Optimal Project Management

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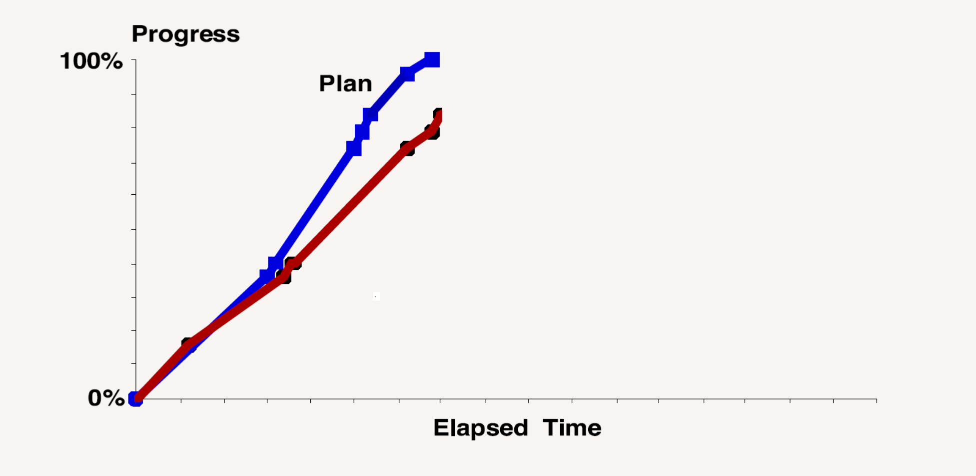
Project Dynamics



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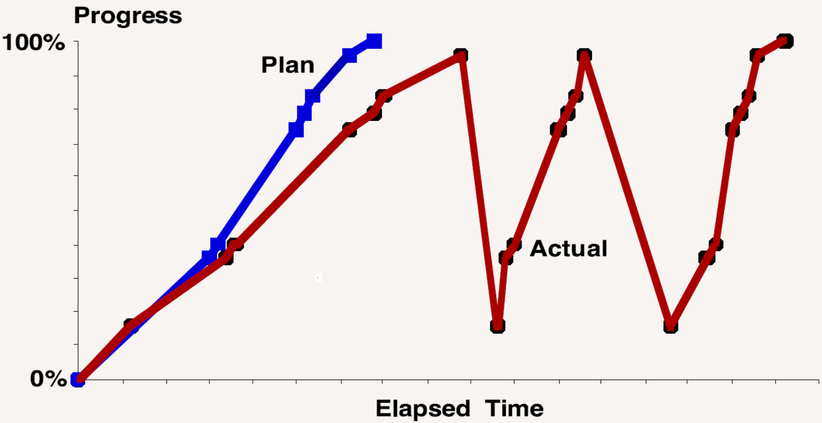
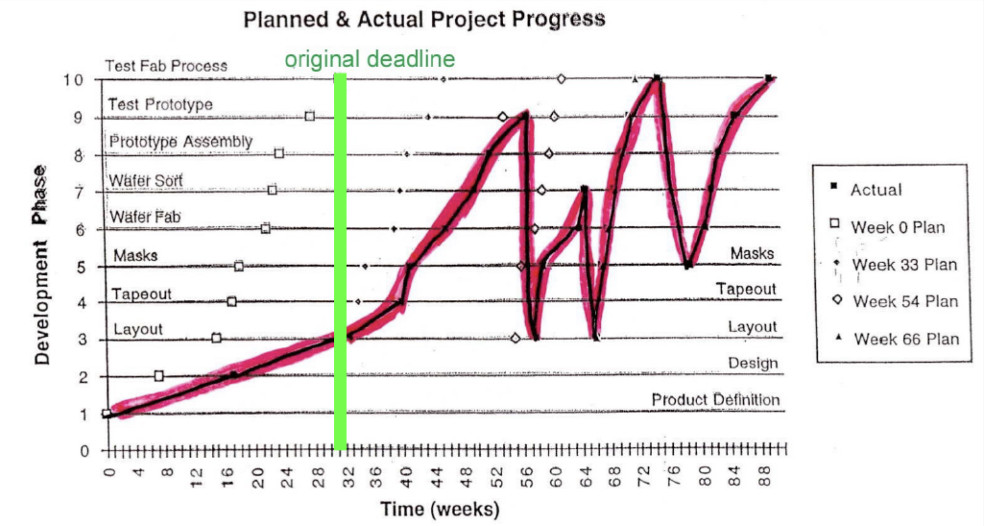


- 79% *apparently* complete by original deadline
- 2 major unplanned iterations requiring redesign
- Actual duration: 208% of schedule

Project Dynamics



Semiconductor Fabrication Plants (Ford and Sterman, 2003)

Introduction

Project Management (R&D, infrastructure, public works . . .):

- Multiple stages, cumulative progress
- Stochastic outcomes; risk of setbacks
- Rewards upon completion

Dynamic moral hazard in a non-stationary environment:

- Progress is slower, success less likely
- Less ambitious, less failure-tolerant projects
- Further delays to deter risk taking

Related Literature

Dynamic moral hazard: lumpy (Poisson) progress

- Mason and Valimaki (2015); Green and Taylor (2016); Guo (2018); Halac, Kartik, Liu (2018); Moroni (2018)

Dynamic moral hazard: continuous progress

- Georgiadis (2015); Georgiadis, Lippman, and Tang (2014)

Dynamic mechanism design: optimal stopping

- Kruse and Strack (2015, 2018); Madsen (2022); McClellan (2023)

Choice of risk

- DeMarzo et al. (2014); Wong (2019); Li and Williams (2023)

Model

Principal (P) hires agent (A) to complete a project

The project's (stochastic) progress is publicly observable

Agent controls the process exerting costly hidden effort

Time is continuous and possibly infinite: $t \in [0, \infty)$

Common discount rate $r \geq 0$

Baseline Model

Project evolves in continuous time according to

$$dX_t = \mu(a_t, X_t)dt + \sigma(X_t)dZ_t$$

For this talk

$$\mu(a_t, X_t) = a_t, \quad \sigma(X_t) = \sigma$$

Threshold structure: project is successful if $X \geq \bar{x}$ (target, ceiling...)

Principal's Payoffs

The project is complete the first time the progress X hits the target $\bar{x}$

The principal can also terminate the project earlier

Let τ denote the termination time. Principal's realized payoff:

$$e^{-r\tau}\left(\mathbf{1}_{\{X_\tau \geq \bar{x}\}} b + \mathbf{1}_{\{X_\tau < \bar{x}\}} s\right) - \left(\int_0^\tau e^{-rt} c_t dt + \frac{e^{-r\tau}}{r} c_\tau\right)$$

$b > 0$ is the benefit from project completion

$s > 0$ is the salvage value of the project

c_t is the flow wage paid to the agent

Agent's Payoffs

Agent's (CARA) flow utility:

$$u(\hat{a}, \hat{c}) = -\frac{1}{\eta} \exp\{-\eta(\hat{c} - h(\hat{a}))\} = -\frac{1}{\eta} \exp\{-\eta(\hat{c} - \frac{1}{2}\hat{a}^2)\}$$

$\hat{c}_t$ is agent's time-t chosen consumption level

Agent can privately borrow and save at rate $r \geq 0$

Savings account with balance S_t at time t

Agent also has outside option W_0

Planner's Problem

The social planner solves the following problem

$$\max_{A,\tau} \mathbb{E}_A \left[\mathbf{1}_{\{X_\tau \geq \bar{x}\}} e^{-r\tau} b + \mathbf{1}_{\{X_\tau < \bar{x}\}} e^{-r\tau} s - \frac{1}{2} \int_0^\tau e^{-rt} a_t^2 dt \right]$$

subject to

$$dX_t = a_t dt + \sigma dZ_t$$

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Recursive Formulation:

$$rV(x) = \max_a \left[-\frac{1}{2} a^2 + aV'(x) + \frac{1}{2} \sigma^2 V''(x) \right]$$

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Optimal Effort: $a(x) = V'(x)$

Optimal Termination: two thresholds $\bar{x}$ and $\underline{x}^{FB}$ that satisfy
Value Matching ($V(\bar{x}) = b, \;\; V(\underline{x}^{FB}) = s$) and Smooth Pasting ($V'(\underline{x}^{FB}) = 0$)

Planner's Value Function

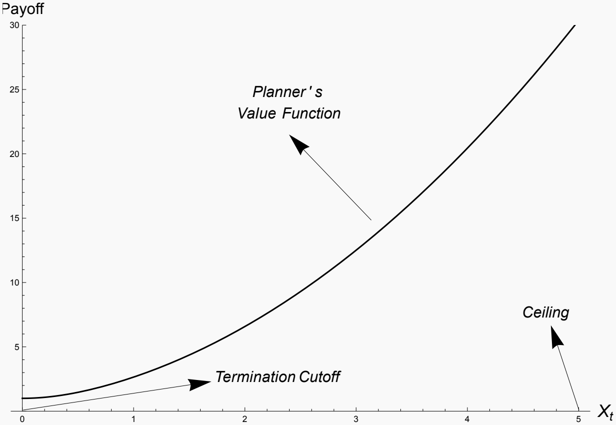
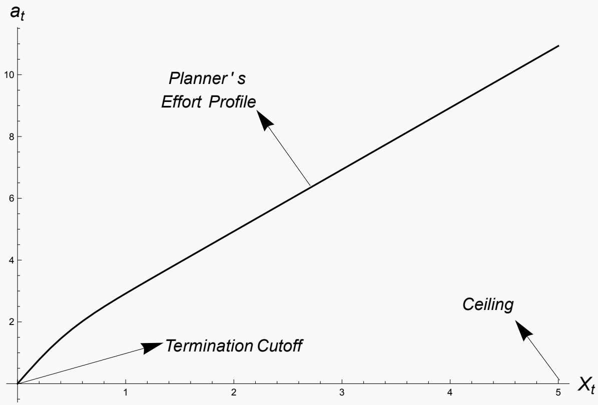


Figure: $(b, \bar{x}, \sigma, s, r, \eta) = (30, 5, 1, 1, 2, 1/2)$

Planner's Effort Profile



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Contracting Problem

A contract is a triple $\mathcal{C} = (A, C, \tau)$ adapted to the public history X^t:

- recommended consumption level C (i.e., flow wage process)
- recommended effort level A (i.e., reference progress path)
- termination policy τ

Savings account evolution:

$$dS_t = rS_t + c_t dt - \hat{c}_t dt, \quad S_0 = 0, \quad \lim_{t\to\infty} \mathbb{E}\left[e^{-rt} S_t\right] \to 0$$

Agent continuation utility representation:

$$dW_t = (rW_t - u(\hat{a}_t, c_t, \hat{c}_t))\, dt + \beta_t(-\eta r W_t)(dX_t - a_t dt)$$

Principal's Problem

Principal maximizes her payoff by choosing $\mathcal{C}$

$$\max_{A,C,\tau} \mathbb{E}_A \left[\mathbf{1}_{\{X_\tau \geq \bar{x}\}} e^{-r\tau} b + \mathbf{1}_{\{X_\tau < \bar{x}\}} e^{-r\tau} s - \int_0^\tau e^{-rt} c dt - \frac{e^{-r\tau}}{r} c_\tau \right]$$

subject to: IC + IR + No Savings

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General Recursive Formulation

$$v(x,w) = \max_{a \in \mathcal{A}, c} \Big[-c + \underbrace{\mu(W)}_{\textit{drift of cont. util.}} v_w + \frac{1}{2} \underbrace{\sigma^2(w)}_{\textit{volatility of cont. util.}} v_{ww} + a v_x + \frac{1}{2}\sigma^2 v_{xx} + \sigma \cdot \sigma(w) v_{xw} \Big]$$

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In general this PDE is difficult to analyze

CARA Properties

Private savings + CARA helps tractability (He, 2011)

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We extend these results to non-stationary environment

1. CARA implies level-invariance of continuation utility
   - Consider a deviating agent with savings S who faces contract $\mathcal{C}$. Denote the agent's continuation value at time t by $W_t(S, \mathcal{C})$. It holds that

     $$W_t(S, \mathcal{C}) = e^{-\eta r S} W_t(0, \mathcal{C}),$$

     where $W_t(0, \mathcal{C})$ is the continuation value along the no-savings path.

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     where $W_t(0, \mathcal{C})$ is the continuation value along the no-savings path.
2. Consumption pinned down $\Rightarrow$ continuation utility is a martingale
   - $c_t = \frac{1}{2}a_t^2 - \frac{1}{\eta}\ln(-\eta r W_t)$, i.e., $u(a_t, c_t) = rW_t$
   - $$\begin{aligned} dW_t &= (rW_t - u(a_t, c_t))\, dt + \beta_t(-\eta r W_t)(dX_t - a_t dt) \\ &= \beta_t(-\eta r W_t)(dX_t - a_t dt) \end{aligned}$$

Incentive Compatability

3. Incentive Compatibility of the agent
   - Agent's problem

$$\max_{\hat{a}} \left[u(\hat{a}, \hat{c}) + dW \right]$$

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   - Agent's problem

     $$\max_{\hat{a}} [u(\hat{a}, \hat{c}) + dW]$$

   - Equivalently

     $$\max_{\hat{a}} \; [u(\hat{a}, \hat{c}) + \beta(-\eta r W)(dX - \hat{a}dt)]$$

     FOC + CARA

     $$u_a(\hat{a}, \hat{c}) = \beta \eta r W \Rightarrow \hat{a} = \beta$$

4. 1+2+3 together yield additively separable solution to the PDE

HJB Equation - Derivation

Let $v(x, w)$ denote the principal's value function

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$$v(x, w) = \underbrace{\pi(x)}_{Project\ Value} - \underbrace{\left[-\frac{1}{\eta r}\ln(-\eta r w)\right]}_{Certainty\ Equivalent\ to\ Agent}$$

CARA utility $\Rightarrow$ the certainty equivalent is all that matters for continuation values

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Using the functional form

$$v_{xw} = 0; \quad v_{ww} = -\frac{1}{\eta r w^2}; \quad \mu(w) = 0; \quad \sigma^2(w) = (-\beta \eta r W)^2;$$

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IC + No savings

$$c = \frac{1}{2}a^2 - \frac{1}{\eta} ln(-\eta r W); \quad \beta = a$$

HJB Equation

Plugging it back to HJB

$$r\pi(x) = \max_{a \in \mathcal{A}} \Big\{ -\frac{1}{2}\kappa^{-1}a^2 + a\pi_x(x) + \frac{1}{2}\sigma^2\pi_{xx}(x) \Big\},$$

where $\kappa := (1 + \eta r \sigma^2)^{-1} < 1$. (Note: the planner's problem corresponds to $\kappa = 1$)

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Intuition: separability + downward sloping Pareto frontier; but IC requires progress-contingent wages, and a risk-averse agent requires compensation; therefore, the principal cannot pay W_0 and induce efficient effort.

Principal's Problem: Solution

$$
\begin{aligned}
r\pi(x) &= \max_{a \in \mathcal{A}} \Big\{ -\frac{1}{2}\kappa^{-1}a^2 + a\pi_x(x) + \frac{1}{2}\sigma^2\pi_{xx}(x) \Big\} \\
\Rightarrow a^*(x) &= \kappa\pi_x(x) \qquad \text{(FOC)}
\end{aligned}
$$

Theorem (Optimal Contract)

The principal's optimal termination policy is given by two thresholds $\underline{x}$ and $\bar{x}$. These thresholds solve the following ODE

$$
r\pi(x) = \frac{1}{2}\kappa\left[\pi_x(x)\right]^2 + \frac{1}{2}\sigma^2\pi_{xx}(x),
$$

with boundary conditions:

Value Matching $\pi(\bar{x}) = b, \quad \pi(\underline{x}) = s$

Smooth Pasting $\pi_x(\underline{x}) = 0.$

Optimal Contract

"Ceiling" model $\Rightarrow$ optimal contract "retires" the agent at the top and at the bottom

Proposition (Properties of the Optimal Contract)

The principal's value function satisfies the following properties:

- $\pi(x)$ *is increasing* ($\pi_x(x) \geq 0$) *and convex* ($\pi_{xx}(x) \geq 0$)
- *Effort is increasing in* x
- *Agent works harder and longer in the planner's solution*
  - $\underline{x} \geq \underline{x}^{FB}$
  - $a^{FB}(x) \geq a(x)$ *for all* $x \in [\underline{x}, \bar{x}]$
- *The planner succeeds with higher probability*

Principal's Value Function

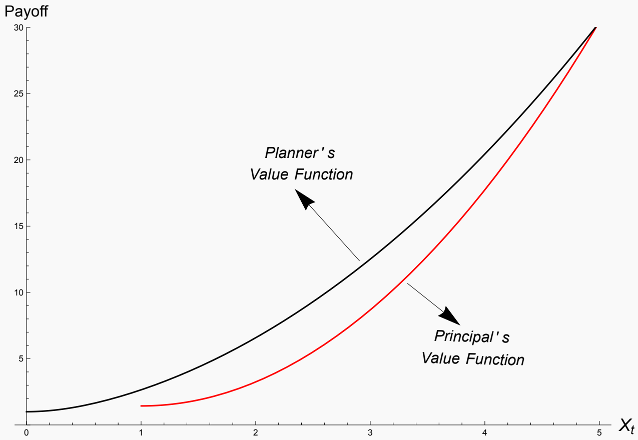
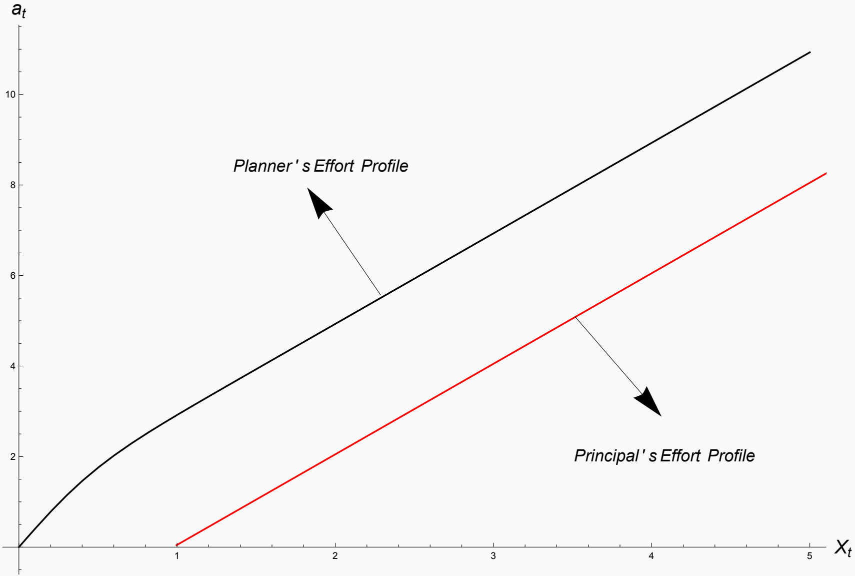


Figure: $(b, \bar{x}, \sigma, s, r, \eta) = (30, 5, 1, 1, 2, 1/2)$

Principal's Effort Profile



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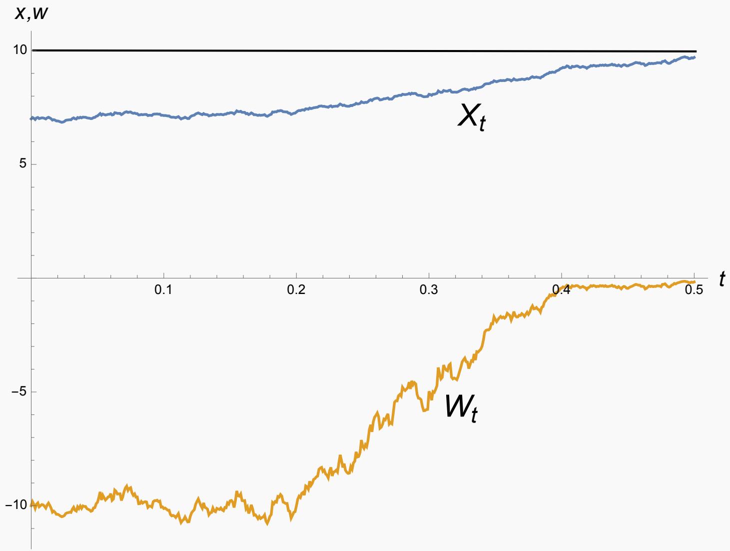
Sample Paths



$$(b, \bar{x}, \sigma, s, r, \eta) = (1, 10, 2, 2, 2, 1/2)$$

Endogenous Project Scope

Suppose now principal can control how ambitious the project is

- endogenize the ceiling; $b(x)$

Smooth pasting needs to hold at the top as well

$$\pi_x(\bar{x}) = b_x(\bar{x})$$

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Distortion at both cut-offs (upper and lower)

- Principal chooses less ambitious projects $\bar{x} < \bar{x}^{FB}$
- Principal terminates earlier $\underline{x} > \underline{x}^{FB}$

Risk Choices

Two dimensional moral hazard

Agent chooses risk level $q_t \in \{0, 1\}$ and the effort a_t

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Two dimensional moral hazard

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Baseline breakdown risk: project terminates at rate λ

$q_t = 1$ implies agent choose risky action

- risk boosts the drift by $g > 0$
- also increases the arrival rate by $\lambda_r > 0$

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Two dimensional moral hazard

Agent chooses risk level $q_t \in \{0, 1\}$ and the effort a_t

Baseline breakdown risk: project terminates at rate λ

$q_t = 1$ implies agent choose risky action

- risk boosts the drift by $g > 0$
- also increases the arrival rate by $\lambda_r > 0$

Formally, project evolves according to following SDE

$$dX_t = \mu(a_t, X_t, q_t)dt + \sigma(X_t)dZ_t - DdN_t,$$

IC with Hidden Risk

Proposition (Continuation Utility)

Agent's continuation utility W under an incentive compatible contract evolves according to following SDE:

$$dW_t = (rW_t - u(a,c))\,dt + \beta_t(-\eta r W_t)(dX_t - (a_t + q_t g)dt) + \psi_t(-\eta r W_t)(dN_t - (\lambda + q_t \lambda_r)\,dt),$$

where β is the process controlling the strength of incentives and ψ is the process controlling the strength of risk taking incentives

IC with Hidden Risk

Proposition (IC for Risk Taking)

The agent chooses the risky regime ($q_t = 1$) if and only if

$$-r\eta W_t \left(\underbrace{g\beta_t}_{boost\ in\ X} + \underbrace{\lambda_r \psi_t}_{boost\ in\ risk} \right) \geq 0 \;\Rightarrow\; \beta_t \geq -\frac{\lambda_r}{g}\psi_t$$

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If she increases β, the principal must also increase the size of the punishment ($\psi < 0$) in order to deter the agent from taking the risky action

Optimal Contract with Hidden Risk

Principal's HJB equation (seperable) can be written as follows:

$$rf(x,w) = \max_{a,c\beta,\psi,q} -c + (a+qg)f_x + (rw - u(a,c) - (\lambda+\lambda_r q)\psi)\, f_w + \frac{1}{2}\beta^2\sigma^2 f_{ww} + (\lambda+\lambda_r q)\,(T(w+\psi) - f(x,w))\,,$$

where $T(w+\psi)$ denotes the termination payoff of the principal when agent has continuation utility of $w+\psi$

Optimal Contract with Hidden Risk

Using No Savings condition + IC for effort

$$rf(x,w) = \max_{a,\psi,q} -\frac{1}{2}a^2 + (a+qg)f_x - (\lambda+\lambda_r q)\psi f_w + \frac{1}{2}a^2\sigma^2 f_{ww} + (\lambda+\lambda_r q)\left(T(w+\psi) - f(x,w)\right),$$

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FOC respect to q,

$$gf_X + \lambda_r\psi + \lambda_r\left(s - f(x,w) + \frac{1}{\eta r}\ln(1-\eta r\psi)\right) \geq 0$$

FOC respect to ψ,

$$(\lambda+\lambda_r)q - (\lambda+\lambda_r)q\frac{1}{1-\eta r\psi} \Rightarrow (\lambda+\lambda_r)q(1-\frac{1}{1-\eta r\psi}) = 0$$

When $q=1$, we have $\psi=0$

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When $q = 1$, we have $\psi = 0 \Rightarrow$ no punishment for risk taking

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IC for risk taking is non binding by construction, $\beta \geq 0$

Optimal Contract with Hidden Risk

When $q = 0$, we have $\psi < 0 \Rightarrow$ punishment for risk taking

IC for risk taking binds

$$\beta = -\frac{\lambda_r}{g}\psi$$

Higher the punishment, higher the $\beta \Rightarrow a$

Contract with Hidden Risk

Theorem (Optimal contract with Hidden Risk)

The optimal contract is characterized by two regions:

- *High risk region* $q_t = 1$, *where* $[\underline{x}, x_c]$
- *Low risk region* $q_t = 0$, *where* $[x_c, \bar{x}]$

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In the high risk region principal's HJB equation solves

$$(r + \lambda + \lambda_r) f(x) = \max_{a \in \mathcal{A}} \left\{ -\frac{1}{2} a^2 + (a + g) f_x - \frac{1}{2} \eta r a^2 \sigma^2 + \frac{1}{2} \sigma^2 f_{xx} + (\lambda + \lambda_r) s \right\},$$

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Boundary condition $f(\underline{x}) = s$

Smooth Pasting $f_x(\underline{x}) = 0$

Switching point $x_c := \inf_{x > \underline{x}} [g f_x(x) + \lambda_r(s - f(x))] = 0$

Contract with Hidden Risk

Theorem (Optimal Contract with Hidden Risk)

In the low risk region principal's HJB equation solves

$$(r+\lambda)f(x) = \max_{a\in\mathcal{A},\,\psi}\left\{-\frac{1}{2}a^2+af_x+\lambda\psi-\frac{1}{2}\eta r a^2\sigma^2+\lambda\left(s+\frac{1}{\eta r}\ln(1-\eta r\psi)\right)+\frac{1}{2}\sigma^2 f_{xx}\right\},$$

where $a = -\frac{\lambda_r \psi}{g}$

Contract with Hidden Risk

Theorem (Optimal Contract with Hidden Risk)

In the low risk region principal's HJB equation solves

$$(r+\lambda)f(x) = \max_{a\in\mathcal{A},\,\psi} \left\{ -\frac{1}{2}a^2 + af_x + \lambda\psi - \frac{1}{2}\eta r a^2\sigma^2 + \lambda\left(s + \frac{1}{\eta r}\ln(1-\eta r\psi)\right) + \frac{1}{2}\sigma^2 f_{xx} \right\},$$

where $a = -\frac{\lambda_r \psi}{g}$

Boundary condition: $f(\bar{x}) = b$

Optimal Contract

Proposition (Properties of the Optimal Contract)

The principal's value function satisfies the following properties:

- $f(x)$ *is increasing* ($f_x(x) \geq 0$) *and convex* ($f_{xx}(x) \geq 0$)
- *Effort is increasing up to* x_c, *jumps down at* x_c, *then keeps increasing*
    - *The planner's effort is increasing and continuous*

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- *Effort is increasing up to* x_c*, jumps down at* x_c*, then keeps increasing*
  - *The planner's effort is increasing and continuous*
- *The agent works harder and longer in the planner's solution*
  - $\underline{x} \geq \underline{x}^{FB}$
  - $a^{FB}(x) \geq a(x)$ *for all* $x \in [\underline{x}, \bar{x}]$
- *The principal induces risk-taking longer than the planner*
  - $x_c \geq x_c^{FB}$

▸ Proof

Value Functions

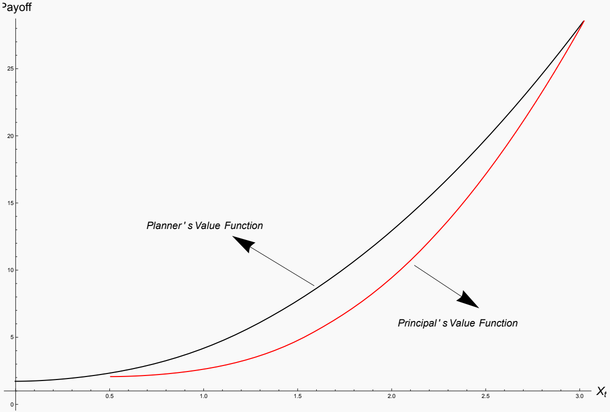

Figure: $(b, \bar{x}, \sigma, s, r, \eta, \lambda, \lambda_r) = (27, 5, 1, 1, 2, 1/2, 3)$

Effort Profiles

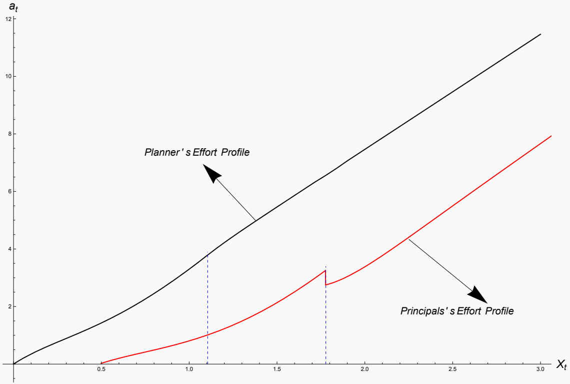

Figure: $(b, \bar{x}, \sigma, s, r, \eta, \lambda, \lambda_r) = (27, 5, 1, 1, 2, 1/2, 3)$

Conclusions

We study the provision of incentives in dynamic project

- Tractable model
- Agent works harder as time passes
- Principal terminates the project before than the designer
- To deter risk taking principal slows down the project

Many possibilities to move forward

- Unobserved progress or success
- Adverse selection

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We study the provision of incentives in dynamic project

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Many possibilities to move forward

- Unobserved progress or success
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This line of research is still far from complete!

DeMarzo, Peter M, Dmitry Livdan, and Alexei Tchistyi (2014) "Risking other people's money: Gambling, limited liability, and optimal incentives," Technical Report 3149, Stanford GSB.

Green, Brett and Curtis R Taylor (2016) "Breakthroughs, deadlines, and self-reported progress: Contracting for multistage projects," *American Economic Review*, 106, 3660–99.

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Proof of Non-monotonicity

Use the fact $a_t = \beta_t = -\frac{\lambda_r}{g}\psi_t$, then FOC respect to ψ when $q_t = 0$

$$-\frac{\lambda_r}{g}\psi - \frac{\lambda_r}{g}f_x - \sigma^2\eta r\left(\frac{\lambda_r}{g}\right)^2\psi + \lambda - \lambda\frac{1}{1-\eta r\psi} = 0$$

Rearranging it we reach

$$a = \kappa\left[f_x(x) - \frac{g}{\lambda_r}\lambda\left(1 - \frac{1}{1+\eta r\frac{g}{\lambda r}a}\right)\right]$$

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Recall when $q_t = 1$

$$a = \kappa\left[f_x(x)\right]$$

Since $f_x(x)$ is continuous at x^c, we have the desired result.

back

Backup

Agent's Problem

Given contract with wages c and recommended effort policy A

$$\max_{\{\hat{c},\hat{a}\}} \mathbb{E}\left[\int_0^T e^{-rt} u(\hat{c}_t, \hat{a}_t) dt\right]$$

subject to

$$\begin{aligned} dX_t &= \hat{a}_t dt + \sigma dZ_t, \quad X_0 = x_0 \\ dS_t &= (rS_t + c_t - \hat{c}_t) dt, \quad S_0 = 0, \quad \lim_{t \to \infty} e^{-rt} S_t = 0 \end{aligned}$$

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w.l.o.g, consider contracts that are IC ($\hat{a}_t = a_t$) + no-savings ($\hat{c}_t = c_t$)

Analysis: Continuation Utility

Let W_t denote the agent's continuation utility at time t

$$W_t = \mathbb{E}\left[\int_t^{\tau} e^{-r(s-t)} u(a_s, c_s) dt + e^{-r(\tau-t)} \frac{u(0, c_\tau)}{r} \,\middle|\, \mathcal{F}_t\right]$$

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Proposition

The agent's continuation utility W under an IC contract (A, C) evolves according to

$$dW_t = (rW_t - u(a_t, c_t))\,dt + \beta_t(-\eta r W_t)(dX_t - a_t dt),$$

where β is the process controlling the strength of incentives

No Savings

Lemma

Consider a deviating agent with saving S who faces contract $\mathcal{C}$ and denote his deviation continuation value at time t by $W_t(S, \mathcal{C})$. It holds that

$$W_t(S, \mathcal{C}) = e^{-\eta r S} W_t(0, \mathcal{C})$$

$W_t(0, \mathcal{C})$ is the agent's continuation value along the no savings path.

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CARA $\Rightarrow$ the agent's problem is translation-invariant to his underlying wealth level

No Savings

Optimality of agent's consumption-savings implies

$$\underbrace{u_c(a,c)}_{marginal\ utility\ of\ consumption} = \underbrace{\frac{d}{dS}W_t(0,\mathcal{C})}_{marginal\ value\ of\ savings}$$

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Therefore, by the above Lemma,

$$u_c(a_t,c_t) = -r\eta W_t \quad \Rightarrow \quad rW_t = u(a_t,c_t)$$

and no savings implies

$$c_t = \frac{1}{2}a_t^2 - \frac{1}{\eta}\ln(-\eta r W_t)$$

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Continuation utility becomes a martingale

$$dW_t = \beta(-\eta r W_t)\,(dX_t - a_t dt)$$

No Savings + IC

Agent's IC

$$\max_{\hat{a}} \; [-\hat{a}\beta\eta r W + u(\hat{a}, c)]$$

Then FOC implies

$$\hat{a} u_a(\hat{a}, c) = \beta\eta r W$$

Using the fact that $\hat{a} u_a(\hat{a}, c) = u_c(\hat{a}, c)$ and $u_c(a, c) = -\eta r W$

That implies

$$\hat{a} = \beta$$

HJB Equation

Let $v(x, w)$ denotes the principal's value function

Using No Savings condition we write

$$
\begin{aligned}
v(x,w) = \max_{a\in\mathcal{A},\beta} \Bigg[-\frac{1}{2}a^2 \quad &+ \quad \frac{1}{\eta}\ln(-\eta r W_t) + a v_x + \frac{1}{2}\,(-\beta\eta r W_t)^2 v_{ww}(x,w) \\
&- \quad \beta\eta r W_t \sigma v_{xw}(x,w) + \frac{1}{2}\sigma^2 v_{xx}(x,w) \Bigg]
\end{aligned}
$$

IC with Hidden Risk

Proposition (Continuation Utility)

Agent's continuation utility W under an incentive compatible contract evolves according to following SDE:

$$dW_t = (rW_t - u(a,c))\,dt + \beta_t(-\eta r W_t)(dX_t - (a_t + q_t g)dt) + \psi_t(-\eta r W_t)(dN_t - (\lambda + q_t \lambda_r)\,dt),$$

where β is the process controlling the strength of incentives and ψ is the process controlling the strength of risk taking incentives

IC with Hidden Risk

Proposition (IC for Risk Taking)

The agent chooses the risky regime ($q_t = 1$) if and only if

$$-r\eta W_t \left(\underbrace{g\beta_t}_{\textit{boost in } X} + \underbrace{\lambda_r \psi_t}_{\textit{boost in risk}} \right) \geq 0 \;\Rightarrow\; \beta_t \geq -\frac{\lambda_r}{g}\psi_t$$

IC with Hidden Risk

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As β increases in order to deter the agent taking risky action principal needs to increase size of the punishment (negative ψ)

Optimal Contract with Hidden Risk

Principal's HJB equation (seperable) can be written as follows:

$$rf(x,w) = \max_{a,c\beta,\psi,q} -c + (a+qg)f_x + (rw_t - u(a,c) - (\lambda + \lambda_r q)\psi_t)\, f_w + \frac{1}{2}\beta^2\sigma^2 f_{ww} + (\lambda + \lambda_r q)\,(T(w+\psi) - f(x,w))\,,$$

where $T(w+\psi)$ denotes the termination payoff of the principal when agent has continuation utility of $w+\psi$

Optimal Contract with Hidden Risk

Using no savings condition + IC for effort

$$rf(x,w) = \max_{a,\psi,q} -\frac{1}{2}a^2 + (a+qg)f_x - (\lambda+\lambda_r q)\psi_t f_w + \frac{1}{2}a^2\sigma^2 f_{ww} + (\lambda+\lambda_r q)\left(T(w+\psi) - f(x,w)\right),$$

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FOC respect to q,

$$gf_X + \lambda_r\psi_t + \lambda_r\left(s - f(x,w) + \frac{1}{\eta r}\ln\left(1-\eta r\psi_t\right)\right) \geq 0$$

FOC respect to ψ,

$$(\lambda+\lambda_r)q - (\lambda+\lambda_r)q\frac{1}{1-\eta r\psi} \Rightarrow (\lambda+\lambda_r)q(1-\frac{1}{1-\eta r\psi}) = 0$$

When $q=1$, we have $\psi = 0$

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$$+ (\lambda+\lambda_r q)\left(T(w+\psi) - f(x,w)\right),$$

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When $q = 1$, we have $\psi = 0 \Rightarrow$ no punishment for risk taking

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IC for risk taking is non binding by construction, $\beta_t \geq 0$

Optimal Contract with Hidden Risk

When $q = 0$, we have $\psi < 0 \Rightarrow$ punishment for risk taking

IC for risk taking binds

$$\beta_t = -\frac{\lambda_r}{g}\psi_t$$

Higher the punishment, higher the $\beta_t \Rightarrow a_t$