# Spoofing in Equilibrium*

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**Abstract**

We present a model of dynamic trading with exogenous and strategic cancellation of orders. We define spoofing as strategically placing and canceling orders in order to move prices and trade later in the opposite direction. We show that spoofing can occur in equilibrium, slowing price discovery and raising spreads and volatility. A novel prediction is that the prevalence of equilibrium spoofing is single-peaked in the measure of informed traders, suggesting that spoofing should be more prevalent in markets of intermediate liquidity. We also consider cross-market spoofing and discuss how regulators should allocate resources towards cross-market surveillance.

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## Conflict of interest disclosure statement

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I have nothing to disclose.

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# 1 Introduction

In September of 2020, U.S. regulators levied a $920 million fine on JP Morgan Chase for eight years of price manipulation in markets for precious metals and treasury bills. The form of manipulation JP Morgan committed is called "spoofing," which involves quickly placing and canceling orders to create an illusion of supply or demand. The size of the fine—comparable to penalties for rate rigging, corruption, and money-laundering—sets a new record in a series of increasingly severe regulatory actions against spoofing since the Dodd-Frank Act made it illegal in 2010.[1] Other high profile cases include Navinder Sarao, who was convicted in 2016 of spoofing U.S. futures markets and thereby contributing to the 2010 Flash Crash, and Michael Coscia, who was fined almost a million dollars for spoofing European futures markets.

The regulatory crackdown on spoofing has necessitated the creation of a special CFTC task force on spoofing in 2018. Thomas LaSala, the CFTC director, said "Policing the market for disruptive trading practices continues to be a huge part of our regulatory investment and effort."[2] Regulators and other commentators worry that spoofing harms markets by exacerbating adverse selection, volatility, and instability, which discourages legitimate traders from participating.[3] However, little scholarship exists to confirm or falsify these effects and guide regulation; empirical literature on spoofing is scant, and formal theory almost nonexistent.

In this paper, we present a dynamic trading model to better understand the economic consequences of spoofing. In the model, there is a mixture of informed and unformed traders. Some traders cancel their orders for exogenous reasons while other traders cancel strategically. We define spoofing as the strategic placing and canceling of orders with the intent to move prices and subsequently trade in the opposite direction.[4] We ask if spoofing can be an equilibrium phenomenon (yes!) and how spoofing affects prices, liquidity, and volatility. We then endogenize the amount of spoofing and ask what markets are most likely to at-

[1] https://www.ft.com/content/cc598fab-3d6a-4dd9-a91d-cebaa4130d03

[2] https://www.wsj.com/articles/u-s-market-manipulation-cases-reach-record-1540983720

[3] See https://www.sec.gov/files/Algo_Trading_Report_2020.pdf, Dalko and M. H. Wang (2019), and https://www.ft.com/content/cbf0aeaa-76ff-11e5-933d-efcdc3c11c89

[4] In practice, spoofing takes many forms, but they have in common that a spoofer takes some insincere action with the intent to move prices and then profit from such manipulation. Our model offers a stylized and tractable version of such phenomena. We discuss later how our insights apply more broadly to other forms of spoofing.

tract such activity. Finally, we employ the model to discuss market regulation, in particular cross-market surveillance.

First, we show that spoofing can occur in equilibrium, and that it slows price discovery, raises bid-ask spreads, and raises return volatility. Second, we show that the prevalence of equilibrium spoofing is non-monotone (single-peaked) in the fraction of informed traders in the market, suggesting that spoofing should be more prevalent in markets of intermediate liquidity. Third, we show that traders may profit by spoofing across distinct markets (for assets with correlated returns) and discuss how the regulators should allocate limited resources between cross-market and within-market surveillance.

Our findings are consistent with the view of regulators and other commentators that spoofing can have harmful effects on markets. Our findings also imply that spoofing may not be just an out-of-equilibrium phenomenon that can be completely neutralized by sophisticated traders once understood by all market participants. As a result, a regulation designed to catch and penalize spoofers, rather than just educating market participants about the possibility of spoofing, can be beneficial.

In practice, spoofing can take many forms, but they all involve the placing and cancelling of limit orders. However, models involving limit orders are known to be intractable, usually requiring either partial equilibrium analysis or numerical solutions.[5] Because we seek a tractable equilibrium model of spoofing, we avoid these difficulties by adapting the market order framework of Glosten and Milgrom (1985). We add to that model the possibility of cancellation. We believe that it allows us to capture the main economic forces present in limit order markets while keeping the analysis tractable. For example, our results that the returns to spoofing depend non-monotonically on the fraction of informed traders in the market are likely to hold in more complex environments.

In our setting, there are three dates, a competitive market maker, and a large set of traders who exchange units of an asset with the market maker. The asset's value may be high or low and is not revealed publicly until the final date. At each date, the market maker posts a bid and an ask price, and a random trader arrives at the market, placing an order to buy or sell the asset. A fraction of these traders are informed of the asset's true value, and the remaining uninformed traders want to buy or sell the asset for liquidity reasons.

To allow for spoofing, we make two modifications to the Glosten and Milgrom (1985)

[5] See the literature review.

framework. First, we assume that traders may cancel their order after placing it. Traders may cancel either because they experience sudden changes to their liquidity needs, or because they never intended to execute the order in the first place; the market maker cannot tell the difference. Second, we assume that a subset of informed and uninformed traders may trade twice in a row anonymously with the market maker; we call these *long-term* traders.

Spoofing occurs in equilibrium because long-term traders find it profitable to place an order at date 1, cancel it, and then place the opposite order at date 2. Doing so allows them to trade at a more favorable price than simply placing and executing an order directly, even though the market maker is aware that spoofing occurs in equilibrium. For example, suppose a long-term trader wants to ultimately buy the asset, either for liquidity reasons or because he is informed that its value is high. If he places a sell order at date 1 and then cancels it, the market maker cannot tell whether he canceled due to a sudden change in liquidity needs or because he never intended to sell in the first place. But because there is a chance that the sell order was sincere, the market maker revises his beliefs about asset quality downward, posting a lower ask at date 2 than at date 1. The long-term trader can then buy the asset at a discount. That discount depends on the fraction of spoofers in equilibrium since the market maker rationally expects some spoofing.

We derive three effects spoofing has on equilibrium prices. First, spoofing inhibits price discovery. That is, the market maker's beliefs about the asset value at the end of period two are, on average, less accurate than in a benchmark equilibrium without spoofing. Intuitively, trading is a signal to the market maker of asset value, so orders which are cancelled exogenously are informative. However, spoofing obscures trading motives and hence reduces the informativeness of canceled orders. Second, we show that spoofing raises both posted bid-ask spreads and the spread between average executed bids and asks. Intuitively, because spoofing inhibits price discovery, the market maker faces more severe adverse selection and must raise spreads to break even in expectation. Third, we show that spoofing raises return volatility. Intuitively, spoofers' canceled orders move prices away from the true value, and such movements are reversed when the asset value is revealed at the last date. This back and forth motion in prices raises return volatility.

These three results are consistent with regulator's concerns about the negative effects of spoofing. But our model also allows us to derive new predictions about which market conditions make spoofing more likely. To do this, we endogenize the measure of spoofers.

We assume an expected penalty for being caught spoofing and let the long-term traders choose whether to spoof or to trade directly. The more spoofers, the less profitable spoofing is compared to direct trading, because the market maker regards canceled orders with more suspicion and therefore moves the price less in the spoofer's desired direction.[6] In equilibrium the measure of spoofers is such that the gains to spoofing equal the expected penalty for being caught.

We find that the equilibrium measure of spoofers is single-peaked in the proportion of informed traders. In the Glosten and Milgrom (1985) framework, a higher proportion of informed traders corresponds with lower liquidity, in the sense that bid-ask spreads are larger and that executed trades have greater price impacts. So this result suggests that spoofing should be most prevalent in markets which are sufficiently illiquid, but not too illiquid.

The intuition for this non-monotone amount of spoofing is as follows. If the likelihood of informed traders is low, the market maker regards orders as largely uninformative of the asset value and therefore adjusts the prices only mildly in response to trades. This makes it harder for spoofers to move the price in their desired direction, which discourages spoofing. On the other hand, if the likelihood of informed traders is high, orders in opposite directions strongly indicate spoofing, because non-spoofing informed traders always place orders in the same direction. This makes the market maker suspicious of order reversals, which again discourages spoofing. In addition, we find that the equilibrium measure of spoofers is increasing in the probability of "legitimate" (exogenous) cancellations, because the spoofers' canceled orders are less suspicious to the market maker, making spoofing more attractive. This result points to one downside of the rise of high-frequency trading: even if HFTs themselves do not intentionally manipulate the market, their tendency to frequently cancel orders creates market conditions which attract spoofing.

Finally, we study the optimal regulation of spoofing. In 2012, the SEC ordered the creation of the Consolidated Audit Trail (CAT), which will aggregate detailed trading data across many markets into a central database. A major reason for the creation of the CAT is to enable surveillance of cross-market price manipulation, in which "trading on one market is used to affect a security's price while trading on another market is used to take advantage

[6] If there were no exogenously canceled orders, the market maker would not be influenced by canceled orders, making spoofing unprofitable.

of that price change."[7] Inspired by the SEC's concerns, we expand our setting to allow for cross-market spoofing and study the extent to which the regulator should allocate resources to cross-market surveillance versus within-market surveillance.

To allow for cross-market spoofing, we expand our environment to include two assets with correlated values, so that orders for one asset are informative of the other's value. First, we show that traders can profit by spoofing across markets—that is, by placing and canceling an order for one asset before executing the opposite order on the other asset—and that cross-market spoofing slows price discovery and raises bid-ask spreads. Second, we endogenize the measures of within- and cross-market spoofing, assuming regulators allocate a fixed amount of resources to within- and cross-market monitoring. Finally, we study the regulator's optimal allocation of resources between within- and cross-market monitoring that accounts for its impact on the equilibrium conduct of long-term traders.

We show first that if the regulator is sufficiently constrained in its monitoring resources, he should allocate a fixed proportion of those resources to cross-market monitoring. That proportion is constant in his total resources but increasing in the degree of correlation between assets. Intuitively, as the correlation increases, orders in one market have a stronger effect on the other market's prices, making cross-market spoofing more tempting and increasing the need for cross-market monitoring. Second, we show that if the regulator's monitoring resources are sufficiently high, he should monitor across markets just enough to eliminate cross-market spoofing, and allocate the rest to within-market monitoring. Intuitively, the imperfect correlation between assets makes cross-market spoofing less effective at price manipulation than within-market spoofing, so it is less tempting to traders. As a result, it is also cheaper for regulators to deter than within-market spoofing, so if the regulator has sufficient resources, he should eliminate it.

**Literature.** To our knowledge, we are the first paper to show in a formal theoretical model that spoofing may occur in equilibrium, and to study its equilibrium consequences. However, many papers have modeled other forms of price manipulation, which can be broadly classified as either information-based or trade-based. Information-based manipulation functions by directly releasing misleading information (Bagnoli and Lipman 1996; Benabou and Laroque 1992; Van Bommel 2003; Vila 1989), whereas trade-based manipulation functions through

[7] FINRA/Euronext letter, as quoted in https://www.sec.gov/rules/final/2012/34-67457.pdf.

buying and selling alone (Fischel and Ross 1991; Hart 1977; Jarrow 1992).[8]

Our paper is most closely related to papers which study trade-based manipulation in a Glosten and Milgrom (1985) setting. Compared to these papers, ours is the first to allow for canceled orders, which is the means by which traders can profitably manipulate prices in our paper. Allen and Gorton (1992) show that manipulation is possible if more liquidity traders want to sell rather than buy the asset, or if informed traders only want to buy. Allen and Gale (1992) show that if good news is revealed later than bad news, uninformed traders can give the false impression of forthcoming good news by placing large buy orders and exiting before the market learns that no news will be revealed. In Chakraborty and Yılmaz (2004a,b), informed traders mimic uninformed traders by initially trading opposite their information, which incurs a loss, but that loss is recouped if the trader may trade sufficiently many periods thereafter. In our paper, however, both informed and uninformed traders manipulate by cancelling orders opposite their desired trade, mimicking traders that cancel for legitimate reasons. Because cancelled orders manipulate prices without incurring losses, profit accrues in the trade immediately following the cancelled order, so multiple trading rounds are not required for manipulation to be profitable. These papers also do not study the impact of manipulation on volatility, nor do they consider cross-market manipulation and its optimal regulation.

Our approach of adding cancellations to the Glosten-Milgrom market order setting is a stylized way of capturing spoofing, which in practice usually takes place with limit orders rather than market orders. There are many papers which model limit orders explicitly,[9] but we have not found any that enable tractable modeling of equilibrium limit order spoofing, which requires dynamics, endogenous prices, and the ability of informed traders to place limit orders so that such orders affect market beliefs about fundamentals. For example, in most static and dynamic limit order models, only uninformed traders place limit orders (Foucault 1999; Foucault, Kadan, and Kandel 2005; Glosten 1994; Goettler, Parlour, and Rajan 2005; Parlour 1998; Rock 1996; Roşu 2009; Seppi 1997). Chakravarty and Holden (1995) and Seppi and Kumar (1994) allow for informed limit orders, but the models are static, so limit orders do not affect future prices. Goettler, Parlour, and Rajan (2005, 2009) present dynamic equilibrium limit order models, but only the 2009 paper allows for informed

[8]See Putniņš (2012) for a comprehensive survey.

[9]See Parlour and Seppi (2008) for a survey.

limit orders, and both papers require numerical solutions. One exception is Kaniel and Liu (2006), but their paper, like ours, also adapts the Glosten-Milgrom market order setting.

There are several papers in the computer science and applied math literature which use numerical simulations to study spoofing. Cartea, Jaimungal, and Y. Wang (2020) study the optimal policy of a trader seeking to liquidate shares, showing that spoofing may be optimal and that it moves prices away from fundamentals. Martínez-Miranda, McBurney, and Howard (2016) use simulations to study the optimal actions of a spoofer, finding that regulators can reduce the incentive to spoof by imposing fees and altering liquidity. X. Wang, Hoang, and Wellman (2019) and X. Wang and Wellman (2019) assume traders adopt heuristic-based learning strategies, numerically simulate equilibrium, and find that spoofing can coexist with sophisticated traders. In contrast to these papers, our simple model allows us to analytically demonstrate the equilibrium existence, market consequences, and optimal regulation of spoofing.

Empirical literature on spoofing is scant. However, Lee, Eom, and Park (2013) use Korean stock data to study the relationship between platform price disclosure and spoofing. They find that the absence of price disclosure makes spoofing more profitable. They also show that spoofing is more common in more volatile stocks, though they do not identify the direction of causation. Y.-Y. Wang (2019) studies spoofing in Taiwan futures markets, finding that spoofing is more likely during periods of high volume and volatility, and that both spreads and volatility tend to rise rise after spoofing takes place. Although they do not identify causation, these results are consistent with our prediction that spoofing raises spreads and volatility.

# 2 Environment

There are three dates: 1, 2, and 3. There is a market maker and a large set of traders who exchange units of an asset with the market maker. The fundamental value of the asset is a random variable, normalized to have mean zero. In particular, the normalized value $V$ takes value 1 or $-1$ with equal probability, and this distribution is common knowledge. The value of the asset is realized at the beginning of date 1, but the realization $v$ is not revealed publicly until date 3.

The market maker is competitive and risk-neutral. At dates 1 and 2, the market maker posts a bid and ask price, with the interpretation that he is willing to buy one unit of the asset at the bid price and sell one unit at the ask price.

There are also a large number of risk-neutral traders. At dates 1 and 2, a single trader is randomly chosen from the pool of all traders and meets anonymously with the market maker. Fraction $\alpha$ of traders are informed—they know the realization $v$ of the asset's value—whereas fraction $1-\alpha$ of traders are uninformed. Among the uninformed traders, half of them want to buy the asset and half of them want to sell; their reasons for trade are left unmodeled, as in Glosten and Milgrom (1985).

Unlike Glosten and Milgrom (1985), we assume that a fraction $\sigma$ of traders are *long-term* traders, and $1-\sigma$ are *short-term*. If a long-term trader meets the market maker at date 1, then this same trader also meets the market maker at date 2; we assume that he can execute at most one order with the market maker. In contrast, if a short-term trader meets the market maker at date 1, then a different short-term trader meets the market maker at date 2. Whether a trader is short- or long-term is independent of whether he is informed or uninformed. A trader's type $j$ is therefore an element of the set $\{l, s\} \times \{i, u_b, u_s\}$, where $l$ is "long-term", $s$ is "short-term", $i$ is "informed", $u_b$ is "uninformed buyer" and $u_s$ is "uninformed seller".

When a trader arrives at the market, he chooses an order $o_t \in \{B, S, N\}$ to place with the market maker, where $B$ is a buy order, $S$ is a sell order, and $N$ is no order. In another departure from Glosten and Milgrom (1985), we assume that the trader may strategically cancel his placed order before it is executed. We denote by $c_t \in \{C, E\}$ his cancellation decision, where $C$ means "cancel" and $E$ means "execute." So the short-term traders' strategy is a pair $(o_t, c_t) \in \{B, S\} \times \{C, E\}$, and the long-term traders' strategy is a sequence $(o_1, c_1, o_2, c_2) \in \{\{B, S\} \times \{C, E\}\}^2$.

If a short-term trader chooses not to cancel, we assume that the order is canceled anyway (exogenously) with probability $\beta \in (0, 1)$; we denote the exogenous cancellation random variable by $c_t^e \in \{C, E\}$. The exogenous cancellation represents unmodeled "legitimate" cancellations.[10] For simplicity, we assume that long-term traders are not subject to exogenous

[10]In Coscia v. United States (2018), the Supreme Court affirmed the 7th circuit's distinction between legitimate cancellations, such as fill-or-kill or stop-loss orders, and illegitimate cancellations involved in spoofing. See also former SEC Chairman Mary Shapiro's comment that "There may, of course, be justifiable explanations for many canceled orders to reflect changing market conditions." https://www.sec.gov/news/speech/2010/spch090710mls.htm. In practice, traders accused of spoofing often say "that they had a

cancellations. We denote the ultimate status of the order by $s_t \in \{C, E\}$, where $C$ means "canceled" and $E$ means "executed." Formally, if $c_t = C$ or $c_t^e = C$ then $s_t = C$ , and otherwise $s_t = E$.

We assume the market maker is competitive, so he sets prices in such a way that his expected profits are zero. That is, he sets each bid and ask equal to the expected value of the asset, where the expectation conditions on both the prior order history and on execution at the posted price. So the date 1 price is a function of the order $o_1$ placed at date 1, and the date 2 price is a function of the date 1 history $(o_1, s_1)$ and the date 2 order $o_2$.

**Definition 1.** An equilibrium is a set of short-term trading strategies $(o_1, c_1)(j)$, long-term trading strategies $(o_1, c_1, o_2, c_2)(j)$, and pricing strategies $p_1(o_1)$ and $p_2(o_1, s_1, o_2)$ such that

1. The short-term strategies are optimal for the short-term traders.
2. The long-term strategies are optimal for the long-term traders.
3. The pricing strategies are such that the market maker breaks even, conditional on order execution: $p_1(o_1) = \mathbb{E}[V|o_1, s_1 = E]$ and $p_2(o_1, s_1, o_2) = \mathbb{E}[V|o_1, s_1, o_2, s_2 = E]$.

Short-term traders have only a single opportunity to trade with the market maker, so they have no incentive to strategically cancel their orders. Long-term traders, however, may try to mislead the market maker by placing orders which they never intend to execute. We formally describe this kind of manipulation as follows.

**Definition 2.**

1. A long-term trader *spoofs* if he places an order at date 1, cancels it strategically, and places the opposite order at date 2.
2. A trader places a *direct order* if he submits an order and does not strategically cancel it.

legitimate reason for canceling orders, such as a client suddenly deciding not to participate." https://www.ft.com/content/cbf0aeaa-76ff-11e5-933d-efcdc3c11c89. The CME said that "Market participants may enter stop orders as a means of minimising potential losses with the hope that the order will not be triggered. However, it must be the intent of the market participant that the order will be executed if the specified condition is met. Such an order entry is not prohibited by this Rule." https://www.fi-desk.com/what-the-regulators-dont-know-about-spoofing

We are interested in whether spoofing can occur in equilibrium. So suppose that there exists an equilibrium in which every short-term trader places a direct order, and every long-term trader spoofs. We consider the payoff of a long-term trader who ultimately wants to buy the asset, either because he knows it is high value or due to liquidity reasons.

First, consider the price a long-term trader pays by deviating from the candidate equilibrium strategy and placing a direct buy order. The ask price charged by the market maker is equal to the expected value of the asset conditional on executing that buy order: $\mathbb{E}[V|BE]$. In the candidate equilibrium, the probability of an executed buy order occuring at date 1 may be decomposed into two subcases: (1) a short-term informed trader who knows the asset is high value arrives and executes a buy order, which occurs with probability $(1-\beta)\alpha 0.5$; or (2) a short-term uninformed liquidity trader arrives and executes a buy order, which occurs with probability $(1-\beta)(1-\alpha)0.5$. Furthermore, conditional on an informed trader placing a buy order, the expected value of the asset must be 1, whereas conditional on an uninformed liquidity trader placing a buy order, the expected value of the asset must be zero, because uninformed trades convey no information. So by the law of iterated expectations, the price paid to execute a direct buy order at date 1 is

$$\begin{aligned}\mathbb{E}[V|BE] &= \mathbb{E}[V|i,BE]P(i|BE) + \mathbb{E}[V|u_b,BE]P(u_b|BE)\\ &= \frac{\mathbb{E}[V|i,BE]P(i,BE) + \mathbb{E}[V|u_b,BE]P(u_b,BE)}{P(i,BE)+P(u_b,BE)}\\ &= \frac{(1)(1-\beta)\alpha 0.5 + (0)(1-\beta)(1-\alpha)0.5}{(1-\beta)\alpha 0.5 + (1-\beta)(1-\alpha)0.5}\\ &= \alpha.\end{aligned}$$

Next, consider the price a long-term buyer pays for following the candidate equilibrium strategy of spoofing; that is, placing a sell order at date 1, canceling it, and executing a buy order at date 2. The ask price charged by the market maker at date 2 is equal to the expected value of the asset conditional on the date 1 order history $SC$ and execution of the buy order $BE$ at date 2. However, the market maker understands that in equilibrium, the order history $SCBE$ could have come from either a single long-term spoofer, an event we denote by $l$, or two short-term direct traders, an event we denote by $\neg l$. By the law of iterated expectations, the price is a weighted sum of the conditional expectations that correspond to these two events:

$$\mathbb{E}[V|SCBE] = \mathbb{E}[V|l, SCBE]P(l|SCBE) + \mathbb{E}[V|\neg l, SCBE]P(\neg l|SCBE).$$

If the market maker knew that the order history $SCBE$ came from a long-term spoofer, the order history would be just as informative for asset value as a direct buy order $BE$: in either case, the market maker is faced with a single trader who wants to buy the asset. So conditional on the order history $SCBE$ coming from a long-term spoofer, the expected value of the asset is $\mathbb{E}[V|l, SCBE] = \mathbb{E}[V|BE] = \alpha$.

On the other hand, conditional on the order $SCBE$ coming from two short-term direct traders, the expected value of the asset $\mathbb{E}[V|\neg l, SCBE]$ must be zero. To see this, note that because the orders are in opposite directions, the two short-term traders cannot both be informed. Furthermore, if only the date 1 seller is informed, any negative information conveyed by that possibility is exactly offset by the possibility that only the date 1 buyer is informed. Therfore, $\mathbb{E}[V|\neg l, SCBE] = 0$, and the price paid for executing the spoofing strategy $SCBE$ is

$$\begin{aligned}
\mathbb{E}[V|SCBE] &= \mathbb{E}[V|l, SCBE]P(l|SCBE) + \mathbb{E}[V|\neg l, SCBE]P(\neg l|SCBE) \\
&= \mathbb{E}[V|BE]P(l|SCBE) + 0 \cdot P(\neg l|SCBE) \\
&= \mathbb{E}[V|BE]P(l|SCBE).
\end{aligned}$$

which is strictly less than the price $\mathbb{E}[V|BE]$ paid for directly buying the asset at date 1. As a consequence, the long-term trader strictly prefers spoofing to direct trading, even if the market maker knows that spoofing may occur. This leads to our first result.

**Proposition 1.** *There exists an equilibrium in which all long-term traders spoof.*

Spoofing occurs in equilibrium because it allows long-term traders to trade at a more favorable price than by simply placing and executing an order directly, even though the market maker is aware that spoofing occurs in equilibrium. The reason is that the market maker cannot tell whether the date 1 trader canceled exogenously or never intended to trade at date 1 in the first place. Because there is a chance that the first order was sincere, the market maker revises his beliefs about asset quality away from the truth, posting a more favorable price at date 2 than at date 1. As a result, spoofing leads to better prices for the spoofer than direct trading.

*Remark* 1. Our model's definition of spoofing is a simplification of the many ways in which spoofing can occur in practice. For example, spoofers might place limit orders far away from the best bid and ask in order to have a low chance of execution, but still mislead other traders. A more elaborate form of spoofing, known as layering, involves placing a sequence of limit orders at prices increasingly close to the best bid or ask, giving the illusion of swelling supply or demand. Spoofers may also choose to cancel their fake limit orders only after executing their desired trade, rather than before, as in our simplified setting. However, models in which privately informed traders place limit orders are complicated, typically allowing for only numerical solutions (Goettler, Parlour, and Rajan 2009). To maintain tractability, instead of modelling a limit order book, we extend the simple market order setting of Glosten and Milgrom (1985) to allow cancellations. So in our setting, orders that can be canceled function as a metaphor for limit orders, permitting analytical characterization of equilibrium spoofing. We believe that the forces driving manipulation in our setting would extend to more elaborate settings that allow traders to place limit orders at multiple price points. In particular, as long as some orders that contain information about the distribution of future prices happen to be cancelled with positive probability even without spoofing, traders will have the incentive to spoof; such spoofing will reduce the informativeness of equilibrium prices by diluting the information content of cancelled orders.

# 3 Consequences of Spoofing

Regulators have expressed concern that spoofing may create adverse market conditions which discourage legitimate traders from participating. Two major functions of markets are to provide price discovery, the process by which information about fundamental values are incorporated into prices, and liquidity, the ease with which traders can buy and sell assets (O'Hara 2003). So in this section, we examine the impact of spoofing on price discovery and liquidity, which we measure with bid-ask spreads. In addition, we study the effect that spoofing has on return volatility.

To examine the impact of spoofing, we compare the prices in the spoofing equilibrium to a benchmark equilibrium without spoofing. The benchmark setting is identical to our main setting in every way except that long-term traders are disallowed from spoofing. Instead, if a long-term trader arrives at the market at date 1, he must wait until date 2 to place an order. So if a long-term trader arrives at the market at date 1, no order at all is placed at date 1, and the market maker learns nothing about the asset value until date 2. We use $N$ to signify the event that no order was placed at date 1.

The results are the same if we assume that long-term traders trade immediately with the market maker and no trader arrives at date 2. The important thing is that direct trades by the long-term trader do not open up opportunities for other traders to arrive at the market. This guarantees that the number of traders faced by the market maker is the same as in our main spoofing setting, so that any effects of spoofing cannot be attributed simply to there being a smaller number of traders.

In the benchmark equilibrium, because each trader has only a single opportunity to trade with the market maker, no trader strategically cancels, instead placing a direct order which reflects his true information and preferences. As a result, order cancellations convey no special information, so the market maker updates his beliefs in response to executed and canceled orders in the same way. That is, the market maker's pricing strategy is the same as in the standard Glosten and Milgrom (1985) setting.

## 3.1 Price Discovery

The first proposition establishes that spoofing inhibits the market's discovery of the true value of the asset. We denote prices in the benchmark equilibrium by $p_b$ and denote the expectation with respect to the distribution of benchmark orders by $\mathbb{E}_b$.

**Proposition 2** (Price Discovery)**.** *Given any initial prior $\pi_0$, the average executed price at date 2 in the spoofing equilibrium is further from the true asset value $v$ than in the benchmark equilibrium:*

$$|v - \mathbb{E}[p(h)|v]| > |v - \mathbb{E}_b[p_b(h)|v]| \qquad \textit{for } v \in \{-1, 1\}.$$

Intuitively, spoofing obscures trading motives by pooling long-term traders with short-term traders. As a result, the market maker cannot tell whether canceled orders at date 1 were sincere or intending to lead him away from the truth, so trading histories involving canceled orders are less informative of the asset's value than under the benchmark.

More formally, viewing trading as a signal of the underlying state $V$, trading in the spoofing equilibrium is a Blackwell garbling of trading under the benchmark equilibrium. To see this, recall that in the spoofing equilibrium, the history $SCB$ can result from either a single long-term spoofer or two short-term traders. But in the benchmark equilibrium, the corresponding histories are $NB$ for a long-term trader and $SCB$ for two short-term traders. So the spoofing equilibrium collapses the two benchmark histories $NB$ and $SCB$ into the single history $SCB$, and is therefore a Blackwell garbling of benchmark trading.

Because the proposition holds for each state realization $v \in \{-1, 1\}$, rather than only in expectation over the state $V$, the proof requires more than a direct application of Blackwell's Theorem. We show that the market maker's posterior belief $\pi(h)$—and therefore, price $p(h)$—is a decreasing convex function of the likelihood ratio $P(h|v = -1)/P(h|v = 1)$. If $v = 1$, then under spoofing, the likelihood ratio of a spoofer's order such as $h = SCB$ is a weighted average of the likelihood ratios of the corresponding benchmark orders $SCB$ and $NB$. So by convexity, the average benchmark price is higher than the average spoofing price.

## 3.2 Bid-Ask Spreads

A common measure of liquidity is the spread between the bid and the ask. The next proposition shows that spoofing raises bid-ask spreads.

**Proposition 3.** *Given $v \in \{-1, 1\}$, the spoofing equilibrium exhibits a higher spread between average executed date 2 bids and asks than the benchmark equilibrium.*

Proposition 3 is closely related to Proposition 2. Because spoofing obscures trading motives, the market maker's beliefs are less accurate (Proposition 2) at the end of date 2; the greater information asymmetry between the market maker and informed traders means the market maker faces greater adverse selection, and must therefore raise spreads compared to the benchmark in order to break even in expectation.[11]

The Proposition is illustrated in Figure 1. If the asset value is high ($v = 1$), then under spoofing, average asks and bids are lower than under the benchmark, because the market maker doesn't update as accurately under spoofing (Proposition 2). However, under spoofing, average bids are *much* lower than under the benchmark, whereas average asks are only *slightly* lower than under the benchmark. This is because if $v = 1$, date 2 sell orders are placed only by uninformed traders, which amplifies the mispricing of sell orders (that

[11]Lemma 4 in the appendix shows that not only executed but also posted spreads are higher under the spoofing equilibrium.

is, bids) in the spoofing equilibrium. As a result, the spread between average bids and asks is larger under spoofing. Similar intuition applies to the case where $v = -1$, except that average bids and asks are higher under spoofing, and the mispricing is worse for asks, because buy orders are placed only by uninformed traders.

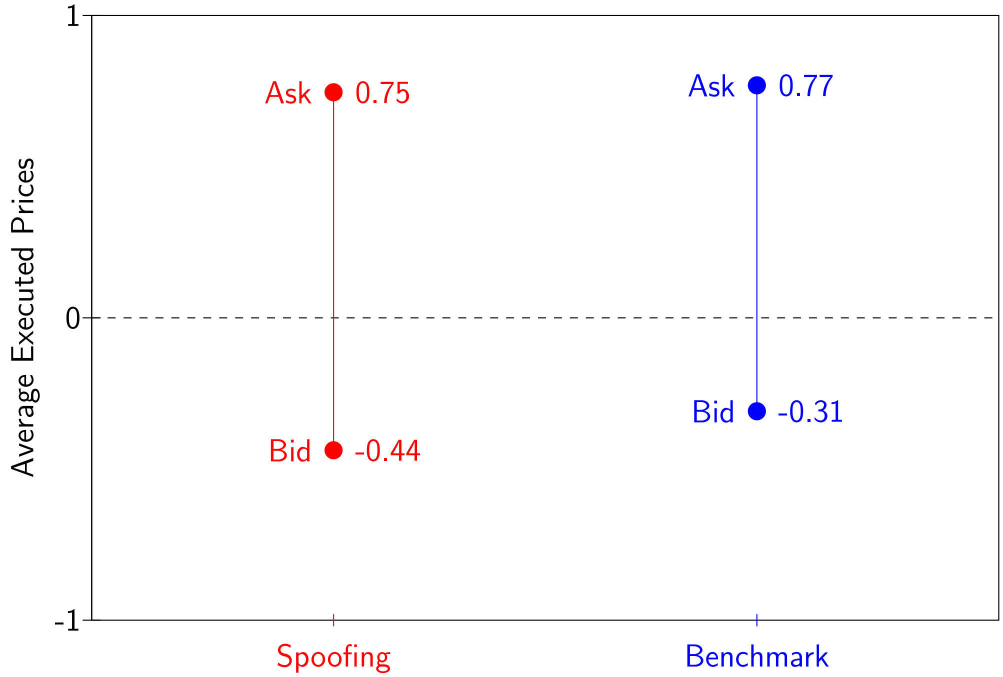


Figure 1: Given $v = 1$, the average executed bids ($o_2 = S$) and asks ($o_2 = B$) for each equilibrium, assuming $\sigma = 0.3$, $\alpha = 0.7$, and $\beta = 0.6$. The spread between average executed bids and asks is higher under spoofing.

## 3.3 Volatility

We next examine the impact of spoofing on return volatility, defined as the variance of log returns to bid-ask midpoints. For each order history, we compute the volatility along that history, average over all order histories, and then compare the average volatility in the spoofing and benchmark equilibria.

Recall that the asset value $V$ is normalized to have expected value 0. As a result, the prices $p_t$ derived in previous sections are also normalized. When computing log returns, we use midpoints of non-normalized bids and asks. In addition, we assume that price movements are small relative to price levels, so that first differences $p_{t+1}-p_t$ approximate non-normalized log returns.[12]

Recall that at date 1, the posted bid is $-\alpha$ and the posted ask is $\alpha$, so the midpoint $p_1$ equals zero and is uncontingent on order history. At date 2, the posted bid and ask are contingent on the date 1 order history $h_1 = (o_1, c_1)$; We denote the midpoint between the date 2 bid and ask by $p_2(h_1)$. Because the value $v$ is publicly known at date 3, the date 3 bid, ask, and midpoint are simply $v$.

The return achieved between date 1 and date 2 is $p_2(h_1) - p_1$, and the return achieved between date 2 and date 3 is $v - p_2(h_1)$. The return volatility is the variance of these two returns:

$$\Sigma(h_1, v) = \left(\frac{(p_2(h_1) - p_1)^2}{2} + \frac{(v - p_2(h_1))^2}{2}\right) - \left(\frac{(p_2(h_1) - p_1)}{2} + \frac{(v - p_2(h_1))}{2}\right)^2.$$

The following result shows that volatility is higher in the spoofing equilibrium.

**Proposition 4.** *Returns on midpoints are more volatile in expectation in the spoofing equilibrium than in the benchmark:*

$$\mathbb{E}[\Sigma(h_1, v)] > \mathbb{E}_b[\Sigma_b(h_1, v)].$$

Intuitively, spoofing leads to a greater likelihood that traders place orders in a direction opposite to the natural order given the value of the asset. For example, suppose the true value of the asset is $v = 1$. Then all informed traders ultimately want to place a buy order with the market maker. But under the spoofing equilibrium, there is a higher probability that traders will place a canceled sell order at date 1. This leads the market maker to post lower bids and asks at date 2, following which the price rises to $v = 1$ at date 3. It is this tendency of spoofing to lead the market maker away from the true value which leads to higher return volatility.

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[12] If the non-normalized expected value of the asset is $\mu$, then non-normalized prices at date $t$ are $p_t + \mu$. As $\mu$ gets large, (scaled) log returns $\mu \ln\left(\frac{p_{t+1}+\mu}{p_t+\mu}\right)$ converge to $p_{t+1} - p_t$.

## 4 Endogenous Measure of Spoofers

We now endogenize the measure of spoofers $\sigma$ in the market. Suppose that $\bar{\sigma} \in (0,1)$ traders are long-term traders and therefore have the opportunity to spoof if they wish. Also suppose that if an trader chooses to spoof, there is some chance that a regulator may catch him and impose some penalty. We let $k > 0$ be the expected penalty of being caught spoofing.

A long-term trader that arrives at the market can choose to either place his order directly or spoof. If he places his order directly, we assume for consistency with the previous section that he places no order at date 1, and places a direct order at date 2; this is without loss of generality. So for example, if he executes a direct buy order, the market maker would observe history $NBE$.

To see how the incentive to spoof relates to the measure of spoofers in the market, suppose that $\sigma$ traders choose to spoof. If the long-term trader executes a direct buy order $NBE$, he pays price

$$\mathbb{E}[V|NBE] = \mathbb{E}[V|BE] = \alpha.$$

If he instead spoofs, he pays price $\mathbb{E}[V|SCBE] = \mathbb{E}[V|BE]P(l|SCBE) = \alpha P(l|SCBE)$, as established in Section 2. The appendix shows that $P(l|SCBE) = \sigma/(\sigma+(1-\bar{\sigma})\beta 0.5(1-\alpha^2))$, so the trader pays price

$$\mathbb{E}[V|SCBE] = \alpha \frac{\sigma}{\sigma + (1-\bar{\sigma})\beta 0.5(1-\alpha^2)}$$

for spoofing. Therefore, his gains to spoofing are

$$G(\sigma) \equiv \mathbb{E}[V|NBE] - \mathbb{E}[V|SCBE] = \alpha\left(1 - \frac{\sigma}{\sigma + \beta(1-\bar{\sigma})0.5(1-\alpha^2)}\right),$$

which strictly decreases from $\alpha$ to $G(\bar{\sigma})$ in the number of spoofers $\sigma \in [0,\bar{\sigma}]$. Similarly, if the trader wishes to ultimately sell the asset, the gains to spoofing are $\mathbb{E}[V|BCSE] - \mathbb{E}[V|NSE] = (-\mathbb{E}[V|SCBE]) - (-\mathbb{E}[V|NBE]) = G(\sigma)$. A potential spoofer will find it profitable to spoof if and only if the gains $G(\sigma)$ from spoofing exceed the expected penalty $k$ of being caught.

**Definition 3.** Given $(\alpha, \beta, \bar{\sigma}, k) \in (0,1)^4$, an equilibrium is a quantity $\sigma^* \in [0, \bar{\sigma}]$ of spoofers that satisfies

$$\begin{cases} G(\sigma^*) \leq k & \text{if } \sigma^* = 0. \\ G(\sigma^*) = k & \text{if } \sigma^* \in (0, \bar{\sigma}) \\ G(\sigma^*) \geq k & \text{if } \sigma^* = \bar{\sigma}. \end{cases} \tag{1}$$

Observe that the gains $G(\sigma)$ from spoofing are decreasing in the number of spoofers $\sigma$, because a prevalence of spoofing makes the market maker suspicious of canceled orders, so his beliefs update less in the spoofers' desired direction. This leads to a unique equilibrium quantity of spoofing.

**Proposition 5.** *There exists a unique pure-strategy equilibrium in which the number of traders $\sigma^*$ who spoof is* single-peaked *in the number of informed traders $\alpha$.*

*In particular, the equilibrium $\sigma^*$ satisfies*

$$\sigma^* = \begin{cases} 0 & \textit{if } \hat{\sigma}(\alpha, \beta, \bar{\sigma}, k) \leq 0 \\ \hat{\sigma}(\alpha, \beta, \bar{\sigma}, k) & \textit{if } \hat{\sigma}(\alpha, \beta, \bar{\sigma}, k) \in (0, \bar{\sigma}) \\ \bar{\sigma} & \textit{if } \hat{\sigma}(\alpha, \beta, \bar{\sigma}, k) \geq \bar{\sigma}, \end{cases} \tag{2}$$

*where*

$$\hat{\sigma}(\alpha, \beta, \bar{\sigma}, k) \equiv \frac{\beta}{2k}(1 - \bar{\sigma})(\alpha - k)(1 - \alpha^2). \tag{3}$$

*Moreover, for parameters such that $\sigma^*$ is interior, the number of spoofers $\sigma^*$ is strictly increasing in the probability $\beta$ of legitimate cancellation, strictly decreasing in the expected penalty $k$ of spoofing, and strictly decreasing in the proportion $\bar{\sigma}$ of potential spoofers.*

Intuitively, if the proportion of informed traders $\alpha$ is low, then trades are not informative of the asset's value, so the spoofer has difficulty moving the price in his desired direction, which makes spoofing less attractive. If $\alpha$ is high, orders in opposite directions strongly indicate spoofing, resulting in a similar price as for a direct order, which again makes spoofing less attractive. It $\alpha$ is intermediate, then spoofers can move the price substantially without being easily identified as spoofers, making spoofing very attractive. Because higher $\alpha$ corresponds with lower liquidity, in the sense of higher spreads and greater price impacts from executed trades, this result suggests that spoofing should be most prevalent in markets of intermediate liquidity.

The conventional wisdom is that manipulation of various kinds is more common in illiquid markets, because illiquidity enables manipulators to more easily move the price. Indeed, Aggarwal and Wu (2006) document that most SEC litigated manipulation cases occur in small, illiquid markets. However, conditional on this set of illiquid markets, their regressions indicate that manipulation is more likely when such markets are *more* liquid. The authors point out that this positive correlation is likely understated, given that higher liquidity enables manipulators to avoid detection by blending in with the trading activity of others. Our model of spoofing captures both forces, showing that spoofers are attracted to markets which are illiquid enough to allow spoofers to move prices, but not so illiquid as to make their behavior stand out from the crowd.[13]

If the number of legitimate cancellations $\beta$ is high, spoofers' canceled orders are less suspicious to the market maker and therefore move the price more in the spoofer's desired direction, making spoofing more attractive. If the expected penalty $k$ for being caught spoofing is high, then only high gains $G(\sigma)$ can justify spoofing, which is possible only if few traders spoof so that the market maker trusts that canceled orders are informative of value. Finally, if the number of potential spoofers $\bar{\sigma}$ is high, then the market maker knows that relatively few canceled orders come from sincere, direct traders. He therefore distrusts canceled orders, updating his beliefs only slightly in the spoofers' desired direction, making spoofing unattractive.

# 5 Cross-Market Spoofing

Suppose there are two assets $a$ and $b$ with fundamental values $v_a \in \{-1, 1\}$ and $v_b \in \{-1, 1\}$. Each asset has equal probability of having value -1 or 1. Asset values are correlated: $P(v_a = v_b) = \phi > 1/2$, so their correlation coefficient is $\rho \equiv 2\phi - 1 > 0$. Each asset is traded in a separate submarket, but both submarkets together constitute the market.

There is a competitive market maker who posts bids and asks for each asset. There are also a large number of risk-neutral traders. A fraction $\alpha/2$ know the value $v_a$ of asset $a$ but not of asset $b$, and fraction $\alpha/2$ know the value $v_b$ of asset $b$, but not of asset $a$. The

[13] In a non-equilibrium numerical simulation with exogenous prices, Withanawasam, Whigham, and Crack (2018) show that pump-and-dump manipulation is easiest in markets of intermediate liquidity, though the mechanism is different from our setting.

remaining $1-\alpha$ fraction of traders are uninformed, half of whom want to buy or sell asset $a$ with equal probability and half of whom want to buy or sell asset $b$ with equal probability.

A fraction $\sigma_w$ of traders are within-market long-term traders; if they arrive at the market at date 1, they can trade twice in a row *within* the same submarket. A fraction $\sigma_c$ of traders are cross-market long-term traders; if they arrive at the market at date 1, they can trade once in one submarket and then again in the opposite submarket. We let $\sigma \equiv \sigma_w + \sigma_c$ denote the total fraction of long-term traders. The remaining $1-\sigma$ traders are short-term; as in Section 2, if they arrive at the market, that is their only opportunity to trade. Whether a trader is within–market long-term, cross-market long-term, or short-term is independent of his private information and liquidity preferences.

At date 1, a trader is randomly drawn from the total pool of traders and arrives at the market. He can choose which asset to trade. As in Section 2, a trader may choose to cancel his order after placing it, and may also get canceled exogenously with probability $\beta$. So a short-term trader's strategy is a pair $(o_t, c_t) \in \{B_a, S_a, B_b, S_b\} \times \{C, E\}$, a within-market long-term trader's strategy is a sequence $(o_1, c_1, o_2, c_2) \in \{\{B_i, S_i\} \times \{C, E\}\}^2$, $i \in \{a, b\}$, and a cross-market long-term trader's strategy is a sequence $(o_1, c_1, o_2, c_2) \in \{B_i, S_i\} \times \{C, E\} \times \{B_{-i}, S_{-i}\} \times \{C, E\}$, $i \in \{a, b\}$.

The market maker observes trades in both markets, using the order history to update his beliefs about the values of both assets. The equilibrium definition is nearly identical to that in Section 2, except that the pricing strategy for each asset conditions on execution of an order *for that asset*. That is, for each asset $i \in \{a, b\}$, pricing strategies satisfy $p_{1i}(o_1) = \mathbb{E}[V_i | o_1, s_1 = E]$, $o_1 \in \{B_i, S_i\}$, and $p_{2i}(o_1, s_2, o_2) = \mathbb{E}[V_i | o_1, s_1, o_2, s_2 = E]$, $o_2 \in \{B_i, S_i\}$.

To see whether there exists an equilibrium with cross-market spoofing, suppose that all cross-market long-term traders spoof, and consider the payoff of a cross-market long-term trader who ultimately wants to buy the asset.

If the trader places a direct trade, he pays price $\mathbb{E}[V_a | B_a E]$. If he spoofs across markets, by first placing a canceled sell in market $b$, the market maker knows that the order history $S_b C B_a E$ could have arisen from two events: either a cross-market long-term trader spoofed, an event we denote with $l_c$, or two short-term traders placed direct trades, an event we denote with $\neg l_c$. The price paid by the cross-market spoofer is therefore a weighted average of the expectations which condition on these two events:

$$\mathbb{E}[V_a|S_bCB_aE] = \mathbb{E}[V_a|l_c, S_bCB_aE]P(l_c|S_bCB_aE) + \mathbb{E}[V_a|\neg l, S_bCB_aE]P(\neg l|S_bCB_aE). \quad (4)$$

As in Section 2, if the market maker knew that the order history $S_bCB_aE$ came from a cross-market spoofer, the order history would be just as informative for asset value as a direct buy order $BE$. So $\mathbb{E}[V|l_c, S_bCB_aE] = \mathbb{E}[V|B_aE]$.

On the other hand, conditional on the order $S_bCB_aE$ coming from two short-term direct traders, the expected value of the asset $\mathbb{E}[V|\neg l_c, S_bCB_aE]$ must be strictly between $\mathbb{E}[V|B_aE]$ and zero. Similar to the within-market reasoning of Section 2, it is strictly less than the price $\mathbb{E}[V|B_aE]$ of a direct order because there is some chance that the date 1 trader who placed the cancelled sell $S_bC$ is informed that the asset is of low value. In contrast to the within-market reasoning of Section 2, it is strictly greater than zero because the assets are only imperfectly correlated, so the negative information about asset $a$ conveyed by an informed seller of the alternative asset $b$ is small compared to the positive information conveyed by an informed buyer of asset $a$.

By (4), the price paid for cross-market spoofing is a weighted average of the price $\mathbb{E}[V|l_c, S_bCB_aE] = \mathbb{E}[V|BE]$ for a direct order and a value $\mathbb{E}[V|\neg l_c, S_bCB_aE]$ strictly less than the price $\mathbb{E}[V|BE]$ for a direct order, so the price $\mathbb{E}[V|S_bCB_a, E]$ for cross-market spoofing must also be strictly less than that for a direct order. This means that cross-market spoofing may be profitable in equilibrium, leading to our next result.

**Proposition 6.** *There exists an equilibrium in which within-market long-term traders always spoof within-market, and cross-market traders always spoof across markets.*

We next confirm that cross-market spoofing has similar adverse market consequences as within-market spoofing. We compare the equilibrium of Proposition 6 to a benchmark without cross-market spoofing but with within-market spoofing. In this benchmark, we assume that if a cross-market long-term trader arrives at the market at date 1, he places no order at date 1 and places a direct order at date 2. The next two propositions demonstrate that cross-market spoofing has similarly negative effects on the market as those demonstrated for within-market spoofing in Section 3.

**Proposition 7.** *The market maker's beliefs about asset $i \in \{a, b\}$ are more accurate at the end of date 2 in the benchmark equilibrium than in the cross-market spoofing equilibrium:* $|v_i - \mathbb{E}[\pi_i(h)|v_i]| > |v_i - \hat{\mathbb{E}}[\hat{\pi}_i(h)|v_i]|$ *for* $v_i \in \{-1, 1\}$.

**Proposition 8.** *Given asset $i \in \{a, b\}$ and value $v_i \in \{-1, 1\}$, the cross-market spoofing equilibrium exhibits a higher spread in market $i$ between average executed bids and asks than the benchmark equilibrium.*

The intuition for the negative effects of cross-market spoofing is the same as for within-market spoofing: they both obscure trading motives, which gives the market maker less information about fundamentals, and necessitates that he raise bid-ask spreads to compensate for the worsened adverse selection. The results are not surprising, but we present them before studying optimal regulation in order to confirm that regulators have the incentive to discourage not only within-market but also cross-market spoofing.

## 5.1 Endogenous Cross-Market Spoofing

We now endogenize the measure of within- and cross-market spoofers, $(\sigma_w, \sigma_c)$. Suppose that $\bar{\sigma} \in (0, 1)$ traders are long-term traders and therefore have the opportunity to spoof if they wish. When such an trader arrives at the market, he can choose not only which submarket to trade in, but also whether to place a direct trade, to spoof within a particular submarket, or to spoof across submarkets.

A regulator monitors the market for spoofing, assigning some resources to monitoring trade in asset $a$, some resources to trade in asset $b$, and some resources to cross-market monitoring. The regulator's monitoring implies an expected penalty $k_{wi}$ for being caught spoofing within market $i \in \{a, b\}$, and expected penalty $k_{ci}$ for cross-market spoofing of asset $i \in \{a, b\}$.

Taking into account the expected penalty of being caught, the trader's expected cost of buying, for example, asset $a$ directly is $\alpha$, of spoofing within market $a$ is $C_a(\sigma_a) \equiv \mathbb{E}[V_a|S_aCB_a] + k_a$, and of spoofing across markets is $C_{ca}(\sigma_{ca}) \equiv \mathbb{E}[V_a|S_bCB_a] + k_c$. The trader takes an action which minimizes his expected costs. This leads to the following equilibrium definition.

**Definition 4.** Given $(\alpha, \beta, \bar{\sigma}, k) \in (0,1)^4$, an equilibrium is a vector $(\sigma_a^*, \sigma_b^*, \sigma_{ca}^*, \sigma_{cb}^*) \in [0,1]^4$, with $\sigma_i^* + \sigma_{ci}^* \leq \bar{\sigma}_i$ for $i \in \{a, b\}$ that satisfies

$$\begin{cases} \sigma_i^* = 0 & \text{if } C_i(\sigma_i^*) > \min\{\alpha, C_{ci}(\sigma_{ci}^*)\} \\ \sigma_{ci}^* = 0 & \text{if } C_{ci}(\sigma_{ci}^*) > \min\{\alpha, C_i(\sigma_i^*)\} \\ \sigma_i^* + \sigma_{ci}^* = \bar{\sigma}_i & \text{if } \alpha > \min\{C_i(\sigma_i^*), C_{ci}(\sigma_{ci}^*)\}. \end{cases} \tag{5}$$

The appendix shows that the cost $C_i(\sigma_i)$ of within-market spoofing is strictly increasing in the measure $\sigma_i$ of within-market spoofers, and the cost $C_{ci}(\sigma_{ci})$ of cross-market spoofing is strictly increasing in the measure $\sigma_{ci}$ of cross-market spoofers. So in equilibrium, the measure of both kinds of spoofing is such that their costs are equal to the cost of a direct trade. The following proposition gives closed form expressions for the equilibrium measure of both kinds of spoofing.

**Proposition 9.** *There exists a unique pure-strategy equilibrium. For high enough $k_i$ and $k_{ci}$, this equilibrium takes the following form for $i \in \{a, b\}$:*

$$\sigma_i^* = \max\left\{0, (1-\bar{\sigma}_i)\frac{\beta}{2}\left(\frac{\alpha}{k_i} - 1\right)(1-\alpha^2)\right\}$$
$$\sigma_{ci}^* = \max\left\{0, (1-\bar{\sigma}_i)\frac{\beta}{2}\left[\left(\rho\frac{\alpha}{k_{ci}} - 1\right)(1-\alpha^2) - (1-\rho)\alpha^2\right]\right\}.$$

The main thing to note is that if the penalties $k_i$ and $k_{ci}$ for both forms of spoofing are equal, there is less cross-market spoofing $\sigma_{ci}$ than within-market spoofing $\sigma_i$. Intuitively, cross-market spoofing is less tempting than within-market spoofing because the assets are imperfectly correlated, which implies that a cancelled order for the opposite asset one intends to trade moves the price of the target asset less than a canceled order for the target asset itself. So for example, if the measures $\sigma_i$ and $\sigma_{ci}$ of both forms of spoofing were equal, the ask paid by a cross-market spoofer would be higher than the ask paid by a within-market spoofer. This makes cross-market spoofing less attractive than within-market spoofing, so if the penalties for both are equal, there must be less cross-market spoofing in equilibrium.

## 5.2 Optimal Regulation

We now consider how a regulator optimally allocates monitoring resources between within- and cross-market spoofing. In practice, regulators do not have sufficient resources to monitor every possible trade, so in our setting we assume that each order is examined only with some probability.

The regulator adopts the following monitoring protocol. With some exogenous probability, he first examines an executed order. For simplicity, we assume that he examines executed orders in markets $a$ and $b$ with equal probability $\kappa \in [0, 1]$. He then checks to see if the same trader previously placed a canceled order, but he must decide whether to check the same or the opposite market as that in which the executed order was placed. Conditional on examining the executed order, the regulator commits to a probability of checking within the same market and the remaining probability of checking the opposite market. We denote the unconditional probability of within-market monitoring by $\kappa_w \in [0, \kappa]$ and of cross-market monitoring by $\kappa_c = \kappa - \kappa_w$.

If the penalty for being caught spoofing is $z > 0$, then the expected penalty for being caught spoofing within-market is $k_w = \kappa_w z$ and across-market is $k_c = \kappa_c z = (\kappa - \kappa_w)z = \kappa z - k_w$. So selecting the monitoring probablities $\kappa_w$ and $\kappa_c$ is equivalent to selecting the expected penalties $k_w$ and $k_c$, subject to the constraint that $k_w + k_c = k \equiv \kappa z$.

Suppose the regulator wishes to minimize the total measure of spoofers $\sigma_a^* + \sigma_b^* = \sigma_{wa}^* + \sigma_{ca}^* + \sigma_{wb}^* + \sigma_{cb}^*$. Because we assume that the regulator adopts a symmetric policy when examining markets $a$ and $b$, long-term traders will also adopt symmetric strategies, so it is sufficient to consider the policy $(\kappa_w, \kappa_c)$ which minimizes the total measure of spoofers $\sigma_i^* = \sigma_{iw}^* + \sigma_{ic}^*$ for either asset $i$. In the following proposition, we assume that the regulator has enough resources $\kappa$ so that Proposition 9 holds. Furthermore, we say that the regulator is *constrained* if he does not have sufficient resources $\kappa$ to eliminate spoofing altogether.

**Proposition 10.** *If the regulator is constrained, there exists a unique threshold $\bar{\kappa}$ such that*

*(i) If $\kappa < \bar{\kappa}$, there is a positive measure of both within- and cross-market spoofing. Furthermore, the regulator allocates monitoring in a fixed proportion between within- and cross-market spoofing:*

$$\frac{\kappa_c^*}{\kappa_w^*} = \sqrt{\rho}.$$

*(ii) If $\kappa > \bar{\kappa}$, the regulator monitors across markets just enough to eliminate cross-market spoofing, and spends his remaining resources on within-market monitoring:*

$$\kappa_c^* = \frac{\alpha(1-\alpha)^2}{z(\rho^{-1}-\alpha^2)} \qquad \kappa_w^* = \kappa - \kappa_c^*.$$

Part (i) states that if the regulator is sufficiently constrained in its monitoring resources, he should allocate a fixed proportion of those resources to cross-market monitoring. That proportion $\sqrt{\rho}$ is constant in his total resources $\kappa$ but increasing in the degree of correlation $\rho$ between assets. Intuitively, as the correlation increases, orders in one market have a stronger effect on the other market's prices, making cross-market spoofing more tempting and increasing the need for cross-market monitoring. The intuition for part (ii) is that the imperfect correlation between assets makes cross-market spoofing less effective at price manipulation than within-market spoofing, so it is less tempting to traders. As a result, it is also cheaper for regulators to deter than within-market spoofing, so if the regulator has sufficient resources, he should eliminate it.

*Remark* 2. The analysis in this section assumes that regulators aim to minimize the measure of spoofers in the market. In practice, regulators may have different objectives, such as minimizing average spreads or volatility, which may imply different optimal regulation. For example, because cross-market spoofing has a smaller impact on prices than within-market spoofing, it is both less tempting to traders and has a smaller impact on spreads. As a result, a regulator seeking to minimize average spreads may choose to never eliminate cross-market spoofing: it may be cheaper to eliminate than within-market spoofing, but its small impact on spreads makes it less important to eliminate.

# 6 Conclusion

In this paper, we have presented a dynamic model of trading in which traders may cancel orders strategically. Our paper extends the market order setting of Glosten and Milgrom (1985) by allowing canceled orders, which enables us to capture the main forces driving spoofing in practice, while avoiding the tractability challenges of information-based limit order models.

We show that in this setting, spoofing may occur in equilibrium, despite the full awareness of other market participants that spoofing may occur. We show that spoofing slows price discovery, raises bid-ask spreads, and raises return volatility, consistent with regulator concerns that spoofing threatens market stability and discourages legitimate traders from participating.

We also endogenize the measure of spoofers, showing that spoofing is more prevalent when the proportion of informed traders is not too high and not too low. Finally, we show that spoofers can profit from spoofing across markets, and that more cross-market surveillance should occur when markets are highly correlated.

While we have presented the economic logic using a simple market order model, we believe that the intuitions are much more robust. We expect incentives to spoof to exist in any model as long as it satisfies three conditions: 1) some traders make legitimate cancellations (for example, traders sometimes cancel limit orders outside current bid and ask for reasons other than an attempt to manipulate the market); 2) legitimately canceled orders affect prices (for example, because they contain information about the fundamental as in our model, or about relative demand and supply by liquidity traders, or about the depth of the market, which entices large traders to execute trades); 3) the ability of the regulator to detect and punish spoofing is somewhat limited. We expect that spoofing would happen in equilibrium in any such model, and spoofing would affect prices, making them less informative and the market less liquid. That said, additional analysis of spoofing (theoretical and empirical) would help us assess the quantitative costs of spoofing and appropriate regulatory responses.

# Appendix

**Proof of Proposition 1**. We claim that the following is an equilibrium: all short-term traders place direct trades, all long-term traders spoof, and prices are such that the market maker breaks even.

First consider short-term traders. If a short-term trader is selected to meet with the market maker at date 1, he has only an infinitesimal chance of also being selected at date 2; that is, he has effectively one chance to trade, so he has no incentive to strategically cancel the order. Therefore, an uninformed short-term trader wishing to buy (sell) the asset will place a buy (sell) order and attempt to execute it. A short term trader informed that the asset is good (bad) will place a buy (sell) order and attempt to execute it, as long as the price of the asset is less (greater) than the asset's fundamental value. We show below that this is always the case in our candidate equilibrium. Similarly, because date 2 is the final trading date of the game, a short-term trader meeting the market maker at date 2 has no incentive to strategically cancel, so he employs the same strategy as if he had met the market maker at date 1.

Next, consider the pricing strategy of the market maker, summarized in the following lemma.

**Lemma 1** (Pricing Strategies)**.** *If the market maker expects that all long-term traders spoof, he sets the following bids and asks:*

*(i) At date 1,*

$$p_1(B) = \alpha, \qquad p_1(S) = -\alpha.$$

*(ii) At date 2,*

$$\begin{aligned}
p_2(NB) &= \alpha & p_2(NS) &= -\alpha \\
p_2(SEB) &= 0 & p_2(BES) &= 0 \\
p_2(BEB) = p_2(BCB) &= \frac{2\alpha}{1+\alpha^2} & p_2(SES) = p_2(SCS) &= \frac{-2\alpha}{1+\alpha^2} \\
p_2(SCB) &= \frac{\sigma\alpha}{\sigma + (1-\sigma)\beta 0.5(1-\alpha^2)} & p_2(BCS) &= \frac{-\sigma\alpha}{\sigma + (1-\sigma)\beta 0.5(1-\alpha^2)}.
\end{aligned}$$

*Proof.* Given some history $h$, the market maker assigns a price equal to his expected value of the asset. A variety of trader types may yield a particular order history $h$. Recall that a trader's type consists of the horizon of his trading (long- or short-term) and his motive for trade (informed, uninformed buying, uninformed selling). Let $M \equiv \{i, u_b, u_s\}$ denote the set of motives for trade.

We first derive prices for date 1 order histories $h_1$. Given an order history $h_1$, the market maker sets the price conditional on order execution. The probability of order history $h_1$ may be decomposed as

$$P(h_1) = \sum_{m \in M} P(h_1, m), \tag{6}$$

so by the Law of Iterated Expectations, the market maker's expected value given order $h_1$ may be decomposed as:

$$\begin{aligned}\mathbb{E}[V|h_1] &= \sum_{m \in M} \mathbb{E}[V|m, h_1] P(m|h_1) = \sum_{m \in M} \mathbb{E}[V|m, h_1] \frac{P(m, h_1)}{P(h_1)} \\ &= \frac{\sum_{m \in M} \mathbb{E}[V|m, h] P(m, h_1)}{\sum_{m \in M} P(m, h_1)}.\end{aligned} \tag{7}$$

For example, what price does the market maker set for a buy order executed at date 1? The buy order could have been placed by an informed trader who knows the asset is high value, which occurs with probability $\alpha 0.5$ or by an uninformed liquidity trader, which occurs with probability $(1-\alpha)0.5$. Furthermore, conditional on an informed trader placing a buy order, the expected value of the asset must be 1, whereas conditional on an uninformed liquidity trader placing a buy order, the expected value of the asset must be zero, because uninformed trades convey no information. So applying (7), the price set for an executed buy order at date 1 is

$$\mathbb{E}[V|BE] = \frac{\mathbb{E}[V|i, BE] P(i, BE) + \mathbb{E}[V|u_b, BE] P(u_b, BE)}{P(i, BE) + P(u_b, BE)} = \frac{(1)\alpha 0.5 + (0)(1-\alpha)0.5}{\alpha 0.5 + (1-\alpha)0.5},$$

which simplifies to

$$\mathbb{E}[V|BE] = \alpha.$$

A symmetric argument shows that

$$\mathbb{E}[V|SE] = -\alpha.$$

We next consider the prices the market maker sets for date 2 order histories $h$. Any order history $h$ was placed either by one long-term trader or by two short-term traders, so letting $l$ denote the event of a long-term trader, and $\neg l$ denote the event of two short term traders, we can decompose the probability of order $h$ as follows:

$$P(h) = P(h,l) + P(h,\neg l) = \sum_{m\in M} P(h,l,m) + \sum_{m\in M^2} P(h,\neg l,m). \tag{8}$$

Similarly, the market maker's expected value given order $h$ may be decomposed as follows:

$$\begin{aligned}
\mathbb{E}[V|h] &= \sum_{m\in M} \mathbb{E}[V|l,m,h]P(l,m|h) + \sum_{m\in M^2} \mathbb{E}[V|\neg l,m,h]P(\neg l,m|h) \\
&= \Big(\sum_{m\in M} \mathbb{E}[V|l,m,h]P(l,m,h) + \sum_{m\in M^2} \mathbb{E}[V|\neg l,m,h]P(\neg l,m,h)\Big)/P(h) \\
&= \frac{\sum_{m\in M} \mathbb{E}[V|l,m,h]P(l,m,h) + \sum_{m\in M^2} \mathbb{E}[V|\neg l,m,h]P(\neg l,m,h)}{\sum_{m\in M} P(h,l,m) + \sum_{m\in M^2} P(h,\neg l,m)}. \qquad (9)
\end{aligned}$$

For example, what ask does a market maker set at date 2, if he observed a canceled sell order at date 1? The ask is the expected value of the asset, conditional on the date 1 history $SC$ and on the execution of a buy order $BE$ at date 2: $\mathbb{E}[V|SCBE]$. In our hypothesized equilibrium, no traders strategically cancel at date 2, so whether the order is executed or not at date 2 contains no information. As a result, $\mathbb{E}[V|SCBE] = \mathbb{E}[V|SCBC]$, and we drop the date 2 $E$ in the conditional expectation to simplify notation in the rest of the proof.

If the market maker observes history $SCB$, he knows that it could have come from either a spoofing long-term trader with probability $\sigma$ or two short-term traders with probability $(1-\sigma)\beta$. If the history came from a long-term trader, then his ultimate intention is to buy the asset, in which case there are two subcases: (1) with probability $\alpha 0.5$, the trader is informed that the asset is good, in which case the expected asset value is 1; (2) with probability $(1-\alpha)0.5$, the trader is uninformed and wants to purchase only for liquidity reasons, in which case the expected asset value is 0. If the history came from two short-term traders, then there are three subcases: (1) with probability $\alpha(1-\alpha)0.5^2$, the first trader is informed that the asset is good (in which case the expected asset value is 1) and the second trader is an uninformed buyer; (2) with probability $\alpha(1-\alpha)0.5^2$, the first trader is an uninformed seller, and the second trader is informed that the asset is bad (in which case the expected asset value is -1); (3) with probability $(1-\alpha)^2 0.5^2$, the first trader is an uninformed seller, and the second trader is an uninformed buyer (in which case the expected asset value is 0). Of course, the two traders cannot both be informed, because they place orders in opposite directions.

Applying Equation (9), we have

$$\mathbb{E}[V|SCB] = \frac{\sigma((1)\alpha 0.5 + (0)(1-\alpha)0.5) + (1-\sigma)\beta((1-1)\alpha(1-\alpha)0.5^2 + (0)(1-\alpha)^2 0.5^2)}{\sigma(\alpha 0.5 + (1-\alpha)0.5) + (1-\sigma)\beta(2\alpha(1-\alpha)0.5^2 + (1-\alpha)^2 0.5^2)},$$

which simplifies to

$$\mathbb{E}[V|SCB] = \frac{\sigma\alpha}{\sigma + (1-\sigma)\beta 0.5(1-\alpha^2)}.$$

A symmetric argument shows that

$$\mathbb{E}[V|BCS] = -\frac{\sigma\alpha}{\sigma + (1-\sigma)\beta 0.5(1-\alpha^2)}.$$

Consider the price the market maker sets on observing history $BEB$. In the proposed equilibrium, long-term traders always spoof, so this order history could not have come from a long-term trader; therefore, this history must have arisen from two short-term traders, which occurs with probability $(1-\sigma)^2(1-\beta)$. This event can further be composed into three possible events: (1) with probability $\alpha^2 0.5$, the two traders are informed that the asset is good, in which case the expected asset value is 1; (2) with probability $2\alpha(1-\alpha)0.5^2$, one trader is informed that the asset is good, and the other is an uninformed buyer, in which case the expected value is 1; (3) with probability $(1-\alpha)^2 0.5^2$ both traders are uninformed buyers, in which case the expected value is zero. Applying (9), we have

$$\mathbb{E}[V|BEB] = \frac{(1-\sigma)^2(1-\beta)[\alpha^2 0.5(1) + 2\alpha(1-\alpha)0.5^2(1) + (1-\alpha)^2 0.5^2(0)]}{(1-\sigma)^2(1-\beta)[\alpha^2 0.5 + 2\alpha(1-\alpha)0.5^2 + (1-\alpha)^2 0.5^2]},$$

which simplifies to

$$\mathbb{E}[V|BEB] = \frac{2\alpha}{1+\alpha^2}.$$

Symmetric reasoning shows that

$$\mathbb{E}[V|SES] = -\frac{2\alpha}{1+\alpha^2}.$$

Note that order history $BCB$ could also have come only from two short term traders, and with probability $(1-\sigma)^2\beta$. As in the case of $BEB$, this decomposes into the same three events and conditional expectations, and applying (9) shows that $(1-\sigma)^2\beta$ cancels, so that $\mathbb{E}[V|BCB] = \mathbb{E}[V|BEB]$.

Finally, consider history $SEB$. Because the first order was executed, it could not have come from a long-term spoofer, so the order history must have come from two short-term traders, with probability $(1-\sigma)^2(1-\beta)$. Because the orders are in opposite directions, the two traders cannot both be informed, so there are three events to consider: (1) with probability $\alpha(1-\alpha)0.5^2$, the first trader is informed that the asset is bad and the second trader is an uninformed buyer, in which case the conditional expected value is -1; (2) with probability $\alpha(1-\alpha)0.5^2$, the first trader is an uninformed seller, and the second trader is informed that the asse tis good, in which case the conditional expected value is 1; (3) with probability $(1-\alpha)^2 0.5^2$, the first is an uninformed seller and the second is an uninformed buyer, in which case the conditional expected value is 0. Applying (9) shows

$$\begin{aligned}\mathbb{E}[V|SEB] &= \frac{(1-\sigma)^2(1-\beta)[\alpha(1-\alpha)0.5^2(1-1) + (1-\alpha^2)0.5^2(0)]}{(1-\sigma)^2(1-\beta)[2\alpha(1-\alpha)0.5^2 + (1-\alpha^2)0.5^2]} \\ &= 0.\end{aligned}$$

Symmetric reasoning shows that $\mathbb{E}[V|BES] = 0$ as well. □

Finally, we consider the strategy of the long-term trader. If date 2 is the first time a long-term trader is selected to meet with the market maker, then because date 2 is the last trading date of the game, it is also the trader's only opportunity to trade; as a result, he trades directly, the same as short-term traders.

Suppose that a long-term trader meets the market maker at date 1. We assume that he can execute only one order, so we can rule out all strategies that entail two executions.

Without loss of generality, suppose the trader seeks to ultimately buy the asset, either for informational or liquidity reasons. So his optimal strategy must entail executing a buy order either at date 1 or date 2; if he executes the buy order at date 1, he pays $p_1(B) = \alpha$. If he executes the buy order at date 2, the price he pays depends on his order at date 1. If he places no order at date 1, he pays price $p_2(NB) = \alpha$; if he places a cancelled buy, he pays price $p_2(BCB) = 2\alpha/(1+\alpha^2)$, which is strictly greater than $p_2(NB) = \alpha$. If he places a cancelled sell, he pays price $p_2(SCB) = \sigma\alpha/(\sigma + (1-\sigma)\beta 0.5(1-\alpha^2))$, which is strictly less than $p_1(B) = p_2(NB) = \alpha$ and therefore also strictly less than $p_2(BCB)$. That is, he gets the lowest price by placing a cancelled sell followed by an executed buy (that is, spoofing), so that strategy must be optimal. A symmetric argument shows that a long-term seller finds it optimal to place a cancelled buy followed by an executed sell order. So there exists an equilibrium in which long-term traders always spoof, and the proposition is proved. □

**Proof of Proposition 2.** Without loss of generality, assume $v = 1$. By Bayes' Rule, the market maker's posterior belief $\pi(h)$ is equal to $P(v = 1|h) = P(h|v = 1)\pi/[P(h|v = 1)\pi + P(h|v = -1)(1-\pi)]$. Let $H^- = \{SCB, BCS\}$ and $H_b^- = H^- \cup \{nS, nB\}$. For all $h \in H \setminus H_b^-$ and $v \in \{-1, 1\}$, we have that $P(h|v) = P_b(h|v)$, and therefore $\pi(h) = \pi_b(h)$. As a result,

$$
\begin{aligned}
E_b[\pi_b(h)|v=1] - E[\pi(h)|v=1] &= \sum_{h\in H_b} \pi_b(h)P_b(h|v=1) - \sum_{h\in H} \pi(h)P(h|v=1) \\
&= \sum_{h\in H_b^-} \pi_b(h)P_b(h|v=1) - \sum_{h\in H^-} \pi(h)P(h|v=1) \\
&= \pi_b(SCB)P_b(SCB|v=1) + \pi_b(BCS)P_b(BCS|v=1) \\
&\quad + \pi_b(nB)P_b(nB|v=1) + \pi_b(nS)P_b(nS|v=1) \\
&\quad - \pi(SCB)P(SCB|v=1) - \pi(BCS)P(BCS|v=1).
\end{aligned}
\tag{10}
$$

For notational convenience, define

$$
\begin{aligned}
&a \equiv P(SCB|sp, v=1) = P(BCS|sp, v=-1) = \sigma 0.5(1+\alpha), \\
&b \equiv P(SCB|sp, v=-1) = P(BCS|sp, v=1) = \sigma 0.5(1-\alpha)), \\
&c \equiv P(h|nsp, v) = (1-\sigma)\beta 0.5^2(1-\alpha^2), \qquad h \in H^-, \quad v \in \{-1, 1\}
\end{aligned}
$$

Then $P(SCB|v=1) = P(BCS|v=-1) = a+c$, $P(BCS|v=1) = P(SCB|v=-1) = b+c$, $P_b(nB|v=1) = P_b(nS|v=-1) = a$, and $P_b(nS|v=1) = P_b(nB|v=-1) = b$, and $P_b(h|v) = c$ for all $h \in H^-$, $v \in \{-1,1\}$. Also let $f(x) \equiv \pi/(\pi + x(1-\pi))$. Applying the above, we can rewrite (10) as follows:

$$
\begin{aligned}
&E_b[\pi_b(h)|v=1] - E[\pi(h)|v=1] \\
&\quad= \frac{c\pi}{c\pi + c(1-\pi)}c + \frac{c\pi}{c\pi + c(1-\pi)}c + \frac{a\pi}{a\pi + b(1-\pi)}a + \frac{b\pi}{b\pi + a(1-\pi)}b \\
&\qquad - \frac{(a+c)\pi}{(a+c)\pi + (b+c)(1-\pi)}(a+c) - \frac{(b+c)\pi}{(b+c)\pi + (a+c)(1-\pi)}(b+c) \\
&\quad= f(1)c + f(1)c + f\left(\frac{b}{a}\right)a + f\left(\frac{a}{b}\right)b \\
&\qquad - f\left(\frac{b+c}{a+c}\right)(a+c) - f\left(\frac{a+c}{b+c}\right)(b+c) \\
&\quad= (a+c)\left[f(1)\frac{c}{a+c} + f\left(\frac{b}{a}\right)\frac{a}{a+c} - f\left(\frac{b+c}{a+c}\right)\right] \\
&\qquad + (b+c)\left[f(1)\frac{c}{b+c} + f\left(\frac{a}{b}\right)\frac{b}{b+c} - f\left(\frac{a+c}{b+c}\right)\right].
\end{aligned} \tag{11}
$$

By Jensen's inequality, the bracketed sums are strictly positive, so $\mathbb{E}_b[\pi_b(h)|v=1] > \mathbb{E}[\pi(h)|v=1]$. Because prices $p$ and $p_b$ are linearly increasing in posteriors $\pi(h)$ and $\pi_b(h)$, respectively, this implies $\mathbb{E}[p|v=1] < \mathbb{E}_b[p_b|v=1] \leq 1 = v$. □

**Proof of Proposition 3.** We wish to show that for $v \in \{-1,1\}$,

$$\mathbb{E}[p|o_2 = B, v] - \mathbb{E}[p|o_2 = S, v] > \mathbb{E}_b[p_b|o_2 = B, v] - \mathbb{E}_b[p_b|o_2 = S, v], \tag{12}$$

where we have suppressed the dependence of prices $p$ and $p_b$ on history $h$.

As a preliminary, we first establish two lemmas.

**Lemma 2.** *Unconditional on $v$, the spoofing and benchmark equilibria exhibit the same average executed bids and asks:*

$$\mathbb{E}[p(h)|o_2 = S] = \mathbb{E}_b[p_b(h)|o_2 = S] \quad \textit{and} \quad \mathbb{E}[p(h)|o_2 = B] = \mathbb{E}_b[p_b(h)|o_2 = B].$$

*Proof.* Let the set of date 2 histories $h$ be denoted by $H \equiv H_1 \times \{B, S\}$. Consider the average ask in the spoofing equilibrium:

$$\mathbb{E}[p(h)|o_2 = B] = \mathbb{E}[\mathbb{E}[V|h]|o_2 = B] = \mathbb{E}[V|o_2 = B].$$

Rewrite the expectation by conditioning on whether or not the trader who places the date 2 order is informed:

$$\begin{aligned}\mathbb{E}[V|o_2 = B] &= \mathbb{E}[V|o_2 = B, i]P(i|o_2 = B) + \mathbb{E}[V|o_2 = B, \neg i]P(\neg i|o_2 = B)\\ &= 1 \cdot P(i|o_2 = B) + 0 \cdot P(\neg i|o_2 = B)\\ &= P(i|o_2 = B).\end{aligned}$$

Unconditional on $v$, an trader's type and date 2 order are independent, so $\mathbb{E}[V|o_2 = B] = P(i|o_2 = B) = P(i) = \alpha$.

For the benchmark equilibrium, identical reasoning shows that $\mathbb{E}_b[p_b(h)|o_2 = B] = \alpha$. And by symmetry, $\mathbb{E}[p(h)|o_2 = S] = \mathbb{E}_b[p_b(h)|o_2 = S] = -\alpha$. □

The next lemma implies that conditional on $v$, mispricing in the spoofing equilibrium (relative to the benchmark) is worse for orders that occur less often.

**Lemma 3.** *For* $v \in \{-1, 1\}$,

$$\begin{aligned}&P(o_2 = B|v)\Big(\mathbb{E}[p|o_2 = B, v] - \mathbb{E}_b[p_b|o_2 = B, v]\Big)\\ =&P(o_2 = S|v)\Big(\mathbb{E}[p|o_2 = S, v] - \mathbb{E}_b[p_b|o_2 = S, v]\Big).\end{aligned} \tag{13}$$

*Proof.* By the Law of Iterated Expectations,

$$\mathbb{E}[p|o_2] = \mathbb{E}[p|o_2, v]P(v|o_2) + \mathbb{E}[p|o_2, \neg v]P(\neg v|o_2). \tag{14}$$

By symmetry, $\mathbb{E}[p|o_2, \neg v] = -E[p|\neg o_2, v]$, and $P(\neg v|o_2) = P(v|\neg o_2)$. Furthermore, by Bayes' Rule,

$$P(v|o_2) = \frac{P(o_2|v)P(v)}{P(o_2|v)P(v) + P(o_2|\neg v)P(\neg v)} = \frac{P(o_2|v)P(v)}{P(o_2|v)P(v) + P(\neg o_2|v)P(v)} = P(o_2|v).$$

Use these facts to rewrite (14):

$$\begin{aligned}\mathbb{E}[p|o_2] &= \mathbb{E}[p|o_2, v]P(v|o_2) - \mathbb{E}[p|\neg o_2, v]P(v|\neg o_2) \\ &= \mathbb{E}[p|o_2, v]P(o_2|v) - \mathbb{E}[p|\neg o_2, v]P(\neg o_2|v). \end{aligned} \tag{15}$$

Similar reasoning shows that

$$\mathbb{E}_b[p_b|o_2] = \mathbb{E}_b[p_b|o_2, v]P_b(o_2|v) - \mathbb{E}_b[p_b|\neg o_2, v]P_b(\neg o_2|v). \tag{16}$$

By Lemma 2, $\mathbb{E}[p|o_2] = \mathbb{E}_b[p_b|o_2]$, and by construction, the probability of a particular date 2 order under spoofing is the same as under the benchmark: $P(o_2|v) = P_b(o_2|v)$. Combining (15) with (16), replacing $P_b(o_2|v)$ with $P(o_2|v)$, and rearranging gives:

$$P(o_2|v)\Big(\mathbb{E}[p|o_2, v] - \mathbb{E}_b[p_b|o_2, v]\Big) = P(\neg o_2|v)\Big(\mathbb{E}[p|\neg o_2, v] - \mathbb{E}_b[p_b|\neg o_2, v]\Big). \tag{17}$$

Setting $o_2 = B$, which implies $\neg o_2 = S$, results in (13) . □

If $v = 1$, then $P(o_2 = B|v) > P(o_2 = S|v)$. Furthermore, applying the proof of Proposition 2 to $o_2 = S$ shows that $\mathbb{E}[p|o_2 = S, v] < \mathbb{E}_b[p_b|o_2 = S, v]$. On the other hand, if $v = -1$, then $P(o_2 = B|v) < P(o_2 = S|v)$, and $\mathbb{E}[p|o_2 = S, v] > \mathbb{E}_b[p_b|o_2 = S, v]$. So whether $v = 1$ or $v = -1$, by Lemma 3,

$$\begin{aligned}\mathbb{E}[p|o_2 = B, v] - \mathbb{E}_b[p_b|o_2 = B, v] &= \frac{P(o_2 = S|v)}{P(o_2 = B|v)}\Big(\mathbb{E}[p|o_2 = S, v] - \mathbb{E}_b[p_b|o_2 = S, v]\Big) \\ &> \mathbb{E}[p|o_2 = S, v] - \mathbb{E}_b[p_b|o_2 = S, v], \end{aligned} \tag{18}$$

which implies (12) for $v \in \{-1, 1\}$, completing the proof of Part (ii). □

**Lemma 4.** *The average spread between posted bids and asks is higher under the spoofing equilibrium than the benchmark equilibrium.*

*Proof.* Let the set of date 1 histories $h_1$ be denoted by $H_1 = \{BE, SE, BC, SC, N\}$, where $N$ means no one arrives at date 1. Given some history $h_1$, the bid-ask spread is $\mathbb{E}[V|h_1, B] - \mathbb{E}[V|h_1, S]$ under spoofing and $\mathbb{E}_b[v|h_1, B] - \mathbb{E}_b[v|h_1, S]$ under the benchmark. We wish to show that

$$\mathbb{E}[\mathbb{E}[V|h_1, B] - \mathbb{E}[V|h_1, S]] > \mathbb{E}_b[\mathbb{E}_b[v|h_1, B] - \mathbb{E}_b[v|h_1, S]].$$

We first show that $\mathbb{E}[\mathbb{E}[V|h_1, B]] > \mathbb{E}_b[\mathbb{E}_b[v|h_1, B]]$; i.e., average posted asks are higher under the spoofing equilibrium. Note that because a spoofer would never place orders $h_1 \in \{BE, SE\}$, posted asks and probabilities are equal in both equilibria for those orders: $\mathbb{E}[V|h_1, B] = \mathbb{E}_b[v|h_1, B]$, and $P(h_1) = P_b(h_1)$. So it remains to compare asks and probabilities for $h_1 \in \{BC, SC, N\}$. In particular, it suffices to show that

$$\begin{aligned}&\mathbb{E}[V|BCB]P(BC) + \mathbb{E}[V|SCB]P(SC)\\ &\qquad > \mathbb{E}_b[v|BCB]P_b(BC) + \mathbb{E}_b[v|NB]P_b(N) + \mathbb{E}_b[v|SCB]P_b(SC). \qquad (19)\end{aligned}$$

Because a spoofer would never place the order $BCB$, it must be that $\mathbb{E}[V|BCB] = \mathbb{E}_b[BCB]$. Also, letting $sp$ denote the event that a spoofer arrives at date 1, observe that $P(BC) = P(sp, BC) + P(\neg sp, BC) = P(sp, BCS) + P(\neg sp, BC) = P_b(NS) + P_b(BC)$. Se we have

$$\mathbb{E}[V|BCB]P(BC) = \mathbb{E}_b[v|BCB]\Big(P_b(NS) + P_b(BC)\Big). \qquad (20)$$

Next, observe that

$$\begin{aligned}\mathbb{E}[V|SCB] &= \mathbb{E}[V|sp, SCB]P(sp|SCB) + \mathbb{E}[V|\neg sp, SCB]P(\neg sp|SCB)\\ &= \frac{1}{P(SCB)}\Big(\mathbb{E}[V|sp, SCB]P(sp, SCB) + \mathbb{E}[V|\neg sp, SCB]P(\neg sp, SCB)\Big).\end{aligned}$$

Furthermore, by construction of the benchmark, $\mathbb{E}[V|sp, SCB] = \mathbb{E}_b[v|NB]$, $P(sp, SCB) = P_b(NB)$, $\mathbb{E}[V|\neg sp, SCB] = \mathbb{E}_b[v|SCB]$, and $P(\neg sp, SCB) = P_b(SCB)$, which implies

$$\begin{aligned}\mathbb{E}[V|SCB]P(SC) &= \frac{P(SC)}{P(SCB)}\Big(\mathbb{E}_b[v|NB]P_b(NB) + \mathbb{E}_b[v|SCB]P_b(SCB)\Big).\\ &> \mathbb{E}_b[v|NB]P_b(NB) + \mathbb{E}_b[v|SCB]P_b(SCB). \qquad (21)\end{aligned}$$

Summing (20) and (21) gives

$$
\begin{aligned}
&\mathbb{E}[V|BCB]P(BC) + \mathbb{E}[V|SCB]P(SC) \\
&\qquad > \mathbb{E}_b[v|BCB]\Big(P_b(NS) + P_b(BC)\Big) + \mathbb{E}_b[v|NB]P_b(NB) + \mathbb{E}_b[v|SCB]P_b(SCB) \\
&\qquad = \mathbb{E}_b[v|BCB]P_b(NS) + E_b[v|BCB]P_b(BC) + \mathbb{E}_b[v|NB]P_b(NB) \\
&\qquad\qquad + \mathbb{E}_b[v|SCB]P_b(SCB).
\end{aligned} \tag{22}
$$

Observe that because $E_b[v|SCB] = 0$, we have $\mathbb{E}_b[v|SCB]P_b(SCB) = \mathbb{E}_b[v|SCB]P_b(SC)$. Also note that $\mathbb{E}_b[v|BCB] = 2\alpha/(\alpha^2+1) > \alpha = \mathbb{E}_b[NB]$. So (22) implies

$$
\begin{aligned}
&\mathbb{E}[V|BCB]P(BC) + \mathbb{E}[V|SCB]P(SC) \\
&\qquad > \mathbb{E}_b[v|NB]P_b(NS) + E_b[v|BCB]P_b(BC) + \mathbb{E}_b[v|NB]P_b(NB) \\
&\qquad\qquad + \mathbb{E}_b[v|SCB]P_b(SC) \\
&\qquad = E_b[v|BCB]P_b(BC) + \mathbb{E}_b[v|NB]\Big(P_b(NS) + P_b(NB)\Big) + \mathbb{E}_b[v|SCB]P_b(SC) \\
&\qquad = E_b[v|BCB]P_b(BC) + \mathbb{E}_b[v|NB]P_b(N) + \mathbb{E}_b[v|SCB]P_b(SC),
\end{aligned}
$$

which establishes (19). As a result, average posted asks are higher under spoofing than under the benchmark. A symmetric argument shows that average posted bids are lower under spoofing than under the benchmark. So the average spread between posted bids and asks is higher under spoofing than under the benchmark. □

**Proof of Proposition 4.** Tedious calculations show that for midpoints, the difference in expected volatility between the two equilibria is:

$$
\begin{aligned}
&\mathbb{E}[\mathrm{Var}(h)] - \mathbb{E}_b[\mathrm{Var}_b(h)] \\
&\qquad = \frac{2\alpha^2\sigma\left(\left(1-\alpha^2\right)^2\beta^2(1-\sigma)^2 + (1-\alpha)(3+\alpha)\sigma^2 + \left(\alpha^6+2\alpha^4-3\alpha^2+4\right)\beta\sigma(1-\sigma)\right)}{\left(\alpha^2+1\right)^2\left(\left(1-\alpha^2\right)\beta(1-\sigma)+2\sigma\right)^2}.
\end{aligned}
$$

The term $\alpha^6 + 2\alpha^4 - 3\alpha^2 + 4$ is strictly positive for all $\alpha \in [0,1]$, as it has only one real root, which root is strictly negative. All other terms are clearly positive, so $\mathbb{E}[\mathrm{Var}(h)] - \mathbb{E}_b[\mathrm{Var}_b(h)] > 0$. □

**Proof of Proposition 5.** For each case, we first show that the quantity $\sigma^*$ presented in the proposition is an equilibrium, and then show that it is unique. Throughout the proof, we suppress the dependence of $\hat{\sigma}(\alpha, \beta, \bar{\sigma}, k)$ on $\alpha$, $\beta$, $\bar{\sigma}$, and $k$. As a preliminary, observe for all $(\alpha, \beta, \bar{\sigma}, k) \in (0,1)^4$, $G(\sigma)$ is strictly decreasing in $\sigma$ and $G(\hat{\sigma}) = k$.

Case 1: $\hat{\sigma} \leq 0$. If $\sigma^* = 0 \geq \hat{\sigma}$, then because $G(\cdot)$ is decreasing in $\sigma$, we have $G(\sigma^*) \leq G(\hat{\sigma}) = k$, which satisfies the equilibrium conditions (1). Furthermore, if $\sigma$ were strictly positive, then $\sigma > 0 \geq \hat{\sigma}$, so $G(\sigma) < G(\hat{\sigma}) = k$, contradicting equilibrium.

Case 2: $\hat{\sigma} \in (0, \bar{\sigma})$. If $\sigma^* = \hat{\sigma}$, then $G(\sigma^*) = G(\hat{\sigma}) = k$, satisfying (1). If $\sigma < \hat{\sigma} \leq 1$, then $G(\sigma) > G(\hat{\sigma}) = k$, contradicting (1), and if $\sigma > \hat{\sigma} \geq 0$, then $G(\sigma) < G(\hat{\sigma}) = k$, also contradicting (1).

Case 3: $\hat{\sigma} \geq \bar{\sigma}$. If $\sigma^* = \bar{\sigma} \leq \hat{\sigma}$, then $G(\sigma^*) \geq G(\hat{\sigma}) = k$, satisfying (1). If $\sigma < \bar{\sigma} \leq \hat{\sigma}$, then $G(\sigma) > G(\hat{\sigma}) = k$, contradicting (1).

The comparative statics with respect to $\beta$, $k$, and $\bar{\sigma}$ follow directly from (3), so we conclude the proof by showing that $\sigma^*$ is single-peaked in $\alpha$.

If $\alpha \leq k$, then $\hat{\sigma} \leq 0$, so $\sigma^* = 0$. As $\alpha \to 1$, $\hat{\sigma} \to 0$, so $\sigma^* \to 0$. If $\alpha \in [k, 1)$, then $\sigma^* = \min\{\hat{\sigma}, \bar{\sigma}\}$. The derivative of $\hat{\sigma}$ with respect to $\alpha$ has the same sign as

$$\frac{\partial}{\partial \alpha}\left[(\alpha - k)(1 - \alpha^2)\right] = -3\alpha^2 + 2k\alpha + 1. \tag{23}$$

When $\alpha = k$, expression (23) is $-k^2 + 1 > 0$, and when $\alpha = 1$, (23) is $2(k-1) < 0$. By the intermediate value theorem, there exists a root of (23) in $(k, 1)$, and because (23) is quadratic, there is only one root in $(k, 1)$, implying that $\hat{\sigma}$ is single-peaked in $\alpha$ over the range $[k, 1)$. Therefore, $\sigma^*$ is constant at 0 over the range $\alpha \in (0, k]$, converges to 0 as $\alpha$ converges to 1, and is single-peaked (equal to $\min\{\hat{\sigma}, \bar{\sigma}\}$) over the range $\alpha \in [k, 1)$. □

**Proof of Proposition 6**. The proposition states that the strategies of short-term traders and within-market long-term traders is the same as in Proposition 1. Furthermore, Given the proposed equilibrium, the proof that those strategies are optimal is the same as in Proposition 1. So it remains to derive the market maker's prices for cross-market order histories, and then show that cross-market long-term traders find it optimal to spoof across markets.

**Lemma 5.** *If the order history occurs within a single market, the market maker prices according to Lemma 1. Otherwise, the market maker sets the following prices for asset a:*

$$p_2(S_bEB_a) = \frac{\alpha(1-\rho)}{1-\rho\alpha^2} \qquad p_2(B_bES_a) = \frac{-\alpha(1-\rho)}{1-\rho\alpha^2}$$

$$p_2(B_bEB_a) = p_2(B_bCB_a) = \frac{\alpha(1+\rho)}{1+\rho\alpha^2} \qquad p_2(S_bES_a) = p_2(S_bCS_a) = \frac{-\alpha(1+\rho)}{1+\rho\alpha^2}$$

$$p_2(S_bCB_a) = \frac{\sigma_c\alpha + (1-\sigma)\beta 0.5\alpha(1-\rho)}{\sigma_c + (1-\sigma)\beta 0.5(1-\rho\alpha^2)} \quad p_2(B_bCS_a) = -\frac{\sigma_c\alpha + (1-\sigma)\beta 0.5\alpha(1-\rho)}{\sigma_c + (1-\sigma)\beta 0.5(1-\rho\alpha^2)}.$$

*Exchanging indices $a$ and $b$ gives prices for asset $b$.*

*Proof.* The derivation of the market maker's within-market pricing strategies is the same as in Lemma 1.

Any cross-market order history $h$ was placed either by one long-term cross-market trader or by two short-term traders, so we can use (9) to derive prices for cross-market histories. Throughout the proof, we suppress the index $a$ on $\sigma_{wa}$, $\sigma_{ca}$, and $\sigma_a = \sigma_{wa} + \sigma_{ca}$.

If the market maker observes history $S_bCB_a$, it could have come from either a cross-market long-term trader with probability $\sigma_c/2$ or two short-term traders with probability $(1-\sigma)\beta/2$. If the history came from a cross-market long-term trader, then his ultimate intention is to buy asset $a$, in which case there are two subcases: (1), with probability $\alpha 0.5$, the trader is informed that asset $a$ is good, in which case the value of asset $a$ is 1; (2) with probability $(1-\alpha)0.5$, the trader is uninformed and wants to purchase only for liquidity reasons, in which case the expected asset value is 0. If the history came from two short-term traders, then there are four subcases: (1) with probability $\alpha^2 0.5(1-\phi)$, the first trader is informed that asset $b$ is bad and the second trader is informed that asset $a$ is good, in which case asset $a$'s expected value is 1; (2) with probability $\alpha(1-\alpha)0.5^2$, the first trader is informed that asset $b$ is bad, and the second trader is uninformed, in which case the expected asset value is $\mathbb{E}[V_a|V_b = -1] = 1 \cdot P(V_a = 1|V_b = -1) + (-1) \cdot P(V_a = -1|V_b = -1) = 1 \cdot (1-\phi) + (-1) \cdot \phi = 1 - 2\phi = -\rho$; (3) with probability $\alpha(1-\alpha)0.5^2$, the first trader is an uninformed seller, and the second trader is informed that asset $a$ is good, in which case the expected value of asset $a$ is 1; (4) with probability $(1-\alpha)^2 0.5^2$, both traders are uninformed, in which case the expected value of asset $a$ is 0.

Applying Equation (9), we have

$$E[V_a|S_bCB_a] = \frac{\sigma_c[\alpha 0.5(1) + (1-\alpha)0.5(0)] + (1-\sigma)\beta[\alpha^2 0.5(1-\phi)(1) + \alpha(1-\alpha)0.5^2(1-\rho) + (1-\alpha)^2 0.5^2(0)]}{\sigma_c[\alpha 0.5 + (1-\alpha)0.5] + (1-\sigma)\beta[\alpha^2 0.5(1-\phi) + 2\alpha(1-\alpha)0.5^2 + (1-\alpha)^2 0.5^2]}$$
$$= \frac{\sigma_c \alpha + (1-\sigma)\beta 0.5\alpha(1-\rho)}{\sigma_c + (1-\sigma)\beta 0.5(1-\rho\alpha^2)}.$$

□

By symmetry, $\mathbb{E}[V_a|B_bCS_a] = -\mathbb{E}[V_a|S_bCB_a]$.

If the market maker observes order history $S_bEB_a$, it could only have come from two short-term traders. In that case, the four possible events and conditional expectations are the same as in the previous paragraph, implying that

$$\mathbb{E}[V_a|B_bCS_a] = \frac{(1-\sigma)\beta[\alpha^2 0.5(1-\phi)(1) + \alpha(1-\alpha)0.5^2[(1-2\phi)+1] + (1-\alpha)^2 0.5^2(0)]}{(1-\sigma)\beta[\alpha^2 0.5(1-\phi) + 2\alpha(1-\alpha)0.5^2 + (1-\alpha)^2 0.5^2]}$$
$$= \frac{\alpha(1-\rho)}{1-\rho\alpha^2}.$$

If the market maker observes order history $B_bCB_a$, it could only have come from two short-term traders, which occurs with probability $(1-\sigma)\beta$. In that case, there are four possible events: (1) with probability, $\alpha^2 0.5\phi$, both traders are informed that both assets are good, in which case the expected value of asset $a$ is 1; (2) with probability $\alpha(1-\alpha)0.5^2$, the first buyer is informed that asset $b$ is good and the second trader is uninformed, in which case the expected value of asset $a$ is $\mathbb{E}[V_a|V_b = 1] = 1 \cdot \phi + (-1) \cdot (1-\phi) = 2\phi - 1 = \rho$; (3) with probability $\alpha(1-\alpha)0.5^2$, the first buyer is uninformed and the second buyer is informed that asset $a$ is good, in which case the expected value of asset $a$ is 1; (4) with probability $(1-\alpha)^2 0.5^2$, both traders are uninformed, in which case the expected value of asset $a$ is 0. Appying Equation (9) gives

$$\mathbb{E}[V_a|B_bCB_a] = \frac{(1-\sigma)\beta}{(1-\sigma)\beta} \cdot \frac{\alpha^2 0.5\phi(1) + \alpha(1-\alpha)0.5^2(1+\rho) + (1-\alpha)^2 0.5^2(0)}{\alpha^2 0.5\phi + 2\alpha(1-\alpha)0.5^2 + (1-\alpha)^2 0.5^2}$$
$$= \frac{\alpha(1+\rho)}{1+\rho\alpha^2}.$$

The price for $B_bEB_a$ is the same. Symmetry gives the remaining prices, and exchanging indices for $a$ and $b$ gives prices for asset $b$. □

We now show that in the proposed equilibrium, a cross-market long-term trader finds it optimal to spoof across markets. Without loss of generality, suppose a cross-market long-term trader wants to ultimately buy asset $a$, either because he knows it is good or for liquidity reasons. If he buys it directly, either at date 1 or date 2, he pays price $p_1(B_a) = p_2(NB_a) = \alpha$. If he spoofs across markets, by first placing a sell order for asset $b$ and then cancelling it, he pays price

$$p_2(S_bCB_a) = \alpha\frac{\sigma_c + (1-\sigma)\beta 0.5(1-\rho)}{\sigma_c + (1-\sigma)\beta 0.5(1-\rho\alpha^2)} < \alpha,$$

so it is strictly optimal for him to spoof across markets. □

**Proof of Proposition 7**. Without loss of generality, we establish the result for asset $a$. Let $H$ be the set of histories that are possible under the spoofing equilibrium, and $\hat{H} = H \cup \{NB_a, NS_a, NB_b, NS_b\}$ be the set of histories that are possible under the benchmark equilibrium. Given some history $h \in \hat{H}$, Bayes' rule gives the market maker's posterior belief that asset $a$ has value $v_a = 1$ :

$$\pi_a(h) = \frac{P(h|v_a = 1)\pi}{P(h|v_a = 1)\pi + P(h|v_a = -1)(1-\pi)}.$$

Let $H^- \equiv \{S_bCB_a, S_aCB_b, B_bCS_a, B_aCS_b\}$ denote the set of histories in the spoofing equilibrium which could have come from a cross-market spoofer and let $\hat{H}^- \equiv H^- \cup \{NB_a, NB_b, NS_a, NS_b\}$ denote the corresponding set of histories in the benchmark equilibrium. By construction of the benchmark, all histories which could *not* have come from a cross-market spoofer in the spoofing equilibrium are just as likely to occur in the benchmark equilibrium. That is, using ˆ to denote benchmark probabilities $\hat{P}$ and posteriors $\hat{\pi}$, for all $h \in H \setminus H^-$ and $v \in \{-1,1\}^2$, we have that $P(h|v) = \hat{P}(h|v)$ and therefore $\pi(h) = \hat{\pi}(h)$. As a result,

$$\begin{aligned}&\hat{\mathbb{E}}[\hat{\pi}(h)|v] - \mathbb{E}[\pi(h)|v] \\ =&\sum_{h\in\hat{H}} \hat{\pi}(h)\hat{P}(h|v) - \sum_{h\in H}\pi(h)P(h|v) = \sum_{h\in\hat{H}^-}\hat{\pi}(h)\hat{P}(h|v) - \sum_{h\in H^-}\pi(h)P(h|v). \qquad (24)\end{aligned}$$

Letting $Y \equiv \{B, S\} \times \{a, b\}$ and $f(x) \equiv \pi/(\pi + x(1-\pi))$, so that $\pi(h) = f(P(h|v_a = -1)/P(h|v_a = 1))$, we can rewrite (24) as follows:

$$\begin{aligned}
&\hat{\mathbb{E}}[\hat{\pi}_a(h)|v_a = 1] - \mathbb{E}[\pi_a(h)|v_a = 1] \\
&= \sum_{(o,i)\in Y} f\left(\frac{\hat{P}(No_i|v_a = -1)}{\hat{P}(No_i|v_a = 1)}\right)\hat{P}(No_i|v_a = 1) \\
&\quad + \sum_{(o,i)\in Y} f\left(\frac{\hat{P}((\neg o)_{\neg i}Co_i|v_a = -1)}{\hat{P}((\neg o)_{\neg i}Co_i|v_a = 1)}\right)\hat{P}((\neg o)_{\neg i}Co_i|v_a = 1) \\
&\quad - \sum_{(o,i)\in Y} f\left(\frac{P((\neg o)_{\neg i}Co_i|v_a = -1)}{P((\neg o)_{\neg i}Co_i|v_a = 1)}\right)P((\neg o)_{\neg i}Co_i|v_a = 1)
\end{aligned}$$

$$\begin{aligned}
&= \sum_{(o,i)\in Y} \frac{1}{P((\neg o_{\neg i}Co_i|v_a = 1))}\Bigg[f\left(\frac{\hat{P}(No_i|v_a = -1)}{\hat{P}(No_i|v_a = 1)}\right)\frac{\hat{P}(No_i|v_a = 1)}{P((\neg o_{\neg i}Co_i|v_a = 1))} \\
&\qquad + f\left(\frac{\hat{P}((\neg o)_{\neg i}Co_i|v_a = -1)}{\hat{P}((\neg o)_{\neg i}Co_i|v_a = 1)}\right)\frac{\hat{P}((\neg o)_{\neg i}Co_i|v_a = 1)}{P((\neg o_{\neg i})Co_i|v_a = 1)} \\
&\qquad - f\left(\frac{P((\neg o)_{\neg i}Co_i|v_a = -1)}{P((\neg o)_{\neg i}Co_i|v_a = 1)}\right)\Bigg] \qquad (25)
\end{aligned}$$

By construction of the benchmark, for all $(o, i, v_a) \in Y \times \{1, -1\}$,

$$\begin{aligned}
P((\neg o)_{\neg i}Co_i|v_a) &= P((\neg o)_{\neg i}Co_i|sp, v_a) + P((\neg o)_{\neg i}Co_i|\neg sp, v_a) \\
&= \hat{P}(No_i|v_a) + \hat{P}((\neg o)_{\neg i}Co_i|v_a),
\end{aligned}$$

which implies

$$\begin{aligned}
&\frac{\hat{P}(No_i|v_a = -1)}{\hat{P}(No_i|v_a = 1)}\frac{\hat{P}(No_i|v_a = 1)}{P((\neg o_{\neg i}Co_i|v_a = 1))} + \frac{\hat{P}((\neg o)_{\neg i}Co_i|v_a = -1)}{\hat{P}((\neg o)_{\neg i}Co_i|v_a = 1)}\frac{\hat{P}((\neg o)_{\neg i}Co_i|v_a = 1)}{P((\neg o_{\neg i})Co_i|v_a = 1)} \\
&= \frac{\hat{P}(No_i|v_a = -1) + \hat{P}((\neg o)_{\neg i}Co_i|v_a = -1)}{P((\neg o_{\neg i})Co_i|v_a = 1)} = \frac{P((\neg o_{\neg i})Co_i|v_a = -1)}{P((\neg o_{\neg i})Co_i|v_a = 1)}.
\end{aligned}$$

So by the convexity of $f(\cdot)$ and Jensen's inequality, the bracketed differences in (25) are strictly positive, which establishes $\hat{\mathbb{E}}[\hat{\pi}_a(h)|v_a = 1] > \mathbb{E}[\pi_a(h)|v_a = 1]$. Because prices

$p_a$ and $\hat{p}_a$ are linearly increasing in posteriors $\pi_a(h)$ and $\hat{\pi}_a(h)$, respectively, this implies $\mathbb{E}[p_a|v_a = 1] < \hat{\mathbb{E}}[\hat{p}_a|v_a = 1] \leq 1 = v_a$. $\square$

**Proof of Proposition 8.** The proof is identical to Proposition 3, part (ii).

**Proof of Proposition 9.** Without loss of generality, we establish the proposition for asset $a$, and drop the index $a$ throughout the proof. Lemmas 1 and 5 imply that

$$C_w(\sigma_w) = \mathbb{E}[V_a|S_aCB_a] + k_w = \frac{\sigma_w\alpha}{\sigma_w + (1-\bar{\sigma})\beta 0.5(1-\alpha^2)} + k_w$$
$$C_c(\sigma_c) = \mathbb{E}[V_a|S_bCB_a] + k_c = \frac{\sigma_c\alpha + (1-\bar{\sigma})\beta 0.5\alpha(1-\rho)}{\sigma_c + (1-\bar{\sigma})\beta 0.5(1-\rho\alpha^2)} + k_c.$$

As $\sigma_w$ increases from 0 to $\bar{\sigma}$, $C_w(\sigma_w)$ strictly increases from 0 to $C(\bar{\sigma})$. As $\sigma_c$ increases from 0 to $\bar{\sigma}$, $C_c(\sigma_c)$ strictly increases from $C_c(0) > 0$ to $C_c(\bar{\sigma}) > C_w(\bar{\sigma})$.

Let $\hat{\sigma}_w$ be the unique $\sigma_w \in \mathbb{R}$ that solves $C_w(\sigma_w) = \alpha$, and let $\hat{\sigma}_c$ be the unique $\sigma_c \in \mathbb{R}$ that solves $C_c(\sigma_c) = \alpha$. Then

$$\hat{\sigma}_w = (1-\bar{\sigma})\frac{\beta}{2}\left(\frac{\alpha}{k_w} - 1\right)(1-\alpha^2)$$
$$\hat{\sigma}_c = (1-\bar{\sigma})\frac{\beta}{2}\left[\left(\rho\frac{\alpha}{k_c} - 1\right)(1-\alpha^2) - (1-\rho)\alpha^2\right].$$

By assumption, $\sigma_w^* + \sigma_c^* \leq \bar{\sigma}$. We first show that for high enough $k_w$ and $k_c$, the inequality is strict. So suppose that $\sigma_w^* + \sigma_c^* = \bar{\sigma}$. Then by the equilibrium definition, $\alpha \geq \min\{C_w(\sigma_w), C_c(\sigma_c)\}$, which implies that $\sigma_w \leq \hat{\sigma}_w$ or $\sigma_c^* \leq \hat{\sigma}_c$. But for high enough $k_w$ and $k_c$, $\hat{\sigma}_w < 0$ and $\hat{\sigma}_c < 0$, contradicting $\sigma_w^* \geq 0$ and $\sigma_c^* \geq 0$. This establishes that $\sigma_w^* + \sigma_c^* < \bar{\sigma}$, ruling out case (3) in the equilibrium definition, so it must be that $\alpha \leq \min\{C_w(\sigma_w^*), C_c(\sigma_c^*)\}$. It also implies that $\sigma_w^* < \bar{\sigma}$ and $\sigma_c^* < \bar{\sigma}$.

The proposition claims that $\sigma_w^* = \max\{0, \hat{\sigma}_w\}$. For each case below, we first show that $\sigma_w^* = \max\{0, \hat{\sigma}_w\}$ is compatible with the equilibrium definition, and then show that it is necessary.

Case 1: $\hat{\sigma}_w \leq 0$. For this case, if $\sigma_w^* = 0$, then because $C_w(\cdot)$ is strictly decreasing, $C_w(\sigma_w^*) \geq C_w(\hat{\sigma}_w) = \alpha$, which satisfies (5), so $\sigma_w^* = 0$ is compatible with equilibrium. If $\sigma_w^* > 0 \geq \hat{\sigma}_w$, then $C_w(\sigma_w^*) > C_w(\hat{\sigma}_w) = \alpha$, so then (5) implies $\sigma_w^* = 0$, a contradiction. Therefore, it must be that $\sigma_w^* = 0$ whenever $\hat{\sigma}_w \leq 0$.

Case 2: $\hat{\sigma}_w > 0$. If $\sigma_w^* = \hat{\sigma}_w > 0$, then $C_w(\sigma_w^*) = C_w(\hat{\sigma}_w) = \alpha$, which satisfies (5), so $\sigma_w^* = \hat{\sigma}_w$ is compatible with equilibrium. If $\sigma_w^* < \hat{\sigma}_w$, then $C_w(\sigma_w^*) < C_w(\hat{\sigma}_w) = \alpha$, contradicting $\alpha \leq \min\{C_w(\sigma_w^*), C_c(\sigma_c^*)\}$. If $\sigma_w^* > \hat{\sigma}_w > 0$, then $C_w(\sigma_w^*) > C_w(\hat{\sigma}_w) = \alpha$, contradicting case 1 of (5). So $\sigma_w^* = 0$ whenever $\hat{\sigma}_w > 0$, establishing $\sigma_w^* = \max\{0, \hat{\sigma}_w\}$.

Identical reasoning shows that $\sigma_c^* = \max\{0, \hat{\sigma}_c\}$ is compatible with the equilibrium definition, and also necessary. Therefore, $\sigma_w^* = \max\{0, \hat{\sigma}_w\}$ and $\sigma_c^* = \max\{0, \hat{\sigma}_c\}$ is the unique equilibrium for high enough $k_w$ and $k_c$. This completes the proof. □

**Proof of Proposition 10**. By assumption, the regulator has sufficient resources $\kappa$ such that Proposition 9 holds. Let $\hat{k}_w \equiv \min\{k_w : \sigma_w^* = 0\}$ and $\hat{k}_c \equiv \min\{k_c : \sigma_c^* = 0\}$ be the minimum resources a regulator must allocate to within- and cross-market monitoring, respectively, in order to completely eliminate them. Therefore, the assumption that the regulator is constrained amounts to $k < \hat{k}_w + \hat{k}_c$. Proposition 9 gives:

$$\hat{k}_w = \alpha \qquad \text{and} \qquad \hat{k}_c = \frac{\rho\alpha(1-\alpha^2)}{1-\rho\alpha^2}.$$

Given total resources $k$, the regulator selects $k_c$ and $k_w$ to minimize $\sigma_c^* + \sigma_w^*$, subject to the constraint that $k_c + k_w \leq k$. Proposition 9 gives piecewise expressions for $\sigma_c^*$ and $\sigma_w^*$, so we first simplify these expressions by showing that $k_c > \hat{k}_c$ or $k_w > \hat{k}_w$ are never optimal.

Observe that for $i \in \{c, w\}$, $\sigma_i^*$ is positive and strictly decreasing in $k_i$ for $k_i < \hat{k}_i$ and zero otherwise. Without loss of generality, suppose that $k_c > \hat{k}_c$, so that $\sigma_c^* = 0$. Then because $k_c + k_w \leq k < \hat{k}_c + \hat{k}_w$, we must have $k_w < \hat{k}_c + \hat{k}_w - k_c < \hat{k}_w$, so $\sigma_w^* > 0$. If the regulator reduced $k_c$ by small $\Delta$ and raised $k_w$ by $\Delta$, then $\sigma_c^*$ would remain unchanged at zero, but $\sigma_w^*$ would strictly decrease. This improves the regulator's payoff, so $k_c > \hat{k}_c$ cannot be optimal. A similar argument shows that $k_w > \hat{k}_w$ cannot be optimal.

Moving forward, we restrict attention to the region $k_c \leq \hat{k}_c$ and $k_w \leq \hat{k}_w$, which by Proposition 9 implies that the total measure of spoofers $\sigma^*$ is

$$\begin{aligned}\sigma_w^* + \sigma_c^* &= (1-\bar{\sigma}_w)\frac{\beta}{2}\left(\frac{\alpha}{k_w} - 1\right)(1-\alpha^2) + (1-\bar{\sigma}_w)\frac{\beta}{2}\left[\left(\rho\frac{\alpha}{k_c} - 1\right)(1-\alpha^2) - (1-\rho)\alpha^2\right] \\ &= (1-\bar{\sigma}_w)\frac{\beta}{2}\left[\left(\alpha\left(\frac{1}{k_w} + \rho\frac{1}{k_c}\right) - 2\right)(1-\alpha^2) - (1-\rho)\alpha^2\right].\end{aligned}$$

The total measure of spoofers $\sigma_w^* + \sigma_c^*$ depends on $k_w$ and $k_c$ only through the term $k_w^{-1} + \rho k_c^{-1}$, so the regulator's problem reduces to

$$\max_{k_w \in [0,\hat{k}_w], k_c \in [0,\hat{k}_c]} -(k_w^{-1} + \rho k_c^{-1})$$
$$k_w + k_c \leq k. \tag{26}$$

The derivatives with respect to the Lagrangian are

$$\frac{\partial \mathscr{L}}{\partial k_w} = k_w^{-2} - \lambda \qquad \text{and} \qquad \frac{\partial \mathscr{L}}{\partial k_c} = \rho k_c^{-2} - \lambda,$$

where $\lambda$ is the Lagrange multiplier on the resource constraint.

We begin by relaxing the constraints $k_w \leq \hat{k}_w$ and $k_c \leq \hat{k}_c$ while leaving the objective function as $-(k_w^{-1} + \rho k_c^{-1})$. We then give conditions such that the solution to the relaxed problem satisfies the constrained problem.

The objective function is strictly increasing in $k_w$ and $k_c$, so the resource constraint binds. The objective function equals $-\infty$ if $k_w$ or $k_c$ are equal to zero, so both $k_w$ and $k_c$ must be strictly positive, and therefore $\partial\mathscr{L}/\partial k_w = \partial\mathscr{L}/\partial k_c = 0$. This implies $k_w^{-2} = \lambda = \rho k_c^{-2}$, and therefore $k_c/k_w = \sqrt{\rho}$. Combining this with the binding resource constraint gives the unique solution $(\tilde{k}_c, \tilde{k}_w)$ to the relaxed problem:

$$\tilde{k}_w = \frac{k}{1+\sqrt{\rho}} \qquad \text{and} \qquad \tilde{k}_c = \frac{k\sqrt{\rho}}{1+\sqrt{\rho}}.$$

We claim that $(\tilde{k}_c, \tilde{k}_w)$ solves the constrained problem (26) if and only if $\tilde{k}_c \leq \hat{k}_c$. First, if $\tilde{k}_c > \hat{k}_c$, then $\tilde{k}_c$ falls outside the constraint set and therefore $(\tilde{k}_c, \tilde{k}_w)$ cannot solve the constrained problem. Second, suppose that $\tilde{k}_c \leq \hat{k}_c$ and observe that

$$\frac{\hat{k}_c}{\hat{k}_w} = \frac{\rho(1-\alpha^2)}{1-\rho\alpha^2} < \rho < \sqrt{\rho} = \frac{\tilde{k}_c}{\tilde{k}_w}.$$

Therefore, $\tilde{k}_c \leq \hat{k}_c$ implies $\tilde{k}_w \leq \hat{k}_w \cdot \tilde{k}_c/\hat{k}_c < \hat{k}_w$. Thus, $(\tilde{k}_c, \tilde{k}_w)$ satisfy the relaxed constraints and therefore solves (26). That is, if $\tilde{k}_c \leq \hat{k}_c$, then $(k_c^*, k_w^*) = (\tilde{k}_c, \tilde{k}_w)$. Computing the ratio of $k_c^*$ to $k_w^*$, we see that $k_c^*/k_w^* = \tilde{k}_c/\tilde{k}_w = \sqrt{\rho}$, establishing part (i) of the proposition.

Finally, we show that if $\tilde{k}_c > \hat{k}_c$, then $k_c^* = \hat{k}_c$. If $\tilde{k}_c > \hat{k}_c$, then by the previous paragraph, the unconstrained solution $(\tilde{k}_c, \tilde{k}_w)$ cannot be optimal. So suppose to the contrary that

$k_c^* < \hat{k}_c$. Then $k_w^* = \hat{k}_w$, as otherwise $(k_c^*, k_w^*)$ is interior and solves the relaxed problem, contradicting the suboptimality of $(\tilde{k}_c, \tilde{k}_w)$. If $k_c^* < \hat{k}_c$ and $k_w^* = \hat{k}_w$, then $\partial \mathscr{L}/\partial k_c = 0$ while $\partial \mathscr{L}/\partial k_w \leq 0$, implying $\rho(k_c^*)^{-2} = \lambda \geq (k_w^*)^{-2}$, which we can rewrite as $k_c^*/k_w^* \geq \sqrt{\rho} > \hat{k}_c/\hat{k}_w = \hat{k}_c/k_w^*$. But this implies $k_c^* > \hat{k}_c$, a contradiction. So if $\tilde{k}_c > \hat{k}_c$, then $k_c^* = \hat{k}_c$, and the binding resource constraint gives $k_w^* = k - k_c^* = k - \hat{k}_c$, establishing part (ii) of the proposition. □

# References


Aggarwal, Rajesh K and Guojun Wu (2006). "Stock market manipulations". *The Journal of Business* 79.4, pp. 1915–1953.

Allen, Franklin and Douglas Gale (1992). "Stock-price manipulation". *The Review of Financial Studies* 5.3, pp. 503–529.

Allen, Franklin and Gary Gorton (1992). "Stock price manipulation, market microstructure and asymmetric information". *European Economic Review* 36, p. 624.

Bagnoli, Mark and Barton L Lipman (1996). "Stock price manipulation through takeover bids". *The RAND Journal of Economics*, pp. 124–147.

Benabou, Roland and Guy Laroque (1992). "Using privileged information to manipulate markets: Insiders, gurus, and credibility". *The Quarterly Journal of Economics* 107.3, pp. 921–958.

Cartea, Álvaro, Sebastian Jaimungal, and Yixuan Wang (2020). "Spoofing and Price Manipulation in Order-Driven Markets". *Applied Mathematical Finance*, pp. 1–32.

Chakraborty, Archishman and Bilge Yılmaz (2004a). "Informed manipulation". *Journal of Economic Theory* 114.1, pp. 132–152.

Chakraborty, Archishman and Bilge Yılmaz (2004b). "Manipulation in market order models". *Journal of Financial Markets* 7.2, pp. 187–206.

Chakravarty, Sugato and Craig W Holden (1995). "An integrated model of market and limit orders". *Journal of Financial Intermediation* 4.3, pp. 213–241.

Coscia v. United States (2018). *Brief of respondent United States in opposition filed*. (no. 17-1099) Retrieved from https://www.scotusblog.com/case-files/cases/coscia-v-united-states/.

Dalko, Viktoria and Michael H Wang (2019). "High-frequency trading: Order-based innovation or manipulation?" *Journal of Banking Regulation*, pp. 1–10.

Fischel, Daniel R and David J Ross (1991). "Should the Law Prohibit" Manipulation" in Financial Markets?" *Harvard Law Review* 105.2, pp. 503–553.

Foucault, Thierry (1999). "Order flow composition and trading costs in a dynamic limit order market". *Journal of Financial markets* 2.2, pp. 99–134.

Foucault, Thierry, Ohad Kadan, and Eugene Kandel (2005). "Limit order book as a market for liquidity". *The review of financial studies* 18.4, pp. 1171–1217.

Glosten, Lawrence R (1994). "Is the electronic open limit order book inevitable?" *The Journal of Finance* 49.4, pp. 1127–1161.

Glosten, Lawrence R and Paul R Milgrom (1985). "Bid, ask and transaction prices in a specialist market with heterogeneously informed traders". *Journal of Financial Economics* 14.1, pp. 71–100.

Goettler, Ronald L, Christine A Parlour, and Uday Rajan (2005). "Equilibrium in a dynamic limit order market". *The Journal of Finance* 60.5, pp. 2149–2192.

Goettler, Ronald L, Christine A Parlour, and Uday Rajan (2009). "Informed traders and limit order markets". *Journal of Financial Economics* 93.1, pp. 67–87.

Hart, Oliver D (1977). "On the profitability of speculation". *The Quarterly Journal of Economics*, pp. 579–597.

Jarrow, Robert A (1992). "Market manipulation, bubbles, corners, and short squeezes". *Journal of Financial and Quantitative Analysis*, pp. 311–336.

Kaniel, Ron and Hong Liu (2006). "So what orders do informed traders use?" *The Journal of Business* 79.4, pp. 1867–1913.

Lee, Eun Jung, Kyong Shik Eom, and Kyung Suh Park (2013). "Microstructure-based manipulation: Strategic behavior and performance of spoofing traders". *Journal of Financial Markets* 16.2, pp. 227–252.

Martínez-Miranda, Enrique, Peter McBurney, and Matthew JW Howard (2016). "Learning unfair trading: A market manipulation analysis from the reinforcement learning perspective". In: *2016 IEEE Conference on Evolving and Adaptive Intelligent Systems (EAIS)*. IEEE, pp. 103–109.

O'Hara, Maureen (2003). "Presidential address: Liquidity and price discovery". *The Journal of Finance* 58.4, pp. 1335–1354.

Parlour, Christine A (1998). “Price dynamics in limit order markets”. *The Review of Financial Studies* 11.4, pp. 789–816.

Parlour, Christine A and Duane J Seppi (2008). “Limit order markets: A survey”. *Handbook of financial intermediation and banking* 5, pp. 63–95.

Putniņš, Tālis J (2012). “Market manipulation: A survey”. *Journal of Economic Surveys* 26.5, pp. 952–967.

Rock, Kevin (1996). “The specialist’s order book and price anomalies”. *Review of Financial Studies* 9, pp. 1–20.

Roşu, Ioanid (2009). “A dynamic model of the limit order book”. *The Review of Financial Studies* 22.11, pp. 4601–4641.

Seppi, Duane J (1997). “Liquidity provision with limit orders and a strategic specialist”. *The Review of Financial Studies* 10.1, pp. 103–150.

Seppi, Duane J and Praveen Kumar (1994). “Limit and market orders with optimizing traders”. *Available at SSRN 5509*.

Van Bommel, Jos (2003). “Rumors”. *The Journal of Finance* 58.4, pp. 1499–1520.

Vila, Jean-Luc (1989). “Simple games of market manipulation”. *Economics Letters* 29.1, pp. 21–26.

Wang, Xintong, Chris Hoang, and Michael P Wellman (2019). “Learning-Based Trading Strategies in the Face of Market Manipulation”. In: *ICML-19 Workshop on AI in Finance*.

Wang, Xintong and Michael P Wellman (2019). “Spoofing the limit order book: An agent-based model.” In: *16th International Conference on Autonomous Agents and Multiagent Systems*, pp. 651–659.

Wang, Yun-Yi (2019). “Strategic Spoofing Order Trading by Different Types of Investors in Taiwan Index Futures Market”. *Journal of Financial Studies* 27.1, pp. 65–103.

Withanawasam, Rasika, Peter Whigham, and Timothy Falcon Crack (2018). “Are Liquid or Illiquid Stocks More Easily Manipulated? The Impact of Manipulator Aggressiveness”. Available at SSRN: https://ssrn.com/abstract=3643650.