Concentration and robustness of discrepancy–based ABC via Rademacher complexity

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Abstract

Classical implementations of approximate Bayesian computation (ABC) employ summary statistics to measure the discrepancy among the observed data and the synthetic samples generated from each proposed value of the parameter of interest. However, finding suitable summary statistics, that are close to sufficiency, is challenging for most of the complex models for which ABC is required. This issue has motivated a growing literature on summary-free versions of ABC that leverage the discrepancy among the empirical distributions of the observed and synthetic data, rather than focusing on pre-selected summaries. The effectiveness of these solutions has led to an increasing interest in the properties of the corresponding ABC posteriors, with a focus on both concentration and robustness in asymptotic regimes. Although recent contributions have made important advancements along these lines, current theory mostly relies on existence arguments which are not immediate to verify and often yield bounds that are not readily interpretable, thereby limiting the methodological implications of such theoretical results. In this article we address these aspects by developing a novel unified and constructive framework, based on the concept of Rademacher complexity, to study concentration and robustness of ABC posteriors within the general class of integral probability semimetrics (IPS), which includes routinely-implemented discrepancies such as Wasserstein distance and maximum mean discrepancy, and naturally extends classical summary-based ABC. For rejection ABC based on such a general class of semimetrics, we prove that the theoretical properties of the approximate posterior in terms of concentration and robustness directly relate to the asymptotic behavior of the Rademacher complexity of the class of test functions associated to each discrepancy. This result yields a novel understanding of the practical performance of ABC with specific discrepancies, as shown also in empirical studies, and allows to develop novel theory guiding calibration of ABC.

Keywords— ABC, integral probability semimetric, maximum mean discrepancy, Rademacher complexity, Wasserstein distance

1 Introduction

The increasing complexity of statistical models in modern applications has led to a boost in the adoption of ABC methods which allow to perform principled Bayesian inference on a vector of parameters $\theta \in \Theta \subseteq \mathbb{R}^p$ even in situations where the likelihood is intractable and cannot be evaluated numerically; see, e.g., Marin et al. (2012) and Sisson et al. (2018). In fact, ABC only requires that simulating from the model is feasible

and, leveraging this, it constructs an approximate posterior distribution for the parameters by retaining all those values of θ , drawn from the prior distribution, which have produced synthetic samples z_1, \ldots, z_m from the model μ_{θ} that are sufficiently *similar* to the observed data y_1, \ldots, y_n .

As highlighted in Section 2, classical implementations of the above procedure measure similarity among the observed data and the synthetic samples via a suitable distance between pre-selected summary statistics. Consequently, the accuracy of the associated ABC posterior and the practical feasibility of sampling from it inherently depend on the selected summaries and on the tolerance threshold used to assess whether the observed and synthetic data are sufficiently close (e.g., Fearnhead and Prangle, 2012). While requiring a perfect match between the observed and synthetic data would yield to samples from the exact posterior distribution, this option is clearly impractical since such a perfect overlap has either zero or negligible probability in routine applications. Conversely, simple summary statistics and wide thresholds generally ensure high acceptance rates and, therefore, reduced running times, but can also deteriorate the accuracy of the induced ABC posterior. This trade-off has motivated extensive literature aimed at developing more sophisticated summary statistics (e.g., Drovandi et al., 2011; Ruli et al., 2016) along with methods to semi-automatically select effective summaries, hopefully close to sufficiency (e.g., Joyce and Marjoram, 2008; Fearnhead and Prangle, 2012). While these advancements have facilitated the practical adoption of ABC in a variety of complex models, summary-based ABC is still generally prone to a loss of efficiency and remains sensitive to the choice of summaries unless the procedure is accurately calibrated, a challenging task in practice, especially under complex models; see Robert et al. (2011); Marin et al. (2014); Frazier et al. (2018); Li and Fearnhead (2018) and Frazier et al. (2020).

The above disadvantages, combined with the recent renewed interest in statistical inference based on discrepancies (e.g., Baraud et al., 2017; Briol et al., 2019; Bernton et al., 2019b; Chérief-Abdellatif and Alquier, 2022; Dellaporta et al., 2022), have motivated a progressive change of perspective away from classical strategies relying on summary statistics and towards summary-free versions of ABC, typically based on a measure of discrepancy between the empirical distributions of the observed data and the synthetic samples. Notable examples within this class are ABC implementations employing maximum mean discrepancy (MMD) (Park et al., 2016), Kullback–Leibler (KL) divergence (Jiang et al., 2018), Wasserstein distance (Bernton et al., 2019a), energy statistic (Nguyen et al., 2020), Hellinger and Cramer-von Mises distances (Frazier, 2020), and γ -divergence (Fujisawa et al., 2021); see also Gutmann et al. (2018) and Forbes et al. (2021) for other examples of summary-free ABC methods. By overcoming the need to preselect summary statistics, these solutions can effectively prevent the potential information-loss problem of classical ABC implementations, thus yielding evidence of improved performance in simulated studies and illustrative applications (e.g., Park et al., 2016; Bernton et al., 2019a; Fujisawa et al., 2021). These promising empirical results have motivated active research on the theoretical properties of discrepancybased ABC posteriors, with a specific focus on concentration and robustness under different asymptotic regimes for the tolerance threshold and the sample size (Jiang et al., 2018; Bernton et al., 2019a; Nguyen et al., 2020; Frazier, 2020; Fujisawa et al., 2021). Among these regimes, of particular interest in recent theoretical investigations is the situation in which the tolerance threshold progressively shrinks as both n and m diverge (Bernton et al., 2019a; Frazier, 2020; Nguyen et al., 2020). Albeit more challenging than classical asymptotic studies fixing either the threshold or the sample size, this setting can have a superior impact in guiding the choice of discrepancies and calibration of thresholds. However, although available results have the merit of providing theoretical support to several versions of discrepancy-based ABC, current theory assumes the validity of suitable concentration inequalities which are often difficult to verify in practice — unless additional and more stringent conditions are imposed on the model or on the data-generating process — and, as mentioned in Bernton et al. (2019a) and Nguyen et al. (2020), it generally provides bounds that are not immediately interpretable, thereby limiting the methodological and practical potentials of these theoretical results.

In this article, we address the aforementioned drawbacks by introducing an innovative bridge between recent theory on concentration and robustness for discrepancy–based ABC (e.g., Bernton et al., 2019a; Frazier, 2020; Nguyen et al., 2020) and the concept of Rademacher complexity (e.g., Wainwright, 2019, Chapter 4) within the general class of integral probability semimetrics (IPS) (e.g., Müller, 1997; Sriperumbudur et al., 2012), which includes routinely–implemented discrepancies such as MMD and Wasserstein,

and naturally extends classical summary-based ABC. As clarified in Sections 2 and 3, such a perspective, which to the best of our knowledge is novel within the ABC context, not only allows to derive unified and generally-interpretable bounds, with broader methodological implications, for both concentration and robustness of the ABC posterior under a variety of discrepancies, but also leverages more constructive arguments than those currently employed. Among such arguments, the core one is that the Rademacher complexity — a general measure of *richness* of the class of test functions characterizing the chosen IPS discrepancy — decays to zero as n and m diverge to infinity. This sufficient condition essentially implies that the class of test functions employed within IPS-ABC is not excessively large for statistical purposes and, as shown in Section 3, is automatically satisfied by Kolmogorov–Smirnov distance and MMD with bounded kernels — e.g., the routinely implemented Gaussian one — without additional assumptions on the statistical model or on the true underlying data-generating process, apart from the data being i.i.d. This allows for misspecified models and heavy-tailed data-generating processes, possibly contaminated with arbitrary distributions. In contrast, Wasserstein distance and MMD with unbounded kernels require further assumptions on the model or on the data-generating process, even in the i.i.d. setting. This allows us to provide additional constructive intuitions for the innovative theory on Wasserstein-ABC in Bernton et al. (2019a), after noticing that the assumed concentration inequalities can be typically obtained as a consequence of bounds on the Rademacher complexity, in the general IPS class.

The empirical studies in Section 4 showcase the potentials of our results in guiding the practical implementation of discrepancy-based ABC. As discussed in Section 5, besides providing novel and interpretable bounds under constructive arguments within the general IPS class, the previously-unexplored bridge that we create between discrepancy-based ABC and the Rademacher complexity has a wider scope and can open the avenues for an even more general theory by leveraging the broad and active literature on Rademacher complexity (e.g., Bartlett and Mendelson, 2002; Bartlett et al., 2005; Koltchinskii, 2006; Mohri and Rostamizadeh, 2008; Sriperumbudur et al., 2012). For example, although Section 3 focuses for simplicity on the i.i.d. setting, the results in Mohri and Rostamizadeh (2008) potentially allow to extend our concentration and robustness results to non-i.i.d. data. More generally, our contribution can have direct implications even beyond discrepancy-based ABC, especially within the framework of generalized likelihood-free Bayesian inference via pseudo-posteriors based on discrepancies (e.g., Bissiri et al., 2016; Jewson et al., 2018; Miller and Dunson, 2019; Knoblauch et al., 2019; Chérief-Abdellatif and Alquier, 2020; Matsubara et al., 2022; Dellaporta et al., 2022).

2 IPS–ABC and Rademacher complexity

Let $y_{1:n} = (y_1, \ldots, y_n) \in \mathcal{Y}^n$ be an i.i.d. sample from a distribution $\mu^* \in \mathcal{P}(\mathcal{Y})$, where $\mathcal{P}(\mathcal{Y})$ is the space of probability measures on \mathcal{Y} , and assume that \mathcal{Y} is a metric space endowed with a distance ρ . Given a statistical model $\{\mu_{\theta} : \theta \in \Theta \subseteq \mathbb{R}^p\}$ within $\mathcal{P}(\mathcal{Y})$ and a prior distribution π on the parameter θ , classical rejection ABC iteratively samples θ from π , generates a synthetic i.i.d. sample $z_{1:m} = (z_1, \ldots, z_m) \in \mathcal{Y}^m$ from μ_{θ} , and retains θ as a sample from the ABC posterior distribution $\pi_n^{(\varepsilon_n)}$ if the discrepancy $\Delta(z_{1:m}, y_{1:n})$ between the synthetic and observed data is below a pre–specified tolerance threshold $\varepsilon_n \geq 0$, which may depend or not on the sample size n. Although the size m of each synthetic sample can differ from n, in this article we follow common practice in theoretical investigations of ABC posteriors (e.g., Bernton et al., 2019a; Frazier, 2020) and consider the setting with m = n, for simplicity.

As mentioned in Section 1, the above procedure does not sample from the exact posterior distribution $\pi_n(d\theta) \propto \pi(d\theta) \mu_{\theta}^n(y_{1:n})$, but rather from the ABC posterior

$$\pi_n^{(\varepsilon_n)}(d\theta) \propto \pi(d\theta) \int_{\mathcal{Y}^n} \mathbb{1}\{\Delta(z_{1:n}, y_{1:n}) \le \varepsilon_n\} \ \mu_{\theta}^n(dz_{1:n}),$$

whose properties clearly depend on the selected discrepancy $\Delta(z_{1:n}, y_{1:n})$. In traditional summary-based ABC, $\Delta(z_{1:n}, y_{1:n})$ is induced by a suitable distance, often Euclidean, among summary statistics computed from the synthetic and observed data. However, recalling, e.g., Robert et al. (2011); Marin et al. (2014); Frazier et al. (2018); Li and Fearnhead (2018) and Frazier et al. (2020), unless such summaries are

sufficient statistics, this procedure inherently leads to a loss of information and inefficient estimates. To overcome these challenges, recent literature has progressively moved toward the adoption of discrepancies $\mathcal{D}: \mathcal{P}(\mathcal{Y}) \times \mathcal{P}(\mathcal{Y}) \to [0, \infty]$ among the empirical distributions of the synthetic and observed data, that is

$$\Delta(z_{1:n}, y_{1:n}) = \mathcal{D}(\hat{\mu}_{z_{1:n}}, \hat{\mu}_{y_{1:n}}) = \mathcal{D}(n^{-1} \sum_{i=1}^{n} \delta_{z_i}, n^{-1} \sum_{i=1}^{n} \delta_{y_i}),$$

where δ_x denotes the Dirac delta at a generic $x \in \mathcal{Y}$. Popular examples of this approach are ABC strategies based on MMD, KL divergence, Wasserstein distance, energy statistic, Hellinger and Cramer-von Mises distances, and γ -divergence, whose concentration and robustness properties have been studied in Park et al. (2016); Jiang et al. (2018); Bernton et al. (2019a); Nguyen et al. (2020); Frazier (2020) and Fujisawa et al. (2021).

Although most of the aforementioned contributions treat the different discrepancies separately, some of the above proposals share, in fact, a common origin. For instance, MMD, Wasserstein distance and energy statistic belong to the general class of integral probability semimetrics (IPS) (e.g., Müller, 1997; Sriperumbudur et al., 2012), which further includes other examples of potential interest, such as total variation (TV) and Kolmogorov–Smirnov (KS) distances, and recovers specific implementations of summary–based ABC as special cases. Motivated by the richness of this class, we consider a general and unified discrepancy– based ABC, named IPS–ABC, in which \mathcal{D} can be any member of the IPS class in Definition 1, instead of a single pre–selected discrepancy.

Definition 1 (Integral probability semimetric (Müller, 1997)) Let $(\mathcal{Y}, \mathscr{A})$ be a measure space and denote with $g: \mathcal{Y} \to [1, \infty)$ a measurable function. Moreover, let \mathfrak{B}_g be the set of measurable functions $f: \mathcal{Y} \to \mathbb{R}$ such that $||f||_g := \sup_{y \in \mathcal{Y}} |f(y)|/g(y) < \infty$. Then, for $\mathfrak{F} \subseteq \mathfrak{B}_g$, an integral probability semimetric $\mathcal{D}_{\mathfrak{F}}$ among any μ_1 and μ_2 in $\mathcal{P}(\mathcal{Y})$ is defined as

$$\mathcal{D}_{\mathfrak{F}}(\mu_1,\mu_2) := \sup_{f\in\mathfrak{F}} \left| \int f d\mu_1 - \int f d\mu_2 \right|.$$

Definition 1 implies that $\mathcal{D}_{\mathfrak{F}}$ satisfies, for any μ_1, μ_2 , and μ_3 in $\mathcal{P}(\mathcal{Y})$, the following properties

- $(S1) \mathcal{D}_{\mathfrak{F}}(\mu_1,\mu_1) = 0,$
- (S2) $\mathcal{D}_{\mathfrak{F}}(\mu_1,\mu_2) = \mathcal{D}_{\mathfrak{F}}(\mu_2,\mu_1),$
- (S3) $\mathcal{D}_{\mathfrak{F}}(\mu_1,\mu_2) \leq \mathcal{D}_{\mathfrak{F}}(\mu_1,\mu_3) + \mathcal{D}_{\mathfrak{F}}(\mu_3,\mu_2),$

thus clarifying why $\mathcal{D}_{\mathfrak{F}}$ is a probability semimetric. When $\mathcal{D}_{\mathfrak{F}}(\mu_1, \mu_2) = 0$ if and only if $\mu_1 = \mu_2$, then $\mathcal{D}_{\mathfrak{F}}$ is also a metric. As discussed below, several discrepancies commonly employed in the context of summaryfree ABC (Park et al., 2016; Bernton et al., 2019a; Nguyen et al., 2020) can be obtained as special examples of the general IPS class in Definition 1, under suitable choices of \mathfrak{F} ; see also Sriperumbudur et al. (2010, 2012) and Birrell et al. (2022) for related derivations outside the ABC context.

Example 1.1 (Total variation distance) Although the TV distance is not a common choice within discrepancy-based ABC, it still provides a notable example of IPS that can be obtained by letting \mathfrak{F} coincide with the class of measurable functions whose sup-norm is bounded by 1, namely $\mathfrak{F} = \{f : ||f||_{\infty} \leq 1\}$.

Example 1.2 (Kolmogorov–Smirnov distance) Set $\mathcal{Y} = \mathbb{R}$ and let $\mathfrak{F} = \{\mathbb{1}_{(-\infty,a]}\}_{a \in \mathbb{R}}$, then $\mathcal{D}_{\mathfrak{F}}$ is the KS distance, which can be also rewritten as $\mathcal{D}_{\mathfrak{F}}(\mu_1, \mu_2) = \sup_{x \in \mathcal{Y}} |F_1(x) - F_2(x)|$, where F_1 and F_2 are the cumulative distribution functions associated with μ_1 and μ_2 , respectively.

Example 1.3 (Kantorovich metric and Wasserstein distance) Consider the Lipschitz seminorm defined as $||f||_L := \sup\{|f(x) - f(x')|/\rho(x,x') : x \neq x' \text{ in } \mathcal{Y}\}$. Setting $\mathfrak{F} = \{f : ||f||_L \leq 1\}$ yields the Kantorovich metric. Recalling the Kantorovich–Rubinstein theorem (e.g., Dudley, 2018), when \mathcal{Y} is separable such a metric is the dual representation of the Wasserstein distance, which is therefore recovered as a member of the IPS class.

Example 1.4 (Maximum mean discrepancy and energy distance) Given a positive-definite kernel $k(\cdot, \cdot)$ on $\mathcal{Y} \times \mathcal{Y}$, let $\mathfrak{F} = \{f : ||f||_{\mathcal{H}} \leq 1\}$, where \mathcal{H} is the reproducing kernel Hilbert space (RKHS) corresponding to $k(\cdot, \cdot)$. Then $\mathcal{D}_{\mathfrak{F}}$ is the MMD; see Muandet et al. (2017) for a detailed treatment of RKHS and MMD. Since several kernels, such as the Gaussian one, are characterized by an infinite-dimensional feature map and the maximum mean discrepancy among distributions is based on the distance between these features, then MMD is also a natural limiting version of summary-based ABC when the vector of summaries is infinite-dimensional. Finally, recalling Sejdinovic et al. (2013), MMD is inherently related to energy distance, due to the direct correspondence between positive-definite kernels and negative-definite functions.

Example 1.5 (Summary-based distance) In addition to the above interpretation of MMD as a limiting version of summary-based ABC, the IPS class also includes other examples of ABC based on simple summaries. For instance, letting $\mathfrak{F} = \{f_1, \ldots, f_K\}$ in Definition 1, with $K < \infty$, would yield an ABC strategy in which $\mathcal{D}_{\mathfrak{F}}(\mu_1, \mu_2)$ is equal to the largest among the K absolute differences between the userselected functionals computed under μ_1 and μ_2 , respectively. This coincides with comparing summaries under the sup-norm distance instead of the classical Euclidean one.

As anticipated, we generally do not require that $\mathcal{D}_{\mathfrak{F}}$ is a metric, but only a semimetric. By Definition 1, all IPS are semimetrics. However, some IPS are also metrics, such as TV, KS and Wasserstein distances in Examples 1.1–1.3. MMD is a metric if and only if the corresponding kernel is *characteristic*; see e.g., Muandet et al. (2017). Examples of characteristic kernels are the Gaussian kernel $k(x, x') = \exp(-||x - x'||^2/\sigma^2)$ and the Laplace kernel $k(x, x') = \exp(-||x - x'||/\sigma)$. We shall also emphasize that Examples 1.1– 1.5 cover only a subset of the interesting semimetrics belonging to the IPS class (e.g., Sriperumbudur et al., 2010, 2012; Birrell et al., 2022), which therefore provides a broad family of discrepancies that motivate the general rejection IPS-ABC illustrated in Algorithm 1.

Although Algorithm 1 overcomes the need to pre-select summary statistics, it still requires the choice of a suitable discrepancy and the calibration of the tolerance threshold ε_n . This motivates careful studies of the theoretical properties of the ABC posterior under different discrepancies and thresholding schemes. Available results focus on concentration and robustness under three main asymptotic regimes, covering fixed n and $\varepsilon_n \to 0$, fixed ε_n and $n \to \infty$, and, finally, $n \to \infty$ and $\varepsilon_n \to \varepsilon^*$, where $\varepsilon^* = \inf_{\theta \in \Theta} \mathcal{D}_{\mathfrak{F}}(\mu_{\theta}, \mu^*)$ denotes the lowest attainable discrepancy between the assumed statistical model and the true datagenerating process (Jiang et al., 2018; Bernton et al., 2019a; Nguyen et al., 2020; Frazier, 2020; Fujisawa et al., 2021). This latter regime has catalyzed a growing interest in recent theoretical studies of both summary-based (Frazier et al., 2018, 2020) and summary-free (Bernton et al., 2019a; Nguyen et al., 2020; Frazier, 2020) ABC, due to its practical potentials in guiding the choice of the discrepancy and the calibration of the tolerance threshold, both in correctly specified models when $\mu^* = \mu_{\theta^*}$ for some $\theta^* \in \Theta$, and in misspecified contexts where μ^* does not belong to $\{\mu_{\theta} : \theta \in \Theta \subseteq \mathbb{R}^p\}$. Extending the reasoning to the general IPS-ABC, this theory investigates whether the control over $\mathcal{D}_{\mathfrak{F}}(\hat{\mu}_{z_{1:n}}, \hat{\mu}_{y_{1:n}})$ established by Algorithm 1 translates into meaningful upper bounds on the rates of concentration for the sequence of

Algorithm 1: Rejection IPS-ABC

Let $y_{1:n}$ be the observed data, $\{\mu_{\theta} : \theta \in \Theta \subseteq \mathbb{R}^p\}$ the assumed statistical model, π the prior distribution for θ , ε_n the tolerance threshold, and $\mathcal{D}_{\mathfrak{F}}$ the selected discrepancy in the IPS class.

for t = 1, ..., T do repeat Sample $\theta \sim \pi$ Generate $z_{1:n} \stackrel{\text{i.i.d.}}{\sim} \mu_{\theta}$ until $\mathcal{D}_{\mathfrak{F}}(\hat{\mu}_{z_{1:n}}, \hat{\mu}_{y_{1:n}}) \leq \varepsilon_n$; Set $\theta_t = \theta$ end output $\theta_1, ..., \theta_T$ from the IPS-ABC posterior distribution $\pi_n^{(\varepsilon_n)}$ IPS-ABC posteriors around the true data-generating process μ^* , under the discrepancy $\mathcal{D}_{\mathfrak{F}}$. A common strategy to derive these bounds is to study the concentration of the empirical measures $\hat{\mu}_{z_{1:n}}$ and $\hat{\mu}_{y_{1:n}}$ around the corresponding *true* distributions μ_{θ} and μ^* , respectively, after noticing that, as a consequence of (S2)-(S3), $\mathcal{D}_{\mathfrak{F}}(\mu_{\theta}, \mu^*) \leq \mathcal{D}_{\mathfrak{F}}(\hat{\mu}_{z_{1:n}}, \mu_{\theta}) + \mathcal{D}_{\mathfrak{F}}(\hat{\mu}_{z_{1:n}}, \hat{\mu}_{y_{1:n}}) + \mathcal{D}_{\mathfrak{F}}(\hat{\mu}_{y_{1:n}}, \mu^*)$. To this end, current theory assumes that suitable convergence results and concentration inequalities for the empirical measures hold; see, e.g., Assumptions 1–2 and Proposition 3 in Bernton et al. (2019a). However, as clarified by the same authors, this direction provides existence results based on conditions which are often difficult to verify, and generally yields bounds that are not immediately interpretable, thus limiting the methodological and practical potentials of the theoretical statements.

Recalling Definition 1, the aforementioned challenges in studying the concentration of the empirical measures — when contextualized within IPS-ABC — are inherently related to the richness of the class of functions \mathfrak{F} which characterizes each IPS. Intuitively, if \mathfrak{F} is excessively wide, it might be always possible to find a function $f \in \mathfrak{F}$ which yields large discrepancies even when $\hat{\mu}_{z_{1:n}}$ and $\hat{\mu}_{y_{1:n}}$ are arbitrarily close to μ_{θ} and μ^* , respectively. As result, $\mathcal{D}_{\mathfrak{F}}(\hat{\mu}_{z_{1:n}}, \mu_{\theta})$ and $\mathcal{D}_{\mathfrak{F}}(\hat{\mu}_{y_{1:n}}, \mu^*)$ remain large with positive probability, thus making the inequality $\mathcal{D}_{\mathfrak{F}}(\mu_{\theta}, \mu^*) \leq \mathcal{D}_{\mathfrak{F}}(\hat{\mu}_{z_{1:n}}, \mu_{\theta}) + \mathcal{D}_{\mathfrak{F}}(\hat{\mu}_{z_{1:n}}, \hat{\mu}_{y_{1:n}}) + \mathcal{D}_{\mathfrak{F}}(\hat{\mu}_{y_{1:n}}, \mu^*)$ of limited interest, since a low $\mathcal{D}_{\mathfrak{F}}(\hat{\mu}_{z_{1:n}}, \hat{\mu}_{y_{1:n}})$ would not necessarily imply that $\mathcal{D}_{\mathfrak{F}}(\mu_{\theta}, \mu^*)$ is also small. This intuition suggests that the theoretical properties of the IPS-ABC posterior in terms of concentration and robustness could be directly, and more generally, related to the richness of the class of test functions \mathfrak{F} . In Section 3 we prove that this is the case when such a richness is measured by the Rademacher complexity (Bartlett and Mendelson, 2002; Bartlett et al., 2002; Koltchinskii and Panchenko, 2002); see Definition 2 below, and Chapter 4 in Wainwright (2019) for an introduction.

Definition 2 (Rademacher complexity) Given an i.i.d. sample $x_{1:n} = (x_1, \ldots, x_n) \in \mathcal{Y}^n$ from a distribution $\mu \in \mathcal{P}(\mathcal{Y})$, and a class \mathfrak{F} of real-valued measurable functions, the Rademacher complexity of \mathfrak{F} is defined as

$$\mathfrak{R}_{\mu,n}(\mathfrak{F}) = \mathbb{E}_{x_{1:n},\epsilon_{1:n}} \left[\sup_{f \in \mathfrak{F}} \left| \frac{1}{n} \sum_{i=1}^{n} \epsilon_i f(x_i) \right| \right],$$

where $\epsilon_1, \ldots, \epsilon_n$ are *i.i.d.* Rademacher random variables, that is $\mathbb{P}(\epsilon_i = 1) = \mathbb{P}(\epsilon_i = -1) = 1/2$.

As is clear from Definition 2, if $\mathfrak{R}_{\mu,n}(\mathfrak{F})$ is high, it means that the class \mathfrak{F} is large enough to deliver sufficiently flexible functions $f \in \mathfrak{F}$ of $x_{1:n} \in \mathcal{Y}^n$ that can closely interpolate, on average, even full-noise vectors $\epsilon_{1:n} = (\epsilon_1, \ldots, \epsilon_n)$, or alternatively $-\epsilon_{1:n}$, thus providing a suitable measure of the richness of \mathfrak{F} . As a consequence, if $\mathfrak{R}_{\mu,n}(\mathfrak{F})$ is bounded away from zero for any n, then \mathfrak{F} might be overly-wide for statistical purposes since it would be able to closely interpolate, on average, noise vectors of any size. Conversely, a Rademacher complexity $\mathfrak{R}_{\mu,n}(\mathfrak{F})$ decaying to 0 as n diverges implies that \mathfrak{F} is a more parsimonious class and, therefore, cannot effectively interpolate noise vectors of growing size n. This intuition is further highlighted in Lemma 1 below, which also clarifies why the concept of Rademacher complexity can open yet unexplored avenues to advance in the analysis of concentration and robustness properties of general discrepancy-based ABC posteriors under the IPS class.

Lemma 1 (Theorem 4.10 and Proposition 4.12 in Wainwright (2019)) Let $x_{1:n}$ be i.i.d. from some distribution $\mu \in \mathcal{P}(\mathcal{Y})$. Then, for any *b*-uniformly bounded class \mathfrak{F} — i.e., any class \mathfrak{F} of functions f such that $\|f\|_{\infty} \leq b$ — any positive integer $n \geq 1$ and any scalar $\delta \geq 0$, it holds that

$$\mathbb{P}_{x_{1:n}}\left[\mathcal{D}_{\mathfrak{F}}(\hat{\mu}_{x_{1:n}},\mu) \le 2\mathfrak{R}_{\mu,n}(\mathfrak{F}) + \delta\right] \ge 1 - \exp(-n\delta^2/2b^2). \tag{1}$$

Moreover, $\mathbb{P}_{x_{1:n}}\left[\mathcal{D}_{\mathfrak{F}}(\hat{\mu}_{x_{1:n}},\mu) \geq \mathfrak{R}_{\mu,n}(\mathfrak{F})/2 - \sup_{f \in \mathfrak{F}} |\mathbb{E}(f)|/2n^{1/2} - \delta\right] \geq 1 - \exp(-n\delta^2/2b^2).$

Lemma 1 provides general tail bounds for the probability that $\mathcal{D}_{\mathfrak{F}}(\hat{\mu}_{x_{1:n}},\mu)$ takes values above a multiple and below a fraction of the Rademacher complexity of \mathfrak{F} . Recalling the previous discussion on the theoretical properties of IPS-ABC posteriors, these results are of fundamental interest to study the concentration of both $\hat{\mu}_{z_{1:n}}$ and $\hat{\mu}_{y_{1:n}}$, when measured via $\mathcal{D}_{\mathfrak{F}}(\hat{\mu}_{z_{1:n}},\mu_{\theta})$ and $\mathcal{D}_{\mathfrak{F}}(\hat{\mu}_{y_{1:n}},\mu^*)$, respectively.

This is paramount to obtain more constructive theoretical statements which rely on general and verifiable assumptions, while yielding unified and interpretable bounds for concentration and robustness. Our novel theory in Section 3 proves that this is possible when

$$\mathfrak{R}_n(\mathfrak{F}) := \sup_{\mu \in \mathcal{P}(\mathcal{Y})} \mathfrak{R}_{\mu,n}(\mathfrak{F})$$

goes to zero as $n \to \infty$. As clarified in Lemma 1, this is a sensible constructive assumption and, in fact, it provides a necessary and sufficient condition for $\mathcal{D}_{\mathfrak{F}}(\hat{\mu}_{x_{1:n}},\mu)$ to converge to zero in $\mathbb{P}_{x_{1:n}}$ -probability, as $n \to \infty$, for any $\mu \in \mathcal{P}(\mathcal{Y})$.

3 Concentration and robustness of the IPS-ABC posterior

In Sections 3.1–3.2, we present our core results on posterior concentration and robustness of IPS–ABC. As anticipated in Section 2, these results leverage Lemma 1 to connect the properties of the IPS–ABC posterior under the discrepancy $\mathcal{D}_{\mathfrak{F}}$ with the behavior of the Rademacher complexity for the associated family \mathfrak{F} . This requires that \mathfrak{F} is a class of *b*–uniformly bounded functions with $\mathfrak{R}_n(\mathfrak{F}) \to 0$ as $n \to \infty$. Crucially, such assumptions are made directly on the user–selected discrepancy $\mathcal{D}_{\mathfrak{F}}$, rather than on the data–generating process or on the assumed statistical model, thereby providing general constructive arguments to select $\mathcal{D}_{\mathfrak{F}}$ in practice and, potentially, develop new discrepancies with theoretical guarantees of posterior concentration and robustness.

For instance, the two conditions above hold for Kolmogorov–Smirnov distance and MMD with bounded kernels without requiring additional assumptions, thus allowing for misspecified models and heavy-tailed data-generating processes, possibly contaminated with arbitrary distributions. In contrast, the currentlyavailable bounds on the Rademacher complexity ensure that MMD with an unbounded kernel and Wasserstein distance meet the above conditions only under specific models and data-generating processes, even in the i.i.d. setting. For example, although to the best of our knowledge general results for the Rademacher complexity of the Wasserstein distance are not yet available, when \mathcal{Y} is bounded the functions having $||f||_L \leq 1$ are also b-uniformly bounded within \mathcal{Y} and, as recently shown in Sriperumbudur et al. (2010) and Sriperumbudur et al. (2012), the Rademacher complexity for the Wasserstein distance in bounded spaces \mathcal{Y} is guaranteed to converge to zero as $n \to \infty$. Interestingly, this boundedness condition relates to Assumptions 1–2 of Bernton et al. (2019a) which have been verified when \mathcal{Y} is bounded by, e.g., Weed and Bach (2019). Alternatively, leveraging results in Fournier and Guillin (2015), it is possible to impose suitable conditions on the existence of exponential moments to ensure that Assumptions 1-2 are met under the Wasserstein distance. This direction is explored in the supplementary materials of Bernton et al. (2019a) to obtain a more explicit concentration inequality, in the i.i.d. case, relative to the one provided in Proposition 3 by the same authors. As clarified in Section 3.1, the bound presented under this specific Wasserstein-ABC context has connections with the more general concentration result we derive in Theorem 1 for the whole IPS class, thus providing additional constructive intuitions also for the novel theory in Bernton et al. (2019a). Finally, note that, by Lemma 1, a necessary and sufficient condition for Assumption 1 in Bernton et al. (2019a) to hold, under the i.i.d. setting, is that $\mathfrak{R}_{\mu^*,n}(\mathfrak{F}) \to 0$ as $n \to \infty$, which is not generally the case under the Wasserstein distance. This result further motivates our novel perspective through Rademacher complexity, which allows for general and constructive assumptions yielding interpretable bounds for a broader class of discrepancies. As mentioned in Examples 1.4 and 1.5, the IPS family additionally includes both classical and limiting versions of summary-based ABC as a special case, thereby establishing connections also with recent theory on ABC relying on summaries (e.g., Frazier et al., 2018, 2020).

As anticipated in Section 2, for both posterior concentration and robustness, we consider the scenario in which the sample size n goes to infinity, while the ABC threshold ε_n shrinks towards $\varepsilon^* = \inf_{\theta \in \Theta} \mathcal{D}_{\mathfrak{F}}(\mu_{\theta}, \mu^*)$. If the model is well–specified, i.e., $\mu^* = \mu_{\theta^*}$ for some, possibly not unique, $\theta^* \in \Theta$, then $\varepsilon^* = 0$. Although the regimes characterized by $n \to \infty$ with fixed ε_n , and $\varepsilon_n \to 0$ with fixed n are also of interest, such settings can be readily addressed under IPS–ABC via direct adaptations of the results in Jiang et al. (2018) and Bernton et al. (2019a); see also Miller and Dunson (2019) for related derivations in the context of

coarsened posteriors under the regime with $n \to \infty$ and fixed ε_n . For both posterior concentration and robustness, we will rely on the following assumptions.

- (C1) the observed data $y_{1:n} \in \mathcal{Y}^n$ are i.i.d. from the data-generating process μ^*
- (C2) there exist positive L and c_{π} such that, for $\bar{\varepsilon}$ small enough, $\pi (\{\theta \in \Theta : \mathcal{D}_{\mathfrak{F}}(\mu_{\theta}, \mu^*) \leq \varepsilon^* + \bar{\varepsilon}\}) \geq c_{\pi} \bar{\varepsilon}^L$
- (C3) \mathfrak{F} is a set of *b*-uniformly bounded functions; i.e., there exist a $b \in \mathbb{R}$ such that $||f||_{\infty} \leq b$ for all $f \in \mathfrak{F}$
- (C4) $\mathfrak{R}_n(\mathfrak{F}) := \sup_{\mu \in \mathcal{P}(\mathcal{V})} \mathfrak{R}_{\mu,n}(\mathfrak{F})$ goes to zero as $n \to \infty$

Assumption (C1) is the only condition which refers explicitly to the data–generating process and is also made, e.g., in Nguyen et al. (2020) and in Section 2.1 of the supplementary materials in Bernton et al. (2019a). Although exploring IPS–ABC properties in non–i.i.d. contexts is also of interest — see Section 5 for possible extensions to this setting — here we consider the i.i.d. case in (C1) which allows to directly inherit the results in Lemma 1 to obtain more constructive and interpretable bounds that apply whenever the data are exchangeable. Note that, as previously discussed, some of the existence assumptions currently made in the literature may not hold even in the i.i.d. context. Assumption (C2) is common in theoretical studies on discrepancy–based ABC posteriors (see e.g., Bernton et al., 2019a; Nguyen et al., 2020; Frazier, 2020) and essentially requires that the prior distribution π places sufficient mass on the set of parameters associated with those models μ_{θ} closest to μ^* , under the selected discrepancy $\mathcal{D}_{\mathfrak{F}}$. Finally, Assumptions (C3) and (C4) provide conditions on the class \mathfrak{F} to ensure that the induced $\mathcal{D}_{\mathfrak{F}}$ admits interpretable and meaningful bounds for concentration and robustness, leveraging the results in Lemma 1. In contrast to current theory, these last two assumptions rely on simpler constructive arguments which can be broadly verified leveraging the form of the assumed \mathfrak{F} and available bounds on the Rademacher complexity; see Examples 2.1–2.5 below.

Example 2.1 (Total variation distance) Recalling Example 1.1, TV satisfies Assumption (C3) by definition. Conversely, (C4) does not generally hold for TV. For instance, when $\mathcal{Y} = \mathbb{R}$ and μ is continuous, the probability that there exists an index $i \neq i'$ such that $x_i = x_{i'}$, is zero. Thus, with probability 1, for any given vector $\epsilon_{1:n}$ of Rademacher variables, there exists a 1-uniformly bounded function $f_{\epsilon}: \mathcal{Y} \to \{0; 1\}$ such that $f_{\epsilon}(x) = \mathbb{1}_{\{x_i:\epsilon_i=1\}}$. Then, $\sup_{f\in\mathfrak{F}} |(1/n) \sum_{i=1}^n \epsilon_i f(x_i)| \geq (1/n) \sum_{i=1}^n \epsilon_i f_{\epsilon}(x_i) = (1/n) \sum_{i=1}^n \mathbb{1}_{\{\epsilon_i=1\}}$, which implies $\mathfrak{R}_{\mu,n}(\mathfrak{F}) \geq \mathbb{E}_{x_{1:n},\epsilon_{1:n}} \left[(1/n) \sum_{i=1}^n \mathbb{1}_{\{\epsilon_i=1\}} \right] = (1/n) \sum_{i=1}^n \mathbb{P}(\epsilon_i = 1) = 1/2$. Although this counterexample proves that (C4) is not generally satisfied under TV, it is still possible to obtain $\mathfrak{R}_n(\mathfrak{F}) \to 0$ also for TV in specific contexts. This is for instance the case when the cardinality $|\mathcal{Y}|$ of \mathcal{Y} is finite. This result follows from the bound in Lemma 5.2 of Massart (2000) after noticing that, when $|\mathcal{Y}|$ is finite, there will be replicates in $[f(x_1), \ldots, f(x_n)]$ whenever $n > |\mathcal{Y}|$. This means that, as n grows, it will be impossible to find a function within \mathfrak{F} which can closely interpolate any noise vector $(\epsilon_1, \ldots, \epsilon_n)$ with $[f(x_1), \ldots, f(x_n)]$, thereby causing \mathfrak{R}_n to decay toward zero as $n \to \infty$.

Example 2.2 (Kolmogorov–Smirnov distance) Since the class of functions \mathfrak{F} which characterizes the Kolmogorov–Smirnov distance is $\mathfrak{F} = \{1_{(-\infty,a]}\}_{a\in\mathbb{R}}$, Assumption (C3) is met by definition and, unlike for TV, also Assumption (C4) is satisfied without the need to impose additional conditions on the statistical model or on the data–generating process. This follows from the inequality $\mathfrak{R}_{\mu,n}(\mathfrak{F}) \leq 2[\log(n+1)/n]^{1/2}$ in Chapter 4.3.1 of Wainwright (2019) for the KS distance. Such a result is a consequence of known bounds on $\mathfrak{R}_{\mu,n}(\mathfrak{F})$ when \mathfrak{F} is a class of b–uniformly bounded functions such that, for some $\nu \geq 1$, $\operatorname{card}[\mathfrak{F}(x_{1:n})] \leq$ $(n+1)^{\nu}$ for any n and $x_{1:n} = (x_1, \ldots, x_n)$ in \mathcal{Y}^n , where $\operatorname{card}[\mathfrak{F}(x_{1:n})]$ counts the number of unique vectors $[f(x_1), \ldots, f(x_n)]$ obtained by evaluating the functions $f \in \mathfrak{F}$ at $x_{1:n}$. When $\mathfrak{F} = \{1_{(-\infty,a]}\}_{a\in\mathbb{R}}$ each $x_{1:n}$ would divide \mathbb{R} in at most n + 1 intervals and any indicator function $\{1_{(-\infty,a]}\}_{a\in\mathbb{R}}$ in \mathfrak{F} will take value 1 for all $x_i \leq a$ and 0 otherwise, meaning that $\operatorname{card}[\mathfrak{F}(x_{1:n})] \leq (n+1)$. Therefore, applying equation (4.24) in Wainwright (2019), with b = 1 and $\nu = 1$, yields to the bound $\mathfrak{R}_{\mu,n}(\mathfrak{F}) \leq 2[\log(n+1)/n]^{1/2}$ for any $\mu \in \mathcal{P}(\mathcal{Y})$, which implies that $\sup_{\mu \in \mathcal{P}(\mathcal{Y})} \mathfrak{R}_{\mu,n}(\mathfrak{F}) = \mathfrak{R}_n(\mathfrak{F})$ goes to zero as $n \to \infty$ as in (C4).

Example 2.3 (Wasserstein distance) The family $\mathfrak{F} = \{f : ||f||_L \leq 1\}$ which defines the Wasserstein distance is not b-uniformly bounded. However, as discussed, e.g., in Theorem 1.14 of Villani (2021),

the supremum in Definition 1 does not change if the functions in $\mathfrak{F} = \{f : ||f||_L \leq 1\}$ are further constrained to be b-uniformly bounded. Hence, assuming (C3) poses no restrictions under the Wasserstein distance. More problematic is (C4) since, as previously mentioned, we are not yet aware of general upper bounds for the Rademacher complexity of the Wasserstein distance which can be applied in broad contexts without further conditions. However, when \mathcal{Y} is a bounded subset of \mathbb{R}^d , Sriperumbudur et al. (2010) and Sriperumbudur et al. (2012) derived useful upper bounds which directly guarantee that $\mathfrak{R}_n \to 0$ as $n \to \infty$. As a consequence, our theory holds also under the Wasserstein distance, whenever \mathcal{Y} is a bounded subset of \mathbb{R}^d . Although the boundedness of \mathcal{Y} is a strong constraint, note that Weed and Bach (2019) rely on this assumption to prove the concentration inequalities assumed in Bernton et al. (2019a). This complements available theory for Wasserstein–ABC.

Example 2.4 (Maximum mean discrepancy) As is clear from Example 1.4, the properties of MMD directly depend on the selected kernel $k(\cdot, \cdot)$. This is evident from the inequalities $\mathfrak{R}_{\mu,n}(\mathfrak{F}) \leq [\mathbb{E}_x k(x,x)/n]^{1/2}$ with $x \sim \mu$ (Bartlett and Mendelson, 2002, Lemma 22), and $|f(x)| \leq [k(x,x)]^{1/2}||f||_{\mathcal{H}}$ for any $x \in \mathcal{Y}$ (e.g., Hofmann et al., 2008, equation (16)). Such bounds also provide constructive sufficient conditions to verify which kernels yield MMD discrepancies satisfying Assumptions (C3) and (C4). In fact, by these two inequalities, any bounded kernel $k(\cdot, \cdot)$ would ensure that $\mathfrak{R}_n \to 0$ as $n \to \infty$ (C4) and that $||f||_{\infty} \leq b$ (C3), under MMD. Commonly-implement kernels, including the Gaussian $k(x,x') = \exp(-||x-x'||^2/\sigma^2)$ and Laplace $k(x,x') = \exp(-||x-x'||/\sigma)$ kernels, meet this condition and are further bounded by 1, namely $|k(x,x)| \leq 1$ for any $x \in \mathcal{Y}$, thus implying $\mathfrak{R}_n(\mathfrak{F}) \leq n^{-1/2}$ and $||f||_{\infty} \leq 1$. When the kernel is unbounded, the inequality $\mathfrak{R}_{\mu,n}(\mathfrak{F}) \leq [\mathbb{E}_x k(x,x)/n]^{1/2}$ is only informative for those μ such that $\mathbb{E}_x k(x,x) \leq \infty$, with $x \sim \mu$, whereas the bound on |f(x)| does not generally hold unless further assumptions are made, e.g., on \mathcal{Y} ; see Section 3.1 and in particular Proposition 2, for details on the unbounded-kernel case.

Example 2.5 (Summary-based distances) Recalling the discussion on the Kolmogorov-Smirnov distance in Example 2.2, when $\mathfrak{F} = \{f_1, \ldots, f_K\}$, with $K < \infty$, there are, for every $x_{1:n}$, at most K unique vectors $[f(x_1), \ldots, f(x_n)]$ obtained by evaluating the functions $f \in \mathfrak{F} = \{f_1, \ldots, f_K\}$ at $x_{1:n}$. This implies that the cardinality $\operatorname{card}[\mathfrak{F}(x_{1:n})]$ is upper-bounded by K and, as a direct consequence, by n + 1 for any $n + 1 \ge K$. Therefore, applying again equation (4.24) in Wainwright (2019) with $\nu = 1$ yields the bound $\mathfrak{R}_{\mu,n}(\mathfrak{F}) \le 2b[\log(n+1)/n]^{1/2}$ for any $\mu \in \mathcal{P}(\mathcal{Y})$ and any $n \ge K-1$, which also includes $n \to \infty$ since K is finite. This result implies that $\sup_{\mu \in \mathcal{P}(\mathcal{Y})} \mathfrak{R}_{\mu,n}(\mathfrak{F}) = \mathfrak{R}_n(\mathfrak{F})$ goes to zero as $n \to \infty$ (C4), when f_1, \ldots, f_K are b-uniformly bounded functions. This latter condition, which coincides with (C3), requires that the search for suitable summaries is explicitly constrained to the class of b-uniformly bounded functions. Albeit feasible, this option would exclude routinely-used summaries, such as the moments. To employ these more general measures, while still meeting (C3), further constraints are required, such as the boundedness of \mathcal{Y} , already discussed in Example 2.3 for the Wasserstein distance.

Examples 2.1-2.5 clarify that assumptions (C3)–(C4) are met under a variety of discrepancies in the IPS class. As shown in Sections 3.1-3.2, when this happens, it is possible to derive informative and interpretable bounds on concentration and robustness of IPS–ABC posteriors.

3.1 Concentration

Theorem 1 states our main result on concentration of the IPS-ABC posterior as $n \to \infty$ and $\varepsilon_n \to \varepsilon^*$ or, equivalently, $\bar{\varepsilon}_n \to 0$, with $\varepsilon_n = \varepsilon^* + \bar{\varepsilon}_n$. As in the proof of Proposition 3 by Bernton et al. (2019a), also our theoretical derivations move from the triangle inequality $\mathcal{D}_{\mathfrak{F}}(\mu_{\theta},\mu^*) \leq \mathcal{D}_{\mathfrak{F}}(\hat{\mu}_{z_{1:n}},\mu_{\theta}) + \mathcal{D}_{\mathfrak{F}}(\hat{\mu}_{z_{1:n}},\hat{\mu}_{y_{1:n}}) + \mathcal{D}_{\mathfrak{F}}(\hat{\mu}_{y_{1:n}},\mu^*)$, but then, instead of assuming that suitable concentration results for $\mathcal{D}_{\mathfrak{F}}(\hat{\mu}_{z_{1:n}},\mu_{\theta})$ and $\mathcal{D}_{\mathfrak{F}}(\hat{\mu}_{y_{1:n}},\mu^*)$ are satisfied, we leverage Lemma 1 to obtain simpler and automatic concentration results for both discrepancies. In fact, Equation (1) within Lemma 1 ensures that the discrepancy $\mathcal{D}_{\mathfrak{F}}(\hat{\mu}_{y_{1:n}},\mu^*)$ exceeds $2\mathfrak{R}_n(\mathfrak{F})$ with vanishing probability, and similarly for $\mathcal{D}_{\mathfrak{F}}(\hat{\mu}_{z_{1:n}},\mu_{\theta})$. Therefore, when $\mathfrak{R}_n(\mathfrak{F})$ shrinks to 0, we have that $\mathcal{D}_{\mathfrak{F}}(\hat{\mu}_{z_{1:n}},\hat{\mu}_{y_{1:n}}) \approx \mathcal{D}_{\mathfrak{F}}(\mu_{\theta},\mu^*)$ by combining the above triangle inequality with the additional one $\mathcal{D}_{\mathfrak{F}}(\mu_{\theta},\mu^*) \geq -\mathcal{D}_{\mathfrak{F}}(\hat{\mu}_{z_{1:n}},\mu_{\theta}) + \mathcal{D}_{\mathfrak{F}}(\hat{\mu}_{z_{1:n}},\hat{\mu}_{y_{1:n}}) - \mathcal{D}_{\mathfrak{F}}(\hat{\mu}_{y_{1:n}},\mu^*)$. This means that if $\mathcal{D}_{\mathfrak{F}}(\hat{\mu}_{z_{1:n}},\hat{\mu}_{y_{1:n}})$ is small then $\mathcal{D}_{\mathfrak{F}}(\mu_{\theta},\mu^*)$ is also small, with high probability. This clarifies the importance of Assumption (C4), which is further supported by the fact that, if $\mathfrak{R}_n(\mathfrak{F})$ does not shrink to zero, by the second inequality in Lemma 1, the two empirical measures $\hat{\mu}_{y_{1:n}}$ and $\hat{\mu}_{z_{1:n}}$ would not concentrate around the corresponding true distributions, μ^* and μ_{θ} , and therefore there is no guarantee that a small $\mathcal{D}_{\mathfrak{F}}(\hat{\mu}_{z_{1:n}}, \hat{\mu}_{y_{1:n}})$ corresponds to a small $\mathcal{D}_{\mathfrak{F}}(\mu_{\theta}, \mu^*)$; see the Appendix for a detailed proof of Theorem 1.

Theorem 1 (Concentration) Under (S2)–(S3) and (C1)–(C4), take $\bar{\varepsilon}_n \to 0$ as $n \to \infty$, with $n\bar{\varepsilon}_n^2 \to \infty$ and $\bar{\varepsilon}_n/\Re_n(\mathfrak{F}) \to \infty$. Then, for any sequence $M_n > 1$, the IPS-ABC posterior with threshold $\varepsilon_n = \varepsilon^* + \bar{\varepsilon}_n$ satisfies the following concentration inequality

$$\pi_n^{(\varepsilon^* + \bar{\varepsilon}_n)} \left(\{ \theta \in \Theta : \mathcal{D}_{\mathfrak{F}}(\mu_\theta, \mu^*) > \varepsilon^* + 4\bar{\varepsilon}_n/3 + 2\mathfrak{R}_n(\mathfrak{F}) + [(2b^2/n)\log(M_n/\bar{\varepsilon}_n^L)]^{1/2} \} \right) \le (2 \cdot 3^L)/(c_\pi M_n),$$

with $\mathbb{P}_{y_{1:n}}$ -probability going to 1 as $n \to \infty$.

Theorem 1 states a general concentration result which holds for any sequence $M_n > 1$, including those guaranteeing effective contraction of the IPS-ABC posterior. To this end, it suffices to let $M_n \to \infty$ as $n \to \infty$, which holds, for example, when $M_n = n$. Under this setting, the concentration result in Theorem 1 becomes $\pi_n^{(\varepsilon^* + \bar{\varepsilon}_n)}(\{\theta \in \Theta : \mathcal{D}_{\mathfrak{F}}(\mu_{\theta}, \mu^*) > \varepsilon^* + 4\bar{\varepsilon}_n/3 + 2\mathfrak{R}_n(\mathfrak{F}) + [(2b^2/n)\log(n/\bar{\varepsilon}_n^L)]^{1/2}\}) \leq (2 \cdot 3^L)/(c_{\pi}n)$. As a consequence, under the settings of Theorem 1, the quantities $4\bar{\varepsilon}_n/3$, $2\mathfrak{R}_n(\mathfrak{F})$, $[(2b^2/n)\log(n/\bar{\varepsilon}_n^L)]^{1/2}$ and $(2 \cdot 3^L)/(c_{\pi}n)$ are guaranteed to converge to zero when $n \to \infty$, meaning that the IPS-ABC posterior asymptotically concentrates on those values of θ yielding a model μ_{θ} within discrepancy ε^* from the datagenerating process μ^* . Since ε^* is the lowest attainable discrepancy among μ_{θ} and μ^* , for $\theta \in \Theta$, this result further implies that the IPS-ABC posterior concentrates on those values of θ yielding $\mathcal{D}_{\mathfrak{F}}(\mu_{\theta}, \mu^*) = \varepsilon^*$.

In contrast to most state–of–the–art theory, this concentration result relies on more explicit bounds which involve interpretable quantities, and are derived under constructive arguments with direct methodological implications. For instance, combining Theorem 1 with Examples 2.1–2.5 provides clear sufficient conditions to ensure concentration of IPS–ABC posteriors under a variety of popular discrepancies, and further guides the choice of the threshold ε_n . In fact, note that to make the bound as tight as possible, we must essentially choose $\bar{\varepsilon}_n$ to ensure that the terms $4\bar{\varepsilon}_n/3$ and $[(2b^2/n)\log(n/\bar{\varepsilon}_n^L)]^{1/2}$ are of the same order of magnitude. Neglecting terms in log log n, this would lead to set $\bar{\varepsilon}_n$ of the order $[\log(n)/n]^{1/2}$. However, it might be that the constraint $\bar{\varepsilon}_n/\Re_n(\mathfrak{F}) \to \infty$ is not satisfied in this case, thereby requiring the choice of a larger $\bar{\varepsilon}_n$, such as $\Re_n(\mathfrak{F}) \log \log(n)$. Summarizing, the setting $\bar{\varepsilon}_n = \max\{[\log(n)/n]^{1/2}, \Re_n(\mathfrak{F}) \log \log(n)\}$ will lead to $4\bar{\varepsilon}_n/3 + 2\Re_n(\mathfrak{F}) + [(2b^2/n)\log(n/\bar{\varepsilon}_n^L)]^{1/2} = \mathcal{O}(\bar{\varepsilon}_n)$, thus providing direct methodological implications, while clarifying that the choice of ε_n can be guided by suitable upper bounds on $\Re_n(\mathfrak{F})$, when available, for any discrepancy in the IPS class.

The above argument is further clarified in Proposition 1 which specializes Theorem 1 to the context of MMD induced by bounded kernels, such as the routinely-used Gaussian and Laplace kernels. As discussed in Example 2.4, the MMD discrepancy associated with such kernels satisfies condition (C3) — with b = 1 — by definition, and further meets (C4) without requiring assumptions on the model μ_{θ} or on the datagenerating process μ^* , such as boundedness of \mathcal{Y} or existence of moments. This allows direct application of Theorem 1 to obtain a concentration result of immediate practical use, leveraging the fact that, under MMD with Gaussian and Laplace kernels, $\Re_n(\mathfrak{F}) \leq n^{-1/2}$; see Example 2.4. This upper bound implies that $\Re_n(\mathfrak{F}) \log \log(n) \leq \log \log(n)/n^{1/2} \leq [\log(n)/n]^{1/2}$ for any $n \geq 1$, and hence, as a consequence of the previous discussion, the rate is essentially minimized by setting $\bar{\varepsilon}_n = [\log(n)/n]^{1/2}$; see Proposition 1.

Proposition 1 (Concentration of MMD with bounded kernels) Consider the MMD discrepancy $\mathcal{D}_{\mathfrak{F}}$ induced by a generic bounded kernel $k(\cdot, \cdot)$ on \mathbb{R}^d , with $|k(x, x)| \leq 1$ for any $x \in \mathbb{R}^d$. Notable examples are the Gaussian $k(x, x') = \exp(-||x-x'||^2/\sigma^2)$ and Laplace $k(x, x') = \exp(-||x-x'||/\sigma)$ kernels on \mathbb{R}^d . Then, for any $\mu_{\theta} \in \mathcal{P}(\mathcal{Y})$ and $\mu^* \in \mathcal{P}(\mathcal{Y})$, under Assumptions (C1)–(C2) and considering the same settings of Theorem 1 with $M_n = n$ and $\bar{\varepsilon}_n = [\log(n)/n]^{1/2}$, it holds that

$$\pi_n^{(\varepsilon^* + \bar{\varepsilon}_n)} \left(\{ \theta \in \Theta : \mathcal{D}_{\mathfrak{F}}(\mu_\theta, \mu^*) > \varepsilon^* + [10/3 + (L+2)^{1/2}] \cdot [\log(n)/n]^{1/2} \right) \le (2 \cdot 3^L)/(c_\pi n),$$

with $\mathbb{P}_{y_{1:n}}$ -probability going to 1 as $n \to \infty$.

To derive the above expression for the bound it is sufficient to first plug $M_n = n$, $\bar{\varepsilon}_n = [(\log n)/n]^{1/2}$ and b = 1 into the inequality in Theorem 1, and then derive a simple upper-bound for the resulting expression of the radius, which is possible by leveraging the inequalities $\Re_n(\mathfrak{F}) \leq n^{-1/2}$ and $\log(n) \geq 1$ — where the last one holds for every $n \geq 3$, and hence for $n \to \infty$. Recall that that $\pi_n^{(\varepsilon^* + \bar{\varepsilon}_n)} (\{\theta \in \Theta : \mathcal{D}_{\mathfrak{F}}(\mu_{\theta}, \mu^*) > c)\} \leq b$ implies $\pi_n^{(\varepsilon^* + \bar{\varepsilon}_n)} (\{\theta \in \Theta : \mathcal{D}_{\mathfrak{F}}(\mu_{\theta}, \mu^*) > c)\} \leq b$, for any $\bar{c} > c$.

Proposition 1 is an effective example of the practical potentials of Theorem 1 which, in the context of MMD with Gaussian and Laplace kernels, ensures posterior concentration without requiring assumptions on μ_{θ} or μ^* , and explicitly guides the choice of the threshold by pointing toward $\bar{\varepsilon}_n = [\log(n)/n]^{1/2}$. Moreover, as expected from our previous discussion on Theorem 1, $[\log(n)/n]^{1/2}$ is also the quantity controlling the rate of contraction of the radius towards the minimum achievable discrepancy between μ_{θ} and μ^* . Similar results can be obtained for all the discrepancies in Examples 1.1–1.5 as long as Assumptions (C3)–(C4) are satisfied and $\mathfrak{R}_n(\mathfrak{F})$ admits explicit upper bounds as for MMD with bounded kernels. For example, when \mathcal{Y} is bounded, this would be possible for the Wasserstein distance in Example 2.3, leveraging the upper bounds for $\mathfrak{R}_n(\mathfrak{F})$ derived by Sriperumbudur et al. (2010) and Sriperumbudur et al. (2012), thus complementing the Wasserstein–ABC theory in Bernton et al. (2019a). In fact, the $[\log(n)/n]^{1/2}$ term that arises from our general derivations, also appear in Section 2.1 of the supplementary materials in Bernton et al. (2019a) which, for Wasserstein–ABC, provide a more explicit bound than the one in their Proposition 3, under i.i.d. settings and leveraging exponential moments conditions (Fournier and Guillin, 2015).

As clarified in Proposition 2 below, related conditions are also required under MMD with unbounded kernels since (C3) and (C4) are not generally satisfied without further assumptions, e.g., on μ_{θ} and μ^* . In this setting it is however possible to revisit the results for the Wasserstein distance in Proposition 3 of Bernton et al. (2019a) under the novel Rademacher complexity framework developed in the present article. In particular, as shown in Proposition 2, under MMD with unbounded kernel, the existence Assumptions 1–2 in Bernton et al. (2019a) can be directly related to constructive conditions on the kernel — inherently related to our Assumption (C4) — which in turn yield explicit bounds that are reminiscent of those provided in Theorem 1 and Proposition 1; see the Appendix for a detailed proof of Proposition 2.

Proposition 2 (Concentration of MMD with unbounded kernels) Consider the MMD discrepancy $\mathcal{D}_{\mathfrak{F}}$ induced by a generic unbounded kernel $k(\cdot, \cdot)$ on \mathbb{R}^d . A key example is the polynomial kernel $k(x, x') = (1 + a \langle x, x' \rangle)^q$ for some integer $q \in \{2, 3, \ldots\}$, and some a > 0, defined for any $x, x' \in \mathbb{R}^d$. In addition, assume (C2) along with (A1) $\mathbb{E}_y[k(y,y)] < \infty$, (A2) $\int_{\Theta} \mathbb{E}_z[k(z,z)] \pi(\mathrm{d}\theta) < \infty$, and (A3) there exists $a \delta_0 > 0$ and $a c_0 > 0$ such that $\mathbb{E}_z[k(z,z)] < c_0$ for any θ satisfying $(\mathbb{E}_{z,z'}[k(z,z')] - 2\mathbb{E}_{z,y}[k(y,z)] + \mathbb{E}_{y,y'}[k(y,y')])^{1/2} \leq \varepsilon^* + \delta_0$, where $z, z' \sim \mu_{\theta}$ and $y, y' \sim \mu^*$. Then, as $n \to \infty$ and $\bar{\varepsilon}_n \to 0$, the IPS-ABC posterior with threshold $\varepsilon_n = \varepsilon^* + \bar{\varepsilon}_n$ satisfies, for some $C \in (0, \infty)$ and any $M_n \in (0, \infty)$, the inequality

$$\pi_n^{(\varepsilon^* + \bar{\varepsilon}_n)} \left(\{ \theta \in \Theta : \mathcal{D}_{\mathfrak{F}}(\mu_\theta, \mu^*) > \varepsilon^* + 4\bar{\varepsilon}_n/3 + M_n^2/(n^2 \bar{\varepsilon}_n^{2L}) \} \right) \le C/M_n,$$

with $\mathbb{P}_{y_{1:n}}$ -probability going to 1 as $n \to \infty$.

Proposition 2 provides a concentration result which is comparable to those in Theorem 1 and Proposition 1, but holds also when (C1), (C3)–(C4) are not necessarily met and can be substituted by (A1)–(A3). For instance, under the polynomial kernel $k(x, x') = (1 + a \langle x, x' \rangle)^q$ it can be easily shown that if $\mathbb{E}_y[||y||^q] \leq \infty$ and $\theta \mapsto \mathbb{E}_z(||z||^p)$ is π -integrable, then (A1)–(A3) are satisfied. Note that Assumptions (A1)–(A3) essentially require that the expected value of the kernel is finite under both μ^* and μ_{θ} for suitable $\theta \in \bar{\Theta} \subseteq \Theta$, and uniformly bounded for those μ_{θ} close to μ^* . Recalling Example 2.4 and the bound $\mathfrak{R}_{\mu,n}(\mathfrak{F}) \leq [\mathbb{E}_x k(x,x)/n]^{1/2}$, with $x \sim \mu$, these two conditions are inherently related to (C4), which however further requires that these expectations are finite for any $\mu \in \mathcal{P}(\mathcal{Y})$. This clarifies again the relevance of the new proposed theoretical framework and its connection with available theory. Finally, note that in this case a sensible setting for $\bar{\varepsilon}_n$ would be $\bar{\varepsilon}_n = (M_n/n)^{2/(2L+1)}$, thereby leading to $\pi_n^{(\varepsilon^* + \bar{\varepsilon}_n)}(\{\theta \in \Theta : \mathcal{D}_{\mathfrak{F}}(\mu_{\theta}, \mu^*) > \varepsilon^* + (7/3)(M_n/n)^{2/(2L+1)}\}) \leq C/M_n$ that is essentially the tighter possible order of magnitude for the bound. In this case, $M_n = n$ is not suitable, but any $M_n \to \infty$ slower than n can work; e.g., $M_n = n^{1/2}$ yields $\pi_n^{(\varepsilon^* + \bar{\varepsilon}_n)}(\{\theta \in \Theta : \mathcal{D}_{\mathfrak{F}}(\mu_{\theta}, \mu^*) > \varepsilon^* + (7/3)(\mu_n, \mu^*) > \varepsilon^* + (7/3)(1/n)^{1/(2L+1)}\}) \leq C/n^{1/2}$.

To conclude our analysis of the concentration properties of IPS-ABC posteriors, we shall emphasize that, although Theorem 1 and Propositions 1-2 are stated for neighborhoods in the space of distributions,

similar results can be readily derived in the space of parameters. To this end, it suffices to adapt Corollary 1 in Bernton et al. (2019a) to our general framework, under the same additional assumptions of Bernton et al. (2019a) and Frazier (2020), which are stated below for the whole IPS class.

- (C5) The minimizer θ^* of $\mathcal{D}_{\mathfrak{F}}(\mu_{\theta},\mu^*)$ exists and is well separated, meaning that for any $\delta > 0$ there is a $\delta' > 0$ such that $\inf_{\theta \in \Theta: \rho(\theta,\theta^*) > \delta} \mathcal{D}_{\mathfrak{F}}(\mu_{\theta},\mu^*) > \mathcal{D}_{\mathfrak{F}}(\mu_{\theta^*},\mu^*) + \delta';$
- (C6) The parameters are identifiable, and there exist K > 0, $\nu > 0$ and an open neighborhood $U \subset \Theta$ of θ^* such that, for any $\theta \in U$, it holds that $\rho(\theta, \theta^*) \leq K \left[\mathcal{D}_{\mathfrak{F}}(\mu_{\theta}, \mu^*) \varepsilon^*\right]^{\nu}$.

Assumptions (C5) and (C6) essentially require that the parameters are identifiable, sufficiently well– separated, and that the distance between the parameters has some reasonable correspondence with the discrepancy among the associated distributions. Notably, these assumptions have been checked in Chérief-Abdellatif and Alquier (2022) for a class of models including Gaussian and Cauchy ones, with $\rho(\theta, \theta')$ being the Euclidean distance, $\mathcal{D}_{\mathfrak{F}}$ the MMD generated by a Gaussian kernel, and exponent $\nu = 1$. Under (C5) and (C6), it is possible to state Corollary 1.

Corollary 1 (Concentration around θ^*) Under (S2)-(S3), (C1)-(C4) and (C5)-(C6), take $\bar{\varepsilon}_n \to 0$ as $n \to \infty$, with $n\bar{\varepsilon}_n^2 \to \infty$ and $\bar{\varepsilon}_n/\Re_n(\mathfrak{F}) \to \infty$. Then, for any sequence $M_n > 1$, the IPS-ABC posterior with threshold $\varepsilon_n = \varepsilon^* + \bar{\varepsilon}_n$ satisfies the following concentration inequality

$$\pi_n^{(\varepsilon^* + \bar{\varepsilon}_n)} \left(\{ \theta \in \Theta : \rho(\theta, \theta^*) > K[4\bar{\varepsilon}_n/3 + 2\mathfrak{R}_n(\mathfrak{F}) + [(2b^2/n)\log(M_n/\bar{\varepsilon}_n^L)]^{1/2}]^\nu \} \right) \le (2 \cdot 3^L)/(c_\pi M_n),$$

with $\mathbb{P}_{y_{1:n}}$ -probability going to 1 as $n \to \infty$.

The proof of Corollary 1 follows directly from Bernton et al. (2019a) after replacing the original bound derived by the authors with the one that we obtained in Theorem 1, thus allowing to inherit the previous discussion after Theorem 1 and Propositions 1–2 also when concentration is measured directly within the parameter space via $\rho(\theta, \theta^*)$. For instance, under the examples mentioned above — where $\rho(\theta, \theta')$ is the Euclidean distance and $\nu = 1$ — this implies that whenever $\Re_n(\mathfrak{F}) = \mathcal{O}(n^{-1/2})$ the contraction rate will be in the order of $\mathcal{O}([\log(n)/n]^{1/2})$, that is the expected rate in parametric models.

Theorem 1, Propositions 1–2 and Corollary 1 provide general concentration results under both well– specified and miss–specified settings. When $\mu^* \in {\mu_{\theta} : \theta \in \Theta \subseteq \mathbb{R}^p}$, that is $\mu^* = \mu_{\theta^*}$ for some, possibly not unique, $\theta^* \in \Theta$, it follows that $\varepsilon^* = \inf_{\theta \in \Theta} \mathcal{D}_{\mathfrak{F}}(\mu_{\theta}, \mu^*) = \inf_{\theta \in \Theta} \mathcal{D}_{\mathfrak{F}}(\mu_{\theta}, \mu_{\theta^*}) = 0$. Therefore, in well– specified settings, the proposed theory ensures that the IPS–ABC posterior with threshold $\varepsilon_n \to \varepsilon^* = 0$ concentrates all its mass at the values of θ yielding $\mu_{\theta} = \mu^*$. When instead the model is miss–specified, Theorem 1 and Propositions 1–2 guarantee that, asymptotically, the IPS–ABC posterior with threshold $\varepsilon_n \to \varepsilon^* > 0$ concentrates on those θ s yielding a model μ_{θ} within discrepancy ε^* from the data–generating process μ^* . Since ε^* is the lowest attainable discrepancy between μ^* and $\mu_{\theta} \in {\mu_{\theta} : \theta \in \Theta \subseteq \mathbb{R}^p}$, this result further implies that the IPS–ABC posterior essentially places all its mass on those values of θ corresponding to a model μ_{θ} which is the closest possible to μ^* , under the discrepancy $\mathcal{D}_{\mathfrak{F}}$ considered. As clarified in Section 3.2 below, such a behavior can be beneficial in terms of robustness.

3.2 Robustness

Discrepancy-based ABC, including IPS-ABC, has direct connections with the concept of coarsened posterior (Miller and Dunson, 2019) and, therefore, is naturally prone to robust inference. Nonetheless, as for any property of ABC posteriors, also in this case the choice of the discrepancy is expected to play a major role in controlling the amount of robustness to some perturbations of the model (e.g., Ruli et al., 2020; Frazier, 2020; Frazier et al., 2020; Frazier and Drovandi, 2021; Fujisawa et al., 2021). This has motivated increasing research on robust discrepancies and on the properties of the induced ABC posterior, with a main focus on local perturbations of the model. Within this context, Frazier (2020) provides effective bounds on asymptotic bias for suitable functionals of the ABC posterior under the Hellinger and Cramer-von Mises distances, whereas Fujisawa et al. (2021) study the sensitivity curve of ABC posterior induced by

 γ -divergence in Huber contamination models and derive further concentration properties under a number of constraints on the contaminating distribution μ_C .

Motivated by the results in Section 3.1, we derive below general robustness properties, under the same constructive assumptions of Theorem 1, with a focus on the whole ABC posterior within the IPS class. Besides studying the robustness of an entire class of discrepancy-based ABC strategies for which similar results are not yet available, this perspective allows to obtain general concentration statements beyond the analysis of asymptotic bias and without constraints on the contaminating distributions μ_C . Following related studies in available literature, these results are provided under the Huber contamination model. In this setting, most of the observations in $y_{1:n}$ are sampled from an uncontaminated distribution μ_{θ^*} which belongs to the assumed statistical model $\{\mu_{\theta} : \theta \in \Theta \subseteq \mathbb{R}^p\}$, whereas the remaining data are drawn from an arbitrary contaminating distribution μ_C . Therefore, the data-generating mechanism μ^* is defined as

$$\mu^* = (1 - \alpha_n)\mu_{\theta^*} + \alpha_n\mu_C,\tag{2}$$

where $\alpha_n \in [0, 1)$ controls the fraction of contaminated data from μ_C . In contrast to the general concentration results provided in Section 3.1 — where the focus is on μ^* — in assessing robustness properties of the ABC posterior under model (2), it is of interest to study whether concentration is achieved in a neighborhood of μ_{θ^*} , rather than μ^* . Although this objective is different from those addressed in Section 3.1, Theorem 1 has direct implications also in terms of robustness under model (2). In fact, the discrepancies $\mathcal{D}_{\mathfrak{F}}(\mu_{\theta}, \mu^*)$ analyzed in Section 3.1 can be suitably related to those of interest in robustness studies, i.e., $\mathcal{D}_{\mathfrak{F}}(\mu_{\theta}, \mu_{\theta^*})$, via simple inequalities. To this end, first notice that, under model (2), Definition 1 and Assumption (C3) it follows that

$$\varepsilon^* \leq \mathcal{D}_{\mathfrak{F}}(\mu_{\theta^*}, \mu^*) = \mathcal{D}_{\mathfrak{F}}(\mu_{\theta^*}, (1 - \alpha_n)\mu_{\theta^*} + \alpha_n\mu_C) = \alpha_n\mathcal{D}_{\mathfrak{F}}(\mu_{\theta^*}, \mu_C) \leq 2b\alpha_n.$$
(3)

The inequality $\alpha_n \mathcal{D}_{\mathfrak{F}}(\mu_{\theta^*}, \mu_C) \leq 2b\alpha_n$ is a consequence of Assumption (C3), which ensures that $-b \leq \int f d\mu_{\theta^*} \leq b$ and $-b \leq \int f d\mu_C \leq b$, and hence $\mathcal{D}_{\mathfrak{F}}(\mu_{\theta^*}, \mu_C) = \sup_{f \in \mathfrak{F}} \left| \int f d\mu_{\theta^*} - \int f d\mu_C \right| \leq 2b$. Equation (3) together with a direct application of the triangle inequality, also imply

$$\mathcal{D}_{\mathfrak{F}}(\mu_{\theta}, \mu^*) \ge \mathcal{D}_{\mathfrak{F}}(\mu_{\theta}, \mu_{\theta^*}) - \mathcal{D}_{\mathfrak{F}}(\mu_{\theta^*}, \mu^*) \ge \mathcal{D}_{\mathfrak{F}}(\mu_{\theta}, \mu_{\theta^*}) - 2b\alpha_n, \tag{4}$$

since, as shown in (3), $\mathcal{D}_{\mathfrak{F}}(\mu_{\theta^*}, \mu^*) \leq 2b\alpha_n$. Combining the two above inequalities with Theorem 1 yields the desired robustness result stated in Theorem 2.

Theorem 2 (Robustness) Consider the Huber contamination model $\mu^* = (1 - \alpha_n)\mu_{\theta^*} + \alpha_n\mu_C$ in (2), where $\alpha_n \in [0, 1)$, and μ_C is an arbitrary distribution. Then, under the same assumptions as in Theorem 1 and for the same choice of $\bar{\varepsilon}_n$, we have that, for every sequence $M_n > 1$ and $\alpha_n \in [0, 1)$, the IPS-ABC posterior with threshold $\varepsilon_n = \varepsilon^* + \bar{\varepsilon}_n$ satisfies the following concentration inequality

$$\pi_n^{(\varepsilon^* + \bar{\varepsilon}_n)} \left(\{ \theta \in \Theta : \mathcal{D}_{\mathfrak{F}}(\mu_\theta, \mu_{\theta^*}) > 4b\alpha_n + 4\bar{\varepsilon}_n/3 + 2\mathfrak{R}_n(\mathfrak{F}) + [(2b^2/n)\log(M_n/\bar{\varepsilon}_n^L)]^{1/2} \} \right) \le (2 \cdot 3^L)/(c_\pi M_n),$$

with $\mathbb{P}_{y_{1:n}}$ -probability going to 1 as $n \to \infty$.

To clarify how Theorem 2 relates to Theorem 1 simply notice that, by inequalities (3)–(4), the statement in Theorem 1 holds also when replacing $\mathcal{D}_{\mathfrak{F}}(\mu_{\theta},\mu^*)$ with $\mathcal{D}_{\mathfrak{F}}(\mu_{\theta},\mu_{\theta^*}) - 2b\alpha_n$, and ε^* with $2b\alpha_n$, respectively, thereby obtaining Theorem 2. Under the same reasoning, it is also possible to extend Corollary 1 to obtain a robustness statement with respect to neighborhoods of θ^* rather than μ_{θ^*} . Interestingly, Theorem 2 ensures that, under the same assumptions underlying the concentration results in Section 3.1, the whole IPS-ABC posterior asymptotically contracts in a neighborhood of the uncontaminated model μ_{θ^*} of radius at most $4b\alpha_n$, thereby guaranteeing robust inference when the perturbation is minor. In particular, when the amount of contamination — measured in model (2) by α_n — decays to zero for $n \to \infty$, as e.g., in Frazier (2020), then Theorem 2 guarantees that the IPS-ABC posterior asymptotically concentrates at the parameter θ^* associated with the uncontaminated model μ_{θ^*} , thereby ensuring robustness.

The practical usefulness of the above results can be further appreciated by combining Theorem 2 with the convergence properties stated in Proposition 1 for MMD–ABC relying on bounded kernels. Since

this divergence meets the assumptions underlying Theorem 1 without requiring constraints on the datagenerating process or on the assumed model, its robustness properties hold regardless the type of contamination μ_C . This is in contrast with the theory in Fujisawa et al. (2021) which imposes moment conditions on μ_C . Similar constraints may still be required under some discrepancies within the IPS class. Nonetheless, these are the same requirements discussed in Examples 2.1–2.5 for Assumptions (C3) and (C4) to hold. It shall also be emphasized that (C3) plays a dual role in Theorem 2, in that it is not only necessary to apply Lemma 1, but it is also a key to find an informative upper bound for ε^* in Equation (3).

Section 4 below illustrates how the novel concentration and robustness results derived in Sections 3.1– 3.2 can yield improved understanding of the empirical performance of ABC with specific discrepancies.

4 Empirical studies

Several empirical studies have appeared recently to quantify both concentration and robustness properties of summary-free ABC under different discrepancy measures (e.g., Jiang et al., 2018; Bernton et al., 2019a; Frazier, 2020; Nguyen et al., 2020; Fujisawa et al., 2021; Drovandi and Frazier, 2022). We contribute to these analyses by focusing on an alternative challenging setting based on a Cauchy-tail contamination, instead of the commonly-assumed Gaussian-location one. This analysis is particularly interesting since, as discussed in Sections 3.1–3.2, available theory often requires assumptions on the existence of moments, which are not met under a Cauchy-type contamination. Therefore, this setting provides another insightful scenario to assess the performance of the most widely-implemented examples in the IPS class, namely MMD with Gaussian kernel (Park et al., 2016; Nguyen et al., 2020) and Wasserstein distance (Bernton et al., 2019a), while comparing results with alternative discrepancies such as the Kullback-Leibler (Jiang et al., 2018) and the γ -divergence (Fujisawa et al., 2021).

In evaluating the above methods, we consider a total of n = 100 data points $y_{1:100}$ from a N(1,1) with an increasing level $\alpha_n = \alpha \in \{0.00, 0.05, 0.10, 0.15\}$ of Cauchy-tail contamination. Specifically, the observed data $y_{1:100}$ are obtained by sampling 100 values from a N(1,1) and then replacing $100 \cdot \alpha$ of them with the most extreme positive realizations within a sample of size 100 from a Cauchy(0,1). For the data we assume a miss-specified $N(\theta, 1)$ model and provide posterior inference via summary-free ABC under a N(0,1) prior for θ , considering the aforementioned discrepancies as implemented in the Python library ABCpy (Dutta et al., 2021). We focus, in particular, on a simple rejection-based ABC with m = n and, following standard practice in comparing different discrepancies, we do not directly specify the tolerance thresholds, but rather set a common computational budget of T = 2500 simulations and then keep the



Figure 1: For rejection-based ABC relying, respectively, on MMD, Wasserstein, KL and γ -discrepancy, graphical representation of the ABC posterior for θ under the Gaussian model with sample size n = 100 and varying fractions of Cauchy-tail contamination. Results are also compared with the "Exact" Gaussian posterior when assuming that all data, including contaminated ones, are from the Gaussian likelihood.

	$\pi_n(\theta \in [0.95, 1.05])$				$\pi_n(\theta \in [0.99, 1.01])$				time
	0.00	0.05	0.10	0.15	0.00	0.05	0.10	0.15	-
MMD	0.216	0.212	0.192	0.172	0.028	0.028	0.024	0.024	3"
Wasserstein	0.208	0.076	0.056	0.044	0.032	0.012	0.012	0.008	6"
KL	0.108	0.112	0.112	0.104	0.016	0.016	0.016	0.016	5"
γ -divergence	0.104	0.112	0.116	0.100	0.012	0.016	0.016	0.016	7"
"Exact" Gaussian	0.342	0.000	0.000	0.000	0.071	0.000	0.000	0.000	

Table 1: For ABC based on the four discrepancies in Figure 1, running times — on a standard laptop — and summaries of the ABC posterior for θ under the Gaussian model with sample size n = 100 and varying fractions of Cauchy-tail contamination. Results are also compared to the "Exact" Gaussian posterior when assuming that all data, including contaminated ones, are from the Gaussian model.

10% simulated values of θ which generated the synthetic data closest to the observed ones, under the corresponding discrepancy.

Figure 1 and Table 1 summarize the results of the simulation experiment, and further clarify the concentration and robustness properties of the discrepancies analyzed by including also the case of no contamination and comparing results with the closed-form Gaussian posterior obtained under the assumption that all data, including contaminated ones, are from a $N(\theta, 1)$. When this is the case, namely $\alpha = 0$, the closed-form Gaussian posterior achieves, as expected, the most rapid concentration around $\theta^* = 1$, followed by Wasserstein-ABC and MMD-ABC. The contraction of MMD with Gaussian kernel is in line with our theory in Proposition 1, whereas the accurate performance of Wasserstein-ABC under the correctly specified Gaussian model seems to suggest that the sufficient condition on the boundedness of \mathcal{Y} , under which bounds on the Rademacher complexity for the Wasserstein distance are currently derived, might be further relaxed for specific models; see Section 5 for further discussion. However, these relaxations may not hold in general. In fact, as illustrated in Figure 1 and Table 1, even a small fraction of Cauchy-tail contamination causes both the closed-form Gaussian posterior and the Wasserstein-ABC one to concentrate far from the truth. According to our theory in Sections 3.1–3.2 this is attributable to the fact that, when \mathcal{Y} is unbounded, Wasserstein-ABC, and the Wasserstein distance itself, require alternative assumptions on the data-generating process, which may not be met under the contaminated model. As clarified in Sections 3.1–3.2 and in Proposition 1, MMD with bounded kernel does not suffer from these limitations and, in fact, the new theory based on Rademacher complexity in Sections 3.1–3.2 matches closely the empirical concentration and robustness behavior of MMD in this experiment. Kullback–Leibler and γ -divergence achieve similar robustness but at the expense of a lower concentration, due to an inflation of the variance. It shall be noticed that the γ -divergence is specifically motivated by robustness arguments (Fujisawa et al., 2021) and, hence, the improved performance of MMD in this Cauchy-tail contamination experiment is remarkable.

5 Discussion

This article provides important advancements with respect to the recent literature on concentration and robustness of discrepancy-based ABC posteriors by connecting such properties with the asymptotic behavior of the Rademacher complexity associated with the chosen discrepancy. Although the concept of Rademacher complexity had never been considered within ABC before, such a quantity and its associated results provide a powerful and promising framework to study the theoretical properties of discrepancy-based ABC posteriors, as clarified in Sections 2 and 3. In fact, when the Rademacher complexity associated to the chosen discrepancy decays to zero as $n \to \infty$, it is possible to obtain more interpretable concentration and robustness results, with direct practical implications and under more constructive assumptions, for a broad class of discrepancies within the IPS family.

Although the aforementioned results already provide important advancements, the Rademacher complexity perspective proposed in this article has a broader scope and sets the premises for additional future research. For example, as clarified in Sections 2–3, any new result and bound on the Rademacher complexity of specific discrepancies can be directly applied to our theory. This may yield tighter or even more explicit bounds, possibly holding under milder assumptions and more general discrepancies. For example, to the best of our knowledge, informative bounds for the Rademacher complexity of the Wasserstein distance are currently available only under the assumption of bounded \mathcal{Y} and, hence, it would be of interest to leverage future findings on the case of unbounded \mathcal{Y} in order to broaden the range of models for which our theory, when specialized to the Wasserstein distance, applies. Under this perspective, it might also be promising to explore available results on local Rademacher complexities (e.g., Bartlett et al., 2005; Koltchinskii, 2006). Similarly, although we focus for simplicity on the i.i.d. case, it would be of interest to extend our theory beyond this setting leveraging, e.g, the results in Mohri and Rostamizadeh (2008) on the Rademacher complexity for non–i.i.d. processes.

The perspective proposed in this paper, based on the Rademacher complexity, applies to the whole class of IPS discrepancies. Although such a class is broad and includes the widely–implemented MMD (Park et al., 2016; Nguyen et al., 2020) and Wasserstein distance (Bernton et al., 2019a), it does not cover all the discrepancies employed in ABC. For example, the Kullback–Leibler divergence (Jiang et al., 2018) and the Hellinger distance (Frazier, 2020), are not IPS but rather belong to the class of f-divergences. While this family has important differences relative to the IPS class, it would be worth investigating f-divergences in the light of the results derived in Sections 3.1–3.2. To accomplish this goal, a possible direction is to exploit the recent unified treatments of these two classes in e.g., Agrawal and Horel (2021) and Birrell et al. (2022). More generally, our results could also stimulate methodological and theoretical advacements even beyond discrepancy–based ABC, especially within the framework of generalized likelihood–free Bayesian inference via discrepancies–based pseudo–posteriors (e.g., Bissiri et al., 2016; Jewson et al., 2018; Miller and Dunson, 2019; Knoblauch et al., 2019; Chérief-Abdellatif and Alquier, 2020; Matsubara et al., 2022; Dellaporta et al., 2022).

Finally, note that for the sake of simplicity and for an ease of comparison with related studies, we have focused on rejection ABC and have constrained the number m of the synthetic samples to be equal to the size n of the observed data. While these settings are standard in state-of-the-art theoretical analyses (e.g., Bernton et al., 2019a; Frazier, 2020), other ABC routines and alternative scenarios where m grows sub-linearly with n deserve further investigation. This latter regime would be especially of interest in settings where the simulation of synthetic data is costly.

A Proofs

Proof of Theorem 1. To prove Theorem 1 first notice that, since Lemma 1 and the inequality $\mathfrak{R}_n(\mathfrak{F}) = \sup_{\mu \in \mathcal{P}(\mathcal{Y})} \mathfrak{R}_{\mu,n}(\mathfrak{F}) \geq \mathfrak{R}_{\mu,n}(\mathfrak{F})$ hold for every $\mu \in \mathcal{P}(\mathcal{Y})$, then, for any integer $n \geq 1$ and any scalar $\delta \geq 0$, Equation (1) also implies $\mathbb{P}_{x_{1:n}} [\mathcal{D}_{\mathfrak{F}}(\hat{\mu}_{x_{1:n}}, \mu) \leq 2\mathfrak{R}_n(\mathfrak{F}) + \delta] \geq 1 - \exp(-n\delta^2/2b^2)$. Moreover, since this result holds for any $\delta \geq 0$, if follows that $\mathbb{P}_{x_{1:n}} [\mathcal{D}_{\mathfrak{F}}(\hat{\mu}_{x_{1:n}}, \mu) \leq 2\mathfrak{R}_n(\mathfrak{F}) + (c_1 - 2\mathfrak{R}_n(\mathfrak{F}))] \geq 1 - \exp[-n(c_1 - 2\mathfrak{R}_n(\mathfrak{F}))^2/2b^2]$, for any $c_1 > 2\mathfrak{R}_n(\mathfrak{F})$. Hence,

$$\mathbb{P}_{x_{1:n}}\left[\mathcal{D}_{\mathfrak{F}}(\hat{\mu}_{x_{1:n}},\mu) \le c_1\right] \ge 1 - \exp[-n(c_1 - 2\mathfrak{R}_n(\mathfrak{F}))^2 / 2b^2].$$
(5)

Recalling the statement of Theorem 1, consider now the sequence $\bar{\varepsilon}_n \to 0$ as $n \to \infty$, with $n\bar{\varepsilon}_n^2 \to \infty$ and $\bar{\varepsilon}_n/\Re_n(\mathfrak{F}) \to \infty$, which is possible by Assumption (C4). These regimes imply that $\bar{\varepsilon}_n$ goes to zero slower than $\Re_n(\mathfrak{F})$ and, therefore, for *n* large enough, $\bar{\varepsilon}_n/3 > 2\Re_n(\mathfrak{F})$. Hence, under Assumptions (C1) and (C3) it is possible to apply Equation (5) to $y_{1:n}$, setting $c_1 = \bar{\varepsilon}_n/3$, which yields

$$\mathbb{P}_{y_{1:n}}\left[\mathcal{D}_{\mathfrak{F}}(\hat{\mu}_{y_{1:n}},\mu^*) \le \bar{\varepsilon}_n/3\right] > 1 - \exp\left[-n(\bar{\varepsilon}_n/3 - 2\mathfrak{R}_n(\mathfrak{F}))^2/2b^2\right].$$

Since $-n(\bar{\varepsilon}_n/3-2\mathfrak{R}_n(\mathfrak{F}))^2 = -n\bar{\varepsilon}_n^2[1/9+4(\mathfrak{R}_n(\mathfrak{F})/\bar{\varepsilon}_n)^2-(4/3)\mathfrak{R}_n(\mathfrak{F})/\bar{\varepsilon}_n]$, it directly follows that $-n(\bar{\varepsilon}_n/3-2\mathfrak{R}_n(\mathfrak{F}))^2 \to -\infty$ as $n \to \infty$; recall that, from the above settings, $n\bar{\varepsilon}_n^2 \to \infty$ and $\mathfrak{R}_n(\mathfrak{F})/\bar{\varepsilon}_n \to 0$, when $n \to \infty$. As a consequence, we have that $1 - \exp[-n(\bar{\varepsilon}_n/3-2\mathfrak{R}_n(\mathfrak{F}))^2/2b^2] \to 1$ as $n \to \infty$. Therefore, in the rest of the proof of Theorem 1 we restrict to the event $\{\mathcal{D}_{\mathfrak{F}}(\hat{\mu}_{y_{1:n}}, \mu^*) \leq \bar{\varepsilon}_n/3\}$. Denote with $\mathbb{P}_{\theta,z_{1:n}}$.

the joint distribution of $\theta \sim \pi$ and $z_{1:n}$ i.i.d. from μ_{θ} . By definition of conditional probability, for any c_2 — including $c_2 > 2\mathfrak{R}_n(\mathfrak{F})$ — it follows that

$$\pi_{n}^{(\varepsilon^{*}+\bar{\varepsilon}_{n})}\left(\left\{\theta\in\Theta:\mathcal{D}_{\mathfrak{F}}(\mu_{\theta},\mu^{*})>\varepsilon^{*}+4\bar{\varepsilon}_{n}/3+c_{2}\right\}\right)$$

$$=\frac{\mathbb{P}_{\theta,z_{1:n}}\left[\mathcal{D}_{\mathfrak{F}}(\mu_{\theta},\mu^{*})>\varepsilon^{*}+4\bar{\varepsilon}_{n}/3+c_{2},\mathcal{D}_{\mathfrak{F}}(\hat{\mu}_{z_{1:n}},\hat{\mu}_{y_{1:n}})\leq\varepsilon^{*}+\bar{\varepsilon}_{n}\right]}{\mathbb{P}_{\theta,z_{1:n}}\left[\mathcal{D}_{\mathfrak{F}}(\hat{\mu}_{z_{1:n}},\hat{\mu}_{y_{1:n}})\leq\varepsilon^{*}+\bar{\varepsilon}_{n}\right]}.$$
(6)

To derive an upper bound for the above ratio, we first seek an upper bound for its numerator. In addressing this goal, we leverage the triangle inequality $\mathcal{D}_{\mathfrak{F}}(\mu_{\theta},\mu^*) \leq \mathcal{D}_{\mathfrak{F}}(\hat{\mu}_{z_{1:n}},\mu_{\theta}) + \mathcal{D}_{\mathfrak{F}}(\hat{\mu}_{z_{1:n}},\hat{\mu}_{y_{1:n}}) + \mathcal{D}_{\mathfrak{F}}(\hat{\mu}_{y_{1:n}},\mu^*)$ — since $\mathcal{D}_{\mathfrak{F}}$ is a semimetric — and the previously-proved result that the event $\{\mathcal{D}_{\mathfrak{F}}(\hat{\mu}_{y_{1:n}},\mu^*) \leq \bar{\varepsilon}_n/3\}$ has $\mathbb{P}_{y_{1:n}}$ -probability going to 1, thereby obtaining

$$\begin{split} \mathbb{P}_{\theta,z_{1:n}} [\mathcal{D}_{\mathfrak{F}}(\mu_{\theta},\mu^{*}) > \varepsilon^{*} + 4\bar{\varepsilon}_{n}/3 + c_{2}, \mathcal{D}_{\mathfrak{F}}(\hat{\mu}_{z_{1:n}},\hat{\mu}_{y_{1:n}}) \leq \varepsilon^{*} + \bar{\varepsilon}_{n}] \\ & \leq \mathbb{P}_{\theta,z_{1:n}} \left[\mathcal{D}_{\mathfrak{F}}(\hat{\mu}_{z_{1:n}},\hat{\mu}_{y_{1:n}}) + \mathcal{D}_{\mathfrak{F}}(\hat{\mu}_{z_{1:n}},\mu_{\theta}) + \mathcal{D}_{\mathfrak{F}}(\hat{\mu}_{y_{1:n}},\mu^{*}) > \varepsilon^{*} + 4\bar{\varepsilon}_{n}/3 + c_{2}, \mathcal{D}_{\mathfrak{F}}(\hat{\mu}_{z_{1:n}},\hat{\mu}_{y_{1:n}}) \leq \varepsilon^{*} + \bar{\varepsilon}_{n} \right] \\ & \leq \mathbb{P}_{\theta,z_{1:n}} \left[\mathcal{D}_{\mathfrak{F}}(\hat{\mu}_{z_{1:n}},\mu_{\theta}) + \mathcal{D}_{\mathfrak{F}}(\hat{\mu}_{y_{1:n}},\mu^{*}) > \bar{\varepsilon}_{n}/3 + c_{2} \right] \leq \mathbb{P}_{\theta,z_{1:n}} \left[\mathcal{D}_{\mathfrak{F}}(\hat{\mu}_{z_{1:n}},\mu_{\theta}) > c_{2} \right]. \end{split}$$

Rewriting the last term $\mathbb{P}_{\theta, z_{1:n}} \left[\mathcal{D}_{\mathfrak{F}}(\hat{\mu}_{z_{1:n}}, \mu_{\theta}) > c_2 \right]$ as $\int_{\theta \in \Theta} \mathbb{P}_{z_{1:n}} \left[\mathcal{D}_{\mathfrak{F}}(\hat{\mu}_{z_{1:n}}, \mu_{\theta}) > c_2 \mid \theta \right] \pi(\mathrm{d}\theta)$ and applying Equation (5) to $z_{1:n}$ yields

$$\int_{\theta\in\Theta} \mathbb{P}_{z_{1:n}} \left[\mathcal{D}_{\mathfrak{F}}(\hat{\mu}_{z_{1:n}}, \mu_{\theta}) > c_{2} \mid \theta \right] \pi(\mathrm{d}\theta) = \int_{\theta\in\Theta} (1 - \mathbb{P}_{z_{1:n}} \left[\mathcal{D}_{\mathfrak{F}}(\hat{\mu}_{z_{1:n}}, \mu_{\theta}) \le c_{2} \mid \theta \right]) \pi(\mathrm{d}\theta)$$
$$\leq \int_{\theta\in\Theta} \exp[-n(c_{2} - 2\mathfrak{R}_{n}(\mathfrak{F}))^{2}/2b^{2}] \pi(\mathrm{d}\theta) = \exp[-n(c_{2} - 2\mathfrak{R}_{n}(\mathfrak{F}))^{2}/2b^{2}].$$

Therefore, the numerator of the ratio in Equation (6) can be upper bounded by $\exp[-n(c_2-2\Re_n(\mathfrak{F}))^2/2b^2]$ for any $c_2 > 2\Re_n(\mathfrak{F})$. As for the denominator, defining the event $\mathbf{E}_n := \{\theta \in \Theta : \mathcal{D}_{\mathfrak{F}}(\mu_{\theta}, \mu^*) \leq \varepsilon^* + \bar{\varepsilon}_n/3\}$ and applying again the triangle inequality, we have that

$$\begin{aligned} \mathbb{P}_{\theta, z_{1:n}} \left[\mathcal{D}_{\mathfrak{F}}(\hat{\mu}_{z_{1:n}}, \hat{\mu}_{y_{1:n}}) \leq \varepsilon^* + \bar{\varepsilon}_n \right] \geq \int_{\mathcal{E}_n} \mathbb{P}_{z_{1:n}} \left[\mathcal{D}_{\mathfrak{F}}(\hat{\mu}_{z_{1:n}}, \hat{\mu}_{y_{1:n}}) \leq \varepsilon^* + \bar{\varepsilon}_n \mid \theta \right] \pi(\mathrm{d}\theta) \\ \geq \int_{\mathcal{E}_n} \mathbb{P}_{z_{1:n}} \left[\mathcal{D}_{\mathfrak{F}}(\hat{\mu}_{y_{1:n}}, \mu^*) + \mathcal{D}_{\mathfrak{F}}(\mu_{\theta}, \mu^*) + \mathcal{D}_{\mathfrak{F}}(\hat{\mu}_{z_{1:n}}, \mu_{\theta}) \leq \varepsilon^* + \bar{\varepsilon}_n \mid \theta \right] \pi(\mathrm{d}\theta) \\ \geq \int_{\mathcal{E}_n} \mathbb{P}_{z_{1:n}} \left[\mathcal{D}_{\mathfrak{F}}(\hat{\mu}_{z_{1:n}}, \mu_{\theta}) \leq \bar{\varepsilon}_n / 3 \mid \theta \right] \pi(\mathrm{d}\theta), \end{aligned}$$

where the last inequality follows from that fact that we can restrict to the events $\{\mathcal{D}_{\mathfrak{F}}(\hat{\mu}_{y_{1:n}}, \mu^*) \leq \bar{\varepsilon}_n/3\}$ and $E_n := \{\theta \in \Theta : \mathcal{D}_{\mathfrak{F}}(\mu_{\theta}, \mu^*) \leq \varepsilon^* + \bar{\varepsilon}_n/3\}$. Applying again Equation (5) to $z_{1:n}$, the last term of the above inequality can be further lower bounded by

$$\int_{\mathbf{E}_n} \left(1 - \exp\left[-n(\bar{\varepsilon}_n/3 - 2\mathfrak{R}_n(\mathfrak{F}))^2/2b^2 \right] \right) \pi(\mathrm{d}\theta) = \pi(\mathbf{E}_n) \left(1 - \exp\left[-n(\bar{\varepsilon}_n/3 - 2\mathfrak{R}_n(\mathfrak{F}))^2/2b^2 \right] \right),$$

where $\pi(\mathbf{E}_n) \geq c_{\pi}(\bar{\varepsilon}_n/3)^L$ by (C2), and, as already shown, $1 - \exp\left[-n(\bar{\varepsilon}_n/3 - 2\mathfrak{R}_n(\mathfrak{F}))^2/2b^2\right] \to 1$, when $n \to \infty$, which implies that, for *n* large enough, $1 - \exp\left[-n(\bar{\varepsilon}_n/3 - 2\mathfrak{R}_n(\mathfrak{F}))^2/2b^2\right] > 1/2$. Leveraging both results, the denominator in (6) can be lower bounded by $(c_{\pi}/2) (\bar{\varepsilon}_n/3)^L$.

To conclude the proof, it is sufficient to combine the upper and lower bounds derived, respectively, for the numerator and the denominator of the ratio in (6), to obtain

$$\pi_n^{(\varepsilon^* + \bar{\varepsilon}_n)} \left(\{ \theta \in \Theta : \mathcal{D}_{\mathfrak{F}}(\mu_\theta, \mu^*) > \varepsilon^* + 4\bar{\varepsilon}_n/3 + c_2 \} \right) \le \frac{\exp[-n(c_2 - 2\mathfrak{R}_n(\mathfrak{F}))^2/2b^2]}{(c_\pi/2)(\bar{\varepsilon}_n/3)^L},\tag{7}$$

with $\mathbb{P}_{y_{1:n}}$ -probability going to 1 as $n \to \infty$. Replacing c_2 in (7) with $2\mathfrak{R}_n(\mathfrak{F}) + \sqrt{(2b^2/n)\log(M_n/\bar{\varepsilon}_n^L)}$, which is never lower than $2\mathfrak{R}_n(\mathfrak{F})$, concludes the proof.

Proof of Proposition 2. To prove Proposition 2, we first show that, under (A1)–(A3), Assumptions 1 and 2 in Bernton et al. (2019a) are satisfied with $f_n(\bar{\varepsilon}_n) = 1/n^2 \bar{\varepsilon}_n^2$ and $c(\theta) = \mathbb{E}_z [k(z, z)]$. To this end, first recall that, by standard properties of MMD,

$$\mathcal{D}^2_{\mathfrak{F}}(\mu_1,\mu_2) = \mathbb{E}_{x_1,x_1'}\left[k(x_1,x_1')\right] - 2\mathbb{E}_{x_1,x_2}\left[k(x_1,x_2)\right] + \mathbb{E}_{x_2,x_2'}\left[k(x_2,x_2')\right],\tag{8}$$

with $x_1, x'_1 \sim \mu_1$ and $x_2, x'_2 \sim \mu_2$; see e.g. Chérief-Abdellatif and Alquier (2022) and Briol et al. (2019). Since $k(x, x') = \langle \phi(x), \phi(x') \rangle_{\mathcal{H}}$ (e.g., Muandet et al., 2017), the above result also implies

$$\mathcal{D}^{2}_{\mathfrak{F}}(\mu_{1},\mu_{2}) = \mathbb{E}_{x_{1},x_{1}'} \left[\left\langle \phi(x_{1}), \phi(x_{1}') \right\rangle_{\mathcal{H}} \right] - 2\mathbb{E}_{x_{1},x_{2}} \left[\left\langle \phi(x_{1}), \phi(x_{2}) \right\rangle_{\mathcal{H}} \right] + \mathbb{E}_{x_{2},x_{2}'} \left[\left\langle \phi(x_{2}), \phi(x_{2}') \right\rangle_{\mathcal{H}} \right] \\ = ||\mathbb{E}_{x_{1}} [\phi(x_{1})]||^{2}_{\mathcal{H}} - 2 \left\langle \mathbb{E}_{x_{1}} [\phi(x_{1})], \mathbb{E}_{x_{2}} [\phi(x_{2})] \right\rangle_{\mathcal{H}} + ||\mathbb{E}_{x_{2}} [\phi(x_{2})]||^{2}_{\mathcal{H}}$$

$$= ||\mathbb{E}_{x_{1}} [\phi(x_{1})] - \mathbb{E}_{x_{2}} [\phi(x_{2})]||^{2}_{\mathcal{H}}.$$

$$(9)$$

Leveraging Equations (8)–(9) and basic Markov inequalities, for any $\bar{\varepsilon}_n \geq 0$, it holds

$$\mathbb{P}_{y_{1:n}} \left[\mathcal{D}_{\mathfrak{F}}(\hat{\mu}_{y_{1:n}}, \mu^*) > \bar{\varepsilon}_n \right] \leq (1/\bar{\varepsilon}_n^2) \mathbb{E}_{y_{1:n}} \left[\mathcal{D}_{\mathfrak{F}}^2(\hat{\mu}_{y_{1:n}}, \mu^*) \right] = (1/\bar{\varepsilon}_n^2) \mathbb{E}_{y_{1:n}} [\|(1/n) \sum_{i=1}^n \phi(y_i) - \mathbb{E}_y \left[\phi(y)\right] \|_{\mathcal{H}}^2]$$

$$= (1/n^2 \bar{\varepsilon}_n^2) \sum_{i=1}^n \mathbb{E}_{y_i} \left[\|\phi(y_i) - \mathbb{E}_y \left[\phi(y)\right] \|_{\mathcal{H}}^2 \right] \leq (1/n^2 \bar{\varepsilon}_n^2) \sum_{i=1}^n \mathbb{E}_{y_i} \left[\|\phi(y_i) \|_{\mathcal{H}}^2 \right]$$

$$\leq (1/n^2 \bar{\varepsilon}_n^2) \mathbb{E}_{y_1} \left[\|\phi(y_1) \|_{\mathcal{H}}^2 \right] = (1/n^2 \bar{\varepsilon}_n^2) \mathbb{E}_{y_1} \left[k(y_1, y_1) \right] = (1/n^2 \bar{\varepsilon}_n^2) \mathbb{E}_y \left[k(y, y) \right],$$

with $y \sim \mu^*$. Since $(1/n^2 \bar{\varepsilon}_n^2) \mathbb{E}_y[k(y, y)] \to 0$ as $n \to \infty$ by (A1), we have that $\mathcal{D}_{\mathfrak{F}}(\hat{\mu}_{y_{1:n}}, \mu^*) \to 0$ in $\mathbb{P}_{y_{1:n}}$ -probability as $n \to \infty$, thereby meeting Assumption 1 in Bernton et al. (2019a). Moreover, as a direct consequence of the above derivations, it also follows that

$$\mathbb{P}_{z_{1:n}}\left[\mathcal{D}_{\mathfrak{F}}(\hat{\mu}_{z_{1:n}},\mu_{\theta}) > \bar{\varepsilon}_n\right] \le (1/n^2 \bar{\varepsilon}_n^2) \mathbb{E}_z\left[k(z,z)\right],$$

with $z \sim \mu_{\theta}$. Thus, setting $1/n^2 \bar{\varepsilon}_n^2 = f_n(\bar{\varepsilon}_n)$ and $\mathbb{E}_z[k(z,z)] = c(\theta)$, ensures that $\mathbb{P}_{z_{1:n}}[\mathcal{D}_{\mathfrak{F}}(\hat{\mu}_{z_{1:n}},\mu_{\theta}] > \bar{\varepsilon}_n] \leq c(\theta)f_n(\bar{\varepsilon}_n)$, with $f_n(\bar{\varepsilon}_n) = 1/n^2 \bar{\varepsilon}_n^2$ strictly decreasing in $\bar{\varepsilon}_n$ for fixed n, and $f_n(\bar{\varepsilon}_n) \to 0$ as $n \to \infty$, for fixed $\bar{\varepsilon}$. Moreover, by assumptions (A2) and (A3), $c(\theta) = \mathbb{E}_z[k(z,z)]$ is π -integrable and there is a $\delta_0 > 0$ and $c_0 > 0$ such that $c(\theta) < c_0$ for any θ satisfying $(\mathbb{E}_{z,z'}[k(z,z')] - 2\mathbb{E}_{z,y}[k(y,z)] + \mathbb{E}_{y,y'}[k(y,y')])^{1/2} = \mathcal{D}_{\mathfrak{F}}(\mu_{\theta},\mu^*) \leq \varepsilon^* + \delta_0$. These results ensure that Assumption 2 in Bernton et al. (2019a) holds.

Under the Assumptions in Proposition 2 it is, therefore, possible to apply Proposition 3 in Bernton et al. (2019a) with $f_n(\bar{\varepsilon}_n) = 1/n^2 \bar{\varepsilon}_n^2$, $c(\theta) = \mathbb{E}_z [k(z, z)]$ and $R = M_n$, which yields the concentration result reported in Proposition 2.

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