Algorithms for Computing Equilibrium Payoffs in Quitting Games*

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Abstract

Characterizing and explicitly computing equilibria of undiscounted dynamic games has been a challenge for many years. In this paper we look at *quitting games*, which are stopping games where the terminal payoff does not depend on the stage of termination. We develop several practical algorithms that compute different classes of subgame perfect equilibria. Our algorithms are based on the novel representation of strategy profiles through *absorption paths*, which was developed in Ashkenazi-Golan, Krasikov, Rainer, and Solan (2021). The baseline algorithm deals with absorption paths in which exactly one player randomizes between quitting and continuing at any point in time. Two additional algorithms extend the baseline algorithm by allowing for multiple players to randomize at the same time. Since quitting games are special case of both stopping games and stochastic games, our approach may be useful in studying more general classes of stopping games and stochastic games.

Keywords: Stochastic games, quitting games, continuous equilibria, perfect monitoring, algorithm, computation, monotone convex set valued operators.
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1 Introduction

Two classes of dynamic games that have been extensively studied in the literature are stochastic games (Shapley, 1953) and stopping games (Dynkin, 1967; Neveu, 1975). In

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stochastic games, the stage payoff depends on a state variable as well as on the actions of the players at that stage, and the state changes as a function of the players' actions. In stopping games, each player decides when to stop the interaction, and the game terminates once the first player decides to stop. These games have diverse applications. Stochastic games have been used to study, e.g., capital accumulation (Levhari and Mirman, 1980, Dutta and Sundaram, 1992, 1993, Amir, 1996, Nowak, 2003c), taxation (Chari and Kehoe, 1990, Phelan and Stacchetti 2001), communication network (Sagduyu and Ephremides, 2003), and queues (Altman and Hordijk, 1995, Altman, 2005), and stopping games have been applied to, e.g., market exit (Ghemawat and Nalebuff, 1985), product innovation and asset sale (Dutta and Rustichini, 1993), and pricing of game options (Kifer, 2000).

When players discount their payoffs, both discounted games and stopping games admit equilibria under fairly general conditions (see, e.g., Fink (1964), Takahashi (1964), Mertens and Parthasarathy (1987), Ferenstein (2007), Jaskiewicz and Nowak (2017)), and undiscounted equilibrium is known to exist only under restricted conditions (for stochastic games, see, e.g., Vrieze and Thuijsman (1989), Solan (1999), Vieille (2000a, 2000b), Simon (2007, 2012), and Flesch, Schoenmakers, and Vrieze (2008, 2009), and for stopping games, see, e.g., Shmaya and Solan (2004) and Laraki and Solan (2005)). Hörner, Rosenberg, Takahashi, and Vieille (2011) proved a Folk Theorem that characterizes the set of discounted equilibrium payoffs in stochastic games with finitely many states and actions.

The problem of calculating an equilibrium also received a lot of attention. Algorithm for approximating discounted equilibria have been developed, see, e.g., Vrieze and Tijs (1982), Breton (1991), Filar and Vrieze (1996), and Herings and Peeters (2004). Algorithms for approximating ε-optimal strategies in two-player zero-sum undiscounted stochastic games with finite sets of states and actions were developed by Chatterjee, Majumdar, and Henzinger (2008), Solan and Vieille (2010), and Oliu-Barton (2021). Efficient algorithms for computing equilibria have been devised for specific classes of zero-sum games and for non-zero-sum stochastic games in which a single player controls the transitions, see, e.g., Breton (1991), Filar and Raghavan (1984), Nowak and Raghavan (1993), Raghavan (2003), and Bourque and Raghavan (2014).

A class of games that lies in the intersection of stochastic games and stopping games is the class of *quitting games*. Quitting games are stopping games where the terminal payoff depends only on the set of players who quit at the termination stage, and not on the stage of termination. Thus, quitting games are stochastic games with a single non-absorbing state.

Flesch, Thuijsman, and Vrieze (1997) studied a specific three-player quitting game and characterized the set of its equilibria. Solan and Vieille (2001) formally defined the family of quitting games and provided a sufficient condition that guarantees the existence of an ε-equilibrium. Solan (2001) and Solan and Vieille (2002) exhibit three- and four-player quitting games, respectively, which show that ε-equilibria in quitting games may have complex structure. Further sufficient conditions for the existence of ε-equilibrium in quitting games have been provided by Simon (2007, 2012), Solan and Solan (2020), and Ashkenazi-Golan, Krasikov, Rainer, and Solan (2021). To date it is not known whether all four-player quitting games admit ε-equilibria.

In this paper we provide the first practical algorithm for computing a class of undiscounted subgame-perfect equilibrium payoffs in multiplayer quitting games. To describe the extent of our algorithm, we recall a new representation of strategy profiles called *absorption paths*, defined in Ashkenazi-Golan, Krasikov, Rainer, and Solan (2021) (AKRS for short). This representation allows for both discrete-time aspects and continuous-time aspects in the players' behavior, and it involves parametrizing time according to the accumulated probability of absorption. In the new representation, discrete-time aspects capture stages where at least one player quits with probability that is bounded away from 0, and continuous-time aspects capture stages where all players quit with small probability.

AKRS introduced a notion of subgame perfectness, that applies to absorption paths, which they called sequential perfectness to distinguish it from subgame perfectness that applies to strategy profiles, and shows that the set of payoffs that correspond to sequentially perfect absorption paths coincides with the set of subgame-perfect equilibrium payoffs, namely, limits of subgame-perfect ε-equilibrium payoffs as ε goes to 0.

We will provide an algorithm that calculates the set of payoffs, which correspond to sequentially perfect absorption paths that involve only continuous-time aspects. This set corresponds to the limit set of subgame-perfect ε-equilibrium payoffs, where along the corresponding subgame-perfect ε-equilibria, players quit with low probability.

The paper is organized as follows. The model is described in Section 2. A naive approach to computing the set of undiscounted equilibrium payoffs appears in Section 3. The more sophisticated approach using the *Essential APS* operator, is described in Section 4. Extensions of the algorithm to equilibria in which a set of players are allowed to randomize quitting in continuous time are presented in Section 5. Discussion on possible extensions of our results to all equilibrium payoffs, as well as conclusions and final remarks, appear in Section 6.

2 Model

Definition 2.1 *A quitting game is a pair* $\Gamma = (I, r)$*, where* I *is a finite set of players and* $r : \prod_{i\in I}\{C^i, Q^i\} \to \mathbb{R}^I$ *is a payoff function.*

Player i's action set is $A^i := \{C^i, Q^i\}$. These actions are interpreted as continue and quit, respectively. Denote by $A := \prod_{i\in I} A^i$ the set of action profiles. The game is played as follows. Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ be the set of non-negative integers. At every stage $n \in \mathbb{N}$ each player $i \in I$ chooses an action $a^i_n \in A^i$. If all players continue, the play continues to the next stage; if at least one of them quits, the play terminates, and the terminal payoff is $r(a_n)$, where $a_n = (a^i_n)_{i\in I}$. If no player ever quits, the payoff is $r(\vec{C})$, where $\vec{C} := (C^i)_{i\in I}$. It is convenient to normalize the payoff function so that $r_i(Q^i, C^{-i}) = 0$ for every $i \in I$.

We denote by $A^* := A \setminus \{\vec{C}\}$ the set of all action profiles in which at least one player quits, by $A^*_1 := \{(Q^i, C^{-i}), i \in N\}$ the set of all action profiles in which exactly one player quits, where $C^{-i} := (C^j)_{j\neq i}$, and by $A^*_{\geq 2} := A^* \setminus A_1$ the set of all action profiles in which at least two players quit.

A *mixed action profile* is a vector $\xi = (\xi^i)_{i\in I} \in [0, 1]^I$, with the interpretation that ξ^i is the probability with which player i quits. The probability of absorption under the mixed action profile ξ is $p(\xi) := 1 - \prod_{i\in I}(1 - \xi^i)$. Extend the absorbing payoff to mixed action profiles that are absorbing with positive probability: for every $\xi \in [0,1]^I$ such that $p(\xi) > 0$, define $r(\xi) := \frac{\sum_{a \in A^*} \xi(a) r(a)}{p(\xi)}$, where $\xi(a) := \left(\prod_{\{i:\, a^i = Q^i\}} \xi^i\right) \cdot \left(\prod_{\{i:\, a^i = C^i\}} (1 - \xi^i)\right)$, for every $a \in A$.

A (behavior) *strategy* of player i is a function $x^i = (x^i_n)_{n \in \mathbb{N}} : \mathbb{N} \to [0,1]$, with the interpretation that x^i_n is the probability that player i quits at stage n if the game did not terminate before that stage. A *strategy profile* is a vector $x = (x^i)_{i \in I}$ of strategies, one for each player.

Denote by $\theta := \inf\{n \in \mathbb{N} \colon a_n \in A^*\}$ the stage of termination; $\theta = \infty$ if all players continue throughout the game. Every strategy profile x induces a probability distribution $\mathbf{P}_x$ over the set of plays. Denote by $\mathbf{E}_x$ the corresponding expectation operator. A strategy profile x is *absorbing* if $\mathbf{P}_x(\theta < \infty) = 1$.

The *payoff* under strategy profile x is

$$\gamma(x) := \mathbf{E}_x \left[\mathbf{1}_{\{\theta < \infty\}} r(a_\theta) + \mathbf{1}_{\{\theta = \infty\}} r(\vec{C}) \right].$$

Let $\varepsilon \geq 0$. A strategy profile x^* is an *ε-equilibrium* if $\gamma^i(x^*) \geq \gamma^i(x^i, x^{*,-i}) - \varepsilon$ for every player $i \in I$ and every strategy x^i of player i. A strategy profile x^* is a *subgame-perfect ε-equilibrium* if for every $n \in \mathbb{N}$, the strategy profile $(x^*_n, x^*_{n+1}, \dots)$ is an ε-equilibrium. When $r^i(\vec{C}) < r^i(Q^i, C^{-i}) = 0$ for some $i \in I$, any subgame-perfect ε-equilibrium is *absorbing*, provided ε is small enough. A payoff vector $w \in \mathbb{R}^I$ is a *subgame-perfect equilibrium payoff* if $w = \lim_{\varepsilon \to 0} \gamma(x^\varepsilon)$, where x^ε is a subgame-perfect ε-equilibrium for every $\varepsilon > 0$, and it is said to be *absorbing* if x^ε is an absorbing strategy profile for every $\varepsilon > 0$.

Three-player quitting games admit subgame-perfect equilibrium payoffs (Solan, 1999). It is not known whether the same applies to quitting games with at least four players. Sufficient conditions that guarantee the existence of a subgame-perfect equilibrium payoff have been provided by Solan and Vieille (2002), Simon (2007, 2012), Solan and Solan (2020), and AKRS. The latter also provided a characterization of the set of limits of absorbing subgame-perfect ε-equilibrium strategy profiles using a novel concept of *absorption paths.* To date there is no algorithm that allows to compute the set of subgame-perfect equilibrium payoffs.

2.1 Motivating example

Suppose that (x^ε) is a sequence of absorbing subgame-perfect ε-equilibria, and suppose that $w = \lim_{\varepsilon \to 0} \gamma(x^\varepsilon)$ exists. To study subgame-perfect equilibrium payoffs one is tempted to study the limit strategy profile x^0 that is defined by $x^0_n := \lim_{\varepsilon \to 0} x^\varepsilon_n$ for every $n \in \mathbb{N}$ (assuming this limit exists). If x^0 is absorbing, then it follows from results in Vrieze and Thuijsman (1989) or Solan (1999) that it is a subgame-perfect 0-equilibrium. However, it might happen that while going to the limit as ε goes to 0, some probability of absorption is lost, and then the limit x^0 is not necessarily a subgame-perfect 0-equilibrium (or even an ε-equilibrium, for $\varepsilon > 0$ sufficiently small). In fact, the following example shows that even the limit of subgame perfect 0-equilibira is not necessarily an ε-equilibrium.

Example 2.2 (Flesch, Thuijsman, and Vrieze (1997)) *Consider the three-player quitting game where the payoff function r is given by the table in Figure 1.*

C^3:

	C^2	Q^2
C^1	$-1,-1,-1$	$-1,0,2$
Q^1	$0,2,-1$	$0,-1,0$

Q^3:

	C^2	Q^2
C^1	$2,-1,0$	$0,0,-1$
Q^1	$-1,0,0$	$-1,-1,-1$

Figure 1: The three-player game in Example 2.2.

We note that $r_i(\vec{C}) = -1 < 0 = r_i(Q^i, C^{-i})$ for all $i \in I$, and therefore every subgame-perfect ε-equilibrium must be absorbing, for all $\varepsilon < 1$.

For each $k \in \mathbb{N}$, let $\delta_k = 1 - (\frac{1}{2})^{1/k}$, so that $(1-\delta_k)^k = \frac{1}{2}$. Consider the strategy profile x^k, which repeats the following block of $3k$ action profiles:

- *In stages $1, 2, \ldots, k$ the players play $(\delta_k, 0, 0)$.*
- *In stages $k+1, k+2, \ldots, 2k$ the players play $(0, \delta_k, 0)$.*
- *In stages $2k+1, 2k+2, \ldots, 3k$ the players play $(0, 0, \delta_k)$.*

By Flesch, Thuijsman, and Vrieze (1997), x^k is a subgame-perfect 0-equilibrium for every $k \in \mathbb{N}$. Since $\delta_k \searrow 0$, the limit of x^k as k goes to 0 is the strategy profile in which all players always continue, which is not even an ε-equilibrium for $\varepsilon \in (0,1)$.

To overcome the difficulty pointed out in Example 2.2, AKRS presented the concept of absorption paths, which does not allow probability of absorption to be lost when taking the limit as ε goes to 0. More precisely, they show that, by re-parametrizing time (originally *equal to* $\mathbb{N}$) by the non-decreasing probability of absorption, the set of all strategy profiles can be embedded in a sequentially compact set of continuous-time, A^*-valued paths, that contains also the limit case where players quit during some determined times with infinitesimal probabilities. The main result of AKRS is that w is an absorbing subgame-perfect 0-equilibrium payoff if and only if there exists an absorption path satisfying a certain notion of subgame perfectness and w is the expected payoff under this absorption path.

We refer to the original paper for an extensive treatment of this approach. Here we focus on strategy profiles where at each time only one player quits with vanishing probability, like in Example 2.2.

In Example 2.2, for each $k \in \mathbb{N}$, the strategy profile x^k can be described as follows: for each $n \geq 1$,

- set $t_n^k := \mathbf{P}_{x^k}(\theta < n)$,
- set $\iota^k(t) := i$ whenever $t \in [t_n^k, t_{n+1}^k)$ and $x_n^{k,i}$, player i's probability of quitting under x^k at stage n, is positive.

Remark that the functions $\iota^k, k \in \mathbb{N}$, -as well as their limit as k tends to $+\infty$- are all equal to the function ι which is piecewise constant and equals to $n+1 \mod (3)$ on the interval $[1-\frac{1}{2^n}, 1-\frac{1}{2^{n+1}})$, for every $n \in \mathbb{N}$.

More generally, in the following section we shall introduce a set of right-continuous I-valued functions ι, that we call *Flesch absorption paths.* A Flesch absorption path is a special case of absorption paths where only continuous-time aspects appear, and where at every time instance, at most one player quits with positive rate. The main result of the paper is an algorithm for computing the set of absorbing subgame-perfect equilibrium payoffs that can be attained by Flesch absorption paths.

In Section 5.2 we show how our approach can be used to go beyond the class of Flesch absorption paths, and characterize the set of absorbing subgame-perfect equilibrium payoffs that can be attained by continuous absorption paths, i.e., those in which multiple players quit in continuous time throughout the play.

2.2 Flesch Absorption Paths

As mentioned in the introduction, we will characterize a set of equilibrium payoffs in the game in discrete time where the per-stage probability of quitting is small. To this end, it will be convenient to study the quitting game in continuous time. In this section we define a concept of strategies that is suitable for quitting games in continuous time. In these strategies, time is divided into intervals, and in each interval a single player quits. In Section 5 we will extend this concept, and allow more than one player to quit with positive rate at each time instance. As shown in AKRS, this concept is useful for studying subgame-perfect ε-equilibrium payoffs in quitting games in discrete time.

Definition 2.3 *A Flesch Absorption Path (FAP) is a right-continuous map $\iota : [0,1) \to I$, such that the set of its discontinuities including 0, denoted by $H(\iota)$, is countable and well-ordered.*

The interpretation of an FAP is as follows. Quitting occurs in continuous time, and at every $t \in [0,1)$, a single player $\iota(t)$ quits with rate one. The parameter t does not represent time but rather the total probability of absorption. For each $t \in H(\iota)$, the *successor* of $\iota(t)$ is the minimal element in $H(\iota)$ larger than t, which exists since $H(\iota)$ is well ordered.

Example 2.4 *The FAP ι that is defined by $\iota([0,1)) = 1$ corresponds to the behavior that player 1 is the only player who quits. The FAP that corresponds to a situation where players 1 and 2 alternately quit in continuous time, each with probability $\frac{1}{2}$, is given by*

$$\iota(t) = \begin{cases} 1, & t \in [0,\frac{1}{2}) \cup [\frac{1}{4},\frac{1}{8}) \cup \cdots, \\ 2, & t \in [\frac{1}{2},\frac{1}{4}) \cup [\frac{1}{8},\frac{1}{16}) \cup \cdots. \end{cases}$$

More generally, suppose that $H(\iota) = (t_n)_{n\in\mathbb{N}}$ where $0 = t_0 < t_1 < t_2 < \cdots$, and $\lim_{n\nearrow\infty} t_n = 1$. This FAP corresponds to a situation where first player $\iota(t_0)$ quits with probability $\frac{t_1-t_0}{1-t_0} = t_1$, then player $\iota(t_1)$ quits with probability $\frac{t_2-t_1}{1-t_1}$, and so on.

Remarks 2.5 *Let ι be an FAP as defined above.*

1. *A point $t \in (0,1)$ (resp. $t \in [0,1)$) is a* left accumulation point *(resp.* right accumulation point*) of $H(\iota)$ if there is a sequence $(t_n)_{n\in\mathbb{N}}$ of points in $H(\iota)$ that increases (resp. decreases) to t. Left accumulation points of $H(\iota)$ allow describing players' behavior where they quit with arbitrarily small probabilities. The assumption that $H(\iota)$ is well-ordered excludes the existence of right accumulation points.*

2. *The assumption that ι is right-continuous guaranties that, for each connected component (t_1, t_2) of $[0,1) \setminus H(\iota)$, the value of ι at t_1 is the same as on (t_1, t_2).*

3. *The set of FAPs is not compact. Indeed, one can devise a sequence of FAPs such that, in their natural limit, two players quit simultaneously in continuous time. For example, for each $k = 2, 3, \ldots$, let ι^k be the FAP defined by $\iota^k(t) := tk + 1 \mod (2)$ so that $H(\iota^k) = \{0, \frac{1}{k}, \ldots, \frac{k-1}{k}\}$. The FAP ι^k corresponds to a situation where player 1 quits with probability $\frac{1}{k}$, then player 2 quits with probability $(\frac{1}{k})/(\frac{k-1}{k}) = \frac{1}{k-1}$, then player 1 quits with probability $\frac{1}{k-2}$, and so on.*

   The natural limit of the sequence of these FAPs is the strategy profile in continuous time in which both players quit simultaneously throughout the game at the same rate, yet this behavior cannot be described by an FAP.

4. *For each $i \in I$, the total probability that player i quits in the interval $[a, b) \subset [0,1)$ is* $\mathrm{Leb}(\{s \in [a,b) | \iota(s) = i\})$*, where* Leb *is the Lebesgue measure. In particular,* $\mathrm{Leb}(\{s \in [t,1) | \iota(s) = i\})$ *for $t \in [0,1)$ is the total probability that player i quits after time t.*

We can then define the expected payoff under an FAP.

Definition 2.6 *For every $t \in [0,1)$, the* expected payoff *after absorption probability t is given by the following:*

$$\gamma_t(\iota) := \sum_{i\in I} \frac{\mathrm{Leb}(\{s \in [t,1) \mid \iota(s) = i\})}{1-t} \cdot R_i, \tag{1}$$

where $R_i := r(Q^i, C^{-i})$ is the payoff when player i quits alone.

Since FAPs model behavior in continuous time, the payoff vector $\gamma_t(\iota)$ depends only on the payoffs when players quit alone. We let R be the *payoff $(|I| \times |I|)$-matrix of single quittings* whose i'th row is R_i. As we normalized payoffs so that $R_{i,i} = 0$ for each $i \in I$, the diagonal of R is $\vec{0} = (0, 0, \ldots, 0)$. We further assume that the matrix R is *generic* in the following sense.

Assumption 2.7 (Genericity of payoffs) *The quitting game $\Gamma = (I, r)$ satisfies the following genericity assumption:*

$$R_{i,j} = 0 \iff i = j.$$

A strategy profile is a subgame-perfect ε-equilibrium if in any subgame no player can profit more than ε by deviating. When time is continuous, players cannot quit simultaneously, hence this requirement translates into two conditions: *i*) a player who quits with positive rate is indifferent between quitting and continuing and *ii*) a player who quits with rate 0 cannot profit by quitting. This leads to the following definition of sequential perfectness for FAPs, which is adapted from AKRS.

Definition 2.8 (AKRS) *An FAP ι is* sequentially perfect *if for every $t \in [0,1)$, $\gamma_t(\iota) \geq \vec{0}$ and $\gamma_t^i(\iota) = 0$ whenever $\iota(t) = i$.*

Remarks 2.9 *Let ι be an FAP.*

1. *Set $i = \iota(0)$ and suppose that $t := \inf\{s \in (0,1) | \iota(s) \neq i\} < 1$. Then,*

$$\gamma_0(\iota) = tR_i + (1-t)\gamma_t(\iota).$$

   In words, this equation says that the payoff (from $t = 0$ and on) is equal to the probability that player $\iota(0)$ quits at $t = 0$ times the payoff if that player quits, plus the probability that player $\iota(0)$ does not quit at $t = 0$ times the continuation payoff.

2. *Given $t \in [0,1)$, the FAP induced by ι in the subgame starting at t is the FAP $\iota^t : [0,1) \to I$ defined by $\iota^t(s) := \iota(t + (1-t)s)$.*
   - *It holds that $\gamma_t(\iota) = \gamma_0(\iota^t)$.*
   - *If ι is sequentially perfect then so is ι^t.*

Denote the *set of sequentially perfect FAPs* by Υ, and let $\mathcal{E}$ be the set of payoffs that can be attained by them, that is

$$\mathcal{E} := \left\{ w \in \mathbb{R}^I \mid \exists\, \iota \in \Upsilon \text{ s.t. } w = \gamma_0(\iota) \right\}.$$

We seek to characterize and compute $\mathcal{E}$.

3 APS approach

In this section we naively apply the approach developed in Abreu, Pearce, and Stacchetti (1986), and bound $\mathcal{E}$ by the union of largest invariant sets of a certain monotone operator.

Let $\mathcal{R}_I$ be the set of all non-negative payoffs that can be attained by FAPs, not necessarily sequentially perfect. By definition, $\mathcal{R}_I$ is an upper bound on $\mathcal{E}$. For later references, it is convenient to define also, for each non-empty subset of players $N \subseteq I$, the set of non-negative payoffs $\mathcal{R}_N$ that can be generated by FAPs in which only players in N can ever quit, that is,

$$\mathcal{R}_N := \left\{ w \in \mathbb{R}^I_+ \mid \exists \text{ FAP } \iota \text{ s.t. } w = \gamma_0(\iota), \iota([0,1)) \subseteq N \right\}, \ \forall N \subseteq I.$$

The following lemma says that $\mathcal{R}_N$ equals the set of non-negative convex combinations of $(R_i)_{i \in N}$.

Lemma 3.1 *For each non-empty $N \subseteq I$, we have*

$$\mathcal{R}_N = co\{R_i, i \in N\} \cap \mathbb{R}^I_+,$$

where co denotes the convex hull. In particular, for any $i \in I$, $\mathcal{R}_{\{i\}} = \{R_i\}$ if $R_i \geq \vec{0}$ and $\emptyset$, otherwise.

Proof. The "$\subseteq$" inclusion follows from the definition of $\mathcal{R}_N$ and Eq. (1).

Conversely, let $w \in co\{R_i, i \in N\} \cap \mathbb{R}^I_+$. By definition, there exists $\lambda \in \mathbb{R}^N_+$ such that $\sum_{i \in N} \lambda_i = 1$ and $\sum_{i \in N} \lambda_i R_i = w$. Define $(t_n)_{n=0}^{|N|}$ by $t_0 := 0$, $t_n := t_{n-1} + \lambda_n$ for $n = 1, \ldots, |N|$. Set $\iota(t) := n$ whenever $t \in [t_{n-1}, t_n)$. By Eq. (1), $\gamma_0(\iota) = w$, and therefore $w \in \mathcal{R}_N$. ■

Remark 3.2 *It is evident that $\mathcal{E} \subseteq \mathcal{R}_I$. On the other hand, for any $i \in I$, we have $\mathcal{R}_{\{i\}} \subseteq \mathcal{E}$. Indeed, if $R_i \in \mathbb{R}^I_+$, then the FAP ι with $\iota(t) = i$ for all $t \in [0,1)$ belongs to Υ; and, if $R_i \notin \mathbb{R}^I_+$, then $\mathcal{R}_{\{i\}} = \emptyset$ and the assertion trivially holds. In general, it is not true that $\mathcal{E} = \mathcal{R}_I$. For instance, consider the three-player game in Example 2.2. In this game, $\mathcal{R}_I$ is the triangle whose extreme points are $(0,0,1)$, $(0,1,0)$, and $(1,0,0)$, while $\mathcal{E}$ is the boundary of this triangle as shown in Flesch, Thuijsman, and Vrieze (1997).*

The following lemma recursively unpacks the set $\mathcal{E}$. It will serve as a basis for our algorithm.

Lemma 3.3

$$\mathcal{E} = \left\{ w \in \mathbb{R}^I_+ \mid \exists\, (\lambda, i, \iota) \in [0,1] \times I \times \iota \in \Upsilon \ \textit{s.t.}\ w = \lambda R_i + (1-\lambda)\gamma_0(\iota), w_i = 0 \right\}. \qquad (2)$$

The lemma states that if $w \in \mathcal{E}$, then (a) $w_i = 0$ for some player i, and (b) w is a convex combination of the payoff if player i quits alone and the payoff induced by some sequentially perfect FAP.

Proof. Taking $\lambda = 0$ and $i = \iota(0)$ in the right side of relation (2) makes it clear that it contains $\mathcal{E}$.

Conversely, let $\lambda \in (0,1]$, $i \in I$, and $\iota \in \Upsilon$ such that $w = \lambda R_i + (1-\lambda)\gamma_0(\iota)$ and $w_i = 0$. If $\lambda = 1$, then $w = R_i \geq \vec{0}$. Thus, w is attained by the FAP $\iota \in \Upsilon$ with $\iota(t) = i$ for all $t \in [0,1)$. If $\lambda \neq 1$, then define a new FAP ι' by setting

$$\iota'(t) := \begin{cases} i, & \text{if } t \in [0, \lambda), \\ \iota\left(\frac{t-\lambda}{1-\lambda}\right), & \text{if } t \in [\lambda, 1). \end{cases}$$

Clearly, $\iota' \in \Upsilon$ and $w = \gamma_0(\iota')$, and thus $w \in \mathcal{E}$. ■

In what follows we shall construct the *APS operator* and show that the set $\mathcal{E}$ is invariant with respect to it. For every subset $E \subseteq \mathcal{R}_I$, define the set of payoffs $\mathbf{T}(E)$ that can be attained with continuation payoffs in E, that is,

$$\mathbf{T}(E) := \left\{ w \in \mathbb{R}^I_+ \mid \exists\, (\lambda, i, \iota) \in [0,1] \times I \times E, \text{ s.t. } w = \lambda R_i + (1-\lambda)v, w_i = 0 \right\}. \qquad (3)$$

We list below several useful properties of the operator $\mathbf{T}$.

Remarks 3.4 *Let* $E \subseteq \mathcal{R}_I$.

1. *If* E *is closed, then* $\mathbf{T}(E) \subseteq \mathbb{R}^I$ *is closed as well.*
2. $\mathbf{T}(E) \subseteq \mathbf{T}(E')$ *for every* E' *such that* $E \subseteq E' \subseteq \mathcal{R}_I$, *i.e.,* $\mathbf{T}$ *is monotone in the set-inclusion order.*
3. *Lemma 3.3 implies that the set* $\mathcal{E}$ *is invariant for the operator* $\mathbf{T}$.

Since $\mathcal{E}$ is invariant for $\mathbf{T}$ and included in $\mathcal{R}_I$, we can follow Abreu, Pearce, and Stacchetti (1986) and bound $\mathcal{E}$ from above by the *largest invariant set* $\overline{\mathcal{E}}$ with respect to $\mathbf{T}$ within $\mathcal{R}_I$. The existence of this largest invariant set is guaranteed by Knaster-Tarski's Theorem (Knaster (1928), Tarski (1955)), which also suggests an algorithm for computing it: $\overline{\mathcal{E}}$ can be obtained by repeatedly applying the operator $\mathbf{T}$ to $\mathcal{R}_I$, that is

$$\overline{\mathcal{E}} = \bigcap_{n=0}^{\infty} \mathbf{T}^n(\mathcal{R}_I), \tag{4}$$

where $\mathbf{T}^n$ is the n-th application of the operator $\mathbf{T}$. Unfortunately, it turns out that the iterations in Eq. (4) always terminate in a single step with $\overline{\mathcal{E}} = \mathcal{R}_I$. As already mentioned, there is no reason to expect that all these payoffs can be attained by a sequentially perfect FAP. For instance, $\mathcal{E} = \emptyset$ but $\mathcal{R}_I \neq \emptyset$ in the following example.

Example 3.5 *Consider the quitting game with four players and the following payoff matrix of single quittings:*

$$R = \begin{pmatrix} 0 & 4 & -\frac{1}{2} & -1 \\ -1 & 0 & 3 & 1 \\ -\frac{1}{8} & -1 & 0 & 4 \\ 4 & \frac{1}{2} & -1 & 0 \end{pmatrix}.$$

The reader can verify that the set $\mathcal{R}_I$ *is the convex hull of* $(0, 0, \frac{173}{87}, \frac{95}{87})$, $(\frac{173}{88}, 0, 0, \frac{14}{11})$, $(0, \frac{1799}{678}, 0, 0)$, $(0, 0, 0, \frac{1799}{635})$, $(0, \frac{11}{6}, 1, 0)$, $(2, \frac{7}{6}, 0, 0)$, $(\frac{11}{4}, \frac{3}{8}, 0, \frac{1}{4})$ *and* $(0, \frac{1}{10}, \frac{11}{5}, \frac{4}{5})$. *Yet, the set* $\mathcal{E}$ *is empty, a fact that will be established on page 20.*

To sum up, the naive APS approach, which is inspired by the classical recursive algorithm of Abreu, Pearce, and Stacchetti (1986), is not applicable in our undiscounted setting — even though the set $\mathcal{E}$ is invariant with respect to $\mathbf{T}$, it might differ from the largest invariant sets $\overline{\mathcal{E}}$. To the best of our knowledge, there is no algorithm that can be used to compute some invariant sets of $\mathbf{T}$, except the largest (and smallest) ones.

4 Essential APS Approach

In this section we modify the APS operator and construct an alternative operator, which we term *essential APS*. We then bound $\mathcal{E}$ by the union of largest invariant sets of this essential APS operator. Finally, we provide a condition for this bound to be tight.

4.1 Graph of play

Recall that every FAP ι indicates for each time instance a player who quits with a positive rate. The following lemma shows that not all quitting orders are compatible with sequential perfectness. Specifically, it establishes that the *set of player i's potential successors S_i* is given by

$$S_i := \{j \in I \mid R_{j,i} < 0 < R_{i,j}\}. \tag{5}$$

Lemma 4.1 *Let ι be a sequentially perfect FAP with $\iota(0) = i$. The following three claims hold:*

(C1) There exists $j \neq i$ such that $0 < R_{i,j}$.

(C2) For each connected component (t, t') of $[0, 1) \setminus H(\iota)$ with $t' \neq 1$, we have $R_{\iota(t'),\iota(t)} < 0 < R_{\iota(t),\iota(t')}$, i.e., $\iota(t') \in S_{\iota(t)}$.

(C3) For each left accumulation point $t \in (0, 1)$ there exists $j \in I$ such that $\iota(t) \in S_j$.

Proof. There are two cases to consider, namely $\iota([0, 1)) = \{i\}$ and $\iota([0, 1)) \neq \{i\}$. Suppose first that $\iota([0, 1)) = \{i\}$. Then, $\gamma_0(\iota) = R_i$. Since ι is sequentially perfect, we must have $R_i \geq \vec{0}$. Claim (C1) follows from Assumption 2.7 on the payoff matrix of single quittings, and (C2) is satisfied vacuously.

Suppose next that $\iota([0, 1)) \neq \{i\}$. Since $H(\iota)$ is well-ordered, a different player gets to quit at $t := \inf\{s \in ([0, 1) \mid \iota(s) \neq i\} > 0$. By Remark 2.9.1,

$$\gamma_0(\iota) = tR_i + (1 - t)\gamma_t(\iota).$$

Let $j := \iota(t)$. Sequential perfectness asks for $\gamma_0^j(\iota) \geq 0$ and $\gamma_t^j(\iota) = \gamma_t^i(\iota) = 0$. From Assumption 2.7, we necessarily have $R_{i,j} > 0$, which proves (C1).

We now show (C2) for the connected component $(0, t)$. First, note that player j cannot be a quitter throughout the play. Indeed, if $\iota([t, 1)) = \{j\}$, then $\gamma_t(\iota) = R_j$. In particular, we have $R_{j,i} = \gamma_t^i(\iota) = 0$, which contradicts Assumption 2.7. It follows from the same argument as above that another player gets to quit at $t' := \inf\{s \geq t|\iota(s) \neq j\} > t$, and

$$(1 - t)\gamma_t(\iota) = (t' - t)R_j + (1 - t')\gamma_{t'}(\iota).$$

Since ι is sequentially perfect, we have $\gamma_t^i(\iota) = 0$ and $\gamma_{t'}^i(\iota) \geq 0$. It follows that $R_{j,i} < 0$, and therefore $j \in S_i$.

As a result, since, for each connected component (t, t') of $[0, 1) \setminus H(\iota)$ with $t' \neq 1$, the FAP ι^t, which is defined in Remark 2.9.2, is sequentially perfect, we must have $\iota(t') \in S_{\iota(t)}$.

We now show that (C3) holds. Recall that for every $s \in [0, t)$, the players' payoffs under ι satisfy the following relationship:

$$(1 - s)\gamma_s(\iota) = \sum_{j \in I} Leb\{l \in [s, t)|\iota(l) = j\} \cdot R_j + (1 - t)\gamma_t(\iota).$$

It follows that $\gamma_s(\iota)$ converges to $\gamma_t(\iota)$ as s goes to t. Since $\gamma_s^{\iota(s)}(\iota) = 0$ for all $s \in [0, t)$, we necessarily have that $\gamma_t^j(\iota) = 0$ for all $j \in \bigcap_{s \in [0,t)} \iota([s, t)) \setminus \{\iota(t)\}$. We claim that $\iota(t)$ is a successor of at least one such player j.

Indeed, by the same argument as above, $R_{\iota(t),j} < 0$ for each such player $j \in \bigcap_{s\in[0,t)} \iota([s,t)) \setminus \{\iota(t)\}$. On the other hand, if $R_{j,\iota(t)} < 0$ for all $j \in \bigcap_{s\in[0,t)} \iota([s,t)) \setminus \{\iota(t)\}$, then for every s sufficiently close to t,

$$\sum_{j\in I} Leb\{l \in [s,t) | \iota(l) = j\} \cdot R_{j,\iota(t)} + (1-t)\gamma_t^{\iota(t)}(\iota) < 0,$$

which contradicts sequential perfectness of ι. ■

It is convenient to visualize the order of players' quittings by a *direct graph* (I, L), where I is the set of vertices (players) and $L := \{(i,j) \in I^2 : j \in S_i\}$ is the set of directed edges. To make further progress we explore the topology of (I, L).

Let us introduce some auxiliary definitions. A *subgraph* (N, L_N) is a directed graph where $N \subseteq I$, $N \neq \emptyset$ and $L_N := \{(i,j) \in N^2 : j \in S_i\}$. To simplify notations we identify a subgraph with its vertices, i.e., we write N instead of (N, L_N). A *directed path in N from $i \in N$ to $j \in N$* is a vector of distinct vertices $(i_1, ..., i_m) \in N^m$ such that $i_1 = i$, $i_m = j$ and $i_k \in S_{i_{k-1}}$ for $k = 2, ..., m$. In particular, there is a trivial directed path from each vertex to itself. A *simple circuit* N is a subgraph with at least two vertices such that for all distinct $i, j \in N$ there exists a unique directed path in N from i to j.

A subgraph N is called a *strongly connected component* if it is a maximal set of vertices, such that there is a directed path between each pair of these vertices. Let $\mathbb{I}$ be the *set of strongly connected components of* (I, L). For each $N \in \mathbb{I}$, we denote by $\widehat{N} \subseteq I$ the set of all vertices not in N that are reachable from N:

$$\widehat{N} := \{j \in I \setminus N \mid \exists \text{ directed path in } I \text{ from some } i \in N \text{ to } j\}.$$

Since the collection of strongly connected components is a partition of I, $\widehat{N}$ is the union of all strongly connected components disjoint of N that are reachable from N.

Remarks 4.2

1. *The set $\mathbb{I}$ is a partition of I.*

2. *Each $N \in \mathbb{I}$ contains either one or more than three elements, because for every distinct $i, j \in I$ we cannot have simultaneously $j \in S_i$ and $i \in S_j$, see Eq.* (5).

3. *The strongly connected components of (I, L) form an acyclic graph. This is the graph whose vertices are the strongly connected components of (I, L), such that there is a directed edge from the component N to the component N' if and only if N' is reachable from N.*

4. *Every player $i \in I$ with $R_i \leq \vec{0}$ forms a singleton strongly connected component; moreover, $\widehat{\{i\}} = \emptyset$.*

5. *Every player $i \in I$ with $R_i \geq \vec{0}$ forms a singleton strongly connected component; moreover, there is no $N \in \mathbb{I}$ such that $i \in \widehat{N}$.*

6. *Since the number of strongly connected components is finite, and they form an acyclic graph, there always exists at least one strongly connected component $N \in \mathbb{I}$ with $\widehat{N} = \emptyset$.*

Example 4.3 *Consider the following payoff matrix of single quittings:*

$$R = \begin{pmatrix} 0 & + & - & \times & \times & \times & \times & \times & - & + \\ - & 0 & + & \times & \times & \times & \times & \times & - & + \\ + & - & 0 & + & \times & \times & + & \times & - & + \\ \times & \times & - & 0 & + & - & \times & - & - & + \\ \times & \times & \times & - & 0 & + & \times & \times & + & + \\ \times & \times & \times & + & - & 0 & \times & \times & - & + \\ \times & \times & - & \times & \times & \times & 0 & \times & - & + \\ \times & \times & \times & + & \times & \times & \times & 0 & - & + \\ - & - & - & - & - & - & - & - & 0 & + \\ + & + & + & + & + & + & + & + & + & 0 \end{pmatrix},$$

where "+" ("-") stands for a positive (negative) entry, "$\times$" at positions (i,j) and (j,i) means that neither $i \in S_j$ nor $j \in S_i$, i.e., $R_{i,j}R_{j,i} > 0$. Figure 4.3 plots the directed graph (I, L) visualizing the order of players' quittings. There are six strongly connected compo-

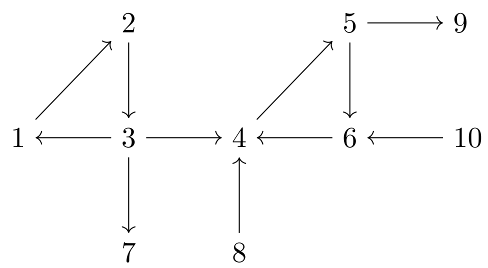

Figure 1: Digraph (I, L).

nents, that is $\mathbb{I} = \{\{1,2,3\},\{4,5,6\},\{7\},\{8\},\{9\},\{10\}\}$ with $\widehat{\{7\}} = \widehat{\{9\}} = \emptyset$, $\widehat{\{4,5,6\}} = \{9\}$, $\widehat{\{8\}} = \widehat{\{10\}} = \{4,5,6,9\}$ and $\widehat{\{1,2,3\}} = \{4,5,6,7,9\}$.

4.2 Constructing the Essential APS Operator

In this section we build on the classical APS operator to construct a tighter bound for the set $\mathcal{E}$ of payoffs attainable by sequentially perfect FAPs. The gist of our construction is to inductively find the largest invariant sets in each strongly connected component of the graph (I, L) following their order prescribed by reachability. For this purpose, we shall use Lemma 4.4 to decompose $\mathcal{E}$. Recall that each set of payoffs $\mathcal{R}_{\{i\}} := \{R_i\} \cap \mathbb{R}^I_+$ corresponds to the set of payoffs attained by the constant FAP $\iota(t) = i$, when it is sequentially perfect, and is empty otherwise.

Lemma 4.4 *It holds that $\mathcal{E} = \cup_{i\in I}\mathcal{E}_i$, where, for all $i \in I$,*

$$\mathcal{E}_i = \mathcal{R}_{\{i\}} \cup \Big\{ w \in \mathbb{R}^I_+ \mid \exists(\lambda, \iota) \in [0,1] \times \Upsilon \text{ s.t. } w = \lambda R_i + (1-\lambda)\gamma_0(\iota), w_i = 0, \\ \iota(0) \in S_i, \iota([0,1)) \subseteq N \cup \widehat{N} \Big\},$$

where N is the strongly connected component containing player i; i.e., $\mathcal{E}_i$ is the set that contains all payoffs that can be attained by sequentially perfect FAPs in which player i receives 0, quits at the outset with some probability and any player who quits next must be reachable from i in the graph (I, L).

Proof. Remark 3.2 and Lemma 3.3 imply that $\mathcal{E}_i \subseteq \mathcal{E}$ for all $i \in I$.

Conversely, for $w \in \mathcal{E}$, let $\iota \in \Upsilon$ be such that $w = \gamma_0(\iota)$, and $i \in I$ such that $w_i = 0$. If $\iota([0,1)) = \{i\}$, then $w = R_i \in \mathcal{R}_{\{i\}} \subseteq \mathcal{E}$. If $\iota([0,1)) \neq \{i\}$, then, since $H(\iota)$ is well-ordered, the value of $t := \inf\{s \in (0,1) | \iota(s) \neq i\}$ is positive. By Lemma 4.1, $\iota(t) \in S_i$. Moreover, (C1) and (C2) of this lemma jointly imply that no player outside of $N \cup \widehat{N}$ will ever get to quit under ι, i.e., $\iota([0,1)) \subseteq N \cup \widehat{N}$. The assertion of the claim then follows from Remark 2.9.1: $\gamma_0(\iota) = tR_i + (1-t)\gamma_0(\iota^t)$, where $\iota^t := \iota(t + (1-t)\cdot)$ belongs to Υ. ■

Recall that the classical APS operator $\mathbf{T}$ maps subsets of $\mathcal{R}_I$ to subsets of $\mathcal{R}_I$, and was defined so that the set $\mathcal{E}$ is invariant under it. In contrast, the *Essential APS* operator takes as an input a collection of subsets of $\mathcal{R}_I$, outputs a collection of subsets of $\mathcal{R}_I$, and it is defined in such a way that $(\mathcal{E}_i)_{i\in I}$ is invariant under this operator. This operator will be used separately on each strongly connected component of (I, L), and it will be defined recursively along the graph of strongly connected components.

To this end note first that, for $N \in \mathbb{I}$ and $i \in N$, we necessarily have $S_i \subsetneq N \cup \widehat{N}$. Then, given an arbitrary collection of sets indexed by $\widehat{N}$, say $(E_j)_{j\in\widehat{N}} \subseteq (\mathcal{R}_I)^{\widehat{N}}$, the following operator is well-defined: for all collections $(E_j)_{j\in N} \subseteq (\mathcal{R}_I)^N$, set

$$\mathbf{T}_{i,N}\left((E_j)_{j\in N} | (E_j)_{j\in\widehat{N}}\right) := \mathcal{R}_{\{i\}} \cup \big\{ w \in \mathbb{R}^I_+ | \exists\, (\lambda, v) \in [0,1] \times \cup_{j\in S_i} E_j, \\ s.t.\ w = \lambda R_i + (1-\lambda)v, w_i = 0 \big\}. \tag{6}$$

The set $\mathbf{T}_{i,N}\left((E_j)_{j\in N} | (E_j)_{j\in\widehat{N}}\right)$ contains all payoffs that can be attained at time 0 when player i quits with non-negative probability, selects a player j in S_i and a continuation payoff in E_j that yields 0 to player i. It is convenient to stack together $\mathbf{T}_{i,N}$ as

$$\mathbf{T}_N\left((E_i)_{i\in N} | (E_i)_{i\in\widehat{N}}\right) := \left(\mathbf{T}_{i,N}\left((E_j)_{j\in N} | (E_j)_{j\in\widehat{N}}\right)\right)_{i\in N}.$$

We next mention a few useful properties of $\mathbf{T}_N$.

Remarks 4.5 *Fix $N \in \mathbb{I}$ and $(E_i)_{i\in N\cup\widehat{N}} \subseteq (\mathcal{R}_I)^{N\cup\widehat{N}}$.*

1. *If the sets $(E_i)_{i\in N\cup\widehat{N}}$ are closed, then $\mathbf{T}_N\left((E_i)_{i\in N} | (E_i)_{i\in\widehat{N}}\right)$ is a collection of closed sets as well.*

2. *For each $i \in N$, we have $\mathbf{T}_{i,N}\left((E_j)_{j\in N}|(E_j)_{j\in\widehat{N}}\right) \subseteq \mathbf{T}_{i,N}\left((E'_j)_{j\in N}|(E'_j)_{j\in\widehat{N}}\right)$ for every $(E'_j)_{j\in N\cup\widehat{N}}$ such that $E_j \subseteq E'_j \subseteq \mathcal{R}_I$ for all $j \in N \cup \widehat{N}$.*

3. *Lemma 4.4 implies that the collection $(\mathcal{E}_i)_{i\in N}$ is a fixed point of the operator $\mathbf{T}_N\left(\cdot|(\mathcal{E}_i)_{i\in\widehat{N}}\right)$, that is,*

$$(\mathcal{E}_i)_{i\in N} = \mathbf{T}_N\left((\mathcal{E}_i)_{i\in N}|(\mathcal{E}_i)_{i\in\widehat{N}}\right). \tag{7}$$

4.3 An algorithm based on the essential APS operator

After a transition out of a strongly connected component N, the play will never return to it, hence we can inductively construct the largest invariant sets $(\overline{\mathcal{E}_i})_{i\in I}$ of the Essential APS operator. These largest invariant sets will be upper bounds of the sets $(\mathcal{E}_i)_{i\in I}$.

- Consider first $N \in \mathbb{I}$ such that $\widehat{N} = \emptyset$. Set $\mathcal{F}_N := \mathcal{R}_N$, and define $(\overline{\mathcal{E}_i})_{i\in N}$ as follows:

$$(\overline{\mathcal{E}}_i)_{i\in N} := \bigcap_{n=0}^{\infty} (\mathbf{T}_N)^n\left((\mathcal{F}_N)^N \mid \emptyset\right). \tag{8}$$

- Consider next $N \in \mathbb{I}$ such that $\widehat{N} \neq \emptyset$, and suppose that the sets $(\overline{\mathcal{E}}_i)_{i\in\widehat{N}}$ are already known. Let $\mathcal{F}_N$ be given by

$$\mathcal{F}_N := co\left\{\{R_i\} \cup \bigcup_{j\in S_i\cap\widehat{N}} \left(\overline{\mathcal{E}}_j \cap \{w \in \mathbb{R}^I \mid w_i = 0\}\right), i \in N\right\} \cap \mathbb{R}^I_+. \tag{9}$$

We can see recursively that, for all $N \in \mathbb{I}$, $\mathcal{F}_N \subset \mathcal{R}_I$ and compact. Then again, we can define the sets $(\overline{\mathcal{E}}_i)_{i\in N}$ by

$$(\overline{\mathcal{E}}_i)_{i\in N} := \bigcap_{n=0}^{\infty} (\mathbf{T}_N)^n\left((\mathcal{F}_N)^N \mid (\overline{\mathcal{E}}_i)_{i\in\widehat{N}}\right). \tag{10}$$

The following lemma unpacks the sets $(\overline{\mathcal{E}}_i)_{i\in I}$ as defined in (8) and (10), and gives a first link between our upper bounds and $\mathcal{E}$:

Lemma 4.6 *Let $N \in \mathbb{I}$. For each $i \in N$ and every $w \in \overline{\mathcal{E}}_i$, there exists a sequence $(w^n, i^n, \lambda^n)_{n\in\mathbb{N}} \subset (\mathbb{R}^I \times (N \cup \widehat{N}) \times [0,1])^{\mathbb{N}}$ with $w^0 = w$ and $i^0 = i$, such that for each $n \in \mathbb{N}$,*

$$\begin{cases} w^n \in \overline{\mathcal{E}}_{i^n}, \\ w^n = \lambda^n R_{i^n} + (1-\lambda^n) w^{n+1}, \\ i^{n+1} \in S_{i^n} \text{ whenever } \lambda^n \neq 1. \end{cases} \tag{11}$$

Moreover, if $w^n \in \mathcal{E}$ for some $n \in \mathbb{N}$, then $w \in \mathcal{E}$.

If $\prod_{n\in\mathbb{N}}(1-\lambda^n) = 0$, then the sequence $(w^n, i^n, \lambda^n)_{n\in\mathbb{N}}$ of Lemma 4.6 naturally defines a sequentially perfect FAP: player i^0 quits first with probability λ^0, and w^1 is the continuation payoff; player i^1 quits with probability λ^1, and w^2 is the continuation payoff, and so on. If $\prod_{n\in\mathbb{N}}(1-\lambda^n) > 0$, then this sequence defines a prefix of an FAP that implements the payoff vector w^0.

Proof. Fix $i \in N$ and let $w \in \overline{\mathcal{E}}_i$. The existence of a sequence $(w^n, i^n, \lambda^n)_{n\in\mathbb{N}}$ satisfying Eq. (11) follows from the fact that for each $M \in \mathbb{I}$ reachable from N, the sets $(\overline{\mathcal{E}}_j)_{j\in M}$ are invariant with respect to $\mathbf{T}_M(\cdot \mid (\overline{\mathcal{E}}_j)_{j\in\widehat{M}})$.

We now prove the second claim. Suppose that there exists $n \in \mathbb{N}$ such that $w^n \in \mathcal{E}$, i.e, there is a sequentially perfect FAP ι with $\gamma_0(\iota) = w^n$. If $n = 0$, then $w = w^0 \in \mathcal{E}$. Since $i^0 = i$, we have $w_i = 0$, and hence $w \in \mathcal{E}_i \subseteq \mathcal{E}$.

If $n \neq 0$, then we know that once the time reaches t^n, a sequentially perfect strategy profile to continue the play and obtain a payoff of w^n is available. We only need to verify that up to time t^n the algorithm yields a beginning that answers the conditions of sequential perfectness when the continuation payoff at time t^n is w^n. Let $(t^k)_{k=0}^n$ be as follows: $t^0 := 0$ and $t^{k+1} := t^k + (1-t^k)\lambda^k$ for $k = 0, \dots, n-1$. Define a new FAP ι' by

$$\iota'(t) := \begin{cases} i^k & \text{if } t \in [t^k, t^{k+1}),\ k = 0, \dots, n-1, \\ \iota\left(\frac{t-t^n}{1-t^n}\right) & \text{if } t \in [t^n, 1). \end{cases}$$

By construction,
$w = \gamma_0(\iota')$ with $w_i = 0$, $w \in \mathbb{R}^I_+$ and ι' is sequentially perfect, thus $w \in \mathcal{E}_i \subseteq \mathcal{E}$. ■

We show now that the sets $(\overline{\mathcal{E}}_i)_{i\in I}$ are supersets of $(\mathcal{E}_i)_{i\in I}$. Our argument is divided into two parts. Lemma 4.7 establishes that $(\overline{\mathcal{E}}_i)_{i\in I}$ is larger than any collection of invariant sets of the Essential APS operator satisfying a certain property. It is then sufficient to show that the collection $(\mathcal{E}_i)_{i\in I}$ satisfies the premise of Lemma 4.7.

Lemma 4.7 *Fix $N \in \mathbb{I}$. Let $(E_i)_{i\in N\cup\widehat{N}}$ be a collection of sets such that for all $M \in \mathbb{I}$ reachable from N, as well as for $M = N$,*

- *$E_i \subseteq \mathcal{F}_M$, for each $i \in M$,*
- *$(E_i)_{i\in M} = \mathbf{T}_M\left((E_i)_{i\in M} \mid (E_i)_{i\in\widehat{M}}\right)$.*

Then, $E_i \subseteq \overline{\mathcal{E}}_i$ for all $i \in N \cup \widehat{N}$.

Proof. We proceed by induction. For $N \in \mathbb{I}$ such that $\widehat{N} = \emptyset$, the result follows directly from Eq. (8) and the fact that $\mathbf{T}_N(\cdot \mid \emptyset)$ is monotone.

Next consider $N \in \mathbb{I}$ such that $\widehat{N} \neq \emptyset$, and suppose that $E_i \subseteq \overline{\mathcal{E}}_i$ for each $i \in \widehat{N}$. The monotonicity of $\mathbf{T}_N$ in all its arguments imply that

$$(E_i)_{i\in N} = \bigcap_{n=0}^{\infty} (\mathbf{T}_N)^n \left((E_i)_{i\in N} \mid (E_i)_{i\in\widehat{N}}\right) \subseteq \bigcap_{n=0}^{\infty} (\mathbf{T}_N)^n \left((\mathcal{F}_N)^N \mid (\overline{\mathcal{E}}_i)_{i\in\widehat{N}}\right) = (\overline{\mathcal{E}}_i)_{i\in N},$$

which completes the inductive step. ■

Proposition 4.8 *$\mathcal{E}_i \subseteq \overline{\mathcal{E}}_i$ for all $i \in I$.*

Proof. We proceed again by induction. In view of Remark 4.5.3 and Lemma 4.7 it is sufficient to prove that $\mathcal{E}_i \subseteq \mathcal{F}_N$ for every strongly connected component N and every $i \in N$. This trivially holds if $\widehat{N} = \emptyset$.

Fix now $N \in \mathbb{I}$ such that $\widehat{N} \neq \emptyset$, and suppose that $(\mathcal{E}_i)_{i\in M} \subseteq (\mathcal{F}_M)^M$ for every $M \in \mathbb{I}$ reachable from N. Let $w \in \mathcal{E}_i$ as defined in Lemma 4.4, i.e., $w = \lambda R_i + (1-\lambda)\gamma_0(\iota) \geq 0$ with $w_i = 0$ for some $(\lambda, \iota) \in [0,1] \times \Upsilon$. We shall show that $w \in \mathcal{F}_N$.

Suppose first that $\iota([0,1)) \subseteq N$. Then, w is a convex combination of $(R_j)_{j\in N}$, and the assertion trivially holds.

Suppose next that $t := \inf\{s \in (0,1) \mid \iota(s) \notin N\} < 1$. By construction, $\gamma_t(\iota) \in \mathcal{E}_{\iota(t)}$, and, by Lemma 4.1, there exists $j \in N$ such that $\gamma_t^j(\iota) = 0$ and $\iota(t) \in S_j$. It follows that w is a convex combination of $\left(\{R_j\} \cup \bigcup_{l\in S_j\cap\widehat{N}} \left(\mathcal{E}_l \cap \{w \in \mathbb{R}^I \mid w_j = 0\}\right)\right)_{j\in N}$.

By the induction hypothesis, $(\mathcal{E}_j)_{j\in M} \subseteq (\mathcal{F}_M)^M$ for every $M \in \mathbb{I}$ reachable from N that implies that $\mathcal{E}_j \subseteq \overline{\mathcal{E}}_j$ for all $j \in \widehat{N}$. As a result, $w \in \mathcal{F}_N$, and the induction step is complete. ■

The following theorem delivers a condition under which the algorithm of this section computes the set $\mathcal{E}$, i.e., every payoff vector in $\bigcup_{i\in I} \overline{\mathcal{E}}_i$ can be attained by some sequentially perfect FAP. This condition says that for every strongly connected component N, the sets $(\overline{\mathcal{E}})_{i\in N}$ are such that there is no subset of players who can quit consecutively after each other with zero probabilities. This is satisfied in Example 2.2, see also Section 4.4 for more computed examples.

Theorem 4.9 *Suppose that for every strongly connected component $N \in \mathbb{I}$ with at least three elements, $\mathcal{F}_N \cap \{w \in \mathbb{R}^I \mid w_i = 0,\ \forall i \in M\} = \emptyset$ for all simple circuits $M \subseteq N$. Then,*

$$\mathcal{E} = \bigcup_{i\in I} \overline{\mathcal{E}}_i.$$

Furthermore, for each $w \in \mathcal{E}$, there exists a sequentially perfect FAP ι with $\gamma_0(\iota) = w$ such that the ordinality of $H(\iota)$ is at most the ordinality of $\mathbb{N}$.

Proof. Lemma 4.4 and Proposition 4.8 imply that $\mathcal{E} = \bigcup_{i\in I} \mathcal{E}_i \subseteq \bigcup_{i\in I} \overline{\mathcal{E}}_i$. In what follows we show the reverse inclusion.

Take $N \in \mathbb{I}$ and suppose that, by induction, we have already shown that for each $i \in \widehat{N}$, (i) the set $\overline{\mathcal{E}}_i$ is closed, (ii) $\overline{\mathcal{E}}_i \subseteq \mathcal{E}$, and (iii) all elements of $\overline{\mathcal{E}}_i$ can be obtained by FAPs ι such that the ordinality of $H(\iota)$ is at most the ordinality of $\mathbb{N}$. Note that the premise is vacuously true whenever $\widehat{N} = \emptyset$.

Property (i) is immediate. Indeed, since $\mathbf{T}_N(\cdot \mid (\overline{\mathcal{E}}_i)_{i\in\widehat{N}})$ maps closed sets to closed sets and $\mathcal{F}_N$ is closed, the sets $(\overline{\mathcal{E}}_i)$ are closed as an intersection of closed sets.

We now establish properties (ii) and (iii). Fix $i \in N$ and $w \in \overline{\mathcal{E}}_i$. Suppose first that there exists a sequence $(w^n, i^n, \lambda^n)_{n\in\mathbb{N}}$ satisfying Eq. (11) such that either $\lambda^n = 1$ or $i^{n+1} \in \widehat{N}$ for some $n \in \mathbb{N}$. In the former case, we must have $w^n = R_{i^n} \geq \vec{0}$, whereas in the latter case, $w^{n+1} \in \overline{\mathcal{E}}_{i^{n+1}} \subseteq \mathcal{E}$. In either case, Lemma 4.6 implies that $w \in \mathcal{E}$.

Suppose now that any sequence $(w^n, i^n, \lambda^n)_{n\in\mathbb{N}}$ satisfying Eq. (11) is such that $\lambda^n < 1$ and $i_{n+1} \in N$ for all $n \in \mathbb{N}$. Define a sequence $(t^n)_{n\in\mathbb{N}}$ by setting $t^0 := 0$ and $t^{n+1} := t^n + (1 - t^n)\lambda^n$ for all $n \in \mathbb{N}$. By construction, $(t^n)_{n\in\mathbb{N}}$ is non-decreasing and bounded from above by 1. We claim that this sequence converges to 1. To establish this claim, consider the following infimum:

$$\begin{aligned}
\nu := \inf_{\left((\widetilde{i}^k)_{k=1}^{|N|}, \widetilde{w}, (\widetilde{\lambda}^k)_{k=1}^{|N|-1}\right) \in N^{|N|} \times \mathcal{F}_N \times [0,1]^{|N|-1}} \quad & \textstyle\sum_{k=1}^{|N|-1} \widetilde{\lambda}^k, \\
\text{s.t.} \quad & \widetilde{i}^{k+1} \in S_{\widetilde{i}^k} \ \text{ for } \ k = 1, \dots, |N| - 1, \ \text{ and} \\
& \begin{cases} \widetilde{w}_{\widetilde{i}^1} = 0, \\ \widetilde{w}_{\widetilde{i}^2} = \widetilde{\lambda}^1 R_{\widetilde{i}^1, \widetilde{i}^2}, \\ \widetilde{w}_{\widetilde{i}^k} = \widetilde{\lambda}^1 R_{\widetilde{i}^1, \widetilde{i}^k} + \prod_{m=1}^{k-2} (1 - \widetilde{\lambda}^m) \widetilde{\lambda}^{k-1} R_{\widetilde{i}^{k-1}, \widetilde{i}^m} \ \text{for } k = 3, \dots, |N|. \end{cases}
\end{aligned} \tag{12}$$

The quantity $\sum_{k=1}^{|N|-1} \widetilde{\lambda}^k$ serves as a proxy to the total probability of quitting of $N - 1$ consecutive players in a sequence $(w^n, i^n, \lambda^n)_{n\in\mathbb{N}}$ satisfying Eq. (11). Thus, ν is the infimum of these quantities.

We will show that the condition in Theorem 4.9 implies that $\nu > 0$, which will imply that the sequence $(t^n)_{n\in\mathbb{N}}$ converges to 1. By way of contradiction, assume that $\nu = 0$. Since $\mathcal{F}_N$ is compact, the infimum is attained by some point, say $\left((\widetilde{i}^k)_{k=1}^{|N|}, \widetilde{w}, (\widetilde{\lambda}^k)_{k=1}^{|N|-1}\right)$. By assumption, $\nu = 0$ which implies that $\widetilde{w}_j = 0$ for every $j \in \{\widetilde{i}^k\}_{k=1}^{|N|}$. Since $\widetilde{i}^{k+1} \in S_{\widetilde{i}^k}$ for $k = 1, \dots, |N| - 1$, there exists a simple circuit M in $(\widetilde{i}^k)_{k=1}^{|N|}$. The existence of a simple circuit and a payoff in which all players in that simple circuit gets zero contradict the assumption of the theorem. It follows that $\nu > 0$.

The reader can verify that, for each $n \in \mathbb{N}$, the point $\left((i^{n+k})_{k=1}^{|N|}, w^n, (\lambda^{n+k})_{k=1}^{|N|-1}\right)$ satisfies the constraints of the auxiliary problem (12). As a result, $\sum_{k=1}^{|N|-1} \lambda^{n+k} \geqslant \nu > 0$ for every $n \in \mathbb{N}$.

Going back to the sequence $(t^n)_{n\in\mathbb{N}}$, note that $t^{n+|N|-1} - t^n = \sum_{k=1}^{|N|-1} \lambda^{n+k}(1 - t^{n+k-1})$ for every $n \in \mathbb{N}$. Since this latter sequence is non-decreasing, for each $n \in \mathbb{N}$, it has to satisfy the following:

$$t^{n+|N|-1} - t^n \geq \sum_{k=1}^{|N|-1} \lambda^{n+k}(1 - t^{n+|N|-1}) \geq \nu(1 - t^{n+|N|-1}).$$

It follows that $(t^n)_{n\in\mathbb{N}}$ converges to 1 as n tends to ∞. Since $(t^n)_{n\in\mathbb{N}}$ converges to 1, the following ι constitutes an FAP:

$$\iota(t) := i^n \text{ whenever } t \in [t^n, t^{n+1}).$$

By construction, ι is sequentially perfect with $w = \gamma_0(\iota)$ and $w \in \mathcal{E}$.

Finally, note that in each of the two cases, the ordinality of the constructed FAP ι is of at most the ordinality of $\mathbb{N}$. This completes the induction step. ■

Theorem 4.9 provides the condition under which the Essential APS operator can be used to compute the set of all payoff vectors attainable by sequentially perfect FAPs. We note that any such payoff vector can be implemented by an FAP without infinite cycles. This is not true in general, see Example 4.11 below. The next corollary of Theorem 4.9 presents a simple sufficient condition, which ensures that $\mathcal{E} = \bigcup_{i \in I} \overline{\mathcal{E}}_i$. It holds in particular in Example 2.2.

Corollary 4.10 *Suppose that for each $i \in I$, $S_i = \{i+1\}$, where the addition is modulo $|I|$. Then $\mathbb{I} = \{I\}$ and, if $\vec{0} \notin \mathcal{R}_I$, then for each $i \in I$,*

$$\overline{\mathcal{E}}_i = \bigcap_{n=0}^{\infty} \Big(\mathbf{T}_{i,I} \circ \mathbf{T}_{i+1,I} \circ \ldots \circ \mathbf{T}_{i_{i+|I|-1},I}\Big)^n (\mathcal{R}_I|\emptyset) \ \ \text{and} \ \ \mathcal{E} = \bigcup_{i \in I} \overline{\mathcal{E}}_i.$$

If the condition of Theorem 4.9 fails, then the upper bounds $(\overline{\mathcal{E}}_i)_{i \in I}$ might not be tight. We now provide an example, which illustrates that the assumption of Theorem 4.9 cannot be easily dispensed with.

Example 4.11 *Consider the quitting game with five players and the following payoff matrix of single quittings:*

$$R = \begin{pmatrix} 0 & 2 & -\frac{1}{2} & 1 & -1 \\ -\frac{1}{2} & 0 & 2 & 1 & -1 \\ 2 & -\frac{1}{2} & 0 & 1 & -1 \\ -1 & -2 & -3 & 0 & \frac{10}{7} \\ 2 & \frac{7}{2} & \frac{47}{8} & -\frac{5}{12} & 0 \end{pmatrix}.$$

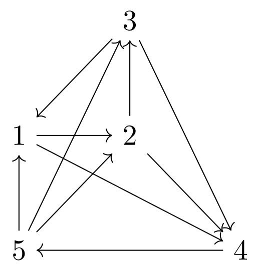

Figure 2: Digraph (I, L).

In this game, $S_1 = \{2,4\}$, $S_2 = \{3,4\}$, $S_3 = \{1,4\}$, $S_4 = \{5\}$, and $S_5 = \{1,2,3\}$, see Figure 2 for an illustration of the graph (I, L). The reader can verify that the unique strongly connected component is I.

We claim that $\bigcup_{i\in I}\overline{\mathcal{E}}_i \neq \mathcal{E}$. *To this end, note that there exists a point* $w \in \mathcal{R}_I$ *such that* $w_1 = w_2 = w_3 = 0$ *and* $w_4 > 0$, *i.e.,*

$$w = \left(0, 0, 0, \frac{20}{71}, \frac{20}{71}\right) = \frac{5}{71}R_1 + \frac{686}{2911}R_2 + \frac{684}{2911}R_3 + \frac{1309}{2911}R_4 + \frac{552}{2911}R_5.$$

On the one hand, this point cannot be eliminated during iterations of the essential APS operator, because $w \in \mathbf{T}_{i,I}(\{w\}|\emptyset)$ *for* $i = 1, 2, 3$ *and* $\{1, 2, 3\}$ *is a simple circuit. As a result,* $w \in \overline{\mathcal{E}}_i$ *for* $i \in \{1, 2, 3\}$.

On the other hand, w *cannot be attained by any sequentially perfect FAP. Indeed, since* $R_{i,i+1} > 0$ *for all* $i \in \{1, 2, 3\}$, *where the addition is modulo 3, there is no* $\lambda \in (0, 1]$ *and* $v \in \mathcal{R}_I \subseteq \mathbb{R}^I_+$ *such that* $w = \lambda R_i + (1-\lambda)v$. *It follows that* $w \notin \mathcal{E}_1 \cup \mathcal{E}_2 \cup \mathcal{E}_3$. *At the same time,* $w \notin \mathcal{E}_4 \cup \mathcal{E}_5$, *because* $w_4 > 0$ *and* $w_5 > 0$. *We conclude that there is no sequentially perfect FAP* ι *such that* $\gamma_0(\iota) = w$.

4.4 Numerical examples

In this section, we illustrate how the Essential APS approach can be used to numerically compute the set $\mathcal{E}$. All computations were performed numerically in Julia, using exact arithmetic and the "Polyhedra" and "LazySets" packages. A basic computational step takes a convex compact set $E \subseteq \mathbb{R}^I_+$ with a finite number of extreme points. Then, it returns $co\{R_i, E\} \cap \{w \geq \vec{0} \mid w_i = 0\}$ as an output, which is a convex compact set with a finite number of extreme points. This construction naturally extends to sets that can be written as a finite union of convex sets.

Example 3.5 continued. *Recall that in this example there are four players with the following payoff matrix of single quittings:*

$$R = \begin{pmatrix} 0 & 4 & -\frac{1}{2} & -1 \\ -1 & 0 & 3 & 1 \\ -\frac{1}{8} & -1 & 0 & 4 \\ 4 & \frac{1}{2} & -1 & 0 \end{pmatrix}.$$

The reader can verify that $S_i = \{i+1\}$ *for every* $i \in I$, *where the addition is modulo* 4, *and the graph* (I, L) *is a simple circuit. Moreover, the condition of Theorem 4.9 is satisfied. Since the graph is a simple circuit, the sets* $(\overline{\mathcal{E}}_i)_{i\in I}$ *can be re-written as follows: for each* $i \in I$,

$$\overline{\mathcal{E}}_i = \mathbf{T}_{i,I}(\overline{\mathcal{E}}_i|\emptyset) = (\mathbf{T}_{i,I} \circ \mathbf{T}_{i+1,I} \circ \mathbf{T}_{i+2} \circ \mathbf{T}_{i+3,I})(\overline{\mathcal{E}}_i|\emptyset).$$

We claim that $\overline{\mathcal{E}}_i = \emptyset$ *for all* $i \in I$. *To show this, we proceed backwards along the simple circuit repeatedly applying* $\mathbf{T}_{i,I}$*'s:*

- *As mentioned on Page 10, the set* $\mathcal{R}_I \cap \{w \geq \vec{0} \mid w_1 = 0\}$ *is the convex hull of* $(0, 0, \frac{173}{87}, \frac{95}{87})$, $(0, \frac{1799}{678}, 0, 0)$, $(0, 0, 0, \frac{1799}{635})$, $(0, \frac{1}{10}, \frac{11}{5}, \frac{4}{5})$, $(0, \frac{1}{10}, \frac{11}{5}, \frac{4}{5})$, *and* $(0, \frac{11}{6}, 1, 0)$.
- *The set* $(\mathbf{T}_{1,I} \circ \mathbf{T}_{2,I} \circ \mathbf{T}_{3,I} \circ \mathbf{T}_{4,I})(\mathcal{R}_I|\emptyset)$ *is the convex hull of* $(0, 0, \frac{267}{193}, \frac{313}{193})$, $(0, \frac{1799}{678}, 0, 0)$, $(0, 0, 0, \frac{1799}{635})$, *and* $(0, \frac{626}{253}, \frac{221}{1012}, 0)$.

- *The set* $(\mathbf{T}_{1,I}\circ\mathbf{T}_{2,I}\circ\mathbf{T}_{3,I}\circ\mathbf{T}_{4,I})^2(\mathcal{R}_I|\emptyset)$ *is the convex hull of* $(0,0,\frac{5349}{14095},\frac{176243}{70475})$, $(0,0,0,\frac{1799}{635})$, *and* $(0,\frac{42795}{24793},0,\frac{122753}{123965})$.

Note that every extreme point of $(\mathbf{T}_{1,I}\circ\mathbf{T}_{2,I}\circ\mathbf{T}_{3,I}\circ\mathbf{T}_{4,I})^2(\mathcal{R}_I|\emptyset)$ *is such that* $w_4>0$, *and therefore* $\mathbf{T}_{4,I}((\mathbf{T}_{1,I}\circ\mathbf{T}_{2,I}\circ\mathbf{T}_{3,I}\circ\mathbf{T}_{4,I})^2(\mathcal{R}_I|\emptyset)|\emptyset)=\emptyset$. *It follows that* $\mathcal{E}=\emptyset$.

We next provide an example in which the set $\mathcal{E}=\bigcup_{i\in I}\overline{\mathcal{E}}_i$ is non-empty and is strictly smaller than $\mathcal{R}_I$. In other words, the set of attainable payoffs cannot be computed by the naive approach, but the Essential APS approach does the job.

Example 4.12 *There are five players with the following payoff matrix of single quittings:*

$$R=\begin{pmatrix}0 & 4 & -\frac{1}{2} & -1 & -1\\ -1 & 0 & 3 & -1 & \frac{1}{2}\\ -\frac{1}{8} & -1 & 0 & 4 & -1\\ -1 & -\frac{1}{2} & -1 & 0 & 4\\ 5 & 1 & -1 & -1 & 0\end{pmatrix}.$$

In this game, $S_i=\{i+1\}$ *for each* $i\in I$, *where the addition is modulo* 5, *and the graph* (I,L) *is a simple circuit. The upper bounds* $(\overline{\mathcal{E}}_i)_{i\in I}$ *satisfy the following:*

$$\overline{\mathcal{E}}_i=\mathbf{T}_{i,I}(\overline{\mathcal{E}}_i|\emptyset)=(\mathbf{T}_{i,I}\circ\mathbf{T}_{i+1,I}\circ\mathbf{T}_{i+2,I}\circ\mathbf{T}_{i+3,I}\circ\mathbf{T}_{i+4,I})(\overline{\mathcal{E}}_i|\emptyset).$$

We now illustrate how to find $\overline{\mathcal{E}}_1$:

- *The set* $\mathcal{R}_I\cap\{w\geq\vec{0}|w_1=0\}$ *is the convex hull of* $(0,0,0,0,\frac{51885}{27356})$, $(0,0,0,\frac{51885}{27356},0)$, $(0,\frac{51885}{27356},0,0,0)$, $(0,\frac{199}{680},\frac{127}{80},0,0)$, $(0,0,\frac{2197}{1362},\frac{199}{681},0)$ *and* $(0,0,\frac{3319}{1840},0,\frac{199}{1840})$.
- *The set* $(\mathbf{T}_{1,I}\circ\mathbf{T}_{2,I}\circ\mathbf{T}_{3,I}\circ\mathbf{T}_{4,I}\circ\mathbf{T}_{5,I})(\mathcal{R}_I|\emptyset)$ *is the convex hull of* $(0,0,0,0,\frac{51885}{27356})$, $(0,0,0,\frac{51885}{27829},0)$, $(0,\frac{51885}{30158},0,0,0)$, $(0,0,\frac{31131}{19010},0,\frac{10377}{38020})$ *and* $(0,\frac{34590}{44969},\frac{190245}{179876},0,0)$.
- *The set* $(\mathbf{T}_{1,I}\circ\mathbf{T}_{2,I}\circ\mathbf{T}_{3,I}\circ\mathbf{T}_{4,I}\circ\mathbf{T}_{5,I})^2(\mathcal{R}_I|\emptyset)$ *is the convex hull of* $(0,\frac{1400895}{876794},0,0,\frac{415080}{3068779})$, $(0,\frac{178715}{152098},\frac{46120}{76049},0,0)$, $(0,\frac{51885}{30158},0,0,0)$, $(0,0,0,\frac{466965}{600227},\frac{9287415}{8403178})$, $(0,\frac{7429932}{10054513},\frac{20432313}{20109026},\frac{736767}{10054513},0)$, $(0,0,\frac{5572449}{4098515},\frac{259425}{819703},\frac{1857483}{8197030})$ *and* $(0,0,0,\frac{51885}{27829},0)$.

It turns out that $(\mathbf{T}_{1,I}\circ\mathbf{T}_{2,I}\circ\mathbf{T}_{3,I}\circ\mathbf{T}_{4,I}\circ\mathbf{T}_{5,I})^3(\mathcal{R}_I|\emptyset)=(\mathbf{T}_{1,I}\circ\mathbf{T}_{2,I}\circ\mathbf{T}_{3,I}\circ\mathbf{T}_{4,I}\circ\mathbf{T}_{5,I})^2(\mathcal{R}_I|\emptyset)$, *as a result, we have* $\overline{\mathcal{E}}_1=(\mathbf{T}_{1,I}\circ\mathbf{T}_{2,I}\circ\mathbf{T}_{3,I}\circ\mathbf{T}_{4,I}\circ\mathbf{T}_{5,I})^2(\mathcal{R}_I|\emptyset)$. *One can similarly obtain the remaining upper bounds* $\overline{\mathcal{E}}_i$ *for* $i=2,\ldots,5$. *The game satisfies the condition of Theorem 4.9, hence* $\mathcal{E}=\bigcup_{i\in I}\overline{\mathcal{E}}_i$

In both examples above, there is only one strongly connected component, which forms a simple circuit. As a result, for each player $i\in I$, the upper bound $\overline{\mathcal{E}}_i$ is convex. This is not necessarily the case when there are multiple strongly connected components as the following example illustrates.

Example 4.13 *Consider the game with six players and the following payoff matrix of single quittings:*

$$R = \begin{pmatrix} 0 & 2 & -\frac{1}{2} & 1 & 1 & 1 \\ -\frac{1}{2} & 0 & 2 & 1 & 1 & 1 \\ 2 & -\frac{1}{2} & 0 & 1 & 1 & 1 \\ -\frac{3}{8} & 1 & 1 & 0 & 2 & -\frac{1}{2} \\ 1 & 2 & 1 & -\frac{1}{2} & 0 & 2 \\ 2 & 1 & 1 & 2 & -\frac{1}{2} & 0 \end{pmatrix}.$$

Figure 3 depicts the graph (I, L). There are two strongly connected components $\mathbb{I} = \{\{1,2,3\},\{4,5,6\}\}$, and $\{4,5,6\}$ is reachable from $\{1,2,3\}$.

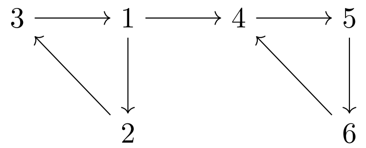

Figure 3: Digraph (I, L).

We now use the Essential APS approach to compute the upper bounds. First, consider the strongly connected component $\{4,5,6\}$. Similarly to Example 2.2, the sets of payoffs attainable by sequentially perfect FAPs in which only players 4, 5, and 6 quit coincide with the boundary of a certain triangle: $\overline{\mathcal{E}}_4 = \mathcal{R}_{\{4,5,6\}} \cap \{w \geq \vec{0} \mid w_4 = 0\}$ is the convex hull of $(0, \frac{25}{21}, 1, 0, \frac{3}{2}, 0)$ and $(\frac{9}{8}, \frac{37}{21}, 1, 0, 0, \frac{3}{2})$, $\overline{\mathcal{E}}_5 = \mathcal{R}_{\{4,5,6\}} \cap \{w \geq \vec{0} \mid w_5 = 0\}$ is the convex hull of $(\frac{9}{8}, \frac{37}{21}, 1, 0, 0, \frac{3}{2})$ and $(\frac{3}{2}, \frac{22}{21}, 1, \frac{3}{2}, 0, 0)$, and $\overline{\mathcal{E}}_6 = \mathcal{R}_{\{4,5,6\}} \cap \{w \geq \vec{0} \mid w_6 = 0\}$ is the convex hull of $(\frac{3}{2}, \frac{22}{21}, 1, \frac{3}{2}, 0, 0)$ and $(0, \frac{25}{21}, 1, 0, \frac{3}{2}, 0)$.

We next consider the strongly connected component $\{1,2,3\}$, where we focus on $\overline{\mathcal{E}}_1$, because the remaining sets can be easily derived from it. Iterate along the simple circuit $\{1,2,3\}$ to obtain the following:

$$\overline{\mathcal{E}}_1 = \mathbf{T}_{1,\{1,2,3\}}(\overline{\mathcal{E}}_2|\overline{\mathcal{E}}_4) = \mathbf{T}_{1,I}(\overline{\mathcal{E}}_2|\emptyset) \cup \mathbf{T}_{1,I}(\overline{\mathcal{E}}_4|\emptyset) = (\mathbf{T}_{1,I} \circ \mathbf{T}_{2,I} \circ \mathbf{T}_{3,I})(\overline{\mathcal{E}}_1|\emptyset) \cup \mathbf{T}_{1,I}(\overline{\mathcal{E}}_4|\emptyset).$$

The set $\mathcal{F}_{\{1,2,3\}} \cap \{w \in \mathbb{R}^I_+ | w_1 = 0\}$ is the convex hull of $(0, \frac{109}{63}, 0, \frac{2}{3}, \frac{7}{6}, \frac{2}{3})$, $(0, \frac{3}{2}, 0, 1, 1, 1)$, $(0, 0, \frac{3}{2}, 1, 1, 1)$, $(0, 0, \frac{421}{271}, \frac{250}{271}, \frac{563}{542}, \frac{250}{271})$ and $(0, \frac{25}{21}, 1, 0, \frac{3}{2}, 0)$. After $n \geq 1$ iterations, the essential APS algorithm outputs the following:

$$(\mathbf{T}_{1,I} \circ \mathbf{T}_{2,I} \circ \mathbf{T}_{3,I})^n(\mathcal{F}_{\{1,2,3\}}|\emptyset) \cup \bigcup_{k=0}^{n-1} (\mathbf{T}_{1,I} \circ \mathbf{T}_{2,I} \circ \mathbf{T}_{3,I})^k(\mathbf{T}_{1,I}(\overline{\mathcal{E}}_4|\emptyset)|\emptyset).$$

We now numerically approximate each term rounding rational numbers to $\frac{1}{1000}$:

- *The set $(\mathbf{T}_{1,I} \circ \mathbf{T}_{2,I} \circ \mathbf{T}_{3,I})^n(\mathcal{F}_{\{1,2,3\}}|\emptyset)$ converges to the convex hull of $(0, 0, \frac{3}{2}, 1, 1, 1)$ and $(0, \frac{3}{2}, 0, 1, 1, 1)$.*
- *$\mathbf{T}_{1,I}(\overline{\mathcal{E}}_4|\emptyset)$ is the convex hull of $(0, \frac{119}{100}, 1, 0, \frac{3}{2}, 1)$ and $(0, \frac{173}{100}, 1, \frac{667}{1000}, \frac{1167}{1000}, \frac{667}{1000})$.*

- $(\mathbf{T}_{1,I} \circ \mathbf{T}_{2,I} \circ \mathbf{T}_{3,I})(\mathbf{T}_{1,I}(\overline{\mathcal{E}}_4|\emptyset)|\emptyset)$ *is the convex hull of* $(0, \frac{1503}{1000}, 0, \frac{199}{200}, \frac{501}{500}, \frac{199}{200})$ *and* $(0, 0, \frac{1513}{1000}, \frac{491}{500}, \frac{1009}{1000}, \frac{491}{500})$.
- $(\mathbf{T}_{1,I} \circ \mathbf{T}_{2,I} \circ \mathbf{T}_{3,I})^2(\mathbf{T}_{1,I}(\overline{\mathcal{E}}_4|\emptyset)|\emptyset)$ *is the convex hull of* $(0, 0, \frac{3}{2}, 1, 1, 1)$ *and* $(0, \frac{3}{2}, 0, 1, 1, 1)$.

The reader can verify that $(\mathbf{T}_{1,I} \circ \mathbf{T}_{2,I} \circ \mathbf{T}_{3,I})^3(\mathbf{T}_{1,I}(\overline{\mathcal{E}}_4|\emptyset)|\emptyset)$*is the same as* $(\mathbf{T}_{1,I} \circ \mathbf{T}_{2,I} \circ \mathbf{T}_{3,I})^2(\mathbf{T}_{1,I}(\overline{\mathcal{E}}_4|\emptyset)|\emptyset)$. *To sum up,* $\overline{\mathcal{E}}_1$ *can be approximated by the union of the three convex combinations shown above. The number of convex combinations will increase if we decrease the rounding error. In general, since the assumptions of Theorem 4.9 are satisfied,* $\mathcal{E}$ *coincides with the union of* $(\overline{\mathcal{E}}_i)_{i \in I}$.

We conclude this section by revisiting Example 4.11. Even though the Essential APS does not produce the set of all payoffs attainable by sequentially perfect FAPs, it can be used to characterize some subsets of it.

Example 4.11 continued. *Recall that in this example there are five players and the payoff matrix of single quittings is*

$$R = \begin{pmatrix} 0 & 2 & -\frac{1}{2} & 1 & -1 \\ -\frac{1}{2} & 0 & 2 & 1 & -1 \\ 2 & -\frac{1}{2} & 0 & 1 & -1 \\ -1 & -2 & -3 & 0 & \frac{10}{7} \\ 2 & \frac{7}{2} & \frac{47}{8} & -\frac{5}{12} & 0 \end{pmatrix}.$$

As pointed out on page 19, the essential APS approach only provides an upper bound to the set $\mathcal{E}$. *We will show that there is a sequentially perfect FAP* ι *with an infinite cycle, which should be contrasted with the claim of Theorem 4.9. Then, we will illustrate how the payoff along this FAP can be numerically approximated using our technique.*

The set $H(\iota)$ *will be order equivalent to* $\mathbb{N} \times \mathbb{N}$, *specifically:* $H(\iota) = \{t^{k\omega+n} \mid k, n \in \mathbb{N}\}$ *for a certain sequence* $(t^{k\omega+n})_{k,n \in \mathbb{N}}$. *To formally define this FAP, we need the following auxilary sequence* $(i^n, \lambda^n)_{n \in \mathbb{N}}$ *given by*

n	0	1	2	3	4	5	6	7	$\cdots$	n	$\cdots$
i^n	4	5	1	2	3	1	2	3	$\cdots$	$(n-2) \mod 3$	$\cdots$
λ^n	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$							$\frac{3}{4+14\cdot 4^{n-3}}$	$\cdots$

Define recursively $(t^{k\omega+n})_{k,n \in \mathbb{N}}$ *as follows:*

$$\begin{aligned} t^{kw} &:= 1 - \left(\frac{5}{12}\right)^k, \\ t^{k\omega+n+1} &:= t^{k\omega+n} + (1 - t^{k\omega+n})\lambda^n, \end{aligned}$$

and set the FAP ι *to be*

$$\iota(t) = i^{k\omega+n} \quad \textit{whenever} \quad t \in [t^{k\omega+n}, t^{k\omega+n+1}).$$

We note that $t^{k\omega+n}$ *converges to* 1 *as* $k,n\to\infty$*. Thus, the function* ι *is, in a sense, periodic with period* ∞*.*

The reader can verify that this FAP is sequentially perfect, i.e., $\gamma_{t^{k\omega+n}}(\iota)\geq\vec{0}$ *and* $\gamma^{i^{k\omega+n}}_{t^{k\omega+n}}(\iota)=0$ *for every* $k,n\in\mathbb{N}$*. In fact,*

$$\begin{array}{rcl}\gamma_0(\iota) & = & (0,0,0,0,\frac{5}{7})\\ \gamma_{t^1}(\iota) & = & (1,2,3,0,0),\\ \gamma_{t^2}(\iota) & = & (0,\frac{1}{2},\frac{1}{8},\frac{5}{12},0),\\ \gamma_{t^n}(\iota) & \to & \gamma_0(\iota),\\ \gamma_{t^{k\omega+n}}(\iota) & = & \gamma_{t^{k'\omega+n}}(\iota),\quad \forall k,k',n\in\mathbb{N}.\end{array}$$

We now numerically approximate the payoffs of this orbit using our approach. Specifically, for some fixed $k\geq 1$*, we solve for the set of payoff vectors which are attainable by sequentially perfect FAPs in which players* 1*,* 2*, and* 3 *cycle for* k *times and then players* 4 *and* 5 *quit consecutively. Let* $\mathcal{U}^k$ *be the set of such equilibrium payoffs at the beginning of the cycle, when player* 1 *is the quitter. Then,*

$$\mathcal{U}^k\subseteq((\mathbf{T}_{1,I}\circ\mathbf{T}_{2,I}\circ\mathbf{T}_{3,I})^k\circ\mathbf{T}_{4,I}\circ\mathbf{T}_{5,I})(\mathcal{U}^k|\emptyset).$$

As before, we calculate an upper bound $\overline{\mathcal{U}^k}$ *of* $\mathcal{U}^k$ *using iterations, starting with the set of feasible and rational payoffs:*

$$\overline{\mathcal{U}}^k:=\bigcap_{n=0}^{\infty}\left((\mathbf{T}_{1,I}\circ\mathbf{T}_{2,I}\circ\mathbf{T}_{3,I})^k\circ\mathbf{T}_{4,I}\circ\mathbf{T}_{5,I}\right)^n(\mathcal{R}_I|\emptyset).$$

Since $\vec{0}\notin\overline{\mathcal{U}}^k$*, by the same argument as the one used in the proof of Theorem 4.9, every point in* $\overline{\mathcal{U}}^k$ *can be attained by a sequentially perfect FAP. We provide the set* $\overline{\mathcal{U}}^k$ *for different values of* k*, rounding rational numbers to* $1/1000$*:*

- *The set* $\overline{\mathcal{U}}^1$ *is the convex hull of* $(0,0,\frac{17}{250},\frac{43}{1000},\frac{641}{1000})$*,* $(0,\frac{241}{1000},0,\frac{79}{500},\frac{443}{1000})$*,* $(0,\frac{127}{200},0,\frac{52}{125},0)$*,* $(0,\frac{77}{500},\frac{101}{200},\frac{207}{500},0)$*, and* $(0,0,\frac{589}{1000},\frac{73}{200},\frac{83}{100})$*.*
- *The set* $\overline{\mathcal{U}}^2$ *is the convex hull of* $(0,0,\frac{313}{500},\frac{417}{1000},0)$*,* $(0,\frac{5}{8},0,\frac{417}{1000},0)$*,* $(0,\frac{59}{250},0,\frac{157}{1000},\frac{89}{200})$*, and* $(0,0,\frac{67}{1000},\frac{9}{200},\frac{319}{500})$*.*

The reader can verify that when the payoffs are rounded up to $1/1000$*,* $\overline{\mathcal{U}}^2=\overline{\mathcal{U}}^3$*. Moreover, the convex combination of* $(0,0,\frac{313}{500},\frac{417}{1000},0)$ *and* $(0,\frac{5}{8},0,\frac{417}{1000},0)$ *with weights* $\frac{1}{5}$ *and* $\frac{4}{5}$ *is approximately equal to* $\gamma_{t^2}(\iota)=(0,\frac{1}{2},\frac{1}{8},\frac{5}{12},0)$*.*

5 Extensions

The algorithm we presented in Section 4 characterizes the set of all payoffs that correspond to sequentially perfect FAPs, and allows to construct these FAPs. In general, we may have sequentially perfect absorption paths in which several players quit with positive rates at the same time and/or the set of discontinuities is not well-ordered. In this section we show that the algorithm can be adapted to handle these cases, which suggests that the algorithm may be useful in characterizing the set of payoffs that correspond to sequentially perfect absorption paths in larger classes of games.

5.1 Generalized FAPs

Recall that under an FAP ι, a single player quits with rate one in each connected component (t, t') of $H(\iota)$. We now define a generalization of FAPs, allowing several players to quit at the same time over (t, t').

Definition 5.1 *A Generalized Flesch Absorption Path (GFAP) is a right-continuous, piecewise constant map $\alpha : [0, 1) \to \Delta(I)$, such that the set of its discontinuities including 0, denoted by $H(\alpha)$, is countable and well-ordered.*

Let α be a GFAP. For every $0 \leq t < 1$ and for each player $i \in I$, the value of $\int_t^1 \alpha_i(s)ds$ represents the probability that the play terminates by the action profile (Q^i, C^{-i}) in the interval $[t, 1)$. Since the probability of absorption in $[t, 1)$ is $1 - t$, the *expected payoff* after absorption probability t is given by

$$\gamma_t(\alpha) := \sum_{i \in I} \frac{\int_t^1 \alpha_i(s)ds}{1 - t} \cdot R_i. \tag{13}$$

The notion of sequential perfectness for GFAPs is exactly the same as the one for FAPs:

Definition 5.2 *A GFAP α is* sequentially perfect *if for every $t \in [0, 1)$, $\gamma_t(\alpha) \geq \vec{0}$ and $\gamma_t^i(\alpha) = 0$ whenever $\alpha_i(t) > 0$.*

Let $\underline{\Upsilon}$ be the set of sequentially perfect GFAPs, and set

$$\underline{\mathcal{E}} := \{w \in \mathbb{R}^I \mid \exists \alpha \in \underline{\Upsilon}, \text{ s.t. } w = \gamma_0(\alpha)\}.$$

We will show how $\underline{\mathcal{E}}$ can be characterized and computed using a variation of the Essential APS algorithm. Since the construction is parallel to the one in Section 4, we will skip most of the intermediate steps.

Let us define the digraph $(\underline{I}, \underline{L})$ we need for the algorithm. The vertices are here the non-empty subsets of players $\underline{I} := 2^I \setminus \{\emptyset\}$, and we add an edge between two sets $N, M \in \underline{I}$ if the players in N can be followed by M as quitters in a sequentially perfect GFAP. With analogous arguments as in Lemma 4.1, this is equivalent to the following: $(N, M) \in \underline{L}$ if and only if $N \neq M$ and there exist $(\lambda_i)_{i \in N} \in (0, 1]^N$ and $(\lambda'_i)_{i \in M} \in (0, 1]^M$ such that

$$\begin{cases} \sum_{i \in N} \lambda_i R_{i,j} \geq 0 \ \ \forall j \in M, \\ \sum_{i \in M} \lambda'_i R_{i,j} \leq 0 \ \ \forall j \in N. \end{cases} \tag{14}$$

For each $N \in \underline{I}$, let $\underline{S}_N$ be the set of subsets M with $(N, M) \in \underline{L}$. Let $\mathbb{I}$ be the set of strongly connected components with a typical element $\underline{N} \in \mathbb{I}$. The union of all strongly connected components reachable from $\underline{N}$ is denoted by $\widehat{\underline{N}}$.

We now define the Essential APS operator similarly to the definition in Section 4. For each strongly connected component $\underline{N} \in \mathbb{I}$, every element $N \in \underline{N}$, and every collection of sets $(E_M)_{M\in\underline{N}\cup\widehat{\underline{N}}} \subseteq (\mathcal{R}_I)^{\underline{N}\cup\widehat{\underline{N}}}$, set

$$\underline{\mathbf{T}}_{N,\underline{N}}\left((E_M)_{M\in\underline{N}}|(E_M)_{M\in\widehat{\underline{N}}}\right) := \mathcal{R}_N^0 \cup \Big\{w \in \mathbb{R}^I | \exists\, (\lambda, v) \in [0,1]^N \times \cup_{M\in\underline{S}_N} E_M \text{ s.t. } w = \sum_{i\in N}\lambda_i R_i + \big(1-\sum_{i\in N}\lambda_i\big)v, w_j = 0\ \forall j \in N\Big\},$$

where $\mathcal{R}_N^0 := \{w \in \mathcal{R}_N | w_j = 0\ \forall j \in N\}$. Stack together $\underline{\mathbf{T}}_{N,\underline{N}}$'s as follows:

$$\underline{\mathbf{T}}_{\underline{N}}\left((E_M)_{M\in\underline{N}}|(E_M)_{M\in\widehat{\underline{N}}}\right) := \left(\underline{\mathbf{T}}_{N,\underline{N}}\left((E_M)_{M\in\underline{N}}|(E_M)_{M\in\widehat{\underline{N}}}\right)\right)_{N\in\underline{N}}.$$

Following the same inductive approach as in Section 4, we built the largest invariant sets of this generalized Essential APS operator: for $\underline{N} \in \mathbb{I}$ such that $\widehat{\underline{N}} = \emptyset$, we set

$$\underline{\mathcal{F}}_{\underline{N}} := co\big\{\mathcal{R}_N^0, N \in \underline{N}\big\} \cap \mathbb{R}_+^I \text{ and } (\overline{\underline{\mathcal{E}}}_N)_{N\in\underline{N}} := \bigcap_{n=0}^{\infty}\left(\underline{\mathbf{T}}_{\underline{N}}\right)^n\left((\underline{\mathcal{F}}_{\underline{N}})^{\underline{N}}|\emptyset\right).$$

Then, for arbitrary $\underline{N} \in \mathbb{I}$ such that the sets $(\overline{\underline{\mathcal{E}}}_N)_{N\in\widehat{\underline{N}}}$ are known, define

$$\underline{\mathcal{F}}_{\underline{N}} := co\left\{\mathcal{R}_N^0 \ \cup \bigcup_{M\in\underline{S}_N\cap\widehat{\underline{N}}}\left(\overline{\underline{\mathcal{E}}}_M \cap \{w \in \mathbb{R}_+^I | w_i = 0\ \forall i \in N\}\right), N \in \underline{N}\right\} \cap \mathbb{R}_+^I,$$

and set

$$(\overline{\underline{\mathcal{E}}}_N)_{N\in\underline{N}} := \bigcap_{n=0}^{\infty}\left(\underline{\mathbf{T}}_{\underline{N}}\right)^n\left((\underline{\mathcal{F}}_{\underline{N}})^{|\underline{N}|}|(\overline{\underline{\mathcal{E}}}_N)_{N\in\widehat{\underline{N}}}\right).$$

The following theorem characterizes the set of payoffs which can be implemented by GFAPs. We omit its proof, because it is parallel to the proof of Theorem 4.9.

Theorem 5.3 *Suppose that for every strongly connected component $\underline{N} \in \mathbb{I}$ with at least two elements, $\underline{\mathcal{F}}_{\underline{N}} \cap \{w \in \mathbb{R}^I | w_i = 0,\ \forall N \in \underline{M},\ \forall i \in N\} = \emptyset$ for all simple circuits $\underline{M} \subseteq \underline{N}$. Then,*

$$\underline{\mathcal{E}} = \bigcup_{N\in\underline{I}} \overline{\underline{\mathcal{E}}}_N.$$

Furthermore, for each $w \in \underline{\mathcal{E}}$, there exists a sequentially perfect GFAP α with $\gamma_0(\alpha) = w$ such that the ordinality of $H(\alpha)$ is at most the ordinality of $\mathbb{N}$.

5.2 Characterizing continuous equilibrium payoffs

In this last section, we consider the most general framework of continuous quitting. We adapt the APS approach to approximate the whole set of subgame-perfect equilibrium payoffs where in the corresponding ε-equilibria, players quit with infinitesimal probabilities throughout the play.

Definition 5.4 *A continuous absorption path (CAP) α is a measurable map from $[0,1)$ to $\Delta(I)$.*

An expected payoff path under CAPs is still given by Eq. (13). The notion of sequential perfectness for CAPs is almost identical to the analogous notion for GFAPs.

Definition 5.5 *A CAP α is* sequential perfect *if for every $t \in [0,1)$, $\gamma_t(\alpha) \geq \vec{0}$ and $\gamma_t^i(\alpha) = 0$ for a.e. $t \in [0,1)$ such that $\alpha_i(t) > 0$.*

We are interested in the set of payoffs $\underline{\mathcal{E}}^*$ in which every element can be attained by a sequential perfect CAP:

$$\underline{\mathcal{E}}^* = \big\{w \in \mathbb{R}^I \,|\, \exists \text{ sequentially perfect } \alpha \text{ s.t. } w = \gamma_0(\alpha)\big\}.$$

We shall show how to characterize the set $\underline{\mathcal{E}}^*$. To overcome the problem that a set of discontinuities of CAPs is not necessarily countable and well-ordered, we approach the elements of $\underline{\mathcal{E}}^*$ by the expected payoffs of GFAPs on which we impose a uniform lower bound on their quitting rates and which satisfy a certain relaxed notion of sequential perfectness. More precisely, we define for all $\varepsilon > 0$ the operator

$$\underline{\mathbf{T}}_\varepsilon(E) := \Big\{w \in \mathbb{R}_+^I \,|\, \exists\, (\lambda, v) \in [0,1]^I \times E \text{ s.t. } w = \textstyle\sum_{i\in I} \lambda_i R_i + (1 - \sum_{i\in I} \lambda_i) v, \\ \textstyle\sum_{i \in I} \lambda_i \geq \varepsilon \text{ and } w_j \leq \varepsilon \|R\| \text{ whenever } \lambda_j > 0\Big\},$$

where $\|R\| := \max_{i,j} |R_{i,j}|$ is the max norm of R.

Theorem 5.6 $\underline{\mathcal{E}}^* = \bigcap_{\varepsilon>0} \bigcap_{n=0}^{\infty} (\underline{\mathbf{T}}_\varepsilon)^n(\mathcal{R}_I)$.

Proof. First of all, we unpack $\bigcap_{\varepsilon>0} \bigcap_{n=0}^{\infty} (\underline{\mathbf{T}}_\varepsilon)^n(\mathcal{R}_I)$ and characterize its elements more explicitly. For all $\varepsilon > 0$, $w \in \bigcap_{\varepsilon>0} \bigcap_{n=0}^{\infty} (\underline{\mathbf{T}}_\varepsilon)^n(\mathcal{R}_I)$ if and only if there exists a sequence $(w^n, \lambda^n)_{n\in\mathbb{N}} \subset (\mathbb{R}_+^I \times [0,1]^I)^{\mathbb{N}}$ with $w^0 = w$ such that for each $n \in \mathbb{N}$,

$$\begin{cases} w^n = \sum_{i\in I} \lambda_i^n R_i + (1 - \sum_{i\in I} \lambda_i^n) w^{n+1}, \\ \sum_{i\in I} \lambda_i^n \in [\varepsilon, 1] \text{ and } w_i^n \leq \varepsilon \|R\| \text{ whenever } \lambda_i^n > 0. \end{cases} \tag{15}$$

We now show that, for any fixed $\varepsilon > 0$, $\underline{\mathcal{E}}^* \subseteq \bigcap_{n=0}^{\infty} (\underline{\mathbf{T}}_\varepsilon)^n(\mathcal{R}_I)$. Let α be a sequentially perfect CAP and define $(t^n)_{n\in\mathbb{N}}$ as follows: $t^0 := 0$, $t^{n+1} := t^n + \varepsilon(1 - t^n)$ for all $n \in \mathbb{N}$. Remark that $t^n \nearrow 1$ as $n \to \infty$. In addition, define a sequence $(w^n, \lambda^n)_{n\in\mathbb{N}}$ by setting for each $n \in \mathbb{N}$, $w^n := \gamma_{t^n}(\alpha)$, and

$$\lambda_i^n := \frac{\int_{t^n}^{t^{n+1}} \alpha_s^i ds}{1 - t^n}, \;\; i \in I.$$

Next, we prove that the sequence $(w^n, \lambda^n)_{n\in\mathbb{N}}$ satisfies the relation (15). Since α is sequentially perfect, $w^n \geq \vec{0}$ for all $n \in \mathbb{N}$. By definition, $\sum_{i\in I} \alpha_t^i = 1$ for all $t \in [0,1)$, thus

$\sum_{i\in I}\lambda_i^n=\varepsilon$ for all $n\in\mathbb{N}$.
In this generalized framework it still holds that, for any $t>t_n$,

$$w^n=\sum_{i\in I}\frac{\int_{t^n}^t\alpha_i(s)ds}{1-t}\cdot R_i+\frac{1-t}{1-t^n}\gamma_t(\alpha). \tag{16}$$

Setting $t=t_{n+1}$ in Eq. (16), we recover the relation between w_n and w_{n+1}. Moreover, for some $i\in I$, if $\lambda_i^n>0$, then there exists $\tilde{t}\in(t^n,t^{n+1}]$ such that $\gamma_{\tilde{t}}^i(\alpha)=0$, because $Leb(\{s\in(t^n,t^{n+1}]|\alpha_s^i>0\}\neq 0$ and α is sequentially perfect. Then, Eq. (16) for $t=\tilde{t}$ gives that $w_i^n\le\varepsilon\|R\|$.

Now we look at the reverse direction and let $w\in\bigcap_{\varepsilon>0}\bigcap_{n=0}^{\infty}(\underline{\mathbf{T}}_\varepsilon)^n(\mathcal{R}_I)$. Then, for all $\varepsilon>0$, there exists a sequence $(w^{n,\varepsilon},\lambda^{n,\varepsilon})\in(\mathbb{R}_+^I\times[0,1])^{\mathbb{N}}$ that satisfies (15). Let $(t^{n,\varepsilon})_{n\in N}$ be given by $t^{0,\varepsilon}:=0$, and $t^{n+1,\varepsilon}:=t^{n,\varepsilon}+(1-t^{n,\varepsilon})\sum_{i\in I}\lambda_i^{n,\varepsilon}$ for all $n\in\mathbb{N}$. Since $\sum_{i\in I}\lambda_i^{n,\varepsilon}\ge\varepsilon>0$, the following CAP $\alpha^\varepsilon:[0,1)\to\Delta(I)$ is well-defined:

$$\alpha_t^{\varepsilon,i}=\frac{\lambda_i^{n,\varepsilon}}{\sum_{j\in I}\lambda_j^{n,\varepsilon}}\ \text{ if }\ t\in[t^{n,\varepsilon},t^{n+1,\varepsilon})\ \text{ for } i\in I. \tag{17}$$

Define $\pi^\varepsilon:[0,1)\times A^*\times\mathbb{R}$ as follows: $\pi_t^\varepsilon(Q^i,C^{-i})=\int_0^t\alpha_s^{\varepsilon,i}ds$ and $\pi_t^\varepsilon(a)=0$ for all $a\in A_{\ge 2}^*$. The function π^* is an absorption path as defined in AKRS. Since, by Proposition 4.11 of AKRS, the space of absorption paths is sequentially compact, the sequence $(\pi^\varepsilon)_{\varepsilon>0}$ admits a convergent subsequence. Let π be the associated limit point. For each $t\in[0,1)$, since $\widehat{\pi}_t^\varepsilon:=\sum_{i\in I}\pi_t((Q^i,C^{-i}))=t$ for all ε, we have also $\widehat{\pi}_t=t$ and it follows that $\alpha_t:=(\dot{\pi}_t((Q^i,C^{-i})))_{i\in I}$ defines a CAP. As shown in AKRS, the expected payoff is a continuous function of absorption paths, thus $\gamma(\alpha)=w$. Finally, remark that α is sequentially perfect, because the sequence $(w^{n,\varepsilon},\lambda^{n,\varepsilon})$ satisfies (15) for each $\varepsilon>0$. Taking all pieces together, we conclude that $w\in\underline{\mathcal{E}}^*$. ■

6 Conclusion

The APS approach is an iterative method to calculate a set of equilibrium payoffs (and strategies that attain them) in discounted games. The Average Cost Optimality Equation (see, e.g., Feinberg and Shwartz, 2002) is an analog of the APS approach for undiscounted games, yet we are not aware of an algorithm that uses it for calculating equilibrium payoffs in undiscounted dynamic games.

In this paper we adapted the APS approach to study undiscounted subgame perfect equilibria in quitting games, and provided a practical algorithm for calculating a class of subgame perfect equilibrium payoffs and the corresponding equilibrium strategy profiles in this class of games. It is interesting to know whether the approach can be extended to find *all* subgame perfect equilibrium payoffs in quitting games, and not only those that are supported by FAPs, GFAPs, or CAPs. Since quitting games are both stopping games and stochastic games, it is also interesting to know whether our approach can be extended to more general classes of stopping games and stochastic games.

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