Gradual Persuasion and Maximal Inequalities

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Economic Theory Seminar - Toulouse, April 2024

Motivating example

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Which signaling policy maximizes the probability of adoption?

What is the maximal probability of adoption?

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Immediate answer: The Sender cannot persuade two receivers with different θ_is with their maximal probability $\frac{p}{\theta_i}$.

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Proposition (Informal)

For every discount factor $\delta < 1$ of the receiver, the sender can reveal information slow enough over time $t \in [0, \infty)$ to incentivize the receiver to adopt slightly above θ.

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- Lower semi-continuity $\Rightarrow \tau = \tau'$.
- $\mathbb{E}[X_\tau] = p \Rightarrow \mathbb{P}[X_\tau = \theta] = \frac{p}{\theta}$. ■

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Maximal Inequalities $\Rightarrow$ Gradual Persuasion.

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Gradual persuasion as a tool for analyzing maximal inequalities; simple proofs for classical results such as

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  New results:

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  - An alternative simple construction for the maximal maximum martingale.

Corollary

The existence of a martingale that persuades any receiver with its maximal possible probability is quite a general phenomenon.

Hardy-Littlewood Inequality

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Hardy-Littlewood Maximal Inequality

For every martingale $(X_t)_{t\in[0,1]}$ with terminal distribution $X_1 = Y$ and every $\theta \in [p,1]$ we have

$$\mathbb{P}\left[\max_{t\in[0,1]} X_t \geq \theta\right] \leq q_Y(\theta).$$

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- $\mathbb{P}[\max_t X_t \geq \theta]$: the probability of adoption in the dynamic model if the sender uses strategy X_t.
- $\mathbb{P}[\max_t X_t \geq \theta] \leq \text{Val(dynamic)} \leq \text{Val(static)} = q_Y(\theta)$. ■

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Theorem: [Dubins-Gilat '78]

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This martingale is called a maximal maximum martingale because its distribution of the maximum FOSD the distribution of the maximum of any other martingale with terminal distribution Y.

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Corollary

Using the Dubins-Gilat martingale, a partially informed sender whose partial information is $Y \in \Delta([0,1])$ persuades every receiver with threshold θ with the maximal possible probability $q_Y(\theta)$ in the gradual persuasion model.

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Our contributions

- A formula for this inequality.
- A different construction of a maximal maximum martingale.
- Simple proofs!

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Proposition (A generalized Hardy-Littlewood inequality)

For every martingale X_t with $X_0 = Y_0$ and $X_1 = Y_1$ we have

$$\mathbb{P}[\max_t X_t \geq \theta] \leq z$$

where z is the fixed point of

$$z = q_{Y_1}\left(\theta + \frac{c}{z}\right)$$

in the range $z \in [\mathbb{P}[Y_0 \geq \theta], q_{Y_1}(\theta)]$.

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Maximization over mean-preserving contraction and mean-preserving spreads have been recently actively studied in the persuasion literature [Dworczak, Martini '19], [Kleiner et. al. '21] [Arieli et. al. '21]. But not both.

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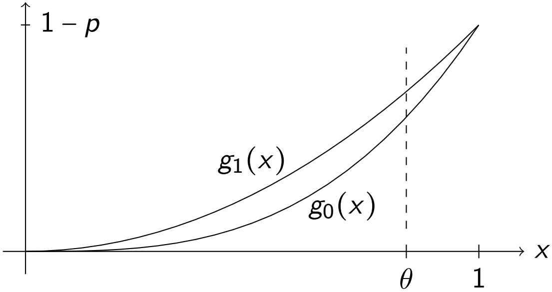
The latter minimization can be solved explicitly:



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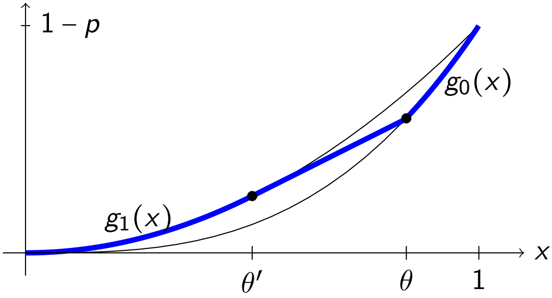

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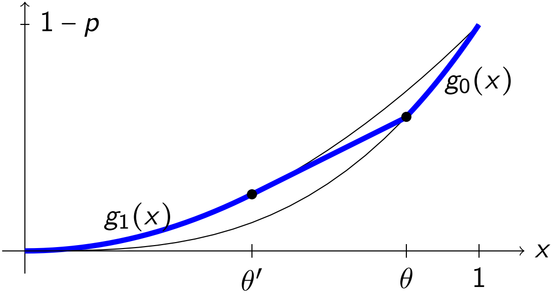
We provide a different, simple, construction of such a martingale.

A Construction for Hobson's Martingale



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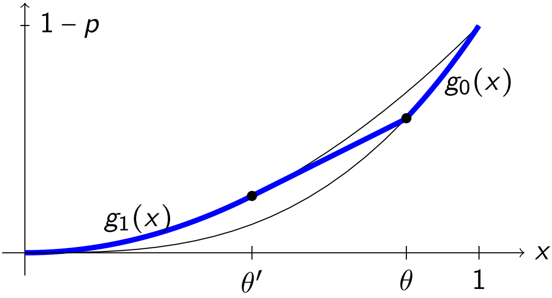

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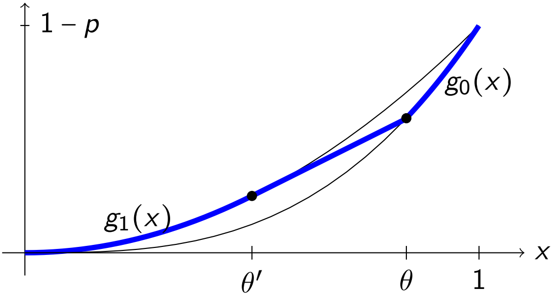

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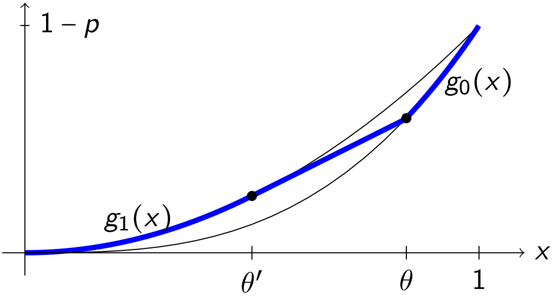

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Namely, a single parametric family (X_t) s.t., $X_t \leq X_{t'}$ for $t < t'$.

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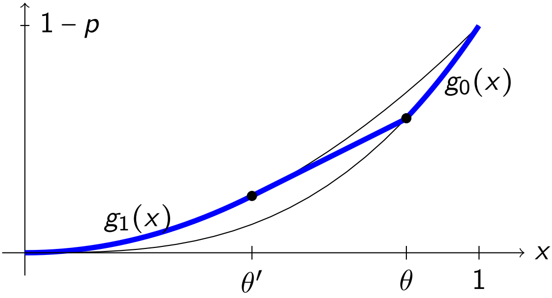

Continuously move θ from 0 to 1.

We get a single parametric family of functions f_t (where $t = \theta$), such that $f_t \leq f_{t'}$ for $t < t'$.

Namely, a single parametric family (X_t) s.t., $X_t \leq X_{t'}$ for $t < t'$.

There exists a martingale whose distribution at time t is X_t [Kellerer '61].

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Sender's utility can be expressed as a monotonic function of $\max_{t \in [0,1]} X_t$. $\Rightarrow$ the same martingales extract the maximal utility from every receiver type $(\theta_i)_{i \in [n]}$.

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This exactly happens in the gradual persuasion model.

Thank You!