i)\!)} = z|_{[\![0,\tau^i)\!)}$, and $s_t(\omega, \eta(\omega, h)) = \eta_t(\omega, h)$ for all (t, ω, h) with $\tau^i(\omega, h) \leq t$.

Suppose that there is $(\omega, h) \in W$ with $\sigma_{n_*}(\omega, h) < \infty$. Then, by Properties 1 and 3, $\eta^{n_*}_{\sigma_{n_*}-} = \mathbf{0}$. Property 2 then yields the contradiction $s_{\sigma_{n_*}(\omega,h)}(\omega, \sigma_{n_*}(\omega, h)) < \mathbf{0}$. Hence, $\sigma_{n_*} = \infty$. The second claim now follows from the initial values of σ_0 and η^0, Properties 5 and 6, as well as the convention that $s_\infty = \mathbf{0} = \eta_\infty$.

Let $\chi = \eta \circ (\mathrm{id}_\Omega \star \tilde{\chi})$. By Lemma 3.11, χ is $\mathscr{F}^\vee$-optional. It is decreasing and right-continuous, and satisfies $\chi_\infty = 0$. $\eta = \eta^{n_*}$ has upper vertical level inferior or equal to α, and hence the same is true of χ. Hence, $(\chi, \chi) \in \mathcal{W}$. By construction of η, for all $\omega \in \Omega$, we have:

$$\chi|_{[0,\tau^i(\omega,\tilde{\chi}(\omega)))_{\overline{\mathbb{T}}}} = \tilde{\chi}|_{[0,\tau^i(\omega,\tilde{\chi}(\omega)))_{\overline{\mathbb{T}}}},$$
$$s_{\llcorner}\chi|_{[\tau^i(\omega,\tilde{\chi}(\omega))\infty]_{\overline{\mathbb{T}}}} = \chi|_{[\tau^i(\omega,\tilde{\chi}(\omega))\infty]_{\overline{\mathbb{T}}}},$$

where we used Property 8. We infer using Proposition 3.4 that $\chi \approx_{i,\tau^i} \tilde{\chi}$.

Let χ' be another state process such that $\chi' \approx_{i,\tau^i} \tilde{\chi}$ satisfying $s_{\llcorner}\chi'(t, \omega) = \chi'(t, \omega)$ for all $(t, \omega) \in [\![\tau^i \circ (\mathrm{id}_\Omega \star \chi(\omega), \infty]\!]$. As both χ' and χ are $\mathscr{F}^\vee$-optional with locally right-constant paths,

$$\sigma = \inf\{t \in \overline{\mathbb{T}} \mid \chi'_t \neq \chi_t\}$$

defines an $\mathscr{F}^\vee$-optional time with $\chi'_\sigma \neq \chi_\sigma$ on $\{\sigma < \infty\}$. Hence, for all $\omega \in \{\sigma < \infty\}$, we have

$$\chi'_\sigma = (s_{\llcorner}\chi')_\sigma = (s_{\llcorner}\chi)_\sigma = \chi_\sigma,$$

again by a scenario- and componentwise application of SPF Axiom 4 combined with Proposition 3.4, Part 1.[71] Thus, $\{\sigma < \infty\} = \emptyset$. As $\chi'_\infty = \mathbf{0} = \chi_\infty$, we infer that $\chi' = \chi$. The proof of well-posedness is complete. □

Proof of Theorem 3.14. Note that, being an action process in $\mathbf{F}$, every component of ξ is an $\mathscr{F}^\vee$-optional decreasing process valued in $\{0, 1\}$ taking the value zero in time ∞. Hence, any component is of the form $1[\![0, \sigma)\!)$ for some $\mathscr{F}^\vee$-decision time σ, by Theorem 2.12 and universal completeness of $(\Omega, \mathscr{E}, \mathscr{F}^\vee)$. Then, the theorem follows directly from Proposition 2.33 in combination with Remark 2.29. □

Proof of Theorem 3.15. (s is a strategy profile): Let $i \in I$. We start with proving all properties not concerning measurability. Regarding lower semicontinuity from the right, let $t \in \overline{\mathbb{T}}$ be a right-limit point, i.e. $\pi(t) = \mathfrak{w}_1$, and let $(\omega, h) \in W$ such that $s^i_t(\omega, h) = 1$. Then, $t < \tau(\omega)$. Hence, there is $u \in \mathbb{R}_+$ such that $t < u < \tau(\omega)$, so that $s^i(\omega, h)$ is constant with value 1 on $[t, u)_{\overline{\mathbb{T}}}$. Thus, s^i is lower semicontinuous from the right. Moreover, $s^i_t(\omega, h) = 0$ for all $(t, \omega, h) \in \overline{\mathbb{T}} \times W$ with $\pi(t) \geq \mathfrak{w}$. Hence, s^i has upper vertical level inferior or equal to $\alpha = \mathfrak{w} + 1$. By construction, we have $s^i_t(\omega, h) \leq h^i(t-)$ for all $(t, \omega, h) \in \overline{\mathbb{T}} \times W$ and $s^i_\infty = 0$.

It remains to verify $\mathscr{H}^i$-progressive measurability and "local" $\mathscr{M}^i$-measurability. We first show that the map

$$f^i \colon \overline{\mathbb{T}} \times W \to \mathbb{R},\ (t, \omega, h) \mapsto \begin{cases} v^{i,\pi(t)}(\omega), & \text{if } \pi(t) < \mathfrak{w}, \\ -1, & \text{else}, \end{cases}$$

is $\mathscr{H}^i$-progressively measurable. As f^i is valued in $\{-1\} \cup [0, 1]_{\mathbb{R}_+}$, it suffices to show that, for all $b \in \mathbb{R}$ with $b > -1$ and $u \in \overline{\mathbb{T}}$, we have $\{f^i > b\} \in \mathscr{M}^i$. Let such b and u be given. Then, we have

$$\{f^i > b\}$$

[71] Namely, for fixed $\omega \in \Omega$, $\sigma(\omega)$ is a deterministic optional time. For any $i \in I$, there is $\mathscr{M}^i$-measurable $\tilde{s}^i$ with $s^i_{\sigma(\omega)} = \tilde{s}^i_{\sigma(\omega)}$. Now, plug in both $(\omega, \chi(\omega))$ and $(\omega, \chi'(\omega))$ and apply Proposition 3.4, Part 1.

$$= \bigcup_{\beta\in\mathfrak{w}} (\rho^{\mathfrak{w}})^{-1}(\overline{\mathbb{R}_+} \times \{\beta\}) \times \{v^{i,\beta} > b\} \times \mathbb{B}^{\overline{\mathbb{T}}} \in \mathscr{P}_{\overline{\mathbb{T}}} \otimes \mathscr{F}^i_0 \otimes \{\emptyset, \mathbb{B}^{\overline{\mathbb{T}}}\}.$$

Now, we have

$$\mathscr{P}_{\overline{\mathbb{T}}} \otimes \mathscr{F}^i_0 \otimes \{\emptyset, \mathbb{B}^{\overline{\mathbb{T}}}\} \subseteq \mathrm{Prg}(\mathscr{F}^i \otimes \{\emptyset, \mathbb{B}^{\overline{\mathbb{T}}}\}) \subseteq \mathrm{Prg}(\mathscr{H}^i)$$

by Remark 2.2, Part 4.

The map $\tilde\tau\colon W \to \overline{\mathbb{T}}$, $(\omega, h) \mapsto \tau(\omega)$, defines an $\mathscr{F}^i \otimes \{\emptyset, \mathbb{B}^{\overline{\mathbb{T}}}\}$-optional time because τ is an $\mathscr{F}^i$-optional time. In particular, $[\![0, \tilde\tau)\!) \in \mathscr{M}^i$.

Moreover, with $\mathbb{B}_{i,0} = \{\boldsymbol{b} = (b^i)_{i\in I} \in \mathbb{B} \mid b^i = 0\}$, we have

$$\tau_{i,0} = \inf\{t \in \overline{\mathbb{T}} \mid z^i_{t-} = 0\} = \inf_{\boldsymbol{b}\in\mathbb{B}_{i,0}} \tau^-_{\boldsymbol{b}}.$$

For any $\boldsymbol{b} \in \mathbb{B}_{i,0}$, $\tau^-_{\boldsymbol{b}}$ is an $\mathscr{H}^i$-optional time with $[\![0, \tau^-_{\boldsymbol{b}})\!) \in \mathrm{Prd}(\mathscr{H}^i) \subseteq \mathscr{M}^i$, in view of Lemma 3.9. By Proposition 2.10, Part 4, infima of finitely many optional times are optional times. Thus, $\tau_{i,0}$ is an $\mathscr{H}^i$-optional time. Moreover,

$$[\![0, \tau_{i,0})\!) = \bigcap_{\boldsymbol{b}\in\mathbb{B}_{i,0}} [\![0, \tau^-_{\boldsymbol{b}})\!) \in \mathscr{M}^i.$$

Let us extend η^j to a process

$$\tilde\eta^j : \overline{\mathbb{T}} \times W \to (0,\infty)_{\mathbb{R}_+},\ (t, \omega, h) \mapsto \begin{cases} \eta^j_{p(t)}(\omega), & \text{if } t < \infty, \\ 1, & \text{else.} \end{cases}$$

As η^j, seen as a classical process with time axis $\mathbb{R}_+$, is $\mathscr{G}^i$-progressively measurable in the classical sense, $\tilde\eta^j$ is $\mathscr{F}^i \otimes \{\emptyset, \mathbb{B}^{\overline{\mathbb{T}}}\}$-optional and thus $\mathscr{M}^i$-measurable in the sense of the present text. With this, we get the representation

$$s^i = 1[\![0, \tau_{i,0})\!) \cdot \Big(1[\![0, \tilde\tau)\!) + 1[\![\tilde\tau, \infty)\!) \cdot 1\Big\{f^i \geq \frac{\tilde\eta^j}{1 + \tilde\eta^j}\Big\}\Big),$$

from which we infer that s^i is $\mathscr{H}^i$-progressively measurable.

It remains to show that s^i is "locally" $\mathscr{M}^i$-measurable, for any $i \in I$. For this let $i \in I$, $\beta \in \mathfrak{w}_1$, and τ^i be an optional time for i. Let

$$\tilde s^i = 1[\![0, \tau_{i,0})\!) \cdot \Big(1[\![0, \tilde\tau)\!) + 1[\![\tilde\tau, \infty)\!) \cdot 1\Big\{v^{i,\beta} \geq \frac{\tilde\eta^j_{\tau^i}}{1 + \tilde\eta^j_{\tau^i}},\ \beta \in \mathfrak{w}\Big\}\Big).$$

As $v^{i,\beta}$, seen as a process constant in time, is $\mathscr{F}^i \otimes \{\emptyset, \mathbb{B}^{\overline{\mathbb{T}}}\}$-optional and thus $\mathscr{M}^i$-measurable, combining with the measurability properties of the other relevant processes in the definition of $\tilde s^i$, we infer that $\tilde s^i$ is $\mathscr{M}^i$-measurable. Moreover, we have $s^i_{\tau^i} = \tilde s^i_{\tau^i}$ on $\{\pi \circ \tau^i = \beta\}$, as wanted.

(Construction of Pr): Let $\Pi = (\mathscr{P}^{i,\tau^i}, \kappa^{i,\tau^i}, p_{i,\mathfrak{p}}, \mathscr{P}_{i,\mathfrak{p}})_{\mathfrak{p}=(\tau^i,\mathfrak{x})\in\mathfrak{P}^i,\, i\in I}$ be given as follows:

- for any $i \in I$ and any optional time τ^i for i, $\mathscr{P}^{i,\tau^i}$ is the coarsest (i.e. two-element) σ-algebra on $\mathfrak{P}^i(\tau^i)$;
- for any $i \in I$, any optional time τ^i for i, any $\mathfrak{p} \in \mathfrak{P}^i(\tau^i)$ and $E \in \mathscr{E}$, let $\kappa^{i,\tau^i}(E, \mathfrak{p}) = \mathbb{P}(E)$;
- for any $i \in I$, any optional time τ^i for i, any $\mathfrak{p} = (\tau^i, \mathfrak{x}) \in \mathfrak{P}^i(\tau^i)$, let $\mathscr{P}_{i,\mathfrak{p}}$ be the coarsest (i.e. two-element) σ-algebra on $\mathfrak{x}$;
- for any $i \in I$, any optional time τ^i for i, any $\mathfrak{p} = (\tau^i, \mathfrak{x}) \in \mathfrak{P}^i(\tau^i)$, and any $\omega \in \Omega$, let $p_{i,\mathfrak{p}}(\omega) = \mathrm{Out}^\star(s \mid \tau^i, \tilde\chi)$ for $\tilde\chi \in \mathfrak{x}$.

Π clearly defines a belief system and $U = (u_{i,\mathfrak{p}})_{i\in I,\,\mathfrak{p}\in\mathfrak{P}^i}$ with $u_{i,\mathfrak{p}} = u_i$ obviously defines a taste system in the sense of Definition 3.6. $\mathscr{W}$ is a σ-algebra on W.

We show that Pr is an EU preference structure. For this, let $i \in I$, τ^i be an optional time for i and $\mathfrak{p} = (\tau^i, \mathfrak{x}) \in \mathfrak{P}^i(\tau^i)$ be a corresponding information set. Regarding Property 3(a), the map u_i is bounded and $\mathscr{W}$-$\mathscr{B}_{\mathbb{R}}$-measurable, because, first, the maps $W \ni (\omega, h) \mapsto \sigma_i(h)$ are measurable functions of the family $(\tau_{\boldsymbol{b}})_{\boldsymbol{b}\in\mathbb{B}}$. And, second, for any $(\omega, h) \in W$,

$$a_i(h) = 1\{\mathrm{prj}_{\mathfrak{w}+1} \circ \rho^{\mathfrak{w}} \circ \sigma_i(h) < \mathfrak{w}\},$$

where $\mathrm{prj}_{\mathfrak{w}+1} \colon \overline{\mathbb{R}_+} \times (\mathfrak{w}+1) \to (\mathfrak{w}+1)$ is the canonical projection. Hence, $W \ni (\omega, h) \mapsto a_i(h)$ is measurable as well. Furthermore, for the proof of Property 3(b), let $s' \in \mathcal{S}$ and denote the state processes induced by s' and s given the information set $\mathfrak{p}$ — or more precisely, given arbitrary $\tilde{\chi} \in \mathfrak{x}$ and time τ^i — by χ' and χ, respectively. This makes sense because information sets are small here and characterised by "equality on $[\![0, \tau^i)\!)$", see Proposition 3.4. Then, $p_{i,\mathfrak{p}}$ is constant with value χ, by construction. Moreover, as $\chi \in \mathfrak{x}$, we get, for all $\omega \in \Omega$,

$$\mathrm{Out}^{s'}_{i,\mathfrak{p}}(\omega) = (\omega, \chi'(\omega)) = (\mathrm{id}_\Omega \star \chi')(\omega).$$

By Axiom 2, $(\mathrm{id}_\Omega \star \chi')$ is $\mathscr{E}$-$\mathscr{H}^j_\infty$-measurable for all $j \in I$. By universal completeness and basic measure theory, then, it is also $\mathscr{E}$-$\mathscr{H}^\vee_\infty$-measurable. Property 3(c) is trivially satisfied because $p_{i,\mathfrak{p}}$ is constant. The constancy of $p_{i,\mathfrak{p}}$ also implies that Property 3(d) holds true. Hence, Pr is an expected utility preference structure.

(Dynamic consistency of (Pr, s)): Property 4(a) in the definition of dynamic consistency is satisfied by construction. For the proof of Property 4(b), let $i \in I$, τ^i, σ^i be optional times for i with $\tau^i \leq \sigma^i$, $\mathfrak{p} = (\tau^i, \mathfrak{x}) \in \mathfrak{P}^i(\tau^i)$ an information set for i with time τ^i and $\omega \in \Omega$. Using Proposition 3.4, let χ denote the state process induced by s given $\mathfrak{p}$.[72] Then, by construction, $p_{i,\mathfrak{p}}(\omega) = \chi$ and $\mathrm{Out}^\star(s \mid \tau^i, p_{i,\mathfrak{p}}(\omega)) = \chi$. Moreover, $\varphi^s_{i,\mathfrak{p},\sigma^i}(\chi)$ equals the information set $\mathfrak{p}' = (\sigma^i, \mathfrak{x}')$ at time σ^i with $\chi \in \mathfrak{x}'$. Hence, by definition of $p_{i,\mathfrak{p}'}$, $p_{i,\varphi^s_{i,\mathfrak{p},\sigma^i}(p_{i,\mathfrak{p}}(\omega))}(\omega)$ equals the state process induced by s given $\mathfrak{p}'$ alias χ at time σ^i. It follows, then, directly from the definition of "state process induced by s" that $p_{i,\varphi^s_{i,\mathfrak{p},\sigma^i}(p_{i,\mathfrak{p}}(\omega))}(\omega) = \chi$. Hence, Property 4(b) is satisfied as well. For the proof of Property 4(c), let $i \in I$, τ^i, σ^i be optional times for i with $\tau^i \leq \sigma^i$, $\mathfrak{p} = (\tau^i, \mathfrak{x}) \in \mathfrak{P}^i(\tau^i)$ and $E \in \mathscr{E}$. Note that $p_{i,\mathfrak{p}}$ is constant, whence $\kappa^{i,\sigma^i}(E, \varphi^s_{i,\mathfrak{p},\sigma^i} \circ p_{i,\mathfrak{p}}) = \mathbb{P}(E) = \mathbb{P}(E \mid \varphi^s_{i,\mathfrak{p},\sigma^i} \circ p_{i,\mathfrak{p}})$. We conclude that (Pr, s) is dynamically consistent.

(Ad dynamic rationality): Let $i, j \in I$ with $i \neq j$, $\mathfrak{p} = (\tau^i, \mathfrak{x}) \in \mathfrak{P}^i$ such that there is $\mathscr{F}^i$-progressively measurable $\chi \in \mathfrak{x}$, and $\tilde{s} \in \mathcal{S}$ such that $\tilde{s}^j = s^j$. Without loss of generality, we can assume that χ has no jumps on $[\![\tau^i \circ (\mathrm{id}_\Omega \star \chi)]\!]$, that is, for any $\omega \in \Omega$, χ is constant on $[\tau^i(\omega, \chi(\omega)), \infty]_{\overline{\mathbb{T}}}$ and, if $\tau^i(\omega, \chi(\omega)) > 0$, there is $u \in [0, \tau^i(\omega, \chi(\omega)))_{\overline{\mathbb{T}}}$ such that χ is even constant on $[u, \infty]_{\overline{\mathbb{T}}}$.

Let $\tilde{\chi}$ denote the state process induced by $\tilde{s}$ given $(\tau^i, \chi_{i,\mathfrak{p}})$. Recall that $\chi_{i,\mathfrak{p}}$, by construction, is the state process induced by s given $(\tau^i, \chi_{i,\mathfrak{p}})$, that is $\chi_{i,\mathfrak{p}} = \mathrm{Out}^s_{i,\mathfrak{p}}$. By Proposition 3.4, χ, $\chi_{i,\mathfrak{p}}$, and $\tilde{\chi}$ cannot be distinguished from another until τ^i. In particular, we have, for all $\omega \in \Omega$,

$$\tau^i(\omega, \chi(\omega)) = \tau^i(\omega, \chi_{i,\mathfrak{p}}(\omega)) = \tau^i(\omega, \tilde{\chi}(\omega)).$$

Let $\hat{\tau}^i = \tau^i(\omega, \chi(\omega))$, an $\mathscr{F}^i$-optional time. Indeed, this can be shown as in Lemma 3.11, using the additional assumption that χ is $\mathscr{F}^i$-progressively measurable and thus even $\mathscr{F}^i$-optional, because it is componentwise decreasing, locally right-constant, and $(\Omega, \mathscr{E}, \mathscr{F}^i)$ is universally complete so that Theorem 2.12 can be applied. Furthermore, let $\tilde{\sigma}_k = \sigma_k \circ \tilde{\chi}$ and $\hat{\sigma}_k = \sigma_k \circ \chi_{i,\mathfrak{p}}$ for both $k = 1, 2$, which are $\mathscr{F}^\vee$-optional times.

[72] More precisely, by the proposition, the induced state process does not depend on the choice of $\tilde{\chi} \in \mathfrak{x}$.

First note that, as $\chi_{i,\mathfrak{p}}$ and $\tilde{\chi}$ coincide on $[\![0, \hat{\tau}^i)\!)$,

$$\mathbb{E}[u_i \circ \mathrm{Out}^{\tilde{s}}_{i,\mathfrak{p}}\ 1\{\tilde{\sigma}_i < \hat{\tau}^i\} \mid \mathscr{F}^*_{i,\mathfrak{p}}] = \mathbb{E}[u_i \circ \mathrm{Out}^{s}_{i,\mathfrak{p}}\ 1\{\hat{\sigma}_i < \hat{\tau}^i\} \mid \mathscr{F}^*_{i,\mathfrak{p}}].$$

Similarly, we get

$$\mathbb{E}[u_i \circ \mathrm{Out}^{\tilde{s}}_{i,\mathfrak{p}}\ 1\{\tilde{\sigma}_j < \hat{\tau}^i\} \mid \mathscr{F}^*_{i,\mathfrak{p}}] = \mathbb{E}[u_i \circ \mathrm{Out}^{s}_{i,\mathfrak{p}}\ 1\{\hat{\sigma}_j < \hat{\tau}^i\} \mid \mathscr{F}^*_{i,\mathfrak{p}}]$$

Furthermore, we have

$$\mathbb{E}[u_i \circ \mathrm{Out}^{\tilde{s}}_{i,\mathfrak{p}}\ 1\{\hat{\tau}^i \leq \tilde{\sigma}_i \wedge \tilde{\sigma}_j < \tau\} \mid \mathscr{F}^*_{i,\mathfrak{p}}] \leq 0,$$

with equality if $\tilde{s} = s$, because s avoids stopping before τ.

It remains to consider the most central event $E = \{\hat{\tau}^i \vee \tau \leq \tilde{\sigma}_i \wedge \tilde{\sigma}_j\}$. Note that E can be decomposed into the three cases: a) $\hat{\tau}^i \vee \tau \leq \tilde{\sigma}_i < \tilde{\sigma}_j$, b) $\hat{\tau}^i \vee \tau \leq \tilde{\sigma}_i = \tilde{\sigma}_j$, c) $\hat{\tau}^i \vee \tau \leq \tilde{\sigma}_j < \tilde{\sigma}_i$. By definition of s^j, we have $p \circ \hat{\tau}^i = p \circ \tilde{\sigma}_j$ and $\pi \circ \tilde{\sigma}_j < \mathfrak{w}$ $\mathbb{P}$-almost surely on $E \cap \{\pi \circ \hat{\tau}^i < \mathfrak{w}\}$. Now, for any optional time σ and $\alpha \in \mathfrak{w}$, let $\sigma \oplus \alpha$ denote the optional time given by $p \circ (\sigma \oplus \alpha) = p \circ \sigma$ and, on $\{\sigma < \infty\}$, $\pi \circ (\sigma \oplus \alpha) = \pi \circ \sigma + \alpha$.[73] Also, let use recall that we extend η^k to a map $\overline{\mathbb{T}} \times \Omega \to \mathbb{R}$, via the requirement $\eta^k = \eta^k \circ (p \times \mathrm{id}_\Omega)$, an $\mathscr{I}_{\overline{\mathbb{T}}}(\mathbb{T}) \otimes \mathscr{F}^i_0$-measurable map.

Then, we get — with conditional probability denoting conditional expectation of the corresponding indicator —:

$$\begin{aligned}
&\mathbb{E}\big[u_i \circ \mathrm{Out}^s_{i,\mathfrak{p}}\ 1_E \mid \mathscr{F}^*_{i,\mathfrak{p}}\big] \\
&= \mathbb{E}\big[\eta^i_{\tilde{\sigma}_i}\ 1\{\hat{\tau}^i \vee \tau \leq \tilde{\sigma}_i < \tilde{\sigma}_j,\, \pi \circ \hat{\tau}^i < \mathfrak{w}\} - 1\{\hat{\tau}^i \vee \tau \leq \tilde{\sigma}_i = \tilde{\sigma}_j,\, \pi \circ \hat{\tau}^i < \mathfrak{w}\} \mid \mathscr{F}^*_{i,\mathfrak{p}}\big] \\
&= \sum_{\beta,\gamma \in \mathfrak{w}\colon \beta < \gamma} \eta^i_{\hat{\tau}^i}\, \mathbb{P}\big(\tilde{\sigma}_i = \hat{\tau}^i \oplus \beta,\, \tilde{\sigma}_j = \hat{\tau}^i \oplus \gamma,\, \tau \leq \hat{\tau}^i \oplus \beta,\, \pi \circ \hat{\tau}^i < \mathfrak{w} \mid \mathscr{F}^*_{i,\mathfrak{p}}\big) \\
&\qquad - \sum_{\beta \in \mathfrak{w}} \mathbb{P}\big(\tilde{\sigma}_i = \hat{\tau}^i \oplus \beta = \tilde{\sigma}_j \geq \tau,\, \pi \circ \hat{\tau}^i < \mathfrak{w} \mid \mathscr{F}^*_{i,\mathfrak{p}}\big).
\end{aligned}$$

We have used that $\tilde{\eta}^i$ is $\mathscr{M}^i$-measurable, whence the $\mathscr{F}^*_{i,\mathfrak{p}}$-measurability of $\eta^i_{\hat{\tau}^i}$.

Now, we have, for $\beta, \gamma \in \mathfrak{w}$ with $\beta < \gamma$:

$$\begin{aligned}
&\big\{\tilde{\sigma}_i = \hat{\tau}^i \oplus \beta,\, \tilde{\sigma}_j = \hat{\tau}^i \oplus \gamma,\, \tau \leq \hat{\tau}^i \oplus \beta,\, \pi \circ \hat{\tau}^i < \mathfrak{w}\big\} \\
&= \Big\{\forall \delta \in [0,\beta)_{\mathfrak{w}}\colon \tilde{s}^i_{\hat{\tau}^i \oplus \delta} \circ (\mathrm{id}_\Omega \star \chi) = 1,\ \tilde{s}^i_{\hat{\tau}^i \oplus \beta} \circ (\mathrm{id}_\Omega \star \chi) = 0, \\
&\qquad \tau \leq \hat{\tau}^i \oplus \beta,\, \pi \circ \hat{\tau}^i < \mathfrak{w}, \\
&\qquad \forall \delta \in [0,\gamma)_{\mathfrak{w}}\colon f^j_{\hat{\tau}^i + \delta} \geq \frac{\eta^i_{\hat{\tau}^i}}{1 + \eta^i_{\hat{\tau}^i}},\ f^j_{\hat{\tau}^i + \gamma} < \frac{\eta^i_{\hat{\tau}^i}}{1 + \eta^i_{\hat{\tau}^i}}\Big\}.
\end{aligned}$$

We have crucially used that $\tilde{\sigma}_i = \hat{\tau}^i \oplus \beta$ and $\tilde{\sigma}_j = \hat{\tau}^i \oplus \gamma$ imply that χ and $\tilde{\chi}$ coincide on $[\![0, \hat{\tau}^i \oplus \beta)\!)$. Now, as χ is $\mathscr{F}^i$-optional, $\tilde{s}^i$ is $\mathscr{H}^i$-progressively measurable, τ is an $\mathscr{F}^i$-optional time, the first two lines of the set on the right-hand side of the equation are $\mathscr{F}^i_\infty$-measurable conditions. The third line on the right-hand side, however, can be written as

$$\Big\{\omega \in \Omega \mid h\big(\hat{\tau}^i, \frac{\eta^i_{\hat{\tau}^i}}{1 + \eta^i_{\hat{\tau}^i}}, \upsilon^j\big) = 0\Big\},$$

for some $\mathscr{P}_{\overline{\mathbb{T}}} \otimes \mathscr{B}_{\mathbb{R}} \otimes (\mathscr{B}_{\mathbb{R}}|_{[0,1]_{\mathbb{R}}})$-measurable map h. As $\mathscr{F}^*_{i,\mathfrak{p}} \subseteq \mathscr{F}^i_\infty$, and as the family $(\upsilon^{j,\ell})_{\ell \in \mathbb{N}}$ is $\mathbb{P}$-independent and $\mathbb{P}$-independent from $\mathscr{F}^i$, an application of the power rule and the basic behaviour of conditional expectation on functions with independent arguments, we get that

$$\mathbb{P}\big(\tilde{\sigma}_i = \hat{\tau}^i \oplus \beta,\, \tilde{\sigma}_j = \hat{\tau}^i \oplus \gamma,\, \tau \leq \hat{\tau}^i \oplus \beta,\, \pi \circ \hat{\tau}^i < \mathfrak{w} \mid \mathscr{F}^*_{i,\mathfrak{p}}\big)$$

[73] We have already used this construction several times in case $\alpha = 1$, for instance, in the proof of Lemma 2.14. The general construction obtains by iterating this process finitely many times.

$$= \lambda^*_{i,\mathfrak{p}} \cdot \frac{\tilde{\eta}^i_{\tau^i}}{(1+\tilde{\eta}^i_{\tau^i})^{\gamma+1}}.$$

where

$$\lambda^*_{i,\mathfrak{p}} = \mathbb{P}\big(\forall \delta \in [0,\beta)_{\mathfrak{w}} \colon \tilde{s}^i_{\hat{\tau}^i \oplus \delta} \circ (\mathrm{id}_\Omega \star \chi) = 1,\, \tilde{s}^i_{\hat{\tau}^i \oplus \beta} \circ (\mathrm{id}_\Omega \star \chi) = 0, \tau \leq \hat{\tau}^i \oplus \beta,\, \pi \circ \hat{\tau}^i < \mathfrak{w} \mid \mathscr{F}^*_{i,\mathfrak{p}}\big).$$

Using a completely analogous argument, we also obtain

$$\begin{aligned}
&\mathbb{P}\big(\tilde{\sigma}_i = \hat{\tau}^i \oplus \beta = \tilde{\sigma}_j \geq \tau,\, \pi \circ \hat{\tau}^i < \mathfrak{w} \mid \mathscr{F}^*_{i,\mathfrak{p}}\big) \\
&= \lambda^*_{i,\mathfrak{p}} \cdot \frac{\tilde{\eta}^i_{\tau^i}}{(1+\tilde{\eta}^i_{\tau^i})^{\beta+1}}
\end{aligned}$$

Hence, we conclude that:[74]

$$\begin{aligned}
&\mathbb{E}\big[u_i \circ \mathrm{Out}^s_{i,\mathfrak{p}}\ 1_E \mid \mathscr{F}^*_{i,\mathfrak{p}}\big] \\
&= \sum_{\beta \in \mathfrak{w}} \lambda^*_{i,\mathfrak{p}} \left(\eta^i_{\tau^i} \sum_{\gamma=\beta+1}^{\infty} \frac{\tilde{\eta}^i_{\tau^i}}{(1+\tilde{\eta}^i_{\tau^i})^{\gamma+1}} - \frac{\tilde{\eta}^i_{\tau^i}}{(1+\tilde{\eta}^i_{\tau^i})^{\beta+1}} \right) \\
&= \sum_{\beta \in \mathfrak{w}} \lambda^*_{i,\mathfrak{p}}\, (1-1) \frac{\eta^i_{\tau^i}}{(1+\eta^i_{\tau^i})^{\beta+1}} \\
&= 0.
\end{aligned}$$

Putting all pieces together, we infer that

$$\pi_{i,\mathfrak{p}}(\tilde{s}) \leq \pi_{i,\mathfrak{p}}(s),$$

with equality if $\tilde{s} = s$. This completes the proof of dynamic rationality on $\tilde{\mathfrak{P}}$. We conclude that (s, Pr) is in equilibrium on $\tilde{\mathfrak{P}}$. □

Stochastic differential games and control.

Proof of Proposition 3.16. Axioms 1 and 4 are satisfied by assumption. Axiom 2 is satisfied automatically, because χ is a stochastic process by assumption and $\mathscr{H}^i_\infty \subseteq [\mathscr{E} \otimes (\mathscr{B}_{\mathbb{B}})^{\otimes \overline{\mathbb{T}}}|_W]^{\mathrm{u}}$ for all $i \in I$. Axiom 3 is also satisfied, as proved next.

Let (ξ, χ) and (ξ', χ') be elements of $\mathcal{W}$, $i \in I$, τ^i be an $\mathscr{H}^i$-optional time such that $[\![0, \tau^i)\!) \in \mathscr{M}^i$ and $\hat{\tau}^i = \tau^i \circ (\mathrm{id} \star \chi)$ be the induced optional time on Ω. Suppose that $\xi|_{[\![0,\hat{\tau}^i]\!]} \cong_{\mathbb{P}} \xi'|_{[\![0,\hat{\tau}^i]\!]}$. χ (resp. χ') is the up to $\mathbb{P}$-indistinguishability, unique solution to System 3.7 associated to ξ (resp. ξ') and the initial data $(0, \hat{\chi}^0)$, by the third axiom defining $\mathcal{W}$. Then, by the fourth axiom, there is an, up to $\mathbb{P}$-indistinguishability unique, solution $\tilde{\chi}$ to System 3.7 for ξ with initial data $(\hat{\tau}^i, \chi'_{[\![0,\hat{\tau}^i]\!]})$ satisfying $(\xi, \tilde{\chi}) \in \mathcal{W}$. As $\chi'_0 = \hat{\chi}^0$ by definition, and $\xi|_{[\![0,\hat{\tau}^i]\!]} \cong_{\mathbb{P}} \xi'|_{[\![0,\hat{\tau}^i]\!]}$ by hypothesis, Assumption SDG, applied to $(\xi, \tilde{\chi})$ and (ξ', χ'), implies that $\tilde{\chi}$ then also solves System 3.7 for ξ with initial data $(0, \hat{\chi}^0)$. Thus, by the third axiom, χ and $\tilde{\chi}$ are $\mathbb{P}$-indistinguishable. Whence, $\chi|_{[\![0,\hat{\tau}^i]\!]} \cong_{\mathbb{P}} \tilde{\chi}|_{[\![0,\hat{\tau}^i]\!]} \cong_{\mathbb{P}} \chi'|_{[\![0,\hat{\tau}^i]\!]}$.

The claim on well-posedness follows directly from the construction of $\mathcal{S}$. □

Appendix B. The smallest completion of posets

In this note, we characterise the Dedekind-MacNeille completion as the smallest completion of a poset using elementary methods. Although the results shown in the following are slightly stronger than those the author could find in the literature (e.g. in [20]), it is to be assumed that the results, tools, and proofs that follow in this subsection are indeed classical and well-known, and not new.

[74] These computations generalise and are related to the well-known verifications for the classical discrete-time grab-the-dollar game as well as the stochastic variant in [58].

In any case, the aim of the present development is to provide a focused and self-contained overview on the mentioned characterisation as simply and clearly as possible.

B.1. **Basic definitions and notation.** A *partially ordered set*, in short *poset*, is a pair $(P, \leq)$ consisting of a set P and a *partial order* $\leq$ on P, that is, a binary relation that is reflexive, antisymmetric, and transitive. For pragmatic reasons, we write P instead of $(P, \leq)$ and do not indicate the dependence of $\leq$ on P, unless strictly necessary. For any poset P with partial order $\leq$, there is a dual poset on P whose partial order is given by $x \geq y$ iff $x \leq y$, for all $x, y \in P$. Moreover, a binary relation $<$ is induced by letting $x < y$ iff $x \leq y$ and $x \neq y$, for all $x, y \in P$. The corresponding dual relation is denoted by $>$. A *totally ordered set*, or *chain*, is a poset T such that for all $x, y \in T$, $x \leq y$ or $y \leq x$. Given two posets P and Q, a set-theoretic map $f\colon P \to Q$ is said *monotone* iff for all $x, y \in P$ with $x \leq y$, we have $f(x) \leq f(y)$. An *embedding* between two posets P and Q, or of P into Q, is a set-theoretic map $f\colon P \to Q$ such that for all $x, y \in P$, $x \leq y$ iff $f(x) \leq f(y)$.

Posets define the objects and monotone set-theoretic maps between posets the morphisms of a category, denoted by **Pos**. It is easily checked that **Pos**-isomorphisms are exactly the surjective embeddings. For a fixed poset P, there is a further category, denoted by P-**Pos**: its objects are given by the **Pos**-morphisms with domain P and the morphisms between two objects $\varphi\colon P \to Q$ and $\psi\colon P \to R$ are given by all **Pos**-morphisms $f\colon Q \to R$ such that $\psi = f \circ \varphi$. By definition, a P-**Pos**-*embedding* is a P-**Pos**-morphism that is an embedding. It is easily checked that P-**Pos**-isomorphisms are exactly the P-**Pos**-morphisms that are **Pos**-isomorphisms, or equivalently, the surjective P-**Pos**-embeddings.

For any poset P and any subset $A \subseteq P$, the sets of lower and upper bounds are defined by

$$A^\ell = \{x \in P \mid \forall a \in A\colon x \leq a\}, \qquad A^u = \{x \in P \mid \forall a \in A\colon x \geq a\}.$$

In case $A = \{a\}$ is a singleton, then we write $\downarrow a = A^\ell$, the *principal down-set of* a, and $\uparrow a = A^u$, the *principal up-set of* a. A is *downward closed* iff for all $x \in A$, $\downarrow x \subseteq A$. A is *upward closed* iff for all $x \in A$, $\uparrow x \subseteq A$. For $n \in \mathbb{N}$ and symbols $i_1, \ldots, i_n = \ell, u$, we write $A^{i_n i_{n-1} \ldots i_2 i_1} = ((\ldots((A^{i_n})^{i_{n-1}})\ldots)^{i_2})^{i_1}$. An *infimum* of A is an element of $A^\ell \cap A^{\ell u}$. If it exists, it is unique and denoted by $\inf A$. Conversely, a *supremum* of A is an element of $A^u \cap A^{u\ell}$. If it exists, it is unique and denoted by $\sup A$. For **Pos**-isomorphisms f, f or $\mathcal{P}f$, respectively, commutes with both $.^u$ and $.^\ell$, and in particular with the existence and, if applicable, the values of suprema and infima. Note also that when passing from the partial order to its dual $\geq$, lower bounds, principal down-sets, downward closed sets, and infima become upper bounds, principal up-sets, upward closed sets, and suprema, and vice versa.

We collect some basic observations whose proofs are direct and therefore omitted (see [20, Section 7.37 and Lemma 7.39] for a discussion of the first four points from that the others follow). If P is a poset and $A, B \subseteq P$ are subsets, then:

1. $A \subseteq A^{u\ell}$ and $B \subseteq B^{\ell u}$;
2. if $A \subseteq B$, then $A^u \supseteq B^u$ and $A^\ell \supseteq B^\ell$;
3. $A^u = A^{u\ell u}$ and $B^\ell = B^{\ell u \ell}$;
4. for all $x \in P$, $(\downarrow x)^u = \uparrow x$ and $(\uparrow x)^\ell = \downarrow x$;
5. A admits a supremum iff A^u admits an infimum, and in that case $\sup A = \inf A^u$;
6. B admits an infimum iff B^ℓ admits a supremum, and in that case $\inf B = \sup B^\ell$;
7. if A, B admit suprema and $A \subseteq B$, then $\sup A \leq \sup B$;
8. if A, B admit infima and $A \subseteq B$, then $\inf A \geq \inf B$.

A *lattice* is a poset L such that for all $x, y \in L$, $\{x, y\}$ admits both an infimum and a supremum. This terminology is reasonable because such L defines an algebraic lattice, whose meet and join operations $\wedge$ and $\vee$ are given by inf and sup, and conversely, any algebraic lattice $(L, \wedge, \vee)$ naturally defines a poset by letting $x \leq y$ iff $x \wedge y = x$, for all $x, y \in L$, and both operations are essentially inverse to each other. A lattice L is *complete* iff any subset of L admits both an infimum and a supremum. A subset D of L is said *meet-dense* iff any $x \in L$ admits a subset $S \subseteq D$ with $x = \sup S$. A subset D of L is said *join-dense* iff any $x \in L$ admits a subset $S \subseteq D$ with $x = \inf S$, or equivalently, iff D is meet-dense with respect to the dual order.

B.2. **Dense completions.** A *completion* of a poset P is a pair (φ, L), given by a complete lattice L and an embedding $\varphi\colon P \hookrightarrow L$ of P into L. However, by slight abuse, we may refer to φ or L as the completion if the other datum is clear from the context. Note that any completion of a poset P is an object of P-**Pos**. Many possible completions exist and are of interest in the literature. Here, we are interested in the following notions, for which we introduce names.

Definition B.1. Let P be a poset and $\varphi\colon P \hookrightarrow L$ be a completion. We call φ *dense* iff $\operatorname{im}\varphi$ is both join- and meet-dense in L. We call φ *small* iff for any completion $\psi\colon P \hookrightarrow M$, there is a P-**Pos**-embedding $(\varphi, L) \hookrightarrow (\psi, M)$.

These notions are invariant under isomorphisms in P-**Pos**.

Lemma B.2. *Let P be a poset and $\varphi\colon P \hookrightarrow L$, $\psi\colon P \hookrightarrow M$ be two P-**Pos**-isomorphic completions. Then, (φ, L) is dense (small) iff (ψ, M) is dense (small, respectively).*

Proof. Let $f\colon (\varphi, L) \to (\psi, M)$ be a P-**Pos**-isomorphism and g its inverse. For symmetry reasons, it suffices to show that if (ψ, M) has the relevant property, then (φ, L) does so, too.

(Ad "dense"): Let (ψ, M) be dense and $x \in L$. Then, there are $A, B \subseteq P$ such that

$$\sup \mathcal{P}\psi(A) = f(x) = \inf \mathcal{P}\psi(B).$$

As g is a **Pos**-isomorphism with $\varphi = g \circ \psi$ and $\mathcal{P}$ a functor, we infer that

$$\begin{aligned}\sup \mathcal{P}\varphi(A) &= \sup \mathcal{P}(g \circ \psi)(A) = g(\sup \mathcal{P}\psi(A)) \\ &= x = g(\inf \mathcal{P}\psi(B)) = \inf \mathcal{P}(g \circ \psi)(B) = \inf \mathcal{P}\varphi(B).\end{aligned}$$

Hence, (φ, L) is dense.

(Ad "small"): Let (ψ, M) be small and $\rho\colon P \hookrightarrow N$ be a completion of P. Then, there is a P-**Pos**-embedding $h\colon (\psi, M) \hookrightarrow (\rho, N)$. Then, $h \circ f$ is a P-**Pos**-embedding $(\varphi, L) \hookrightarrow (\rho, N)$. Hence, (φ, L) is small. ☐

In [20, Theorem 7.4.1], it is shown that for any poset P there is an up to P-**Pos**-isomorphism unique dense completion, and a representative of it is constructed explicitly, namely the Dedekind-MacNeille completion. In this note, we show the following stronger result: a completion is dense iff it is small, for any poset P there is an up to unique P-**Pos**-isomorphism unique small completion, and it can be represented by the Dedekind-MacNeille completion. Moreover, our method of proof is more elementary.

The building block of the proof will be the following lemma. Given a (dense) completion $\varphi\colon P \hookrightarrow L$ of a poset P, we try to find a representative system for the expression of elements $x \in L$ as suprema and infima over sets in $\operatorname{im}\varphi$.

Lemma B.3. *Let P be a poset, $\varphi\colon P \hookrightarrow L$ be a completion of P, $x \in L$, and $A, B \subseteq P$ such that*

$$\sup \mathcal{P}\varphi(A) = x = \inf \mathcal{P}\varphi(B).$$

Then,

$$A^{u\ell} = \{y \in P \mid \varphi(y) \leq x\}, \qquad B^{\ell u} = \{y \in P \mid \varphi(y) \geq x\}, \tag{B.1}$$

and

$$\sup \mathcal{P}\varphi(A^{u\ell}) = x = \inf \mathcal{P}\varphi(B^{\ell u}). \tag{B.2}$$

Proof. In view of duality, it suffices to show the left-hand equalities in Equations B.1 and B.2. We start with a preliminary observation, namely that

$$(*) \qquad \mathcal{P}\varphi(B^\ell) \subseteq [\mathcal{P}\varphi(B)]^\ell, \qquad \mathcal{P}\varphi(A^u) \subseteq [\mathcal{P}\varphi(A)]^u.$$

Indeed, if $c \in P$ satisfies with $c \leq b$ for all $b \in B$, then $\varphi(c) \leq \varphi(b)$ for all $b \in B$, which shows the left-hand statement. The right-hand one follows from the left-hand one by duality.

(Ad Equation B.2): Let $a \in A$ and $b \in B$. Then, $\varphi(a) \leq x \leq \varphi(b)$, whence $a \leq b$. Thus, $B \subseteq A^u$, and, by Observation 2,

$$(\dagger) \qquad A^{u\ell} \subseteq B^\ell.$$

Hence, using $(*)$ and Observation 1, we get

$$\mathcal{P}\varphi(A) \subseteq \mathcal{P}\varphi(A^{u\ell}) \subseteq \mathcal{P}\varphi(B^\ell) \subseteq [\mathcal{P}\varphi(B)]^\ell.$$

By Observations 6 and 7, this implies

$$x = \sup \mathcal{P}\varphi(A) \leq \sup \mathcal{P}\varphi(A^{u\ell}) \leq \sup[\mathcal{P}\varphi(B)]^\ell = \inf \mathcal{P}\varphi(B) = x,$$

whence $x = \sup \mathcal{P}(A^{u\ell})$.

(Ad Equation B.1): Regarding "$\supseteq$", let $y \in P$ be such that $\varphi(y) \leq x$. Let $z \in A^u$. Then, by Observations $(*)$, 5, and 8

$$x = \sup \mathcal{P}\varphi(A) = \inf[\mathcal{P}\varphi(A)]^u \leq \inf \mathcal{P}\varphi(A^u) \leq \varphi(z).$$

Hence, $\varphi(y) \leq \varphi(z)$. As φ is an embedding, $y \leq z$. We have shown that $y \in A^{u\ell}$.

Regarding "$\subseteq$", let $y \in A^{u\ell}$. By $(\dagger)$, $y \in B^\ell$. In view of $(*)$, thus, $\varphi(y) \in [\mathcal{P}\varphi(B)]^\ell$, and, by Observation 6,

$$\varphi(y) \leq \sup[\mathcal{P}\varphi(B)]^\ell = \inf \mathcal{P}\varphi(B) = x.$$

□

As a consequence, if φ is a dense completion, then for any $x \in L$ there are $A, B \subseteq P$ with $A = A^{u\ell}$ and $B = B^{\ell u}$ – namely the sets on the right-hand sides of the equations in Equation B.1 – such that Equation B.2 holds true. Together with Observation 4, this motivates the following, classical construction. For any poset P, let

$$\mathbf{DM}(P) = \{A \subseteq P \mid A^{u\ell} = A\}, \quad \varphi_{\mathbf{DM}} \colon P \to \mathbf{DM}(P),\ x \mapsto \downarrow x,$$

and equip $\mathbf{DM}(P)$ with the partial order "$\subseteq$". This is well-defined by Observation 4.

Theorem B.4. *Let P be a poset. Then, $\varphi_{\mathbf{DM}} \colon P \to \mathbf{DM}(P)$ is a dense completion of P.*

This classical theorem can be found, for instance, in [20, Theorems 7.40, 7.41]. $\varphi_{\mathbf{DM}}$ is called the *Dedekind-MacNeille completion of P*. It generalises the construction of the extended real line out of the rationals by Dedekind cuts.

Proof. It is clear that $\mathbf{DM}(P)$ is a poset and that $\varphi_{\mathbf{DM}}$ is an embedding. To show that $\mathbf{DM}(P)$ is a complete lattice, let $S \subseteq \mathbf{DM}(P)$. In P, define the sets $A = [\bigcup S]^{u\ell}$ and $B = \bigcap S$. By Observation 3, $A \in \mathbf{DM}(P)$. Moreover, for any $C \in S$, $B \subseteq C$. Applying Observation 2 twice, we get $B^{ul} \subseteq C$. Hence, $B^{ul} \subseteq \bigcap S = B$. Using Observation 1, we infer $B \in \mathbf{DM}(P)$. We claim that

$$\sup S = A, \qquad \inf S = B.$$

Regarding the first equality, we have to show that $A \in S^u \cap S^{u\ell}$. For this, note first that, for all $C \in \mathbf{DM}(P)$, we have $C \in S^u$ iff $\bigcup S \subseteq C$. By Observation 1, $\bigcup S \subseteq [\bigcup S]^{u\ell} = A$, whence $A \in S^u$. Further, let $C \in S^u$. Then $\bigcup S \subseteq C$. Applying Observation 2 twice yields $A = [\bigcup S]^{u\ell} \subseteq C$. Thus, $A \in S^{u\ell}$.

Regarding the second equality, we have to show that $B \in S^\ell \cap S^{\ell u}$. For this, note first that, for all $C \in \mathbf{DM}(P)$, we have $C \in S^\ell$ iff $C \subseteq \bigcap S$. For $C = B$ the latter condition is evidently satisfied, whence $B \in S^\ell$. Further, let $C \in S^\ell$. Then, $C \subseteq \bigcap S = B$. Hence, $B \in S^{\ell u}$. We have shown that $\mathbf{DM}(P)$ is a complete lattice.

It remains to show that $\varphi_{\mathbf{DM}}$ is dense. For this, let $A \in \mathbf{DM}(P)$. Let $B = A^u$. Then, we claim that

$$\sup \mathcal{P}\varphi_{\mathbf{DM}}(A) = A = \inf \mathcal{P}\varphi_{\mathbf{DM}}(B).$$

Regarding the first equality, let $x \in A$. Then, clearly $\varphi_{\mathbf{DM}}(x) = \downarrow x \subseteq A$, because $A = A^{u\ell}$ is downward closed. Thus, $A \in [\mathcal{P}\varphi_{\mathbf{DM}}(A)]^u$. Let $C \in [\mathcal{P}\varphi_{\mathbf{DM}}(A)]^u$. Then,

$$C \supseteq \bigcup [\mathcal{P}\varphi_{\mathbf{DM}}(A)] = \{x \in P \mid \exists a \in A \colon x \leq a\} \supseteq A.$$

Hence, $A \in [\mathcal{P}\varphi_{\mathbf{DM}}(A)]^{u\ell}$. We conclude that $\sup \mathcal{P}\varphi_{\mathbf{DM}}(A) = A$.

Regarding the second equality, let $x \in A$ and $y \in B = A^u$. Then, $x \leq y$, whence $x \in \downarrow y = \varphi_{\mathbf{DM}}(y)$. Thus, $A \subseteq \varphi_{\mathbf{DM}}(y)$. We infer that $A \in [\mathcal{P}\varphi_{\mathbf{DM}}(B)]^\ell$. Next, let $C \in [\mathcal{P}\varphi_{\mathbf{DM}}(B)]^\ell$. That is,

$$C \subseteq \bigcap [\mathcal{P}\varphi_{\mathbf{DM}}(B)] = \{x \in P \mid \forall b \in B \colon x \leq b\} = B^\ell = A^{u\ell} = A.$$

Thus, $A \in [\mathcal{P}\varphi_{\mathbf{DM}}(B)]^{\ell u}$. We conclude that $\inf \mathcal{P}\varphi_{\mathbf{DM}}(B) = A$. □

B.3. **Small completions.** We begin with analysing the extension of morphisms into complete lattices onto a dense completion of the domain, thereby establishing that any dense completion is small.

Proposition B.5. *Let P be a poset, $\varphi \colon P \hookrightarrow L$ be a dense completion and $f \colon P \to M$ be an object in P-**Pos** for some complete lattice M. Then, the set-theoretic map $f_L \colon L \to M$ given by*

$$f_L(x) = \sup\{f(y) \mid y \in P \colon \varphi(y) \leq x\}, \qquad x \in L,$$

*is a P-**Pos**-morphism $(\varphi, L) \to (f, M)$. If f is an embedding, then f_L is so as well.*

Combining the two statements from this proposition, we directly obtain:

Corollary B.6. *Any dense completion $\varphi \colon P \hookrightarrow L$ of a poset P is small.* □

Corollary B.6 and Theorem B.4 directly imply the following.

Corollary B.7. *For any poset P, there is a small completion, namely $\mathbf{DM}(P)$.* □

Proof of the proposition. Regarding the first claim, let $x, x' \in L$ be such that $x \leq x'$. Then, all $y \in P$ with $\varphi(y) \leq x$ satisfy $\varphi(y) \leq x'$. Hence, by Observation 7, $f_L(x) \leq f_L(x')$.

Further, let $z \in P$. Then, for all $y \in P$ with $\varphi(y) \leq \varphi(z)$, we have $y \leq z$, because φ is an embedding, whence $f(y) \leq f(z)$. Furthermore, if $w \in M$ is such that $f(y) \leq w$ for all $y \in P$ with $\varphi(y) \leq \varphi(z)$, then, in particular, $f(z) \leq w$. Thus

$$f(z) = \sup\{f(y) \mid y \in P \colon \varphi(y) \leq \varphi(z)\} = f_L(\varphi(z)).$$

Regarding the second claim, suppose that f is an embedding. Let $x, x' \in L$ with $f_L(x) \leq f_L(x')$. We show the auxiliary Claim 1 that for all $y \in P$ with $\varphi(y) \leq x$ we also have $\varphi(y) \leq x'$. For this, let $y \in P$ with $\varphi(y) \leq x$. By assumption on x and x', we have $f(y) \leq f_L(x) \leq f_L(x')$, whence by transitivity

$$(*) \qquad f(y) \in [\mathcal{P}f(\{y' \in P \mid \varphi(y') \leq x'\})]^{u\ell}.$$

We infer that

$$(\dagger) \qquad f(y) \in \mathcal{P}f(\{y' \in P \mid \varphi(y') \leq x'\}^{u\ell}) = \mathcal{P}f(\{y' \in P \mid \varphi(y') \leq x'\}).$$

For the proof, note that the equality follows from Lemma B.3. Indeed, there is $A \subseteq P$ with $\sup \mathcal{P}\varphi(A) = x'$ because φ is dense. Then, by the lemma, $\sup \mathcal{P}\varphi(A^{u\ell}) = x'$ and $A^{u\ell} = \{y' \in P \mid \varphi(y') \leq x'\}$. Hence, by Observation 3,

$$\{y' \in P \mid \varphi(y') \leq x'\}^{u\ell} = A^{u\ell u\ell} = A^{u\ell} = \{y' \in P \mid \varphi(y') \leq x'\},$$

which implies the equality. For the proof of the $\in$-relation, let $z \in \{y' \in P \mid \varphi(y') \leq x'\}^u$. As f is monotone, we infer that $f(z) \in [\mathcal{P}f(\{y' \in P \mid \varphi(y') \leq x'\})]^u$. Hence, by $(*)$, $f(y) \leq f(z)$. As f is an embedding, $y \leq z$. Hence, $y \in \{y' \in P \mid \varphi(y') \leq x'\}^{u\ell}$, whence $f(y) \in \mathcal{P}f(\{y' \in P \mid \varphi(y') \leq x'\}^{u\ell})$, as claimed.

As an embedding, f is injective. By $(\dagger)$, then, $\varphi(y) \leq x'$. This shows our auxiliary Claim 1. Hence, by Observation 7 and Lemma B.3, using density of φ, we infer

$$x = \sup \mathcal{P}\varphi(\{y \in P \mid \varphi(y) \leq x\}) \leq \sup \mathcal{P}\varphi(\{y' \in P \mid \varphi(y') \leq x'\}) = x'.$$

This shows that f_L is an embedding. $\square$

Next, we analyse the uniqueness of dense completions. From this, all remaining open claims follow easily, as shown afterwards.

Proposition B.8. *Let P be a poset and $\varphi\colon P \hookrightarrow L$, $\psi\colon P \hookrightarrow M$ be dense completions. Then, there is a unique P-**Pos**-morphism $f\colon (\varphi, L) \to (\psi, M)$, and f is a P-**Pos**-isomorphism.*

Proof. By Proposition B.5, there are P-**Pos**-embeddings $\psi_L\colon (\varphi, L) \hookrightarrow (\psi, M)$ and $\varphi_M\colon (\psi, M) \hookrightarrow (\varphi, L)$. For symmetry reasons, it remains to show that any P-**Pos**-morphism $f\colon (\varphi, L) \hookrightarrow (\psi, M)$ is equal to ψ_L and that φ_M is surjective.

To show this, let $f\colon (\varphi, L) \hookrightarrow (\psi, M)$ be a P-**Pos**-morphism and $x \in L$. By density, there are $A, B \subseteq P$ such that

$$\sup \mathcal{P}\varphi(A) = x = \inf \mathcal{P}\varphi(B).$$

As f is a P-**Pos**-morphism and $\mathcal{P}$ is a functor, we infer

$$\sup \mathcal{P}\psi(A) = \sup \mathcal{P}(f \circ \varphi)(A) \leq f(x) \leq \inf \mathcal{P}(f \circ \varphi)(B) = \inf \mathcal{P}\psi(B).$$

As a consequence, the fact that φ_M is a P-**Pos**-morphism and $\mathcal{P}$ a functor, implies

$$x = \sup \mathcal{P}\varphi(A) = \sup \mathcal{P}(\varphi_M \circ \psi)(A) \leq \varphi_M \circ f(x) \leq \inf \mathcal{P}(\varphi_M \circ \psi)(B) = \inf \mathcal{P}\varphi(B) = x.$$

Hence, $\varphi_M \circ f(x) = x$. Thus, φ_M is surjective. As ψ_L is a P-**Pos**-morphism $(\varphi, L) \hookrightarrow (\psi, M)$, this result can be applied to ψ_L (that is, we can plug in ψ_L for f). Then, we get $\varphi_M(f(x)) = x = \varphi_M(\psi_L(x))$, whence $f(x) = \psi_L(x)$ because φ_M is an embedding. $\square$

Corollary B.9. *For any poset P, a completion of P is dense iff it is small.*

Proof. Let (φ, L) be a completion of a poset P. If it is dense, then it is small, by Corollary B.6. For the converse implication, suppose it to be small. Then, $\mathbf{DM}(P)$ is a small completion of P as well, by Corollary B.7. Hence, there are P-**Pos**-embeddings $f\colon (\varphi, L) \hookrightarrow (\varphi_{\mathbf{DM}}, \mathbf{DM}(P))$ and $g\colon (\varphi_{\mathbf{DM}}, \mathbf{DM}(P)) \hookrightarrow (\varphi, L)$. Thus, $f \circ g$ defines a P-**Pos**-embedding of $\mathbf{DM}(P)$ into itself. As $\mathbf{DM}(P)$ is dense, by Theorem B.4, Proposition B.8 implies that $f \circ g = \mathrm{id}_{\mathbf{DM}(P)}$. Hence, f is surjective and, thus, a P-**Pos**-isomorphism with inverse g. By Lemma B.2, (φ, L) is dense. $\square$

Theorem B.10. *For any poset P, there is an up to unique P-**Pos**-isomorphism unique small completion (φ, L), given by the Dedekind-MacNeille completion $\mathbf{DM}(P)$.*

Proof. Let P be a poset. By Corollary B.7, $\mathbf{DM}(P)$ is a small completion of P. By Proposition B.8 and Corollary B.9, between any to small completions there is a unique P-**Pos**-isomorphism. $\square$

In that sense, the Dedekind-MacNeille completion $\mathbf{DM}(P)$ is the smallest completion of a poset P, the use of the definite article being completely specified.

B.4. **Further results.** We have discussed small and dense completions through general embeddings, and the Dedekind-MacNeille completion is given by a specific embedding. If some complete lattice L is fixed as a reference, and all posets under consideration are embedded into L, then we may wish to construct the small completion as a subset of L. This is discussed next.

Proposition B.11. *Let L be a complete lattice and $(P_i)_{i\in I}$ be a family of subsets of L. Then, there is a family $(L_i)_{i\in I}$ of subsets of L such that for each $i \in I$, set-theoretic inclusion $P_i \hookrightarrow L_i$ is a small completion of P_i. This statement also holds true if the property "small" is replaced with "dense" or with "P_i-**Pos**-isomorphic to $\mathbf{DM}(P_i)$". The completions satisfy*

$$\{x \in L \mid \exists A, B \subseteq P_i\colon \sup A = x = \inf B\} \subseteq L_i, \qquad i \in I. \tag{B.3}$$

Proof. For any $i \in I$, set-theoretic inclusion $\iota_i\colon P_i \hookrightarrow L$ defines a completion. Denote the Dedekind-MacNeille completion of P_i by $\varphi^i_{\mathbf{DM}}\colon P_i \hookrightarrow \mathbf{DM}(P_i)$. Then, by Theorem B.10, there is a P_i-**Pos**-embedding $f_i\colon (\varphi^i_{\mathbf{DM}}, \mathbf{DM}(P_i)) \hookrightarrow (\iota_i, L)$. Let $L_i = \operatorname{im} f_i$. Then, (ι_i, L_i) is P_i-**Pos**-isomorphic to the Dedekind-MacNeille completion $(\varphi^i_{\mathbf{DM}}, \mathbf{DM}(P_i))$ of P_i – in particular, it is a small and dense completion of P_i, by Lemma B.2, Theorem B.4, and Corollary B.6.

For the proof of "$\subseteq$" in Equation B.3, let $x \in L$ be such that there are $A, B \subseteq P_i$ with $\sup A = x = \inf B$. Then, $A, B \subseteq L_i$, and, as L_i is a complete lattice, A has a supremum and B an infimum in L_i, denoted by $\sup^{L_i} A$ and $\inf^{L_i} B$, respectively. As $L_i \subseteq L$, $\inf^{L_i} B \le x \le \sup^{L_i} A$. But for all $a \in A$ and $b \in B$, we have $a \le x \le b$. Whence $a \le \inf^{L_i} B$ for all $a \in A$ and thus $\sup^{L_i} A \le \inf^{L_i} B$. Hence, $\inf^{L_i} B = x = \sup^{L_i} A$ and $x \in L_i$. □

Remark B.12. The embedded completions L_i, $i \in I$, need not be unique and the inclusion in Equation B.3 can be strict. To see this, consider the real interval $L = [0, 3]_{\mathbb{R}}$, a complete lattice, and $P_i = [0, 1)_{\mathbb{R}} \cup (2, 3]_{\mathbb{R}}$. Then, there exists an uncountable set of small completions embedded into L, namely $L_i^x = P_i \cup \{x\}$, for any $x \in [1, 2]_{\mathbb{R}}$. Obviously, in any of the cases, the inclusion in Equation B.3 is strict.

The small completion restricts to the subcategory of chains, as the following result implies.

Proposition B.13. *For any chain T, its Dedekind-MacNeille completion $\mathbf{DM}(T)$ is a chain as well.*

Proof. We have to show that for all $A, B \in \mathbf{DM}(T)$ with $B \nsubseteq A$, we have $A \subseteq B$.

Let $A, B \in \mathbf{DM}(T)$ such that there is $b \in B \setminus A$, and let $a \in A$. As $A = A^{u\ell}$ und $B = B^{u\ell}$, A and B are downward closed. Hence, $b \nleq a$, and thus, by the assumption on T, $a \le b$. Thus, $a \in B$. □

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