Taxing Wealth and Capital Income when Returns are Heterogeneous*

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Abstract

When is a wealth tax preferable to a capital income tax? We study this question theoretically in an infinite-horizon model with entrepreneurs and workers, in which entrepreneurial firms are subject to idiosyncratic productivity shocks and collateral constraints. We focus on the steady state equilibrium that features heterogeneous returns and misallocation of capital. In this equilibrium, increasing the wealth tax always increases aggregate productivity if and only if entrepreneurial productivity is positively auto-correlated. The gains result from the use-it-or-lose-it effect, which causes a reallocation of capital from entrepreneurs with low productivity to those with high productivity. Furthermore, if the capital income tax is adjusted to balance the government's budget, aggregate capital, output, and wages increase. We provide necessary and sufficient conditions for a switch to wealth taxes to imply higher average welfare, which amount to a lower bound on the capital-elasticity of output, α—around 1/3 for most parameter combinations. We then study the optimal tax mix when both instruments can be used to maximize welfare. Optimal policy depends on two thresholds. When α is sufficiently high, optimal policy involves a positive wealth tax and a negative capital income tax (a subsidy); the sign flips when α is sufficiently low, and both taxes are positive between these two thresholds. Finally, we consider extensions that introduce rent-seeking behavior and endogenous entrepreneurial effort.

JEL Codes: E21, E22, E62, H21.

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1 Introduction

When is a wealth tax preferable to a capital income tax? When is the opposite true? What is the optimal mix of capital income and wealth taxes when it is feasible to use both? While these and related questions dominate policy debates, some standard frameworks used by economists to study capital taxation are largely silent on them. This is because capital income and wealth taxation are equivalent under the standard assumption that all individuals earn the same return on wealth. However, a growing body of empirical work documents large and persistent heterogeneity in returns across individuals, which challenges this assumption and opens the door for differences in the aggregate and distributional outcomes of these two forms of taxation.[1]

In this paper, we study capital income and wealth taxation when returns are heterogeneous across individuals. We establish conditions under which replacing capital income taxes with wealth taxes generates efficiency and welfare gains. We also study the more general problem of the optimal mix of wealth and capital income taxes that maximizes average welfare. The framework we employ is fairly standard: an analytical model with infinitely lived entrepreneurs and workers, in which entrepreneurial firms are subject to idiosyncratic productivity shocks and collateral constraints, à la Moll (2014).

Entrepreneurs produce a final good, sold to consumers in a perfectly competitive market, using a common constant-returns-to-scale technology that combines capital and labor but they differ in their productivity, which changes stochastically. There is a bond market, with zero net supply, through which entrepreneurs can borrow from each other subject to a collateral constraint. Entrepreneurs with high productivity borrow to invest in their own firms, while those with low productivity lend at least part of their wealth. Workers are hand-to-mouth and consume their wages, so all the wealth is held by entrepreneurs.

We show four main results. First, we prove that two types of steady-state equilibria can emerge across the parameter space. The first equilibrium is inefficient and exhibits capital misallocation. In this equilibrium collateral constraints bind for more productive entrepreneurs, who then earn higher returns on wealth than less productive ones. Return heterogeneity makes this the empirically relevant equilibrium. We show that this equilibrium emerges under a wide set of plausible parameter values. We solve in closed

[1] See Fagereng, Guiso, Malacrino and Pistaferri (2020), Bach, Calvet and Sodini (2020), Campbell, Ramadorai and Ranish (2019), and Smith, Zidar and Zwick (2021) for empirical evidence on persistent return heterogeneity. See Chari and Kehoe (1999), Golosov, Tsyvinski and Werning (2006), and Stantcheva (2020) for reviews of the literature on capital taxation.

form for the maximum borrowing limit below which the heterogeneous return equilibrium arises and show that it implies debt-to-GDP ratios larger than the U.S. levels. Thus, generating return heterogeneity does not require imposing strict borrowing limits to entrepreneurs. The second equilibrium is efficient but requires implausibly high borrowing limits. In this equilibrium the most productive entrepreneurs employ all the capital, and all entrepreneurs earn the same return. We focus on the heterogeneous returns equilibrium in the rest of the paper.

Second, we show that a marginal increase in wealth taxes increases aggregate productivity (TFP) if and only if entrepreneurial productivity is persistent. We focus on reallocation effect of wealth taxes coming the use-it-or-lose-it effect studied quantitatively in Guvenen et al. (2019). All the allocative effects of wealth taxes come from the change in after-tax returns as there is no behavioral response in the model, saving rates are (endogenously) constant. Wealth taxes trigger a reallocation of wealth because they place a similar tax-burden on entrepreneurs with similar wealth levels regardless of their productivity, unlike capital income taxes that place a higher tax burden on more productive entrepreneurs.

We show that the use-it-or-lose-it effect operates by decreasing after-tax returns of entrepreneurs who earn below the wealth-weighted average returns in the economy, while increasing the returns for those who earn above it. This increase in return dispersion allows high-productivity entrepreneurs to accumulate more wealth. When entrepreneurial productivity is persistent, the wealth share of more productive entrepreneurs increases over time, increasing aggregate productivity.[2] Moreover, when capital income taxes decrease in response to the increase in wealth taxes, capital, output and wages increase.

Third, we study the welfare implications of a marginal increase in wealth taxes (matched by an an adjustment in capital income taxes) and provide necessary and sufficient conditions for the change in taxes to increase welfare. Workers unambiguously benefit from higher wealth taxes through the rise in wages, while entrepreneurs have welfare losses because higher wealth taxes imply higher dispersion and lower expected value of returns. Thus, the aggregate welfare gains of an increase in wealth taxes depend on the strength of the increase in wages, which is determined by the output elasticity with respect to capital. We show that the conditions for welfare gains amount to a lower bound on this elasticity, which is

[2]Our results build on those of Moll (2014) on the role of the persistence of entrepreneurial productivity in determining aggregate productivity. Higher persistence allows productive entrepreneurs to save and relax their financial constraints, increasing efficiency. We show that wealth taxes can achieve the same objectives through their heterogeneous effects across entrepreneurs (for a given degree of persistence).

close to one-third for most parameter combinations.

We also study how the welfare of entrepreneurs varies depending on their productivity and consider alternative welfare measures that take into account wealth accumulation. Higher wealth taxes increase wealth accumulation, ameliorating (and potentially overturning) the welfare loss of entrepreneurs. High-productivity entrepreneurs benefit unambiguously from the increase in wealth taxes once we account for wealth accumulation or if the initial dispersion of returns is not too large. Consequently, taking into account wealth accumulation relaxes the conditions for welfare gains from wealth taxation.

Fourth, we study the optimal combination of capital income and wealth taxes. We derive an optimal tax formula for wealth taxes as a function of the output elasticity with respect to capital, α. The optimal tax weighs the benefit to workers from the increase in wages against the cost to entrepreneurs from higher dispersion and lower expected value of returns. A larger value of α implies a larger response of wages to increases in TFP coming from the increase in wealth taxes, resulting on a higher optimal wealth tax. Accordingly, we characterize the optimal taxes as functions of a lower bound and an upper bound on the elasticity. If the elasticity is above the upper bound, the optimal wealth tax is positive and the capital income tax is negative (a subsidy), the signs flip when the elasticity is below the lower bound, and both taxes are positive in the narrow range between the thresholds.

We consider two separate extensions of our framework: incorporating entrepreneurial effort in the entrepreneurs' production function and incorporating excess returns in the sense that higher returns do not necessarily imply higher entrepreneurial productivity. Although the main results of the analysis remain unchanged, these extensions are informative of the main mechanisms operating in the model and highlight the overall appeal of our theoretical framework.

Finally, we study an alternative perpetual-youth model where all entrepreneurs survive to the next period with a constant probability. There are no annuity markets and total accidental bequests are distributed equally among the newborn. Each newborn entrepreneur draws a productivity type, which is permanent over their lifespan. This alternative model exhibits a stationary wealth distribution that we solve for in closed form. We show that all results from our benchmark model carry on. We also show that the top wealth shares increase after an increase in wealth taxes and high-productivity entrepreneurs unambiguously benefit from a shift from capital income to wealth taxes.

Related literature. An important common element in most of the previous studies on capital taxation is the assumption of homogenous returns across the population. Because capital income and wealth taxes are equivalent under this assumption, an analysis of the differences between the two taxes is naturally absent from this earlier literature, which focuses on capital income taxation (a short list includes Judd 1985; Chamley 1986; Aiyagari, 1995; Imrohoroglu, 1998; Erosa and Gervais, 2002; Garriga, 2003; Conesa, Kitao and Krueger, 2009; Kitao, 2010; Saez and Stantcheva, 2018; Straub and Werning, 2020). That said, a series of recent empirical papers analyze the behavioral savings response to changes in wealth taxes (Seim, 2017; Jakobsen, Jakobsen, Kleven and Zucman, 2019; Londoño-Vélez and Ávila-Mahecha, 2021; Ring, 2021; Brulhart, Gruber, Krapf and Schmidheiny, 2022).

By contrast, there have been few theoretical studies of wealth taxation (and its comparison to capital income taxation) when returns are heterogeneous and, to our knowledge, no analysis of the use-it-or-lose it effect of wealth taxes until very recently (Guvenen et al., 2019).[3] Allais (1977) and Piketty (2014) are partial exceptions, they describe the use-it-or-lose-it mechanism but do not study it. Guvenen et al. (2019) build a rich overlapping generations model that matches the distribution of cross-sectional and lifetime rates of returns, as well as the extreme concentration and the Pareto tail of the wealth distribution. They show quantitatively that there are large efficiency and distributional welfare gains from using wealth taxes instead of capital income taxes.

Relative to Guvenen et al. (2019), here we consider an analytical framework—an infinite-horizon entrepreneur-worker model with heterogeneous/stochastic productivities, which has been widely used in the literature.[4] We use this set up to establish theoretically the conditions under which a wealth tax generates higher aggregate efficiency and welfare than a capital income tax, and vice versa. We also study the optimal mix of the two taxes, which is not studied in Guvenen et al. (2019). Overall, we show that efficiency and welfare gains from wealth taxation arise as a robust outcome under reasonable and large range of parameter values when there is return heterogeneity, and the optimal combination of capital income and wealth taxes depend on the elasticity of output with respect to capital.

Our focus on (persistent) return heterogeneity is motivated by strong empirical evidence for it (that we mentioned earlier) and theoretical work showing the importance of return heterogeneity for generating the dynamics and the Pareto tail of the wealth distribution

[3]Scheuer and Slemrod (2021) provide an excellent survey of the debate on the implementation and optimality of wealth taxes.

[4]This framework is closest to Moll (2014), Buera and Moll (2015) and Itskhoki and Moll (2019). Some quantitative papers that feature similar entrepreneurial heterogeneity and financial frictions are Quadrini (2000), Cagetti and De Nardi (2006), Buera, Kaboski and Shin (2011), and Boar and Midrigan (2020).

observed in the data (Benhabib, Bisin and Zhu 2011, 2013, 2014; Gabaix, Lasry, Lions and Moll 2016; Jones and Kim 2018; Stachurski and Toda 2019, among others).

2 Benchmark Model

Time is discrete. There are two types of infinitely-lived agents: homogenous workers of size L and heterogenous entrepreneurs of size 2. Workers supply one unit of labor inelastically, behave as hand-to-mouth agents, and hold no wealth. Workers' and entrepreneurs' preferences take the form $E_0 \sum_{t=1}^{\infty} \beta^{t-1} \log(c_t)$, where $0 < \beta < 1$.

Each entrepreneur produces a homogenous good combining capital, k, and labor, n, using a constant-returns-to-scale technology

$$y = (zk)^{\alpha} n^{1-\alpha}. \tag{1}$$

We assume that capital does not depreciate.

Entrepreneurs differ in their productivity $z \in \{z_\ell, z_h\}$, where $0 \leq z_\ell < z_h$. Individual productivity follows a Markov process with transition matrix

$$\mathbb{P} = \begin{bmatrix} p & 1-p \\ 1-p & p \end{bmatrix}, \tag{2}$$

where $0 < p < 1$ is the probability that an entrepreneur retains their productivity across periods. The autocorrelation coefficient of the productivity process is $\rho = 2p - 1$, so that if $p > 1/2$ productivity is persistent across time. There is always a mass 1 of high-productivity entrepreneurs $(z = z_h)$ and a mass 1 of low-productivity entrepreneurs $(z = z_\ell)$.

Entrepreneurs hire labor at a wage w and can borrow through a bond market at an interest rate r to invest in their firm over and above their own wealth a. Both markets are perfectly competitive. The same bonds, which are in zero net supply, can be used as a savings device, which will be optimal for entrepreneurs whose return is lower than the interest rate r. Thus, k can be greater or smaller than a. However, entrepreneurs' borrowing is subject to a collateral constraint that depends on beginning-of-period wealth (a), so that

$$k \leq \lambda a, \tag{3}$$

where $\lambda \geq 1$. If $\lambda = 1$ an entrepreneur can use only their wealth in production.

The government taxes capital income at a rate τ_k and (beginning-of-period) wealth at a rate τ_a to finance an exogenous expenditure G.

2.1 Entrepreneur's Problem

We now summarize the entrepreneur's problem. We present it in detail in Appendix A. We start with the choice of capital and labor to maximize entrepreneurial income:

$$\Pi^{\star}(z,a) = \max_{k \leq \lambda a, n \geq 0} (zk)^{\alpha} n^{1-\alpha} - rk - wn. \tag{4}$$

The constant-return-to-scale technology with which the entrepreneur produces implies that entrepreneurs whose marginal return to capital is greater than the interest rate borrow up to their limit and set $k^{\star} = \lambda a$, while those whose return is below the interest rate do not produce and instead earn the return r in the bond market on their wealth a. Consequently, the optimal entrepreneurial income can be written as $\Pi^{\star}(z,a) = \pi^{\star}(z)\, a$, where

$$\pi^{\star}(z) = \begin{cases} \left(\alpha \left(\frac{1-\alpha}{w}\right)^{\frac{1-\alpha}{\alpha}} z - r\right)\lambda & \text{if } \alpha \left(\frac{1-\alpha}{w}\right)^{\frac{1-\alpha}{\alpha}} z > r \\ 0 & \text{if } \alpha \left(\frac{1-\alpha}{w}\right)^{\frac{1-\alpha}{\alpha}} z \leq r \end{cases} \tag{5}$$

is the excess return an entrepreneur earns above the interest rate r.

We now turn to the entrepreneurs' optimal savings problem

$$V(a,z) = \max_{a'} \log\left(R(z)\, a - a'\right) + \beta \sum_{z'} \mathbb{P}(z' \mid z)\, V(a', z')$$

where $R(z) \equiv (1-\tau_a) + (1-\tau_k)(r + \pi^{\star}(z))$ is the after-tax gross return on savings. In solving this problem we take as given time-invariant taxes τ_a and τ_k, and prices r and w, anticipating the results of Section 2.2. The solution is the following optimal savings rule

$$a' = \beta R(z)\, a. \tag{6}$$

Importantly, the saving rate of entrepreneurs is constant and independent of their productivity. In particular, saving rates do not respond to policy, such as changes in wealth taxes, implying that all the reallocation effects of changes in taxation operate through their effect on returns.

Finally, the value of an entrepreneur with assets a and productivity z_i, $i \in \{\ell, h\}$, is

$$V_i(a) = m_i + \frac{1}{1-\beta}\log(a), \tag{7}$$

where $m_i \equiv \frac{\log(1-\beta)}{1-\beta} + \frac{\beta}{(1-\beta)^2}\log(\beta) + \frac{(1-\beta)\log R_i + \beta(1-p)(\log R_\ell + \log R_h)}{(1-\beta)^2(1-\beta(2p-1))}$ and $R_i \equiv R(z_i)$. As for workers, they hold no assets and consume all of their labor income, hence, their value is

$$V_w = \frac{1}{1-\beta}\log w. \tag{8}$$

2.2 Equilibrium

We characterize equilibrium by focusing on aggregate quantities and prices. For the capital market to clear, the equilibrium interest rate must be between the marginal return to capital of the low and high productivity entrepreneurs, i.e.,

$$\alpha\left(\frac{1-\alpha}{w}\right)^{\frac{1-\alpha}{\alpha}} z_\ell \le r \le \alpha\left(\frac{1-\alpha}{w}\right)^{\frac{1-\alpha}{\alpha}} z_h. \tag{9}$$

In fact, the equilibrium interest rate must equal one of the two bounds depending on the net demand for assets in the economy.[5] This gives rise to two possible equilibria.

Let A_i for $i \in \{h, \ell\}$ be each group's aggregate (beginning-of-period) wealth. If $(\lambda - 1) A_h > A_\ell$, the high-productivity entrepreneurs demand more funds than can be supplied by low types and bid up the equilibrium interest rate to their marginal product (the upper bound in equation 9). In this case, all entrepreneurs earn the same rate of return and the equilibrium aggregates coincide with the (efficient) complete markets allocation. Moreover, capital income and wealth taxes are equivalent.

If $(\lambda - 1) A_h < A_\ell$, there are more funds available in the hands of low-productivity entrepreneurs than the amount demanded by high-productivity entrepreneurs, so that the

[5] We can equivalently introduce a corporate sector that faces no collateral constraints and provides entrepreneurs with an alternative use for their wealth. The corporate sector's technology is $Y_c = (z_c K_c)^\alpha L_c^{1-\alpha}$. The marginal return of capital in the corporate sector imposes a lower bound on the equilibrium interest rate: $r \ge \alpha z_c \left(\frac{1-\alpha}{w}\right)^{\frac{1-\alpha}{\alpha}}$. If $z_\ell < z_c < z_h$ the corporate sector and the high-productivity entrepreneurs operate in equilibrium, while the low-productivity entrepreneurs do not produce and instead lend all of their assets. This delivers the same result as our benchmark model with z_c taking the role of z_ℓ. We discuss this further in Appendix D.1.

interest rate is bid down to the marginal product of the low-productivity entrepreneurs,

$$r = \alpha \left(\frac{1-\alpha}{w} \right)^{\frac{1-\alpha}{\alpha}} z_\ell. \tag{10}$$

In this case, high-productivity entrepreneurs are constrained and their aggregate capital is $K_h = \lambda A_h$. Low-productivity entrepreneurs are indifferent between using their assets in their firm or lending them in the bond market. Their capital is $K_\ell = A_\ell - (\lambda - 1) A_h > 0$. This equilibrium features return heterogeneity, with $R_h > R_\ell$, and capital misallocation. Moreover, capital income and wealth taxes are not equivalent.

We focus on the heterogeneous return equilibrium. We provide a necessary and sufficient condition for $(\lambda - 1) A_h < A_\ell$ to hold in steady state in Proposition 1 and show that it is satisfied under a wide range of parameter values.

In the heterogeneous return equilibrium, the aggregate productivity of capital is endogenous and depends on the wealth share of high-productivity entrepreneurs, $s_h \equiv A_h/K$. We denote aggregate productivity as

$$Z \equiv s_h z_\lambda + (1 - s_h) z_\ell, \tag{11}$$

where $z_\lambda \equiv z_h + (\lambda - 1)(z_h - z_\ell)$ denotes the effective productivity of high-productivity entrepreneurs (return on own wealth plus excess return on borrowed funds). By contrast, in the homogenous return equilibrium ($(\lambda - 1) A_h > A_\ell$), high-productivity entrepreneurs use all the capital and $Z = Z^\star = z_h$. This lets us measure the TFP loss due to the collateral constraints and the resulting misallocation of capital as the ratio $\frac{TFP^\star}{TFP} = \left(\frac{z_h}{s_h z_\lambda + (1-s_h) z_\ell} \right)^\alpha$, which declines with the wealth share of h-type s_h and the borrowing limit λ, and increases with the productivity gap $\frac{z_h}{z_\ell}$.

We solve for the aggregate variables in equilibrium in terms of effective capital $Q \equiv ZK \left(= z_\lambda A_h + z_\ell A_\ell\right)$ as in Guvenen et al. (2019). In particular, aggregate output is $Y = Q^\alpha L^{1-\alpha}$ and the interest rate is $r = \alpha \left(Q/L\right)^{\alpha - 1} z_\ell$. We provide details and all relevant expressions in Lemma 1 in Appendix B. We also derive the law of motion of capital and effective capital using the saving rules in equation (6) and the transitions implied by $\mathbb{P}$.

2.3 Characterizing the Steady State

We focus on the steady state of the economy given time-invariant tax rates τ_a and τ_k, where the aggregates, A_h, A_ℓ, K, Q, w, r, and Z, are constant. We start by characterizing the steady state of capital taking productivity as given. As is standard in a wide range of models, the steady state level of capital is such that the marginal (after-tax) return on capital equates the discount factor

$$(1-\tau_a) + (1-\tau_k)\,\alpha Z^{\alpha}\left({}^{K}\!/_{L}\right)^{\alpha-1} = \frac{1}{\beta}. \tag{12}$$

Equation (12) allows us to focus on the determinants of aggregate productivity by expressing other variables in terms of Z. For instance, we write after tax returns as

$$R_h = (1-\tau_a) + \left(\frac{1}{\beta} - (1-\tau_a)\right)\frac{z_\lambda}{Z} \quad \text{and} \quad R_\ell = (1-\tau_a) + \left(\frac{1}{\beta} - (1-\tau_a)\right)\frac{z_\ell}{Z}. \tag{13}$$

Crucially, equation (12) pins down the wealth-weighted return in the economy $(s_h R_h + (1-s_h)\,R_\ell)$, which is therefore constant in steady state and only affected by preferences.

We arrive at an equation characterizing the steady state value of Z from the law of motion for $Q = ZK$ and equation (12): model,

$$(1-\rho\beta\,(1-\tau_a))\,Z^2 - \frac{z_\ell + z_\lambda}{2}\,(1+\rho\,(1-2\beta\,(1-\tau_a)))\,Z + z_\ell z_\lambda \rho\,(1-\beta\,(1-\tau_a)) = 0. \tag{14}$$

Studying this quadratic equation, we show that there is a unique steady state and obtain necessary and sufficient conditions for it to feature heterogeneous returns (see Figure 2). Before providing the formal statement of our result in Proposition 1, we discuss the logic behind the proof. Existence and uniqueness follow from analyzing the solution to equation (14). For $\rho \leq 0$, there is a unique solution. For $\rho > 0$, there are two positive roots. However, only the larger root satisfies $z_\ell < Z < z_\lambda$. Then, there is always a unique equilibrium.

Then, we turn to whether the equilibrium features return heterogeneity with $R_h > R_\ell$. This necessarily implies that there is misallocation, therefore Z is below its efficient level z_h. We obtain an upper bound on the collateral constraint parameter, $\overline{\lambda}$, that guarantees that $Z < z_h$. This upper bound turns out to be not only sufficient but also necessary for the result. The proof for these and all other results is presented in the Appendix.

Proposition 1. ***(Existence and Uniqueness of Steady State)*** *There exists a unique steady state that features heterogenous returns* ($R_h > R_\ell$) *if and only if*

$$\lambda < \overline{\lambda} \equiv 1 + \frac{1-\rho}{1+\rho\left(1-2\left(\beta\left(1-\tau_a\right)+\left(1-\beta\left(1-\tau_a\right)\right)\frac{z_\ell}{z_h}\right)\right)}. \tag{15}$$

Corollary 1. *The condition for the steady state to feature heterogeneous returns can be restated as an upper bound on wealth taxes:*

$$\lambda < \overline{\lambda} \longleftrightarrow \tau_a \leq \overline{\tau}_a \equiv 1 - \frac{1}{\beta\left(1-\frac{z_\ell}{z_h}\right)}\left[\frac{(\lambda-1)(\rho+1)-(1-\rho)}{2(\lambda-1)\rho} - \frac{z_\ell}{z_h}\right]. \tag{16}$$

Corollary 1 gives us a condition for the heterogeneous return equilibrium to arise in terms of the level of wealth taxes, $\tau_a < \overline{\tau}_a$. Intuitively, neither λ nor τ_a can be too high for there to be heterogeneous returns because they both reduce misallocation, λ by loosening the entrepreneurs' collateral constraint and τ_a by increasing the dispersion of (after-tax) returns, as we show in Proposition 2 below. We assume that $\lambda < \overline{\lambda}(\tau_a = 0)$ in the rest of the analysis, so that the steady state of the economy without wealth taxes features return heterogeneity and later analyze what happens to the economy as we change wealth taxes for $\tau_a < \overline{\tau}_a$ without further changes to λ.

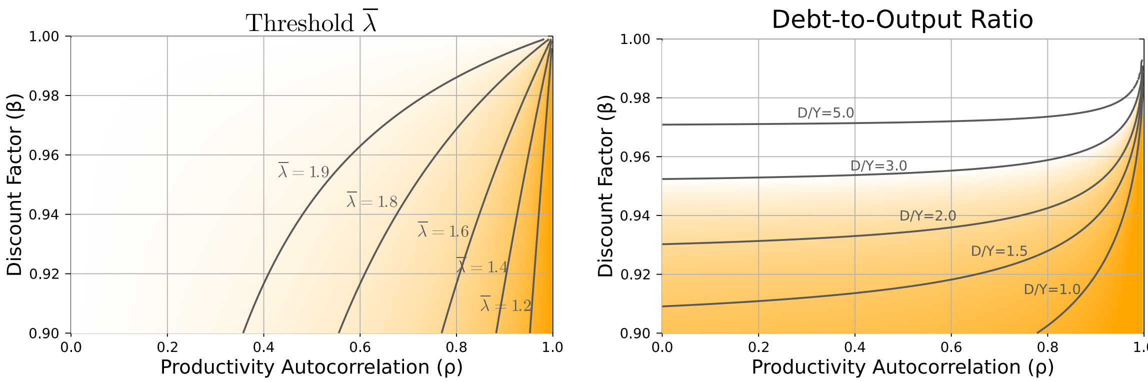
The value of $\overline{\lambda}$ for an economy without wealth taxes allows for sensible levels of borrowing in the economy. One way to gauge this is to compare the entrepreneurial debt-to-GDP ratio from the model when we set $\lambda = \overline{\lambda}(\tau_a = 0)$ to the ratio in the data. Guvenen et al. (2019) compute this ratio for the US as 1.52. In Figure 1, we report $\overline{\lambda}(\tau_a = 0)$ and the debt-to-GDP ratio $\left({}^{(\lambda-1)A_h}/_Y\right)$ for different β and ρ values when the borrowing limit λ is set to $\overline{\lambda}$.[6] The bottom line is that the debt-to-GDP ratio associated with the $\overline{\lambda}$ limit is typically much higher than the data counterpart of 1.52, implying that the λ needed for the heterogeneous return equilibrium is not restrictive at all. For example, for $\beta = 0.96$ and $\rho = 0.90$, $\overline{\lambda} = 1.68$ and the debt-to-GDP ratio is 2.90.[7]

Finally, we discuss the properties of saving rates in the heterogeneous return equilibrium.

[6] $\overline{\lambda}$ increases with savings $\left({}^{d\overline{\lambda}}/_{d\beta} > 0\right)$, decreases with the persistence of productivity $\left({}^{d\overline{\lambda}}/_{d\rho} < 0\right)$, and decreases if the productivity gap between types widens $\left({}^{d\overline{\lambda}}/_{d(z_h/z_\ell)} < 0\right)$. We consider in Appendix E the behavior of $\overline{\tau}_a$: it gets tighter as either λ or ρ increase and capital misallocation decreases, and gets looser if ${}^{z_h}/_{z_\ell}$ decreases, as long as $\rho > 0$. Figure E.4 illustrates these results along with the behavior of the implied debt-to-GDP ratios.

[7] These results are not driven by unreasonable dispersion in returns. The return gap, $R_h - R_\ell$, is between 2 and 10 percentage points for the relevant combinations of parameters, see Figure E.3, Appendix E.

Figure 1: Conditions for Steady State with Heterogeneous Returns



Note: *Figure 1a reports the value of $\overline{\lambda}$ found in Proposition 1 for combinations of the autocorrelation of productivity (ρ) and the discount factor (β). Figure 1b reports the debt-to-output ratio when $\lambda = \overline{\lambda}$ computed as $(\overline{\lambda}-1)A_h/Y$. In both figures we set the remaining parameters as follows: $z_\ell = 0$, $z_h = 2$, $\tau_k = 25\%$, and $\alpha = 0.4$.*

When $\tau_a < \overline{\tau}_a$, high-productivity entrepreneurs accumulate wealth, while low-productivity entrepreneurs dissave. Consequently, the high-productivity entrepreneurs hold most of the wealth ($s_h > 1/2$) if and only if entrepreneurial productivity is persistent ($\rho > 0$).

Corollary 2. ***(Saving Rates and Wealth Shares)*** *For all $\tau_a < \overline{\tau}_a$, the steady state saving rate of high-productivity entrepreneurs is positive and the saving rate of low-productivity entrepreneurs is negative: $\beta R_h > 1 > \beta R_\ell$. Furthermore, $s_h > 1/2$ if and only if $\rho > 0$.*

2.4 Government Budget Constraint

The government uses capital income and wealth tax revenues to finance non-productive government expenditures G. In steady state we can simplify this expression by substituting equation (12) to obtain

$$G = \tau_k \alpha Y + \tau_a K = \left(\tau_k + \tau_a \frac{\beta(1-\tau_k)}{1-\beta(1-\tau_a)}\right)\alpha Y. \tag{17}$$

Next, we make an assumption that greatly simplifies our upcoming analysis.

Assumption 1. *G is a constant fraction $\theta\alpha$ of aggregate output: $G = \theta\alpha Y$.*

Assumption 1 requires tax revenue to increase with the size of the economy. We will present conditions under which increasing wealth taxes delivers higher output and average welfare, while meeting the increased revenue requirements. The output and welfare gains we find would have likely been higher if we had imposed revenue neutrality. Finally, under Assumption 1, equation (17) implies a tight link between τ_k and τ_a:[8]

$$\frac{1-\tau_k}{1-\beta(1-\tau_a)} = \frac{1-\theta}{1-\beta}. \tag{18}$$

3 Tax Reform

We now consider a tax reform where the wealth tax is increased gradually. All comparisons are across steady states. The results here are global in nature—they hold for any starting level for τ_k and τ_a. We study the optimal combination of capital income and wealth taxes that maximizes average welfare in Section 4. We abstract from other taxes and transfers to focus on the comparison between these two taxes. As mentioned earlier, we set $\lambda < \overline{\lambda}(\tau_a = 0)$ so that the economy features heterogeneous returns.

3.1 Productivity Effects of an Increase in the Wealth Tax Rate

We start by proving a general result describing how aggregate productivity varies with wealth taxes. The result is independent of whether the government balances its budget constraint and states that Z is increasing in the wealth tax rate (τ_a) as long as $\rho > 0$.[9]

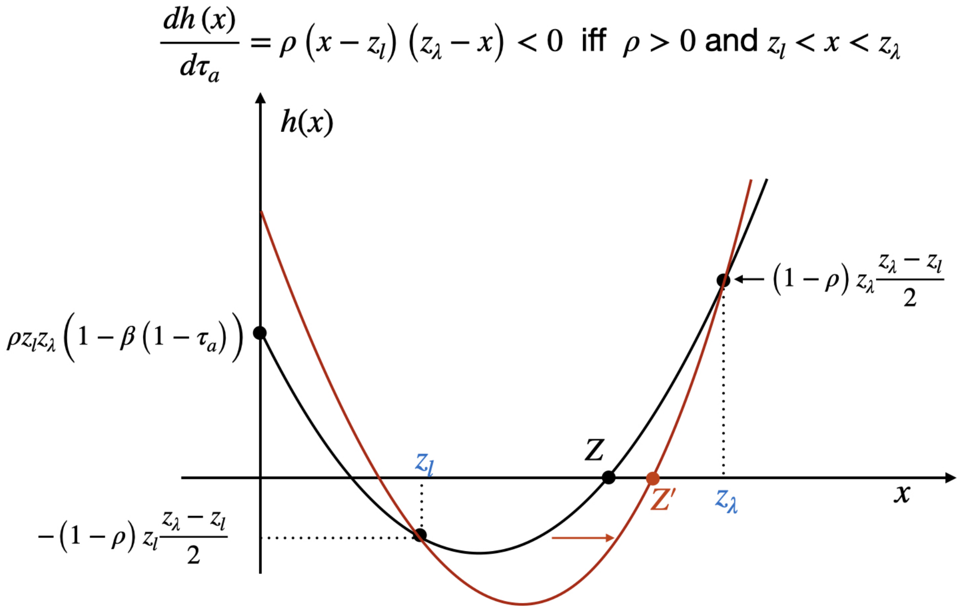
Proposition 2. ***(Efficiency Gains from Wealth Taxation)*** *For all $\tau_a < \overline{\tau}_a$, a marginal increase in wealth taxes (τ_a) increases aggregate productivity (Z), $\frac{dZ}{d\tau_a} > 0$, if and only if entrepreneurial productivity is persistent, $\rho > 0$.*

Why does productivity increase? Higher productivity is necessarily a consequence of the reallocation of wealth towards high-productivity entrepreneurs, that is, an increase in s_h (see equation 11). In the model, this reallocation is exclusively the consequence of the use-it-or-lose-it effect of wealth taxes changing entrepreneurial returns. However, the direction of the change in after-tax returns is not immediate because it involves two effects. First, a direct effect through change in tax rates. Second, a general equilibrium effect triggered by an increase in the effective capital stock Q (see Lemma 2 below), which reduces returns

[8] $\tau_k = \theta$ if there is only capital income tax ($\tau_a = 0$) and $\tau_a = \frac{\theta(1-\beta)}{\beta(1-\theta)}$ if there is only wealth tax ($\tau_k = 0$).

[9] As Figure 2 shows, an increase in τ_a shifts the steady state value of Z to the right, marked by the largest root of equation (14). Geometrically, when $\rho > 0$, an increase in τ_a increases the y-intercept of h, defined in Figure 2. The values of the parabola are fixed at z_ℓ and z_λ, forcing the x-intercepts (the roots of h) to shift right.

Figure 2: Tax Reform: Switch from Capital Income Tax to Wealth Tax



Note: *The figure plots* $h(x) = (1-\rho\beta(1-\tau_a))\,x^2 - (z_\ell+z_\lambda)/2\,(1+\rho(1-2\beta(1-\tau_a)))\,x + z_\ell z_\lambda \rho(1-\beta(1-\tau_a))$ *for two levels of wealth taxes. The steady state productivity corresponds to the larger root of* h*, marked with a circle on the horizontal axis. The red curve corresponds to an increase in wealth taxes* τ_a*.*

because of decreasing marginal returns to capital. We can decompose the change in after-tax returns into the two effects using equation (13):

$$\frac{dR(z)}{d\tau_a} = \underbrace{\left(\frac{z}{Z} - 1\right)}_{\text{use-it-}\ell\text{ose-it}\lesseqgtr 0} \underbrace{- \left(\frac{1}{\beta} - (1-\tau_a)\right) \frac{z}{Z^2} \frac{dZ}{d\tau_a}}_{\text{G.E. effect}<0} \tag{19}$$

While the general equilibrium effect is always negative, the direct use-it-or-lose-it effect depends on entrepreneurial productivity z.

We show that the use-it-or-lose-it effect operates relative to the wealth-weighted average of returns, further increasing above-average returns and decreasing below-average returns. This implies that the returns of high-productivity entrepreneurs increase while the returns of low-productivity entrepreneurs decrease, in turn increasing the dispersion of returns. The increase in the dispersion of returns translates into higher wealth accumulation by the high-productivity entrepreneurs only when productivity is persistent. Importantly, these results do not depend on how (or whether) the government's budget is balanced. This is because the level of capital adjusts in steady state according to equation (12) in such a way that returns depend only on productivity and wealth taxes.

Lemma 1. ***(Use-it-or-Lose-it)*** *For all $\tau_a < \overline{\tau}_a$, a marginal increase in wealth taxes increases entrepreneurial returns that are above the wealth-weighted average return and vice versa. That is, for any z, ${}^{dR(z)}/_{d\tau_a} \geq 0$ if and only if $z \geq Z = (s_h z_\lambda + (1 - s_h) z_\ell)$ and $\rho > 0$.*

Two important consequences of Lemma 1 are that an increase in wealth taxes does not change wealth-weighted returns and that, as the dispersion of returns increases, the arithmetic and geometric average returns decrease, ${}^{d(R_\ell + R_h)}/_{d\tau_a} < 0$ and ${}^{d(R_h R_\ell)}/_{d\tau_a} < 0$.[10]

3.2 Steady State Effects of an Increase in the Wealth Tax Rate

We now tackle the effect of an increase in wealth taxes on aggregate variables. We show that aggregate capital (K) and effective capital (Q) increase with aggregate productivity (Z), and therefore increase in response to an increase in wealth taxes. The effects on other variables depend on the effect of taxes on the government's budget. We use Assumption 1 to express the steady state values of aggregates in terms of the relative size of government spending (captured by θ) and Z. This result implies that output (Y) and wages (w) also increase in Z. Finally, we show that A_h increases in Z and that the response of A_ℓ depends on the elasticity of output with respect to capital. We group all these results in Lemma 2.

Lemma 2. ***(Aggregate Variables in Steady State)*** *If $\tau < \overline{\tau}_a$ and under Assumption 1, the steady state level of aggregate capital is*

$$K = \left(\frac{\alpha \beta (1 - \theta)}{1 - \beta} \right)^{\frac{1}{1-\alpha}} L Z^{\frac{\alpha}{1-\alpha}} \tag{20}$$

and the steady state elasticities of aggregate variables with respect to productivity are

$$\xi_K = \xi_Y = \xi_w = \xi \equiv \frac{\alpha}{1 - \alpha} \qquad \text{and} \qquad \xi_Q = 1 + \xi, \tag{21}$$

where $\xi_x \equiv \frac{d \log x}{d \log Z}$ is the elasticity of variable x with respect to Z.

Moreover, the wealth levels of each entrepreneurial type in steady state are

$$A_h = \frac{Z - z_\ell}{z_\lambda - z_\ell} K \qquad \qquad \frac{dA_h}{dZ} \propto Z^{\frac{2\alpha - 1}{1-\alpha}} (Z - \alpha z_\ell) > 0 \tag{22}$$

$$A_\ell = \frac{z_\lambda - Z}{z_\lambda - z_\ell} K \qquad \qquad \frac{dA_\ell}{dZ} \propto Z^{\frac{2\alpha - 1}{1-\alpha}} (\alpha z_\lambda - Z), \tag{23}$$

[10] Recall that $s_h > {}^1/_2$ when $\rho > 0$, so that $Z > {}^{(z_\lambda + z_\ell)}/_2$ and therefore the use-it-or-lose-it effect is negative for average returns. More generally, the effects on returns can be shown to follow any change in productivity Z, not necessarily triggered by a change in wealth taxes. See Lemma 2 in Appendix B.

where $\frac{dA_\ell}{dZ} < 0$ *if and only if* $\alpha z_\lambda < Z$.

Remark. We can interpret the condition $\alpha z_\lambda < Z$ as a threshold level for α because the steady state Z is independent of α (see equation 14).

The gains from wealth taxes arise despite the fact that wasteful government expenditure G increases. K, Q, Y, and w could be increased further if G were kept constant in a revenue-neutral fashion. On the other hand, if $\rho < 0$, an increase in wealth taxes not only decreases these variables but also G.

We conclude this section by considering a sustained increase in wealth taxes. As τ_a increases towards $\overline{\tau}_a$, Z gets closer to its upper bound z_h. It might be tempting to conclude that we should therefore increase τ_a as much as possible. However, maximizing productivity does not maximize welfare. While Z increases with wealth tax, the dispersion in rates of return also increases, decreasing the welfare of entrepreneurs. We turn to the welfare consequences of an increase in τ_a next.

3.3 Welfare Effects

We first discuss our welfare measures. Let B denote the initial benchmark economy with a given capital income and wealth taxes and C denote a counterfactual economy with a higher wealth tax and a lower capital income tax. Define $\left\{c_t^{\mathrm{j}}(a,i)\right\}$ as the consumption path and $V^{\mathrm{j}}(a,i)$ as the value function of an individual of type $i \in \{w,h,\ell\}$ under economy $\mathrm{j} \in \{\mathrm{B},\mathrm{C}\}$. We ask each individual how much they value being dropped from B to C in terms of a consumption-equivalent welfare measure $\mathrm{CE}_1(a,i)$, which is defined by

$$E\sum_t \beta^{t-1}\log\left(\left(1+\mathrm{CE}_1(a,i)\right)c_t^{\mathrm{B}}(a,i)\right) \;=\; E\sum_t \beta^{t-1}\log\left(c_t^{\mathrm{C}}(a,i)\right). \tag{24}$$

We solve for $\mathrm{CE}_1(a,i)$ using equations (7) and (8). All terms containing wealth cancel, so drop wealth from the arguments and write[11]

$$\log\left(1+\mathrm{CE}_{1,i}\right) \;=\; \begin{cases} \log\left(\frac{w^{\mathrm{C}}}{w^{\mathrm{B}}}\right) & \text{if } i=w \\ \frac{(1-\beta)\log\left(\frac{R_i^{\mathrm{C}}}{R_i^{\mathrm{B}}}\right)+\beta(1-p)\left(\log\left(\frac{R_\ell^{\mathrm{C}}}{R_\ell^{\mathrm{B}}}\right)+\log\left(\frac{R_h^{\mathrm{C}}}{R_h^{\mathrm{B}}}\right)\right)}{(1-\beta)(1-\beta(2p-1))} & \text{if } i\in\{\ell,h\}. \end{cases} \tag{25}$$

[11]The independence from wealth allows us to compute welfare even though there is no stationary wealth distribution in models with constant saving rates and no reflecting barrier or resetting mechanisms like ours (Gabaix, 2009). We study a model with a stationary wealth distribution in Section 5.

We compute the aggregate welfare gain as the population-weighted average of welfare gains. Letting $n_w \equiv L/(L+2)$ be the population share of workers and $n_h = n_\ell \equiv 1/(L+2)$ the share of entrepreneurs, we write

$$\log(1 + \mathrm{CE}_1) = \sum_{i \in \{w,h,\ell\}} n_i \log(1 + \mathrm{CE}_{1,i}). \tag{26}$$

We also define the average entrepreneurial welfare gain ($\mathrm{CE}_1^{\mathrm{e}}$) as

$$\log(1 + \mathrm{CE}_1^{\mathrm{e}}) = \sum_{i \in \{h,\ell\}} \frac{1}{2} \log(1 + \mathrm{CE}_{1,i}) = \frac{1}{1-\beta} \left(\log\left(\frac{R_\ell^{\mathrm{C}}}{R_\ell^{\mathrm{B}}}\right) + \log\left(\frac{R_h^{\mathrm{C}}}{R_h^{\mathrm{B}}}\right) \right). \tag{27}$$

Having defined our welfare measures, we now use our previous results to determine the welfare implications of a marginal increase in the wealth tax.

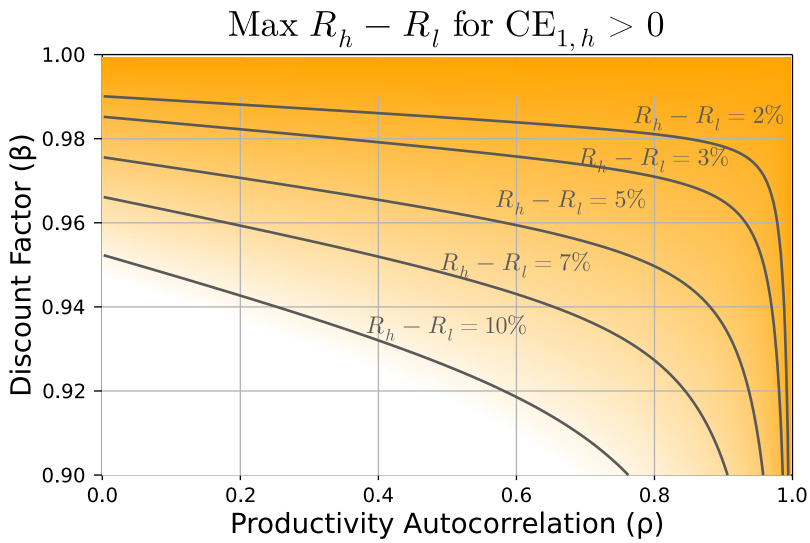
Proposition 3. ***(Welfare Gain by Agent Type)*** *For all $\tau_a < \overline{\tau}_a$, if Assumption 1 holds and $\rho > 0$, a marginal increase in wealth taxes (τ_a) increases the welfare of workers ($CE_{1,w} > 0$) and decreases the welfare of low-productivity entrepreneurs ($CE_{1,\ell} < 0$) and the average welfare of entrepreneurs ($CE_1^{\mathrm{e}} < 0$). Furthermore, there exists an upper bound on the dispersion of returns (κ_R) such that an increase in wealth taxes increases the welfare of high-productivity entrepreneurs ($CE_{1,h} > 0$) if and only if $R_h - R_\ell < \kappa_R$.*

Workers gain from an increase in wealth taxes because wages increase. For entrepreneurs, the welfare effects of the increase in wealth taxes come from changes in after-tax returns. There are two effects. First, higher wealth taxes reduce the current returns of low-productivity entrepreneurs and increase those of high-productivity entrepreneurs. Second, (log-)average of returns decrease with wealth taxes, decreasing entrepreneurs' expectations over future returns and reducing their welfare. The net result of these effects is a lower welfare for the low-productivity entrepreneurs and for entrepreneurs as a group.

The welfare gain for the high-productivity entrepreneurs depends on the magnitude of the decrease in average returns, that in turn depends on the initial return dispersion. The upper bound on the dispersion of returns (κ_R) ensures that the loss from lower expected returns is low relative to the increase in R_h. The upper bound is a function of only β and ρ and does not change with wealth taxes. Figure 3 presents the upper bound on the dispersion of returns for which $\mathrm{CE}_{1,h}$ is positive for different combinations of β and p.[12]

[12]The $\mathrm{CE}_{1,h}$ welfare measure we consider here ignores the effects of the increase in the assets of high-

Figure 3: Dispersion of Returns and Welfare Gains for High-Productivity Entrepreneurs



Note: *The figure reports the upper bound on the steady state dispersion of returns, $R_h - R_\ell$, for which $CE_{1,h} > 0$. The upper bound is a function of only ρ and β and is obtained by finding the upper bound on the wealth share of high-productivity entrepreneurs implied by equation (82) and evaluating the returns at that level using the results from Lemma 2. We set the remaining parameters as follows: $z_\ell = 0$, $z_h = 2$, $\tau_k = 25\%$, $\lambda = 1.32$, and $\alpha = 0.4$.*

4 Optimal Taxation: Combining Wealth and Capital Income Taxes

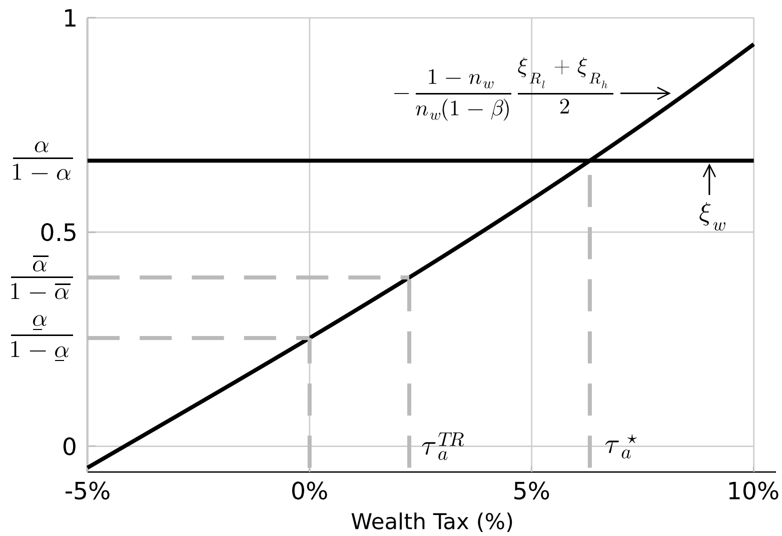
We now characterize the optimal tax combination $(\tau_a^\star, \tau_k^\star)$ that maximizes the utilitarian welfare measure CE_1. Proposition 3 makes clear the key tradeoff when considering the welfare effects of wealth taxation: Higher wealth taxes increase the welfare of workers by increasing wages through productivity gains, but they reduce the welfare of entrepreneurs by increasing the dispersion of returns and decreasing their expected value. As we show in Proposition 4 below, the tradeoff is captured by the elasticities of wages and returns to changes in productivity. The welfare gain of workers is proportional to the wage elasticity with respect to productivity, $\xi_w = \frac{\alpha}{1-\alpha}$, while the welfare loss of entrepreneurs is proportional to the average elasticity of returns with respect to productivity, $(\xi_{R_\ell} + \xi_{R_h})/2$.[13]

Proposition 4. ***(Optimal** CE_1 **Taxes)*** *Under Assumption 1, there exist a unique tax combination $(\tau_a^\star, \tau_k^\star)$ that maximizes the utilitarian welfare measure CE_1. An interior*

productivity entrepreneurs brought about by the increase in wealth taxes. We show in Appendix C that taking the change in assets into account makes the welfare change unambiguously positive for them.

[13] A similar result follows when wealth accumulation is taken into account. The main change comes from the added benefits for welfare of higher capital. See Appendix C.

Figure 4: Optimal Wealth Tax



Note: *The figure shows the conditions satisfied by the optimal wealth tax, defined as the tax that maximizes* CE_1*. The horizontal line is the elasticity of wages with respect to productivity* (ξ_w)*. The increasing line is proportional to the average elasticity of returns with respect to productivity* (ξ_R)*.* $\tau_a^\star$ *denotes the optimal wealth tax.* $\tau_a^{TR} = \theta(1-\beta)/\beta(1-\theta)$ *denotes the tax reform tax, the level at which* $\tau_k = 0$*. The remaining parameters are as follows:* $\beta = 0.96$, $p = 0.9$, $z_\ell = 1/2$, $z_h = 3/2$, $\theta = 25\%$, *and* $\lambda = 1.2$.

solution, $\tau_a^\star < \overline{\tau}_a$*, is the solution to:*

$$n_w \xi_w + \frac{1-n_w}{1-\beta}\left(\frac{\xi_{R_\ell}+\xi_{R_h}}{2}\right) = 0, \tag{28}$$

where $\xi_x \equiv \frac{d\log x}{d\log Z}$ *is the elasticity of variable* x *with respect to* Z*. Furthermore, there exist two cutoff values for* α*,* $\underline{\alpha}$ *and* $\overline{\alpha}$*, such that* $(\tau_a^\star, \tau_k^\star)$ *satisfies the following properties:*

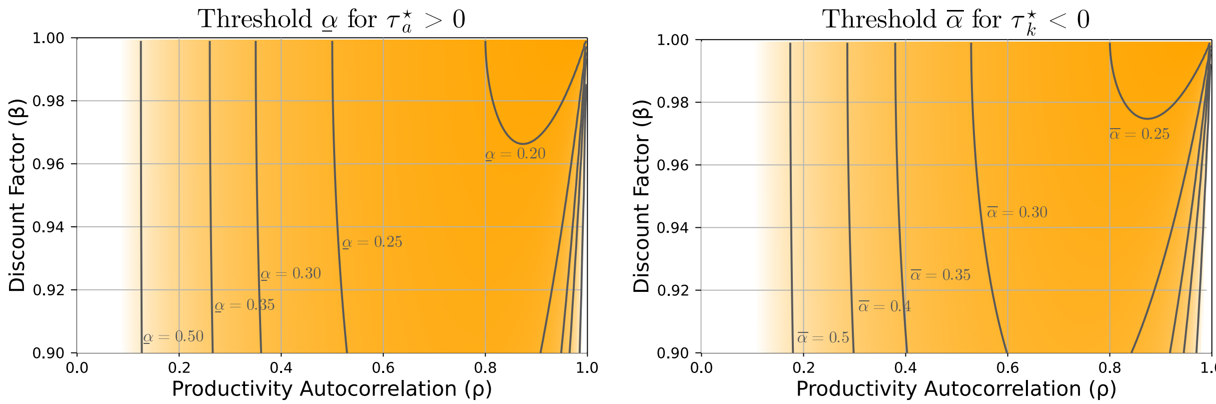
$$\begin{aligned}
\tau_a^\star &\in \left[1-\frac{1}{\beta}, 0\right) \text{ and } \tau_k^\star > \theta && \text{if } \alpha < \underline{\alpha} \\
\tau_a^\star &\in \left[0, \frac{\theta(1-\beta)}{\beta(1-\theta)}\right] \text{ and } \tau_k^\star \in [0,\theta] && \text{if } \underline{\alpha} \le \alpha \le \bar{\alpha} \\
\tau_a^\star &\in \left(\frac{\theta(1-\beta)}{\beta(1-\theta)}, \tau_a^{\max}\right) \text{ and } \tau_k^\star < 0 && \text{if } \alpha > \bar{\alpha},
\end{aligned}$$

where $\underline{\alpha}$ *and* $\overline{\alpha}$ *are the solutions to equation (28) with* $\tau_a = 0$ *and* $\tau_a = \tau^{TR} \equiv \frac{\theta(1-\beta)}{\beta(1-\theta)}$*, respectively. Recall from Lemma (2) that* $\xi_w = \xi \equiv \alpha/1-\alpha$*.*

Figure 4 illustrates the forces at play. The elasticity of wages with respect to productivity (ξ_w) gives the (percentage) gain in workers' welfare as wealth taxes increase

Figure 5: Conditions for Welfare Gains from Wealth Taxation

(a) (b)



Note: *Figure 5a reports the value of $\underline{\alpha}$ found in Proposition 4 for combinations of the autocorrelation of productivity (ρ) and the discount factor (β). Positive wealth taxes induce welfare gains (as measured by CE_1) if $\alpha \geq \underline{\alpha}$. Figure 5b reports the value of $\overline{\alpha}$ found in Proposition 4. Positive wealth taxes and capital income subsidies induce welfare gains (as measured by CE_1) if $\alpha \geq \overline{\alpha}$. In both figures we set the remaining parameters as follows: $z_\ell = 0$, $z_h = 2$, $\theta = 25\%$, and $\lambda = 1.32$.*

(raising productivity). This elasticity is constant for the Cobb-Douglas production function in equation (1), $\xi_w = \alpha/1-\alpha$. On the other hand, the (negative) average elasticity of returns is increasing, reflecting the widening gap between low- and high-returns as wealth taxes increase. The decrease in entrepreneurial welfare is proportional to the average elasticity of returns, see equation (27). The intersection of the two lines marks the optimal wealth tax.

Figure 4 also clarifies the role of the thresholds $\underline{\alpha}$ and $\overline{\alpha}$.[14] The lower threshold $\underline{\alpha}$ marks the level of ξ_w for which $\tau_a = 0$ would be optimal, any $\alpha > \underline{\alpha}$ implies a higher scope for wages to rise as $Q = ZK$ increases with the wealth tax and thus a positive optimal wealth tax. The upper threshold $\overline{\alpha}$ is similarly defined by the level of ξ_w for which $\tau_a = \tau_a^{TR} \equiv \theta(1-\beta)/\beta(1-\theta)$ is optimal. At that level, wealth taxes finance all government spending and $\tau_k = 0$. Consequently, any $\alpha > \overline{\alpha}$ implies that the optimal tax combination is one of wealth taxes and capital income subsidies. Finally, the upper bound on the wealth tax ($\tau_a^{\max}$) ensures that R_ℓ remains positive.

To give an idea of the level of the thresholds $\underline{\alpha}$ and $\overline{\alpha}$, Figure 5 presents values for different combinations of β and ρ. We keep the dispersion of productivities as in Figure 1 and set $\lambda = 1.32$ and $\theta = 0.25$, we also set $L = 18$ so that 10% of agents are entrepreneurs. $\underline{\alpha}$ and $\overline{\alpha}$ are lower than $1/3$ if productivity is sufficiently persistent ($\rho > 0.6$), although, the

[14]The value of the thresholds depend on s_h and Z, which are endogenous but independent of α (equation 14), so they can be used to define the threshold.

thresholds are not monotone and they go up as ρ approaches 1. The non-monotonicity arises due to two opposing forces as ρ increases. First, there is less capital misallocation because wealth is more concentrated in the hands of high-productivity entrepreneurs (Moll, 2014). Thus, there is less scope for improvement from wealth taxes. Second, the effect of the use-it-or-lose-it mechanism on misallocation gets stronger as ρ increases.

The thresholds are also affected by other aspects of fiscal policy. For instance, $\bar{\alpha}$ increases with θ, the relative size of the government spending. However, the optimal τ_a is independent of θ. Thus, the optimal combination of taxes is more likely to involve positive capital income and wealth taxes when government spending is relatively high.

The optimal level of wealth taxes exhibits the same pattern of increasing in the persistence of individual productivity up until ρ approaches 1. Welfare gains are close to zero if persistence is relatively low ($\rho < 0.6$), but grow rapidly as ρ increases with the highest levels associated with high levels of β. We report the optimal tax levels and welfare gains for combinations of β and ρ in Figure E.5 of Appendix E.

5 Extensions

5.1 Entrepreneurial Effort

We now consider the role of entrepreneurial effort in shaping the productivity of private enterprises, as well as the role of the tax system in affecting entrepreneurs' incentives to exert effort. Both capital income and wealth taxes can lower wealth accumulation, reducing the amount of capital entrepreneurs use in their firms and the marginal product of entrepreneurial effort. However, unlike wealth taxes, capital income taxes also reduce the profits retained by entrepreneurs, further lowering the benefits from entrepreneurial effort.

We introduce effort in a tractable manner that allows us to identify its core implications for wealth taxation. Effort, e, affects production according to

$$y = (zk)^{\alpha} e^{\gamma} n^{1-\alpha-\gamma}, \tag{29}$$

where $0 \leq \gamma < 1 - \alpha$. Exerting effort has a utility cost that we capture by modifying the utility function to

$$u(c, e) = \log(c - h(e)), \tag{30}$$

where $h(e) = \psi e$ and $\psi > 0$.[15] Tractability depends on preserving the constant-returns-to-scale in production and abstracting from income effects in the effort choice as in Greenwood, Hercowitz and Huffman (1988).[16] Together, these properties allow us to solve the model analytically. The solution inherits the properties of our benchmark model after a suitable change of variables. We define consumption and profits net of effort as $\hat{c} = c - h(e)$ and

$$\hat{\pi}(z,k) = \max_{n,e} y - wn - rk - \frac{1}{1-\tau_k} h(e). \tag{31}$$

Crucially, capital income taxes have a direct effect on effort. Labor and capital rental costs can be deducted from taxes, while effort costs are paid privately by the entrepreneur and are not deductible. Because of this, the effective cost per unit of effort is $\psi/_{1-\tau_k}$.

We obtain closed form expressions for equilibrium quantities as a function of aggregate capital, K, and productivity, Z, paralleling the results of Lemma 1, see Appendix D.2. The main difference is of course the introduction of effort. Aggregate effort is

$$E = \left(\frac{(1-\tau_k)\gamma}{\psi}\right)^{\frac{1}{1-\gamma}} Q^{\frac{\alpha}{1-\gamma}} L^{\frac{1-\alpha-\gamma}{1-\gamma}}. \tag{32}$$

There are two different, but related, forces shaping aggregate entrepreneurial effort. First, effort is increasing in effective capital, $Q = ZK$, because it raises the marginal product of effort. Second, effort is disincentivized by capital income taxes, that reduce the after-tax marginal product of effort, effectively making effort more costly. Consequently, capital income taxes also reduce aggregate output and wages, through their effect on effort

$$Y = \left(\frac{(1-\tau_k)\gamma}{\psi}\right)^{\frac{\gamma}{1-\gamma}} Q^{\frac{\alpha}{1-\gamma}} L^{\frac{1-\alpha-\gamma}{1-\gamma}}, \tag{33}$$

$$w = (1-\alpha-\gamma)\left(\frac{(1-\tau_k)\gamma}{\psi}\right)^{\frac{\gamma}{1-\gamma}} \left(\frac{Q}{L}\right)^{\frac{\alpha}{1-\gamma}}. \tag{34}$$

By contrast, wealth taxes do not directly affect the effort choice because they do not affect the fraction of profits retained by the entrepreneur.

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[15] In general we can let effort affect production according to an increasing function $g(e)$ and we only require that the ratio $h'(e)/g'(e)$ is constant. See Appendix D.2.

[16] Abstracting from the income effect on the entrepreneur's effort choice potentially leads to an overstatement of the response of effort to wealth taxes. Wealth taxes increase returns and incentivize effort, but wealthier entrepreneurs may want to exert less effort in the presence of income effects.

The steady state behavior of aggregate productivity remains unchanged (equation 14). Even though the relationship between productivity, taxes, and steady state capital in equation (12) changes, the relationship between productivity and the after-tax return (net of effort costs) is the same as in equation (13),

$$\hat{R}(z) = (1-\tau_a) + \left(\frac{1}{\beta} - (1-\tau_a)\right)\frac{z}{Z} \tag{35}$$

This is because the steady state level of capital adjusts so that its marginal product is equal to $\frac{1}{\beta} - (1-\tau_a)$ as in equation (12). Consequently, the results of our benchmark model regarding the existence of the steady state and the efficiency gains from wealth taxation (Propositions 1 and 2) remain unchanged.

Proposition 5. *A steady state equilibrium with heterogeneous returns exists if and only if $\lambda < \overline{\lambda}$, and a marginal increase in wealth taxes in such an equilibrium increases productivity Z if and only if $\rho > 0$.*

Nevertheless, introducing entrepreneurial effort does change the response of aggregates to wealth taxes and the optimal tax combination. As wealth taxes increase, productivity rises, along with capital, output, and wages as described in Lemma 2. But, higher wealth taxes also reduce the level of capital income taxes (equation 18), incentivizing entrepreneurial effort and, trough it, increasing aggregate output, capital, and wages further, as equations (32)-(34) make clear.

Lemma 3. *For all $\tau_a < \overline{\tau}_a$, a marginal increase in wealth taxes (τ_a) increases aggregate entrepreneurial effort, capital, output, and wages, $\frac{dE}{d\tau_a}, \frac{dK}{d\tau_a}, \frac{dY}{d\tau_a}, \frac{dw}{d\tau_a} > 0$, if $\rho > 0$.*

As for the optimal tax choice, the reduction of distortions on effort adds a motive for replacing capital income taxes with wealth taxes. Just as in Proposition 4, the optimal tax combination balances the gains to workers from a higher wage with the reduction in average after-tax returns (now net of effort costs). The response of the after-tax returns to taxes is not affected by effort, as implied by equation (35), but the increase in wages is now augmented via an increase in entrepreneurial effort. Because of this, the optimal tax combination now involves higher wealth taxes and lower capital income taxes.

Proposition 6. *The optimal wealth tax with entrepreneurial effort is higher than in*

Proposition 4. Moreover, if the optimal wealth tax is interior ($\tau_a^\star < \overline{\tau}_a$) it satisfies

$$n_w \frac{\gamma}{1-\alpha-\gamma}\left(\frac{\frac{\beta\tau_a}{1-\beta(1-\tau_a)}}{\frac{d\log Z}{d\log\tau_a}} + \frac{\alpha}{1-\alpha}\right) = -\left(n_w\frac{\alpha}{1-\alpha} + \frac{1-n_w}{2}\left(\xi_{\hat{R}_\ell} + \xi_{\hat{R}_h}\right)\right), \tag{36}$$

where $\xi_{\hat{R}_i} \equiv \frac{d\log\hat{R}_i}{d\log Z}$ for $i \in \{h,\ell\}$.

5.2 Excess Returns from Rent Seeking

We now consider the possibility that returns do not capture entrepreneurial productivity by introducing return wedges: an entrepreneur who earns more (less) than their marginal product is assumed to face a positive (negative) wedge. These wedges reallocate resources across groups. We assume that the net reallocation is equal to zero. Entrepreneurs face wedges ω_h and ω_ℓ respectively, so that their after-tax returns become

$$R_\ell = (1-\tau_a) + (1-\tau_k)(1+\omega_\ell)\,\alpha\left(^{ZK}/_L\right)^{\alpha-1} z_\ell \tag{37}$$
$$R_h = (1-\tau_a) + (1-\tau_k)(1+\omega_h)\,\alpha\left(^{ZK}/_L\right)^{\alpha-1} z_\lambda. \tag{38}$$

The zero-sum condition for the wedges implies $\omega_\ell z_\ell A_\ell + \omega_h z_\lambda A_h = 0$.

The return wedges do not affect the law of motion of capital, so that the steady state condition for aggregate capital is still given by equation (12), while the steady state condition for aggregate productivity, Z, becomes

$$\begin{aligned}(1-\rho\beta(1-\tau_a))\,Z^2 - \frac{z_\lambda+z_\ell}{2}\left(1+\rho\left(1-2\beta(1-\tau_a)\right)\right)Z \\ +z_\ell z_\lambda\rho\left(1-\beta(1-\tau_a)\right)\left(1+\omega_h-\omega_h\frac{Z}{z_\ell}\right) = 0.\end{aligned} \tag{39}$$

The wedges impose new conditions for the existence of a steady state equilibrium with heterogeneous returns by modifying the upper bounds of λ and τ_a. We present these conditions in Proposition 11 in Appendix D.3. Relative to our benchmark (equivalent to $\omega_h = 0$), an increase in ω_h makes the bound on λ more stringent, $^{d\overline{\lambda}}/_{d\omega_h} < 0$, because ω_h increases the effective return of high-productivity entrepreneurs.[17]

The wedges also modify the effect of an increase in wealth taxes on productivity. There

[17] Additionally, the wedges imply an upper (lower) bound on τ_a when ω_h is positive (negative). The upper bound is positive if and only if $\omega_h < \frac{1-\rho}{2\rho(1-\beta)}$, which is always true if $\rho \leq \beta$ because $\omega_h < 1$. The lower bound is always negative as long as $\rho > 0$.

are two cases to consider. If entrepreneurial productivity is persistent, $\rho > 0$, Z increases with wealth taxes if and only if the return of high-productivity entrepreneurs is higher than the return of low-productivity entrepreneurs ($R_h(\omega_h) > R_\ell(\omega_\ell)$). This is the case if the subsidies to low-productivity entrepreneurs are not too large, $\omega_h > \underline{\omega}_h$ for $\underline{\omega}_h$ defined below. Intuitively, wealth taxes benefit the agents with higher returns, regardless of their productivity. So, wealth taxes increase aggregate productivity when high-return individuals have high productivity.

The second case arises if productivity is negatively auto-correlated, $\rho < 0$. Then, low-productivity entrepreneurs today are likely to become more productive. Therefore, an increase in wealth taxes increases Z if and only if $R_\ell(\omega_\ell) > R_h(\omega_h)$. This is the opposite of the condition in the first case. However, the rationale is the same, wealth taxes increase the returns of currently unproductive (but high-return) entrepreneurs, increasing the wealth share of high-productivity entrepreneurs in the future.

The following proposition formalizes the analysis. We abuse notation by referring to the modified upper bound on taxes as $\overline{\tau}_a$, even though it now depends on the return wedges.

Proposition 7. ***(Efficiency Gains from Wealth Taxation)*** *For all $\tau_a < \overline{\tau}_a$, a marginal increase in wealth taxes (τ_a) increases aggregate productivity (Z), $\frac{dZ}{d\tau_a} > 0$, if entrepreneurial productivity is autocorrelated, $\rho > 0$, and $R_h > R_\ell$, or if entrepreneurial productivity is negatively autocorrelated, $\rho < 0$, and $R_h < R_\ell$.*

Corollary 3. *For all $\tau_a < \overline{\tau}_a$, the steady state returns satisfy $R_h > R_\ell$ if and only if :*

$$\omega_h > \underline{\omega}_h \equiv -\frac{1}{2}\left(\frac{1-\rho}{3+\rho}\right)\left(\frac{z_\lambda - z_\ell}{z_\lambda}\right).$$

5.3 Stationary Wealth Distribution

The model in Section 2, as well as the extensions presented above, do not have a stationary wealth distribution. Here we consider an alternative model with a perpetual-youth demographic structure where entrepreneurial productivity is fixed over the life cycle but varies stochastically between generations. Entrepreneurs die with a constant probability, $1-\delta$, and are replaced by a new entrepreneur who is endowed with wealth $\overline{a}$, equal to the average bequest in the economy (which coincides with average wealth) and draws a productivity $z_i \in \{z_h, z_\ell\}$ with equal probability.

This perpetual-youth model has a stationary wealth distribution that allows us to study distributional outcomes. Higher wealth taxes increase wealth inequality, increasing concentration at the top and the bottom, reflecting the higher dispersion in returns. In terms of welfare, all entrepreneurs benefit from the wealth accumulation that follows an increase in wealth taxes, but they are also more sensitive to changes in returns that compound to higher or lower wealth as they age, amplifying the positive and negative effects of wealth taxation.

We also show in Appendix D.4 that the mechanism behind the efficiency gains from wealth taxation remains active. In fact, an increase in wealth taxes always leads to an increase in productivity because individual productivity is permanent (within a generation). The responses of aggregate variables to changes in equilibrium Z (and hence to τ_a) follow the same patterns as in Section 3.

Proposition 8. ***(Efficiency Gains from Wealth Taxation)*** *For all $\tau_a < \overline{\tau}_a^p$, a marginal increase in wealth taxes (τ_a) increases productivity, $\frac{dZ}{d\tau_a} > 0$.*

We now characterize the stationary wealth distribution, Γ.[18] High-productivity entrepreneurs save at a (gross) rate $\beta\delta R_h > 1$ while low-productivity entrepreneurs dissave at a (gross) rate $\beta\delta R_\ell < 1$. The wealth distribution is discrete, with endogenous mass points at $\left\{\overline{a}, \beta\delta R_i\overline{a}, (\beta\delta R_i)^2\,\overline{a}, \ldots\right\}$ for $i \in \{h, \ell\}$ respectively. The share of entrepreneurs of type i with wealth $a = (\beta\delta R_i)^t\,\overline{a}$ is equal to the share who has lived exactly t periods:

$$\Gamma_i\left((\beta\delta R_i)^t\,\overline{a}\right) = \Pr\left(\text{age} = t\right) = \delta^t\left(1 - \delta\right) \tag{40}$$

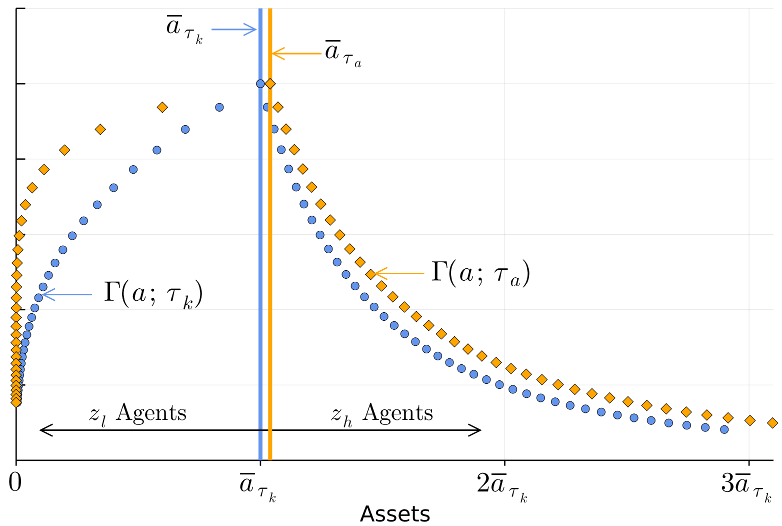
So, the wealth distribution is a geometric distribution with parameter δ.

Figure 6 presents the stationary wealth distribution. Entrepreneurs are born with initial wealth $\overline{a}$ and save or dissave at constant rates depending on their productivity. A change in taxes affects the location of the mass points of the distribution. In the figure, we contrast an economy without wealth taxes (labeled τ_k) with one with wealth taxes (labeled τ_a). The wealth tax economy has higher aggregate wealth ($\overline{a}_{\tau_a} > \overline{a}_{\tau_k}$). The change in $\overline{a}$ shifts all mass points rightwards. The increase in the dispersion of wealth is explained by the increase in the dispersion of returns, something reminiscent of Lemma 2.

We characterize top wealth concentration using the wealth distribution. Because wealth is determined by productivity and age, we focus on the fraction of wealth held by high-productivity entrepreneurs age t and older. This corresponds to the wealth share of the top

[18] The characterization of the distribution follows Jones (2015) adapted to a discrete-time setting.

Figure 6: Stationary Wealth Distribution



Note: *The figure reports the stationary wealth distribution for two economies. The blue circles correspond to an economy with only capital income taxes ($\tau_k = \theta$ and $\tau_a = 0$) and its values are labeled with τ_k. The orange diamonds correspond to an economy with wealth taxes (τ_k set to satisfy Assumption 1) and its values are labeled with τ_a. The horizontal axis is presented in units of average assets in the capital income tax economy ($\overline{a}_{\tau_k}$).*

$100\times(1-\delta)\sum_{s=t}^{\infty}\delta^s = 100\times\delta^t$ percent. Their total wealth is $A_{h,t} \equiv (1-\delta)\sum_{s=t}^{\infty}\left(\beta\delta^2 R_h\right)^s \overline{a}$ and their top wealth share is

$$s_{h,t} \equiv \frac{A_{h,t}}{K} = \left(\beta\delta^2 R_h\right)^t s_h. \tag{41}$$

These wealth shares increase with the wealth tax along with s_h and R_h.

Lemma 4. ***(Top-Wealth Shares and Wealth Taxes)*** *For all $\tau_a < \overline{\tau}_a^p$, a marginal increase in wealth taxes increases the top-wealth-shares as in equation (41). The percentage increase in the wealth share is higher for higher wealth levels.*

Finally, we use the wealth distribution to study the welfare implications of wealth accumulation. We show in Appendix D.4 that low- and high-productivity entrepreneurs benefit from wealth accumulation following an increase in wealth taxes. However, the effect on the optimal combination of taxes is ambiguous because entrepreneurs are more sensitive to changes in returns (that are fixed in their lifetimes) due to the compounding effect of returns on individuals' asset accumulation.

6 Conclusions

We studied the taxation of capital through capital income and wealth taxes. In the heterogeneous-returns equilibrium that emerges under a broad set of parameter choices, an increase in wealth taxes leads to higher aggregate productivity and output when returns are positively auto-correlated. The empirical evidence shows robustly positive autocorrelation (persistence) of returns both from year to year as well as from one generation to the next (although the correlation is lower). Higher wealth taxes benefit workers through the increase in wages that follows the increases in productivity and output. High-productivity entrepreneurs also benefit under reasonable parameter combinations and low-productivity entrepreneurs lose, reflecting the shift in the tax burden from high-return individuals under capital income taxes to low-return (but wealthy) individuals under wealth taxes.

Turning to optimal taxation, when the government can use both tools simultaneously, the optimal policy depends on the model parameters in the form of thresholds in the capital intensity of production. For sufficiently high capital share parameters (that are well within the standard empirical values used in the literature), the optimal policy is a positive wealth tax combined with a capital income subsidy.

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APPENDIX

A Entrepreneur's problem

We start with an entrepreneur's labor choice given her capital:

$$\pi(z,k) = \max_n (zk)^\alpha n^{1-\alpha} - wn,$$

which gives the following labor demand

$$n^\star(z,k) = \left(\frac{1-\alpha}{w}\right)^{1/\alpha} zk. \tag{42}$$

Substituting the optimal labor demand into the profit, the entrepreneur's capital choice is given by

$$k^\star(z,a) = \arg\max_{0\le k\le \lambda a}\left[\alpha\left(\frac{1-\alpha}{w}\right)^{(1-\alpha)/\alpha} z - r\right]k.$$

The optimal capital decision of the entrepreneur is characterized by the following function:

$$k^\star(z,a) = \begin{cases} \lambda a & \text{if } \alpha\left(\frac{1-\alpha}{w}\right)^{(1-\alpha)/\alpha} z > r \\ [0,\lambda a] & \text{if } \alpha\left(\frac{1-\alpha}{w}\right)^{(1-\alpha)/\alpha} z = r \\ 0 & \text{if } \alpha\left(\frac{1-\alpha}{w}\right)^{(1-\alpha)/\alpha} z < r. \end{cases}$$

Thus, entrepreneurs whose marginal return to capital is greater than the interest rate borrow up to the limit and sets λa and those whose return is below the interest rate does not produce zero output and earns the return r in the bond market on wealth a.

The optimal profit of the entrepreneur can be written as

$$\pi^\star(z)\,a = \begin{cases} \left(\alpha\left(\frac{1-\alpha}{w}\right)^{(1-\alpha)/\alpha} z - r\right)\lambda a & \text{if } \alpha\left(\frac{1-\alpha}{w}\right)^{(1-\alpha)/\alpha} z > r \\ 0 & \text{if } \alpha\left(\frac{1-\alpha}{w}\right)^{(1-\alpha)/\alpha} z \le r \end{cases} \tag{43}$$

Given taxes τ_a and τ_k and constant prices, an entrepreneur's optimal savings problem can be written as

$$V(a,z) = \max_{a'} \log(c) + \beta\sum_{z'} \Pi\left(z' \mid z\right) V\left(a', z'\right)$$

subject to

$$c + a' = R(z)\,a,$$

where $R(z) = 1 - \tau_a + (1-\tau_k)(r + \pi^\star(z))$ as in the main text.

We solve the dynamic programming problem of the entrepreneur via guess and verify. To this end, we guess that the value function of an entrepreneur of type $i \in \{\ell, h\}$ has the form

$$V_i(a) = m_i + n \log(a),$$

where $m_\ell, m_h, n \in \mathbb{R}$ are coefficients. Under this guess the optimal savings choice of the entrepreneur is characterized by

$$\frac{1}{R_i a - a_i'} = \frac{\beta n}{a_i'}.$$

Solving for savings gives:

$$a_i' = \frac{\beta n}{1 + \beta n} R_i a.$$

Replacing the savings rule into the value function gives:

$$\begin{aligned}
V_i(a) &= \log\left(R_i a - a_i'\right) + \beta\left(pV_i\left(a_i'\right) + (1-p)V_j\left(a_i'\right)\right) \\
m_i + n\log(a) &= \log\left(R_i a - a_i'\right) + \beta\left(pm_i + (1-p)m_j\right) + \beta n \log\left(a_i'\right) \\
m_i + n\log(a) &= \beta n \log(\beta n) + (1+\beta n)\log\left(\frac{R_i}{1+\beta n}\right) + \beta\left(pm_i + (1-p)m_j\right) + (1+\beta n)\log(a)
\end{aligned}$$

Matching coefficients:

$$\begin{aligned}
n &= 1 + \beta n \\
m_i &= \beta n \log(\beta n) + (1+\beta n)\log\left(\frac{R_i}{1+\beta n}\right) + \beta\left(pm_i + (1-p)m_j\right),
\end{aligned}$$

where $j \neq i$. The solution to the first equation implies:

$$n = \frac{1}{1-\beta},$$

which in turn delivers the optimal saving decision of the entrepreneur:

$$a' = \beta R(z)\, a. \tag{44}$$

Finally, we solve for the remaining coefficients from the system of linear equations:

$$m_i = \frac{\beta}{1-\beta}\log\left(\frac{\beta}{1-\beta}\right) + \frac{1}{1-\beta}\log\left((1-\beta)R_i\right) + \beta\left(pm_i + (1-p)m_j\right)$$

The solution is given by:

$$m_i = \frac{\log(1-\beta)}{1-\beta} + \frac{\beta}{(1-\beta)^2}\log(\beta) + \frac{(1-\beta p)\log R_i + \beta(1-p)\log R_j}{(1-\beta)^2\left(1-\beta(2p-1)\right)}$$

B Proofs

This appendix presents the proofs for the results listed in the paper. We reproduce the statement of all results for the reader's convenience.

Lemma 1. ***(Aggregate Variables in Equilibrium)*** *In the heterogenous return equilibrium* $((\lambda-1)\,A_h < A_\ell)$*, output, wages, interest rate, and gross returns are:*

$$
\begin{aligned}
Y &= (ZK)^{\alpha}\, L^{1-\alpha} &(45)\\
w &= (1-\alpha)\left({}^{ZK}\!/_{L}\right)^{\alpha} &(46)\\
r &= \alpha \left({}^{ZK}\!/_{L}\right)^{\alpha-1} z_\ell &(47)\\
R_\ell &= (1-\tau_a) + (1-\tau_k)\,\alpha \left({}^{ZK}\!/_{L}\right)^{\alpha-1} z_\ell &(48)\\
R_h &= (1-\tau_a) + (1-\tau_k)\,\alpha \left({}^{ZK}\!/_{L}\right)^{\alpha-1} z_\lambda. &(49)
\end{aligned}
$$

Remark. Aggregate output can be rewritten in a familiar Cobb-Douglas form by defining effective capital $Q \equiv ZK\,(= z_\lambda A_h + z_\ell A_\ell)$ as in Guvenen et al. (2019), $Y = Q^{\alpha} L^{1-\alpha}$.

Proof. We start by considering the labor market clearing condition

$$n^{\star}(z_h, K_h) + n^{\star}(z_\ell, K_\ell) = L.$$

Replacing for the optimal labor demand (42) we get

$$
\begin{aligned}
\left(\frac{1-\alpha}{w}\right)^{1/\alpha} (z_h K_h + z_\ell K_\ell) &= L\\
\left(\frac{1-\alpha}{w}\right)^{1/\alpha} Q &= L
\end{aligned}
$$

Manipulating this expression we get wages as:

$$w = (1-\alpha)\left({}^{Q}\!/_{L}\right)^{\alpha}. \tag{50}$$

Replacing into the equilibrium interest rate we get:

$$r = \alpha\left(\frac{1-\alpha}{w}\right)^{\frac{1-\alpha}{\alpha}} z_\ell = \alpha\left({}^{Q}\!/_{L}\right)^{\alpha-1} z_\ell \tag{51}$$

These two expressions also let us rewrite the profit rate of the high-productivity entrepreneurs (from (5)):

$$\pi^{\star}(z_h) = \left(\alpha\left(\frac{1-\alpha}{w}\right)^{(1-\alpha)/\alpha} z_h - r\right)\lambda = \alpha\left({}^{Q}\!/_{L}\right)^{\alpha-1}(z_h - z_\ell)\,\lambda \tag{52}$$

We can then use the equilibrium profit rates of entrepreneurs to rewrite the gross returns of entrepreneurs:

$$R_\ell = (1-\tau_a) + (1-\tau_k)\left(r + \pi^\star(z_\ell)\right)$$

and

$$\begin{aligned} R_h &= (1-\tau_a) + (1-\tau_k)\left(r + \pi^\star(z_h)\right) \\ &= (1-\tau_a) + (1-\tau_k)\,\alpha\left(Q/L\right)^{\alpha-1}\left(z_\ell + \lambda\left(z_h - z_\ell\right)\right) \\ &= (1-\tau_a) + (1-\tau_k)\,\alpha\left(Q/L\right)^{\alpha-1} z_\lambda \end{aligned}$$

Finally we consider aggregate output, for this note that the ratio of labor to capital is constant across entrepreneurs which allows us to aggregate in terms of the total capital of each type. From (42) we can express the output of an individual entrepreneur with productivity z and capital k as:

$$y(z,k) = \left(\frac{1-\alpha}{w}\right)^{(1-\alpha)/\alpha} zk = \left(Q/L\right)^{\alpha-1} zk,$$

where the second equality comes after replacing the wage from (50). Aggregate output is the sum of the total output produced by each type of entrepreneur:

$$\begin{aligned} Y &= \left(Q/L\right)^{\alpha-1}\left(z_h K_h + z_\ell K_\ell\right) \\ Y &= Q^\alpha L^{1-\alpha} \end{aligned} \tag{53}$$

Alternatively we can write:

$$Y = (ZK)^\alpha L^{1-\alpha} \tag{54}$$

This completes the derivation of the results.

□

Evolution of aggregates. Using the saving rules in equation (6), we derive the law of motion for the aggregate wealth of each group and for the aggregate capital ($K \equiv A_\ell + A_h$)

$$A_h' = p\beta R_h A_h + (1-p)\,\beta R_\ell A_\ell \quad \text{and} \quad A_\ell' = (1-p)\,\beta R_h A_h + p\beta R_\ell A_\ell, \tag{55}$$

$$K' = \beta(1-\tau_a)K + \beta(1-\tau_k)\,\alpha(ZK)^\alpha L^{1-\alpha}. \tag{56}$$

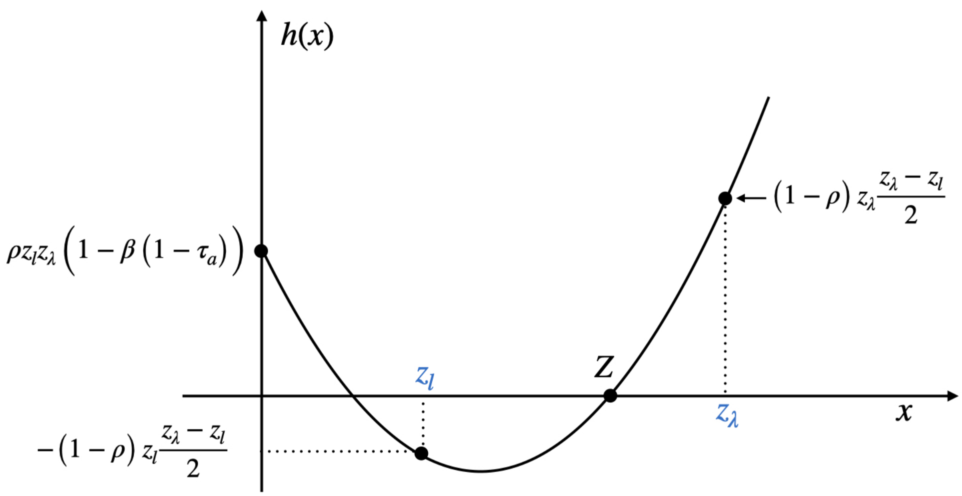
The law of motion for Q follows from $Q' = z_\lambda A_h' + z_\ell A_\ell'$ after substituting A_h' and A_ℓ',

$$\begin{aligned} Q' = \beta(1-\tau_a)&\left(\rho Q + \frac{z_\ell + z_\lambda}{2}(1-\rho)K\right) \\ &+ \beta(1-\tau_k)\,\alpha\left(Q/L\right)^{\alpha-1}\left(\frac{z_\ell + z_\lambda}{2}(1+\rho)Q - z_\ell z_\lambda \rho K\right). \end{aligned} \tag{57}$$

Proposition 1. ***(Existence and Uniqueness of Steady State)*** *There exists a unique steady state that features heterogenous returns ($R_h > R_\ell$) if and only if*

$$\lambda < \overline{\lambda} \equiv 1 + \frac{(1-p)}{p - (2p-1)\left(\beta(1-\tau_a) + (1-\beta(1-\tau_a))\frac{z_\ell}{z_h}\right)}.$$

Figure B.1: Steady State Productivity (Z)



Note: *The figure plots* $h(x) = (1-\rho\beta(1-\tau_a))\,x^2 - {}^{(z_\ell+z_\lambda)}/_2\,(1+\rho(1-2\beta(1-\tau_a)))\,x + z_\ell z_\lambda \rho(1-\beta(1-\tau_a))$. *The steady state productivity corresponds to the larger root of h, marked with a circle on the horizontal axis.*

Proof. First, we show that the steady state is unique when $(\lambda-1)\,A_h < A_\ell$. In this case, the steady state Z is the solution to equation (14). We will show that the larger root of that equation is the steady state Z. For this, let $h(z)$ be a function defined as

$$h(z) = (1-\beta(1-\tau_a)(2p-1))\,z^2 - (z_\ell+z_\lambda)(p-\beta(1-\tau_a)(2p-1))\,z + (2p-1)\,z_\ell z_\lambda (1-\beta(1-\tau_a)) = 0.$$

It is easy to show that $h(z_\ell) = (1-p)\,z_\ell(z_\ell - z_\lambda) < 0$ and $h(z_\lambda) = (1-p)\,z_\lambda(z_\lambda - z_\ell) > 0$. Since $h(z)$ is a quadratic function and $z_\ell < Z < z_\lambda$, this implies that there is a unique steady state Z as shown in Figure B.1.

□

Proof. Next, we prove that $(\lambda-1)\,A_h < A_\ell$ (excess supply of funds) iff $\lambda < \overline{\lambda}$ where

$$\overline{\lambda} \equiv 1 + \frac{(1-p)}{p-(2p-1)\left(\beta(1-\tau_a) + (1-\beta(1-\tau_a))\frac{z_\ell}{z_h}\right)}.$$

The proof involves two steps. First, we show that $(\lambda-1)\,A_h < A_\ell$ iff $Z < z_h$. Second, we find the condition on λ so that $Z < z_h$. For the first step, substituting the definition of $Z = \frac{(z_h+(\lambda-1)(z_h-z_\ell))A_h + z_\ell A_\ell}{A_h+A_\ell}$ into $Z < z_h$ and some algebra gives $(\lambda-1)\,A_h < A_\ell$. For the second step, we derive the condition on λ so that $h(z_h) > 0$ in equation (14). Thus to complete the proof, we evaluate $h(z_h)$:

$$\begin{aligned} h(z_h)/z_h^2 &= 1-(2p-1)\beta(1-\tau_a)-\frac{(z_\ell+z_\lambda)}{z_h}(p-(2p-1)\beta(1-\tau_a)) \\ &+ (2p-1)\frac{z_\ell z_\lambda}{z_h^2}(1-\beta(1-\tau_a)). \end{aligned}$$

Inserting $z_\lambda = z_h + (\lambda-1)(z_h - z_\ell)$ gives

$$\begin{aligned} h(z_h)/z_h^2 &= 1-(2p-1)\beta(1-\tau_a)-\frac{(z_\ell+z_h+(\lambda-1)(z_h-z_\ell))}{z_h}(p-(2p-1)\beta(1-\tau_a)) \\ &+ (2p-1)\frac{z_\ell(z_h+(\lambda-1)(z_h-z_\ell))}{z_h^2}(1-\beta(1-\tau_a)). \end{aligned}$$

Next we combine the terms that include $\lambda-1$:

$$\begin{aligned} h(z_h)/z_h^2 &= \frac{(1-p)(z_h-z_\ell)}{z_h} \\ &- \frac{(\lambda-1)(z_h-z_\ell)}{z_h}\left(p-(2p-1)\left(\beta(1-\tau_a)+(1-\beta(1-\tau_a))\frac{z_\ell}{z_h}\right)\right). \end{aligned}$$

Since $p-(2p-1)\left(\beta(1-\tau_a)+(1-\beta(1-\tau_a))\frac{z_\ell}{z_h}\right) > 0$ for all p, then, $h(z_h) > 0$ iff $\lambda - 1 < \frac{1-p}{p-(2p-1)\left(\beta(1-\tau_a)+(1-\beta(1-\tau_a))\frac{z_\ell}{z_h}\right)}$. Finally, recall that this equilibrium can only exist if $\lambda \leq 2$ (this gives $K_\ell \geq 0$). Inspecting the previous result it is immediate that $\overline{\lambda} \leq 2$ iff $p \geq 1/2$.

□

Corollary 2. ***(Savings Rates and Wealth Shares)*** *For all $\tau_a < \overline{\tau}_a$, the steady state saving rate of high-productivity entrepreneurs is positive and the saving rate of low-productivity entrepreneurs is negative: $\beta R_h > 1 > \beta R_\ell$. Furthermore, $s_h > 1/2$ if and only if $\rho > 0$.*

Proof. The gross saving rate of the entrepreneurs is βR_i. We first show that $\beta R_i > 1$ if and only if $\overline{z}_i > Z$, where we slightly abuse notation by letting $\overline{z}_\ell = z_\ell$. The result follows immediately from expressing the savings rate in terms of Z by substituting R_i's from equation (13):

$$\begin{aligned} \beta R_i &> 1 \\ \beta(1-\tau_a)+(1-\beta(1-\tau_a))\overline{z}_i/Z &> 1 \\ \overline{z}_i &> Z \end{aligned}$$

To finalize the proof recall from Proposition 1 that the steady state Z satisfies $z_\ell < Z < z_\lambda$, this gives the desired result.

Now, consider $s_h \geq 1/2$. We know that $s_h = \frac{Z-z_\ell}{z_\lambda - z_\ell}$, so $s_h > 1/2$ is equivalent to $Z > \frac{z_\lambda + z_\ell}{2}$. We can verify if this is the case by evaluating the residual of (14) at $\frac{z_\lambda+z_\ell}{2}$:

$$\begin{aligned}
h\left(\frac{z_\lambda+z_\ell}{2}\right) &= -\left(2p-1\right)\left(1-\beta\left(1-\tau_a\right)\right)\left(\frac{z_\lambda+z_\ell}{2}\right)^2+\left(2p-1\right)\left(1-\beta\left(1-\tau_a\right)\right)z_\ell z_\lambda \\
&= -\left(2p-1\right)\left(1-\beta\left(1-\tau_a\right)\right)\left[\left(\frac{z_\lambda+z_\ell}{2}\right)^2-z_\ell z_\lambda\right] \\
&= -\left(2p-1\right)\left(1-\beta\left(1-\tau_a\right)\right)\left(\frac{z_\lambda-z_\ell}{2}\right)^2<0
\end{aligned}$$

The residual is negative if and only if $p \geq 1/2$. So it must be that $Z > \frac{z_\lambda+z_\ell}{2}$ and thus $s_h > 1/2$ for $p \geq 1/2$.

□

Before proving Lemma 1 we provide a result that explicitly relates the change in returns to changes in productivity.

Lemma 2. ***(Wealth Shares and Returns in Steady State)*** *For all $\tau_a < \overline{\tau}_a$, the following equations and inequalities hold in steady state:*

$$s_h = \frac{1-\beta R_\ell}{\beta\left(R_h-R_\ell\right)} = \frac{Z-z_\ell}{z_\lambda-z_\ell} \qquad \frac{ds_h}{dZ} = \frac{1}{z_\lambda-z_\ell} > 0 \tag{58}$$

$$R_h = \frac{1}{\beta\left(2p-1\right)}\left(1-\frac{1-p}{s_h}\right) \qquad \frac{dR_h}{dZ} > 0 \tag{59}$$

$$R_\ell = \frac{1}{\beta\left(2p-1\right)}\left(1-\frac{1-p}{1-s_h}\right) \qquad \frac{dR_h}{dZ} < 0. \tag{60}$$

Moreover, the average returns are always decreasing with productivity, $\frac{d(R_\ell+R_h)}{dZ} < 0$, and the geometric average of returns decreases, $\frac{d(R_h R_\ell)}{dZ} < 0$, if and only if $\rho > 0$.

Proof. Using equation (55) and imposing steady state, we obtain

$$A_\ell\left(1-\beta R_\ell\right) = \left(\beta R_h-1\right)A_h, \tag{61}$$

Manipulating equation (61) directly we can express the ratio of wealth of the high types to total wealth:

$$\begin{aligned}
A_\ell\left(1-\beta R_\ell\right) &= \left(\beta R_h-1\right)A_h \\
\left(1-\beta R_\ell\right)\left(A_\ell+A_h\right) &= \beta\left(R_h-R_\ell\right)A_h \\
\frac{A_h}{A_\ell+A_h} &= \frac{1-\beta R_\ell}{\beta\left(R_h-R_\ell\right)}
\end{aligned} \tag{62}$$

The ratio depends on the returns of both types. We can further express the ratio in terms of Z by substituting R_i's from equation (13):

$$s_h = \frac{Z - z_\ell}{z_\lambda - z_\ell} \tag{63}$$

To finalize the proof take the derivative of s_h with respect to Z: $\frac{ds_h}{dZ} = \frac{1}{z_\lambda - z_\ell} > 0$.

Now we consider what happens to R_h as Z increases. We start by considering the evolution equation for A_h in steady state (55)

$$(1 - p\beta R_h) A_h \quad = \quad (1-p)\,\beta R_\ell A_\ell.$$

Manipulating this expression gives

$$R_h = \frac{1}{p\beta} - \left(\frac{1-p}{p}\right)\left(\frac{1-s_h}{s_h}\right) R_\ell.$$

We can also use the law of motion for A_ℓ in steady state to obtain an expression for R_ℓ in terms of R_h and s_h:

$$R_\ell = \frac{1}{p\beta} - \left(\frac{1-p}{p}\right)\left(\frac{s_h}{1-s_h}\right) R_h$$

Replacing we can solve for R_h as a function of s_h:

$$R_h = \frac{1}{\beta\,(2p-1)}\left(1 - \frac{1-p}{s_h}\right) \tag{64}$$

We can now obtain the derivative of the high-type returns with respect to Z:

$$\frac{dR_h}{dZ} = \frac{1-p}{\beta\,(2p-1)}\frac{1}{s_h^2}\frac{ds_h}{dZ} > 0 \tag{65}$$

The sign follows from Proposition 2.

We can also obtain an expression for R_ℓ in terms of s_h:

$$R_\ell = \frac{1}{\beta\,(2p-1)}\left(1 - \frac{1-p}{1-s_h}\right) \tag{66}$$

This expression allows to obtain an alternative expression for the derivative of the low-type returns with respect to Z:

$$\frac{dR_\ell}{dZ} = -\frac{(1-p)}{\beta\,(2p-1)}\frac{1}{(1-s_h)^2}\frac{ds_h}{dZ} < 0 \tag{67}$$

Using the results in (64), (65), (66), and (67) we can obtain expressions for the derivative of the sum and product of returns with respect to wealth taxes:

$$\frac{d\left(R_h+R_\ell\right)}{dZ}=\frac{-\left(2s_h-1\right)\left(1-p\right)}{\beta\left(2p-1\right)\left(\left(1-s_h\right)^2 s_h^2\right)}\frac{ds_h}{dZ} \tag{68}$$

$$\frac{d\left(R_hR_\ell\right)}{dZ}=\frac{-\left(2s_h-1\right)p\left(1-p\right)}{\left[\left(1-s_h\right)s_h\beta\left(2p-1\right)\right]^2}\frac{ds_h}{dZ} \tag{69}$$

$\frac{d(R_h+R_\ell)}{dZ}$ is always negative because $s_h \geq 1/2$ if and only if $p \geq 1/2$ (see Corollary 2). $\frac{d(R_hR_\ell)}{dZ}$ is negative if and only if $s_h \geq 1/2$, again, this happens if and only if $p \geq 1/2$.

□

We now proceed to the proof of Lemma 1:

Lemma 3. ***(Use-it-or-Lose-it)*** *For all $\tau_a < \overline{\tau}_a$, a marginal increase in wealth taxes increases entrepreneurial returns that are above the wealth-weighted average return and vice versa. That is, for any z, $dR(z)/d\tau_a \geq 0$ if and only if $z \geq Z = \left(s_h z_\lambda + \left(1-s_h\right) z_\ell\right)$ and $\rho > 0$.*

Proof. Let z be the level of entrepreneurial productivity. The after tax entrepreneurial returns are

$$R\left(z\right)=\left(1-\tau_a\right)+\left(1-\tau_k\right)\left(r+\pi^\star\left(z\right)\right), \tag{70}$$

where $\pi^\star\left(z\right)$ is as in equation 5. If $z \leq z_\ell$ then $\pi^\star\left(z\right)=0$ and it follows that $dR(z)/d\tau_a < 0$. If $z > z_\ell$ then we can write $R\left(z\right)$ as

$$\begin{aligned} R\left(z\right) &= \left(1-\tau_a\right)+\left(1-\tau_k\right)\alpha\left(\frac{1-\alpha}{w}\right)^{\frac{1-\alpha}{\alpha}}\left(z_\ell+\lambda\left(z-z_\ell\right)\right) && (71)\\ &= \left(1-\tau_a\right)+\left(\frac{1}{\beta}-\left(1-\tau_a\right)\right)\left(\frac{z_\ell+\lambda\left(z-z_\ell\right)}{Z}\right) && (72)\\ &= R_\ell+\left(\frac{1}{\beta}-\left(1-\tau_a\right)\right)\lambda\frac{z-z_\ell}{Z}, && (73)\end{aligned}$$

where the first line follows from equation (10) and the second line from Lemma 1 and equation (12).

The effect of an increase in wealth taxes is

$$\frac{dR\left(z\right)}{d\tau_a}=\frac{dR_\ell}{d\tau_a}+\lambda\left[\frac{z-z_\ell}{Z}-\left(\frac{1}{\beta}-\left(1-\tau_a\right)\right)\frac{z-z_\ell}{Z^2}\frac{dZ}{d\tau_a}\right]. \tag{74}$$

The first term, $dR_\ell/d\tau_a$, is negative because we already know that the returns of low-productivity entrepreneurs decrease with wealth taxes. The second term is monotonically increasing in the individual entrepreneurial productivity z. This has to be the case for $dR_h/d\tau_a > 0$ ad in Lemma 1.

Finally, the returns of someone with the wealth weighted productivity $Z = s_h z_\lambda + \left(1-s_h\right) z_\ell$ do not change, $\frac{dR(Z)}{d\tau_a}=0$, because wealth weighted returns are constant in steady state. From

Lemma 2 we get

$$s_h R_h + (1 - s_h) R_\ell = \frac{1}{\beta(2p-1)} \left[s_h \left(1 - \frac{1-p}{s_h} \right) + (1 - s_h) \left(1 - \frac{1-p}{1-s_h} \right) \right] = \frac{1}{\beta}. \tag{75}$$

□

Lemma 4. ***(Aggregate Variables in Steady State)*** *If $\tau < \overline{\tau}_a$ and under Assumption 1, the steady state level of aggregate capital is*

$$K = \left(\frac{\alpha\beta(1-\theta)}{1-\beta} \right)^{\frac{1}{1-\alpha}} L Z^{\frac{\alpha}{1-\alpha}} \tag{76}$$

and the steady state elasticities of aggregate variables with respect to productivity are

$$\xi_K = \xi_Y = \xi_w = \xi \equiv \frac{\alpha}{1-\alpha} \qquad \text{and} \qquad \xi_Q = 1 + \xi \tag{77}$$

where $\xi_x \equiv \frac{d\log x}{d\log Z}$ is the elasticity of variable x with respect to Z. Effective capital is $Q = ZK$, aggregate output is $Y = Q^\alpha L^{1-\alpha}$, and wage is $w = (1-\alpha)\frac{Y}{N}$ from Lemma 1. Moreover, the wealth levels of each entrepreneurial type in steady state are

$$A_h = \frac{Z - z_\ell}{z_\lambda - z_\ell} K \qquad \frac{dA_h}{dZ} \propto Z^{\frac{2\alpha-1}{1-\alpha}} (Z - \alpha z_\ell) > 0 \tag{78}$$

$$A_\ell = \frac{z_\lambda - Z}{z_\lambda - z_\ell} K \qquad \frac{dA_\ell}{dZ} \propto Z^{\frac{2\alpha-1}{1-\alpha}} (\alpha z_\lambda - Z) \tag{79}$$

where $\frac{dA_\ell}{dZ} < 0$ if and only if $\alpha z_\lambda < Z$.

Proof. Combining (12) with (18) gives $K = \left(\frac{\alpha\beta(1-\theta)}{1-\beta}\right)^{\frac{1}{1-\alpha}} L Z^{\frac{\alpha}{1-\alpha}}$. Inserting this into $Q = KZ$ gives Q, output Y and wage w follow from Lemma 1 The elasticity of of aggregate capital to productivity Z is

$$\xi_K \equiv \frac{d\log K}{d\log Z} = \frac{\alpha}{1-\alpha}$$

For convenience we define $\xi \equiv \alpha/1-\alpha$. The elasticities of output, wage, and effective capital with respect to productivity follow immediately.

From equation (63) we can express A_h in terms of Z and total capital K:

$$A_h = \frac{Z - z_\ell}{z_\lambda - z_\ell} K,$$

then we can replace K for its value in terms of Z to get

$$A_h = \frac{Z - z_\ell}{z_\lambda - z_\ell} \left(\frac{\alpha\beta(1-\theta)}{1-\beta} \right)^{\frac{1}{1-\alpha}} L Z^{\frac{\alpha}{1-\alpha}}. \tag{80}$$

The result follows from differentiating with respect to Z:

$$\frac{dA_h}{dZ} \propto Z^{\frac{2\alpha-1}{1-\alpha}} \left(Z - \alpha z_\ell\right) > 0,$$

where the inequality follows from $z_\ell < Z$.

A similar process lets us express A_ℓ in terms of Z and total capital K:

$$A_\ell = \frac{z_\lambda - Z}{z_\lambda - z_\ell} K,$$

which gives

$$A_\ell = \frac{z_\lambda - Z}{z_\lambda - z_\ell} \left(\frac{\alpha\beta\left(1-\theta\right)}{1-\beta}\right)^{\frac{1}{1-\alpha}} L Z^{\frac{\alpha}{1-\alpha}}. \tag{81}$$

The result follows from differentiating with respect to Z:

$$\frac{dA_\ell}{dZ} \propto Z^{\frac{2\alpha-1}{1-\alpha}} \left[\alpha z_\lambda - Z\right]$$

which is negative if $\alpha z_\lambda < Z$.

$\square$

Proposition 2. ***(Efficiency Gains from Wealth Taxation)*** *For all $\tau_a < \overline{\tau}_a$, a marginal increase in wealth taxes (τ_a) increases aggregate productivity (Z), $\frac{dZ}{d\tau_a} > 0$, if and only if entrepreneurial productivity is persistent, $\rho > 0$.*

Proof. The steady state Z is given by the solution of $h\left(Z\right) = 0$ where $h\left(z\right)$ is defined in equation (14). Differentiating $h\left(z\right)$ with respect to τ_a gives

$$\begin{aligned} \frac{d}{d\tau_a} h\left(z\right) &= \left(2p-1\right)\beta z^2 - \left(2p-1\right)\beta\left(z_\ell + z_\lambda\right) z + \left(2p-1\right)\beta z_\ell z_\lambda \\ &= \left(2p-1\right)\beta z_\ell z_\lambda \left(z - z_\ell\right)\left(z - z_\lambda\right). \end{aligned}$$

We know that the steady state Z satisfies $z_\ell < Z < z_\lambda$, so we have $\left(z - z_\ell\right)\left(z - z_\lambda\right) < 0$. Thus, $\frac{d}{d\tau_a} h\left(z\right) < 0$ iff $p > 1/2$. As shown in Figure 2, the steady state Z increases when τ_a increases. Notice also that $\frac{d}{d\tau_a} h\left(z\right) < 0$ for all τ_a if $z_\ell < Z < z_\lambda$. Thus, $\frac{dZ}{d\tau_a} > 0$ for all τ_a as long as the economy is in the first equilibrium which happens if and only if $\lambda \leq \overline{\lambda}$. Notice that the bound $\overline{\lambda}$ is an increasing function of τ_a.

$\square$

Proposition 3. ***(Welfare Gain by Agent Type)*** *For all $\tau_a < \overline{\tau}_a$, if Assumption 1 holds and $\rho > 0$, a marginal increase in wealth taxes (τ_a) increases the welfare of workers ($CE_{1,w} > 0$) and decreases the welfare of low-productivity entrepreneurs ($CE_{1,\ell} < 0$) and the average welfare of entrepreneurs ($CE_1^e < 0$). Furthermore, there exists a upper bound on the dispersion of returns (κ_R) such that an increase in wealth taxes increases the welfare of high-productivity entrepreneurs ($CE_{1,h} > 0$) if and only if $R_h - R_\ell < \kappa_R$.*

Proof. We start by stating the welfare gain measure for each type of agent as in (25):

$$\log\left(1+\mathrm{CE}_{1,i}\right) = \begin{cases} \log {w_a}/{w_k} & \text{if } i = w \\ \frac{(1-\beta)\log {R_{a,i}}/{R_{k,i}} + \beta(1-p)\left(\log {R_{a,\ell}}/{R_{k,\ell}} + \log {R_{a,h}}/{R_{k,h}}\right)}{(1-\beta)(1-\beta(2p-1))} & \text{if } i \in \{\ell, h\}\,. \end{cases}$$

For the workers' welfare note that:

$$\frac{d\log\left(1+\mathrm{CE}_{1,w}\right)}{d\tau_a} = \frac{d\frac{\alpha}{1-\alpha}\log\left(Z_a/Z_k\right)}{d\tau_a} = \frac{\alpha}{1-\alpha}\frac{1}{Z}\frac{dZ}{d\tau_a} > 0 \longleftrightarrow p > {1}/{2}$$

The welfare gain is positive if and only if productivity is persistent because of Proposition (2).

The welfare of the low-productivity entrepreneurs decreases unambiguously:

$$\frac{d\log\left(1+\mathrm{CE}_{1,\ell}\right)}{d\tau_a} \propto \frac{1-\beta}{R_\ell}\frac{dR_\ell}{d\tau_a} + \frac{\beta\left(1-p\right)}{R_\ell R_h}\frac{dR_\ell R_h}{d\tau_a} < 0$$

which follows from Corollary (**??**) $\left(\frac{dR_\ell}{d\tau_a}, \frac{dR_\ell R_h}{d\tau_a} < 0\right)$.

The welfare of entrepreneurs as a group also decreases unambiguously.

$$\frac{d\log\left(1+\mathrm{CE}_1^{\mathrm{e}}\right)}{d\tau_a} = \frac{\beta\left(1-p\right)}{1-\beta}\frac{1}{R_\ell R_h}\frac{dR_\ell R_h}{d\tau_a} < 0$$

Finally, for the high-productivity entrepreneurs:

$$\begin{aligned} \frac{d\log\left(1+\mathrm{CE}_{1,h}\right)}{d\tau_a} &\propto \frac{1-\beta}{R_h}\frac{dR_h}{d\tau_a} + \frac{\beta\left(1-p\right)}{R_\ell R_h}\frac{dR_\ell R_h}{d\tau_a} \\ &= \left[(1-\beta) - \frac{1}{R_\ell}\frac{\left(2s_h-1\right)p\left(1-p\right)}{\left(1-s_h\right)^2\left(2p-1\right)}\right]\frac{\left(1-p\right)}{\beta\left(2p-1\right)s_h^2 R_h}\frac{ds_h}{d\tau_a} \\ &= \left[(1-\beta) - \frac{\beta\left(2s_h-1\right)p\left(1-p\right)}{\left(p-s_h\right)\left(1-s_h\right)}\right]\frac{\left(1-p\right)}{\beta\left(2p-1\right)s_h^2 R_h}\frac{ds_h}{d\tau_a} \end{aligned}$$

We maintain the assumption that $p \geq {1}/{2}$, and from Corollary **??** we know that $\frac{ds_h}{d\tau_a} > 0$. So, the sign of derivative of interest depends on the sign of the term in square brackets.

$$\frac{d\log\left(1+\mathrm{CE}_{1,h}\right)}{d\tau_a} \geq 0 \longleftrightarrow 1-\beta \geq \frac{\beta\left(2s_h-1\right)p\left(1-p\right)}{\left(p-s_h\right)\left(1-s_h\right)}$$

It is easy to verify that in steady state $s_h < p$, which together with Corollary 2 implies that the right hand side of the inequality is always positive. To verify that $s_h < p$ holds in steady steady note that this condition is equivalent to $Z < pz_\lambda + (1-p)\,z_\ell$, then evaluate function h defined in (14) at $pz_\lambda + (1-p)\,z_\ell$. The value of h (the residual of the quadratic equation) is always positive, so it must be that $Z < pz_\lambda + (1-p)\,z_\ell$ and thus $s_h < p$.

Then, the high-type entrepreneurs' welfare gain is positive if and only if

$$g\left(s_h\right) \equiv (1-\beta)\left(p-s_h\right)\left(1-s_h\right) - \beta\left(2s_h-1\right)p\left(1-p\right) \geq 0. \tag{82}$$

Evaluating at $s_h = 1/2$

$$g(s_h) = (1-\beta)\left(p - \frac{1}{2}\right)\frac{1}{2} > 0.$$

Evaluating at $s_h = p$

$$g(s_h) \equiv -\beta(2p-1)p(1-p) < 0.$$

Moreover, g is continuous for $s_h \in [1/2, p]$ and monotonically decreasing:

$$g^{'}(s_h) = -(1-\beta)[(1-s_h) + (p-s_h)] - 2\beta p(1-p) < 0$$

So, there exists an upper bound $\bar{s}_h$ such that

$$\frac{d\log(1+\mathrm{CE}_{1,h})}{d\tau_a} \geq 0 \longleftrightarrow s_h \in \left[\frac{1}{2}, \bar{s}_h\right]$$

The upper bound for z_ℓ is characterized by the solution to

$$(p-\bar{s}_h)(1-\bar{s}_h) - \beta(2\bar{s}_h - 1)p(1-p) = 0$$

Alternatively, we can make us of the link between s_h and the dispersion of returns:

$$R_h - R_\ell = \frac{(1-p)(2s_h-1)}{\beta(2p-1)(1-s_h)s_h}$$

So the high-productivity entrepreneurs benefit from an increase in wealth taxes if and only if the dispersion of returns is low enough:

$$\frac{d\log(1+\mathrm{CE}_{1,h})}{d\tau_a} \geq 0 \longleftrightarrow s_h \in \left[\frac{1}{2}, \bar{s}_h\right] \longleftrightarrow R_h - R_\ell \in [0, \kappa_R]$$

where $\kappa_R \equiv \frac{(1-p)(2\bar{s}_h-1)}{\beta(2p-1)(1-\bar{s}_h)\bar{s}_h}$. Note that $\bar{s}_h$ depends only on p and β, therefore the upper bound for the dispersion of returns is also a function of p and β alone.

□

Proposition 4. ***(Optimal CE$_1$ Taxes)*** *Under Assumption 1 and if $\rho > 0$, there exist a unique tax combination $(\tau_a^\star, \tau_k^\star)$ that maximizes the utilitarian welfare measure CE$_1$. An interior solution, $\tau_a^\star < \bar{\tau}_a$, is the solution to:*

$$n_w\xi_w = -\frac{1-n_w}{1-\beta}\left(\frac{\xi_{R_\ell} + \xi_{R_h}}{2}\right) \tag{83}$$

where $\xi_x \equiv \frac{d\log x}{d\log Z}$ is the elasticity of variable x with respect to Z. Furthermore, there exist two *cutoff values for α, $\underline{\alpha}$ and $\overline{\alpha}$, such that $(\tau_a^\star, \tau_k^\star)$ satisfies the following properties:*

$$\begin{aligned}
\tau_a^\star &\in \left[1-\frac{1}{\beta}, 0\right) \text{ and } \tau_k^\star > \theta && \text{if } \alpha < \underline{\alpha} \\
\tau_a^\star &\in \left[0, \frac{\theta(1-\beta)}{\beta(1-\theta)}\right] \text{ and } \tau_k^\star \in [0,\theta] && \text{if } \underline{\alpha} \le \alpha \le \bar{\alpha} \\
\tau_a^\star &\in \left(\frac{\theta(1-\beta)}{\beta(1-\theta)}, \tau_a^{\max}\right) \text{ and } \tau_k^\star < 0 && \text{if } \alpha > \bar{\alpha}
\end{aligned}$$

where $\underline{\alpha}$ and $\overline{\alpha}$ are the solutions to equation (83) with $\tau_a = 0$ and $\tau_a = \tau^{TR} = \frac{\theta(1-\beta)}{\beta(1-\theta)}$, respectively. Recall from Lemma (2) that $\xi_w = \xi \equiv \alpha/1-\alpha$.

Proof. We start from the definition of aggregate CE_1 welfare which gives us:

$$\text{CE}_1 > 0 \longleftrightarrow \sum_{i\in\{w,h,\ell\}} n_i \log(1+\text{CE}_{1,i}) > 0$$

replacing from (25) gives us:

$$\sum_{i\in\{w,h,\ell\}} n_i \log(1+\text{CE}_{1,i}) = n_w \log\frac{w_a}{w_k} + \frac{1-n_w}{2(1-\beta)}\log\frac{R_{a,\ell}R_{a,h}}{R_{k,\ell}R_{k,h}}$$

The optimal tax is characterized by first order condition:

$$\begin{aligned}
n_w \frac{d\log w}{d\tau_a} + \frac{1-n_w}{2(1-\beta)}\frac{d\log R_\ell R_h}{d\tau_a} &= 0 \\
\left[n_w \frac{d\log w}{d\log Z} + \frac{1-n_w}{2(1-\beta)}\frac{d\log R_\ell R_h}{d\log Z}\right]\frac{d\log Z}{d\tau_a} &= 0 \\
\left[n_w \xi_w + \frac{1-n_w}{1-\beta}\left(\frac{\xi_{R_\ell}+\xi_{R_h}}{2}\right)\right]\frac{d\log Z}{d\tau_a} &= 0
\end{aligned}$$

From Proposition (2) we know that $\frac{d\log Z}{d\tau_a} > 0$ under the sustained assumptions that $p > 1/2$ and $\lambda < \overline{\lambda}$. Then the above equation is satisfied if and only if

$$n_w \xi_w = -\frac{1-n_w}{1-\beta}\left(\frac{\xi_{R_\ell}+\xi_{R_h}}{2}\right)$$

That is, for τ_a such that the steady state values of the elasticities above satisfy the equation. The elasticity of wages with respect to productivity is constant, while the average elasticity of returns is negative (because the geometric average of returns decreases with taxes). Further, we can show that the average elasticity of returns is increasing in wealth taxes (this follows immediately from the explicit solution below). So there exists at most one solution to the optimal wealth taxes.

Note that elasticity of wages depends only on α, while the elasticities of returns are independent of α. Because of this we can define cutoffs for α such by evaluating the right hand side of the equation at $\tau_a = 0$ and $\tau_a = \frac{\theta(1-\beta)}{\beta(1-\theta)}$. If α is exactly equal to the cutoff then the optimal τ_a is either 0 or $\frac{\theta(1-\beta)}{\beta(1-\theta)}$. The monotonicity of the right hand side lets us define the intervals shown in the proposition.

Finally, we can replace to get a more explicit solution using Lemmas **??** and 2:

$$\begin{aligned}
n_w \frac{\alpha}{1-\alpha} &= -\frac{1-n_w}{2(1-\beta)} \frac{Z}{R_\ell R_h} \frac{dR_\ell R_h}{dZ} \\
n_w \frac{\alpha}{1-\alpha} &= \frac{1-n_w}{2(1-\beta)} \frac{Z}{R_\ell R_h} \frac{(2s_h-1)\, p\,(1-p)}{\left[(1-s_h)\, s_h \beta\,(2p-1)\right]^2} \frac{ds_h}{dZ} \\
n_w \frac{\alpha}{1-\alpha} &= \frac{1-n_w}{2(1-\beta)} \frac{1}{R_\ell R_h} \frac{(2s_h-1)\, p\,(1-p)}{\left[(1-s_h)\, s_h \beta\,(2p-1)\right]^2} \frac{Z}{(z_\lambda - z_\ell)} \\
n_w \frac{\alpha}{1-\alpha} &= \frac{1-n_w}{2(1-\beta)} \frac{(2s_h-1)\, p\,(1-p)}{(p-s_h)(p+s_h-1)(1-s_h)\, s_h} \frac{Z}{(z_\lambda - z_\ell)}
\end{aligned}$$

□

Corollary 7. ***(α-Thresholds)*** *If $z_\ell = 0$, the thresholds $\underline{\alpha}$ and $\bar{\alpha}$ are explicitly given by $\frac{\underline{\alpha}}{1-\underline{\alpha}} = \frac{1}{L}\frac{p}{1-p}\frac{(1-\beta(2p-1))^2}{\beta(2p-1)(1-p\beta)}$ and $\frac{\bar{\alpha}}{1-\bar{\alpha}} = \frac{p(1-\theta-(2p-1)(\beta-\theta))^2}{L(1-p)(2p-1)(1-\theta)(\beta-\theta)((1-\theta)-p(\beta-\theta))}$.*

Proof. When $z_\ell = 0$ we can solve for Z and s_h explicitly as:

$$Z = \frac{z_\lambda\left(p-(1-\tau_a)\,\beta\,(2p-1)\right)}{1-(1-\tau_a)\,\beta\,(2p-1)} \qquad s_h = \frac{Z}{z_\lambda}$$

The value of $\underline{\alpha}$ is obtained when $\tau_a = 0$, so $Z = \frac{z_\lambda(p-\beta(2p-1))}{1-\beta(2p-1)}$ and $s_h = \frac{p-\beta(2p-1)}{1-\beta(2p-1)}$. We can then evaluate the expression:

$$\begin{aligned}
\frac{\underline{\alpha}}{1-\underline{\alpha}} &= \frac{1-n_w}{n_w} \frac{Z}{2(1-\beta)} \frac{(2s_h-1)\, p\,(1-p)}{(p-s_h)(p+s_h-1)(1-s_h)\, s_h} \frac{1}{z_\lambda} \\
&= \frac{1}{L} \frac{1}{(1-\beta)} \frac{(2s_h-1)\, p\,(1-p)}{(p-s_h)(p+s_h-1)(1-s_h)} \\
&= \frac{1}{L} \frac{p}{1-p} \frac{(1-\beta\,(2p-1))^2}{\beta\,(2p-1)(1-p\beta)}
\end{aligned}$$

The value of $\overline{\alpha}$ is obtained when $\tau_a = \frac{\theta(1-\beta)}{\beta(1-\theta)}$, so $Z = \frac{z_\lambda((1-\theta)p-(\beta-\theta)(2p-1))}{(1-\theta)-(\beta-\theta)(2p-1)}$ and

$s_h = \frac{(1-\theta)p-(\beta-\theta)(2p-1)}{(1-\theta)-(\beta-\theta)(2p-1)}$. We can then evaluate the expression:

$$\begin{aligned}\frac{\overline{\alpha}}{1-\overline{\alpha}} &= \frac{1-n_w}{n_w}\frac{Z}{2(1-\beta)}\frac{(2s_h-1)\,p\,(1-p)}{(p-s_h)(p+s_h-1)(1-s_h)\,s_h}\frac{1}{z_\lambda}\\ &= \frac{1}{L}\frac{1}{(1-\beta)}\frac{(2s_h-1)\,p\,(1-p)}{(p-s_h)(p+s_h-1)(1-s_h)}\\ &= \frac{1}{L}\frac{p}{1-p}\frac{1}{(2p-1)(1-\theta)(\beta-\theta)}\frac{((1-\theta)-(\beta-\theta)(2p-1))^2}{((1-\theta)-p(\beta-\theta))}.\end{aligned}$$

We can also compute the optimal wealth tax for this case inserting $z_\ell = 0$, $Z = \frac{z_\lambda(p-(1-\tau_a)\beta(2p-1))}{1-(1-\tau_a)\beta(2p-1)}$, and $s_h = \frac{p-(1-\tau_a)\beta(2p-1)}{1-(1-\tau_a)\beta(2p-1)}$ into

$$n_w\frac{\alpha}{1-\alpha} = \frac{1-n_w}{2(1-\beta)}\frac{(2s_h-1)\,p\,(1-p)}{(p-s_h)(p+s_h-1)(1-s_h)\,s_h}\frac{Z}{(z_\lambda-z_\ell)},$$

which gives the following non-linear equation which uniquely determines the optimal wealth tax as functions of parameters:

$$\frac{n_w}{(1-n_w)/2}\frac{\alpha}{1-\alpha}(1-\tau_a) = \frac{p\,(1-(1-\tau_a)\,\beta)\,(1-(1-\tau_a)\,\beta\,(2p-1))^2}{\beta\,(1-\beta)(1-p)(2p-1)(1-p\,(1-\tau_a)\,\beta)}.$$

□

C Taking wealth accumulation into account

The welfare of entrepreneurs depends on three different but intertwined components: The level of their after-tax returns (R_i), the (log-)average returns, and their asset holdings of the entrepreneurs (see equation 7). However, the CE_1 measures used above capture only the first two components, ignoring the effects of the increase in aggregate capital (K) and the wealth share of high-productivity entrepreneurs (s_h) brought about by the tax reform. Leading to potential welfare losses from wealth taxation being measured for high-productivity entrepreneurs, as we proved in Lemma 3.

As an alternative to the CE_1 measure used above, we consider the welfare gain of a stand-in representative entrepreneur of each type. We compare the values assigned by a type-i entrepreneur of being in the capital income or wealth tax economy while holding the average type-i wealth level in that economy. We denote this welfare measure as $\text{CE}_{2,i}$:

$$\log(1+\text{CE}_{2,i}) = (1-\beta)(V_a(A_{i,a},i) - V_k(A_{i,k},i)) = \log(1+\text{CE}_{1,i}) + \log(A_{a,i}/A_{k,i}). \quad (84)$$

For low-productivity entrepreneurs, $\text{CE}_{2,\ell}$ is likely to be lower than $\text{CE}_{2,\ell}$ because A_ℓ goes down with the tax reform if α is not too high (Corollary ??). In contrast, high-productivity entrepreneurs unambiguously benefit from wealth taxes once the increase asset holdings is taken into account, as we show in Lemma 5.

Lemma 5. ***(Entrepreneurial Welfare Gains with Asset Accumulation)*** *For all $\tau_a < \overline{\tau}_a$, if Assumption 1 holds and $\rho > 0$, a marginal increase in wealth taxes (τ_a) increases the welfare of high-productivity entrepreneurs ($CE_{2,h} > 0$). The welfare of low-productivity entrepreneurs decreases ($CE_{2,\ell} < 0$) if $\alpha z_\lambda < Z$.*

Proof. The welfare gain of the average low-productivity entrepreneur is:

$$\log(1 + \text{CE}_{2,\ell}) = (1-\beta)\left(V_a\left(A_{\ell,a},\ell\right) - V_k\left(A_{\ell,k},\ell\right)\right) = \log\left(1 + \text{CE}_{1,\ell}\right) + \log\left(A_{a,\ell}/A_{k,\ell}\right)$$

From Lemma 3 we know that $\text{CE}_{1,\ell} < 0$ and from Lemma 2 we know that $A_{a,\ell} < A_{k,\ell}$ if $\alpha z_\lambda < Z$.

Now, we turn to the CE_2 welfare measure for the high-type entrepreneurs:

$$\log(1 + \text{CE}_{2,h}) = (1-\beta)\left(V_a\left(A_{h,a},h\right) - V_k\left(A_{h,k},h\right)\right) = \log\left(1 + \text{CE}_{1,h}\right) + \log\left(A_{a,h}/A_{k,h}\right)$$

Substituting $A_h = s_h K$ and substituting K from Lemma 2 and taking derivative with respect to τ_a gives:

$$\begin{aligned}
\frac{d\log(1+\text{CE}_{2,h})}{d\tau_a} &\propto \frac{1-\beta}{R_h}\frac{dR_h}{d\tau_a} + \frac{\beta(1-p)}{R_\ell R_h}\frac{dR_\ell R_h}{d\tau_a} + \frac{1}{s_h}\frac{ds_h}{d\tau_a} + \frac{\alpha}{1-\alpha}\frac{1}{Z}\frac{dZ}{d\tau_a} \\
&= \left[\frac{1-\beta}{R_h}\left(\frac{1-p}{\beta(2p-1)}\frac{1}{s_h^2}\right) - \frac{\beta(1-p)}{R_\ell R_h}\left(\frac{(2s_h-1)p(1-p)}{\left[(1-s_h)s_h\beta(2p-1)\right]^2}\right)\right]\frac{ds_h}{d\tau_a} \\
&\qquad + \frac{1}{s_h}\frac{ds_h}{d\tau_a} + \frac{\alpha}{1-\alpha}\frac{1}{Z}\frac{dZ}{d\tau_a} \\
&= \left[\frac{(1-\beta)(1-p)}{(p+s_h-1)} - \frac{\beta(2s_h-1)p(1-p)^2}{(p-s_h)(s_h+p-1)(1-s_h)} + 1\right]\frac{1}{s_h}\frac{ds_h}{d\tau_a} + \frac{\alpha}{1-\alpha}\frac{1}{Z}\frac{dZ}{d\tau_a} \\
&= \left[\left(s_h - \beta(1-p)\frac{2p - s_h + s_h^2 - 3ps_h + (2s_h-1)p^2}{(p-s_h)(1-s_h)}\right)\frac{1}{(s_h+p-1)(Z-z_\ell)} \right. \\
&\qquad \left. + \frac{\alpha}{1-\alpha}\frac{1}{Z}\right]
\end{aligned}$$

A sufficient condition for this to be positive is that

$$s_h > \beta(1-p)\left(\frac{2p - s_h + s_h^2 - 3ps_h + (2s_h-1)p^2}{(p-s_h)(1-s_h)}\right)$$

We know that $s_h < p$ and that $s_h \geq 1/2$. So a sufficient condition is

$$\begin{aligned}
\frac{1}{2} &> \beta\left(1-p\right)\left(\frac{2p - s_h + s_h^2 - 3ps_h + \left(2s_h - 1\right)p^2}{\left(p - s_h\right)\left(1 - s_h\right)}\right) \\
\frac{1}{2} &> \beta\left(1-s_h\right)\left(\frac{2p - s_h + s_h^2 - 3ps_h + \left(2s_h - 1\right)p^2}{\left(p - s_h\right)\left(1 - s_h\right)}\right) \\
p - s_h &> 4p - 2s_h + 2s_h^2 - 6ps_h + 2\left(2s_h - 1\right)p^2 \\
0 &> 3p\left(1 - 2s_h\right) - s_h\left(1 - 2s_h\right) + 2\left(2s_h - 1\right)p^2 \\
0 &> -\left[2p\left(1-p\right) + \left(p - s_h\right)\right]\left(2s_h - 1\right)
\end{aligned}$$

which is verified for all values of $p > s_h > 1/2$.

□

We can also ask each entrepreneur how much they value being in the wealth tax economy with its average wealth (K_a) relative to being in the capital income tax economy with its average wealth (K_k). The welfare gain for a type-i entrepreneur is

$$\log\left(1 + \widetilde{\mathrm{CE}}_{2,i}\right) = \left(1-\beta\right)\left(V_a\left(K_a, i\right) - V_k\left(K_k, i\right)\right) = \log\left(1 + \mathrm{CE}_{1,i}\right) + \log\left({}^{K_a}/_{K_k}\right), \tag{85}$$

and the aggregate (or expected) welfare is

$$\log\left(1 + \widetilde{\mathrm{CE}}_{2}\right) = \sum_i n_i \log\left(1 + \widetilde{\mathrm{CE}}_{2,i}\right) = \log\left(1 + \mathrm{CE}_{1,i}\right) + \log\left({}^{K_a}/_{K_k}\right). \tag{86}$$

Welfare is higher when measured with $\widetilde{\mathrm{CE}}_2$ relative to CE_1 because of the increase in K with the wealth tax. Consequently, the thresholds $\underline{\alpha}$ and $\overline{\alpha}$ are lower under $\widetilde{\mathrm{CE}}_2$. Interestingly, this does not alter the nature of the key tradeoff we described in Proposition 4. Optimal taxes are still trading off the efficiency gains from wealth taxation with the loses to entrepreneurs from higher return dispersion and lower expected value. In fact, we know from Lemma 2 that $\xi_w = \xi_K = \xi$, so that incorporating the gains from higher capital accumulation acts by re-weighting the gains from wealth taxes. In Proposition 5, we present the optimal tax outcomes when maximizing $\widetilde{\mathrm{CE}}_2$.

Proposition 5. ***(Optimal $\widetilde{CE}_2$ Taxes)*** *Under Assumption 1, there exist a unique tax combination $\left(\tau_{a,2}^{\star}, \tau_{k,2}^{\star}\right)$ that maximizes the utilitarian welfare measure $\widetilde{CE}_2$, an interior solution $\tau_{a,2}^{\star} < \overline{\tau}_a$ is the solution to:*

$$n_w \xi_w + \left(1 - n_w\right)\xi_K = \xi = -\frac{1 - n_w}{1 - \beta}\left(\frac{\xi_{R_\ell} + \xi_{R_h}}{2}\right) \tag{87}$$

where $\xi_x \equiv \frac{d \log x}{d \log Z}$ is the elasticity of variable x with respect to Z. Furthermore, there exist two *cutoff values for* α, $\underline{\alpha}_2$ *and* $\overline{\alpha}_2$, *such that* $\left(\tau_{a,2}^{\star}, \tau_{k,2}^{\star}\right)$ *satisfies the following properties:*

$$
\begin{aligned}
\tau_{a,2}^{\star} &\in \left[1-\frac{1}{\beta}, 0\right) \text{ and } \tau_{k,2}^{\star} > \theta && \text{if } \alpha < \underline{\alpha}_2 \\
\tau_{a,2}^{\star} &\in \left[0, \frac{\theta(1-\beta)}{\beta(1-\theta)}\right] \text{ and } \tau_{k,2}^{\star} \in [0, \theta] && \text{if } \underline{\alpha}_2 \leq \alpha \leq \bar{\alpha}_2 \\
\tau_{a,2}^{\star} &\in \left(\frac{\theta(1-\beta)}{\beta(1-\theta)}, \tau_{a,2}^{\max}\right) \text{ and } \tau_{k,2}^{\star} < 0 && \text{if } \alpha > \bar{\alpha}_2
\end{aligned}
$$

where $\underline{\alpha}_2$ *and* $\overline{\alpha}_2$ *are the solutions to equation (87) with* $\tau_a = 0$ *and* $\tau_a = \frac{\theta(1-\beta)}{\beta(1-\theta)}$, *respectively. Recall from Lemma (2) that* $\xi = \alpha/1-\alpha$.

Proof. From (86) we obtain the first order condition to maximize $\widetilde{\mathrm{CE}}_2$:

$$
\begin{aligned}
\frac{d\log(1+\mathrm{CE}_1)}{d\tau_a} + (1-n_w)\frac{d\log K}{d\tau_a} &= 0 \\
\left[\frac{d\log(1+\mathrm{CE}_1)}{d\log Z} + (1-n_w)\frac{d\log K}{d\log Z}\right]\frac{d\log Z}{d\tau_a} &= 0 \\
\left[n_w\xi_w + \frac{1-n_w}{1-\beta}\left(\frac{\xi_{R_\ell}+\xi_{R_h}}{2}\right) + (1-n_w)\xi_K\right]\frac{d\log Z}{d\tau_a} &= 0
\end{aligned}
$$

As in the proof of Proposition 4 this leads to the optimality condition:

$$
n_w\xi_w + (1-n_w)\xi_K = -\frac{1-n_w}{1-\beta}\left(\frac{\xi_{R_\ell}+\xi_{R_h}}{2}\right).
$$

Further, we know from Lemma 2 that $\xi_w = \xi_K = \xi = \alpha/1-\alpha$. The right hand side of the equation is the same as in Proposition 4 and an explicit formula can be found in the proof to that proposition. The uniqueness of the solution and the definition of the thresholds for α and its implications for the optimal taxes follow from the same arguments as in Proposition 4.

□

Taking into account the role of capital accumulation results in a higher optimal tax level, and lower thresholds $\underline{\alpha}$ and $\overline{\alpha}$:

Corollary 8. ***(Comparison of CE*** $_1$ ***and CE*** $_2$ ***Taxes)*** *Optimal wealth taxes are higher when taking the wealth accumulation into account* $\left(\tau_{a,2}^{\star} > \tau_a^{\star}\right)$. *Moreover, the* α*-thresholds are lower* $\underline{\alpha}_2 < \underline{\alpha}$ *and* $\overline{\alpha}_2 < \overline{\alpha}$.

D Extensions

D.1 Corporate sector

We now introduce a corporate sector that faces no collateral constraints and provides entrepreneurs with an alternative use for their wealth. The corporate sector produces the same good as the entrepreneurs using a constant-returns-to-scale technology:

$$Y_c = (z_c K_c)^{\alpha} L_c^{1-\alpha}. \tag{88}$$

The lack of constraints implies that the marginal return of capital in the corporate sector imposes a lower bound on the equilibrium rental rate of capital r,

$$r \geq \alpha z_c \left(\frac{1-\alpha}{w}\right)^{\frac{1-\alpha}{\alpha}}. \tag{89}$$

We focus on the case where $z_\ell < z_c < z_h$ and the corporate sector and the high-productivity entrepreneurs operate in equilibrium, while the low-productivity entrepreneurs do not produce and instead lend all of their assets.[19] As in Section 2, we focus on the equilibrium with heterogeneous return, where the high-productivity entrepreneurs are constrained in their demand for capital and demand $K_h = \lambda A_h$, but now the remaining capital is used by the corporate sector rather than by the low-productivity entrepreneurs ($K_c = K - K_h$ and $K_\ell = 0$). The corporate sector makes zero profits and $r = \alpha z_c \left({}^{(1-\alpha)}/_{w}\right)^{(1-\alpha)/\alpha}$. The outcome is that z_c takes the place of z_ℓ in determining the model's aggregates. For instance, the wealth-weighted productivity of capital is now $Z = s_h z_\lambda + s_\ell z_c$, where $z_\lambda = z_h + (\lambda - 1)(z_h - z_c)$.

These changes do not affect the derivation of any of our results. In particular, Propositions 1, 2, and 4 characterizing the steady state, the efficiency gains from wealth taxation and the optimal tax schedule apply unchanged. The main consequence of the change from z_ℓ to z_c is that the relevant dispersion of productivities is now lower, z_h versus $z_c\,(> z_\ell)$. This reduces misallocation and thus the scope for efficiency gains. Lemma 8 in Appendix D.1 formalizes these results.

Derivations. Consider a model like that in Section 2 where there is also a corporate sector that produces the same final good as the entrepreneurs using a constant returns to scale technology:

$$Y_c = (z_c K_c)^{\alpha} L_c^{1-\alpha}. \tag{90}$$

The conditional demand for labor of the corporate sector is characterized by

$$w = (1-\alpha)\left(\frac{z_c K_c}{L_c}\right)^{\alpha}$$

as is standard.

[19] If $z_c < z_\ell$, the corporate sector does not operate and the economy works as in Section 2. If $z_c > z_h$, only the corporate sector operates using the assets of all entrepreneurs. This equilibrium is efficient. We discuss the knife-edge cases of $z_c = z_h$ and $z_c = z_\ell$ in Appendix D.1.

Unlike the entrepreneurs, the corporate sector faces no collateral constraints and thus the demand for capital is:

$$K_c = \begin{cases} \infty & \text{if } \alpha z_c \left(\frac{1-\alpha}{w}\right)^{\frac{1-\alpha}{\alpha}} > r \\ [0,\infty) & \text{if } \alpha z_c \left(\frac{1-\alpha}{w}\right)^{\frac{1-\alpha}{\alpha}} = r \\ 0 & \text{if } \alpha z_c \left(\frac{1-\alpha}{w}\right)^{\frac{1-\alpha}{\alpha}} < r \end{cases}$$

So, any equilibrium must satisfy:

$$r \leq \alpha z_c \left(\frac{1-\alpha}{w}\right)^{\frac{1-\alpha}{\alpha}}.$$

If $r < \alpha z_c \left(1-\alpha/w\right)^{1-\alpha/\alpha}$ the corporate sector does not operate and the economy works as in Section 2. This happens if z_c is too low relative to the productivity of entrepreneurs, $z_c < z_\ell$. The more interesting case is when $r = \alpha z_c \left(1-\alpha/w\right)^{1-\alpha/\alpha}$ in equilibrium which happens if $z_\ell \leq z_c$. The behavior of the entrepreneurs depends then on how high the corporate sector's productivity is relative to that of the entrepreneurs. We will focus on the most relevant scenario where $z_\ell < z_c < z_h$. In this scenario both the corporate sector and the high-productivity entrepreneurs operate in equilibrium, while the low-productivity entrepreneurs do not produce and instead lend all of their funds.[20]

Even though the corporate sector is operating in equilibrium, there are no real changes in the aggregates of the economy. In fact, the equilibrium looks just like that of Section 2 with the corporate sector's productivity z_c taking the place of the z_ℓ. As in Section 2, the high-productivity entrepreneurs are constrained in their demand for capital and demand $K_h = \lambda A_h$, but now the remaining capital is used by the corporate sector rather than by the low-productivity entrepreneurs. This is only sustainable in equilibrium if $r = \alpha z_c \left(1-\alpha/w\right)^{1-\alpha/\alpha}$ as noted above, and in this way z_c takes the place of z_ℓ in determining the interest rate in the economy. The main consequence of this change is that the relevant dispersion of productivities is now $z_h - z_c$ which is lower than it was in Section 2 (recall that $z_c > z_\ell$). This reduces the range of parameters for which the heterogeneous return equilibrium applies, and reduces the scope for misallocation and thus for efficiency gains. Lemma 8 makes the above results precise:

Lemma 8. *If $z_\ell < z_c < z_h$ and*

$$\lambda < \lambda_c^\star \equiv 1 + \frac{(1-p)}{p - (2p-1)\left(\beta\left(1-\tau_a\right) + \left(1-\beta\left(1-\tau_a\right)\right)\frac{z_c}{z_h}\right)} < \overline{\lambda},$$

the economy is in the heterogeneous return equilibrium with both the corporate sector and high-productivity entrepreneurs operating and output, wages, interest rate, and gross returns on savings

[20] If $z_c > z_h$, only the corporate sector operates in equilibrium and it is optimal for all entrepreneurs to lend their assets to the corporate sector where they will receive a higher return. The equilibrium is efficient and total productivity Z is equal to z_c. The knife-edge case with $z_c = z_h$ has the same result but the distribution of capital between the high-productive entrepreneurs and the corporate sector is indeterminate. Finally, the knife-edge case with $z_\ell = z_c$ is identical to the model in Section 2 with the low-productivity entrepreneurs being indifferent between producing themselves or lending to the corporate sector. All aggregates remain unchanged.

are the same as in Lemma 1 with z_c taking the place of z_ℓ:

$$Y = (ZK)^\alpha L^{1-\alpha} \tag{91}$$
$$w = (1-\alpha)\left({}^{ZK}/_{L}\right)^\alpha \tag{92}$$
$$r = \alpha\left({}^{ZK}/_{L}\right)^{\alpha-1} z_c \tag{93}$$
$$R_\ell = (1-\tau_a) + (1-\tau_k)\,\alpha\left({}^{ZK}/_{L}\right)^{\alpha-1} z_c \tag{94}$$
$$R_h = (1-\tau_a) + (1-\tau_k)\,\alpha\left({}^{ZK}/_{L}\right)^{\alpha-1} z_\lambda. \tag{95}$$

where $Z \equiv s_h z_\lambda + s_\ell z_c$, $z_\lambda \equiv z_h + (\lambda-1)(z_h - z_c)$ and $s_h = {}^{A_h}/_{K}$.

Proof. If $z_\ell < z_c < z_h$, $\alpha z_\ell \left({}^{1-\alpha}/_{w}\right)^{1-\alpha/\alpha} < r = \alpha z_c \left({}^{1-\alpha}/_{w}\right)^{1-\alpha/\alpha} < \alpha z_h \left({}^{1-\alpha}/_{w}\right)^{1-\alpha/\alpha}$ and thus $K_\ell = 0$, $K_h = \lambda A_h$, and $K_c = A_\ell - (\lambda-1)A_h$ to guarantee that the capital market clears.

Given the wage rate w, the labor demand of the corporate and private sectors are:

$$n_i^\star(K_i) \;=\; \left(\frac{1-\alpha}{w}\right)^{\frac{1}{\alpha}} z_i K_i\,; \quad i \in \{\ell, h, c\}\,.$$

The labor market clearing condition gives

$$\left(\frac{1-\alpha}{w}\right)^{\frac{1}{\alpha}} (z_h K_h + z_\ell K_\ell + z_c K_c) = L$$
$$\left(\frac{1-\alpha}{w}\right)^{\frac{1}{\alpha}} Q = L$$
$$(1-\alpha)\left(\frac{Q}{L}\right)^\alpha = w$$

where $Q = z_h K_h + z_\ell K_\ell + z_c K_c = ZK$ with $Z = s_h z_\lambda + s_\ell z_c$ after replacing for the equilibrium capital demand. Then, the equilibrium interest rate is

$$r = \alpha z_c \left(\frac{Q}{L}\right)^{\alpha-1}.$$

In equilibrium, low productivity entrepreneurs and the corporate sector do not generate profits, while high-productivity entrepreneurs do:

$$\pi^\star(z_h) = \left(\alpha\left(\frac{1-\alpha}{w}\right)^{\frac{1-\alpha}{\alpha}} z_h - r\right)\lambda = \alpha\left({}^{Q}/_{L}\right)^{\alpha-1}(z_h - z_c)\,\lambda$$

Total private output corresponds to the output of high-productivity entrepreneurs. Note that the output of an individual entrepreneur is proportional to their capital, so total private output is:

$$Y_p = Y_h = \left(\frac{1-\alpha}{w}\right)^{\frac{1-\alpha}{\alpha}} z_h K_h$$

Total output is then:

$$\begin{aligned} Y \equiv Y_c + Y_p &= \left(\frac{1-\alpha}{w}\right)^{\frac{1-\alpha}{\alpha}} (z_c K_c + z_h K_h) \\ &= Q^\alpha L^{1-\alpha} \end{aligned}$$

Finally we derive the after-tax returns on savings for low- and high-productivity entrepreneurs. Low-productivity entrepreneurs do not produce so they have:

$$R_\ell = (1-\tau_a) + (1-\tau_k) \underbrace{\alpha \left({}^{Q}\!/\!{}_{L}\right)^{\alpha-1} z_c}_{r}.$$

High productivity entrepreneurs have:

$$R_h = (1-\tau_a) + (1-\tau_k) \underbrace{\alpha \left({}^{Q}\!/\!{}_{L}\right)^{\alpha-1} z_\lambda}_{r+\pi^\star(z_h)}$$

where $z_\lambda \equiv z_h + (\lambda - 1)(z_h - z_c)$.

All aggregates are then as in Lemma 1 with z_c taking the role of z_c. Consequently, Proposition 1 applies with the only modification of z_c replacing z_ℓ in the condition that characterizes the steady state value of Z and the upper bound for λ.

□

D.2 Entrepreneurial effort

Consider a model like that in Section 2 where entrepreneurs can exert effort to increase their productivity. We capture the effect of effort as modifying the production function of entrepreneurs to:

$$y = (zk)^{\alpha} g(e)^{\gamma} n^{1-\alpha-\gamma}. \tag{96}$$

where $\gamma \in [0,1)$.

Exerting effort has a utility cost of $h(e)$, where $h^{'}(e) > 0$ and $h^{''}(e) \geq 0$ but no dynamic effects. The utility function is

$$u(c,e) = \log(c - h(e)).$$

D.2.1 Entrepreneur's problem

We can solve the entrepreneur's static effort choice. The solution is characterized by the following first order conditions:

$$u_e h^{'}(e) = (1-\tau_k) u_c \cdot \gamma (zk)^{\alpha} g(e)^{\gamma-1} n^{1-\alpha-\gamma} g^{'}(e) \qquad w = (1-\alpha-\gamma)(zk)^{\alpha} g(e)^{\gamma} n^{-\alpha-\gamma}$$

which imply:

$$n = \left[\frac{(1-\alpha-\gamma)(zk)^{\alpha} g(e)^{\gamma}}{w}\right]^{\frac{1}{\alpha+\gamma}}$$

replacing:

$$\begin{aligned}
\frac{u_e}{u_c}\frac{h^{'}(e)}{g^{'}(e)} &= (1-\tau_k)\gamma (zk)^{\frac{\alpha}{\alpha+\gamma}} g(e)^{\frac{-\alpha}{\alpha+\gamma}} \left(\frac{1-\alpha-\gamma}{w}\right)^{\frac{1-\alpha-\gamma}{\alpha+\gamma}} \\
g(e) &= \left(\frac{(1-\tau_k)\gamma}{\psi}\right)^{\frac{\alpha+\gamma}{\alpha}} \left(\frac{1-\alpha-\gamma}{w}\right)^{\frac{1-\alpha-\gamma}{\alpha}} zk
\end{aligned}$$

So we get the desired result if it so happens that $\frac{h^{'}(e)}{g^{'}(e)}$ is constant, say ψ with $h(e) = \psi e$ and $g(e) = e$. If that is the case we can write labor demand as:

$$n = \left(\frac{(1-\tau_k)\gamma}{\psi}\right)^{\frac{\gamma}{\alpha}} \left(\frac{1-\alpha-\gamma}{w}\right)^{\frac{1-\gamma}{\alpha}} zk$$

and profits as:

$$\begin{aligned}
\pi(z,k) &= (zk)^{\alpha} g(e)^{\gamma} n^{1-\alpha-\gamma} - wn - rk \\
&= \left[\underbrace{(\alpha+\gamma)\left(\frac{(1-\tau_k)\gamma}{\psi}\right)^{\frac{\gamma}{\alpha}} \left(\frac{1-\alpha-\gamma}{w}\right)^{\frac{1-\alpha-\gamma}{\alpha}} z - r}_{\pi^{\star}(z)}\right] k
\end{aligned}$$

Both profits and effort are proportional to how much capital the entrepreneur uses. The entrepreneur will only demand capital and operate their firm if the (after-tax) profits net of the effort cost are positive, that is:

$$k \geq 0 \longleftrightarrow (1-\tau_k)\,\pi^{\star}(z) - \underbrace{\frac{u_e h^{'}(e)}{u_c}}_{\text{Shadow Price}} \varepsilon(z) \geq 0,$$

where the shadow price of the effort cost is equal to ψ given our assumptions and

$$\varepsilon(z) \equiv \frac{e(z,k)}{k} = \left(\frac{(1-\tau_k)\,\gamma}{\psi}\right)^{\frac{\alpha+\gamma}{\alpha}} \left(\frac{1-\alpha-\gamma}{w}\right)^{\frac{1-\alpha-\gamma}{\alpha}} z.$$

In order to demand capital the entrepreneur must make profits to cover the cost of effort.

The optimal demand for capital is then:

$$k^{\star}(z,a) = \begin{cases} \lambda a & \text{if } \alpha \left(\frac{(1-\tau_k)\gamma}{\psi}\right)^{\frac{\gamma}{\alpha}} \left(\frac{1-\alpha-\gamma}{w}\right)^{\frac{1-\alpha-\gamma}{\alpha}} z > r \\ [0,\lambda a] & \text{if } \alpha \left(\frac{(1-\tau_k)\gamma}{\psi}\right)^{\frac{\gamma}{\alpha}} \left(\frac{1-\alpha-\gamma}{w}\right)^{\frac{1-\alpha-\gamma}{\alpha}} z = r \\ 0 & \text{if } \alpha \left(\frac{(1-\tau_k)\gamma}{\psi}\right)^{\frac{\gamma}{\alpha}} \left(\frac{1-\alpha-\gamma}{w}\right)^{\frac{1-\alpha-\gamma}{\alpha}} z < r \end{cases}$$

With this demand for capital we can replace back and get the level of profits, effort and labor demand.

Before proceeding to the optimal savings choice of the agent we need to determine the level of the capital demand for each type of entrepreneur. The relevant case has high-productivity entrepreneurs demanding $k^{\star}(z_h,a) = \lambda a$ for a total demand of $K_h = \lambda A_h$. The remaining assets are used by the low-productivity entrepreneurs who will be indifferent between any production level. The total demand for capital required to clear the market is $K_L = A_L - (\lambda-1)\,A_h$. Let $\lambda_{\ell,\iota} \equiv \frac{k_\iota}{a_\iota}$ be the ratio of capital to assets of low-productivity entrepreneur ι, for $\iota \in [0,1]$. We will show that the savings choice of the entrepreneur is independent of the value of $\lambda_{\ell,\iota}$.

Now we turn to the value function:

$$V_\iota(a,z) = \max_{\{c,a'\}} \ln(c - h(e_\iota)) + \beta E\left[V_\iota\left(a^{'},z^{'}\right)|z\right]$$
$$V_\iota(a,z) = \max_{\{c,a'\}} \ln(c - \psi e_\iota(z,a)) + \beta E\left[V_\iota\left(a^{'},z^{'}\right)|z\right]$$

Subject to:

$$c + a^{'} = R_\iota(z)\,a$$

where $R(z) \equiv (1-\tau_a) + (1-\tau_k)\,(r + \pi^{\star}(z)\,\lambda_\iota(z))$, and where $e_\iota(z,a) = \varepsilon(z)\,\lambda_\iota(z)\,a$. The value of $\lambda_\iota(z)$ satisfies:

$$\lambda_\iota(z) = \begin{cases} \lambda & \text{if } z = z_h \\ \lambda_{\iota,\ell} & \text{if } z = z_\ell. \end{cases}$$

We solve the dynamic programming problem of the entrepreneur via guess and verify. To this end, we guess that the value function of an entrepreneur of type $i \in \{\ell, h\}$ has the form

$$V_{i,\iota}(a) = m_{i,\iota} + n \log(a),$$

where $\{m_{\ell,\iota}, m_{h,\iota}\}_{\iota \in \{0,1\}}, n \in \mathbb{R}$ are coefficients. Under this guess the optimal savings choice of the entrepreneur is characterized by

$$\frac{1}{\left(R_{i,\iota} - \psi \varepsilon_i \lambda_{i,\iota}\right) a - a_i^{'}} = \frac{\beta n}{a_i^{'}}.$$

Solving for savings gives:

$$a_i^{'} = \frac{\beta n}{1 + \beta n}\left(R_{i,\iota} - \psi \varepsilon_i \lambda_{i,\iota}\right) a.$$

Replacing the savings rule into the value function gives:

$$\begin{aligned}
V_{i,\iota}(a) &= \log\left(\left(R_{i,\iota} - \psi \varepsilon_i \lambda_{i,\iota}\right) a - a_i^{'}\right) + \beta\left(p V_{i,\iota}\left(a_i^{'}\right) + (1-p) V_{j,\iota}\left(a_i^{'}\right)\right) \\
m_{i,\iota} + n \log(a) &= \log\left(\left(R_{i,\iota} - \psi \varepsilon_i \lambda_{i,\iota}\right) a - a_i^{'}\right) + \beta\left(p m_{i,\iota} + (1-p) m_{j,\iota}\right) + \beta n \log\left(a_i^{'}\right) \\
m_i + n \log(a) &= \beta n \log(\beta n) + (1 + \beta n) \log\left(\frac{R_{i,\iota} - \psi \varepsilon_i \lambda_{i,\iota}}{1 + \beta n}\right) + \beta\left(p m_{i,\iota} + (1-p) m_{j,\iota}\right) + (1 + \beta n) \log(a)
\end{aligned}$$

Matching coefficients:

$$\begin{aligned}
n &= 1 + \beta n \\
m_{i,\iota} &= \beta n \log(\beta n) + (1 + \beta n) \log\left(\frac{R_{i,\iota} - \psi \varepsilon_i \lambda_{i,\iota}}{1 + \beta n}\right) + \beta\left(p m_{i,\iota} + (1-p) m_{j,\iota}\right),
\end{aligned}$$

where $j \neq i$. The solution to the first equation implies:

$$n = \frac{1}{1 - \beta},$$

which in turn delivers the optimal saving decision of the entrepreneur:

$$a' = \beta\left(R_\iota(z) - \psi \varepsilon(z) \lambda_\iota(z)\right) a. \tag{97}$$

Finally, we solve for the remaining coefficients for the relevant case in which high-productivity entrepreneurs are all constrained and low-productivity entrepreneurs are indifferent between any level of production. In that case, it holds that:

$$\begin{aligned}
R_\iota(z_\ell) - \psi \varepsilon(z_\ell) \lambda_\iota(z_\ell) &= (1 - \tau_a) + (1 - \tau_k) r + \left[(1 - \tau_k) \pi^\star(z) - \psi \varepsilon(z_\ell)\right] \lambda_\iota(z_\ell) \\
&= (1 - \tau_a) + (1 - \tau_k) r
\end{aligned}$$

which is independent of the identity of the entrepreneur. It also holds that

$$\begin{aligned}R_{\iota}(z_h) - \psi\varepsilon(z_h)\lambda_{\iota}(z_h) &= (1-\tau_a) + (1-\tau_k)(r + \pi^{\star}(z_h)\lambda) - \psi\varepsilon(z_h)\lambda \\ &= (1-\tau_a) + (1-\tau_k)\left((1-\lambda)r + \alpha\left(\frac{(1-\tau_k)\gamma}{\psi}\right)^{\frac{\gamma}{\alpha}}\left(\frac{1-\alpha-\gamma}{w}\right)^{\frac{1-\alpha-\gamma}{\alpha}} z\lambda\right),\end{aligned}$$

which is also independent of the identity of the entrepreneur. Consequently, we can write without loss:

$$R_{\iota}(z) - \psi\varepsilon(z)\lambda_{\iota}(z) = R(z) - \psi\varepsilon(z)\lambda \equiv \hat{R}(z)$$

Having established these results, we can solve for m_ℓ and m_h from the system of linear equations:

$$m_i = \frac{\beta}{1-\beta}\log\left(\frac{\beta}{1-\beta}\right) + \frac{1}{1-\beta}\log\left((1-\beta)\hat{R}(z)\right) + \beta(pm_i + (1-p)m_j)$$

The solution is given by:

$$m_i = \frac{\log(1-\beta)}{1-\beta} + \frac{\beta}{(1-\beta)^2}\log(\beta) + \frac{(1-\beta p)\log\hat{R}(z) + \beta(1-p)\log\hat{R}(z)}{(1-\beta)^2(1-\beta(2p-1))}$$

D.2.2 Equilibrium and aggregation

In equilibrium the interest rate is such that the low-productivity entrepreneurs are indifferent between lending their assets or using them in their own firm. Lending the assets gives them a (before-tax) return of r, using them gives them $\pi^{\star}(z_\ell)$ but it also entails a utility cost because of effort, which we know from the previous results is proportional to assets, same as returns and profits. The agents will be indifferent if the (after-tax) profits net of effort costs are zero:

$$\begin{aligned}0 &= (1-\tau_k)\pi^{\star}(z_\ell) - \underbrace{\frac{u_e h'(e)}{u_c}}_{\text{Shadow Price}}\varepsilon(z_\ell) \\ 0 &= (1-\tau_k)\pi^{\star}(z_\ell) - \psi\varepsilon(z_\ell)\end{aligned}$$

replacing for the optimal solution of the entrepreneur's problem:

$$r = \alpha\left(\frac{(1-\tau_k)\gamma}{\psi}\right)^{\frac{\gamma}{\alpha}}\left(\frac{1-\alpha-\gamma}{w}\right)^{\frac{1-\alpha-\gamma}{\alpha}} z_\ell \tag{98}$$

We can then exploit the linearity of the savings function to aggregate results:

Lemma 9. *In the heterogenous return equilibrium ($(\lambda-1)A_h < A_\ell$), output, wages, interest rate,* *and gross returns on savings are:*

$$Y = \left(\frac{(1-\tau_k)\,\gamma}{\psi}\right)^{\frac{\gamma}{1-\gamma}} (ZK)^{\frac{\alpha}{1-\gamma}} L^{\frac{1-\alpha-\gamma}{1-\gamma}} \tag{99}$$

$$E = \left(\frac{(1-\tau_k)\,\gamma}{\psi}\right)^{\frac{1}{1-\gamma}} (ZK)^{\frac{\alpha}{1-\gamma}} L^{\frac{1-\alpha-\gamma}{1-\gamma}} \tag{100}$$

$$w = (1-\alpha-\gamma)\left(\frac{(1-\tau_k)\,\gamma}{\psi}\right)^{\frac{\gamma}{1-\gamma}} \left(\frac{ZK}{L}\right)^{\frac{\alpha}{1-\gamma}} \tag{101}$$

$$r = \alpha\left(\frac{(1-\tau_k)\,\gamma}{\psi}\right)^{\frac{\gamma}{1-\gamma}} \left(\frac{L}{ZK}\right)^{\frac{1-\alpha-\gamma}{1-\gamma}} z_\ell \tag{102}$$

$$R_{\ell,\iota} = (1-\tau_a) + (1-\tau_k)\left(\frac{(1-\tau_k)\,\gamma}{\psi}\right)^{\frac{\gamma}{1-\gamma}} \left(\frac{L}{ZK}\right)^{\frac{1-\alpha-\gamma}{1-\gamma}} (\alpha+\gamma\lambda_\iota)\, z_\ell \tag{103}$$

$$R_h = (1-\tau_a) + (1-\tau_k)\left(\frac{(1-\tau_k)\,\gamma}{\psi}\right)^{\frac{\gamma}{1-\gamma}} \left(\frac{L}{ZK}\right)^{\frac{1-\alpha-\gamma}{1-\gamma}} (\alpha z_\lambda + \gamma\lambda z_h) \tag{104}$$

and

$$\hat{R}(z) = R(z) - \psi\varepsilon(z)\,\lambda = \begin{cases} (1-\tau_a) + (1-\tau_k)\,\alpha\left(\frac{(1-\tau_k)\gamma}{\psi}\right)^{\frac{\gamma}{1-\gamma}} \left(\frac{L}{ZK}\right)^{\frac{1-\alpha-\gamma}{1-\gamma}} z_\ell & \textit{if } z = z_\ell \\ (1-\tau_a) + (1-\tau_k)\,\alpha\left(\frac{(1-\tau_k)\gamma}{\psi}\right)^{\frac{\gamma}{1-\gamma}} \left(\frac{L}{ZK}\right)^{\frac{1-\alpha-\gamma}{1-\gamma}} z_\lambda & \textit{if } z = z_h \end{cases} \tag{105}$$

Proof. We start by considering the labor market clearing condition, we get

$$\begin{aligned} n^\star(z_h, K_h) + n^\star(z_\ell, K_\ell) &= L \\ \left(\frac{(1-\tau_k)\,\gamma}{\psi}\right)^{\frac{\gamma}{\alpha}} \left(\frac{1-\alpha-\gamma}{w}\right)^{\frac{1-\gamma}{\alpha}} (z_h K_h + z_\ell K_\ell) &= L \\ \left(\frac{(1-\tau_k)\,\gamma}{\psi}\right)^{\frac{\gamma}{\alpha}} \left(\frac{1-\alpha-\gamma}{w}\right)^{\frac{1-\gamma}{\alpha}} Q &= L \\ (1-\alpha-\gamma)\left(\frac{(1-\tau_k)\,\gamma}{\psi}\right)^{\frac{\gamma}{1-\gamma}} \left(\frac{Q}{L}\right)^{\frac{\alpha}{1-\gamma}} &= w \end{aligned}$$

Turning to the total effort we get:

$$
\begin{aligned}
\left(\frac{E}{Q}\right)^{\alpha} &= \left(\frac{(1-\tau_k)\,\gamma}{\psi}\right)^{\alpha+\gamma}\left(\frac{1-\alpha-\gamma}{w}\right)^{1-\alpha-\gamma} \\
\left(\frac{E}{Q}\right)^{\alpha} &= \left(\frac{(1-\tau_k)\,\gamma}{\psi}\right)^{\alpha}\left(\frac{1-\alpha-\gamma}{w}\right)^{-\alpha}\left(\frac{L}{Q}\right)^{\alpha} \\
\left(\frac{E}{L}\right) &= \left(\frac{(1-\tau_k)\,\gamma}{\psi}\right)\left(\frac{1-\alpha-\gamma}{w}\right)^{-1} \\
E &= \left(\frac{(1-\tau_k)\,\gamma}{\psi}\right)^{\frac{1}{1-\gamma}} Q^{\frac{\alpha}{1-\gamma}} L^{\frac{1-\alpha-\gamma}{1-\gamma}} \qquad (106)
\end{aligned}
$$

replacing back and then applying the result to the interest rate we get the usual Cobb-Douglas expressions:

$$
w = (1-\alpha-\gamma)\,\frac{Q^{\alpha}E^{\gamma}L^{1-\gamma-\alpha}}{L} \qquad (107)
$$

$$
r = \alpha\frac{Q^{\alpha}E^{\gamma}L^{1-\gamma-\alpha}}{Q}z_{\ell} \qquad (108)
$$

We can go further by replacing E which itself depends on other aggregates:

$$
r = \alpha\left(\frac{(1-\tau_k)\,\gamma}{\psi}\right)^{\frac{\gamma}{1-\gamma}}\left(\frac{L}{Q}\right)^{\frac{1-\alpha-\gamma}{1-\gamma}} z_{\ell} \qquad (109)
$$

$$
w = (1-\alpha-\gamma)\left(\frac{(1-\tau_k)\,\gamma}{\psi}\right)^{\frac{\gamma}{1-\gamma}}\left(\frac{Q}{L}\right)^{\frac{\alpha}{1-\gamma}} \qquad (110)
$$

These two expressions also let us rewrite the profit rate (of capital) of entrepreneurs:

$$
\pi^{\star}(z) = \left(\frac{(1-\tau_k)\,\gamma}{\psi}\right)^{\frac{\gamma}{1-\gamma}}\left(\frac{L}{Q}\right)^{\frac{1-\alpha-\gamma}{1-\gamma}}\left(\alpha\,(z-z_{\ell})+\gamma z\right) > 0 \qquad (111)
$$

Notice that profits are always positive for both types of entrepreneurs.

We can then use the equilibrium profit rates of entrepreneurs to rewrite the gross returns of entrepreneurs:

$$
\begin{aligned}
R(z) &= (1-\tau_a) + (1-\tau_k)\,(r+\pi^{\star}(z)\,\lambda) \\
&= (1-\tau_a) + (1-\tau_k)\left(\frac{(1-\tau_k)\,\gamma}{\psi}\right)^{\frac{\gamma}{1-\gamma}}\left(\frac{L}{Q}\right)^{\frac{1-\alpha-\gamma}{1-\gamma}}\left(\alpha\,(z_{\ell}+\lambda\,(z-z_{\ell}))+\gamma\lambda z\right)
\end{aligned}
$$

we can express this as:

$$
R(z) = \begin{cases} (1-\tau_a) + (1-\tau_k)\left(\frac{(1-\tau_k)\gamma}{\psi}\right)^{\frac{\gamma}{1-\gamma}}\left(\frac{L}{Q}\right)^{\frac{1-\alpha-\gamma}{1-\gamma}}(\alpha+\gamma\lambda)\,z_{\ell} & \text{if } z=z_{\ell} \\ (1-\tau_a) + (1-\tau_k)\left(\frac{(1-\tau_k)\gamma}{\psi}\right)^{\frac{\gamma}{1-\gamma}}\left(\frac{L}{Q}\right)^{\frac{1-\alpha-\gamma}{1-\gamma}}(\alpha z_{\lambda}+\gamma\lambda z_h) & \text{if } z=z_h \end{cases} \qquad (112)
$$

We are loosely referring as λ to the ratio of capital to assets of the entrepreneur. This ratio can vary by entrepreneur for the low-productivity entrepreneurs.

The return net of effort cost is:

$$\hat{R}(z) = R(z) - \psi\varepsilon(z)\lambda = (1-\tau_a) + (1-\tau_k)\alpha\left(\frac{(1-\tau_k)\gamma}{\psi}\right)^{\frac{\gamma}{1-\gamma}}\left(\frac{L}{Q}\right)^{\frac{1-\alpha-\gamma}{1-\gamma}}(\lambda z + (1-\lambda)z_\ell)$$

More explicitly:

$$\hat{R}(z) = \begin{cases}(1-\tau_a) + (1-\tau_k)\alpha\left(\frac{(1-\tau_k)\gamma}{\psi}\right)^{\frac{\gamma}{1-\gamma}}\left(\frac{L}{Q}\right)^{\frac{1-\alpha-\gamma}{1-\gamma}} z_\ell & \text{if } z = z_\ell \\ (1-\tau_a) + (1-\tau_k)\alpha\left(\frac{(1-\tau_k)\gamma}{\psi}\right)^{\frac{\gamma}{1-\gamma}}\left(\frac{L}{Q}\right)^{\frac{1-\alpha-\gamma}{1-\gamma}} z_\lambda & \text{if } z = z_h\end{cases} \tag{113}$$

Finally we consider aggregate output, for this note that the ratio of labor to capital is constant across entrepreneurs which allows us to aggregate in terms of the total capital of each type. We can express the output of an individual entrepreneur with productivity z and capital k as:

$$y(z,k) = \left(\frac{(1-\tau_k)\gamma}{\psi}\right)^{\frac{\gamma}{\alpha}}\left(\frac{1-\alpha-\gamma}{w}\right)^{\frac{1-\alpha-\gamma}{\alpha}} zk$$

Aggregate output is the sum of the total output produced by each type of entrepreneur:

$$\begin{aligned} Y &= \left(\frac{(1-\tau_k)\gamma}{\psi}\right)^{\frac{\gamma}{\alpha}}\left(\frac{1-\alpha-\gamma}{w}\right)^{\frac{1-\alpha-\gamma}{\alpha}}(z_h K_h + z_\ell K_\ell) \\ &= \left(\frac{(1-\tau_k)\gamma}{\psi}\right)^{\frac{\gamma}{\alpha}\left(1-\frac{1-\alpha-\gamma}{1-\gamma}\right)}\left(\frac{Q}{L}\right)^{-\frac{1-\alpha-\gamma}{1-\gamma}} Q \\ &= \left(\frac{(1-\tau_k)\gamma}{\psi}\right)^{\frac{\gamma}{1-\gamma}}(ZK)^{\frac{\alpha}{1-\gamma}} L^{\frac{1-\alpha-\gamma}{1-\gamma}} \end{aligned}$$

For completeness we also consider the aggregate effort of high- and low-productivity entrepreneurs:

$$E_i \equiv \int e(z, k_{\iota,i})\,d\iota = \varepsilon(z_i)\int k_{\iota,i}d\iota = \left[\frac{(1-\tau_k)\gamma}{\psi}Q^{-(1-\alpha-\gamma)}L^{1-\alpha-\gamma}\right]^{\frac{1}{1-\gamma}} z_i K_i$$

This completes the derivation of the results.

□

We now turn to the evolution of aggregates:Using the savings decision rules of each type, we can obtain the law of motions for aggregate wealth held by each type as

$$A'_i = p\beta\hat{R}_i A_i + (1-p)\beta\hat{R}_j A_j. \tag{114}$$

Then the law of motion for aggregate wealth/capital ($K \equiv A_\ell + A_h$) becomes

$$K' = \beta \left[(1-\tau_a) K + (1-\tau_k) \alpha \left(\frac{(1-\tau_k)\gamma}{\psi} \right)^{\frac{\gamma}{1-\gamma}} (ZK)^{\frac{\alpha}{1-\gamma}} \left(\frac{L}{Q} \right)^{\frac{1-\alpha-\gamma}{1-\gamma}} \right]. \tag{115}$$

D.2.3 Steady state and changes in taxes

In steady state it must be that:

$$\frac{1}{\beta} = (1-\tau_a) + (1-\tau_k) \alpha \left(\frac{(1-\tau_k)\gamma}{\psi} \right)^{\frac{\gamma}{1-\gamma}} Z^{\frac{\alpha}{1-\gamma}} \left(\frac{L}{K} \right)^{\frac{1-\alpha-\gamma}{1-\gamma}} \tag{116}$$

We can use this to simplify the returns net of effort cost:

$$\hat{R}(z) = \begin{cases} (1-\tau_a) + \left(\frac{1}{\beta} - (1-\tau_a) \right) \frac{z_\ell}{Z} & \text{if } z = z_\ell \\ (1-\tau_a) + \left(\frac{1}{\beta} - (1-\tau_a) \right) \frac{z_\lambda}{Z} & \text{if } z = z_h \end{cases} \tag{117}$$

which is the same as in Section 2.

Finally, from the individual law of motions of aggregate assets it must be that:

$$\frac{1 - p\beta \hat{R}_h}{(1-p)\beta \hat{R}_\ell} = \frac{1-s_h}{s_h} = \frac{(1-p)\beta \hat{R}_h}{1 - p\beta \hat{R}_\ell}$$

After some algebra, this implies:

$$1 - p\beta \left[\hat{R}_\ell + \hat{R}_h \right] + (2p-1)\beta^2 \hat{R}_\ell \hat{R}_h = 0$$

we can further express this condition in terms of Z by replacing $R(z) - \psi\varepsilon(z)$:

$$(1 - (1-\tau_a)\beta(2p-1)) Z^2 - (p - (1-\tau_a)\beta(2p-1))(z_\ell + z_\lambda) Z + (2p-1)(1 - \beta(1-\tau_a)) z_\ell z_\lambda = 0$$

which is the same expression for steady productivity as in Section 2.

Consequently, Propositions 1 and 2 apply to this economy without modifications:

Proposition 6. *Propositions 1 and 2 apply to this economy, so that a steady state equilibrium with heterogeneous returns exists if and only if $\lambda < \overline{\lambda}$, and a marginal increase in wealth taxes in such an equilibrium increases productivity Z if and only if $\rho > 0$.*

The difference between the model in Section 2 and the model with effort is in the response of aggregate variables other than Z to changes in taxes. It turns out that all directions are maintained, but there are now two sources of changes on aggregates. The first source is, as in Section 2, a change in productivity. The second source is a direct effect of taxes on the effort of entrepreneurs. An increase in wealth taxes reduces capital income taxes which in turn reduces the distortions on the effort choice of entrepreneurs.

Before establishing the effects of a change in taxes on aggregate variables we revisit the role of government spending. The Government's constraint can be expressed just as before:

$$G = \tau_k \alpha Y + \tau_a K.$$

Assumption 1 still implies that:

$$\frac{1-\tau_k}{1-\beta\left(1-\tau_a\right)} = \frac{1-\theta}{1-\beta}$$

Then, steady state capital is, under Assumption 1:

$$K = \left(\frac{\alpha\beta\left(1-\theta\right)}{1-\beta}\right)^{\frac{1-\gamma}{1-\alpha-\gamma}} \left(\frac{\left(1-\tau_k\right)\gamma}{\psi}\right)^{\frac{\gamma}{1-\alpha-\gamma}} Z^{\frac{\alpha}{1-\alpha-\gamma}} L \tag{118}$$

Note that the level of capital depends directly on capital income taxes through their effect on effort. Alternatively, we can write the value of capital in terms of the level of wealth taxes:

$$K = \left(\alpha\beta\right)^{\frac{1-\gamma}{1-\alpha-\gamma}} \left(\frac{1-\theta}{1-\beta}\right)^{\frac{1}{1-\alpha-\gamma}} \left(\frac{\left(1-\beta\left(1-\tau_a\right)\right)\gamma}{\psi}\right)^{\frac{\gamma}{1-\alpha-\gamma}} Z^{\frac{\alpha}{1-\alpha-\gamma}} L \tag{119}$$

This makes it clear that aggregate capital increases with wealth taxes both through the efficiency gains (higher Z) and the decrease in distortions, lower τ_k.

Lemma 5. *For all $\tau_a < \overline{\tau}_a$, if $\rho > 0$and after an increase in wealth taxes, the wealth share of high-productivity entrepreneurs increases, $\frac{ds_h}{d\tau_a} > 0$, the after-tax return net of effort costs of high-productivity entrepreneurs also increases, $\frac{d\hat{R}_h}{d\tau_a} > 0$, while the after-tax returns net of effort costs of low-productivity entrepreneurs decreases, $\frac{d\hat{R}_\ell}{d\tau_a} < 0$. The aggregate capital stock and increases, $\frac{dK}{d\tau_a} > 0$, as do total effort, output, and wages, $\frac{dE}{d\tau_a}, \frac{dY}{d\tau_a}, \frac{dw}{d\tau_a} > 0$.*

Proof. The wealth share of high-productivity entrepreneurs is tied to productivity by:

$$s_h = \frac{Z - z_\ell}{z_\lambda - z_\ell}$$

so that the wealth share changes in the same direction as productivity. Productivity increases following Proposition 2.

The results for after-tax returns net of effort costs follow from a straightforward modification of Lemma 2 which gives:

$$\frac{d\left(R_h - \psi\varepsilon_h\right)}{d\tau_a} > 0 \qquad \text{and} \qquad \frac{d\left(R_\ell - \psi\varepsilon_\ell\right)}{d\tau_a} < 0$$

Total capital increases with wealth taxes:

$$\frac{d\log K}{d\log \tau_a} = \frac{\gamma}{1-\alpha-\gamma}\frac{\beta\tau_a}{1-\beta\left(1-\tau_a\right)} + \frac{\alpha}{1-\alpha-\gamma}\frac{d\log Z}{d\log \tau_a} > 0$$

It follows immediately that output, wages, and total effort increase since they depend positively on $Q = ZK$ and negatively on capital income taxes τ_k.

□

D.2.4 Welfare and optimal taxes

Introducing an effort choice for entrepreneurs changes the choice of optimal taxes in two direct ways. First, the equilibrium level of wages, and hence workers' welfare, depend on taxes directly through the effect of taxes on effort. Second, entrepreneurial welfare depends now on after-tax returns net of effort cost. However, only the first effect has an effect on the choice of optimal taxes. This is because in steady state the after-tax returns net of effort cost behave exactly like after-tax returns did in the model of Section 2. This leads to the following result:

Proposition 7. *The optimal wealth tax with entrepreneurial effort is higher than in Proposition 4. Moreover, if the optimal wealth tax is interior ($\tau_a^\star < \overline{\tau}_a$) it satisfies*

$$n_w \frac{\gamma}{1-\alpha-\gamma}\left(\frac{\frac{\beta\tau_a}{1-\beta(1-\tau_a)}}{\frac{d\log Z}{d\log\tau_a}} + \frac{\alpha}{1-\alpha}\right) = -\left(n_w\frac{\alpha}{1-\alpha} + \frac{1-n_w}{2}\left(\xi_{\hat{R}_\ell} + \xi_{\hat{R}_h}\right)\right)$$

where $\xi_{\hat{R}_i} \equiv \frac{d\log \hat{R}_i}{d\log Z}$ for $i \in \{h, \ell\}$.

Proof. The relevant welfare measures are:

$$\log\left(1+\text{CE}_{1,i}\right) = \begin{cases} \log {}^{w_a}/_{w_k} & \text{if } i = w \\ \frac{(1-\beta)\log {}^{\hat{R}_{a,i}}/_{\hat{R}_{k,i}} + \beta(1-p)\left(\log {}^{\hat{R}_{a,\ell}}/_{\hat{R}_{k,\ell}} + \log {}^{\hat{R}_{a,h}}/_{\hat{R}_{k,h}}\right)}{(1-\beta)(1-\beta(2p-1))} & \text{if } i \in \{\ell, h\} \end{cases} \tag{120}$$

and optimal taxes are set to maximize

$$\log\left(1+\text{CE}_1\right) = \sum_i n_i \log\left(1+\text{CE}_{1,i}\right).$$

The optimal is characterized by the following equation:

$$\frac{d\log\left(1+CE_1\right)}{d\tau_a} = 0$$

$$n_w\frac{d\log w}{d\tau_a} + \frac{1-n_w}{2}\left(\xi_{\hat{R}_\ell} + \xi_{\hat{R}_h}\right)\frac{d\log Z}{d\tau_a} = 0$$

where $\xi_{R_i - \psi \epsilon_i}$ are found just as in Lemma 2. The wage satisfies:

$$\frac{d \log w}{d \tau_a} = \frac{\gamma}{1-\alpha-\gamma} \frac{\beta}{1-\beta\left(1-\tau_a\right)} + \frac{\alpha}{1-\alpha-\gamma} \frac{d \log Z}{d \tau_a}$$

we can replace back to get:

$$n_w \left(\frac{\gamma}{1-\alpha-\gamma} \frac{\beta \tau_a}{1-\beta\left(1-\tau_a\right)} + \xi_w \frac{d \log Z}{d \log \tau_a} \right) = -\frac{1-n_w}{2} \left(\xi_{\hat{R}_\ell} + \xi_{\hat{R}_h} \right) \frac{d \log Z}{d \log \tau_a}$$

We do not have a closed form expression for the elasticity of productivity (Z) with respect to wealth taxes (τ_a), but this expression gives the optimal taxes.

We can also express the condition as:

$$n_w \frac{\gamma}{1-\alpha-\gamma} \left(\frac{\frac{\beta \tau_a}{1-\beta(1-\tau_a)}}{\frac{d \log Z}{d \log \tau_a}} + \frac{\alpha}{1-\alpha} \right) = -\left(n_w \frac{\alpha}{1-\alpha} + \frac{1-n_w}{2} \left(\xi_{\hat{R}_\ell} + \xi_{\hat{R}_h} \right) \right)$$

The right hand side is increasing in wealth taxes and it is in fact identical to the result in Proposition 4, while the left hand side is always positive. This leads to the conclusion that wealth taxes are higher with effort. The level of optimal taxes in Proposition 4 makes the right hand side of the equation zero, but the left hand side is still positive, so taxes must be higher.

$\square$

D.3 Excess Return

Proposition 11. ***(Existence and Uniqueness of Steady State)*** *There exists a unique steady state that features heterogenous returns if and only if*

$$\lambda < \overline{\lambda} \equiv 1 + \frac{(1-p) - \omega_h (2p-1)(1-\beta(1-\tau_a))}{p - (2p-1)\left(\beta(1-\tau_a) + (1-\beta(1-\tau_a))\frac{z_\ell}{z_h}\left(1-\omega_h\left(\frac{z_h}{z_\ell}-1\right)\right)\right)}$$

and either $\tau_a < \frac{1}{\beta}\left(\frac{1-p}{2p-1}\frac{1}{\omega_h} - (1-\beta)\right)$ *and* $\omega_h > 0$, *or* $\tau_a > \frac{1}{\beta}\left(\frac{1-p}{2p-1}\frac{1}{\omega_h} - (1-\beta)\right)$ *and* $\omega_h < 0$.

Proof. We evaluate

$$\begin{aligned} h(x) &= (1-(1-\tau_a)\beta(2p-1))x^2 - (z_\lambda + z_\ell)(p - \beta(1-\tau_a)(2p-1))x \\ &+ (2p-1) z_\ell z_\lambda (1-\beta(1-\tau_a))\left(1+\omega_h - \omega_h \frac{x}{z_\ell}\right) \end{aligned}$$

at $x = z_\ell$ and $x = z_\lambda$ as before since $Z = s_h z_\lambda + (1-s_h) z_\ell$. For $x = z_\ell$ we obtain:

$$h(z_\ell) = (1-p) z_\ell (z_\ell - z_\lambda) < 0$$

and for $x = z_\lambda$ we obtain:

$$\begin{aligned} h(z_\lambda) &= (1-p) z_\lambda (z_\lambda - z_\ell) - (2p-1) z_\ell z_\lambda (1-\beta(1-\tau_a))\left(\frac{z_\lambda}{z_\ell} - 1\right)\omega_h \\ &= z_\lambda (z_\lambda - z_\ell)((1-p) - (2p-1)(1-\beta(1-\tau_a))\omega_h) \end{aligned}$$

Thus, there is a unique steady equilibrium Z iff $h(z_\lambda) > 0$, that is

$$\begin{aligned} 0 &< (1-p) - (2p-1)(1-\beta(1-\tau_a))\omega_h \\ 0 &< (1-p) - (2p-1)(1-\beta+\beta\tau_a)\omega_h \\ (2p-1)\beta\omega_h\tau_a &< (1-p) - (2p-1)(1-\beta)\omega_h \end{aligned}$$

There are two cases of interest:

$$\begin{aligned} \tau_a &< \frac{1-p}{(2p-1)\beta\omega_h} - \frac{(1-\beta)}{\beta} \text{ if } \omega_h > 0 \\ \tau_a &> \frac{1-p}{(2p-1)\beta\omega_h} - \frac{(1-\beta)}{\beta} \text{ if } \omega_h < 0. \end{aligned}$$

For returns to be heterogeneous before taxes and wedges we need that $Z < z_h$. To get a bound for this to be the case we evaluate:

$$h\left(z_h\right)=0$$

$$\left(1-\left(1-\tau_a\right)\beta\left(2p-1\right)\right)z_h^2-\left(z_\lambda+z_\ell\right)\left(p-\beta\left(1-\tau_a\right)\left(2p-1\right)\right)z_h$$

$$+\left(2p-1\right)z_\ell z_\lambda\left(1-\beta\left(1-\tau_a\right)\right)\left(1+\omega_h-\omega_h\frac{z_h}{z_\ell}\right)=0$$

$$\left(1-\left(1-\tau_a\right)\beta\left(2p-1\right)\right)\frac{z_h^2}{z_\ell^2}-\left(\frac{z_h}{z_\ell}+1\right)\left(p-\beta\left(1-\tau_a\right)\left(2p-1\right)\right)\frac{z_h}{z_\ell}$$

$$+\left(2p-1\right)\frac{z_h}{z_\ell}\left(1-\beta\left(1-\tau_a\right)\right)\left(1+\omega_h-\omega_h\frac{z_h}{z_\ell}\right)$$

$$-\left(\lambda-1\right)\left(\frac{z_h}{z_\ell}-1\right)\left[\left(p-\beta\left(1-\tau_a\right)\left(2p-1\right)\right)\frac{z_h}{z_\ell}+\left(2p-1\right)\left(1-\beta\left(1-\tau_a\right)\right)\left(1+\omega_h-\omega_h\frac{z_h}{z_\ell}\right)\right]=0$$

$$\frac{z_h}{z_\ell}\left[\left(1-p\right)-\omega_h\left(2p-1\right)\left(1-\beta\left(1-\tau_a\right)\right)\right]$$

$$-\left(\lambda-1\right)\frac{z_h}{z_\ell}\left[\left(p-\beta\left(1-\tau_a\right)\left(2p-1\right)\right)+\left(2p-1\right)\left(1-\beta\left(1-\tau_a\right)\right)\frac{z_\ell}{z_h}\left(1-\omega_h\left(\frac{z_h}{z_\ell}-1\right)\right)\right]=0$$

$$\left(1-p\right)-\omega_h\left(2p-1\right)\left(1-\beta\left(1-\tau_a\right)\right)$$

$$-\left(\lambda-1\right)\left[p-\left(2p-1\right)\left(\beta\left(1-\tau_a\right)+\left(1-\beta\left(1-\tau_a\right)\right)\frac{z_\ell}{z_h}\left(1-\omega_h\left(\frac{z_h}{z_\ell}-1\right)\right)\right)\right]=0$$

So the threshold is:

$$\overline{\lambda}<1+\frac{\left(1-p\right)-\omega_h\left(2p-1\right)\left(1-\beta\left(1-\tau_a\right)\right)}{p-\left(2p-1\right)\left(\beta\left(1-\tau_a\right)+\left(1-\beta\left(1-\tau_a\right)\right)\frac{z_\ell}{z_h}\left(1-\omega_h\left(\frac{z_h}{z_\ell}-1\right)\right)\right)}$$

which is (greater? lower?) than the one we had before:

$$\frac{\partial\log\lambda}{\partial\omega_h}=-\frac{\left(2p-1\right)\left(1-\beta\left(1-\tau_a\right)\right)}{\left(1-p\right)-\omega_h\left(2p-1\right)\left(1-\beta\left(1-\tau_a\right)\right)}$$

$$-\frac{\left(2p-1\right)\left(1-\beta\left(1-\tau_a\right)\right)\frac{z_\ell}{z_h}\left(\frac{z_h}{z_\ell}-1\right)}{p-\left(2p-1\right)\left(\beta\left(1-\tau_a\right)+\left(1-\beta\left(1-\tau_a\right)\right)\frac{z_\ell}{z_h}\left(1-\omega_h\left(\frac{z_h}{z_\ell}-1\right)\right)\right)}$$

$$=-\left(2p-1\right)\left(1-\beta\left(1-\tau_a\right)\right)\left[\frac{1}{\left(1-p\right)-\omega_h\left(2p-1\right)\left(1-\beta\left(1-\tau_a\right)\right)}\right.$$

$$\left.+\frac{\frac{z_\ell}{z_h}\left(\frac{z_h}{z_\ell}-1\right)}{p-\left(2p-1\right)\left(\beta\left(1-\tau_a\right)+\left(1-\beta\left(1-\tau_a\right)\right)\frac{z_\ell}{z_h}\left(1-\omega_h\left(\frac{z_h}{z_\ell}-1\right)\right)\right)}\right]$$

We can evaluate this derivative at $\omega_h=0$:

$$\frac{\partial\log\lambda}{\partial\omega_h}\left(0\right)=-\left(2p-1\right)\left(1-\beta\left(1-\tau_a\right)\right)\left[\frac{1}{1-p}+\frac{\frac{z_\ell}{z_h}\left(\frac{z_h}{z_\ell}-1\right)}{p-\left(2p-1\right)\left(\beta\left(1-\tau_a\right)+\left(1-\beta\left(1-\tau_a\right)\right)\frac{z_\ell}{z_h}\right)}\right]<0$$

So an increase in ω_h makes the bound more stringent, which makes sense as it increases the effective return of high-productivity entrepreneurs. A decrease of ω_h does the opposite.

□

Lemma 10. *If $z_h = z_\ell = z$, then the steady state $Z = z$ and an increase in wealth tax increases the after-tax return R_i iff $\omega_{-i} < 0 < \omega_i$ but it does not increase TFP. Thus, an increase in the wealth tax does not increase the utility of workers and decreases the total utility of entrepreneurs. Overall, the optimal policy is to tax capital $\tau_k^\star > 0$ and subsidize wealth $\tau_a^\star < 0$.*

(Efficiency Gains from Wealth Taxation) For all $\tau_a < \overline{\tau}_a$, a marginal increase in wealth taxes (τ_a) increases aggregate productivity (Z), $\frac{dZ}{d\tau_a} > 0$, if entrepreneurial productivity is autocorrelated, $\rho > 0$, and $R_h > R_\ell$, or if entrepreneurial productivity is negatively autocorrelated, $\rho < 0$, and $R_h < R_\ell$.

Corollary 6. *The wedges (ω_ℓ, ω_h) satisfy $(1+\omega_\ell) z_\ell < (1+\omega_h) z_\lambda$ equilibrium if and only if :*

$$\omega_h > \underline{\omega}_h = -\frac{1}{2}\left(\frac{1-p}{1+p}\right)\left(\frac{z_\lambda - z_\ell}{z_\lambda}\right).$$

Proof. In order to see the effect of a higher wealth tax, take derivative of $h(x)$ with respect to τ_a:

$$\begin{aligned}
\frac{1}{\beta(2p-1)}\frac{dh(x)}{d\tau_a} &= x^2 - (z_\lambda + z_\ell)x + z_\ell z_\lambda\left(1 + \omega_h - \omega_h \frac{x}{z_\ell}\right) \\
&= x^2 - (z_\lambda + z_\ell)x + z_\ell z_\lambda + z_\ell z_\lambda\left(1 - \frac{x}{z_\ell}\right)\omega_h \\
&= (x - z_\ell)(x - z_\lambda) + z_\lambda(z_\ell - x)\omega_h \\
&= (x - z_\ell)(x - (1+\omega_h) z_\lambda)
\end{aligned}$$

Since $Z = s_h z_\lambda + (1-s_h) z_\ell$, we know that $Z - z_\ell > 0$ and, under $p > 1/2$, $\frac{dh(x)}{d\tau_a} < 0$ iff $Z - (1+\omega_h) z_\lambda < 0$. If high-productivity entrepreneurs are earning excess return $\omega_h > 0$, then $Z - (1+\omega_h) z_\lambda < 0$ and a higher wealth tax increases efficiency. If however, $\omega_h < 0$ then a higher wealth tax increases productivity iff $Z - (1+\omega_h) z_\lambda < 0$. Substituting $Z = s_h z_\lambda + (1-s_h) z_\ell$, we obtain

$$\begin{aligned}
s_h z_\lambda + (1-s_h) z_\ell - (1+\omega_h) z_\lambda &> 0 \\
(1-s_h)(z_\ell - z_\lambda) - \omega_h z_\lambda &> 0 \\
\omega_h &> -\frac{(1-s_h)(z_\lambda - z_\ell)}{z_\lambda}.
\end{aligned}$$

We will show that in equilibrium$\omega_h > -\frac{(1-s_h)(z_\lambda - z_\ell)}{z_\lambda}$ if and only if $(1+\omega_h) z_\lambda > (1+\omega_\ell) z_\ell$. First, we evaluate $\omega_\ell z_\ell A_\ell + \omega_h z_\lambda A_h = 0$, which implies

$$\omega_\ell z_\ell = -\omega_h z_\lambda \frac{s_h}{1-s_h}.$$

Then, we can then replace into $(1+\omega_h) z_\lambda > (1+\omega_\ell) z_\ell$ to obtain the result:

$$\begin{aligned} -\omega_h z_\lambda - \omega_h z_\lambda \frac{s_h}{1-s_h} &< z_\lambda - z_\ell \\ -\omega_h z_\lambda \left(\frac{1}{1-s_h} \right) &< z_\lambda - z_\ell \\ \omega_h &> -(1-s_h) \frac{z_\lambda - z_\ell}{z_\lambda}. \end{aligned}$$

So we have that if $p > 1/2$:

$$\frac{dZ}{d\tau_a} > 0 \longleftrightarrow Z < (1+\omega_h) z_\lambda \longleftrightarrow \omega_h > -(1-s_h) \frac{z_\lambda - z_\ell}{z_\lambda} \longleftrightarrow (1+\omega_h) z_\lambda > (1+\omega_\ell) z_\ell \longleftrightarrow R_h > R_\ell$$

Finally, we can check conditions that guarantee that $Z < (1+\omega_h) z_\lambda$ in terms of parameters, for this we evaluate $h((1+\omega_h) z_\lambda) = 0$ which, after some algebra, results in:

$$\omega_h > \underline{\omega}_h = -\frac{1}{2} \left(\frac{1-p}{1+p} \right) \left(\frac{z_\lambda - z_\ell}{z_\lambda} \right).$$

□

D.4 Stationary wealth distribution

The model presented in Section 2 as well as the extensions presented above do not have a stationary wealth distribution. Here we consider an alternative version of the model in which entrepreneurs have a permanent productivity type but are subject to mortality risk. In particular assume that entrepreneurs die with a constant probability $1-\delta$, upon death they are replaced by a new entrepreneur with initial assets $\overline{a}$ and whose productivity is z_i ($i \in \{h, \ell\}$) with probability $1/2$. The value of $\overline{a}$ is determined endogenously in equilibrium as the average bequest in the economy (which coincides with the average wealth). With respect to the main model of Section 2, this model loses the variation in productivity.[21] In exchange, this alternative version of the model exhibits a stationary wealth distribution that allows to better study how changes in taxes affect wealth inequality and welfare.

D.4.1 Entrepreneur's problem

The problem of an entrepreneur is now

$$\begin{aligned} V(a,z) &= \max_{a'} \log(c) + \beta\delta V\left(a', z\right) \\ &\text{s.t. } c + a' = R(z)\, a, \end{aligned} \tag{121}$$

where $R(z)$ takes the same form as in Section 2.1. The solution takes the form $V_i(a) = m_i + n\log(a)$, where $n = \frac{1}{1-\beta\delta}$ and $m_i = \frac{1}{(1-\beta\delta)^2}\left[\beta\delta\log\beta\delta + (1-\beta\delta)\log(1-\beta\delta) + \log R_i\right]$, and implies an optimal savings rule

$$a' = \beta\delta R(z)\, a, \tag{122}$$

D.4.2 Evolution of aggregates and steady state

The savings choices of agents are still linear in assets, which lets us express the evolution of aggregate wealth as:

$$A_i^{'} = \beta\delta^2 R_i A_i + (1-\delta)\,\overline{a}, \tag{123}$$

so that in steady state:

$$A_i = \frac{1-\delta}{1-\beta\delta^2 R_i}\overline{a}, \tag{124}$$

where $\overline{a} \equiv K/2 = (A_\ell + A_h)/2$. We later show that $\beta\delta R_\ell < 1 < \beta\delta R_h < 1/\delta$, so that in equilibrium low types dissave and high types save, but not at rate that prevents the existence of a stationary equilibrium for the economy.

From the evolution of the low- and high-type assets we get:

$$K^{'} = \beta\delta^2\left(R_\ell A_\ell + R_h A_h\right) + (1-\delta)\, K$$

[21] It is straightforward to keep the variation in productivities across generations by redefining p as the probability that an individual entrepreneur keeps the productivity of the previous generation and setting two initial values of assets $\overline{a}_\ell$ and $\overline{a}_h$ for entrepreneurs born with productivity z_ℓ and z_h respectively. The values of $\overline{a}_\ell$ and $\overline{a}_h$ are determined endogenously in equilibrium as the average bequest of each group of entrepreneurs, which are functions of the average wealth of low- and high-productivity entrepreneurs in steady state. We opt to abstract from this to keep the presentation of this new model as simple as possible.

replacing by the equilibrium value of returns $\left(R_i = (1-\tau_a) + (1-\tau_k)\,\alpha\,({}^{ZK}/_L)^{\alpha-1} z_i\right)$ and evaluating in steady state we get:

$$(1-\tau_a) + (1-\tau_k)\,\alpha Z^{\alpha}\left(\frac{K}{L}\right)^{\alpha-1} = \frac{1}{\beta\delta}, \tag{125}$$

which characterizes the level of steady state capital K given productivity, just as in (12).[22] Finally, joining the steady state conditions for low- and high- productivity entrepreneurs in equation (124) gives us a condition that characterizes the equilibrium of the model in terms of returns:

$$1 = \frac{(1-\delta)\left(1-\beta\delta^2\left(\frac{R_\ell+R_h}{2}\right)\right)}{(1-\beta\delta^2 R_\ell)(1-\beta\delta^2 R_h)}. \tag{126}$$

We can express this condition in terms of steady state productivity Z using (125) to get:

$$\left(1-\beta\delta^2(1-\tau_a)\right)Z^2 - (1+\delta(1-2\beta\delta(1-\tau_a)))\left(\frac{z_\lambda+z_\ell}{2}\right)Z + \delta(1-\beta\delta(1-\tau_a))\,z_\ell z_\lambda = 0 \tag{127}$$

which is again a quadratic equation in Z, as (14) in Section 2.3. The solution to this equation determines the steady state of the economy as well as the upper bound on the collateral constraint parameter λ that ensures that the economy is in the heterogeneous returns equilibrium:

Proposition 12. ***(Existence and Uniqueness of Steady State)*** *There exists a unique steady state. The steady state equilibrium features heterogenous returns ($R_h > R_\ell$) if and only if $\lambda < \lambda_p^\star \equiv 1 + \frac{1-\delta}{1-\delta+2\delta(1-\beta\delta(1-\tau_a))\left(1-\frac{z_\ell}{z_h}\right)}$. Moreover, $\beta\delta R_\ell < 1 < \beta\delta R_h < {}^{1}/_{\delta}$ in steady state.*

Proof. We proceed in three steps. First, we determine conditions on the steady state value of Z that guarantee that $\beta\delta R_\ell < 1 < \beta\delta R_h < {}^{1}/_{\delta}$. Second, we verify that there exists a solution to equation (127) satisfying those conditions. Finally, we prove that there is a unique root of (127) satisfying those conditions.

We start by showing that $R_\ell < {}^{1}/_{\beta\delta} < R_h$. We verify this directly using equation (125) and the fact that $z_\ell < Z < z_\lambda$:

$$R_\ell = (1-\tau_a) + (1-\tau_k)\,\alpha Z^{\alpha}\left(\frac{K}{L}\right)^{\alpha-1}\frac{z_\ell}{Z} < (1-\tau_a) + (1-\tau_k)\,\alpha Z^{\alpha}\left(\frac{K}{L}\right)^{\alpha-1} = \frac{1}{\beta\delta}$$

and

$$\frac{1}{\beta\delta} = (1-\tau_a) + (1-\tau_k)\,\alpha Z^{\alpha}\left(\frac{K}{L}\right)^{\alpha-1} < (1-\tau_a) + (1-\tau_k)\,\alpha Z^{\alpha}\left(\frac{K}{L}\right)^{\alpha-1}\frac{z_\lambda}{Z} = R_h$$

[22] After imposing the steady state condition for capital, the after-tax rates of return become $R_i = 1-\tau_a + \frac{1-\beta\delta(1-\tau_a)}{\beta\delta}\frac{z_i}{Z}$.

Letting $\eta = \beta\delta(1-\tau_a)$, we can also show that $\beta\delta R_h < 1/\delta$ if $\frac{\delta(1-\eta)}{1-\delta\eta} z_\lambda < Z$. Thus,

$$\beta\delta R_\ell < 1 < \beta\delta R_h < {}^1\!/\!_\delta \longleftrightarrow Z \in \left(\max\left\{ z_\ell, \frac{\delta(1-\eta)}{1-\delta\eta} z_\lambda \right\}, z_\lambda \right) \tag{128}$$

Note that the interval for Z is non-empty. This is immediate because:

$$z_\ell < z_\lambda \qquad \text{and} \qquad \frac{\delta(1-\eta)}{1-\delta\eta} < 1.$$

Moreover, the lower bound depends on the ratio of productivities. We have $\max\left\{ z_\ell, \frac{\delta(1-\eta)}{1-\delta\eta} z_\lambda \right\} = z_\ell$ if and only if $\frac{\delta(1-\eta)}{1-\delta\eta} \leq \frac{z_\ell}{z_\lambda}$.

Next, we show there exists a unique solution to equation (127) in the interval of equation (128). For this define

$$H(x) = (1-\delta\eta) - (1-\delta(2\eta-1)) \frac{\left(\frac{z_\lambda + z_\ell}{2}\right)}{x} + \delta(1-\eta) \frac{z_\ell z_\lambda}{x^2}$$

as the residual of equation (127) at x. We verify directly that H has a root in the interval $\left(\max\left\{ z_\ell, \frac{\delta(1-\eta)}{1-\delta\eta} z_\lambda \right\}, z_\lambda \right)$:

$$\begin{aligned} H(z_\ell) &= -\frac{1-\delta}{z_\ell}[z_\lambda - z_\ell] < 0 \\ H\left(\frac{\delta(1-\eta)}{1-\delta\eta} z_\lambda\right) &= -\frac{1-\delta\eta}{2\delta(1-\eta)} \frac{1-\delta}{z_\lambda}[z_\lambda - z_\ell] < 0 \\ H(z_\lambda) &= \frac{1-\delta}{2z_\lambda}[z_\lambda - z_\ell] > 0 \end{aligned}$$

The existence of the unique root is guaranteed by the intermediate value theorem and the fact that the function is quadratic.

Now we derive sufficient conditions for the economy to be in the equilibrium with with excess supply of funds: ($A_\ell > (\lambda-1) A_h$). This happens if and only if $Z \leq z_h$. So now we find conditions that guarantee that $H(z_h) > 0$ which implies that $Z \leq z_h$ since $H(Z) = 0$ and $H(z)$ is increasing in in $z \geq Z$.

$$H(z_h) = (1-\delta\eta) - (1-\delta(2\eta-1)) \frac{\left(\frac{z_\lambda + z_\ell}{2}\right)}{z_h} + \delta(1-\eta) \frac{z_\ell z_\lambda}{z_h^2} > 0$$

which after some manipulation gives:

$$\lambda < \overline{\lambda} \equiv 1 + \frac{1-\delta}{1-\delta+2\delta(1-\eta)\left(1-\frac{z_\ell}{z_h}\right)}$$

□

Just as in Section 3, we show that an increase in wealth taxes raises steady state productivity Z. Note that Z always increases with τ_a, that is because productivity is persistent by construction.

We will maintain the assumption that $\lambda < \lambda_p^\star$ in all the results that follow.

Proposition 9. ***(Efficiency Gains from Wealth Taxation)*** *For all $\tau_a < \overline{\tau}_a^p$, a marginal increase in wealth taxes (τ_a) increases productivity, $\frac{dZ}{d\tau_a} > 0$.*

Proof. We first define the auxiliary function

$$H(x;\tau_a) = \left(1 - \beta\delta^2(1-\tau_a)\right) - \left(1 + \delta\left(1 - 2\beta\delta(1-\tau_a)\right)\right)\frac{\left(\frac{z_\lambda + z_\ell}{2}\right)}{x} + \delta\left(1 - \beta\delta(1-\tau_a)\right)\frac{z_\ell z_\lambda}{x^2}$$

which characterizes the steady state if and only if $\lambda < \overline{\lambda}$. Simple manipulation of the function gives:

$$H(x;\tau_a) = 1 - \left[\left(1 + \frac{1}{\delta} - \frac{z_\lambda}{x}\right)z_\ell + \left(1 + \frac{1}{\delta} - \frac{z_\ell}{x}\right)z_\lambda\right]\frac{\delta}{2x} - \left(1 - \frac{z_\ell}{x}\right)\left(1 - \frac{z_\lambda}{x}\right)\beta\delta^2(1-\tau_a)$$

This function is decreasing in τ_a for $x \in (z_\ell, z_\lambda)$, which is the interval of the steady state value of Z similar to the one in Figure 2:

$$\frac{d\tilde{H}(x,\tau_a)}{d\tau_a} = \underbrace{\left(1 - \frac{z_\ell}{x}\right)}_{(+)}\underbrace{\left(1 - \frac{z_\lambda}{x}\right)}_{(-)}\beta\delta^2.$$

This gives the desired result: $\frac{dZ}{d\tau_a} > 0$.

□

The response of aggregate variables to changes in equilibrium Z (and hence to τ_a) follow the same patterns as in Section 3. In Appendix D.4.5 we present the equivalent results to Lemmas 2 and **??** describing the response of all aggregate variables in steady state.

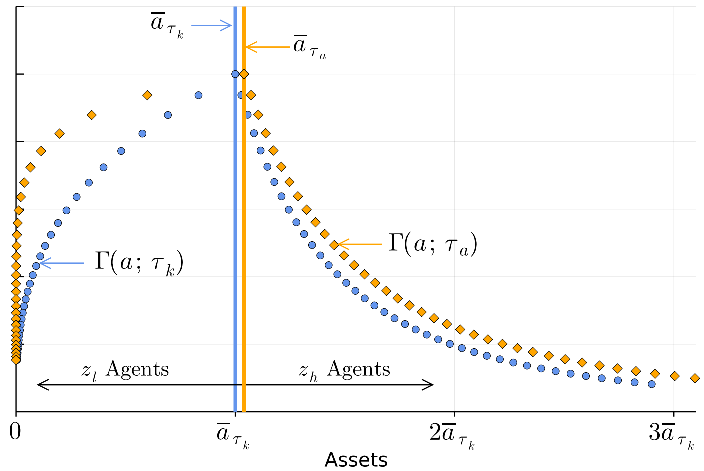
D.4.3 Stationary distribution of assets

We now derive the stationary distribution of assets. Note that all entrepreneurs are born with the same level of wealth $\overline{a}$ and then save at a constant rate during their lifetimes. In particular, high-types save at a (gross) rate $\beta\delta R_h > 1$ and low-types dissave at a (gross) rate $\beta\delta R_\ell < 1$. So, in the stationary equilibrium the wealth distribution of high-types has support in the interval $[\overline{a}, \infty)$ and the distribution of low-types in the interval $(0, \overline{a}]$. Moreover, the distribution of wealth is discrete, with endogenous mass points at $\left\{\overline{a}, \beta\delta R_h\overline{a}, (\beta\delta R_h)^2\overline{a}, \ldots\right\}$ for the high-types and $\left\{\overline{a}, \beta\delta R_\ell\overline{a}, (\beta\delta R_\ell)^2\overline{a}, \ldots\right\}$ for the low-types.

The share of entrepreneurs of type i with wealth $a = (\beta\delta R_i)^t\,\overline{a}$ is given by the share of agents who have lived exactly t periods:

$$\Gamma_i\left((\beta\delta R_i)^t\,\overline{a}\right) = \Pr(\text{age} = t) = \delta^t(1-\delta) \tag{129}$$

Figure D.2: Stationary Distribution of Assets



Note: *The figure reports the stationary distribution of assets for two economies. The blue circles correspond to an economy with only capital income taxes ($\tau_k = \theta$ and $\tau_a = 0$) and its values are labeled with τ_k. The orange diamonds correspond to an economy with wealth taxes ($\tau_a = 10\%$ and τ_k set to satisfy Assumption 1) and its values are labeled with τ_a. The horizontal axis is presented in units of average assets in the capital income tax economy ($\overline{a}_{\tau_k}$). In both economies we set the remaining parameters as follows: $\beta = 0.96$, $\delta = 1 - 1/80$, $z_\ell = 1/2$, $z_h = 3/2$, $\theta = 25\%$, and $\lambda = 1.2$.*

So the distribution of wealth is a geometric distribution with parameter δ.[23]

Figure D.2 illustrates the behavior of the stationary distribution of assets. Agents are born with initial wealth $\overline{a}$ and save or dissave at constant rates depending on their productivity. A change in taxes affects the location of the mass-points of the distribution. In the figure, we contrast and economy without wealth taxes (that we label as τ_k) with one with wealth taxes (that we label as τ_a). The wealth tax economy has a higher level of overall wealth and hence $\overline{a}_{\tau_a}$ is to the right of $\overline{a}_{\tau_k}$. The change in $\overline{a}$ impacts all mass points (which are proportional to $\overline{a}$), shifting them rightwards. Then the increase in the dispersion of wealth is explained by the increase in the dispersion of returns, something reminiscent of the results in Lemma **??** and that we verify below for this economy.

Finally, we define a convenient measure of wealth concentration in the economy. Since wealth is determined by type and age, we can define the top wealth share as the fraction of wealth held by high types above an age t. This would correspond to the wealth share of the top $100 \times (1-\delta) \sum_{s=t}^{\infty} \delta^s = 100 \times \delta^t$ percent. Their total wealth is given by

$$A_{h,t} \equiv (1-\delta) \sum_{s=t}^{\infty} \left(\beta\delta^2 R_h\right)^s \overline{a} = \left(\beta\delta^2 R_h\right)^t A_h.$$

Then the top wealth shares are

$$s_{h,t} \equiv \frac{\left(\beta\delta^2 R_h\right)^t A_h}{K} = \left(\beta\delta^2 R_h\right)^t s_h. \tag{130}$$

[23]The characterization of the stationary distribution of assets mimics the derivations in Jones (2015) adapted to the discrete time setting.

After an increase in wealth taxes the dispersion of returns increases, this affects the distribution by shifting the mass points, although it does not affect the mass associated with each point, as shown in Figure D.2. Because s_h and R_h increase with the wealth tax, the top wealth share $s_{h,t} = \left(\beta\delta^2 R_h\right)^t s_h$ increase with the wealth tax.

Lemma 6. ***(Top-Wealth Shares and Wealth Taxes)*** *For all $\tau_a < \overline{\tau}_a^p$, a marginal increase in wealth taxes increases the top-wealth-shares as in equation (41). The percentage increase in the wealth share is higher for higher wealth levels.*

Proof. The result is immediate from the definition of wealth shares as a function of after-tax returns (equation 130) and the fact that R_h increases with wealth taxes (see Lemma 12 in Appendix D.4). An increase in wealth taxes increases the returns of high-productivity entrepreneurs (R_h), which in turn increases their savings rate and asset holdings. The effect is compounded with age because savings rate are constant, increasing more the wealth holdings of older/wealthier entrepreneurs.

□

D.4.4 Welfare and optimal taxes

We focus on the average welfare gain taking advantage of the characterization of the stationary wealth distribution. This leads to a welfare measure that is closely related to the CE_2 measure presented in equation (84). However, just as in Section 3.3 we can also define an individual welfare measure equivalent to the CE_1 measure defined in equation (25). Workers' welfare behaves just as in Section 3.3, but the welfare measures of low- and high-productivity entrepreneurs now depend only on their own returns. Hence, $\text{CE}_{1,h}$ is always positive and $\text{CE}_{1,\ell}$ is always negative. Total entrepreneurial welfare still depends on (log-)average returns (which decrease with wealth taxes) and optimal taxes are characterized similarly to Proposition 4. We provide details for these results in Appendix D.4.

We compute the average welfare gain by each type (denoting it as $\text{CE}_{2,i}$) in the following way

$$\sum_a \left(V_k(a,i) + \frac{\log\left(1+\text{CE}_{2,i}\right)}{1-\beta\delta}\right)\Gamma_k(a,i) \quad = \quad \sum_a V_a(a,i)\,\Gamma_a(a,i)\,.$$

This average measure depends on the assets of each agent through their distribution, and thus captures the effects of higher capital accumulation triggered by the tax reform. Using the age distribution, we obtain:

$$\log\left(1+\text{CE}_{2,i}\right) = \frac{1-\beta\delta^2}{(1-\delta)(1-\beta\delta)} \log\frac{R_{a,i}}{R_{k,i}} + \log\frac{K_a}{K_k}. \tag{131}$$

The utilitarian welfare gain for the whole population CE_2 is given by $\log\left(1+\text{CE}_2\right) = \sum_i n_i \log\left(1+\text{CE}_{2,i}\right)$. For entrepreneurs, $\text{CE}_{2,i}$ welfare gains depend on the accumulation of capital in the economy. Interestingly, the effect of aggregate capital is the same for both types of entrepreneurs. This is because they both benefit from starting their lives at a higher level of initial assets (recall that $\overline{a} = K/2$) and their future asset levels are all proportional to their initial wealth (as discussed in Section D.4.3). This makes it possible even for low-productivity entrepreneurs to benefit from the increase in wealth taxes if elasticity of output with respect to capital is sufficiently high. We summarize these results in Lemma 7.

Lemma 7. ***(Welfare Gain by Agent Type)*** *For all $\tau_a < \overline{\tau}_a^p$ and under Assumption 1, a marginal increase in wealth taxes (τ_a) increases the welfare of high-productivity entrepreneurs always, $CE_{1,h}, CE_{2,h} > 0$, and $CE_{2,h} > CE_{1,h}$. For low-productivity entrepreneurs, $CE_{1,\ell} < 0$ always and $CE_{2,\ell} > 0$ if and only if*

$$\xi_K \geq \frac{1-\beta\delta^2}{(1-\delta)(1-\beta\delta)}\xi_{R_\ell}$$

where $\xi_x \equiv \frac{d\log x}{d\log Z}$ is the elasticity of variable x with respect to Z.

Proof. From the definition of $\mathrm{CE}_{2,i}$ in (131) we get:

$$\log\left(1+\mathrm{CE}_{2,i}\right) = \frac{1-\beta\delta^2}{(1-\delta)(1-\beta\delta)}\log\frac{R_{a,i}}{R_{k,i}} + \log\frac{K_a}{K_k}$$

It is immediate that $\mathrm{CE}_{2,h} > 0$ because K and R_h are both increasing in wealth taxes (Lemmas 11 and 12). Moreover, $\mathrm{CE}_{2,h} - \mathrm{CE}_{1,h} = \log {}^{K_a}/_{K_k} > 0$ because K is increasing in wealth taxes.

For $\mathrm{CE}_{2,\ell}$ consider the derivative of the welfare measure with respect to wealth taxes:

$$\begin{aligned}
\frac{d\log\left(1+\mathrm{CE}_{2,\ell}\right)}{d\tau_a} &= \frac{1-\beta\delta^2}{(1-\delta)(1-\beta\delta)}\frac{1}{R_\ell}\frac{R_\ell}{d\tau_a} + \frac{\alpha}{1-\alpha}\frac{1}{Z}\frac{dZ}{d\tau_a} \\
&= \left[-\frac{1-\beta\delta^2}{2\beta\delta^2(1-\beta\delta)}\frac{1}{R_\ell}\frac{z_\lambda - z_\ell}{(z_\lambda - Z)^2}Z + \frac{\alpha}{1-\alpha}\right]\frac{1}{Z}\frac{dZ}{d\tau_a} \\
&= \left[-\frac{1-\beta\delta^2}{(1-\beta\delta)}\frac{(1-s_h)}{(2(1-s_h)-(1-\delta))}\frac{z_\lambda - z_\ell}{(z_\lambda - Z)^2}Z + \frac{\alpha}{1-\alpha}\right]\frac{1}{Z}\frac{dZ}{d\tau_a} \\
&= \left[-\frac{\left(1-\beta\delta^2\right)(z_\lambda - z_\ell)Z}{(1-\beta\delta)(2(1-s_h)-(1-\delta))(1-s_h)} + \frac{\alpha}{1-\alpha}\right]\frac{1}{Z}\frac{dZ}{d\tau_a}
\end{aligned}$$

This defines bounds on α above which there are welfare gains for the low-productivity entrepreneurs.

□

We next study the optimal tax problem of the government using the CE_2 measure. Taking the changes in the wealth distribution into account changes the optimal combination of taxes, but, as in Section (3.3), does not change the key tradeoffs at play. However, unlike in Section (3.3), taking wealth accumulation into account does not necessarily lead to higher optimal wealth taxes or lower $\underline{\alpha}$ and $\overline{\alpha}$ thresholds. Higher initial wealth increases the benefits from the reform (we again have $\xi_w = \xi_K = {}^{\alpha}/_{1-\alpha}$), but also increases the losses from the lower expected returns due to the compounding effect of returns on individual asset accumulation (which are suffered by the low-productivity entrepreneurs). In proposition (10) we characterize the optimal tax levels that maximizes CE_2.

Proposition 10. ***(Optimal CE$_2$ Taxes)*** *Under Assumption 1, there exist a unique tax combination $\left(\tau^{\star}_{a,2},\tau^{\star}_{k,2}\right)$ that maximizes the utilitarian welfare measure CE$_2$, an interior solution $\tau^{\star}_{a,2}<\overline{\tau}^{p}_{a}$ is the solution to:*

$$n_w\xi_w+(1-n_w)\,\xi_K=\xi=-\,(1-n_w)\,\frac{1-\beta\delta^2}{(1-\delta)\,(1-\beta\delta)}\left(\frac{\xi_{R_\ell}+\xi_{R_h}}{2}\right)\tag{132}$$

where $\xi_x\equiv\frac{d\log x}{d\log Z}$ is the elasticity of variable x with respect to Z. Furthermore, there exist two cutoff values for α, $\underline{\alpha}_2$ and $\overline{\alpha}_2$, such that $\left(\tau^{\star}_{a,2},\tau^{\star}_{k,2}\right)$ satisfies the following properties:

$$\begin{aligned}
\tau^{\star}_{a,2}&\in\left[1-\frac{1}{\beta\delta},0\right) \text{ and } \tau^{\star}_{k,2}>\theta && \text{if } \alpha<\underline{\alpha}_2\\
\tau^{\star}_{a,2}&\in\left[0,\frac{\theta\,(1-\beta\delta)}{\beta\delta\,(1-\theta)}\right] \text{ and } \tau^{\star}_{k,2}\in[0,\theta] && \text{if } \underline{\alpha}_2\le\alpha\le\bar{\alpha}_2\\
\tau^{\star}_{a,2}&\in\left(\frac{\theta\,(1-\beta\delta)}{\beta\delta\,(1-\theta)},\tau^{\max}_{a,2}\right) \text{ and } \tau^{\star}_{k,2}<0 && \text{if } \alpha>\bar{\alpha}_2
\end{aligned}$$

where $\tau^{\max}_{a,2}\ge 1$, $\underline{\alpha}_2$ and $\overline{\alpha}_2$ are the solutions to equation (132) with $\tau_a=0$ and $\tau_a=\frac{\theta(1-\beta\delta)}{\beta\delta(1-\theta)}$, respectively. Recall from Lemma (2) that $\xi={}^{\alpha}/_{1-\alpha}$.

Proof. For aggregate welfare:

$$\begin{aligned}
\frac{\log(1+\text{CE}_2)}{(1-\beta\delta)}&=n_w\left(V_a(w)-V_k(w)\right)+\sum_{i\in\{\ell,h\}}n_i\left(\sum_a V_a(a,i)\,\Gamma_a(a,i)-\sum_a V_k(a,i)\,\Gamma_k(a,i)\right)\\
\frac{\log(1+\text{CE}_2)}{(1-\beta\delta)}&=n_w\left(\frac{\log(1+\text{CE}_{2,w})}{1-\beta\delta}\right)+\sum_{i\in\{\ell,h\}}n_i\frac{\log(1+\text{CE}_{2,i})}{1-\beta\delta}\\
\log(1+\text{CE}_2)&=\sum_{i\in\{w,\ell,h\}}n_i\log(1+\text{CE}_{2,i})
\end{aligned}$$

where $\text{CE}_{2,w}=\text{CE}_{1,w}$. The optimal wealth tax is characterized by:

$$\begin{aligned}
\frac{d\log(1+\text{CE}_2)}{d\tau_a}&=0\\
\left[\frac{d\log(1+\text{CE}_2)}{d\log Z}\right]\frac{d\log Z}{d\tau_a}&=0\\
\left[n_w\frac{d\log w}{d\log Z}+\frac{1-n_w}{2}\frac{1-\beta\delta^2}{(1-\delta)\,(1-\beta\delta)}\left(\frac{d\log R_\ell}{d\log Z}+\frac{d\log R_h}{d\log Z}\right)+(1-n_w)\frac{d\log K}{d\log Z}\right]\frac{d\log Z}{d\tau_a}&=0\\
\left[n_w\xi_w+(1-n_w)\,\xi_K+(1-n_w)\,\frac{1-\beta\delta^2}{(1-\delta)\,(1-\beta\delta)}\left(\frac{\xi_{R_\ell}+\xi_{R_h}}{2}\right)\right]\frac{d\log Z}{d\tau_a}&=0
\end{aligned}$$

As in the proof of Proposition 4 the above condition is satisfied if and only if

$$n_w \xi_w + (1 - n_w)\, \xi_K = -\,(1 - n_w)\, \frac{1 - \beta\delta^2}{(1-\delta)(1-\beta\delta)} \left(\frac{\xi_{R_\ell} + \xi_{R_h}}{2} \right)$$

and from Lemma 2 we get $\xi_w = \xi_K = \xi = \alpha/1-\alpha$. The uniqueness of the solution and the definition of the thresholds for α and its implications for the optimal taxes follow from the same arguments as in Proposition 4. We can further replace to get a more explicit formula:

$$\frac{\alpha}{1-\alpha} + \frac{1-n_w}{2} \frac{1-\beta\delta^2}{(1-\delta)(1-\beta\delta)} \frac{Z}{R_h R_\ell} \frac{dR_h R_\ell}{dZ} = 0$$
$$\frac{\alpha}{1-\alpha} + \frac{1-n_w}{2} \frac{1-\beta\delta^2}{(1-\delta)(1-\beta\delta)} \frac{Z}{R_h R_\ell} \frac{\left(1-\delta^2\right)(1-2s_h)}{\left(2\beta\delta^2 s_h (1-s_h)\right)^2} \frac{ds_h}{dZ} = 0$$
$$\frac{\alpha}{1-\alpha} + \frac{1-n_w}{2} \frac{1-\beta\delta^2}{(1-\delta)(1-\beta\delta)} \frac{1}{R_h R_\ell} \frac{\left(1-\delta^2\right)(1-2s_h)}{\left(2\beta\delta^2 s_h (1-s_h)\right)^2} \frac{Z}{z_\lambda - z_\ell} = 0$$

Relative to the condition for CE_1 in Appendix D.4, the first term is now larger (multiplied by 1 instead of n_w), the second term is also larger (multiplied by a factor $\frac{1-\beta\delta^2}{1-\delta} > 1$).

□

Corollary 7. ***(Comparison of CE_1 and CE_2 Taxes)*** *In the perpetual youth model, optimal wealth taxes are higher when taking the wealth accumulation into account $\left(\tau^\star_{a,2} > \tau^\star_a\right)$ and the α-thresholds are lower ($\underline{\alpha}_2 < \underline{\alpha}$ and $\overline{\alpha}_2 < \overline{\alpha}$) if $\frac{1-\beta\delta^2}{1-\delta} < \frac{1}{n_w}$.*

D.4.5 Perpetual youth model: Additional results

Aggregate variables in steady state We will use the fact that under Assumption 1, the government budget constraint reduces to

$$\frac{1-\theta}{1-\beta\delta} = \frac{1-\tau_k}{1-\beta\delta(1-\tau_a)}. \tag{133}$$

Lemmas 11 and 12 parallel Lemmas 2 and Corollary **??**:

Lemma 11. ***(Aggregate Variables in Steady State)*** *The steady state aggregate variables satisfy*

$$s_h = \frac{Z - z_\ell}{z_\lambda - z_\ell} > \frac{1}{2} \qquad \frac{ds_h}{dZ} > 0 \tag{134}$$
$$R_h = \frac{1}{\beta\delta^2}\left(1 - \frac{1-\delta}{2s_h}\right) \qquad \frac{dR_h}{dZ} > 0 \tag{135}$$
$$R_\ell = \frac{1}{\beta\delta^2}\left(1 - \frac{1-\delta}{2(1-s_h)}\right) \qquad \frac{dR_\ell}{dZ} < 0. \tag{136}$$

Moreover, under Assumption 1

$$K = \left(\alpha\frac{\beta\delta\left(1-\theta\right)}{1-\beta\delta}\right)^{\frac{1}{1-\alpha}} Z^{\frac{\alpha}{1-\alpha}} L \qquad \frac{dK}{dZ} \propto \frac{\alpha}{1-\alpha} Z^{\frac{2\alpha-1}{1-\alpha}} < 0 \tag{137}$$

$$Q = \left(\alpha\frac{\beta\delta\left(1-\theta\right)}{1-\beta\delta}Z\right)^{\frac{1}{1-\alpha}} L \qquad \frac{dQ}{dZ} \propto \frac{1}{1-\alpha} Z^{\frac{\alpha}{1-\alpha}} > 0 \tag{138}$$

$$Y = \left(ZK\right)^{\alpha} L^{1-\alpha} \qquad \frac{dY}{dZ} \propto \frac{\alpha}{1-\alpha} Z^{\frac{2\alpha-1}{1-\alpha}} > 0 \tag{139}$$

$$A_h = \frac{Z-z_\ell}{z_\lambda - z_\ell} K \qquad \frac{dA_h}{dZ} \propto \frac{Z^{\frac{2\alpha-1}{1-\alpha}}}{1-\alpha}\left(Z - \alpha z_\ell\right) > 0 \tag{140}$$

$$A_\ell = \frac{z_\lambda - Z}{z_\lambda - z_\ell} K \qquad \frac{dA_\ell}{dZ} \propto \frac{Z^{\frac{\alpha}{1-\alpha}}}{1-\alpha}\left(\alpha z_\lambda - Z\right). \tag{141}$$

Proof. We know that $s_h = \frac{Z-z_\ell}{z_\lambda - z_\ell}$, so $s_h > 1/2$ is equivalent to $Z > \frac{z_\lambda + z_\ell}{2}$. We can verify if this is the case by evaluating the residual of (127) at $\frac{z_\lambda+z_\ell}{2}$:

$$\begin{aligned} H\left(\frac{z_\lambda + z_\ell}{2}\right) &= \left(1-\delta\eta\right) - \left(1 - \delta\left(2\eta - 1\right)\right) + \delta\left(1-\eta\right)\frac{z_\ell z_\lambda}{\left(\frac{z_\lambda+z_\ell}{2}\right)^2} \\ &= -\delta\left(1-\eta\right) + \delta\left(1-\eta\right)\frac{z_\ell z_\lambda}{\left(\frac{z_\lambda+z_\ell}{2}\right)^2} \\ &= -\delta\left(1-\eta\right)\left(\frac{z_\lambda - z_\ell}{z_\lambda + z_\ell}\right)^2 < 0 \end{aligned}$$

The residual is always negative. So it must be that $Z > \frac{z_\lambda+z_\ell}{2}$ and thus $s_h > 1/2$.The result for the wealth share of the high-types is immediate from the definition of Z.

From the steady state level of wealth of high-productivity entrepreneurs we know that:

$$R_h = \frac{1}{\beta\delta^2}\left(1 - \frac{1-\delta}{2s_h}\right)$$

which implies:

$$\frac{dR_h}{dZ} = \frac{1-\delta}{2\beta\delta^2}\frac{1}{s_h^2}\frac{ds_h}{dZ} > 0$$

A similar calculation delivers:

$$R_\ell = \frac{1}{\beta\delta^2}\left(1 - \frac{1-\delta}{2\left(1-s_h\right)}\right) \qquad \frac{dR_\ell}{dZ} = -\frac{1-\delta}{2\beta\delta^2}\frac{1}{\left(1-s_h\right)^2}\frac{ds_h}{dZ} < 0$$

From (125) we can express government spending as:

$$G = \left(\tau_k + \tau_a \frac{\beta\delta\left(1-\tau_k\right)}{1-\beta\delta\left(1-\tau_a\right)}\right)\alpha Y,$$

and under Assumption 1 we get:

$$\frac{1-\theta}{1-\beta\delta} = \frac{1-\tau_k}{1-\beta\delta\left(1-\tau_a\right)}.$$

Replacing in (125) we get:

$$K = \left(\alpha\frac{\beta\delta\left(1-\theta\right)}{1-\beta\delta}\right)^{\frac{1}{1-\alpha}} Z^{\frac{\alpha}{1-\alpha}} L$$

which is increasing in Z. From the same expression we get:

$$Q = ZK = \left(\alpha\frac{\beta\delta\left(1-\theta\right)}{1-\beta\delta}\right)^{\frac{1}{1-\alpha}} Z^{\frac{1}{1-\alpha}} L$$

which is also increasing in Z. From this it is immediate that $Y = Q^{\alpha}L^{1-\alpha}$ is also increasing in Z.

Since K and s_h increase it must be the case that $A_h = s_h K$ increases as well. We are left with the response of A_ℓ. To get it we first write A_ℓ in terms of Z using the definition of the wealth share of the high-types:

$$\begin{aligned} A_\ell &= \left(1-s_h\right)A \\ &= \left(1-\frac{Z-z_\ell}{z_\lambda - z_\ell}\right)A \\ &= \left(\alpha\frac{\beta\delta\left(1-\theta\right)}{1-\beta\delta}\right)^{\frac{1}{1-\alpha}} L\frac{z_\lambda - Z}{z_\lambda - z_\ell} Z^{\frac{\alpha}{1-\alpha}} \end{aligned}$$

Taking derivatives shows that A_ℓ decreases with Z (and hence with τ_a):

$$\frac{dA_\ell}{dZ} \propto \frac{Z^{\frac{\alpha}{1-\alpha}-1}}{z_\lambda - z_\ell}\left[\alpha z_\lambda - Z\right]$$

which is negative if $\alpha z_\lambda < Z$.

□

Returns can be expressed in terms of Z as before

$$R_\ell = 1-\tau_a + \left(\frac{1}{\beta\delta} - \left(1-\tau_a\right)\right)\frac{z_\ell}{Z} \quad \text{and} \quad R_h = 1-\tau_a + \left(\frac{1}{\beta\delta} - \left(1-\tau_a\right)\right)\frac{z_\lambda}{Z}$$

Similarly, the change in returns can be divided into the use-it-or-lose-it effect ($-\left(1-{z_\ell}/{Z}\right) < 0$ and $-\left(1-{z_\lambda}/{Z}\right) > 0$) and a negative general equilibrium effect.

Lemma 12. *$\frac{dR_\ell}{d\tau_a} < 0$ and $\frac{dR_h}{d\tau_a} > 0$, $\frac{d(R_h+R_\ell)}{d\tau_a} < 0$ and $\frac{d(R_h R_\ell)}{d\tau_a} < 0$.*

Proof. We know the share of wealth of the high-types is increasing along with the overall wealth in the economy, so A_h must increase as well, this will imply that R_h must have risen. From Lemma 11:

$$\frac{dR_h}{d\tau_a} = \frac{1-\delta}{2\beta\delta^2}\frac{1}{s_h^2}\frac{ds_h}{d\tau_a} = \frac{1-\delta}{2\beta\delta^2}\frac{z_\lambda - z_\ell}{(Z - z_\ell)^2}\frac{dZ}{d\tau_a} > 0,$$

and:

$$\frac{dR_\ell}{d\tau_a} = -\frac{1-\delta}{2\beta\delta^2}\frac{1}{(1-s_h)^2}\frac{ds_h}{d\tau_a} = \frac{-(1-\delta)}{2\beta\delta^2}\frac{z_\lambda - z_\ell}{(z_\lambda - Z)^2}\frac{dZ}{d\tau_a} < 0.$$

With this we get:

$$\frac{d(R_h + R_\ell)}{d\tau_a} = \frac{1-\delta}{2\beta\delta^2}\left(\frac{1-2s_h}{s_h^2(1-s_h)^2}\right)\frac{ds_h}{d\tau_a} = \frac{1-\delta}{2\beta\delta^2}\frac{(z_\lambda - z_\ell)^2(z_\lambda + z_\ell - 2Z)}{(Z - z_\ell)^2(z_\lambda - Z)^2}\frac{dZ}{d\tau_a}$$

so that $\frac{d(R_h+R_\ell)}{d\tau_a} \geq 0$ if and only if $s_h \leq 1/2$. Since $s_h > 1/2$ then $\frac{d(R_h+R_\ell)}{d\tau_a} < 0$.

Finally, we consider the product of returns, which is also decreasing in taxes.

$$\begin{aligned}
\frac{dR_hR_\ell}{d\tau_a} &= R_h\frac{dR_\ell}{d\tau_a} + R_\ell\frac{dR_h}{d\tau_a} \\
&= \frac{1-\delta}{2(\beta\delta^2)^2}\frac{ds_h}{d\tau_a}\left[-\left(1 - \frac{1-\delta}{2s_h}\right)\frac{1}{(1-s_h)^2} + \left(1 - \frac{1-\delta}{2(1-s_h)}\right)\frac{1}{s_h^2}\right] \\
&= \frac{1-\delta}{(2\beta\delta^2 s_h(1-s_h))^2}\frac{ds_h}{d\tau_a}\left[-s_h(2s_h - (1-\delta)) + (1-s_h)(2(1-s_h) - (1-\delta))\right] \\
&= \frac{1-\delta^2}{(2\beta\delta^2 s_h(1-s_h))^2}(1-2s_h)\frac{ds_h}{d\tau_a} \\
&= \frac{1-\delta^2}{(2\beta\delta^2(Z - z_\ell)(z_\lambda - Z))^2}(z_\lambda + z_\ell - 2Z)(z_\lambda - z_\ell)^2\frac{dZ}{d\tau_a} < 0.
\end{aligned}$$

This completes the proof.

□

Individual welfare comparisons Just as in Section 3.3 we can define the welfare change for each individual of type i asking how much they value being dropped from the capital income tax economy with $\tau_a = 0$ and $\tau_k = \theta$ to the economy with a positive wealth tax $\tau_a > 0$ in terms of lifetime consumption. We denote this consumption equivalent welfare measure as $\text{CE}_1(a,i)$ and it is given by

$$\log(1 + \text{CE}_1(a,i)) = (1-\beta\delta)(V_a(a,i) - V_k(a,i)) = (1-\beta\delta)\Delta V(a,i). \quad (142)$$

All the terms containing wealth cancel, thus, the welfare gain depends only on the individual's type i. Consequently, we drop wealth "a" from the welfare measure below and write

$$\log(1 + \text{CE}_{1,i}) = \begin{cases} \log w_a/w_k & \text{if } i = w \\ \frac{1}{1-\beta\delta}\log R_{a,i}/R_{k,i} & \text{if } i \in \{\ell, h\}. \end{cases} \quad (143)$$

Workers' welfare increases with wealth taxes because of the effect of taxes on productivity Z and through it on wages (Lemma 11). From Lemma 12 we conclude that the welfare of high-productivity entrepreneurs goes up if wealth taxes increase, while the welfare of low-productivity entrepreneurs goes down. Recall that productivity types are permanent in this economy, because of that high-types do not take into account the effect of taxes on the returns of low-types, and vice versa. We summarize these results in Lemma 13.

Lemma 13. *$CE_{1,w} > 0$, $CE_{1,h} > 0$, and $CE_{1,\ell} < 0$.*

Proof. The result is immediate from (143) and Lemmas **??** and 12. As wealth taxes increase wages increase which increases the welfare of workers. The returns of high-productivity entrepreneurs increase, which increases their welfare. The returns of low-productivity entrepreneurs decrease, which decreases their welfare.

□

We next turn to aggregate welfare. As before the aggregate welfare measure can be computed as $\log(1+\mathrm{CE}_1) = \sum_i n_i \log(1+\mathrm{CE}_{1,i})$. Substituting in $\mathrm{CE}_{1,i}$'s gives

$$\log(1+\mathrm{CE}_1) = n_w \frac{\alpha}{1-\alpha}\log(Z_a/Z_k) + \frac{1-n_w}{2(1-\beta\delta)}\left(\log {}^{R_{a,\ell}}/_{R_{k,\ell}} + \log {}^{R_{a,h}}/_{R_{k,h}}\right). \tag{144}$$

Again, this result parallels that of our benchmark model, and the same forces are in play when determining the optimal taxes. The total welfare of entrepreneurs decreases with wealth taxes because of lower average returns while workers' welfare increases because of higher wages. In proposition 13 we characterize the optimal tax problem of the government that maximizes the utilitarian welfare by choosing the optimal combination (τ_a, τ_k). The optimal tax combination equates the elasticity of wages with respect to productivity with the average elasticity of returns, weighted by population.

Proposition 13. *The optimal tax combination (τ_a^*, τ_k^*) that maximizes the utilitarian welfare measure CE_1 is unique and given by the solution to the following equation:*

$$n_w \xi_w = -\frac{1-n_w}{1-\beta\delta}\left(\frac{\xi_{R_\ell} + \xi_{R_h}}{2}\right) \tag{145}$$

where $\xi_x \equiv \frac{d\log x}{d\log Z}$ is the elasticity of variable x with respect to Z. Applying Lemma 11 gives

$$n_w \frac{\alpha}{1-\alpha} = \frac{1}{2}\frac{1-n_w}{1-\beta\delta}\frac{1}{R_h R_\ell}\frac{\left(1-\delta^2\right)(2s_h-1)}{\left(2\beta\delta^2 s_h(1-s_h)\right)^2}\frac{Z}{z_\lambda - z_\ell}, \tag{146}$$

where τ_k is given from equation (133), $s_h = \frac{Z-z_\ell}{z_\lambda - z_\ell}$, Z is the solution to equation (14) and R_i's are given by equations (135) and (136). Furthermore, there exist two cutoff values for α which we *denote $\underline{\alpha}_p$ and $\overline{\alpha}_p$ such that $(\tau_a^\star, \tau_k^\star)$ satisfies the following properties:*

$$\begin{aligned}
\tau_a^\star &\in \left[1-\frac{1}{\beta\delta}, 0\right) \text{ and } \tau_k^\star > \theta && \text{if } \alpha < \underline{\alpha}_p \\
\tau_a^\star &\in \left[0, \frac{\theta(1-\beta\delta)}{\beta\delta(1-\theta)}\right] \text{ and } \tau_k^\star \in [0,\theta] && \text{if } \underline{\alpha}_p \le \alpha \le \bar{\alpha}_p \\
\tau_a^\star &\in \left(\frac{\theta(1-\beta\delta)}{\beta\delta(1-\theta)}, \tau_{a,p}^{\max}\right) \text{ and } \tau_k^\star < 0 && \text{if } \alpha > \bar{\alpha}_p
\end{aligned}$$

where $\tau_{a,p}^{\max} \geq 1$, $\underline{\alpha}_p$ is the solution to equation (146) with $\tau_a = 0$ and $\overline{\alpha}_p$ is the solution to equation (146) with $\tau_a = \frac{\theta(1-\beta\delta)}{\beta\delta(1-\theta)}$.

Proof. From (144) we get an expression for the total CE_1 welfare measure:

$$\sum_i n_i \log(1+\mathrm{CE}_{1,i}) = n_w \frac{\alpha}{1-\alpha}\log(Z_a/Z_k) + \frac{1-n_w}{2(1-\beta\delta)}\log\left(\frac{R_{a,h}R_{a,\ell}}{R_{k,h}R_{k,\ell}}\right)$$

This is equal to 0 when $\tau_a = 0$ by construction. For welfare to increase we need the derivative of (144) with respect to τ_a to be positive. The derivative is:

$$\begin{aligned}
n_w \frac{\alpha}{1-\alpha}\frac{1}{Z}\frac{dZ}{d\tau_a} + \frac{1}{2}\frac{1-n_w}{1-\beta\delta}\frac{1}{R_h R_\ell}\frac{dR_h R_\ell}{d\tau_a} &> 0 \\
n_w \frac{\alpha}{1-\alpha}\frac{1}{Z}\frac{dZ}{d\tau_a} + \frac{1}{2}\frac{1-n_w}{1-\beta\delta}\frac{1}{R_h R_\ell}\frac{(1-\delta^2)(1-2s_h)}{(2\beta\delta^2 s_h(1-s_h))^2}\frac{ds_h}{d\tau_a} &> 0 \\
\left[n_w \frac{\alpha}{1-\alpha} + \frac{1}{2}\frac{1-n_w}{1-\beta\delta}\frac{1}{R_h R_\ell}\frac{(1-\delta^2)(1-2s_h)}{(2\beta\delta^2 s_h(1-s_h))^2}\frac{Z}{z_\lambda - z_\ell}\right]\frac{1}{Z}\frac{dZ}{d\tau_a} &> 0
\end{aligned}$$

where $s_h = \frac{Z-z_\ell}{z_\lambda - z_\ell}$, $1-s_h = \frac{z_\lambda - Z}{z_\lambda - z_\ell}$, $R_\ell = \frac{1}{\beta\delta^2}\left(1-\frac{1-\delta}{2(1-s_h)}\right)$, and and $R_h = \frac{1}{\beta\delta^2}\left(1-\frac{1-\delta}{2s_h}\right)$. From Proposition 8 we know that $\frac{dZ}{d\tau_a} > 0$, so an increase in wealth taxes increases welfare if and only if

$$n_w \frac{\alpha}{1-\alpha} - \frac{1}{2}\frac{1-n_w}{1-\beta\delta}\frac{1}{R_h R_\ell}\frac{(1-\delta^2)(2s_h-1)}{(2\beta\delta^2 s_h(1-s_h))^2}\frac{Z}{z_\lambda - z_\ell} \geq 0.$$

Moreover, the optimal level of taxes is given by $\tau_a^\star$ for which this equation holds with equality.

□

The optimal tax combination has a positive wealth tax if $\alpha > \underline{\alpha}_p$ and the optimal wealth tax is greater than the tax reform level $\tau_a = \frac{\theta(1-\beta\delta)}{\beta\delta(1-\theta)}$ if $\alpha > \overline{\alpha}_p$, in that case the optimal capital income tax is a subsidy. The thresholds $\underline{\alpha}_p$ and $\bar{\alpha}_p$ can be solved explicitly in terms of parameters when $z_\ell = 0$. In this case $Z = \frac{(1+\delta)/2 - \beta\delta^2(1-\tau_a)}{1-\beta\delta^2(1-\tau_a)} z_\lambda$, $s_h = \frac{Z}{z_\lambda}$, $R_\ell = 1-\tau_a$, and $R_h = \frac{1-\frac{\delta\beta(1+\delta)(1-\tau_a)}{2}}{\delta\beta((1+\delta)/2 - \beta\delta^2(1-\tau_a))}$. Substituting these variables into the first order condition of the government and evaluating at $\tau_a = 0$ and $\tau_a = \frac{\theta(1-\beta\delta)}{\beta\delta(1-\theta)}$ gives the two α thresholds.

Corollary 8. $\frac{\underline{\alpha}_p}{1-\underline{\alpha}_p} = \frac{1-n_w}{2n_w}\frac{(1+\delta)\left(1-\beta\delta^2\right)^2}{(1-\delta)\beta\delta^2\left(1-\frac{\delta\beta(1+\delta)}{2}\right)}$ *and* $\frac{\bar{\alpha}_p}{1-\bar{\alpha}_p} = \frac{1-n_w}{2n_w}\frac{(1+\delta)\left(1-\theta-\beta\delta^2+\theta\delta\right)^2}{(1-\delta)(\beta\delta-\theta)\delta\left((1-\theta)-\frac{(1+\delta)(\beta\delta-\theta)}{2}\right)(1-\theta)}$ *if* $z_\ell = 0$.

Proof. We substitute the expressions $Z = \frac{(1+\delta)/2-\beta\delta^2(1-\tau_a)}{1-\beta\delta^2(1-\tau_a)}z_\lambda$, $s_h = \frac{Z}{z_\lambda} = \frac{(1+\delta)/2-\beta\delta^2(1-\tau_a)}{1-\beta\delta^2(1-\tau_a)}$, $1-s_h = \frac{(1-\delta)/2}{1-\beta\delta^2(1-\tau_a)}$, $2s_h - 1 = \frac{\delta-\beta\delta^2(1-\tau_a)}{1-\beta\delta^2(1-\tau_a)}$, $R_\ell = 1-\tau_a$, and $R_h = \frac{1-\frac{\delta\beta(1+\delta)(1-\tau_a)}{2}}{\delta\beta((1+\delta)/2-\beta\delta^2(1-\tau_a))}$ into

$$\begin{aligned}\frac{\alpha}{1-\alpha} &= \frac{1}{2}\frac{1-n_w}{(1-\beta\delta)\left(2\beta\delta^2\right)^2 n_w}\frac{1}{R_h R_\ell}\frac{\left(1-\delta^2\right)(2s_h-1)}{\left(s_h\left(1-s_h\right)\right)^2}\frac{Z}{z_\lambda} \\ &= \frac{1-n_w}{2n_w}\frac{(1+\delta)\left(1-\beta\delta\left(1-\tau_a\right)\right)\left(1-\beta\delta^2\left(1-\tau_a\right)\right)^2}{(1-\beta\delta)\,\beta\delta^2\left(1-\frac{\delta\beta(1+\delta)(1-\tau_a)}{2}\right)\left(1-\tau_a\right)(1-\delta)}.\end{aligned}$$

Evaluating this expression at $\tau_a = 0$ gives

$$\begin{aligned}\frac{\underline{\alpha}_p}{1-\underline{\alpha}_p} &= \frac{1-n_w}{2n_w}\frac{(1+\delta)\left(1-\beta\delta^2\right)^2}{(1-\delta)\,\beta\delta^2\left(1-\frac{\delta\beta(1+\delta)}{2}\right)} \\ \underline{\alpha}_p &= \frac{\frac{1-n_w}{2n_w}\frac{(1+\delta)\left(1-\beta\delta^2\right)^2}{(1-\delta)\beta\delta^2\left(1-\frac{\delta\beta(1+\delta)}{2}\right)}}{1+\frac{1-n_w}{2n_w}\frac{(1+\delta)(1-\beta\delta^2)^2}{(1-\delta)\beta\delta^2\left(1-\frac{\delta\beta(1+\delta)}{2}\right)}}. \\ \underline{\alpha}_p &= 1-\frac{1}{1+\frac{1-n_w}{2n_w}\frac{(1+\delta)(1-\beta\delta^2)^2}{(1-\delta)\beta\delta^2\left(1-\frac{\delta\beta(1+\delta)}{2}\right)}}.\end{aligned}$$

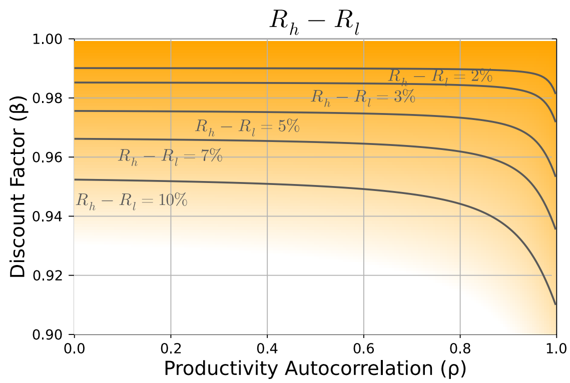
Evaluating at $\tau_a = \frac{\theta(1-\beta\delta)}{\beta\delta(1-\theta)}$ $\left(1-\tau_a = 1-\frac{\theta(1-\beta\delta)}{\beta\delta(1-\theta)} = \frac{\beta\delta-\theta}{\beta\delta(1-\theta)}\right)$

$$\begin{aligned}\frac{\bar{\alpha}_p}{1-\bar{\alpha}_p} &= \frac{1-n_w}{2\,(1-\beta\delta)\,\beta\delta^2 n_w}\frac{(1+\delta)\left(1-\beta\delta\frac{\beta\delta-\theta}{\beta\delta(1-\theta)}\right)\left(1-\beta\delta^2\frac{\beta\delta-\theta}{\beta\delta(1-\theta)}\right)^2}{\left(1-\frac{\delta\beta(1+\delta)}{2}\frac{\beta\delta-\theta}{\beta\delta(1-\theta)}\right)\frac{\beta\delta-\theta}{\beta\delta(1-\theta)}(1-\delta)} \\ &= \frac{1-n_w}{2n_w}\frac{(1+\delta)\left(1-\theta-\beta\delta^2+\theta\delta\right)^2}{(1-\delta)\,(\beta\delta-\theta)\,\delta\left(1-\theta-\frac{(1+\delta)(\beta\delta-\theta)}{2}\right)(1-\theta)} \\ \bar{\alpha}_p &= 1-\frac{1}{1+\frac{1-n_w}{2n_w}\frac{(1+\delta)(1-\theta-\beta\delta^2+\theta\delta)^2}{(1-\delta)(\beta\delta-\theta)\delta\left((1-\theta)-\frac{(1+\delta)(\beta\delta-\theta)}{2}\right)(1-\theta)}}.\end{aligned}$$

□

E Extra Tables and Figures

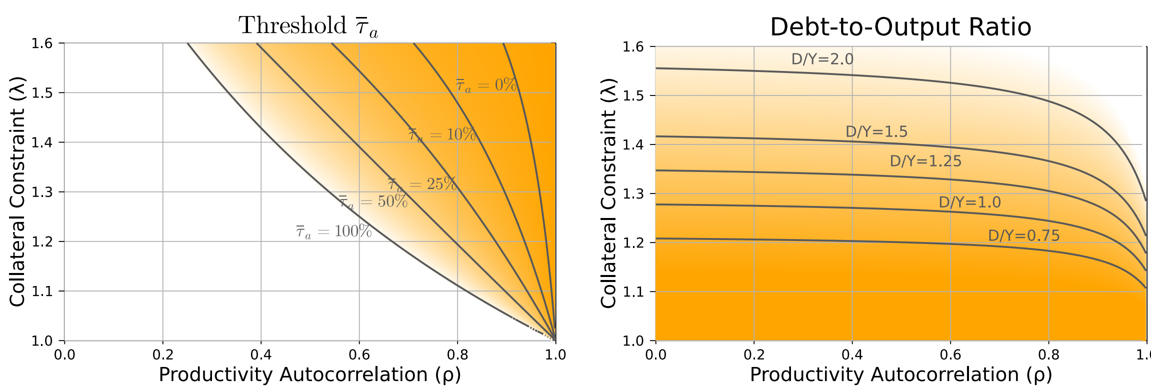
Figure E.3: Steady State Return Dispersion



Note: *Figure E.3 reports the steady state return dispersion $R_h - R_\ell$ for combinations of the autocorrelation of productivity (ρ) and the discount factor (β). We set the remaining parameters as follows: $z_\ell = 0$, $z_h = 2$, $\tau_k = 25\%$, $\lambda = 1.32$, and $\alpha = 0.4$.*

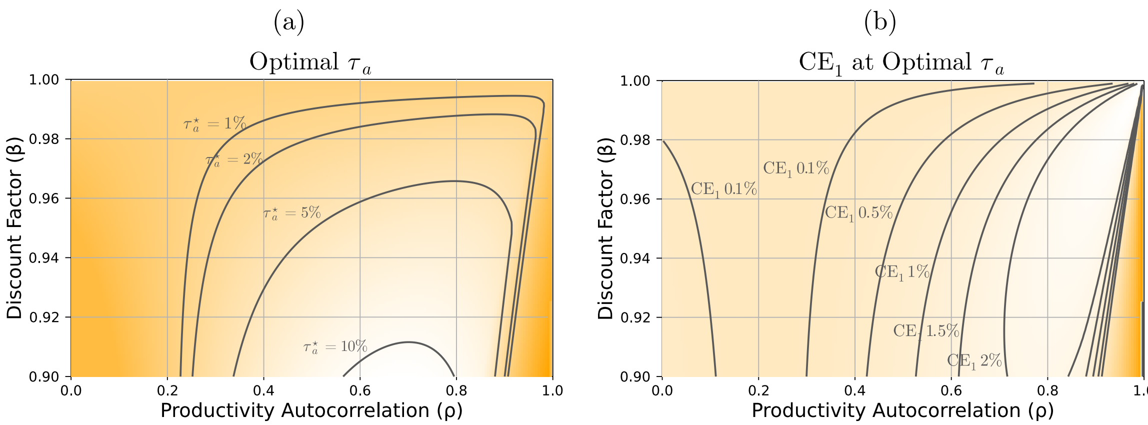
Figure E.4: τ_a Conditions for Steady State with Heterogeneous Returns

(a) (b)



Note: *Figure E.4a reports the value of $\overline{\tau}_a$ found in Corollary 1 for combinations of the autocorrelation of productivity (ρ) and the collateral constraint parameter (λ). The steady state exhibits heterogeneous returns if and only if $\tau_a \leq \overline{\tau}_a$. Figure E.4b reports the debt-to-output ratio when $\tau_a = 0$ computed as $(\lambda-1)A_h/Y$ for the same combinations of ρ and λ. In both figures we set the remaining parameters as follows: $z_\ell = 0$, $z_h = 2$, $\tau_k = 25\%$, $\beta = 0.96$, and $\alpha = 0.4$.*

Figure E.5: Optimal Wealth Taxes and Welfare Gain



Note: *Figure E.5a reports the value of the wealth taxes that maximize CE_1 welfare as described in Proposition 4 for combinations of the autocorrelation of productivity (ρ) and the discount factor (β). Figure E.5b reports the value of CE_1 welfare at the optimal wealth taxes. The value of τ_a is found by finding the root of equation (28). The value of τ_k satisfies equation (18). In both figures we set the remaining parameters as follows: $z_\ell = 0$, $z_h = 2$, $\theta = 25\%$, and $\lambda = 1.32$.*