Nonparametric Identification of First-Price Auction with Unobserved Competition: A Density Discontinuity Framework

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Abstract

We consider nonparametric identification of independent private value first-price auction models, in which the analyst only observes winning bids. Our benchmark model assumes an exogenous number of bidders N. We show that, if the bidders observe N, the resulting discontinuities in the winning bid density can be used to identify the distribution of N. The private value distribution can be nonparametrically identified in a second step. This extends, under testable identification conditions, to the case where N is a number of potential buyers, who bid with some unknown probability. Identification also holds in presence of additive unobserved heterogeneity drawn from some parametric distributions. A last class of extensions deals with cartels which can change size across auctions due to varying bidder cartel membership. Identification still holds if the econometrician observes winner identities and winning bids, provided a (unknown) bidder is always a cartel member. The cartel participation probabilities of other bidders can also be identified. An application to USFS timber auction data illustrates the usefulness of discontinuities to analyze bidder participation.

Keywords: Auction models, unobserved competition, nonparametric identification, density discontinuities, bidder uncertainty, unobserved heterogeneity, collusion.

JEL classification: C14, C57, D44

1 Introduction

1.1 Motivation

There exists a large literature on nonparametric identification of auction models; see, e.g., Athey and Haile (2007) or Hendricks and Porter (2007) for a review. In the case of sealed-bid first-price auctions, a vast majority of work assumes that the analyst can observe all of the bids, or both the winning bid and the number of competitors. This may not always be observed. In French timber auctions, for example, only the winning bid may be available to researchers to preserve bidder anonymity (Lamy, 2012). Indeed, it is common practice in many markets that can be treated as auctions for only the winning bid (i.e. the transaction price) to be recorded. For instance, a company soliciting price quotes for a task to be completed is implicitly organizing a first-price auction. While the company may not record all quotes or the number of responses, the price charged to the winning bidder is likely to appear in accounting records. "Bidding wars" are becoming commonplace in housing markets, where houses are sold through a competitive bidding process resembling an informal first-price auction, as noted in Han and Strange (2015). Governments may offer subsidies to attract firms, as recently considered by Kim (2020) and Slattery (2020) using an auction framework. Observing all the subsidy offers may be difficult, because states or firms may both have some interest in confidentiality. Hence, in many economic situations of interest, the records may only contain the final winning bid. Therefore, the ability to identify auction primitives solely from winning bid data may enlarge the scope of auction theory applications.

A second motivation stems from misspecification considerations. Indeed, structural estimation procedures of auctions crucially depend on the number of active buyers, which may differ from the observed number. For instance, Laffont, Ossard and Vuong (1995) consider an application where bidders are agents of several retail sellers, in which case the number of bids underestimates the level of competition. Given 11 observed bids, they authors estimate the number of active buyers to be 18, causing important changes in the estimated structural parameters. Alternatively, some buyers may enter an auction simply to gain information, in which case their bids would be dominated and never impact the winning bid. Some bidders may collude and place phantom bids in an attempt to hide their cartel memberships; counting these bids would be misleading as it does not account for collusion. For instance, Imhof, Karagök and Rutz (2018) mention a setup where sellers are legally constrained to stop auctions with less than three bidders in attendance. In this case, two buyers may be tempted to contact a third to submit a cover bid but ultimately allow the auction to go through. Econometric methods that check the effectiveness of bidder attendance can be a useful tool for regulators before proceeding to further possibly costly investigations. More generally, using only winning bids provides a robust approach for identifying primitives of interest when the truthfulness of observed bids is dubious.

Last, participation is a parameter of interest in itself. As noted by Bulow and Klemperer (1996), increasing competitive participation would yield higher seller expected revenues than choosing an optimal reserve price under the symmetric independent private value paradigm. A lack of competitiveness, as induced by collusion, decreases the seller's revenue. In our setup, the number of active buyers is viewed as a latent random variable which can vary across auctions. Estimating its distribution and comparing it with the observed number of bids may be useful to detect participation anomalies. For instance, nearly one in three three bidders does not bid competitively in our USFS timber auction application. In the absence of further information, whether this is due to inexperienced buyers making dominated bids or bidder collusion can be analyzed, among others, by testing the fit of collusive or competitive structural auction models, as attempted in Aryal and Gabrielli (2013), or the effectiveness of bidder training programmes, as in De Silva, Li and Zhao (2019).

1.2 Overview of the results

Baseline model. We develop a new approach for identification of first-price auction models that exploits discontinuities in the density function of the winning bid. First, we identify the distribution of unknown competition. In particular, we build on an important restriction that first-price auction models impose on the data: the bid quantile function must be strictly increasing with respect to the number of bidders. Therefore, the upper boundary of the bid distribution, conditioning on the number of bidders, is strictly increasing, as well. We show that this creates discontinuities, or jumps, in the winning bid density function at these upper boundaries. A novel result of the paper is that these jumps identify the distribution of the number of bidders.

Second, we identify the value distribution function by iteratively exploiting two equilibrium mappings that relate the value and bid quantile functions. Based on the location of the discontinuities in the winning bid density function, we create a sequence of expanding quantile intervals over which the private value quantile is identified. For every iteration, we start by identifying the bid quantile function in the most competitive auction, which has the largest number of bidders. This information can then be used to identify the value quantile function in the same quantile interval and further calculate the corresponding bid quantile for other competition levels.

Buyer uncertainty and unobserved hetoregenity. The paper considers several extensions of the baseline model. Section 3 focuses on buyer uncertainty and auction heterogeneity. As the econometrician in the baseline model, buyers may also face unknown competition, as considered in Harstad, Kagel and Levin (1990) and Kong

(2020), among others. Such uncertainty may arise due to the presence of a reserve price, as considered in Guerre, Perrigne and Vuong (2000), or entry costs, as in Marmer, Shneyerov and Xu (2013) and Gentry and Li (2014). We consider a setup where the number of potential bidders, N, is known by the buyers but not the effective one. We allow N to vary across auctions, and give conditions on its support ensuring identification of its distribution together with the private value one and the probability that a potential buyer bids.

First-price auctions with unobserved heterogeneity affecting the auctioned good are considered in Krasnokutskaya (2011), and is especially challenging in our framework. Indeed, the presence of a continuous unobserved heterogeneity component washes discontinuities out of the winning bid density. Fortunately, considering its first and second derivatives allows for the identification of the participation distribution. We also show that some features of the unobserved component distribution can be recovered, so that it can be parametrically identified, raising hope for nonparametric identification of the private value distribution. An application of the homogenized bid approach of Haile, Hong and Shum (2003) for partially observed auction heterogeneity is also discussed.

Varying cartel size and discontinuities. Section 4 illustrates how varying asymmetry can create winning bid density discontinuities. We consider auctions attended by a fixed number of competitive buyers and a cartel of randomly varying size across auctions, as in Asker (2010). The buyer cartel membership can vary across auctions and the probability that a given buyer colludes is a parameter of interest for regulation authorities. Identification of these parameters, together with the private value distribution, which is essential to evaluate collusion losses, is obtained when observing the winning bid and winner identity, provided that a buyer, unknown to the econometrician, belongs to the cartel in all auctions.

Application to participation analysis. As discussed in the final remarks of Section 6, estimation of all considered specifications is out of the scope of the present paper. Chu and Cheng (1996), among other statisticians, have developed nonparametric methods for the detection of several discontinuities and estimation of jump locations and intensities. These nonparametric approaches satisfy the simplicity requirement advocated by Imhof et al. (2018) in a collusion detection framework. A simulation experiment detailed in Appendix A shows that estimation of the support of the number of bidders contributing to the winning bid works reasonably well even in small samples. When applied in Section 5 to the Lu and Perrigne (2008) Timber first-price auctions sample with three bids, an unexpected discontinuity in the winning bid distribution and lower tail analysis leads to conclude that it could not be ruled out that only two buyers are active in some auctions, in a proportion which is estimated to be high. Estimating the support of the number of active bidders seems to work reasonably well in small samples, as requested in Imhof et al. (2018) for collusion detection empirical tools. The application also aims to illustrate that statistical and econometric extensions of our approach can help in that respect.

Remainder of the paper. The next paragraph discusses various streams of the econometric literature potentially related with our approach. Potential estimation methods of interest are discussed in the Final remarks Section 6, and proofs are gathered in Section 7, which is complemented by the online Appendix B devoted to differentiability at the lowest bid of the inverse Bayesian Nash Equilibrium strategies. Appendix A describes the estimation procedures used in Section 5 and reports the results of related simulation experiments.

1.3 Related literature

Auctions with unobserved competition. Allowing for unknown competition started early in the empirical auction literature. Laffont et al. (1995) estimate the number of buyers N as a parameter that they take to be constant across auctions. Paarsch (1997) treats unknown competition as a nuisance parameter, which is eliminated using conditional likelihood estimation. For ascending eBay auctions, Song (2004) shows that the private value distribution and a constant number of buyers are identified from winning and second-highest bids, but not from winning bids alone when N is random. More pertinent to our paper is the misclassification approach of An, Hu and Shum (2010), who study identification from the winning bid using a proxy $N^* \leq N$ for the number of buyers and an instrument that can be a discretized second bid. Shneyerov and Wong (2011) suppose that only winning bids and the number of active bidders are observed. Recent work for ascending auctions include Freyberger and Larsen (2017) and Hernández, Quint and Turansick (2020).

Collusion. Studying participation can be a preliminary step for collusion detection. Abrantes-Metz and Bajari (2009) and Imhof et al. (2018) review behavorial and some structural methods to screen collusion. For instance, Imhof et al (2018) propose to analyze buyer joint participation across auctions: this may not imply collusion if it does impact the winning bid distribution as apparently detected in our application. This suggests that detecting bidder participation anomalies through winning bid p.d.f discontinuities can be a useful new tool for regulation purposes.

Chassang, Kawai, Nakabayashi and Ortner (2021) design a procedure that gives a lower bound for the proportion of bids by a given buyer, which does not satisfy some incentive-compatible and markup constraints in a very general information framework. This translates here to phantom bids and is probably less relevant than the buyer cartel membership probabilities that are identified in our more restrictive independent private value framework. The authors also use huge datasets, which may be too rich compared to most collusion cases investigated by regulation agencies. They do not attempt to back out the primitives of the model, so that their approach cannot be used to evaluate collusion cost impacts, which can be important information for a regulator in the decision to conduct further investigations.

Closer to our paper is Schurter (2020); see also Marmer, Shneyerov and Kaplan (2017) for the case of English auctions. As done here, Schurter (2020) uses winning bids and winner identity to identify model primitives. His approach differs from ours when identifying cartels with an instrument, as in Porter and Zona (1993). Schurter (2020) considers cartels with stable membership across auctions while we allow for varying cartel involvement.

Mixture distribution. The present paper contributes to the literature on nonparametric identification of finite mixtures; see for instance the review of Compiani and Kitamura (2016). Existing identification results require either exclusion restrictions or multiple independent measurements. A first-price auction example of the latter can be found in D'Haultfœuille and Février (2015), who recover the distribution of an unobserved continuous auction characteristic from three bids. Hu and Sasaki (2017) obtain identification from two measurements in a model with discrete unobserved heterogeneity. In our setting, the number of buyers N can be viewed as unobserved heterogeneity while the winning bid is a unique bid. However, identification is possible because the mixture components are the bid distribution given $N = n$, issued from the same private value distribution and constrained by an optimal bidding condition. When the buyers do not observe the number of competitors, this is restrictive enough to ensure identification in the presence of a reserve price or entry cost instrument, without the exclusion restrictions of Compiani and Kitamura (2016).

Discontinuity design. The discontinuity design (DD) literature has expanded rapidly in recent years; interested readers are encouraged to refer to review papers by Imbens and Lemieux (2008), Kleven (2016) and Jales and Yu (2017). Recent auction applications include Coviello and Marinello (2014), Choi, Neisheim and Razul (2016) and Kawai, Nakabayashi, Ortner and Chassang (2021). As in the DD literature, this paper employs jump sizes for identification purposes — more specifically, to identify the participation distribution. However, this paper departs from the DD literature by considering an unknown number of density discontinuities at unknown location.

2 The benchmark model

In this section, we start by describing the benchmark auction model and introduce two equilibrium mappings that are convenient for describing our discontinuity identification strategy. Next, we derive the restrictions that the model imposes on the observed winning bids, especially with respect to the formation of discontinuities. Finally, we describe our identification strategy in two steps. First, we identify the distribution of the number of buyers from the discontinuities in the winning bid density function. Second, we identify the value distribution function using the two equilibrium mappings iteratively.

2.1 The symmetric independent private values paradigm

Suppose there is a single item for sale with N active symmetric buyers bidding for the item. All buyers observe N. In contrast, the analyst does not observe N, which causes auction-specific unobserved heterogeneity. Each buyer i also observes her private valuation V_i, which is unknown to other buyers. The private values V_i are *i.i.d.* draws from a distribution $F(\cdot)$, which is known to all the buyers and is independent of N.

The buyers are risk neutral and their bids B_i are formed according to a symmetric best-response strategy. In sum, the primitives are the distribution of the number of buyers N and the private value distribution.

We assume that the analyst only observes the winning bid W, i.e., the maximum bid among the N buyers in the set $\mathcal{N}$ of active buyers

$$W = \max_{i \in \mathcal{N}} B_i.$$

Hence, the analyst observes draws from the unconditional cumulative probability distribution of the winning bid $G(\cdot)$, which is a mixture of the conditional winning bid distributions given N:

$$G(b) \;=\; \sum_{n=2}^{+\infty} \mathbb{P}\left(\max_{1 \leq i \leq n} B_i \leq b\right) \times \mathbb{P}\left(N = n\right) = \sum_{n=2}^{+\infty} G_n^n\left(b\right) \mathbb{P}\left(N = n\right), \qquad (1)$$

where $G_n(\cdot)$ is the conditional bid distribution given $N = n$.

The following two assumptions introduce some additional conditions for the distribution of N and for the private value distribution $F\left(\cdot\right)$.

Assumption N. *The number of active buyers N is a discrete random variable with support $\{\underline{n}, \ldots, \overline{n}\}$ for some integers $2 \leq \underline{n} \leq \overline{n} < \infty$, i.e., $p_n = \mathbb{P}\left(N = n\right) > 0$ for $n = \underline{n}, \underline{n} + 1, \ldots, \overline{n}$ with $\sum_{n=\underline{n}}^{\overline{n}} p_n = 1$.*

Assumption IPV. *Buyers' private values V_i are i.i.d. draws from a common knowledge distribution $F\left(\cdot\right)$ and are unknown to competitors. The cumulative distribution function $F\left(\cdot\right)$ has a compact support $[\underline{v}, \overline{v}]$. Its probability density function $f\left(\cdot\right)$ is continuous and strictly positive over $[\underline{v}, \overline{v}]$.*

Both theoretical and empirical literatures adopt the assumption of a private value distribution with compact support. In particular, it rules out multiple asymmetric equilibria; see Maskin and Riley (1984, Remark 2.3), who also establish that symmetric Bayesian Nash Equilibrium bids are given by a strictly increasing and continuously differentiable function of private values.

For our discontinuity approach, the compact support assumption ensures the existence of discontinuities in the density of unconditional winning bids that we exploit in this paper. In particular, the winning bid densities $g_n(\cdot)$ given $N = n$ stay bounded away from 0 at its upper boundary; see (7) below.

That the private value p.d.f $f(\cdot)$ is positive over $[\underline{v}, \overline{v}]$ is a current assumption in the auction literature, as seen in Maskin and Riley (1984), Lebrun (1999), Guerre et al. (2000), among others. It is used here to identify the lowest number $\underline{n}$ of active buyers; see Lemma 2.1-(iii) below. The alternative identification method of Section 3.1 allows us to relax this condition as noted in Footnote 4.

2.2 Bid and value quantile equilibrium mappings

In this subsection, we describe two equilibrium mappings that are repeatedly used in our identification procedure. Specifically, there is an equilibrium mapping from the value distribution to the bid distribution, and vice versa. Our discontinuity identification strategy is conveniently described using the quantile framework as in Guerre, Perrigne and Vuong (2009), Liu and Luo (2017), and Guerre and Gimenes (2022), that we recount below.

Let $V(\alpha) = F^{-1}(\alpha)$ represent the private value quantile function, where $\alpha \in [0, 1]$ is the quantile level. Let $B_n(\alpha)$ denote the bid quantile function given that n buyers participate in the auction. Following Milgrom (2001)'s exposition of the identification strategy of Guerre, Perrigne and Vuong (2000), the private value quantile function $V(\cdot)$ can be viewed as the common valuation function of buyers who receive independent uniform private signals

$$A_i = F\left(V_i\right),$$

which determines their private values $V_i = V\left(A_i\right)$. By Assumption IPV, $B_i = \beta_n\left(A_i\right)$ for all i, where $\beta_n\left(\cdot\right)$ is strictly increasing and continuously differentiable. It follows that for any b in the range of $\beta_n\left(\cdot\right)$,

$$G_n\left(b\right) = \mathbb{P}\left(\beta_n\left(A_i\right) \leq b\right) = \mathbb{P}\left(A_i \leq \beta_n^{-1}\left(b\right)\right) = \beta_n^{-1}\left(b\right),$$

because A_i is uniformly distributed over $[0,1]$. Hence, the best-response strategy is the bid quantile function

$$\beta_n\left(\alpha\right) = B_n\left(\alpha\right) \text{ for all } \alpha \in [0,1].$$

Now, let us relate the bid and private value quantile functions. Suppose that buyer i receives signal α but makes a suboptimal bid $B_n\left(a\right)$ for some $a \in [0,1]$. Since her opponents bid $B_n\left(A_j\right)$, the probability that her bid $B_n\left(a\right)$ wins the auction is given by $\mathbb{P}(\max_{j\neq i} A_j \leq a)$, which is equal to a^{n-1} as the signals of the $n-1$ opponents A_j, where $j \neq i$, are independent and uniform. It follows that the expected payoff of buyer i is $\left(V\left(\alpha\right) - B_n\left(a\right)\right) a^{n-1}$, which is maximized when $a = \alpha$. Since

$$\begin{aligned}\frac{\partial}{\partial a}\left[\left(V\left(\alpha\right) - B_n\left(a\right)\right) a^{n-1}\right]\Big|_{a=\alpha} &= V\left(\alpha\right)\left(n-1\right)\alpha^{n-2} - \frac{\partial\left[B_n\left(\alpha\right)\alpha^{n-1}\right]}{\partial\alpha} \\ &= \left(n-1\right)\alpha^{n-2}\left(V\left(\alpha\right) - B_n\left(\alpha\right) - \frac{\alpha B_n^{(1)}\left(\alpha\right)}{n-1}\right),\end{aligned}$$

setting this derivative to 0 gives

$$V\left(\alpha\right) = B_n\left(\alpha\right) + \frac{\alpha B_n^{(1)}\left(\alpha\right)}{n-1}. \tag{2}$$

This constitutes the equilibrium mapping from the bid quantile function to the value quantile function, which is the basis of the identification of $V(\cdot)$ with knowledge of $B_n(\cdot)$.

Now, let us consider the inverse of the mapping (2). Indeed, (2) is equivalent to $\frac{\partial\left[B_n(\alpha)\alpha^{n-1}\right]}{\partial\alpha} = V(\alpha)(n-1)\alpha^{n-2}$, and it follows

$$B_n(\alpha) = \frac{n-1}{\alpha^{n-1}} \int_0^{\alpha} t^{n-2} V(t) dt. \tag{3}$$

For convenience of identification that will be clarified later on, let us introduce the conditional bid upper bound

$$\bar{b}_n = B_n(1) = (n-1) \int_0^1 t^{n-2} V(t) dt,$$

which gives

$$B_n(\alpha) = \frac{n-1}{\alpha^{n-1}} \left[\frac{\bar{b}_n}{n-1} - \int_{\alpha}^1 t^{n-2} V(t)\, dt \right]. \tag{4}$$

This constitutes the equilibrium mapping from the value quantile to the bid quantile function. The two mappings represented in (2) and (4) are repeatedly used in our identification procedure.

2.3 Structure of the winning bid distribution

The structure of winning bid distributions compatible with a first-price auction where buyers observe N follows from the mixture expression of $G(\cdot)$ in Equation (1) and the best-response differential equation (2).

Proposition 2.1 *A c.d.f $G(\cdot)$ is rationalized by a first-price auction model satisfying Assumptions N, IPV, and observability of N by buyers if and only if*

(i). The c.d.f $G(\cdot)$ has a mixture structure

$$G(\cdot) = \sum_{n=\underline{n}}^{\overline{n}} p_n G_n^n(\cdot), \tag{5}$$

where the $G_n(\cdot)$ are c.d.f, $2 \le \underline{n} \le \overline{n}$, and the positive p_n satisfy $\sum_{n=\underline{n}}^{\overline{n}} p_n = 1$.

(ii). The quantile functions $B_n(\cdot) = G_n^{-1}(\cdot)$ *are continuously differentiable over* $[0,1]$ *and satisfy the compatibility conditions*

$$B_n(\alpha) + \frac{\alpha B_n^{(1)}(\alpha)}{n-1} = B_m(\alpha) + \frac{\alpha B_m^{(1)}(\alpha)}{m-1}$$

for all $\underline{n} \leq n, m \leq \overline{n}$ *and all* $\alpha \in [0,1]$. *Moreover, the function* $V(\alpha) = B_n(\alpha) + \frac{\alpha B_n^{(1)}(\alpha)}{n-1}$ *is continuously differentiable over* $[0,1]$ *with* $V^{(1)}(\cdot) > 0$.

In short, a c.d.f $G(\cdot)$ as in Proposition 2.1 is a mixture with components constrained by compatibility conditions driven by the best response differential equation (2). The compatibility conditions of Proposition 2.1-(ii) reflect that the mixture components $G_n(\cdot)$ are generated by the same private value distribution, an important feature for identification. In particular, our identification results rely on the constraints it imposes on the extremities of the conditional bid p.d.f $g_n(\cdot)$, as illustrated in the next corollary. Recall $\overline{b}_n = B_n(1)$, $\underline{v} = V(0) = \underline{b}_n$, and $\overline{v} = V(1)$.

Corollary 2.1 *Suppose that the compatibility conditions of Proposition 2.1-(ii) hold. Then, for all* $n = \underline{n}, \ldots, \overline{n}$, $\overline{b}_n < \overline{v}$, *and*

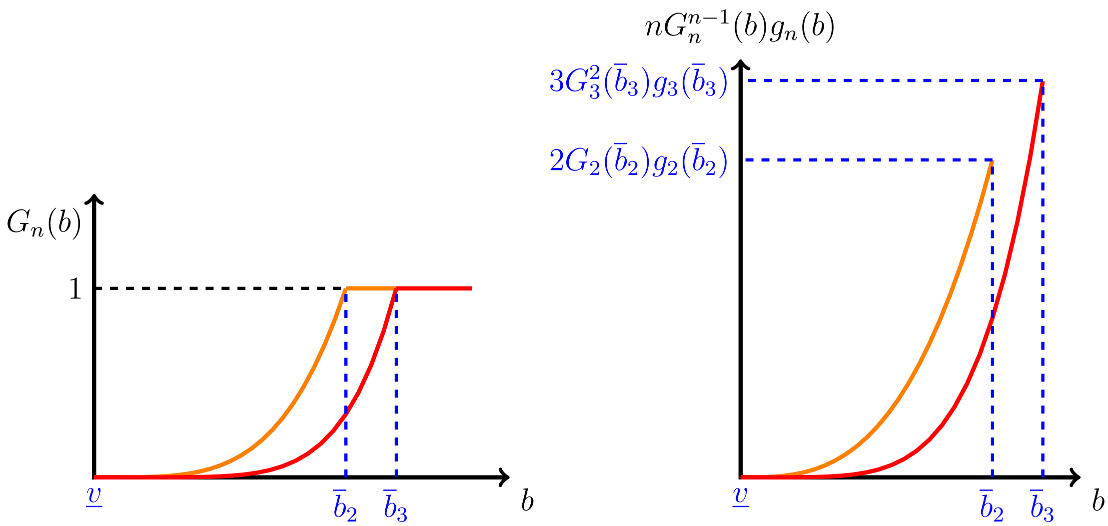
$$g_n(\underline{v}) = \frac{n}{n-1} f(\underline{v}), \text{ with} \tag{6}$$

$$g_n(\overline{b}_n) = \frac{1}{(n-1)(\overline{v} - \overline{b}_n)}. \tag{7}$$

Equation (7) implies that $g_n(\overline{b}_n)$ is strictly positive. It turns out from (1) that this causes discontinuities in the winning bid p.d.f $g(\cdot)$ at each $\overline{b}_n$, as studied in the next section. As it follows that $\overline{b}_n$ is identified, (7) shows that $g_n(\overline{b}_n)$, where $n \in \{\underline{n}, \ldots, \overline{n}\}$, are determined by the common unknown parameter $\overline{v}$. We employ this consequence of the compatibility conditions of Proposition 2.1-(ii) later on to identify the distribution of N.

2.4 Winning bid density discontinuities

In this subsection, we introduce a numerical example to illustrate the discontinuity features of the winning bid p.d.f that follows from Corollary 2.1. This example will also be useful for introducing our identification procedure. A general lemma completes the example.



(a) c.d.f $G_n^n(\cdot)$

(b) p.d.f $nG_n^{n-1}(\cdot)g_n(\cdot)$

Figure 1: Conditional winning bid distribution, where $N = \{2, 3\}$ and $V(\alpha) = \sqrt{\alpha}$

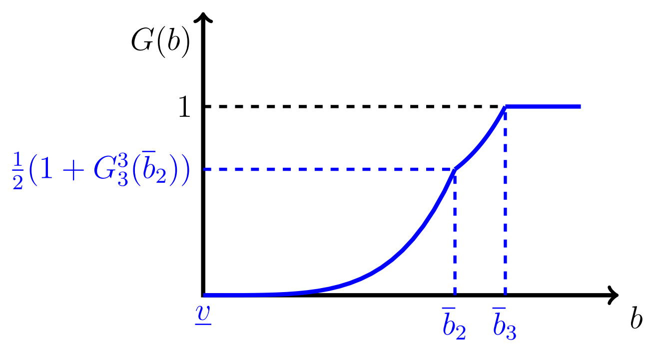
2.4.1 Numerical example

Consider the private value c.d.f $F(v) = v^2$ for all v in $[0, 1]$ and a number of buyers $N = \{2, 3\}$ with equal probability. As $V(\alpha) = \alpha^{1/2}$, it follows that

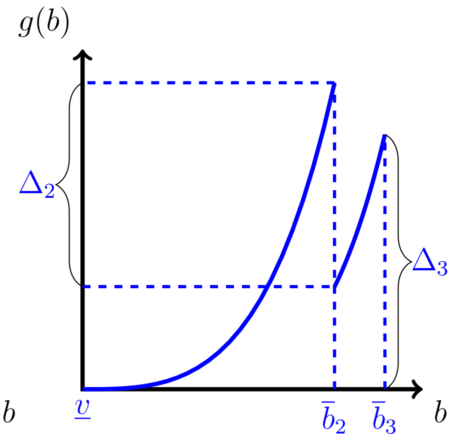
$$B_n(\alpha) = \frac{n-1}{\alpha^{n-1}} \int_0^\alpha t^{n-2+\frac{1}{2}} dt = \frac{n-1}{n-\frac{1}{2}} \alpha^{1/2}.$$

Hence, $\bar{b}_n = \frac{n-1}{n-\frac{1}{2}}$ yields the conditional bid p.d.f $g_n(b)$, given $N = n$, is equal to $2b/\bar{b}_n^2$ on $[0, \bar{b}_n]$ and vanishes outside this interval. Figure 1 displays the conditional c.d.f and p.d.f of the winning bid when $N = \{2, 3\}$. Note that the support of the conditional density function increases with the number of buyers. Both densities jump to zero at their upper boundaries as expected from (7).

Let us now turn to the winning bid, the observation of the analyst. As expected from Figure 1b, the unconditional p.d.f $g(b) = \frac{1}{2} \cdot 2G_2(b)g_2(b) + \frac{1}{2} \cdot 3G_3^2(b)g_3(b)$ displayed in Figure 2b is discontinuous at $\bar{b}_2$ and $\bar{b}_3$, with jump sizes Δ_2 and Δ_3, respectively. The resulting winning bid c.d.f exhibits kinks at these values, as illustrated in Figure 2a. In this example, Figure 2b exhibits two discontinuities (and Figure 2a exhibits two kinks) because N takes two potential values here.



(a) c.d.f $G(\cdot)$



(b) p.d.f $g(\cdot)$

Figure 2: Winning bid distribution ($V(\alpha) = \sqrt{\alpha}$ and $\mathbb{P}(N = 2) = \mathbb{P}(N = 3) = 1/2$)

2.4.2 The general case

The increasing support property observed in Figure 1 and the winning bid p.d.f discontinuities in Figure 2b are generic, as shown in the upcoming lemma. Lemma 2.1-(i) states more generally that bids increase with competition — a key feature of first-price auctions that does not hold in ascending ones, or when buyers do not observe N. Lemma 2.1-(ii) focuses on the winning bid p.d.f discontinuities and its jumps. The identification of $\underline{n}$ in Lemma 2.1-(iii) uses that the lower tail index of $G(b)$ is the one of $G_{\underline{n}}^{n}(b)$.

Lemma 2.1 *Suppose Assumptions N and IPV hold. Then, all of the following hold.*

(i). For all α *in* $(0,1]$, $B_{\underline{n}}(\alpha) < \cdots < B_{\overline{n}}(\alpha) < V(\alpha)$ *with* $B_n(0) = V(0)$ *for all* n. *In particular, for* $\overline{b}_n = B_n(1)$, $\overline{b}_{\underline{n}} < \cdots < \overline{b}_{\overline{n}} < \overline{v}$.

(ii). The c.d.f $G(\cdot)$ *has a p.d.f* $g(\cdot)$ *with* $g(\underline{v}) = 0$, *which is continuous over the straight line with the exception of the* $\overline{n} - \underline{n} + 1$ *discontinuity points* $\overline{b}_{\underline{n}} < \cdots < \overline{b}_{\overline{n}}$, *with the interval* $\left[\underline{v}, \overline{b}_{\overline{n}}\right]$ *being the support of* $G(\cdot)$. *For* $\underline{n} \leq n \leq \overline{n}$, *the jumps* $\Delta_n = \lim_{t \downarrow 0}\left(g(\overline{b}_n - t) - g(\overline{b}_n + t)\right)$ *satisfy*

$$\Delta_n = \frac{np_n}{(n-1)(\overline{v} - \overline{b}_n)}. \tag{8}$$

(iii). It holds that $\underline{n} = \lim_{t \downarrow 0} \frac{\log G(\underline{v}+t)}{\log t}$.

Lemma 2.1 is an important building block for identifying the competition distribution. Part (iii) is a tail identification result for $\underline{n}$ as in Hill and Shneyerov (2013). Lemma 2.1-(ii) shows that the jumps in the winning bid p.d.f identify $\mathbb{P}(N = n)$ up to the unknown $\overline{v}$.

2.5 Identification

Here, we first focus on identification of the participation distribution and then turn to the private values.

2.5.1 Identification of the distribution of N

In this subsection, we describe the identification of the support of N and its distribution using the discontinuity points and jump sizes. To identify the support, we exploit two implications of Lemma 2.1: (a) the minimum number of buyers $\underline{n}$ is identified from the winning bid distribution tail near the lower boundary; (b) each number of buyers n generates a discontinuity in the winning bid distribution, which identifies the difference $\overline{n} - \underline{n}$. More specifically, Lemma 2.1-(ii) identifies $\underline{n}$ and $\overline{n}$ through $\underline{n} = \lim_{t \downarrow 0} \frac{\log G(\underline{v}+t)}{\log t}$ and

$$\overline{n} = \underline{n} + \mathsf{Card}\left\{b; g\left(\cdot\right) \text{ is discontinuous at } b\right\} - 1.$$

This also identifies the support of the distribution of N as $\mathbb{P}(N = n) > 0$ for all n with $\underline{n} \leq n \leq \overline{n}$ by Assumption N.

Next, we exploit the jumps in the p.d.f to identify $p_n = \mathbb{P}(N = n)$. Recall that Equation (8) identifies p_n up to the private value upper bound $\overline{v}$,

$$p_n = \frac{n-1}{n} \Delta_n \left(\overline{v} - \overline{b}_n\right).$$

But $\sum_{\underline{n}}^{\overline{n}} p_n = 1$ implies

$$\overline{v} = \frac{1 + \sum_{n=\underline{n}}^{\overline{n}} \frac{n-1}{n} \Delta_n \overline{b}_n}{\sum_{\underline{n}}^{n=\overline{n}} \frac{n-1}{n} \Delta_n}. \tag{9}$$

Hence, p_n satisfies

$$p_n = \frac{\frac{n-1}{n}\Delta_n}{\sum_{k=\underline{n}}^{\overline{n}} \frac{k-1}{k} \Delta_k} + \frac{n-1}{n} \Delta_n \left(\frac{\sum_{k=\underline{n}}^{\overline{n}} \frac{k-1}{k} \Delta_k \overline{b}_k}{\sum_{k=\underline{n}}^{\overline{n}} \frac{k-1}{k} \Delta_k} - \overline{b}_n \right), \quad n = \underline{n}, \ldots, \overline{n}, \tag{10}$$

and is identified because the discontinuity points $\overline{b}_k$ and jump sizes Δ_k are identified. We summarize these identification results in the next lemma.

Lemma 2.2 *Suppose Assumptions N and IPV hold. Then $\underline{v}$, $\overline{n}$, $\underline{n}$, $\overline{b}_{\underline{n}} < \cdots < \overline{b}_{\overline{n}}$, $\overline{v}$, and the probabilities p_n, $n = \underline{n}, \ldots, \overline{n}$, are identified.*

The identifying equations (9) and (10) can also be used to derive inequality constraints satisfied by the jumps sizes Δ_n, discontinuity locations $\overline{b}_n$, and the lowest and largest numbers of bidders $\underline{n}$ and $\overline{n}$. Indeed $\overline{v} > \overline{b}_{\overline{n}}$ and $0 \leq p_n \leq 1$ are equivalent to the following inequalities[1]

$$\sum_{k=\underline{n}}^{\overline{n}} \frac{k-1}{k} \Delta_k \left(\overline{b}_{\overline{n}} - \overline{b}_k\right) \leq 1,$$

$$1 + \sum_{k=\underline{n}}^{\overline{n}} \frac{k-1}{k} \Delta_k \left(\overline{b}_k - \overline{b}_n\right) \leq \frac{\sum_{k=\underline{n}}^{\overline{n}} \frac{k-1}{k} \Delta_k}{\frac{n-1}{n} \Delta_n}, \quad n = \underline{n}, \ldots, \overline{n},$$

given that $\Delta_n > 0$ must also hold by Lemma 2.1-(ii). A violation of any of these inequalities indicate that the model is misspecified.

2.5.2 Identification of the private value quantile function

We first return to the numerical example to illustrate our iterative identification procedure for the private value distribution.

Numerical example (cont'd). By Lemma 2.2, $\underline{n} = 2$, $\overline{n} = 2$ and $p_1 = p_2 = \frac{1}{2}$ are identified. Let us now turn to the identification of the private value distribution, which is based on the winning bid c.d.f

$$G(b) = \frac{1}{2}\left(G_2^2(b) + G_3^3(b)\right)$$

displayed in Figure 2a. Since $G_2^2(b) = 1$ on $[\overline{b}_2, \overline{b}_3]$,

$$G_3(b) = (2G(b) - 1)^{\frac{1}{3}}, \quad b \in [\overline{b}_2, \overline{b}_3].$$

It follows that $B_3(\cdot)$ is identified on $[\alpha_1, 1]$, where $\alpha_1 = G_3(\overline{b}_2)$, using the top portion of the winning bid distribution; see Figure 2a when $G(b) \in [\frac{1}{2}(1 + G_3^3(\overline{b}_2)), 1]$. Using the mapping (2) from the bid quantile function to the private value one gives

$$V(\alpha) = B_3(\alpha) + \frac{1}{2}\alpha B_3^{(1)}(\alpha),$$

[1] It can be easily seen that $\overline{b}_{\overline{n}} < \overline{v}$, which is equivalent to the first inequality, implying $0 \leq p_n$, $n = \underline{n}, \ldots, \overline{n}$. The second inequality is equivalent to $p_n \leq 1$, $n = \underline{n}, \ldots, \overline{n}$.

and $V(\cdot)$ is identified on $[\alpha_1, 1]$. Additionally, using the mapping (4) from the private value quantile function gives

$$B_2(\alpha) = \frac{1}{\alpha}\left[\bar{b}_2 - \int_\alpha^1 V(t)dt\right]$$

so that $B_2(\cdot)$ is also identified on $[\alpha_1, 1]$. The identified $B_2(\alpha)$, $B_3(\alpha)$, and $V(\alpha)$, where $\alpha \in [\alpha_1, 1]$, are displayed in blue in Figure 3.

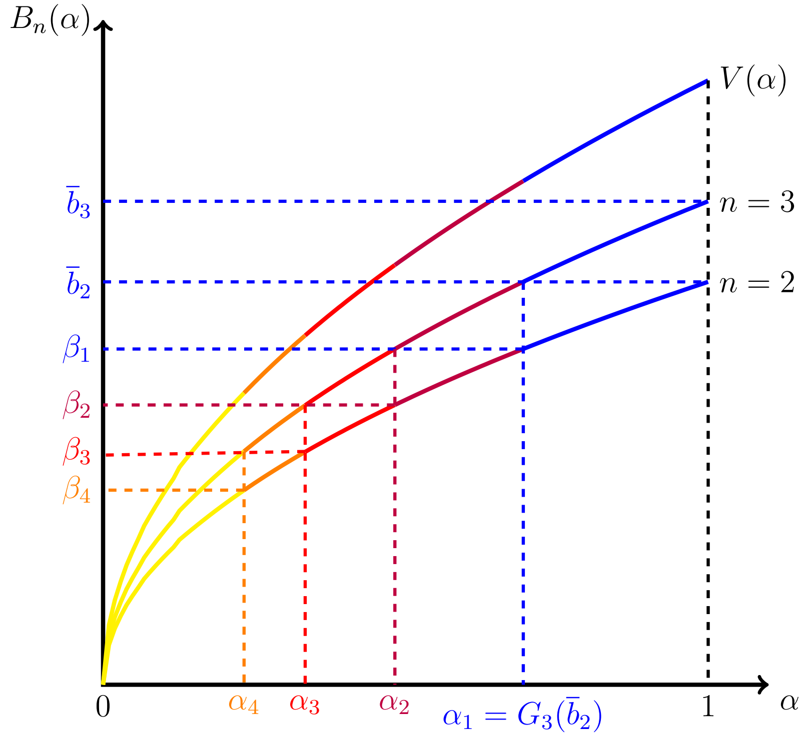


Figure 3: Iterative identification of $B_n(\alpha)$ and $V(\alpha)$ from $G(\cdot)$, as in Figure 2.

Next, we enlarge the interval $[\alpha_1, 1]$ over which $V(\cdot)$ is identified. For this purpose, let $\beta_1 = B_2(\alpha_1) < \bar{b}_2$ and observe that $G_2(b)$ is identified for $b \geq \beta_1$. Given that

$$G_3(b) = \left(G_2^2(b) - 2G(b)\right)^{\frac{1}{3}},$$

$G_3(b)$ is identified for $b \geq \beta_1$, as $B_3(\alpha)$ is identified for $\alpha \geq \alpha_2 = G_3(\beta_1)$. Figure 3 shows that $\alpha_2 < \alpha_1$ and arguing as above gives us identification of $V(\cdot)$ and $B_2(\cdot)$ on

$[\alpha_2, 1]$. Three portions of $V(\cdot)$, $B_3(\cdot)$, and $B_2(\cdot)$ are identified through three iterations and plotted in Figure 3 in purple, red, and orange, respectively. Furthermore, Figure 3 suggests that additional iterations of this identification procedure should allow us to recover any $V(\alpha)$.

The general case. The iterative identification described above can be easily generalized. Showing the convergence of the quantile-level sequence $\{\alpha_k\}$ to 0 can be done using the important fact that the bid quantile functions $B_n(\cdot)$ decrease with n and only cross at the origin. This implies identification of the private value distribution when only observing the winning bid, as stated in the next general result.

Theorem 2.1 *Suppose Assumptions N and IPV hold and that the buyers observe the number of active buyers N. Then $F(\cdot)$ and the distribution of N are identified.*

3 Buyer uncertainty and auction heterogeneity

3.1 Buyer uncertainty

Setup and assumptions. Consider an environment where there are N potentially active buyers, who submit a bid with probability d, an additional parameter to be identified. Buyer uncertainty then arises from the fact that the total number of active buyers is not known but follows a binomial distribution of parameter (N, d) given N, assuming from now on that the bidding decisions and N are independent. This auction setup is summarized in the next assumption.

Assumption BU. *There are N potentially active buyers, observable to the buyers but not the econometrician. Given N, each buyer decides to participate in the auction with probability d in $(0, 1)$, privately and independently of the other buyers. Each* *active buyer draws a private value from the common knowledge distribution $F(\cdot)$. The econometrician observes the winning bid W or that the auction failed if none of the buyers attend.*[2]

In addition, the private value distribution $F(\cdot)$ satisfies Assumption IPV, and N can vary across auctions with $p_n = \mathbb{P}(N = n)$, $n = \underline{n}, \ldots, \overline{n}$ as in Assumption N.

Model primitives. Without loss of generality, assume $\underline{v} > 0$, so that the minimal bid is $\underline{b} = \underline{v} > 0$ when at least one bidder attends the auction, as shown in Harstad et al. (1990). Let $G_n(\cdot|d)$ be the bid distribution given $N = n > 0$, of which support $[\underline{b}, \overline{b}_n]$ is from the same authors.

As a convention, the winning bid is set to 0 when there is no bid. It follows that the winning bid c.d.f, given that no buyers attend the auction and $N = n$, can be written as $G_n^0(\cdot|d)$ over $[0, \infty)$. Since the winning bid distribution given $N = n$ and $0 < m \leq n$ buyers attend the auction is $G_m^0(\cdot|d)$, the unconditional winning bid distribution is

$$
\begin{aligned}
G(b|d) &= \sum_{n=\underline{n}}^{+\overline{n}} p_n \sum_{m=0}^{n} \begin{pmatrix} m \\ n \end{pmatrix} d^m (1-d)^{m-n} G_n^m\,(b|d) \\
&= \sum_{n=\underline{n}}^{+\overline{n}} p_n \left(1 - d + d \cdot G_n(b|d)\right)^n . \qquad (11)
\end{aligned}
$$

A key difference with the baseline model is that

$$
G(\underline{b}|d) = \sum_{n=\underline{n}}^{+\overline{n}} p_n \,(1-d)^n
$$

[2] For auctions with a reserve price R, private values are drawn before the participation decision is made, active buyers being those with a private value larger than R. Changing $F(\cdot)$ into the private value distribution given $V \geq R$ covers auctions with reserve price. See Guerre and Luo (2019) who also consider entry cost, but assume N is fixed across auctions.

is now the probability that no buyers attend the auction, which is positive. The lowest number of bidders $\underline{n}$ cannot be identified from the lower tail of $G(\cdot|d)$ as done in the baseline model, where $G(\cdot|d)$ was vanishing and behaving locally as a power $(b - \underline{b})^{\underline{n}}$ near $\underline{b}$.

Some closed-form expressions can be easily obtained for the equilibrium bid quantile function $B_n(\cdot|d) = G_n^{-1}(\cdot|d)$. Given $N = n$ and assuming that a buyer with private value $V(\alpha) = F^{-1}(\alpha)$ attends the auction, the expected profit generated by a bid $B_n(a|d)$ is

$$
\begin{aligned}
&(V(\alpha) - B_n(a|d)) \sum_{m=0}^{n-1} \begin{pmatrix} m \\ n-1 \end{pmatrix} d^m (1-d)^{n-1-m} a^m \\
&= (V(\alpha) - B_n(a|d)) \left(1 - d + d \cdot a\right)^{n-1}.
\end{aligned}
$$

This expression is obtained noting that, when n bidders are potentially active and at least one is, the distribution of the remaining number of bids is a binomial with parameter $(n-1, d)$. The first-order condition characterizing the Bayesian Nash Equilibrium is then

$$
\frac{d}{d\alpha}\left[\left(1 - d + d \cdot \alpha\right)^{n-1} B_n(\alpha|d)\right] = (n-1)d\left(1 - d + d \cdot \alpha\right)^{n-2} V(\alpha),
$$

implying, given $B_n(0|d) = \underline{v}$,

$$
V(\alpha) = B_n(\alpha|d) + \frac{(1 - d + d \cdot \alpha)\, B_n^{(1)}(\alpha|d)}{(n-1) \cdot d}, \tag{12}
$$

$$
B_n(\alpha|d) = \frac{(1-d)^{n-1}\underline{v} + (n-1) \cdot d \cdot \int_0^\alpha (1 - d + d \cdot t)^{n-2} V(t)dt}{(1 - d + d \cdot \alpha)^{n-1}}, \tag{13}
$$

which are the counterparts of (2) and (3). In particular, integrating by parts in (13) shows that

$$
B_n(\alpha|d) = V(\alpha) - \int_0^\alpha \left(\frac{1 - d + d \cdot t}{1 - d + d \cdot \alpha}\right)^{n-1} V^{(1)}(t)dt,
$$

which implies that $B_n(\alpha|d)$ is strictly increasing with respect to n for all quantile levels $\alpha > 0$. Hence the largest bids $\overline{b}_n = B_n(1|d)$ satisfy $\overline{b}_{\underline{n}} < \overline{b}_{\underline{n}+1} < \cdots < \overline{b}_{\overline{n}} < \overline{v}$ as in the baseline model.

The next lemma describes implications of (12) and (13) for the conditional bid p.d.f $g_n(b|d) = \frac{d}{db} G_n(b|d)$, which parallels Corollary 2.1. Henceforth, we shall consider the additional parameter

$$\overline{v}^{(1)} = V^{(1)}(1) = \frac{1}{f(\overline{v})}.$$

Lemma 3.1 *Suppose Assumptions BU, N, and IPV hold. Then,*

$$g_n(b|d) = \left(\frac{2f(\underline{v})(1-d)}{(n-1)\cdot d\cdot(b-\underline{b})}\right)^{\frac{1}{2}}(1+o(1)) \text{ when } b \downarrow \underline{b}, \tag{14}$$

$$g_n(\overline{b}_n|d) = \frac{1}{(n-1)\cdot d\cdot(\overline{v}-\overline{b}_n)}, \tag{15}$$

$$g_n^{(1)}(\overline{b}_n|d) = \frac{n\cdot d\cdot\left(\overline{v}-\overline{b}_n\right)-\overline{v}^{(1)}}{((n-1)\cdot d)^2\left(\overline{v}-\overline{b}_n\right)^3}. \tag{16}$$

As the expression of the winning bid distribution (11) gives a p.d.f of

$$g(b|d) = \sum_{n=\underline{n}}^{+\overline{n}} np_n \cdot d\cdot(1-d+d\cdot G_n(b|d))^{n-1} g_n(b|d),$$

(15) shows that $g(\cdot|d)$ exhibits downward jumps Δ_n at each $\overline{b}_n$, $n = \underline{n}, \ldots, \overline{n}$ with

$$\Delta_n = \lim_{t\downarrow 0}\left(g(\overline{b}_n - t|d) - g(\overline{b}_n + t|d)\right) = \frac{np_n}{(n-1)(\overline{v}-\overline{b}_n)},$$

an expression identical to the jumps of the baseline model (8). As (13) ensures that $g(\cdot|d)$ is continuous at other points of $(\underline{b}, \overline{b}_{\overline{n}})$, this can be used to identify $\overline{n} - \underline{n}$ and $p_n = \frac{n-1}{n}\Delta_n\left(\overline{v}-\overline{b}_n\right)$ up to n and $\overline{v}$.

Equation (14) shows that the conditional bid p.d.f $g_n(\cdot|d)$ and the winning bid p.d.f $g(\cdot|d)$ both diverge with a $-\frac{1}{2}$ power rate at the lowest bid $\underline{b} = \underline{v}$. This is intuitively due to buyer uncertainty, as a bidder can win with a very low bid if no others attend the auction. As a consequence, this can be used to check the presence of bidder uncertainty. On the other hand, it does not allow for the identification of $\underline{n}$ using the lower tail behavior of $G(\cdot|d)$ as simply as in the baseline model.

Discontinuities and identification strategies. A possible way to recover the minimal number $\underline{n}$ of potentially active bidders relies on the derivative p.d.f discontinuities, as permitted by (16). The winning bid derivative p.d.f is

$$g^{(1)}|(b|d) = \sum_{n=\underline{n}}^{+\overline{n}} np_n \cdot d \cdot (1 - d + d \cdot G_n(b|d))^{n-2}$$
$$\times \left[(1 - d + d \cdot G_n(b|d))\, g_n^{(1)}(b|d) + (n-1) \cdot d \cdot g_n^2(b|d)\right],$$

which, by (13), is continuous over $(\underline{b}, \overline{b}_{\overline{n}})$, with the possible exception of the identified $\overline{b}_n$, $n = \underline{n}, \ldots, \overline{n}$, where it may exhibit jumps

$$\begin{aligned}\Delta_n^{(1)} &= \lim_{t \downarrow 0} \left(g^{(1)}(\overline{b}_n - t|d) - g^{(1)}(\overline{b}_n + t|d)\right) = ndg_n^{(1)}(\overline{b}_n|d) + n(n-1)d^2 g_n^2(\overline{b}_n|d) \\ &= p_n \frac{n}{n-1} \frac{1}{(\overline{v} - \overline{b}_n)^2} \left(2 + \frac{1}{n-1} - \frac{1}{n-1} \frac{\overline{v}^{(1)}}{d} \frac{1}{\overline{v} - \overline{b}_n}\right),\end{aligned}$$

by (15), (16), and a little algebra. To take advantage of the fact that $n - \underline{n}$ is identified as the rank of $\overline{b}_n$,[3], set

$$m = n - \underline{n}, \quad \overline{b}(m) = \overline{b}_{\underline{n}+m}, \quad \varrho(m) = \frac{\Delta_{\underline{n}+m}^{(1)}}{\Delta_{\underline{n}+m}},$$

which are all identified. It then follows from the expression of $\Delta_n^{(1)}$ and Δ_n given the above that

$$\varrho(m)(\underline{n} + m - 1)(\overline{v} - \overline{b}(m))^2 - (2\underline{n} + 2m - 1)\,(\overline{v} - \overline{b}(m)) - \frac{\overline{v}^{(1)}}{d} = 0. \qquad (17)$$

As this equation includes the three unknowns $\underline{n}$, $\overline{v}$, and $\overline{v}^{(1)}/d$, three of these equations are, in principle, needed for identification from (17). The presence of nonlinearities can also complicate this identification strategy.

[3]This follows from $\overline{b}_{\underline{n}} < \cdots < \overline{b}_{\overline{n}}$, defining the rank of $\overline{b}_{\underline{n}}$ as 0, the one of $\overline{b}_{\overline{n}}$ being $\overline{n} - \underline{n}$.

A simple overidentification strategy introduces as additional unknowns some well chosen nonlinear functions, such as $\underline{n}\left(\overline{v}-\overline{b}(0)\right)^2$, $\left(\overline{v}-\overline{b}(0)\right)^2$, and $\underline{n}\left(\overline{v}-\overline{b}(0)\right)$, to transform (17) into a linear equation. This allows us to obtain a condition ensuring identification of the considered auction specification as stated in the next proposition.

Proposition 3.1 *Suppose Assumptions BU, N, and IPV hold, and that* $\overline{n}-\underline{n}\geq 5$. *Assume there is a subset* $\mathcal{M}$ *of* $\{0,\ldots,\overline{n}-\underline{n}\}$ *with six elements, and let* $I_{\mathcal{M}}$ *be the* 6×6 *matrix with the following row entries, for* m *in* $\mathcal{M}$,

$$\Big[1,\ \varrho(m),\ m\varrho(m),\ \overline{b}(m)-\overline{b}(0),\ 2m+(m-1)\varrho(m)\left(\overline{b}(m)-\overline{b}(0)\right),\\ \left(\overline{b}(m)-\overline{b}(0)\right)\left(\varrho(m)\left(\overline{b}(m)-\overline{b}(0)\right)+2\right)\Big].$$

Then, if $\det(I_{\mathcal{M}})\neq 0$, *the uncertainty probability* d, $\underline{n}$, $\overline{n}$, *and* p_n *for* $n=\underline{n},\ldots,\overline{n}$ *together with the private value distribution* $F(\cdot)$ *are identified.*

As $\varrho(m)$ and $\overline{b}_n$ can be estimated from the data, the overidentification condition $\det(I_{\mathcal{M}})\neq 0$ is testable. Proposition 3.1 holds under the condition $\overline{n}-\underline{n}\geq 5$, a condition that can be weakened by introducing fewer additional unknowns in (17) and by taking into account that $\underline{n}$ is an integer number.[4]

3.2 Unobserved auction heterogeneity

Setup and assumptions. Consider now a setup with auction heterogeneity, where for each buyer i,

$$\widetilde{V}_i=\chi+V_i, \tag{18}$$

χ being an auction-specific variable which is not observed by the econometrician but common knowledge to buyers, and V_i is an i.i.d. private value component drawn from $F(\cdot)$ satisfying Assumption IPV.

[4]Note that Proposition 3.1 also applies when buyers are certain about participation (i.e. $d=1$), in which case $\underline{n}$ is identified without relying on the tail argument used in Lemma 2.1-(iii).

Assumption UAH. *The unobserved auction heterogeneity component* χ *is independent of* N *and all the private values* V_i*. The p.d.f* $\varphi(\cdot)$ *of* χ *has a compact support* $[0, \overline{\chi}] \subset [0, \infty)$*, over which it is strictly positive and twice continuously differentiable.* $\overline{\chi} \neq \overline{b}_n - \overline{b}_m$ *for all* $\underline{n} \leq n, m \leq \overline{n}$*.*

The restriction on $\overline{\chi}$ shortens some proofs but can be easily removed.

Model primitives. Under (18), the bids $\widetilde{B}_i$ are equal to $\chi + B_i$, where the conditional quantile function $B_n(\cdot)$ of the i.i.d. B_i given $N = n$ is given by (3) and satisfies (2). It follows that the winning bid $\widetilde{W}$ is now

$$\widetilde{W} = \chi + W, \text{ where } W = \max_{i \in \mathcal{N}} B_i.$$

The conditional c.d.f of W is the one of the baseline model, so that, $\Phi(\cdot)$ being the c.d.f of χ,

$$\widetilde{G}_n^n(b) = \mathbb{P}\left(\widetilde{W} \leq b \middle| N = n\right) = \mathbb{P}\left(\chi \leq b - W \middle| N = n\right) = \int_{\underline{b}}^{\overline{b}} \Phi(b - t) n G_n^{n-1}(t) g_n(t) dt,$$

as the p.d.f of W is $nG_n^{n-1}(b)g_n(b)$. It follows that the p.d.f of $\widetilde{W}$ is, by Assumption N and recalling that χ belongs to $[0, \overline{\chi}]$,

$$\begin{aligned} \widetilde{g}(b) &= \sum_{n=\underline{n}}^{\overline{n}} p_n \int_{b-\overline{\chi}}^{b} \varphi(b - t) n G_n^{n-1}(t) g_n(t) dt \\ &= \int_{b-\overline{\chi}}^{b} \varphi(b - t) g(t) dt \text{ where } g(t) = \sum_{n=\underline{n}}^{\overline{n}} p_n n G_n^{n-1}(t) g_n(t), \end{aligned} \quad (19)$$

noting that $g(\cdot)$ is the winning bid p.d.f of the baseline model.

Winning bid p.d.f derivatives discontinuities. Integrating out $g(\cdot)$ in (19) gives a smooth p.d.f $\widetilde{g}(\cdot)$. However discontinuities arise when differentiating $\widetilde{g}(\cdot)$. It indeed holds that applying the Liebnitz rule for integral differentiation to (19) yields

$$\widetilde{g}^{(1)}(b) = \sum_{n=\underline{n}}^{\overline{n}} p_n n \left[\varphi(0) G_n^{n-1}(b) g_n(b) - \varphi(\overline{\chi}) G_n^{n-1}(b-\overline{\chi}) g_n(b-\overline{\chi})\right] \quad (20)$$
$$+ \sum_{n=\underline{n}}^{\overline{n}} p_n \int_{b-\overline{\chi}}^{b} \varphi^{(1)}(b-t) n G_n^{n-1}(t) g_n(t) dt.$$

As the integral expression above remains continuous, the discontinuities of $\widetilde{g}^{(1)}(\cdot)$ are given by equation (20) and arise at each $\overline{b}_n$ and $\overline{b}_n + \overline{\chi}$, with jumps that are opposite in sign but of proportional magnitude. The next lemma summarizes some properties of $\widetilde{g}(\cdot)$ and its first and second derivatives. Recall that $\overline{v}^{(1)} = V^{(1)}(0) = 1/f(\underline{v})$.

Lemma 3.2 *Suppose Assumptions BU, N, and IPV hold. Then:*

(i). $\widetilde{g}(\cdot)$ *is continuous over* $[0, \infty)$ *with* $\underline{n} = \lim_{t \downarrow 0} \frac{\log \widetilde{g}(\underline{b}+t)}{\log t}$;

(ii). $\widetilde{g}^{(1)}(\cdot)$ *is continuous over* $[0, \infty)$*, except at* $\overline{b}_n$ *and* $\overline{b}_n + \overline{\chi}$, $n = \underline{n}, \ldots, \overline{n}$*. It has a downward jump at* $\overline{b}_n$ *of size*

$$\widetilde{\Delta}_n = \varphi(0) p_n \frac{n}{n-1} \frac{1}{\overline{v} - \overline{b}_n}$$

and an upward jump at $\overline{b}_n + \overline{\chi}$ *of size* $\frac{\varphi(\overline{\chi})}{\varphi(0)} \widetilde{\Delta}_n$;

(iii). $\widetilde{g}^{(2)}(\cdot)$ *is continuous over* $(0, \infty)$*, except at* $\overline{b}_n$ *and* $\overline{b}_n + \overline{\chi}$, $n = \underline{n}, \ldots, \overline{n}$*. It has a downward jump at* $\overline{b}_n$ *of size*

$$\widetilde{\Delta}_n^{(1)} = p_n \left[\varphi(0) n \frac{(2n-1)\left(\overline{v} - \overline{b}_n\right) - \overline{v}^{(1)}}{(n-1)^2 \left(\overline{v} - \overline{b}_n\right)^3} + \varphi^{(1)}(0) \frac{n}{n-1} \frac{1}{\overline{v} - \overline{b}_n} \right]$$

and an upward jump at $\overline{b}_n + \overline{\chi}$.

Identification of the participation distribution p_n**.** Lemma 3.2-(i) ensures that $\underline{n}$ is identified while (ii) implies that the number of jumps of $\widetilde{g}^{(1)}(\cdot)$ above $\underline{b}$ is $2(\overline{n} - \underline{n})$

under the restriction on $\overline{\chi}$ of Assumption AUH, so that $\overline{n}$ can also be recovered. It also shows that $\overline{b}_{\underline{n}} < \cdots < \overline{b}_{\overline{n}}$ are identified as locations of downward jumps. As upward jumps are located at $\overline{b}_{\underline{n}} + \overline{\chi} < \cdots < \overline{b}_{\overline{n}} + \overline{\chi}$, the unobserved heterogeneity upper bound support $\overline{\chi}$ is also identified. This may help identify parametric heterogeneity distributions that depend upon a unique parameter.

Identifying the participation distribution is more difficult than in the baseline model because the downward jumps $\widetilde{\Delta}_n$ in Lemma 3.2-(ii) now depend upon two unknown parameters, $\overline{v}$ and $\varphi(0)$, unless the latter is identified. Introducing the upward jumps do not help, as they depend on the unknown $\overline{v}$ and $\varphi(\overline{\chi})$. To address this issue, one can try to identify $\overline{v}$ using the discontinuity ratio $\widetilde{\varrho}_n = \frac{\widetilde{\Delta}_n^{(1)}}{\widetilde{\Delta}_n}$, which, by Lemma 3.2-(ii,iii), satisfies

$$(n-1)\left(\widetilde{\varrho}_n - \frac{\varphi^{(1)}(0)}{\varphi(0)}\right)\left(\overline{v} - \overline{b}_n\right)^2 - (2n-1)\left(\overline{v} - \overline{b}_n\right) - \overline{v}^{(1)} = 0. \tag{21}$$

This equation involves three unknowns, $\overline{v}$, $\overline{v}^{(1)}$, and the ratio $\varphi^{(1)}(0)/\varphi(0)$. Three of these equations are, in principle, needed for identification. Similar to equation (17), it can be used as for (over)identification purposes, considering several values of n and introducing extra variables that are nonlinear functions of the initial unknowns to back out $\overline{v}$, $\overline{v}^{(1)}$, and $\varphi^{(1)}(0)/\varphi(0)$ as the unique solution of an extended linear system.

Proposition 3.2 *Suppose Assumptions BU, N, and IPV hold, and that* $\overline{n} - \underline{n} \geq 5$. *Assume there is a subset* $\widetilde{\mathcal{N}}$ *of* $\{\underline{n}, \ldots, \overline{n}\}$ *with six elements, and let* $I_{\widetilde{\mathcal{N}}}$ *be the* 6×6 *matrix with the following row entries, for* n *in* $\widetilde{\mathcal{N}}$,

$$\left[1,\ n,\ (n-1)\widetilde{\varrho}_n\ ,(n-1)\left(\overline{b}_n - \overline{b}_{\underline{n}}\right),\ (n-1)\left(\overline{b}_n - \overline{b}_{\underline{n}}\right)^2,\ (n-1)\left(\overline{b}_n - \overline{b}_{\underline{n}}\right)\widetilde{\varrho}_n\right].$$

Then, if $\det(I_{\widetilde{\mathcal{N}}}) \neq 0$, *the participation distribution* $\{p_n, n = \underline{n}, \ldots, \overline{n}\}$ *is identified, as* $\underline{v}$, $\overline{v}$, *and* $\overline{v}^{(1)}$.

In addition, $\overline{\chi}$, $\varphi(0)$, $\varphi^{(1)}(0)$, $\varphi(\overline{\chi})$, *and* $\varphi^{(1)}(\overline{\chi})$ *are also identified.*

Proposition 3.2 gives a testable condition, ensuring that the participation distribution can be identified. It also leaves the door open for identification of parametric private value distributions with a parameter in a one-to-one correspondence with $\left(\underline{v}, \overline{v}, \overline{v}^{(1)}\right)$, as the latter can be identified.

Similarly, the parametric unobserved heterogeneity distribution that can be uniquely recovered from $\left(\overline{\chi}, \varphi(0), \varphi^{(1)}(0), \varphi(\overline{\chi}), \varphi^{(1)}(\overline{\chi}),\right)$ can also be identified. As the expression of the winning bid p.d.f $\widetilde{g}(\cdot)$ in (19) shows that it is the convolution of the baseline winning bid p.d.f $g(\cdot)$ by $\varphi(\cdot)$, the deconvolution technique of Krasnokutskaya (2011) using the identified $\varphi(\cdot)$ allows for the recovery of $g(\cdot)$. If so, the identification procedure developed for the baseline model can be applied to nonparametrically recover the private value distribution.

3.3 Observed auction heterogeneity

An alternative to the unobserved auction heterogeneity specification (18) is to assume that χ is observed up to an unknown parameter, as in Haile, Hong and Shum (2003) or Rezende (2008), who consider a regression specification $\chi = X'\theta$, where X does not include a constant, so that

$$\widetilde{V}_i = X'\theta + V_i.$$

It follows that

$$\widetilde{W} = X'\theta + W, \text{ where } W = \max_{i \in \mathcal{N}} B_i,$$

where the distribution of the bid B_i is identical to the one of the baseline model, provided the private values and the set $\mathcal{N}$, and in particular the number of active buyers N, are independent of X. Under this assumption, θ is identified regressing $\widetilde{W} - \mathbb{E}[\widetilde{W}]$ on X, allowing for the recovery of W and the identification approach of the baseline model to proceed.

It is however crucial here that the number of potential buyers, N, is independent of X. Otherwise, the regression above may identify a biased version $\theta^\star$ of θ, and a $W^\star = \chi^\star + W$ with $\chi^\star = X'(\theta - \theta^\star)$. If X has a continuous distribution, so will $\chi^\star$. The expression (19) of the density of the winning bid contaminated with continuous unobserved heterogeneity then suggests that the p.d.f of $W^\star$ will have no discontinuities even if N varies, so that the proposed identification approach does not apply. Similar issues arise when the private value regression $\widetilde{V}_i = X'\theta + V_i$ is functionally misspecified, in which case $W^\star = \widetilde{W} - X'\theta^\star$ will depend upon a misspecification variable $\chi^\star$ which may also wash out discontinuities.

4 A varying-asymmetry cartel framework

We consider asymmetric auctions confronting with a cartel $\mathcal{C}$ of K buyers and $n-1$ competitive symmetric buyers, where n is fixed across auctions but unknown. As in Pesendorfer (2000) and Schurter (2020), the cartel is efficient, i.e. the sole cartel bidder is the cartel leader who has the highest private value in $\mathcal{C}$. In this setting, K can vary randomly across auctions, as observed for a stamp cartel in Asker (2010). Asymmetry is due to the presence of the cartel, of which size variation will cause discontinuities in the winning bid p.d.f. Buyer status can also vary, from cartel member in a given auction to competitive participant in another, or to non-participant or dominated bidder. As in Schurter (2020), it is assumed that the winner's identity is observed together with the winning bid.

4.1 Main assumptions

Cartel size. In what follows, we also allow for the absence of a cartel, i.e. $K = 1$.

Assumption K. *The cartel size K is common knowledge and distributed over* *the integer numbers* $1, \underline{k}, \ldots, \overline{k}$*,* $\underline{k} \geq 2$*, with probabilities* π_k *satisfying* $\pi_{\underline{k}}, \pi_{\overline{k}} > 0$*. If* $K > 1$*, the leader of the cartel is bidding together with* $n-1$ *competitive buyers. If* $K = 1$*, all* n *buyers are competitive.*

In addition, passive buyers, participating the cartel or not, can make dominated bids. The buyer cartel participation decision can be purely random and exogenous across auctions, generating a random cartel size of K. It may also be the result of anti-collusion measures enforced with varying intensity across time, or correspond to strategic behaviors aiming to mask cartel participation.

Asymmetric private values. We assume a common private value distribution $F(\cdot)$ for cartel members and competitive buyers. The cartel leader is the cartel member with the highest private value, as in Pesendorfer (2000) and Schurter (2020), leading to the asymmetric private value distributions specified in the next assumption.

Assumption CIPV. *Conditional on cartel size* K *and cartel participation status, private values are drawn independently in* $F(\cdot)$ *for competitive buyers and from* $F_c(\cdot|K) = F^K(\cdot)$ *for the cartel leader. The c.d.f* $F(\cdot)$ *has support* $[\underline{v}, \overline{v}]$*, with continuously differentiable p.d.f* $f(\cdot)$ *bounded away from* 0 *on* $[\underline{v}, \overline{v}]$*.*

Cartel membership. A new aspect developed here is the possibility that a buyer can be a cartel member in a given auction, behave competitively in an another, and not participate in a third. For a bidder i, let

- $\gamma_{i|k}^{c} = \mathbb{P}(i \in \mathcal{C}|K = k)$ be the probability that i is in the cartel given that the cartel size is k, and
- $\gamma_{i|k}^{nc}$ be the probability that i is a competitive bidder given that $K = k$.

Note that $\gamma^c_{i|k} + \gamma^{nc}_{i|k} \leq 1$, as $1 - (\gamma^c_{i|k} + \gamma^{nc}_{i|k})$ is the probability that buyer i is not an active bidder, i.e. makes dominated bids or do not attend the auction.[5]

The next assumption is a key condition to ensure identification in our framework.

Assumption E. *There is a permanent cartel member buyer e, such that*

$$\mathbb{P}\left(e \in \mathcal{C} \middle| K > 1\right) = 1 \text{ and } \mathbb{P}\left(e \text{ is an active buyer} | K = 1\right) = 0.$$

Assumption E restricts the cartel participation probabilities $\gamma^c_{e|k}$ of buyer e, which must all be equal to 1 for $k > 1$ (and $\pi_k > 1$), therefore imposing some stability in the cartel composition, not conditional on its size. Note that the identity of buyer e does not need to be known a priori by the Econometrician. Testability of Assumption E, as well as identification of permanent cartel members, are discussed in the next section; see Proposition 4.2 below. As also explained below and because the winner's identity is observed, Assumption E allows for the identification of the winning bid distribution given that e wins, which is essential for identification.

4.2 Bid properties and testability of Assumption E

Inverse strategies. Let $\xi_c(\cdot|k)$ and $\xi(\cdot|k)$ respectively be the cartel leader and competitive bidder inverse bid strategies given $K = k$. Lebrun (1999, Corollary 4) shows existence and uniqueness of the optimal inverse strategies generated by the Bayesian Nash Equilibrium.[6] $\xi_c(\cdot|k)$ and $\xi(\cdot|k)$ are strictly increasing over a common

[5] In our framework, cartel formation works by selecting a subset K of buyers in a larger set, so that the buyer decisions are not independent. The probabilities $\gamma^c_{i|k}$ are then a byproduct of a cartel formation procedure and should not be confused with the probability that a buyer decides to enter the cartel, independent of the other buyers, given $K = k$. The same holds for the probabilities $\gamma^{nc}_{i|k}$.

[6] Uniqueness is also derived in Theorem B.1 of Appendix A, which considers a more general setup than Corollary 4 of Lebrun (1999).

support $[\underline{b}, \overline{b}_k]$ and satisfy $\xi_c(\underline{b}|k) = \xi(\underline{b}|k) = \underline{b} = \underline{v}$, $\xi_c(\overline{b}_k|k) = \xi(\overline{b}_k|k) = \overline{v}$. The inverse strategies are not as explicit as the bid quantile function (3) of the baseline model but can be characterized using a differential system derived now; see also Lebrun (1999, Theorem 1).

Considering private values $\xi_c(b|k)$ and $\xi(b|k)$ with b in $(\underline{b}, \overline{b}_k]$ give, respectively,

$$b = \arg\min_x \left(\xi_c(b|k) - x\right) F^{n-1}\left(\xi(x|k)\right),$$
$$b = \arg\min_x \left(\xi(b|k) - x\right) F^{n-2}\left(\xi(x|k)\right) F^k\left(\xi_c(x|k)\right).$$

The associated first-order conditions, setting $\zeta_c(b|k) = \ln F\left(\xi_c(b|k)\right)$ and $\zeta(b|k) = \ln F\left(\xi(b|k)\right)$, then give

$$(n-1)\zeta^{(1)}(b|k) = \frac{1}{\xi_c(b|k) - b},$$
$$(n-2)\zeta^{(1)}(b|k) + k\zeta_c^{(1)}(b|k) = \frac{1}{\xi(b|k) - b},$$

recalling that $\xi(b|k) > b$ and $\xi_c(b|k) > b$ over $(\underline{b}, \overline{b}_k)$ with equality at the two interval extremities, where $\overline{b}_k$ is the common largest possible cartel and non-cartel bids; see Lebrun (1999). Rearranging then gives, for $k > 0$,

$$\zeta^{(1)}(b|k) = \frac{1}{n-1}\frac{1}{\xi_c(b|k) - b}, \tag{22}$$
$$\zeta_c^{(1)}(b|k) = \frac{1}{k(n-1)}\left(\frac{n-1}{\xi(b|k) - b} - \frac{n-2}{\xi_c(b|k) - b}\right). \tag{23}$$

The differential equations (22) and (23) play a key role in the proof of our identification results, replacing for such purposes equations (2) and (3) of the baseline model.

Issues specific to asymmetry. A difference with the baseline model, where the bid quantile function increases with the number of buyers n, is that ordering the inverse strategies is much more involved. While feasible for $\xi(\cdot|k)$, as seen from Proposition 4.1-(ii), it is difficult to study the variations of the cartel inverse strategy $\xi_c(\cdot|k)$

with respect to cartel size k. A third difficulty is that Lebrun (1997,1999) does not established the differentiability of the inverse strategies at $\underline{b}$, and this cannot be easily derived from the differential equations (22) and (23) as $\xi(\underline{b}|k) - \underline{b} = \xi_c(\underline{b}|k) - \underline{b} = 0$. This issue is addressed by Proposition B.1 in Appendix B.

Conditional bid distributions. Let $G_c^k(\cdot|k)$ and $G(\cdot|k)$ respectively be the cartel and competitive buyer bid distribution given cartel size is k; that is:

$$G_c^k(b|k) = F^k\left(\xi_c(\cdot|k)\right) \text{ and } G(b|k) = F\left(\xi(\cdot|k)\right).$$

Let us first state some important properties of the bid distributions.

Proposition 4.1 *Suppose that Assumption CIPV holds. Then, for any* $k = 1, \underline{k}, \dots, \overline{k}$, *the Bayesian Nash Equilibrium exists and is unique, generating strategies* $\xi^{-1}(\cdot|k)$ *and* $\xi_c^{-1}(\cdot|k)$, *which are increasing and continuously differentiable over* $[\underline{v}, \overline{v}]$, *with strictly positive derivatives. Moreover:*

(i). $\xi^{-1}(\underline{v}|k) = \xi_c^{-1}(\underline{v}|k) = \underline{v} = \underline{b}$ *and* $\xi^{-1}(\overline{v}|k) = \xi_c^{-1}(\overline{v}|k) = \underline{v} = \overline{b}_k$ *for all* k *with strictly increasing* $\overline{b}_k$*:*

$$\underline{b} < \overline{b}_1 < \overline{b}_{\underline{k}} < \cdots < \overline{b}_{\overline{k}} < \overline{v};$$

(ii). *For any* $k_2 > k_1 \geq 1$ *and* $k \geq 2$, $G(\cdot|k_2) < G(\cdot|k_1)$ *over* $(\underline{b}, \overline{b}_{k_1}]$ *and* $G(\cdot|k) < G_c(\cdot|k)$ *over* $(\underline{b}, \overline{b}_k)$.

(iii). $g(\overline{b}_k|k) = \frac{1}{n-1}\frac{1}{\overline{v}-\overline{b}_k}$ *and* $g_c(\overline{b}_k|k) = \frac{1}{k(n-1)}\frac{1}{\overline{v}-\overline{b}_k}$ *for all* k;

(iv). $g(\underline{b}|k) = \frac{n+k-1}{n+k-2} f(\underline{v}) > 0$ *and* $g_c(\underline{b}|k) = \frac{n}{n-1} f(\underline{v}) > 0$ *for all* k.

Proposition 4.1-(i) shows that the bid support increases with the cartel size k. Comparing with Lemma 2.1 shows that the cartel size plays a similar role to the number of active buyers in the baseline model. There are, however, some important differences induced by asymmetry. Lemma 7.3 in the Proof Section shows that the competitive buyer strategy $\xi^{-1}(\cdot|k)$ increases with k, but this does not need to be true for the cartel buyer strategy $\xi_c^{-1}(\cdot|k)$. This implies that the bid c.d.f $G(\cdot|k)$ decreases with k, a result that parallels Lemma 2.1-(i). A difficulty is that this may not hold for $G_c(\cdot|k)$. Ordering the competitive buyer and cartel bid distributions $G(\cdot|k)$ and $G_c(\cdot|k)$, as in Proposition 4.1-(ii), is shown to be sufficient for identification. Proposition 4.1-(iii) will be used to identify the cartel size distribution through discontinuities in the winning bid distribution. Proposition 4.1-(iv) allows for the identification of n and $\underline{k}$ using lower tail features.

Testability of Assumption E. A first application of Proposition 4.1 is to show that Assumption E is testable. In addition, Proposition 4.2 shows that permanent cartel members can be identified under some reasonable assumptions on the cartel size and cartel membership distributions.

Proposition 4.2 *Suppose Assumptions K, with $\pi_1, \pi_{\underline{k}}, \ldots, \pi_{\overline{k}} > 0$, and CIPV hold and that the cartel membership probabilities $\gamma_{i|k}^c$ are such that*

$$\text{either } \gamma_{i|\underline{k}}^c = \cdots = \gamma_{i|\overline{k}}^c = 1 \text{ or } \gamma_{i|\underline{k}}^c, \ldots, \gamma_{i|\overline{k}}^c < 1 \text{ for all } i \text{ in } \mathcal{E}.$$

Then, presence of permanent cartel members e in $\mathcal{E}$, as well as their identities, are identified.

Proposition 4.2 holds assuming that the cartel is absent in some auctions, and that the cartel size can take all successive values between $\underline{k}$ and $\overline{k}$, a condition comparable to Assumption N used for the baseline model. The restrictions on the cartel membership probabilities naturally hold when the cartel size K can only take one value, i.e. $\underline{k} = \overline{k}$. If $\underline{k} < \overline{k}$, it implies that a buyer cannot be a permanent cartel member for some cartel size and not for others. The proof of Proposition 4.2 makes use of p.d.f discontinuities and lower tail behaviors of the unconditional winning bid distribution and of the winning bid distribution given that a buyer j wins, as permitted by Proposition 4.1-(iii,iv).

4.3 Identification results

Tail identification of n and $\underline{k}$. The Econometrician first identifies the unconditional winning bid distribution

$$\begin{aligned} G_c(b) = \mathbb{P}\left(\max_{1\le i\le n} B_i \le n\right) &= \mathbb{P}\left(\max_{1\le i\le n} B_i \le n \,\middle|\, K=1\right)\mathbb{P}(K=1) \\ &\quad + \sum_{k=\underline{k}}^{\overline{k}} \mathbb{P}\left(\max_{1\le i\le n} B_i \le n \,\middle|\, K=k\right)\mathbb{P}(K=k) \\ &= \pi_1 G^{n-1}(b|1)G_c(b|1) + \sum_{k=\underline{k}}^{\overline{k}} \pi_k G^{n-1}(b|k)G_c^k(b|k). \end{aligned} \tag{24}$$

A first key difficulty for identification is that $G_c(\cdot)$ involves a product of powers of two different bid c.d.f, which cannot be easily disentangled, contrasting with the winning bid c.d.f (1) of the baseline model. Second, while the expression above suggests that n can be identified via the lower tail behavior of $G_c(\cdot)$ as $\underline{n}$ in Lemma 2.1-(iii), it is also necessary to identify $\underline{k}$. This is permitted by Assumption E as explained below.

Let $g(\cdot|k)$ and $g_c(\cdot|k)$ be the derivatives of $G(\cdot|k)$ and $G_c(\cdot|k)$, respectively. Given $K=k$, the probability that a bid B_j less than b, by a buyer j who is either competitive or a cartel member, wins the auction is:

$$\int_{\underline{b}}^{b} G^{n-2}(t|k)G_c^k(t|k)g(t|k)dt = G_{enc}(b|k) \text{ if } j \text{ is a competive bidder,}$$
$$\int_{\underline{b}}^{b} G^{n-1}(t|k)G_c^{k-1}(t|k)g_c(t|k)dt = G_{ec}(b|k) \text{ if } j \text{ is a cartel member,}$$

where the last expression takes into account the fact that j must be the cartel leader to win the auction as a cartel member.[7] Hence, the probability $G_{ec}(b)$ that a bid b made by the permanent cartel member e is

$$G_{ec}(b) = \mathbb{P}\left(\max_{1\leq i\neq e\leq n} B_i\right) = \sum_{k=\underline{k}}^{\bar{k}} \pi_k G_{ec}(b|k). \tag{25}$$

Now (24) and (25), Proposition 4.1 gives, when b decreases to $\underline{b}$,

$$G_c(b) = \pi_1 \left(\frac{n}{n-1} f(\underline{v})\right)^n (b-\underline{b})^n (1+o(1)),$$
$$G_{ec}(b) = \pi_{\underline{k}} \left(\frac{n}{n-1} f(\underline{v})\right)^{n-1} \left(\frac{n+\underline{k}-1}{n+\underline{k}-2} f(\underline{v})\right)^{\underline{k}} \frac{(b-\underline{b})^{n+\underline{k}-1}}{n+\underline{k}-1} (1+o(1)),$$

as $\pi_{\underline{k}} > 0$ by Assumption K and assuming $\pi_1 > 0$ as in Proposition 4.2. Hence, the lower tail behaviors of $G_c(\cdot)$ and $G_{ec}(\cdot)$ identify n and $\underline{k}$.

Cartel size distribution. Discontinuities in the derivatives $g_c(\cdot)$ and $g_{ec}(\cdot)$ of $G_c(\cdot)$ and $G_{ec}(\cdot)$, respectively, identify π_k's as shown below.

The derivatives $g_c(\cdot)$ and $g_{ec}(\cdot)$ are given by

$$g_c(b) = \pi_1 G^{n-2}(b|1)\left((n-1)G_c(b|1)g(b|1) + G(b|1)g_c(b|1)\right)$$
$$+ \sum_{k=\underline{k}}^{\bar{k}} \pi_k G^{n-2}(b|k) G_c^{k-1}(b|k)\left((n-1)G_c(b|k)g(b|k) + G(b|k)g_c(b|k)\right),$$
$$g_{ec}(b) = \sum_{k=\underline{k}}^{\bar{k}} \pi_k G^{n-1}(b|k) G_c^{k-1}(b|k) g_c(b|k),$$

[7] This follows from

$$\mathbb{P}\left(\max_{1\leq i\leq n} B_i \leq b,\ j \text{ wins}\middle| K=k\right) = \mathbb{P}\left(\max_{i\notin\mathcal{C}} B_i \leq \xi_c^{-1}(V_j|k) \leq b,\ \max_{m\neq j\in\mathcal{C}} V_m \leq V_j \middle| K=k\right)$$
$$= \mathbb{P}\left(\max\left\{\max_{i\notin\mathcal{C}} B_i,\ \max_{m\neq j\in\mathcal{C}} \xi_c^{-1}(V_m|k)\right\} \leq \xi_c^{-1}(V_j|k) \leq b \middle| K=k\right)$$
$$= \int_{\underline{b}}^{b} G^{n-1}(t|k) G_c^{k-1}(t|k) g_c(t|k) dt.$$

which are continuous over $[\underline{b}, \overline{b}_{\overline{k}}]$, except at those $\overline{b}_k$ such that π_k is positive. The potential discontinuities

$$\Delta_k^{\ell} = \lim_{t \downarrow 0} \left(g_{\ell}(\overline{b}_k - t) - g_{\ell}(\overline{b}_k + t) \right), \quad \ell = c, ec$$

are therefore

$$\Delta_k^c = \pi_k \frac{n}{n-1} \frac{1}{\overline{v} - \overline{b}_k} \geq 0, \quad k = 1, \underline{k}, \ldots, \overline{k}, \tag{26}$$

$$\Delta_k^{ec} = \pi_k \frac{1}{k(n-1)} \frac{1}{\overline{v} - \overline{b}_k} \geq 0, \quad k = \underline{k}, \ldots, \overline{k}. \tag{27}$$

since $(n-1)g(\overline{b}_k|k) + kg_c(\overline{b}_k|k) = \frac{n}{n-1}\frac{1}{\overline{v}-\overline{b}_k}$ by Proposition 4.1-(iii). The ratio Δ_k^{ec}/Δ_k^c identifies those k with $\pi_k > 0$, that is the support $\mathcal{K}$ of K, the corresponding discontinuity locations $\overline{b}_k$ being also identified. The largest private value can then be recovered through

$$\overline{v} = \frac{\frac{n}{n-1} + \sum_{k \in \mathcal{K}} \Delta_k^c \overline{b}_k}{\sum_{k \in \mathcal{K}} \Delta_k^c} = \frac{\frac{1}{n-1} + \sum_{k \in \mathcal{K}} k \Delta_k^{ec} \overline{b}_k}{\sum_{k \in \mathcal{K}} k \Delta_k^{ec}},$$

observing that the denominators do not vanish as discontinuities are nonnegative and strictly positive at $\overline{b}_{\underline{k}}$ and $\overline{b}_{\overline{k}}$. Then, (26) or (27) allow for the recovery of π_k for k in $\mathcal{K}$, so that the cartel size distribution is identified.

Private value distribution. The probability functions $G_c(\cdot)$ and $G_{ec}(\cdot)$ allows for the recovery of the cartel leader and competitive buyers bid distribution $G_c(\cdot|\overline{k})$ and $G(\cdot|\overline{k})$ in the vicinity of $\overline{b}_{\overline{k}}$. It indeed holds, assuming $\pi_{\overline{k}}$ has been preliminarily identified,

$$\frac{g_{ec}(b)}{G_c(b) - (1 - \pi_{\overline{k}})} = \frac{g_c(b|\overline{k})}{G_c(b|\overline{k})} \text{ for } b \text{ in } (\overline{b}_{\overline{k}-1}, \overline{b}_{\overline{k}}], \tag{28}$$

and integrating yields $G_c(\cdot|\overline{k})$. Given that n, $\overline{k}$, and $\pi_{\overline{k}}$ have been identified, the ratio

$$\left(\frac{G_c(b)}{\pi_{\overline{k}} G_c^{\overline{k}}(b|\overline{k})} \right)^{1/(n-1)}$$

identifies $G(\cdot|\overline{k})$ in a vicinity of $\overline{b}_{\overline{k}}$. This is sufficient to start an iterative identification process as the one illustrated by Figure 3 for the baseline model.

The next result parallels Theorem 2.1 and recasts the identification results stated above. As permitted by Proposition 4.2, Theorem 4.1 assumes that $G_{ec}(\cdot)$ is identified.

Theorem 4.1 *Suppose Assumptions E, CIPV, and K with $\pi_1 > 0$ hold, and that $G_{ec}(\cdot)$ is identified. Then, the number of active buyers n, the cartel size, and private value distributions, $\mathbb{P}(K = \cdot)$ and $F(\cdot)$, are identified.*

A possible extension would be to assume that the cartel members draw their private values from a specific c.d.f $F_c(\cdot)$, in which case identification would hold provided the cartel inverse strategies $G_c(\cdot|k)$ decrease with k. If it does not hold, our approach may deliver identification of $F_c(\cdot)$ only over a subset of its support, because the counterpart of the sequence $\{\alpha_k\}$ of Figure 3 would converge to the first crossing location of the $G_c^{-1}(\cdot|k)$ and not to 0.

Cartel membership distribution. The next proposition identifies the proportion of auctions in which a given buyer appears as a cartel member or a competitive bidder.

Proposition 4.3 *Suppose Assumptions E, CIPV, and K hold, and that $G_{ec}(\cdot)$ is identified. Then, for each buyer i, the probabilities $\gamma^c_{i|k}$ and $\gamma^{nc}_{i|k}$ that i enters the auction as a cartel member or as a competitive bidder are identified for $k = \underline{k}, \ldots, \overline{k}$ with $\pi_k > 0$.*

Proposition 4.3 is specific to our framework, which allows for switching cartel participation. As analyzing the cartel participation probabilities $\gamma^c_{i|k}$ across buyers can help assess the prevalence of collusion, it is important to stress the constructive aspects of the proof of this result. Let $G_i(b) = \mathbb{P}\left(\max_{1\leq j\neq i\leq n} B_j \leq B_i \leq b\right)$ be the probability that the winning bid is below b when i wins the auction,

$$G_i(b) = \pi_1 \gamma_{i|1}^{nc} G_{enc}(b|1) + \sum_{k=\underline{k}}^{\overline{k}} \pi_k \left(\gamma_{i|k}^{nc} G_{enc}(b|k) + \gamma_{i|k}^{c} G_{ec}(b|k) \right),$$

which is identified as the Econometrician observes the winning bid and the identity of the winner. By Theorem 4.1, the probability functions $G_{ec}(\cdot|\cdot)$ and $G_{enc}(\cdot|\cdot)$ as well as π_k are identified. Regressing $G_i(\cdot)$ on $\pi_k G_{enc}(\cdot|k)$ and $\pi_k G_{ec}(\cdot|k)$ then identifies the probabilities $\gamma_{i|k}^{nc}$ and $\gamma_{i|k}^{c}$.

The probability that i belongs to the cartel $\gamma_i^c = \sum_{k=\underline{k}}^{\overline{k}} \pi_k \gamma_{i|k}^c$, or the probability that i wins the auction as a cartel member

$$\Gamma_i^c = \sum_{k=\underline{k}}^{\overline{k}} \pi_k \gamma_{i|k}^c G_{ec}(\overline{b}_k|k)$$

can be used to analyze how strongly buyer i is involved in collusion.

5 Participation in USFS timber auctions

Assuming that the observed bids do not correspond to effective ones leads us to introduce an additional parameter in the standard first-price auction model, the participation or competition distribution. As shown in Bulow and Klemperer (1996), participation is an important component of the auction design, so that estimating participation is an econometric issue of interest. As argued below, this can be done in principle following our identification results and without estimating the private value distribution. By contrast, the latter is identified using an iterative argument, which is difficult to implement at this stage. Developing related estimation techniques is out of the scope of the present paper.

A goal of this section is therefore to give first indications on the components of participation that can be recovered with existing statistical methods from a first-price auction sample of moderate size. This is done using the USFS timber data of Lu and Perrigne (2008), and a simulation experiment reported in Appendix A using first-price auctions with two or three bidders and uniform private values. Athey, Levin and Seira (2011) have analyzed buyer competitiveness in timber ascending auctions, using estimates from first-price ones which assume competitive bidding. Our participation analysis therefore differs, as it questions buyer competitiveness in first-price auctions.

A formal description of the statistical techniques used in the participation analysis can be found in Appendix A. Hill and Shneyerov (2013) have pioneered the application of tail analysis in the econometrics of auctions, and we rely on their insights to estimate the lowest number of effective bidders $\underline{n}$. Its highest number, $\overline{n}$, can then be derived counting discontinuities, while the participation distribution, which is the distribution of the number of bidders contributing to the winning bid N, can be recovered via (10), discontinuity locations, and corresponding jumps estimates. While most econometric applications using discontinuity analysis consider known jump locations, as reviewed in Jales and Yu (2017), few statistical works have considered unknown locations in a nonparametric contexts. Gayraud (2002) shows that the optimal nonparametric rate to estimate location discontinuities coincides with the parametric one, given by the inverse of the number of observations. This establishes the near optimality of the nonparametric approach proposed in Chu and Cheng (1996), who also consider nonparametric estimation of the corresponding discontinuity jumps. See also Oudshoorn (1998) and the references listed in these papers. We rely on a nearest-neighbor version of Chu and Cheng (1996).

5.1 The data

The Lu and Perrigne (2008) USFS Timber first-price auction data contain 107 two-bid and 108 three-bid auctions, and report the appraisal value and timber volume of each auctioned lot. See Lu and Perrigne (2008) for further information on this dataset. Since the literature argues that the USFS reserve prices are too low, we likewise assume they are nonbinding.

Homogenized bids. Dealing with covariates is difficult in nonparametric techniques. The homogenized bid approach proposed by Haile, Hong and Shum (2003) assumes a regression model with i.i.d. errors for the private values. As a consequence, the private value quantile function is $V(\alpha|x) = x\theta + v(\alpha)$, where x stands for volume and appraisal value, and $v(\cdot)$ is the quantile function of the error term plus the constant term. Then, (3) gives that the bid quantile function takes the form $B_n(\alpha|x) = x\theta + b_n(\alpha)$, where $b_n(\alpha) = \frac{n-1}{\alpha^{n-1}} \int_0^\alpha t^{n-2} v(t) dt$. It follows that, for a given number of bidders n, the bids and the winning bids satisfy a regression with i.i.d. errors, where the covariate slope equals the one of the private value model, with $b_n(\cdot)$ capturing variation in competition. The next table displays the results of separate homogenized bid regressions for $N = 2$ and $N = 3$, using the bid observations as a dependent variable. Standard deviations are calculated using the standard variance OLS formula, as permitted by the specification.

	Bids			Winning bids		
	$N = 2$	$N = 3$	$\theta_3 - \theta_2$	$N = 2$	$N = 3$	$\theta_3 - \theta_2$
Intercept	-1.07 (8.46)	-20.79 (9.06)	-19.72 (12.40)	4.34 (13.59)	-13.99 (17.18)	-18.33 (21.91)
(Log)Volume	4.06 (1.25)	7.10 (1.24)	3.03 (1.76)	4.18 (2.01)	8.09 (2.35)	3.90 (3.09)
Appraisal Value	1.01 (.038)	1.15 (.042)	0.14 (.058)	1.03 (.062)	1.22 (.081)	0.19 (.102)

Table 1: Homogenized bid slope estimates for $N = 2$ and $N = 3$, with standard deviations in brackets.

Table 1 reveals that the appraisal value slope significantly increases with participation, using either the bids and the winning bids. A chi-squared test similarly gives a small p-value of 0.03 for the null hypothesis of joint (log)volume and appraisal value bid slope constancy. Using volume in levels instead of natural logarithm gives less but still significant variation of the appraisal value coefficient. While the intercept, which is equal to $\int_0^1 b_n(\alpha)d\alpha$ for the bid regression, should increase with n, its estimation is decreasing in a statistically significant way when using bids. The Table 1 dependence of the appraisal value slope on participation can be a consequence of homogenized bid misspecification that, as discussed at the end of Section 3.3, can affect discontinuity detections.[8] We therefore adopt a nonparametric approach, which relies on a covariate partition.

Covariate-quality partition. In principle, discontinuities of the conditional winning p.d.f $g(\cdot|x)$ can be analyzed using observations with covariate X_ℓ close to x, where $\ell = 1, \ldots, L$ is the auction index. In view of our small sample size, we consider three subsamples labeled "Low", "Medium," and "High" defined as follows:

- Low: auctions with appraisal value and volume both smaller or equal to their respective median values (45 auctions, among which 23 have two bidders);

- Medium: auctions with appraisal value and volume strictly above their 25% respective quantiles and below or equal to their 75% ones (53 auctions, among which 28 have two bidders);

- High: auctions with covariates strictly above their median values (44 auctions, among which 20 have two bidders).

[8] This also holds if the model is correctly specified, as estimation can give a parametric rate $1/\sqrt{L}$, where L is the sample size, for estimating jump locations. By contrast, grouping observations in bins of size h may give a better rate $1/(Lh)$ if $1/h = o(\sqrt{L})$, provided the jump location does not vary too much across each bin.

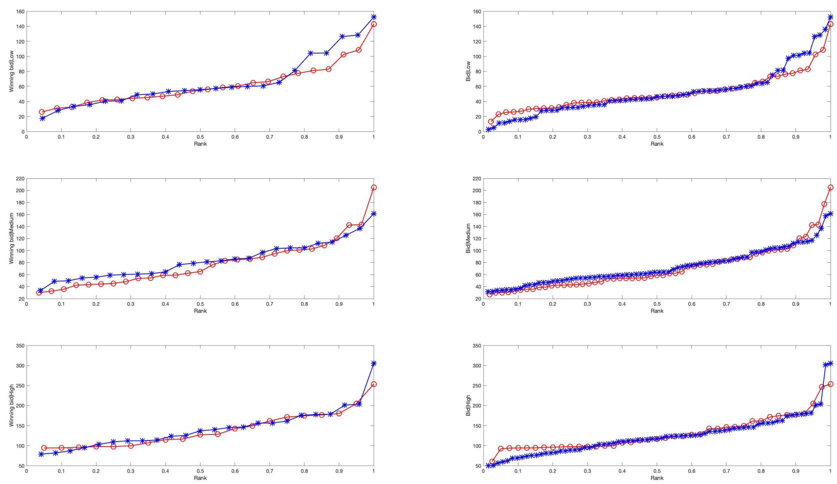


Figure 4: Sample quantile functions for winning bids (left) and bids (right) given $N = 2$ (red) and $N = 3$ (blue). Low quality subsample on top, medium in the middle and high on the bottom.

Figure 4 details various sample quantile functions given competition across these subsamples. At first sight, the winning bid and bid sample quantile functions for $N = 2$ and $N = 3$ look similar, especially in their central part. A more formal Kolmogorov-Smirnov test rejects at the 10% level the null hypothesis that bids for $N = 2$ and $N = 3$ are drawn from the same distribution for high-quality auctions. For the other subsamples, the null is not rejected at all standard statistical levels.[9] For all subsamples, the Kolmogorov-Smirnov test does not reject the null that winning bids for $N = 2$ and $N = 3$ are drawn from the same distribution at all usual

[9] The Kolmogorov-Smirnov test may have low power in small samples, so that related inference may contradict the conclusion drawn from Table 1, which shows a difference between the bid distributions when $N = 2$ and $N = 3$. Participation analysis below suggests that although these distributions should be close, some differences may exist as only two buyers may be active in most but not all three bid auctions.

statistical levels. For the low-quality and high-quality subsamples, the upper parts of the quantile functions given $N = 3$ are above the ones for $N = 2$, as predicted by the theory. But the lower portion of the bid quantile functions with $N = 2$ are above the ones for $N = 3$ for the same subsamples, suggesting that dominated bidding may take place when $N = 3$.

Figure 4 allows for the recovery of the largest bid or winning bid for $N = 2$ and $N = 3$. Increasing N increases the largest bid and winning bid in both the high-quality sample and, to a lesser extent, the low-quality sample. However, increasing N decreases the largest bid and winning bid in the medium-quality sample. This may be due to not only a lack of observations in the upper tail but also possible dependence of private values and N, which may also cause the significant appraisal value slope difference in Table 1.

Participation and covariates. To maintain a reasonable number of winning bid observations, we will sometimes merge the samples with $N = 2$ and $N = 3$, but our main object of interest is the participation distribution given that three bids are observed. In the rest of the application, we allow for the participation distribution to depend upon volume and appraisal value, and will estimate the conditional distribution of N given the level of quality considered above. However, we assume that the lowest number of bidders $\underline{n}$ is independent of the covariate. As the nonparametric method we use for discontinuities is based on the winning bid distribution given quality, the upper number of bidders $\overline{n}$ could, in principle, depend on the covariate.

5.2 Lowest number of bidders and discontinuities

Estimation of the lowest number of bidders $\underline{n}$. A semiparametric estimation of the lowest number of bidders directly relies on Lemma 2.1-(iii), which relates $\underline{n}$

to the lower tail behavior of the winning bid c.d.f $G(\cdot)$. In the vicinity of the lowest bid $\underline{b}(X)$, $G(b|X) = g_{\underline{n}}^{\underline{n}}(0|X)\,(b - \underline{b}(X))^{\underline{n}}\,(1 + o(1))$, so that $\underline{n}$ can be estimated using tail-index methods. Assuming that $\underline{b}(\cdot)$ is known and following Hill and Shneyerov (2013) suggest estimating $\underline{n}$ using the integer part $\widehat{\underline{n}}$ of $\widetilde{\underline{n}}$ with

$$\frac{1}{\widetilde{\underline{n}}} = \ln W^{\dagger}_{(M)} - \frac{1}{M-1}\sum_{m=2}^{M} \ln W^{\dagger}_{(m)}, \quad M = M_L = o(L) \geq 2 \text{ with } W^{\dagger}_{\ell} = \frac{W_{\ell}}{\underline{b}(X_{\ell})} - 1, \tag{29}$$

and where $W^{\dagger}_{(m)}$ is the m-th $W^{\dagger}_{\ell}$ taken in ascending order.

However, the lower boundary $\underline{b}(\cdot)$ is unknown. Several estimation strategies can be used to tackle the dependence. First, $\underline{b}(X_{\ell})$ can be replaced by its minimum over the sample, which can be estimated using the minimum bid. It amounts to consider the covariate as unobserved heterogeneity, as in Lemma 3.2. Under additional conditions including Assumption UAH, the unconditional winning bid c.d.f behaves as $(b-\underline{b})^{\underline{n}+1}$, as shown in Lemma 3.2-(i) under additivity of unobserved heterogeneity. In this approach, the estimator $\widehat{\underline{n}}$ should be redefined as the integer part of $\widetilde{\underline{n}} - 1$, using the minimum bid instead of $\underline{b}$ in (29). Alternatively, $\underline{b}$ can be replaced by a nonparametric estimator of $\underline{b}(X_{\ell})$ based upon bids, as defined in the Appendix.[10]

The two Hill estimation procedures give coherent results. In the two-bidder sample, the lowest number of bidders $\underline{n}$ estimated is 2. For the three bidder-sample, the estimated $\underline{n}$ is again 2, suggesting that the observed number of bids may differ across auctions from the number of active bidders.

Discontinuities and winning bid p.d.f. Although winning bids have been grouped in quality bins, it is still possible to detect discontinuities, as easily guessed from the

[10] As our focus is on participation, bids can be used to estimate $\underline{b}$ or $\underline{b}(\cdot)$, provided the competitive bids share a common lower support bound with non competitive ones. This allows for the estimation of these parameters with superefficient rates, which can be used to show consistency of the considered Hill procedure.

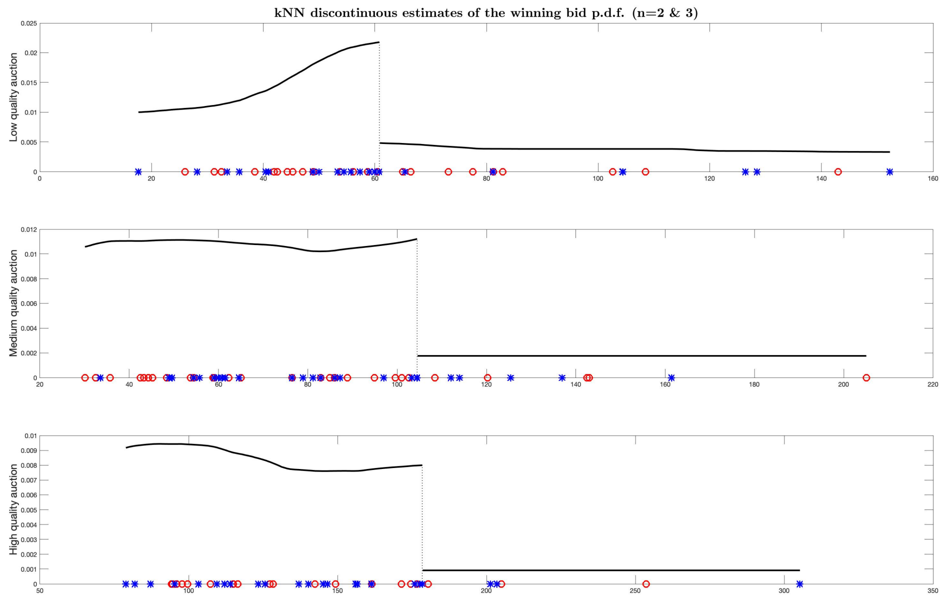


Figure 5: k-NN winning-bid density estimator with $N = 2$ and 3, for the quality subsamples 'Low' (top), 'Medium' (middle), 'High' (bottom). Marks on the x-axis indicate observations with $N = 2$ (red) and $N = 3$ (blue).

case where the jump locations are independent of the covariates. If the jump locations do not vary too much across a bin, the winning bid density will exhibit sharp changes around these locations, which will be considered as jumps by most discontinuity detection procedures as Chu and Cheng (1996). Indeed, jumps are detected when there is a statistically significant difference between left-hand and right-hand side kernel estimators. Because these one-sided density estimators use smoothing, their difference will be high, even in the case of smooth but sharp variation in the underlying density.

The appendix describes the Nearest-Neighbor procedure applied to detect discontinuities in the winning bid p.d.f. In short, difference in k-NN density estimators using observations below and above the considered b is used to estimate a potential discontinuity at b. When this difference falls outside a uniform confidence band obtained under the null of a continuous p.d.f, a discontinuity is detected. In the simulation experiment, detecting such interior discontinuity only occurs with a high probability $\mathbb{P}(N = 2)$.

A winning bid p.d.f interior discontinuity is detected for all quality subsamples with $N = 3$, or when merging $N = 3$ and $N = 2$.[11] This reinforces the possibility of dominated bidding in three-bidder auctions.

The discontinuity detection procedure also delivers an estimated discontinuity location which can be used to compute a discontinuous nearest-neighbor winning-bid density estimator, as detailed in Appendix A. See Figure 5 for the results using the quality subsamples merging $N = 2$ and $N = 3$. Discontinuities look substantial for all cases, suggesting a high $\mathbb{P}(N = 2)$.

Figure 5 positions winning bids with $N = 2$ (red) and $N = 3$ (blue) on the x-axis. Some winning bids with $N = 2$ are above the discontinuity location, which is an estimation of the highest bid given $N = 2$. Hence, most winning bids with $N = 2$ should be above the estimated discontinuity location. Simulations reveal that the corresponding estimator is downward biased, so that observing some winning bids with $N = 2$ above the estimated discontinuity location is possible. It could also be the consequence of dependence of the private value distribution on participation.

5.3 Participation distribution

The simulation experiment of the Appendix suggests that the estimation of $\mathbb{P}(N = 2)$ obtained from (10) by plugging in estimated discontinuity locations and corresponding

[11] No interior discontinuities are found when $N = 2$, except maybe in the low-quality subsample for which the evidence is borderline.

	Two-bid and three-bid samples			Three-bid sample		
	Low	Medium	High	Low	Medium	High
From W	.95	.90	.92	.98	.75	.91
Observed	.51	.53	.45	-	-	-

Table 2: Estimated $\mathbb{P}(N = 2|A)$, $A =$'Low', 'Medium', and 'High' using both two-bid and three-bid auctions (left) and only three-bid ones (right).

jumps can be unreliable. However, the ones obtained here are consistent with Figure 4, which suggests that higher participation might not affect the winning bid distribution.

The three estimates are high and differ from the proportion of auctions with two bidders, which are around 0.5. Restricting to auctions with three observed bids gives similar results as also reported in Table 2. The high probabilities obtained for the event $N = 2$ in all subsamples suggest that most auctions with three bids have in fact only two active bidders.

Further analysis. Figure 6 represents the bids of the three-bid samples, plotting on the y-axis the winning bid $B_{(3)}$, middle bid $B_{(2)}$ and losing bid $B_{(1)}$ for each auction, represented by $B_{(3)}$ on the x-axis. In each subsample, the middle bid is always very close to the winning or losing bids, suggesting some joint bidding. Chassang et al. (2021) and Imhof et al. (2018) report that the winning bid is often isolated in auctions suspected to be rigged, as observed more specifically here in the low-quality subsample.

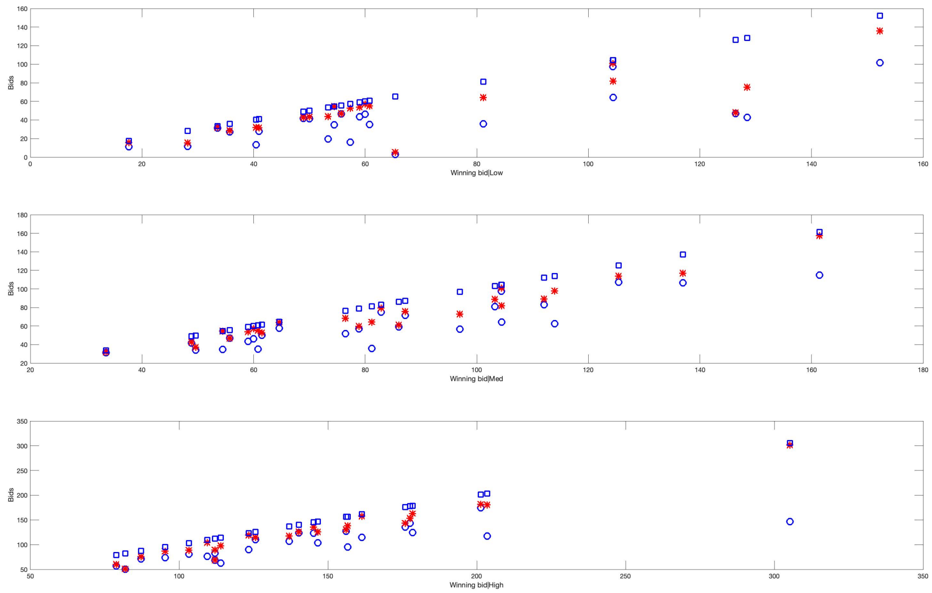


Figure 6: Three-bid auctions given quality levels, low (top), medium (middle) and high (bottom). Auctions are represented by the winning bid on the x-axis, and the associated highest bid (blue square), second-highest bid (red star) and smallest bid (blue circle) on the y-axis.

6 Final remarks

This paper shows that, under the independent symmetric private value paradigm, the first-price winning bid is sufficient to identify model primitives when competition is observed by the buyers but not the econometrician. This is suitable in the presence of phantom bids or when the number of observed bids does not reflect participation. To some extent, buyers can be uncertain about their competitors and auction-specific unobserved heterogeneity can be present. Assuming that winning bids and winner identities are observed may allow for the identification of asymmetric specification, as exemplified here with a varying-size cartel model.

Our theoretical results shed light on new identification arguments for discrete mixture models, which are widely used in economic applications, in particular when unobserved heterogeneity is plausible. In our model, the mixture components are generated by the same function. The components are ordered according first-order stochastic dominance and their supports are nested. These two features may appear in other relevant economic mixtures. See, for instance, An (2017), who studies non equilibrium bids from heterogeneous agents whose beliefs follow from level-k thinking, and where k is unobserved. A key ingredient is that these support components can be identified here through discontinuities of the mixture p.d.f, but many other characteristics can be used for such a purpose. How essential these features are for identification can also be of interest for future extensions.

Inference issues are mostly out of the scope of the present paper. Tail and discontinuity detection methods perform reasonably well in small samples for estimating the support of the distribution of the number of active bidders, as shown in Appendix A. However, they are far less successful regarding estimation of its distribution. Table 3 in Appendix A reports some simulation results suggesting that estimating interior jump locations can be more difficult than previously thought. As jump locations are generated by the same private value distribution and the last one seems to be well estimated, using full estimation methods may provide better results by imposing constraints across discontinuities. For instance, using a sieve specification for the private value distribution as in Grundl and Yu (2018), nonparametric maximum likelihood estimation can be used to estimate a mixture model incorporating participation probabilities. However, this can be numerically involved, and a nonparametric approach may not be appropriate for small samples. Previous attempts for standard auction models, such as Hirano and Porter (2003) and Chernozhukov and Hong (2004) have preferred Bayesian approaches, for which numerical issues can be tackled using simulation methods. Extending general results by Ibragimov and Has'minskii (1981), Hirano and Porter (2003) have established that Bayesian methods can be more efficient when estimating parametric auction models than standard likelihood estimation. Aryal, Charankevich, Jeong and Kim (2021) have recently developed Bayesian algorithms that can tackle asymmetry in first-price auctions. This approach looks reliable and sufficiently flexible for closing the gap between applications and our identification results.

7 Proofs

7.1 Proof of Proposition 2.1

It remains to be shown that (i) and (ii) are sufficient. The function $V(\cdot)$ in (ii) is a quantile function associated with a c.d.f $F(\cdot)$ satisfying the requirements of Assumption IPV, while the mixture weights p_n define a distribution for N, as in Assumption N. These $\{p_n, \underline{n} \leq n \leq \overline{n}\}$ and private value quantile function $V(\cdot)$ generate a distribution for N and best response bidding strategy functions $B_n(\cdot)$ by (3), with $G(\cdot)$ as a winning bid c.d.f $\square$

7.2 Proof of Corollary 2.1

The compatibility conditions imply that (3) holds, and integrating by parts gives

$$B_n(\alpha) = \frac{1}{\alpha^{n-1}} \int_0^\alpha V(t)\, d\left[t^{n-1}\right] = V(\alpha) - \int_0^\alpha \left(\frac{t}{\alpha}\right)^{n-1} V^{(1)}(t)\, dt.$$

Hence, $\overline{b}_n = \overline{v} - \int_0^\alpha t^{n-1} V^{(1)}(t)\, dt < \overline{v}$ as $V^{(1)}(\cdot) > 0$. Note that this also gives $B_n(\alpha) < V(\alpha)$ for all $\alpha > 0$, and then $B^{(1)}(\alpha) > 0$ by (2). When α goes to 0, the following holds

$$\begin{aligned} B_n(\alpha) &= V(0) + V^{(1)}(0)\alpha + o(\alpha) - \int_0^\alpha \left(\frac{t}{\alpha}\right)^{n-1} \left(V^{(1)}(0) + o(1)\right) dt \\ &= V(0) + \frac{n-1}{n} V^{(1)}(0)\alpha + o(\alpha), \end{aligned}$$

which shows that $B_n^{(1)}(0) = \frac{n-1}{n} V^{(1)}(0)$. As $B_n^{(1)}(\cdot) > 0$, the conditional bid p.d.f $g_n(\cdot)$ satisfies

$$g_n(b) = \frac{1}{B_n^{(1)}(G_n(b))} \text{ for all } b \in \left[\underline{v}, \overline{b}_n,\right]. \tag{30}$$

Hence, $g_n(\underline{v}) = 1/B_n^{(1)}(0) = \frac{n}{n-1} 1/V^{(1)}(0) = \frac{n}{n-1} f(\underline{v})$, which is (6). For (7), (2) and (30) give

$$g_n\left(\overline{b}_n\right) = \frac{G_n\left(\overline{b}_n\right)}{(n-1)\left(V\left(G_n\left(\overline{b}_n\right)\right) - \overline{b}_n\right)} = \frac{1}{(n-1)\left(\overline{v} - \overline{b}_n\right)}$$

as $G_n\left(\overline{b}_n\right) = 1$, so that (7) holds. □

7.3 Proof of Lemma 2.1

As

$$B_n(\alpha) = \frac{1}{\alpha^{n-1}} \int_0^\alpha V(t)\, d\left[t^{n-1}\right] = V(\alpha) - \int_0^\alpha \left(\frac{t}{\alpha}\right)^{n-1} V^{(1)}(t)\, dt,$$

differentiating with respect to n gives

$$\frac{\partial B_n(\alpha)}{\partial n} = -\int_0^\alpha \left(\frac{t}{\alpha}\right)^{n-1} \log\left(\frac{t}{\alpha}\right) V^{(1)}(t)\, dt \geq 0.$$

The inequality is strict except when $\alpha = 0$, in which case $B_n(0) = \underline{v}$ for all n. It follows that the bid c.d.f given that $N = n$, $G_n(\cdot)$, has a support $[\underline{v}, \overline{b}_n]$, with an upper bound $\overline{b}_n = B_n(1)$, which is strictly increasing with respect to n and strictly smaller than $\overline{v} = \lim_{n\uparrow\infty} \overline{b}_n$. Hence, this proves Part (i). For part (ii), the expression for jumps (8) follows from (5), which shows that the winning bid p.d.f is

$$g(b) = \sum_{k=\underline{n}}^{\overline{n}} p_k k G_k^{k-1}(b)\, g_k(b), \tag{31}$$

with $g_k(b) = 0$ for $b > \overline{b}_n$ when $k \leq n$ by Lemma 2.1-(i). This gives

$$\begin{aligned}
&g(\overline{b}_n - t) - g(\overline{b}_n + t) \\
&= \sum_{k=n}^{\overline{n}} p_k k G_k^{k-1}\left(\overline{b}_n - t\right) g_k\left(\overline{b}_n - t\right) - \sum_{k=n+1}^{\overline{n}} p_k k G_k^{k-1}\left(\overline{b}_n + t\right) g_k\left(\overline{b}_n + t\right) \\
&\to n p_n g_n\left(\overline{b}_n\right) = \Delta_n,
\end{aligned}$$

when t goes to 0. The equality (7) for $g_n\left(\overline{b}_n\right)$ then gives (8). For part (iii), continuity of $B_n^{(1)}(\cdot)$, which is bounded away from 0 and infinity, and (30) shows that $g_n(\cdot)$ is continuous with $g_n(\underline{v}) > 0$ by (6). When t goes to 0, this gives

$$\begin{aligned}
G\left(\underline{v} + t\right) &= \sum_{n=\underline{n}}^{\overline{n}} p_n \left(\int_{\underline{v}}^{\underline{v}+t} g_n(u) du\right)^n = \sum_{n=\underline{n}}^{\overline{n}} p_n g_n^n(\underline{v}) t^n \left(1 + o(1)\right) \\
&= p_{\underline{n}} g_{\underline{n}}^{\underline{n}}(\underline{v}) t^{\underline{n}} \left(1 + o(1)\right),
\end{aligned}$$

as $p_{\underline{n}} g_{\underline{n}}^{\underline{n}}(\underline{v}) > 0$, which implies $\underline{n} = \lim_{t \downarrow 0} \frac{\log G(\underline{v}+t)}{\log t}$. □

7.4 Proof of Theorem 2.1

We now obtain identification of the value quantile function by iteratively exploiting the two equilibrium mappings in (2) and (4). We proceed in three steps:

Step 1. Note that the winning bid distribution satisfies

$$G\left(b\right) = 1 - p_{\overline{n}} + p_{\overline{n}} G_{\overline{n}}^{\overline{n}}\left(b\right) \text{ for all } b \text{ in } \left[\overline{b}_{\overline{n}-1}, \overline{b}_{\overline{n}}\right]$$

so that $G_{\overline{n}}\left(\cdot\right)$ is identified over $\left[\overline{b}_{\overline{n}-1}, \overline{b}_{\overline{n}}\right]$ as follows:

$$G_{\overline{n}}\left(b\right) = \left(\frac{G\left(b\right) - \left(1 - p_{\overline{n}}\right)}{p_{\overline{n}}}\right)^{1/\overline{n}} \text{ for } b \text{ in } \left[\overline{b}_{\overline{n}-1}, \overline{b}_{\overline{n}}\right].$$

Set

$$\alpha_1 = G_{\overline{n}}\left(\overline{b}_{\overline{n}-1}\right).$$

It follows that $B_{\overline{n}}(\cdot)$ is identified on $[\alpha_1, 1]$, i.e.,

$$B_{\overline{n}}(\alpha) = W\left[(1-p_{\overline{n}}) + p_{\overline{n}}\alpha^{\overline{n}}\right],$$

where $W(\cdot) = G^{-1}(\cdot)$ is the winning bid quantile function.

Using the mapping from the bid quantile function to the value quantile function (2) shows that the private value quantile function satisfies, for all $\alpha \in [\alpha_1, 1]$,

$$\begin{aligned} V(\alpha) &= B_{\overline{n}}(\alpha) + \frac{1}{\overline{n}-1}\alpha B_{\overline{n}}^{(1)}(\alpha) \\ &= W\left[(1-p_{\overline{n}}) + p_{\overline{n}}\alpha^{\overline{n}}\right] + \frac{\overline{n}p_{\overline{n}}}{\overline{n}-1}\alpha^{\overline{n}}W^{(1)}\left[(1-p_{\overline{n}}) + p_{\overline{n}}\alpha^{\overline{n}}\right], \end{aligned}$$

and $V(\cdot)$ is identified over $[\alpha_1, 1]$.

Using the mapping from the value quantile function to the bid quantile function (4) shows that the bid quantile functions $B_n(\cdot)$, $n = \underline{n}, \ldots, \overline{n}-1$ are also identified over $[\alpha_1, 1]$. Hence, $\{B_n(\alpha), \alpha \in [\alpha_1, 1]\}$ and $\left\{G_n(b), b \in [B_n(\alpha_1), \overline{b}_n]\right\}$ are identified, for all $n = \underline{n}, \ldots, \overline{n}$.

Step 2. We now expand the interval over which $G_{\overline{n}}(\cdot)$ is identified using an iterative argument. Define

$$\beta_1 = B_{\overline{n}-1}(\alpha_1),$$

which is identified from the last step. Note that $\beta_1 < \overline{b}_{\overline{n}-1}$ whenever $\alpha_1 > 0$ because, by Lemma 2.1-(i),

$$\beta_1 = B_{\overline{n}-1}\left[G_{\overline{n}}\left(\overline{b}_{\overline{n}-1}\right)\right] < B_{\overline{n}}\left[G_{\overline{n}}\left(\overline{b}_{\overline{n}-1}\right)\right] = \overline{b}_{\overline{n}-1}.$$

The definition of $G(\cdot)$ implies that

$$G_{\overline{n}}(b) = \left(\frac{G(b) - \sum_{n=\underline{n}}^{\overline{n}-1} p_n G_n^n(b)}{p_{\overline{n}}}\right)^{1/\overline{n}}, \tag{32}$$

where $G(\cdot)$ and p_n are identified, and $G_n(\cdot)$ are identified on $[B_n(\alpha_1), \overline{b}_{\overline{n}}]$ for all $n = \underline{n}, \ldots, \overline{n}-1$. Since $B_{\underline{n}}(\alpha_1) < \ldots < B_{\overline{n}-1}(\alpha_1) = \beta_1$, $[\beta_1, \overline{b}_{\overline{n}}] \subseteq [B_n(\alpha_1), \overline{b}_{\overline{n}}]$ for all n. Therefore, the conditional bid distribution $G_{\overline{n}}(b)$ is identified on $[\beta_1, \overline{b}_{\overline{n}}]$.

Step 3. We now identify $V(\cdot)$ on a growing interval $[\alpha_k, 1]$ using an induction argument and the identified $V(\cdot)$ on $[\alpha_1, 1]$. For an integer $k \geq 2$, define

$$\alpha_k = G_{\overline{n}}\left(\beta_{k-1}\right) = G_{\overline{n}}\left[B_{\overline{n}-1}\left(\alpha_{k-1}\right)\right], \quad \beta_k = B_{\overline{n}-1}\left(\alpha_k\right).$$

Identification of $V(\cdot)$ on the growing interval $[\alpha_k, 1]$ is established in Lemma 7.1 below.

Lemma 7.1 *Suppose Assumptions N and IPV hold. Then,*

(i). the sequences $\{\alpha_k, k \geq 1\}$ *and* $\{\beta_k, k \geq 1\}$ *are decreasing sequences with*

$$\lim_{k\to\infty} \alpha_k = 0.$$

(ii). $\{\alpha_k, k \geq 1\}$ *is identified. For any integer number* $k \geq 2$*,* $\{V(\alpha), \alpha \in [\alpha_k, 1]\}$ *is identified if* $\{V(\alpha), \alpha \in [\alpha_{k-1}, 1]\}$ *is identified.*

The proof of Lemma 7.1 is given at the end of this section. Let us now return to the identification of $V(\alpha)$ for any arbitrary $\alpha > 0$. By Lemma 7.1-(i), there exists k such that $\alpha > \alpha_k$ and Lemma 7.1-(ii) yields identification of $V(\alpha)$. Given that $V(0) = \underline{v}$ is identified by Lemma 2.2, the theorem is proven. $\square$

Proof of Lemma 7.1. Consider (i) first. As $\alpha_k = G_{\overline{n}}\left[B_{\overline{n}-1}\left(\alpha_{k-1}\right)\right]$ with $B_{\overline{n}-1}(\alpha) \leq B_{\overline{n}}(\alpha)$,

$$\alpha_k = G_{\overline{n}}\left[B_{\overline{n}-1}\left(\alpha_{k-1}\right)\right] \leq G_{\overline{n}}\left[B_{\overline{n}}\left(\alpha_{k-1}\right)\right] = \alpha_{k-1},$$

which implies that α_k decreases. Moreover, $\beta_k = B_{\overline{n}-1}(\alpha_k)$ decreases because $B_{\overline{n}-1}(\cdot)$ is strictly increasing. Since $\alpha_k \geq 0$, α_k converges to a limit α, which satisfies $\alpha = G_{\overline{n}}\left[B_{\overline{n}-1}(\alpha)\right]$ under Assumption IPV. In other words, the limit α satisfies $B_{\overline{n}}(\alpha) = B_{\overline{n}-1}(\alpha)$. This gives $\alpha = 0$ as $B_{\overline{n}}(\alpha) > B_{\overline{n}-1}(\alpha)$, except for $\alpha = 0$.

Now, consider (ii). That α_k is identified for all k follows from an induction argument, observing α_1 is identified. Suppose then that α_k and $\{V(\alpha), \alpha \in [\alpha_k, 1]\}$ are identified. Recall

$$\alpha_{k+1} = G_{\overline{n}}(\beta_k) = G_{\overline{n}}\left[B_{\overline{n}-1}(\alpha_k)\right], \quad \beta_{k+1} = B_{\overline{n}-1}(\alpha_{k+1}).$$

Then, (4) and Lemma 2.2 give that $\{B_n(\alpha); \alpha \in [\alpha_k, 1]\}$, for all $n = \underline{n}, \ldots, \overline{n}-1$ are identified, as β_k. Now (32) and Lemma 2.2 show that $G_{\overline{n}}(b)$ is identified for all $b \geq \beta_k$, and then $\alpha_{k+1} = G_{\overline{n}}(\beta_k)$ is identified. (2) then gives that $\{V(\alpha); \alpha \in [\alpha_{k+1}, 1]\}$ is identified. This ends the proof of the lemma. □

7.5 Proof of Lemma 3.1

Consider (14) first. Note that (13) shows that $B_n(\cdot|d)$ is continuously differentiable over $[0,1]$. Expanding (12) gives, when α goes to 0,

$$B_n^{(1)}(\alpha|d) = (n-1) \cdot d \frac{V(\alpha) - B_n(\alpha|d)}{1 - d + d \cdot \alpha} = (n-1) \cdot d \cdot \alpha \frac{V^{(1)}(0) - B_n^{(1)}(0|d)}{1-d} + o(\alpha),$$

which implies $B^{(1)}(0|d) = 0$ and then

$$B_n^{(1)}(\alpha|d) = \frac{(n-1)dV^{(1)}(0)}{1-d}\alpha + o(\alpha) = \frac{(n-1)d}{(1-d)f(\underline{v})}\alpha + o(\alpha),$$
$$B_n(\alpha) = \underline{b} + \frac{(n-1)d}{(1-d)f(\underline{v})}\frac{\alpha^2}{2} + o(\alpha^2) \text{ so that when } b \downarrow \underline{b}$$
$$G_n(b|d) = \left(\frac{2f(\underline{v})(1-d)}{(n-1)d}(b - \underline{b})\right)^{\frac{1}{2}}(1 + o(1)).$$

This gives (14), noting

$$g_n(b|d) = \frac{1}{B_n^{(1)}(G_n(b|d)|\, d)} = \left(\frac{2f(\underline{v})(1-d)}{(n-1)d(b-\underline{b})}\right)^{\frac{1}{2}}(1 + o(1)).$$

(15) also follows from $g_n(b|d) = 1/B_n^{(1)}(G_n(b|d)|\, d)$ and (12), which gives

$$g_n(\overline{b}_n|d) = \frac{1}{(n-1)d}\frac{1 - d + d \cdot G_n(b|d)}{V(G_n(b|d)) - b}\bigg|_{b=\overline{b}_n} = \frac{1}{(n-1)d(\overline{v} - \overline{b}_n)}.$$

For (16), first observe that

$$g_n^{(1)}(b|d) = \frac{d}{db}\left[\frac{1}{B_n^{(1)}\left(G_n(b|d)|\, d\right)}\right] = -\frac{B_n^{(2)}\left(G_n(b|d)|\, d\right) g_n(b|d)}{\left(B_n^{(1)}\left(G_n(b|d))|\, d\right)^2}$$
$$= -\frac{B_n^{(2)}\left(G_n(b|d)|\, d\right)}{\left(B_n^{(1)}\left(G_n(b|d))|\, d\right)^3},$$

where

$$B_n^{(2)}(\alpha|d) = \frac{d}{d\alpha}\left[(n-1)\cdot d\frac{V(\alpha)-B_n(\alpha|d))}{1-d+d\cdot\alpha}\right]$$
$$= -(n-1)\cdot d^2\frac{V(\alpha)-B_n(\alpha|d)}{(1-d+d\cdot\alpha)^2} + (n-1)\cdot d\frac{V^{(1)}(\alpha)-B_n^{(1)}(\alpha|d)}{1-d+d\cdot\alpha}$$
$$= -\frac{n(n-1)\cdot d^2\left(V(\alpha)-B_n(\alpha|d)\right)}{(1-d+d\cdot\alpha)^2} + (n-1)\cdot d\frac{V^{(1)}(\alpha)}{1-d+d\cdot\alpha}.$$

Hence,

$$g_n^{(1)}(\bar{b}_n|d) = -\frac{B_n^{(2)}(1|d)}{\left(B_n^{(1)}(1|d)\right)^3} = \frac{(n-1)\cdot d\cdot\left[n\cdot d\cdot\left(\overline{v}-\bar{b}_n\right)-\overline{v}^{(1)}\right]}{\left((n-1)\cdot d\cdot(\overline{v}-\bar{b}_n)\right)^3}$$
$$= \frac{n\cdot d\cdot\left(\overline{v}-\bar{b}_n\right)-\overline{v}^{(1)}}{((n-1)\cdot d)^2\left(\overline{v}-\bar{b}_n\right)^3}.$$

This ends the proof of the lemma. □

7.6 Proof of Proposition 3.1

Set $x_1 = \overline{v}-\bar{b}(0)$, $x_2 = \underline{n}$, and $x_3 = \frac{\overline{v}^{(1)}}{d}$. Additionally, define the extra unknowns

$$x_4 = \underline{n}\left(\overline{v}-\bar{b}(0)\right)^2, \qquad x_5 = \left(\overline{v}-\bar{b}(0)\right)^2, \qquad x_6 = \underline{n}\left(\overline{v}-\bar{b}(0)\right),$$

and set $y_m = -(m-1)\varrho(m)(\bar{b}(m)-\bar{b}(0))^2 - (2m-1)(\bar{b}(m)-\bar{b}(0))$.

It follows that

$$
\begin{aligned}
&(\underline{n}+m-1)(\overline{v}-\overline{b}(m))^2 = (\underline{n}+m-1)\left(\overline{v}-\overline{b}(0)-(\overline{b}(m)-\overline{b}(0))\right)^2 \\
&= \underline{n}\left(\overline{v}-\overline{b}(0)\right)^2 + (\underline{n}+m-1)\left(\overline{b}(m)-\overline{b}(0)\right)^2 \\
&\quad + (m-1)\left(\overline{v}-\overline{b}(0)\right)^2 - 2(\overline{b}(m)-\overline{b}(0))\left[\underline{n}\left(\overline{v}-\overline{b}(0)\right)+(m-1)\left(\overline{v}-\overline{b}(0)\right)\right] \\
&= -2(\overline{b}(m)-\overline{b}(0))(m-1)\cdot x_1 + (\overline{b}(m)-\overline{b}(0))^2\cdot x_2 + x_4 + (m-1)\cdot x_5 \\
&\quad - 2(\overline{b}(m)-\overline{b}(0))\cdot x_6 + (m-1)(\overline{b}(m)-\overline{b}(0))^2, \\
&(2\underline{n}+2m-1)\left(\overline{v}-\overline{b}(m)\right) = (2\underline{n}+2m-1)\left(\overline{v}-\overline{b}(0)-(\overline{b}(m)-\overline{b}(0))\right) \\
&\quad = (2m-1)\cdot x_1 - 2(\overline{b}(m)-\overline{b}(0))\cdot x_2 + 2\cdot x_6 - (2m-1)(\overline{b}(m)-\overline{b}(0)).
\end{aligned}
$$

Hence, using these new notations shows that (17) is equivalent to

$$
\begin{aligned}
-\left(2m-1+(m-1)\varrho(m)(\overline{b}(m)-\overline{b}(0))\right)\cdot x_1 \\
+(\overline{b}(m)-\overline{b}(0))\left(\varrho(m)(\overline{b}(m)-\overline{b}(0))+2\right)\cdot x_2 \\
-x_3+\varrho(m)\cdot x_4+(m-1)\varrho(m)\cdot x_5 - 2\left(\overline{b}(m)-\overline{b}(0)+1\right)\cdot x_6 = y_m.
\end{aligned}
$$

Elementary determinant algebra shows that, when $\det(I_{\mathcal{M}}) \neq 0$, the corresponding linear system obtained for m varying across $\mathcal{M}$ uniquely determines $x_1, \ldots, x_6$, and then $\underline{n}$, $\overline{v}$.

Hence $\overline{n}$ and $\overline{v}$ are identified using the number of discontinuities. $p_n = \frac{n-1}{n}\Delta_n\left(\overline{v}-\overline{b}_n\right)$ then identifies p_n. As $\underline{b}$ and $G(\underline{b}|d) = \sum_{n=\underline{n}}^{+\overline{n}} p_n\,(1-d)^n$ are identified, d is identified, since $\sum_{n=\underline{n}}^{+\overline{n}} p_n x^n$ is an identified polynomial function which is strictly increasing in x over $[0,1]$.

Identification of $F(\cdot)$ can then be established as in the baseline model, using that $B_n(\alpha|d)$ strictly increases with n for α in $(0,1]$ with $B_n(0|d) = \underline{v}$ for all n. □

7.7 Proof of Lemma 3.2

Continuity of $\widetilde{g}(\cdot)$ follows from (19). As $g_n(0) > 0$ by (6), (19) implies, for b sufficiently close to $\underline{b}$,

$$\widetilde{g}(b) = \int_{\underline{b}}^{b} (\varphi(0) + o(1)) \left(\sum_{n=\underline{n}}^{n=\overline{n}} g_n^n(0)(t-\underline{b})^{n-1}(1+o(1)) \right) dt$$
$$= \varphi(0) g_{\underline{n}}^{\underline{n}}(0)(b-\underline{b})^{\underline{n}}(1+o(1)),$$

which implies $\underline{n} = \lim_{t\downarrow 0} \frac{\log \widetilde{g}(\underline{b}+t)}{\log t}$.

(ii) follows from (20) and (7), which states that $g_n(\overline{b}_n) = 1/\left((n-1)(\overline{v}-\overline{b}_n)\right)$. For (iii), differentiating (20) gives

$$\widetilde{g}^{(2)}(b) = \sum_{n=\underline{n}}^{\overline{n}} p_n n \left[\varphi(0) G_n^{n-1}(b) g_n^{(1)}(b) - \varphi(\overline{\chi}) G_n^{n-1}(b-\overline{\chi}) g_n^{(1)}(b-\overline{\chi})\right]$$
$$+ \sum_{n=\underline{n}}^{\overline{n}} p_n n(n-1) \left[\varphi(0) G_n^{n-2}(b) g_n^2(b) - \varphi(\overline{\chi}) G_n^{n-2}(b-\overline{\chi}) g_n^2(b-\overline{\chi})\right]$$
$$+ \sum_{n=\underline{n}}^{\overline{n}} p_n n \left[\varphi^{(1)}(0) G_n^{n-1}(b) g_n(b) - \varphi^{(1)}(\overline{\chi}) G_n^{n-1}(b-\overline{\chi}) g_n(b-\overline{\chi})\right]$$
$$+ \sum_{n=\underline{n}}^{\overline{n}} p_n \int_{b-\overline{\chi}}^{b} \varphi^{(2)}(b-t) n G_n^{n-1}(t) g_n(t) dt.$$

Hence,

$$\widetilde{\Delta}_n^{(1)} = p_n \left[n\varphi(0)\left(g_n^{(1)}(\overline{b}_n) + (n-1) g_n^2(\overline{b}_n)\right) + n\varphi^{(1)}(0) g_n(\overline{b}_n)\right].$$

Let us now compute $g_n^{(1)}(\overline{b}_n) = -B_n^{(2)}(1)/\left(B_n^{(1)}(1)\right)^3$. (2) implies

$$B_n^{(2)}(\alpha) = -n(n-1)\frac{(V(\alpha) - B_n(\alpha))}{\alpha^2} + (n-1)\frac{V^{(1)}(\alpha)}{\alpha}$$

so that

$$B_n^{(2)}(1) = -n(n-1)\left(\overline{v} - \overline{b}_n\right) + (n-1)\overline{v}^{(1)}.$$

Hence,

$$g_n^{(1)}(\bar{b}_n) = -\frac{B_n^{(2)}(1)}{\left(B_n^{(1)}(1)\right)^3} = \frac{n\left(\overline{v}-\bar{b}_n\right) - \overline{v}^{(1)}}{(n-1)^2\left(\overline{v}-\bar{b}_n\right)^3}$$

by computations similar to the ones at the end of Section 7.5. This gives

$$\begin{aligned}
\widetilde{\Delta}_n^{(1)} &= \varphi(0)p_n\left(n\frac{n\left(\overline{v}-\bar{b}_n\right)-\overline{v}^{(1)}}{(n-1)^2\left(\overline{v}-\bar{b}_n\right)^3} + n(n-1)\left(\frac{1}{(n-1)\left(\overline{v}-\bar{b}_n\right)}\right)^2\right) \\
&\quad + \varphi^{(1)}(0)p_n\frac{n}{n-1}\frac{1}{\overline{v}-\bar{b}_n} \\
&= p_n\left[\varphi(0)\frac{n(2n-1)\left(\overline{v}-\bar{b}_n\right)-n\overline{v}^{(1)}}{(n-1)^2\left(\overline{v}-\bar{b}_n\right)^3} + \varphi^{(1)}(0)\frac{n}{n-1}\frac{1}{\overline{v}-\bar{b}_n}\right].
\end{aligned}$$

This ends the proof of the lemma. □

7.8 Proof of Proposition 3.2

Set $x_1 = \overline{v}-\bar{b}_{\underline{n}}$, $x_2 = \frac{\varphi^{(1)}(0)}{\varphi(0)}$, $x_3 = \overline{v}^{(1)}$ and

$$x_4 = \left(\overline{v}-\bar{b}_{\underline{n}}\right)^2, \qquad x_5 = \frac{\varphi^{(1)}(0)}{\varphi(0)}\left(\overline{v}-\bar{b}_{\underline{n}}\right)^2, \qquad x_6 = \frac{\varphi^{(1)}(0)}{\varphi(0)}\left(\overline{v}-\bar{b}_{\underline{n}}\right).$$

Set $y_n = -(n-1)\widetilde{\varrho}_n(\bar{b}_n-\bar{b}_{\underline{n}})^2 - (2n-1)(\bar{b}_n-\bar{b}_{\underline{n}})$.

As

$$\begin{aligned}
(n-1)&\left(\widetilde{\varrho}_n - \frac{\varphi^{(1)}(0)}{\varphi(0)}\right)\left(\overline{v}-\bar{b}_n\right)^2 = (n-1)\left(\widetilde{\varrho}_n - \frac{\varphi^{(1)}(0)}{\varphi(0)}\right)\left(\overline{v}-\bar{b}_{\underline{n}}-(\bar{b}_n-\bar{b}_{\underline{n}})\right)^2 \\
&= (n-1)\left(\widetilde{\varrho}_n - \frac{\varphi^{(1)}(0)}{\varphi(0)}\right)\left(\overline{v}-\bar{b}_{\underline{n}}\right)^2 - 2(n-1)(\bar{b}_n-\bar{b}_{\underline{n}})\left(\widetilde{\varrho}_n - \frac{\varphi^{(1)}(0)}{\varphi(0)}\right)\left(\overline{v}-\bar{b}_{\underline{n}}\right) \\
&\quad - (n-1)(\bar{b}_n-\bar{b}_{\underline{n}})^2\frac{\varphi^{(1)}(0)}{\varphi(0)} + (n-1)\widetilde{\varrho}_n(\bar{b}_n-\bar{b}_{\underline{n}})^2 \\
&= -2(n-1)(\bar{b}_n-\bar{b}_{\underline{n}})\widetilde{\varrho}_n\cdot x_1 - (n-1)(\bar{b}_n-\bar{b}_{\underline{n}})^2\cdot x_2 + (n-1)\widetilde{\varrho}_n\cdot x_4 \\
&\quad - (n-1)\cdot x_5 + 2(n-1)(\bar{b}_n-\bar{b}_{\underline{n}})\cdot x_6 + (n-1)\widetilde{\varrho}_n(\bar{b}_n-\bar{b}_{\underline{n}})^2, \\
-(2n-1)&\left(\overline{v}-\bar{b}_n\right) = -(2n-1)\cdot x_1 + (2n-1)(\bar{b}_n-\bar{b}_{\underline{n}}),
\end{aligned}$$

(21) is equivalent to

$$-\left[2(n-1)(\overline{b}_n-\overline{b}_{\underline{n}})\widetilde{\varrho}_n+(2n-1)\right]\cdot x_1-(n-1)(\overline{b}_n-\overline{b}_{\underline{n}})^2\cdot x_2+x_3$$
$$(n-1)\widetilde{\varrho}_n\cdot x_4-(n-1)\cdot x_5+2(n-1)(\overline{b}_n-\overline{b}_{\underline{n}})\cdot x_6=y_n.$$

Stacking these equations for n in $\widetilde{M}$ gives a linear system with a unique solution when $\det(I_{\widetilde{\mathcal{M}}})\neq 0$. Hence, the initial parameters are $\left(\overline{v},\overline{v}^{(1)},\frac{\varphi^{(1)}(0)}{\varphi(0)}\right)$. The identity $\widetilde{\Delta}_n=\varphi(0)p_n\frac{n}{n-1}\frac{1}{\overline{v}-\overline{b}_n}$ in Lemma 3.2 and $\sum_{n=\underline{n}}^{\overline{n}}p_n=1$ allow for the recovery of $\varphi(0)$ and p_n, $n=\underline{n},\ldots,\overline{n}$. $\overline{\chi}$ has already been identified, and $\varphi(\overline{\chi})$ can be recovered from the upward jump size $\widetilde{\Delta}_n=\varphi(\overline{\chi})p_n\frac{n}{n-1}\frac{1}{\overline{v}-\overline{b}_n}$. As $\varphi^{(1)}(\overline{\chi})/\varphi(\overline{\chi})$ satisfies an equation similar to (21), $\varphi^{(1)}(\overline{\chi})$ is also identified. □

7.9 Preliminary results for Section 4

Recall $\xi_c(\cdot|k)$ and $\xi(\cdot|k)$ are the cartel and non-cartel Nash equilibrium bidding inverse strategies given the cartel size K is k. Proposition B.1 ensures they are continuously differentiable over the whole bid support $[\underline{b},\overline{b}_k]$.

Lemma 7.2 *Suppose Assumption CIPV holds. Then, $\overline{b}_k$ increases with $k>0$.*

Proof of Lemma 7.2. Set $k_1=k>0$ and $k_2=k+1$. We show $\overline{b}(k_1)=\overline{b}_{k_1}<\overline{b}(k_2)=\overline{b}_{k_2}$ by contradiction. Suppose then $\overline{b}(k_2)\leq\overline{b}(k_1)$. Proposition 4.1-(iv) gives $\xi^{(1)}(\underline{b}|k_2)=1+\frac{1}{n+k_2-2}<\xi^{(1)}(\underline{b}|k_1)$ so that $\xi(b|k_2)<\xi(b|k_1)$ on some $(\underline{b},\underline{b}+\epsilon]$. Hence, as $\xi(\overline{b}(k_2)|k_1)\leq\overline{v}=\xi(\overline{b}(k_2)|k_2)$, $\xi(\cdot|k_2)$ and $\xi(\cdot|k_1)$ must cross on $[\underline{b}+\epsilon,\overline{b}(k_2)]$. Let $b^\star$ be the last contact location

$$b^\star=\sup\left\{b\in(\underline{b},\overline{b}(k_2)];\xi(b|k_2)=\xi(b|k_1)\right\},$$

which is such that $b^\star>\underline{b}$. Note that, if $b^\star<\overline{b}(k_1)$ as it follows from Step 1, $b^\star$ satisfies

$$\xi(b^\star|k_2)=\xi(b^\star|k_1),\ \xi^{(1)}(b^\star|k_2)\geq\xi^{(1)}(b^\star|k_1)\text{ so that }b^\star<\xi_c(b^\star|k_2)\leq\xi_c(b^\star|k_1)\quad(33)$$

by (22) for the last inequality.[12]

Step 1: $b^\star < \overline{b}(k_2)$ and $\xi(\cdot|k_2) > \xi(\cdot|k_1)$ on $(b^\star, \overline{b}(k_2))$. Note that $b^\star < \overline{b}(k_2)$ if $\overline{b}(k_2) < \overline{b}(k_1)$, because it implies $\xi(\cdot|k_1) < \xi(\cdot|k_2)$ in the vicinity of $\overline{b}(k_2)$. Suppose now $\overline{b}(k_2) = \overline{b}(k_1)$. Taylor expansion in (22) gives, by Proposition 4.1 and $\zeta^{(1)}(b|k) = \frac{f(\xi(b|k))}{F(\xi(b|k))}\xi^{(1)}(b|k)$,

$$f(\overline{v})\xi^{(1)}(b|k) = \frac{1}{n-1}\frac{1+o(1)}{\overline{v} + \frac{1}{k(n-1)f(\overline{v})}\frac{b-\overline{b}(k_2)}{\overline{v}-\overline{b}(k_2)}(1+o(1)) - b}, \ b \uparrow \overline{b}(k_1) = \overline{b}(k_2), \ k = k_1, k_2.$$

As a consequence of $\xi(b|k) = \overline{v} + \frac{1}{k(n-1)f(\overline{v})}\frac{b-\overline{b}(k_2)}{\overline{v}-\overline{b}(k_2)}(1+o(1)) \geq b$ and $b - \overline{b}(k_2) < 0$ in the expansion above, there is a vicinity $[\overline{b}(k_2) - \epsilon, \overline{b}(k_2))$ of $\overline{b}(k_2)$ over which $\xi^{(1)}(\cdot|k_2) < \xi^{(1)}(\cdot|k_1)$, and then

$$\xi(\cdot|k_2) > \xi(\cdot|k_1) \text{ on } [\overline{b}(k_2) - \epsilon, \overline{b}(k_2)) \text{ if } \overline{b}(k_1) = \overline{b}(k_2), \tag{34}$$

since

$$\xi(b|k_2) - \xi(b|k_1) = -\int_b^{\overline{b}(k_2)} \left(\xi^{(1)}(t|k_2) - \xi^{(1)}(t|k_1)\right) dt.$$

As $\xi(\cdot|k_2) < \xi(\cdot|k_1)$ over some $(\underline{b}, \underline{b}+\epsilon']$, it must be that $\underline{b}+\epsilon' < b^\star < \overline{b}(k_2) - \epsilon$ and then $b^\star < \overline{b}(k_2)$. The inequality (34) and the definition of $b^\star$ yield that $\xi(\cdot|k_2) > \xi(\cdot|k_1)$ on $[b^\star, \overline{b}(k_2))$.

Step 2: the contradiction. If $\overline{b}(k_1) > \overline{b}(k_2)$, then $\xi_c(\cdot|k_1) < \xi_c(\cdot|k_2)$ in a vicinity $[\overline{b}(k_2) - \epsilon, \overline{b}(k_2))$ of $\overline{b}(k_2)$. If $\overline{b}(k_1) = \overline{b}(k_2)$, Proposition 4.1-(iii) ensures that $\xi_c^{(1)}(\cdot|k_1) > \xi_c^{(1)}(\cdot|k_2)$ in a vicinity of $\overline{b}(k_2)$ and argues that, as for (34), gives $\xi_c(\cdot|k_1) < \xi_c(\cdot|k_2)$ over $[\overline{b}(k_2) - \epsilon, \overline{b}(k_2))$. Hence, $\xi_c(b^\star|k_2) \leq \xi_c(b^\star|k_1)$ in (33) and Step 1 imply

[12] $\xi^{(1)}(b^\star|k_2) \geq \xi^{(1)}(b^\star|k_1)$ is true if $\overline{b}(k_1) = \overline{b}(k_2)$, in which case $b^\star = \overline{b}(k_1) = \overline{b}(k_2)$; see Proposition 4.1-(iv). If $\overline{b}(k_1) > \overline{b}(k_2)$, it follows from Taylor expansion $\xi(b^\star + h|k) = \xi(b^\star|k_1) + \xi^{(1)}(b^\star|k)h(1 + o(1))$, with $\xi(b^\star + h|k_1) < \xi(b^\star + h|k_2)$ for $h > 0$.

that $\xi_c(\cdot|k_1)$ and $\xi_c(\cdot|k_2)$ are crossing each other on $[b^\star, \overline{b}(k_2))$. Let $b^\dagger$ be the location of the last crossing, i.e.

$$b^\dagger = \sup\left\{b \in [b^\star, \overline{b}(k_2); \xi_c(b|k_1) = \xi_c(b|k_2)\right\},$$

which yields

$$b^\star \leq b^\dagger < \overline{b}(k_2),\ \xi_c(b^\dagger|k_1) = \xi_c(b^\dagger|k_2) \text{ and } \xi_c^{(1)}(b^\dagger|k_1) \leq \xi_c^{(1)}(b^\dagger|k_2).$$

But $k_2 > k_1$ and (23) implies that $\xi(b^\dagger|k_1) \geq \xi(b^\dagger|k_2)$, while Step 1 shows that $\xi(b^\dagger|k_1) < \xi(b^\dagger|k_2)$, a contradiction. Hence, $\overline{b}(k_2) \leq \overline{b}(k_1)$ is impossible. □

Lemma 7.3 *Suppose Assumption CIPV holds, and consider some integer numbers* $k_2 > k_1 \geq 1$. *Then,* $\xi(b|k_2) < \xi(b|k_1)$ *for all* b *in* $(\underline{b}, \overline{b}_{k_1}]$.

Proof of Lemma 7.3. The proof is by contradiction. Suppose that $\xi(b|k_2) > \xi(b|k_1)$ for some b of $(0, \overline{b}_{k_1}]$. Lemma 7.2 yields that $\xi(\overline{b}_{k_1}|k_2) < \overline{v} = \xi(\overline{b}_{k_1}|k_1)$, since $\xi(\cdot|k)$ are strictly increasing. Define

$$b^\star = \sup\left\{b \in (\underline{b}, \overline{b}_{k_1}]; \xi(b|k_2) > \xi(b|k_1)\right\}, \quad b_\star = \sup\left\{b \in (\underline{b}, b^\star]; \xi(b|k_2) < \xi(b|k_1)\right\},$$

which are such that $\underline{b} < b_\star < b^\star < \overline{b}_{k_1}$ as $\xi^{(1)}(\underline{b}|k_2) = 1 + \frac{1}{n+k_2-1} < \xi^{(1)}(\underline{b}|k_1)$, $\xi(\underline{b}|k_2) = \xi(\underline{b}|k_1) = \underline{b}$, and

$$\xi(b|k_1) \leq \xi(b|k_2) \text{ for all } b \text{ in } (b_\star, b^\star). \tag{35}$$

Continuity and elementary Taylor expansions then give

$$\xi(b_\star|k_1) = \xi(b_\star|k_2) \text{ and } \xi^{(1)}(b_\star|k_1) \leq \xi^{(1)}(b_\star|k_2),$$
$$\xi(b^\star|k_1) = \xi(b^\star|k_2) \text{ and } \xi^{(1)}(b^\star|k_1) \geq \xi^{(1)}(b^\star|k_2).$$

By (22), as

$$\xi^{(1)}(b|k) = \frac{F(\xi(b|k))}{(n-1)f(\xi(b|k))}\frac{1}{\xi_c(b|k) - b},$$

it holds that

$$\xi_c(b_\star|k_2) \leq \xi_c(b_\star|k_1) \text{ and } \xi_c(b^\star|k_2) \geq \xi_c(b^\star|k_1).$$

We now discuss the contradictions when the inequalities above are strict or equalities.

- $\xi_c(b_\star|k_2) < \xi_c(b_\star|k_1)$ **and** $\xi_c(b^\star|k_2) > \xi_c(b^\star|k_1)$**.** Then, there exists $b^\dagger$ in $(b_\star, b^\star)$ such that $\xi_c(b^\dagger|k_2) = \xi_c(b^\dagger|k_1)$ and $\xi_c^{(1)}(b^\dagger|k_2) \geq \xi_c^{(1)}(b^\dagger|k_1)$, i.e. $\xi_c(\cdot|k_1)$ crosses $\xi_c(\cdot|k_2)$ downwards at $b^\dagger$. But (23) yields
$$\xi_c^{(1)}(b^\dagger|k) = \frac{1}{k}\left(\frac{n-1}{\xi(b^\dagger|k) - b^\dagger} - \frac{n-2}{\xi_c(b^\dagger|k) - b^\dagger}\right)\frac{F\left(\xi_c(b^\dagger|k)\right)}{(n-1)f\left(\xi_c(b^\dagger|k)\right)}$$
with $\frac{F_c\left(\xi_c(b^\dagger|k_1)\right)}{(n-1)f\left(\xi_c(b^\dagger|k_1)\right)} = \frac{F\left(\xi_c(b^\dagger|k_2)\right)}{(n-1)f\left(\xi_c(b^\dagger|k_2)\right)}$, since $\xi_c(b^\dagger|k_1) = \xi_c(b^\dagger|k_2)$. As $k_2 > k_1$, it must be that $\xi(b^\dagger|k_2) < \xi(b^\dagger|k_1)$, which contradicts (35). Note that this argument shows that $\xi_c(\cdot|k_1)$ cannot cross $\xi_c(\cdot|k_2)$ downwards over $(b_\star, b^\star)$.

- $\xi_c(b_\star|k_2) = \xi_c(b_\star|k_1)$ **and** $\xi_c(b^\star|k_2) > \xi_c(b^\star|k_1)$**.** Then, since $\xi_c(\cdot|k_1)$ cannot cross $\xi_c(\cdot|k_2)$ downwards over $(b_\star, b^\star)$, it must be that $\xi_c(\cdot|k_2) > \xi_c(\cdot|k_1)$ over $(b_\star, b^\star)$. Hence, Taylor expansion shows $\xi_c^{(1)}(b_\star|k_2) \geq \xi_c^{(1)}(b_\star|k_1)$. Arguing as above shows that it must be that $\xi(b_\star|k_2) < \xi(b_\star|k_1)$, which implies $\xi(b|k_2) < \xi(b|k_1)$ for b in $(b_\star, b^\star)$ sufficiently close to $b_\star$, contradicting (35).

- $\xi_c(b_\star|k_2) < \xi_c(b_\star|k_1)$ **and** $\xi_c(b^\star|k_2) = \xi_c(b^\star|k_1)$**.** It then must be that $\xi_c(\cdot|k_2) < \xi_c(\cdot|k_1)$ over $(b_\star, b^\star)$, so that Taylor expansion shows $\xi_c^{(1)}(b^\star|k_2) \geq \xi_c^{(1)}(b^\star|k_1)$. Arguing as above shows that it must be that $\xi(b^\star|k_2) < \xi(b^\star|k_1)$, which implies $\xi(b|k_2) < \xi(b|k_1)$ for b in $(b_\star, b^\star)$ sufficiently close to $b^\star$, contradicting (35).

The last case is $\xi_c(b_\star|k_2) = \xi_c(b_\star|k_1)$ and $\xi_c(b^\star|k_2) = \xi_c(b^\star|k_1)$. This case also yields a contradiction, which arises as in the two cases above, depending on whether $\xi_c(\cdot|k_2) > \xi_c(\cdot|k_1)$ or $\xi_c(\cdot|k_2) < \xi_c(\cdot|k_1)$ over $(b_\star, b^\star)$. □

Lemma 7.4 *Suppose Assumption CIPV holds. Then, for any integer number $k \geq 2$, $G(b|k) < G_c(b|k)$, or equivalently $\xi(b|k) < \xi_c(b|k)$, for all b in $(\underline{b}, \overline{b}_k)$.*

Proof of Lemma 7.4. Let $V(\cdot) = F^{-1}(\cdot)$ be the private value quantile function. As $G(\cdot|k) = F(\xi(\cdot|k))$, $G_c(\cdot|k) = F(\xi_c(\cdot|k))$, $\zeta(\cdot|k) = \ln G(\cdot|k)$ and $\zeta_c(\cdot|k) = \ln G_c(\cdot|k)$, (23) is equivalent to

$$\frac{d}{db}\left[\ln G_c(b|k)\right] = \frac{1}{k(n-1)}\left(\frac{n-1}{V\left(G(b|k)\right)-b} - \frac{n-2}{V\left(G_c(b|k)\right)-b}\right).$$

As $G(\overline{b}_k|k) = G_c(\overline{b}_k|k) = 1$ with $g(\overline{b}_k|k) = \frac{1}{(n-1)(\overline{v}-\overline{b}_k)} > g_c(\overline{b}_k|k) = \frac{1}{k(n-1)(\overline{v}-\overline{b}_k)}$ $G(\cdot|k) < G_c(\cdot|k)$ in a vicinity of $(\overline{b}_k-\epsilon, \overline{b}_k)$, of $\overline{b}_k$. Suppose $G(\cdot|k)$ and $G_c(\cdot|k)$ cross in $(\underline{b}, \overline{b}_k)$, and let $b^\star > \underline{b}$ be the first crossing location, which must be such that

$$\frac{d}{db}\left[\ln G(b^\star|k)\right] \leq \frac{d}{db}\left[\ln G_c(b^\star|k)\right]. \tag{36}$$

We show that the equation above cannot hold, discussing the relative position of $G(b^\star|k)$ and $G_c(b^\star|k)$.

- If $V(G(b^\star|k)) \geq V(G_c(b^\star|k)) > b^\star$, then
$$\begin{aligned}\frac{d}{db}\left[\ln G_c(b^\star|k)\right] &\leq \frac{1}{k(n-1)}\left(\frac{n-1}{V\left(G_c(b^\star|k)\right)-b} - \frac{n-2}{V\left(G_c(b^\star|k)\right)-b}\right)\\ &= \frac{1}{k(n-1)}\frac{n-1}{V\left(G_c(b^\star|k)\right)-b} < \frac{d}{db}\left[\ln G(b^\star|k)\right] \text{ contradicting (36).}\end{aligned}$$

- If $V(G_c(b^\star|k)) \geq V(G(b^\star|k)) > b^\star$, then
$$\begin{aligned}\frac{d}{db}\left[\ln G(b^\star|k)\right] &= \frac{1}{n-1}\left(\frac{n-1}{V\left(G_c(b^\star|k)\right)-b} - \frac{n-2}{V\left(G_c(b^\star|k)\right)-b}\right)\\ &\geq \frac{1}{n-1}\left(\frac{n-1}{V\left(G(b^\star|k)\right)-b} - \frac{n-2}{V\left(G_c(b^\star|k)\right)-b}\right)\\ &> \frac{d}{db}\left[\ln G_c(b^\star|k)\right] \text{ contradicting (36). } \square\end{aligned}$$

7.10 Proof of the main results

7.10.1 Proof of Proposition 4.1

Existence, uniqueness, and smoothness of the Bayesian Nash Equilibrium strategies follow from Lebrun (1999), Theorem B.1, and Proposition B.1. In (i), $\xi^{-1}(\underline{v}|k) =$

$\xi_c^{-1}(\underline{v}|k) = \underline{v} = \underline{b}$ and $\xi^{-1}(\overline{v}|k) = \xi_c^{-1}(\overline{v}|k) = \underline{v} = \overline{b}_k < \overline{v}$ are from Lebrun (1999), while the fact that $\overline{b}_k$ is increasing is established in Lemma 7.2 in Proofs. For (iii), suppose that the cartel leader and a non cartel buyer have private value $\overline{v}$. As

$$\overline{b}_k = \arg\max_x (\overline{v} - x) G^{n-1}(x|k) = \arg\max_x (\overline{v} - x) G^{n-2}(x|k) G_c^k(x|k),$$

it follows from the associated first-order condition,

$$(\overline{v} - \overline{b}_k)(n-1) g(\overline{b}_k|k) - 1 = 0,$$

$$(\overline{v} - \overline{b}_k)\left((n-2) g(\overline{b}_k|k) + k g(\overline{b}_k|k)\right) - 1 = 0.$$

Solving gives $g(\overline{b}_k|k) = \frac{1}{n-1}\frac{1}{\overline{v}-\overline{b}_k}$ and $g_c(\overline{b}_k|k) = \frac{1}{k(n-1)}\frac{1}{\overline{v}-\overline{b}_k}$. For (iv), Proposition B.1 gives $\xi^{(1)}(\underline{b}) = \frac{n+k-1}{n+k-2}$ and $\xi_c^{(1)}(\underline{b}) = \frac{n}{n-1}$, which gives the result as $g(b|k) = f\left(\xi(b|k)\right)\xi^{(1)}(b|k)$ and $g_c(b|k) = f_c\left(\xi_c(b|k)\right)\xi_c^{(1)}(b|k)$. □

7.10.2 Proof of Proposition 4.2

Consider first the case where the cartel is not present, $K = 1$. Set $\gamma_{i|1}^c = 0$ and $G_{ec}(\cdot|1) = G_{enc}(\cdot|1)$. Recall that $\gamma_{i|k}^c + \gamma_{i|k}^{nc} \leq 1$. The candidate buyers e are such that

$$\gamma_{i|1}^c = \gamma_{i|1}^{nc} = 0, \quad \gamma_{i|\underline{k}}^c = \cdots = \gamma_{i|\overline{k}}^c = 1, \quad \gamma_{i|\underline{k}}^{nc} = \cdots = \gamma_{i|\overline{k}}^{nc} = 0.$$

Let $G_i(b)$ be $\mathbb{P}\left(W \leq b,\ i \text{ wins}\right)$. That is

$$G_i(b) = \pi_1 \gamma_{i|1}^{nc} G_{enc}(b|1) + \sum_{k=\underline{k}}^{\overline{k}} \pi_k \left(\gamma_{i|k}^c G_{ec}(b|k) + \gamma_{i|k}^{nc} G_{enc}(b|k)\right).$$

If i enters the cartel, i has a non-zero probability of being the cartel leader and $G_i(\cdot) > 0$ over $(\underline{b}, \infty)$. So, if $G_i(\cdot) = 0$ for all i, Assumption E cannot hold. Assume without loss of generality that $G_i(\cdot) \neq 0$ for all i. Let $g_i(b) = \frac{d}{db} G_i(b)$. That is

$$g_i(b) = \pi_1 \gamma_{i|1}^{nc} G^{n-1}(b|1) g(b|1) + \sum_{k=\underline{k}}^{\overline{k}} \pi_k \left(\gamma_{i|k}^c G^{n-1}(b|k) G_c^{k-1}(b|k) g_c(b|k) + \gamma_{i|k}^{nc} G^{n-2}(b|k) G_c^k(b|k) g(b|k)\right),$$

and observe that the potential discontinuity of $g_i(\cdot)$ at $\bar{b}_k$ is

$$\Delta_{ik} = \lim_{t \downarrow 0} \left(g_i \left(\bar{b}_k - t \right) - g_i \left(\bar{b}_k - t \right) \right) = \pi_k \left(\frac{\gamma^c_{i|k}}{k} + \gamma^{nc}_{i|k} \right) \frac{1}{n-1} \frac{1}{\overline{v} - \bar{b}_k}$$

by definition of $G_{ec}(\cdot|k)$, $G_{enc}(\cdot|k)$ and Proposition 4.1-(iii). Because

$$g_c(b) = \pi_1 \left((n-1) G^{n-2}(b|1) G_c(b|1) g(b|1) + G^{n-1}(b|1) g_c(b|1) \right)$$
$$\sum_{k=\underline{k}}^{\bar{k}} \pi_k \left((n-1) G^{n-2}(b|k) G^k_c(b|k) g(b|k) + G^{n-1}(b|k) k G^{k-1}_c(b|k) g_c(b|k) \right)$$

with $(n-1) g(\bar{b}_k|k) + k g_c(\bar{b}_k|k) = \frac{n}{n-1} \frac{1}{\overline{v} - \bar{b}_k}$ by Proposition 4.1-(iii), it follows that

$$\Delta^c_k = \lim_{t \downarrow 0} \left(g_c \left(\bar{b}_k - t \right) - g_c \left(\bar{b}_k - t \right) \right) = \pi_k \frac{n}{n-1} \frac{1}{\overline{v} - \bar{b}_k}.$$

Proposition 4.1-(iv) gives, when b goes to $\underline{b}$,

$$G_i(b) = \pi_1 \gamma^{nc}_{i|1} (1 + o(1)) \left(\frac{n}{n-1} f(\underline{v}) \right)^n \frac{(b - \underline{b})^n}{n}$$
$$+ \pi_{\underline{k}} \left(\gamma^c_{i|\underline{k}} + \gamma^{nc}_{i|\underline{k}} + o(1) \right) \left(\frac{n}{n-1} f(\underline{v}) \right)^{n-1} \left(\frac{n + \underline{k} - 1}{n + \underline{k} - 2} f(\underline{v}) \right)^{\underline{k}} \frac{(b - \underline{b})^{n + \underline{k} - 1}}{n + \underline{k} - 1},$$
$$G_c(b) = \pi_1 \left(\frac{n}{n-1} f(\underline{v}) \right)^n (1 + o(1)) (b - \underline{b})^n,$$

recalling $\pi_1 > 0$, $\pi_{\underline{k}} > 0$ and $\gamma^c_{i|\underline{k}} + \gamma^{nc}_{i|\underline{k}} > 0$. Hence, the lower tail behavior of $G_c(\cdot)$ identifies n. All i such that $G_i(b)$ is of exact order $(b - \underline{b})^n$ when b goes to $\underline{b}$ are not e candidates, as their $\gamma^{nc}_{i|1}$ is strictly positive $\pi_1 > 0$. Otherwise, the lower tail behaviors of those $G_i(\cdot)$ with $\gamma^{nc}_{i|1} = 0$ then identify $n + \underline{k} - 1$ as $\gamma^c_{i|\underline{k}} + \gamma^{nc}_{i|\underline{k}} > 0$, so that $\underline{k}$ is identified, and $\bar{k}$ as well, using the number of discontinuities of $G(\cdot)$. As

$$\pi_k = \frac{n-1}{n} \Delta^c_k \left(\overline{v} - \bar{b}_k \right),$$

arguing as in Lemma 2.2 shows that $\overline{v}$ and $\pi_1, \pi_{\underline{k}}, \ldots, \pi_{\bar{k}}$ are identified. Then, the expression of Δ_{ik} shows that

$$\frac{\gamma^c_{i|k}}{k} + \gamma^{nc}_{i|k}, \quad k = \underline{k}, \ldots, \bar{k} \text{ with } \underline{k} \geq 2$$

are identified. Note that, since π_1, n, and $\underline{k}$ are identified and because the lower tail behavior of $G_c(\cdot)$ identifies $f(\underline{v})$, then the lower tail behavior of $G_i(\cdot)$ identifies $\gamma^c_{i|\underline{k}} + \gamma^{nc}_{i|\underline{k}}$. This implies that $\gamma^c_{i|\underline{k}}$ is identified. As, under the condition of the Proposition, it must be that either $\gamma^c_{i|\underline{k}} = \cdots = \gamma^c_{i|\overline{k}} = 1$ or $\gamma^c_{i|\underline{k}}, \ldots, \gamma^c_{i|\overline{k}} < 1$; i.e., whether or not buyer i is a permanent cartel member is revealed. □

7.10.3 Proof of Theorem 4.1

The cartel size distribution and n are identified as shown in the corresponding paragraphs of Section 4.3. We now turn to the private value distribution.

Initial step. Let $\mathcal{K} = \{k = 1, \underline{k}, \ldots, \overline{k}; \pi_k > 0\}$, $\overline{k}_\star = \max\{k \in \mathcal{K}; k < \overline{k}\}$ be the integer number of $\mathcal{K}$ preceding $\overline{k}$, and $\mathcal{K}_\star = \{k \in \mathcal{K}; k < \overline{k}\}$, which are all identified. Let $V(\cdot) = F^{-1}(\cdot)$ be the private value quantile function. As the distribution of K and n are identified, and since (28) holds over $[\overline{b}_{\overline{k}-1}, \overline{b}_{\overline{k}}]$, it follows that $G_c(\cdot|\overline{k})$ and $G(\cdot|\overline{k})$ are also identified over $[\beta_0, \overline{b}_{\overline{k}}]$, where $\beta_0 = \overline{b}_{\overline{k}-1}$, by the argument following (28). Now, as Bayesian Nash Equilibrium bids satisfy

$$b = \arg\max_x \left(\xi_c(b|k) - x\right) G^{n-1}(x|k), \quad b = \arg\max_x \left(\xi(b|k) - x\right) G^{n-2}(x|k) G^k_c(x|k),$$

the corresponding first-order conditions give

$$\left\{ \begin{array}{lcl} \xi_c(b|k) & = & b + \frac{1}{n-1}\frac{G(b|k)}{g(b|k)}, \\ \xi(b|k) & = & b + \frac{1}{(n-2)\frac{g(b|k)}{G(b|k)} + k\frac{g_c(b|k)}{G_c(b|k)}}, \end{array} \right. \qquad k \in \mathcal{K}, \tag{37}$$

where $\xi_c(\cdot|k)$ and $\xi(\cdot|k)$ are strictly increasing. It follows that $\xi_c(\cdot|\overline{k})$ and $\xi(\cdot|\overline{k})$ are identified over $[\beta_0, \overline{b}_{\overline{k}}]$. As

$$V(\cdot) = \xi\left(G^{-1}(\cdot|k)\big|\, k\right) \quad k \in \mathcal{K}, \tag{38}$$

$V(\cdot)$ is identified over $[\alpha_0, 1]$ with

$$\alpha_0 = G(\beta_0|\overline{k}).$$

Iterations. As $G(\cdot|k) = F(\xi(\cdot|k))$, $G_c(\cdot|k) = F(\xi_c(\cdot|k))$, $\zeta(\cdot|k) = \ln G(\cdot|k)$, and $\zeta_c(\cdot|k) = \ln G_c(\cdot|k)$, (22) and (23) are equivalent to

$$\begin{cases} \frac{d}{db}\left[\ln G_c(b|k)\right] = \frac{1}{k(n-1)}\left(\frac{n-1}{V(G(b|k))-b} - \frac{n-2}{V(G_c(b|k))-b}\right), & G_c(\overline{b}_k|k) = 1, \\ \frac{d}{db}\left[\ln G(b|k)\right] = \frac{1}{n-1}\frac{1}{V(G_c(b|k))-b}, & G(\overline{b}_k|k) = 1, \\ \text{where } V(\cdot) = \xi\left(G^{-1}(\cdot|\overline{k})\middle|\,\overline{k}\right), & \end{cases} \tag{39}$$

using identified terminal conditions instead of initial ones. As $V(\cdot)$ is continuously differentiable, the differential system (39) has a unique set of solutions $[G(\cdot|k), G_c(\cdot|k)]$ satisfying $V(G(b|k)) > b$ and $V(G_c(b|k)) > b$, over any interval $[b_\dagger, \overline{b}_k] \subset (\underline{b}, \overline{b}_k]$.

As $V(\cdot)$ is identified over $[\alpha_0, 1]$, $\alpha_0 = G(\beta_0|\overline{k})$, solving (39) shows that $G_c(\cdot|k)$ and $G(\cdot|k)$, for all k in $\mathcal{K}_\star$, are identified over $[\beta_1, \overline{b}_{\overline{k}}]$, where $\beta_1 = \max_{k\in\mathcal{K}_\star}\beta_{1,k}$ with

$$\begin{aligned} \beta_{1,k} &= \inf\left\{b; G_c(b|k) \geq G(\beta_0|\overline{k}) \text{ and } G(b|k) \geq G(\beta_0|\overline{k})\right\} \\ &= \min\left\{b; G(b|k) \geq G(\beta_0|\overline{k})\right\} = G^{-1}\left(G(\beta_0|\overline{k})\middle|\,k\right), \end{aligned}$$

using that $G(\cdot|k) \leq G_c(\cdot|k)$ in Proposition 4.1-(ii) yields $G_c^{-1}(\cdot|k) \leq G^{-1}(\cdot|k)$. Proposition 4.1-(ii) also implies that $G^{-1}(\cdot|k) \leq G^{-1}(\cdot|\overline{k}_\star)$ for all k of $\mathcal{K}_\star$, so that

$$\beta_1 = G^{-1}\left(G(\beta_0|\overline{k})\middle|\,\overline{k}_\star\right).$$

Observe now that

$$\begin{aligned} G_{1,c}(b) &= \frac{1}{\pi_{\overline{k}}}\left\{G(b) - \sum_{k\in\mathcal{K}_\star}\pi_k G^{n-1}(b|k)G_c^k(b|k)\right\}\mathbb{I}(b \geq \beta_1) \\ &= G^{n-1}(b|\overline{k})G_c^{\overline{k}}(b|\overline{k})\mathbb{I}(b \geq \beta_1), \\ G_{1,ec}(b) &= \frac{1}{\pi_{\overline{k}}}\left\{G_{ec}(b) - G_{ec}(\beta_1) - \sum_{k\in\mathcal{K}_\star}\pi_k\left(G_{ec}(b|k) - G_{ec}(\beta_1|k)\right)\right\}\mathbb{I}(b \geq \beta_1) \\ &= \left(G_{ec}(b|\overline{k}) - G_{ec}(\beta_1|\overline{k})\right)\mathbb{I}(b \geq \beta_1), \end{aligned}$$

are both identified at this stage. Applying (28) yields that $G_c(\cdot|\overline{k})$ and $G(\cdot|\overline{k})$ are identified over $[\beta_1, 1]$. Applying (37) and (38) for $k = \overline{k}$ then gives that $V(\cdot)$ is identified over $[\alpha_1, 1]$, $\alpha_1 = G(\beta_1|\overline{k})$. Solving (39) shows that $G_c(\cdot|k)$ and $G(\cdot|k)$, for all k in $\mathcal{K}_\star$, are identified over $[\beta_2, \overline{b}_{\overline{k}}]$, where $\beta_2 = G^{-1}\left(G(\beta_1|\overline{k})\middle|\,\overline{k}_\star\right) < \beta_1$ by Proposition 4.1-(ii).

Iterating allows for the identification of $V(\cdot)$ over an increasing sequence of intervals $[\alpha_m, 1]$, where

$$\alpha_m = G(\beta_m|\overline{k}), \qquad \beta_m = G^{-1}\left(G(\beta_{m-1}|\overline{k})\middle|\,\overline{k}_\star\right) < \beta_{m-1}.$$

Since the sequence β_m is decreasing and bounded from below by $\underline{b}$, β_m converges to a limit β_∞ when m grows. As β_∞ satisfies $G(\beta_\infty|\overline{k}_\star) = G(\beta_\infty|\overline{k})$ by continuity, and since $G(\underline{b}|\overline{k}_\star) = G(\underline{b}|\overline{k})$ with $G(\cdot|\overline{k}_\star) < G(\cdot|\overline{k})$ over $[\underline{b}, \overline{b}_{\overline{k}_\star}]$, it must be that $\beta_\infty = 0$. Continuity then ensures that α_m converges to $0 = G(\underline{b}|\overline{k})$ so that $V(\cdot)$ is identified over the whole quantile level set $[0, 1]$. □

7.10.4 Proof of Proposition 4.3

The proposition will follow from the fact that π_k are identified and that $G_{ec}(\cdot|k)$ and $G_{enc}(\cdot|k)$, which are all identified at this stage, form a system of independent functions as established below. Suppose there exist λ_1 and $\lambda_k^c, \lambda_k^{nc}$, $k = \underline{k}, \ldots, \overline{k}$, such that

$$\lambda_1 G_{enc}(\cdot|1) + \sum_{k=\underline{k}}^{\overline{k}} \left(\lambda_k^c G_{ec}(\cdot|k) + \lambda_k^{nc} G_{enc}(\cdot|k)\right) = 0 \text{ over } [\underline{b}, \overline{b}_{\overline{k}}].$$

Then, differentiating and specializing to $(\overline{b}_{\overline{k}-1}, \overline{b}_{\overline{k}}]$ gives that

$$\lambda_{\overline{k}}^c \frac{g_c(\cdot|\overline{k})}{G_c(\cdot|\overline{k})} + \lambda_{\overline{k}}^{nc} \frac{g(\cdot|\overline{k})}{G(\cdot|\overline{k})} = 0 \text{ over } (\overline{b}_{\overline{k}-1}, \overline{b}_{\overline{k}}].$$

Now, note that either $\lambda_{\overline{k}}^c = \lambda_{\overline{k}}^{nc} = 0$ or $\lambda_{\overline{k}}^c, \lambda_{\overline{k}}^{nc} \neq 0$. Integrating in the latter case gives that there is a κ such that

$$G_c(b|\overline{k}) = G^\kappa(b|\overline{k}) \text{ for all } b \text{ in } (\overline{b}_{\overline{k}-1}, \overline{b}_{\overline{k}}],$$

and Proposition 4.1-(iii) implies that $\kappa = 1/\overline{k}$. That is

$$G_c(b|\overline{k}) = G^{1/\overline{k}}(b|\overline{k}) \text{ for all } b \text{ in } (\overline{b}_{\overline{k}-1}, \overline{b}_{\overline{k}}].$$

Hence, (22) and (23) yield

$$\begin{aligned} \frac{d}{db}\left[\ln G(b|\overline{k})\right] &= \frac{1}{n-1}\frac{1}{\xi_c(b|\overline{k})-b}, \\ \frac{d}{db}\left[\ln G(b|\overline{k})\right] &= \frac{1}{n-1}\left(\frac{n-1}{\xi(b|\overline{k})-b} - \frac{n-2}{\xi_c(b|\overline{k})-b}\right), \end{aligned}$$

which implies

$$\xi_c(b|\overline{k}) = \xi(b|\overline{k}) \text{ for all } b \text{ in } (\overline{b}_{\overline{k}-1}, \overline{b}_{\overline{k}}].$$

Hence,

$$F(v) = G^{1/\overline{k}}\left(\xi^{-1}(b|\overline{k})\right) = F^{1/\overline{k}}(v) \text{ for all } v \text{ in } (\xi^{-1}(\overline{b}_{\overline{k}-1}|\overline{k}), \overline{v}],$$

which is not possible as $\overline{k} \geq 2$. Hence, it must be $\lambda^c_{\overline{k}} = \lambda^{nc}_{\overline{k}} = 0$. Iterating then shows $\lambda^c_k = \lambda^{nc}_k = 0$, $k = \overline{k}-1, \ldots, \underline{k}$ and $\lambda_1 = 0$. This ends the proof of the proposition. □

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Appendix A: Statistical procedures, simulation experiments, and additional results — not for publication

A.1 Simulation design

We consider a simulation design where the number of bidders N is either 2 with probability $p = 0, .2, .4, .6, .8, 1$, or 3. The private values of the N active bidders are uniform over $[1, 2]$ so that the optimal bid is $1 + (1 - 1/N)(v - 1)$ for a private value v. Hence, conditional on N, the bids are uniform over $[1, 2 - 1/N]$. The wining bid c.d.f is then

$$G(b|p) = p\left(2(b-1)\right)^2 \mathbb{I}\left(b \in \left[1, \frac{3}{2}\right]\right) + (1-p)\left(\frac{3}{2}(b-1)\right)^3 \mathbb{I}\left(b \in \left[1, \frac{5}{3}\right]\right) \quad (40)$$

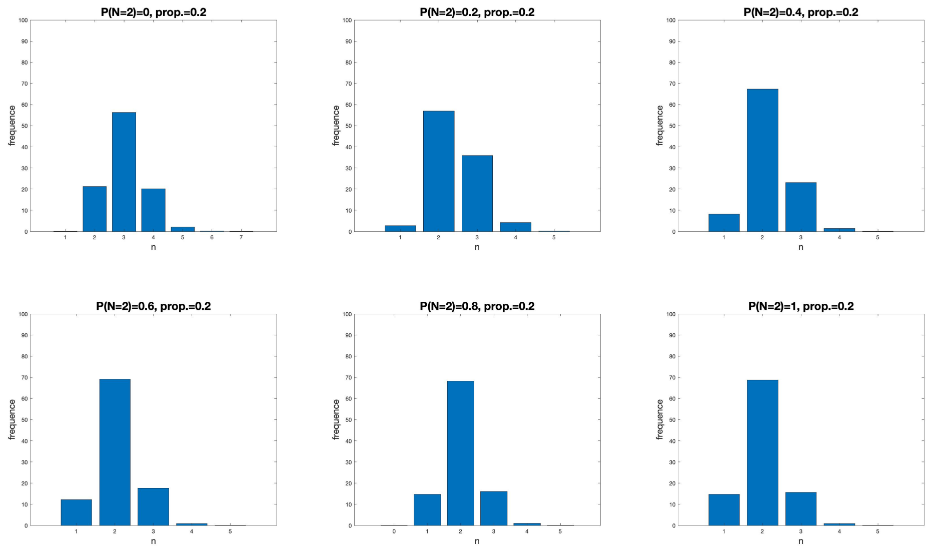
over $\left[1, \frac{5}{3}\right]$, and is equal to 0 if $b \leq 1$ and equal to 1 if $b \geq \frac{5}{3}$. Hence, the winning bid p.d.f is

$$g(b|p) = p8(b-1)\mathbb{I}\left(b \in \left[1, \frac{3}{2}\right]\right) + (1-p)\frac{81}{8}(b-1)^2\mathbb{I}\left(b \in \left[1, \frac{5}{3}\right]\right),$$

with support $\left[1, \frac{5}{3}\right]$, a first jump at $\frac{3}{2}$ of size $4p$, and a final jump at $\frac{5}{3}$ of size $\frac{9}{2}(1-p)$.

Simulations use $10,000$ replications. When the estimated quantity is common to all quality subsamples in the application as for $\underline{n}$, the sample size is set to $L = 200$. When the estimated quantity is specific to quality, as the number of discontinuities and p, the sample size is set to $L = 50$.

$M = .2 \times L$, $p = 0, .2,$ and $.4$ (top row), $p = .6, .8,$ and 1 (bottom row).



$M = .3 \times L$, $p = 0, .2,$ and $.4$ (top row), $p = .6, .8,$ and 1 (bottom row).

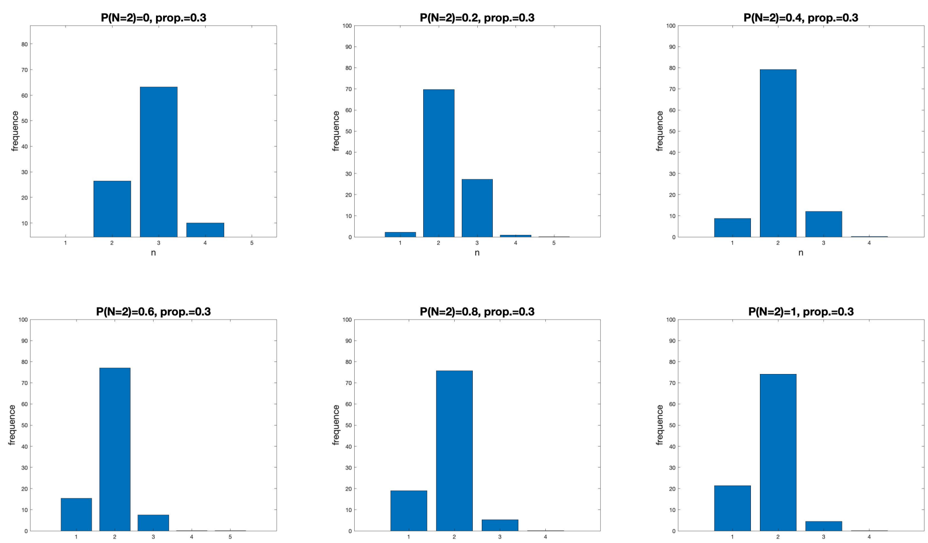
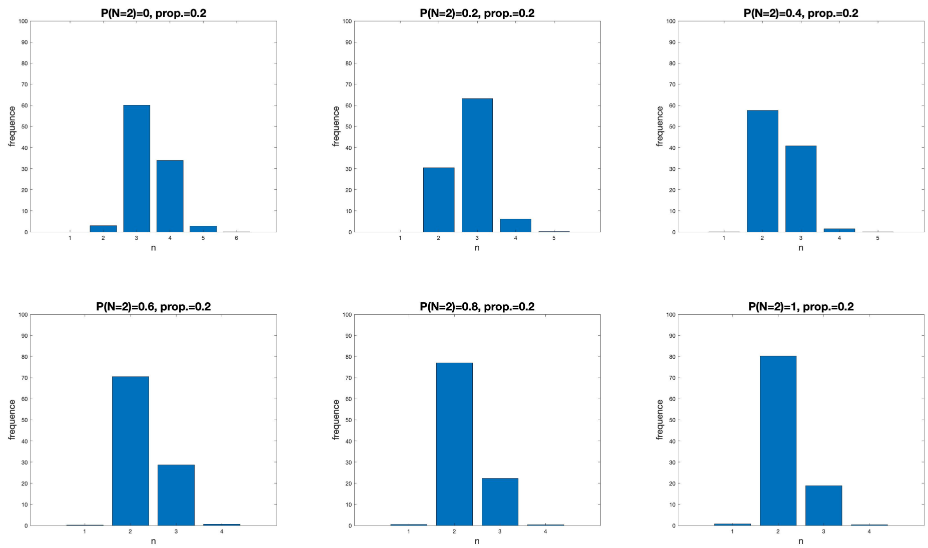


Figure 7: Hill estimation using estimated unconditional lowest bids as in (i). Sample size $L = 200$, with $10,000$ replications.

$M = .2 \times L$, $p = 0, .2$, and $.4$ (top row), $p = .6, .8$, and 1 (bottom row).



$M = .3 \times L$, $p = 0, .2$, and $.4$ (top row), $p = .6, .8$, and 1 (bottom row).

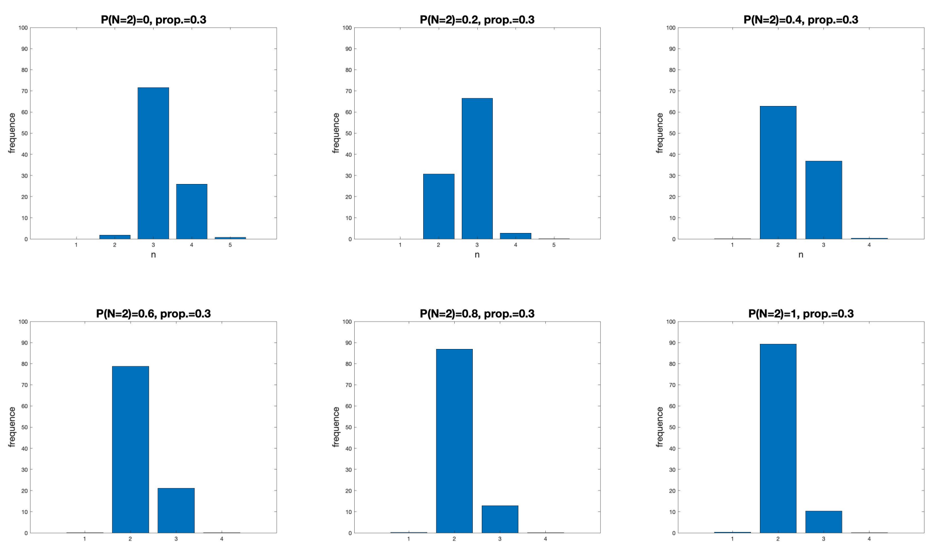


Figure 8: Hill estimation using estimated conditional lowest bids as in (ii). Sample size $L = 200$, with $10,000$ replications.

A.2 Estimation of the lowest number of bidders $\underline{n}$

Methodology. As explained in the main body of the paper, we consider two versions of the Hill estimator (29) corresponding to two different approaches to keep with the nuisance lower boundary $\underline{b}(\cdot)$:

(i). The unconditional lowest bid $\underline{b} = \min_x \underline{b}(x)$ can be estimated by $\widehat{\underline{b}} = \min_{i\ell} B_{i\ell}$. In (29), replace $b(X_\ell)$ by $\widehat{\underline{b}}$, and let $\widehat{\underline{n}}$ be the integer part of $\widetilde{\underline{n}} - 1$ as suggested by Lemma 3.2-(i);

(ii). Alternatively, estimate $\underline{b}(\cdot)$ using a covariate partition. Recall that auction-specific covariates consist of the appraisal value and volume, say $X_\ell = (X_{1\ell}, X_{2\ell})$. For a growing $K = K_L = o(L)$ and $k = 0, \dots, K$, let $\hat{X}_1(k/K)$ and $\hat{X}_2(k/K)$ be the sample k/K-th quantiles of $X_{1\ell}$ and $X_{2\ell}$. Let $\mathcal{X}_\ell$ be the set $(\hat{X}_1((k - 1/K), \hat{X}_1(k/K)] \times (\hat{X}_2((k-1/K), \hat{X}_2(k/K)]$ that contains X_ℓ and set

$$\widehat{\underline{b}}_\ell = \min_{(i,l):X_l \in \mathcal{X}_\ell} B_{il}.$$

In (29), replace $b(X_\ell)$ by $\widehat{\underline{b}}_\ell$, and let $\widehat{\underline{n}}$ be the integer part of $\widetilde{\underline{n}}$. In the simulations and application, K is set to 2.

As the bid density is bounded away from 0 at its lower bound, the proposed boundary estimators converge with superefficient rates, which should ensure consistency of the resulting Hill estimator. Alternative Hill estimation procedures for the conditional tail index, which does not use such normalization, can be found in Gardes and Stuffler (2014) and the references therein. Hill and Shneyerov (2013) show that $\sqrt{M}\,(\widetilde{\underline{n}} - \underline{n})$ converges in distribution to a centred normal with variance $\underline{n}$, a result that can be reasonably conjectured to hold when lower boundaries are estimated.

Simulation results. Figures 7 and 8 illustrate the behaviour of the proposed Hill procedures. We add $.1 \times X_{1\ell} + .1 \times X_{2\ell}$ for independent i.i.d. uniform covariates

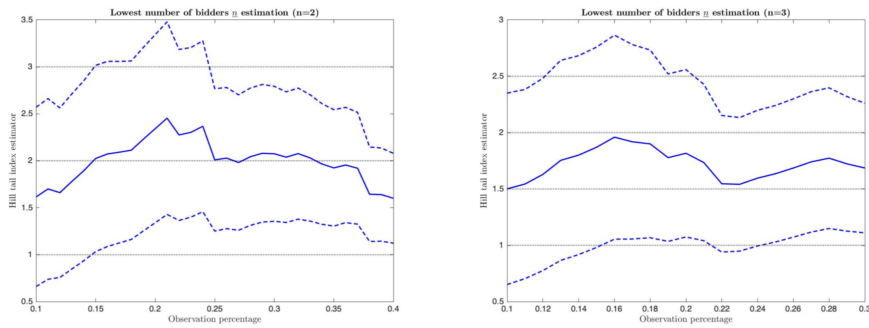


Figure 9: Hill estimates (ii) as a function of M/L. Two bidder (top) and three bidder (bottom) samples.

to the simulated private values and then bids to incorporate the effects of covariate variation. Two choices of M are considered, $.2 \times L$ and $.3 \times L$.

Both procedures work well, but using a nonparametric estimation of $\underline{b}(\cdot)$ as in (ii) seems to give a more concentrated distribution for $\widehat{\underline{n}}$, especially for $M = .3 \times L$ which gives better results and will be used in the applications. The nonparametric procedure gives higher $\widehat{\underline{n}}$ when p is low while the mode of the Hill procedure (i) is at $\widehat{\underline{n}} = 2$ for $p = .2$. When $p = 0$ so that $\underline{n} = 3$, the Hill procedure (i) has a 65% mode at $\underline{n} = 3$ and $\widehat{\underline{n}} \geq 3$ in 80% of the simulation draws. The nonparametric Hill procedure (ii) has a 70% mode at $\underline{n} = 3$ and $\widehat{\underline{n}} \geq 3$ in nearly 98% of the simulation draws. The nonparametric Hill estimator dominates the parametric procedure when $p \geq .6$, finding correctly that $\underline{n} = 2$ in nearly 90% of the simulation draws.

Application results. Figure 9 gives the unrounded Hill estimates $\widetilde{\underline{n}}$ as a function of the percentage of observations M/L, for the winning bid sample with $N = 2$ and $N = 3$. As the two procedures give similar results, we focus on the nonparametric version detailed in (ii). The Hill estimate's path looks reasonably stable. As expected, rounding $\widetilde{\underline{n}}$ when $M = .3 \times M$ gives an estimation $\widehat{\underline{n}} = 2$ for the lowest number of bidders in the two-bidder sample. The same estimate carries over when mixing the two-bidder and three-bidder samples. When focusing on the three-bidder sample, the estimate for the lowest number of bidders remains at $\widehat{\underline{n}} = 2$.

A.3 Detection of discontinuities

Discontinuity detection and jump size estimation methodology. Consider a subsample $A =$ Low, Medium, or High. Hereafter, we omit A for convenience. Let W_ℓ, $\ell = 1, \ldots, L$ be winning bids. First, we estimate a tentative discontinuity at each data point as the difference between the density estimates on its left and right sides. Consider a first "small" bandwidth h_0, set to 0.2 in the applications. The tentative discontinuity at $W_{(\ell)}$ is estimated using the difference of left and right k-NN density estimators

$$\widehat{\delta}_{h_0}(W_{(\ell)}) = \frac{\ell - \max\left(\ell - h_0 L/2, 1\right)}{L\left(W_{(\ell)} - W_{(\ell - h_0 L/2)}\right)} - \frac{\min\left(\ell + h_0 L/2, L\right) - \ell}{L\left(W_{(\ell + h_0 L/2)} - W_{(\ell)}\right)},$$

where $\ell - h_0 L/2$ is truncated to 1 if negative and $\ell + h_0 L/2$ to L if larger than L. Second, we estimate the magnitude of the density at each point and calculate a threshold. Define also the k-NN p.d.f estimator and the critical value[13]

$$\begin{aligned}
\widehat{g}_{h_0}(W_{(\ell)}) &= \frac{\min\left(\ell + h_0 L/2, L\right) - \max\left(\ell - h_0 L/2, 1\right)}{L\left(W_{(\ell + h_0 L/2)} - W_{(\ell - h_0 L/2)}\right)}, \\
C_{(\ell)}(\epsilon; h_0) &= \widehat{g}_{h_0}(W_{(\ell)}) \frac{c(\epsilon; h_0)}{\sqrt{L h_0}} \text{ with} \\
c(\epsilon; h_0) &= \sqrt{\ln(1/h_0)} + \frac{\ln\ln(1/h_0) - \ln(\pi) + 2\epsilon}{2\sqrt{\ln(1/h_0)}},
\end{aligned}$$

and where $\epsilon = \epsilon_L$ goes to 0 with L, and is set to 0.01 here.

[13]Note that the critical value $C_{(\ell)}(\epsilon; h_0)$ is proportional to the estimated p.d.f, as standard for k-NN estimation. This contrasts with the square root estimated p.d.f used in Chu and Cheng (1996) for their kernel approach.

We now use the tentative discontinuity estimates and thresholds to estimate locations of jump points and jump sizes. If

$$\widehat{\delta}_{h_0}(W_{(\ell_1^\star)}) = \max_{1 \leq \ell \leq L} \widehat{\delta}_{h_0}(W_{(\ell)})$$

is smaller than the critical value $C_{(\ell_1^\star)}(\epsilon; h_0)$, then the conditional winning bid p.d.f $g(\cdot)$ has no discontinuities. Otherwise, a discontinuity is found at $W_{(\ell_1^\star)}$, with an estimated jump $\widehat{\delta}_{h_0}(W_{(\ell_1^\star)})$. The next jump is searched for using the same procedure but excluding the indexes ℓ between $\ell_1^\star - h_0 L/2$ and $\ell_1^\star + h_0 L/2$. The procedure is then iterated until iteration $\widehat{q}$, such that the potential jump is smaller than $C_{(\ell_{\widehat{q}}^\star)}(\epsilon; h_0)$. The number of jumps is then $\widehat{q} - 1$, so that the estimation of the largest number $\overline{n}$ of bidders is $\widehat{\overline{n}} = \widehat{\underline{n}} + \widehat{q}$. Ordering the jumps locations $W_{(\ell_q^\star)}$ gives an estimation of the conditional bid support boundary $\widehat{\overline{b}}_{\widehat{\underline{n}}+q}$ and of the discontinuity jumps $\widehat{\Delta}_{\widehat{\underline{n}}+q}$. A conditional winning bid density estimator incorporating discontinuities is then, for $\ell_q^\star < \ell \leq \ell_{q+1}^\star$ with $\ell_0^\star = 1$,

$$\widehat{g}_{h_1}^d(W_{(\ell)}) = \frac{\min\left(\ell + h_1 L/2, \ell_{q+1}^\star\right) - \max\left(\ell - h_1 L/2, \ell_q^\star + 1\right)}{L\left(W_{\left(\min\left(\ell + h_1 L/2, \ell_{q+1}^\star\right)\right)} - W_{\left(\max\left(\ell - h_1 L/2, \ell_q^\star + 1\right)\right)}\right)},$$

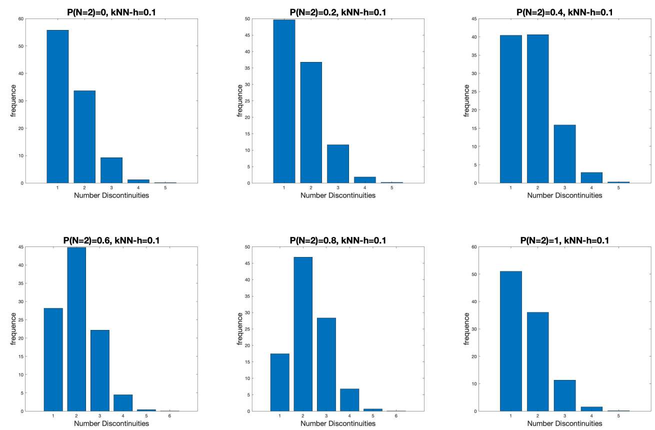
where the bandwidth $h_1 > h_0$ is set to 0.5 in our application. This is extended to the straight line using an additional smoothing step,

$$\widehat{g}^d(b) = \frac{\sum_{W_{(\ell)} \in \left[\widehat{\overline{b}}_{\widehat{\underline{n}}+q}, \widehat{\overline{b}}_{\widehat{\underline{n}}+q+1}\right]} \widehat{g}_{h_1}^d(W_{(\ell)}) K\left(\frac{w - W_{(\ell)}}{h_2}\right)}{\sum_{W_{(\ell)} \in \left[\widehat{\overline{b}}_{\widehat{\underline{n}}+q}, \widehat{\overline{b}}_{\widehat{\underline{n}}+q+1}\right]} K\left(\frac{w - W_{(\ell)}}{h_2}\right)} \text{ for } b \in \left[\widehat{\overline{b}}_{\widehat{\underline{n}}+q}, \widehat{\overline{b}}_{\widehat{\underline{n}}+q+1}\right],$$

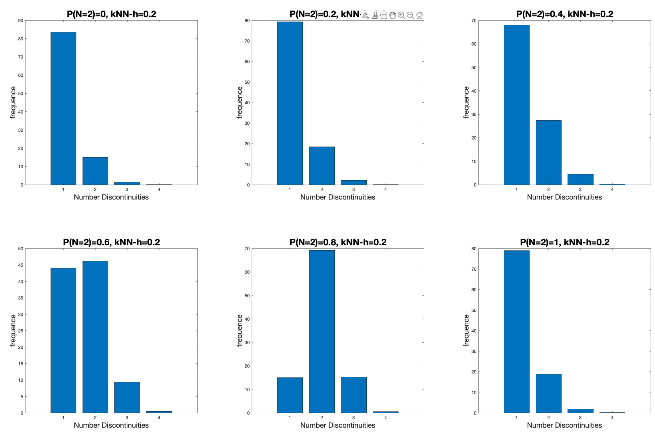
where $K(t) = (1 - t^2)\mathbb{I}(|t| \leq 1)$ is the Epanechnikov and h_2 is set to the median of $W_{(\ell + h_0 L/2)} - W_{(\ell - h_0 L/2)}$, $\ell = 1, \ldots, L$.

Simulation experiment: number of discontinuities Figure 10 reports simulation results for the estimation of the number of discontinuities of the winning bid p.d.f. As there is automatically one discontinuity at the upper support bound, the correct number of discontinuities is 1 if $\mathbb{P}(N = 2) = p$ is 0 or 1, and 2 for $p = .2$, .4, .6, or .8. The considered k-NN bandwidths are $h_0 = .1$, .2, and .3.

k-NN bandwidth $h_0 = .1$, $p = 0, .2$, and $.4$ (top row), $p = .6, .8$, and 1 (bottom row).



k-NN bandwidth $h_0 = .2$, $p = 0, .2$, and $.4$ (top row), $p = .6, .8$, and 1 (bottom row).



k-NN bandwidth $h_0 = .3$, $p = 0, .2$, and $.4$ (top row), $p = .6, .8$, and 1 (bottom row).

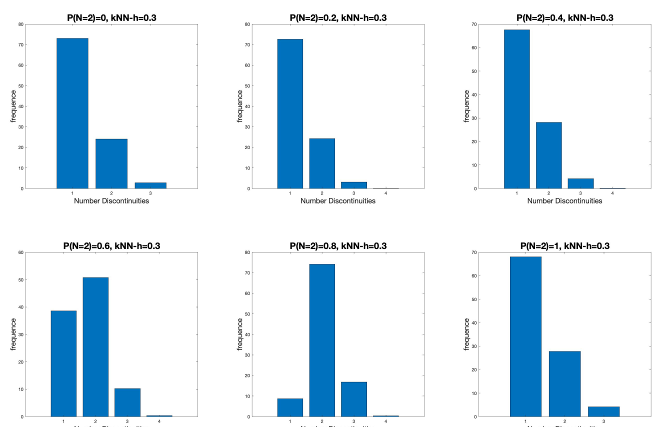


Figure 10: Estimation of the number of discontinuities. Sample size $L = 50$; $10,000$ replications.

$\mathbb{P}(N = 2)$	first location bias	std	second location bias	std
.1	-.31	.16	-.005	.005
.2	-.32	.17	-.005	.005
.3	-.32	.18	-.006	.006
.4	-.31	.19	-.007	.008
.5	-.29	.20	-.009	.009
.6	-.24	.21	-.011	.011
.7	-.18	.20	-.015	.016
.8	-.12	.18	-.023	.025
.9	-.12	.17	-.048	.045

Table 3: Simulation results for the estimation of the first discontinuity location 1/2 and the second 2/3. Bias and standard deviation for each. 50 observations and $h_0 = .2$

The k-NN bandwidth $h_0 = .1$ gives an unstable estimator, while using $h_0 = .3$ leads to an overestimation of the number of discontinuities compared to $h_0 = .2$, which is used in the applications. However, the corresponding estimator tends to underestimate the number of discontinuities for not only low values of $p > 0$, as expected, but also for intermediate ones. For instance, when $p = .6$, it only correctly finds the two discontinuities in 46% of the draws, detecting only one in 45% of the simulated samples. When all auctions have three bidders ($p = 0$), two discontinuities are misleadingly found in only 14% of the simulation draws.

Simulation experiment: discontinuity locations. Table 3 reports the simulation results for the estimation of discontinuity locations. The sample size is set to 50, a rounding of the sample size of the quality samples, and the bandwidths are as in

$\mathbb{P}(N=2)$	0	.1	.2	.3	.4	.5	.6	.7	.8	.9	1
Mean	.02	.12	.25	.37	.43	.48	.48	.48	.48	.51	.71
Standard deviation	.15	.34	.45	.50	.51	.50	.48	.43	.46	.44	.50

Table 4: Simulation results for the estimation of $\mathbb{P}(N = 2)$. Simulation mean and standard deviation for each value of this parameter. 50 observations and $h_0 = .2$

the application. Recall that the first discontinuity location is estimated using a k-NN version of Chu and Cheng (1996), while the second is estimated using the maximum winning bid. As the latter estimator is superefficient, the associated bias and standard deviation are small. However, they substantially increase when the proportion of auctions with three active bidders decrease, as expected. The first location estimator is less precise, with a larger standard deviation that depends not on participation but a bias which decreases with the proportion of auctions with two active bidders, as expected, as it increases the corresponding jump size.

Simulation experiment: participation distribution. We report now the simulation performance of the estimator of $\mathbb{P}(N = 2)$ used in the application. This estimator is obtained by plugging the location discontinuity, jump size, and $\underline{n}$ and $\overline{n}$ estimators into (10). Chu and Cheng (1996) did not report any simulation results for the jump size estimator. In an intermediary simulation experiment, we find it does not perform well for moderate sample sizes. This may also explain the large bias and variance observed in Table 4, especially for large values of $\mathbb{P}(N = 2)$.

Appendix B: Strategy uniqueness and differentiability — not for publication

Suppose there are $n \geq 2$ buyers. One draws his private value from the c.d.f $F_c^k(\cdot)$, where $k > 0$ does not need to be an integer number, and the other buyers draws from $F(\cdot)$. $F(\cdot)$ and $F_c(\cdot)$ have a common support $[\underline{v}, \overline{v}]$, over which their p.d.f $f(\cdot)$ and $f_c(\cdot)$ are continuous and bounded away from 0. These conditions follow from Assumption CIPV if $F_c(\cdot) = F(\cdot)$.

Let $\xi(\cdot)$ and $\xi_c(\cdot)$ be inverse strategies generated by a Bayesian Nash Equilibrium in the first-price auction. Lebrun (1997,1999) shows their existence, and that $\xi(\cdot)$ and $\xi_c(\cdot)$ are differentiable over $(\underline{b}, \overline{b}]$, where $[\underline{b}, \overline{b}]$ is the common bid support. Some works gives the values of $\xi^{(1)}(\underline{b})$ and $\xi_c(\underline{b})$ assuming existence of these derivatives, in some specific setups. See for instance Fibich, Gavious and Sela (2002). A purpose of this Appendix is first to show that the strategies are continuously differentiable over $[\underline{b}, \overline{b}]$, in which case explicit expressions for $\xi^{(1)}(\underline{b})$ and $\xi_c(\underline{b})$ are easily obtained.

Proposition B.1 *In the framework described above, any first-price auction inverse strategies $\xi(\cdot)$ and $\xi_c(\cdot)$ generated by a Bayesian Nash Equilibrium are continuously differentiable over $[\underline{b}, \overline{b}]$ with $\xi^{(1)}(b) > 0$ and $\xi_c^{(1)}(b) > 0$ for all b in $[\underline{b}, \overline{b}]$ and*

$$\xi^{(1)}(\underline{b}) = \frac{n+k-1}{n+k-2}, \quad \xi_c^{(1)}(\underline{b}) = \frac{n}{n-1}.$$

Proposition B.1 implies that $s^{(1)}(\underline{v}) = \frac{n+k-2}{n+k-1}$ and $s_c^{(1)}(\underline{v}) = \frac{n-1}{n}$.

When $k = 1$, the two initial derivatives are both equal to $\frac{n}{n-1}$, as obtained in Fibich et al. (2002, Proposition 2) when assuming differentiability at $\underline{b}$. The case $n = 2$ corresponds to a two-buyers auctions as considered in Lizzerri and Persico (2000), who allow in addition for common values but consider a binding reserve price.

Lebrun (1997,1999) obtains uniqueness of the Bayesian Nash Equilibrium bidding strategies when private value distribution have an "atom" at $\underline{v}$, $F(\underline{v}), F_c(\underline{v}) > 0$, or for $n = 2$ if $\frac{d}{dv}\frac{F(v)}{F_c(v)}$ keeps a constant sign over $[\underline{v}, \overline{v}]$. In the atomless case, Lebrun (2006) obtains uniqueness assuming $F(\cdot)$ and $F_c(\dot{})$ are strictly log-concave in the vicinity of $\underline{v}$. Proposition B.1 allows to establish uniqueness using the identical initial derivative value common to all bidding strategies generated by a Bayesian Nash Equilibrium. This reinforces the initial condition $\xi(\underline{b}) = \xi_c(\underline{b}) = \underline{b}$, allowing so to circumvent the induced singularities in the differential system (22)-(23).

Theorem B.1 *In the framework described above and if $f(\cdot)$ and $f_c(\cdot)$ are continuously differentiable over $[\underline{v}, \overline{v}]$, the Bayesian Nash Equilibrium, and then the optimal inverse strategies $(\xi(\cdot), \xi_c(\cdot))$, are unique.*

It is likely that, continuous differentiability of the private value p.d.f can be weakened to their continuity by reparametrizing the differential system (22)-(23) in term of $\zeta(\cdot) = \log F(\xi(\cdot))$ and $\zeta_c(\cdot) = \log F(\xi_c(\cdot))$ only. This is not attempted here for the sake of brevity.

B.1 A grow rate approach

Lebrun (1997,1999) establishes continuity over $[\underline{b}, \overline{b}]$ and differentiability over $(\underline{b}, \overline{b}]$ of the inverse strategies $\xi(\cdot)$ and $\xi_c(\cdot)$, which also satisfy $\xi(\underline{b}) = \xi_c(\underline{b}) = \underline{b}$, $\xi(b) > b$ and $\xi_c(b) > b$ for all b in $(\underline{b}, \overline{b}]$. The latter together with (23) and (22) implies $\xi^{(1)}(\cdot) > 0$ and $\xi_c^{(1)}(\cdot) > 0$ over $(\underline{b}, \overline{b}]$ as claimed in the Proposition.

To study differentiability at $\underline{b}$, define

$$\tilde{\xi}(t) = \xi\left(\underline{b} + \frac{1}{t}\right) - \underline{b}, \quad \tilde{\xi}_c(t) = \xi_c\left(\underline{b} + \frac{1}{t}\right) - \underline{b}, \quad t \geq \frac{1}{\overline{b}(k) - \underline{b}},$$
$$x(t) = t\tilde{\xi}(t) = t\left(\xi\left(\underline{b} + \frac{1}{t}\right) - \underline{b}\right) \text{ and } y(t) = t\tilde{\xi}(t).$$

The change of variable $t = \frac{1}{\delta}$ gives

$$x\left(\frac{1}{\delta}\right) = \frac{\xi\left(\underline{b}+\delta\right) - \xi\left(\underline{b}\right)}{\delta} \text{ and } y\left(\frac{1}{\delta}\right) = \frac{\xi_c\left(\underline{b}+\delta\right) - \xi_c\left(\underline{b}\right)}{\delta},$$

showing that $x\left(\frac{1}{\delta}\right)$ and $y\left(\frac{1}{\delta}\right)$ are the growth rates of $\xi(\cdot)$ and $\xi_c(\cdot)$ at $\underline{b}$, so that establishing differentiability amounts to show existence of limits when $\delta = 0$, in which case

$$\lim_{\delta\downarrow 0}\left(x\left(\frac{1}{\delta}\right), y\left(\frac{1}{\delta}\right)\right) = \lim_{t\uparrow\infty}\left(x(t), y(t)\right) = \left(\xi^{(1)}(\underline{b}), \xi_c^{(1)}(\underline{b})\right).$$

Observe also that, as $\xi(b), \xi_c(b) > b$ on $(\underline{b}, \overline{b}]$

$$x(t), y(t) > 1 \text{ for } t > \frac{1}{\overline{b}(k) - \underline{b}},\ \lim_{t\uparrow\infty}\tilde{\xi}(t) = \lim_{t\uparrow\infty}\tilde{\xi}_c(t) = 0. \tag{41}$$

Let $\xi^0(\cdot)$ and $\xi_c^0(\cdot)$ be the considered inverse strategies, see Lebrun (1999), and define $\tilde{\xi}^0(\cdot)$ and $\tilde{\xi}_c^0(\cdot)$ accordingly. Define also the "linearization" term errors

$$\varepsilon(t) = \frac{F\left(\underline{b} + \tilde{\xi}^0(t)\right)}{\tilde{\xi}^0(t) f\left(\underline{b} + \tilde{\xi}^0(t)\right)} - 1, \quad \varepsilon_c(t) = \frac{F\left(\underline{b} + \tilde{\xi}_c^0(t)\right)}{\tilde{\xi}_c^0(t) f\left(\underline{b} + \tilde{\xi}_c^0(t)\right)} - 1$$

which are both $o(1)$ when t grows by (41). In what follows, $\varepsilon(\cdot)$ and $\varepsilon_c(\cdot)$ are given, determined by the considered $\xi^0(\cdot)$ and $\xi_c^0(\cdot)$. As $\tilde{\xi}^{(1)}(t) = -\frac{1}{t^2}\xi^{(1)}\left(\underline{b} + \frac{1}{t}\right)$, (22) and (23) gives when t goes to infinity

$$\tilde{\xi}^{(1)}(t) = -\frac{1}{(n-1)t^2}\frac{F\left(\underline{b} + \tilde{\xi}(t)\right)}{f\left(\underline{b} + \tilde{\xi}(t)\right)}\frac{1}{\tilde{\xi}_c(t) - \frac{1}{t}} = -\frac{\left(1+\varepsilon(t)\right)t\tilde{\xi}(t)}{(n-1)t^2}\frac{1}{t\tilde{\xi}_c(t) - 1}, \tag{42}$$

$$\tilde{\xi}_c^{(1)}(t) = -\frac{\left(1+\varepsilon_c(t)\right)t\tilde{\xi}_c(t)}{k(n-1)t^2}\left(\frac{n-1}{t\tilde{\xi}(t) - 1} - \frac{n-2}{t\tilde{\xi}_c(t) - 1}\right). \tag{43}$$

A differential system for the growth rates. Because $\frac{d}{dt}[t\tilde{\xi}(t)] = t\tilde{\xi}^{(1)}(t) + \tilde{\xi}(t)$, it follows

$$\frac{d}{dt}[t\tilde{\xi}(t)] = -\frac{t\tilde{\xi}(t)}{(n-1)t}\left(\frac{1+\varepsilon(t)}{t\tilde{\xi}_c(t) - 1} - (n-1)\right),$$

$$\frac{d}{dt}[t\tilde{\xi}_c(t)] = -\frac{t\tilde{\xi}_c(t)}{k(n-1)t}\left(\frac{\left(1+\varepsilon_c(t)\right)(n-1)}{t\tilde{\xi}(t) - 1} - \frac{\left(1+\varepsilon_c(t)\right)(n-2)}{t\tilde{\xi}_c(t) - 1} - k(n-1)\right),$$

or equivalently

$$\frac{x^{(1)}(t)}{x(t)} = \frac{D\left(x(t), y(t)|\varepsilon(t)\right)}{t}, \tag{44}$$

$$\frac{y^{(1)}(t)}{y(t)} = (1+\varepsilon_c(t))\,\frac{D_c\left(x(t), y(t)|\varepsilon_c(t)\right)}{kt}, \tag{45}$$

with

$$D(x,y|\varepsilon) = \frac{1}{n-1}\left(n-1-\frac{1+\varepsilon}{y-1}\right),$$

$$D_c(x,y|\varepsilon_c) = \frac{1}{n-1}\left(\frac{k(n-1)}{1+\varepsilon_c}+\frac{n-2}{y-1}-\frac{n-1}{x-1}\right).$$

As elementary algebra gives

$$D(x,y|\varepsilon) = \frac{1}{y-1}\left(y-1-\frac{1+\varepsilon}{n-1}\right) = \frac{1}{y-1}\left(y-\frac{n+\varepsilon}{n-1}\right),$$

$$\begin{aligned} D_c(x,y|\varepsilon_c) &= \frac{n-2}{n-1}\left(\frac{1}{y-1}-(n-1)\right)-\left(\frac{1}{x-1}-(n-2)-\frac{k}{1+\varepsilon_c}\right) \\ &= \frac{n-2}{y-1}\left(\frac{1}{n-1}-(y-1)\right) \\ &\quad -\frac{(n-2)(1+\varepsilon_c)+k}{(1+\varepsilon_c)(x-1)}\left(\frac{1+\varepsilon_c}{(n-2)(1+\varepsilon_c)+k}-(x-1)\right) \\ &= \frac{(n-2)(1+\varepsilon_c)+k}{(1+\varepsilon_c)(x-1)}\left(x-\frac{(n-1)(1+\varepsilon_c)+k}{(n-2)(1+\varepsilon_c)+k}\right)-\frac{n-2}{y-1}\left(y-\frac{n}{n-1}\right), \end{aligned}$$

setting the linearization terms to 0 shows that $D(x,y|0)=0$ and $D_c(x,y|0)=0$ has a unique solution $\left(\frac{n+k-1}{n+k-2}, \frac{n}{n-1}\right)$, which corresponds to the derivative values stated in the Lemma.

Moreover, the curve $D(x,y|\varepsilon)=0$ is the horizontal straight line $y=\frac{n+\varepsilon}{n-1}$, above which $D(x,y|\varepsilon)$ is positive, being negative below. The curve $D_c(x,y|\varepsilon_c)=0$ is such that

$$x = 1+\frac{1}{\frac{k}{1+\varepsilon_c}+\frac{n-2}{(n-1)(y-1)}}$$

showing $D_c(x, y|\varepsilon_c) = 0$ lies between $(1, 1)$ and the asymptote $x = 1 + \frac{1+\varepsilon_c}{k}$ achieved when y goes to infinity. The function $D_c(x, y|\varepsilon_c)$ is positive on the right of $D_c(x, y|\varepsilon_c) = 0$, negative on its left. Figure 11 gives the corresponding asymptotic phase diagram of the differential system (44)-(45). Following (41), we restrict to the solutions $(x(\cdot), y(\cdot))$ which are such that $x(t) > 1$ and $y(t) > 1$.

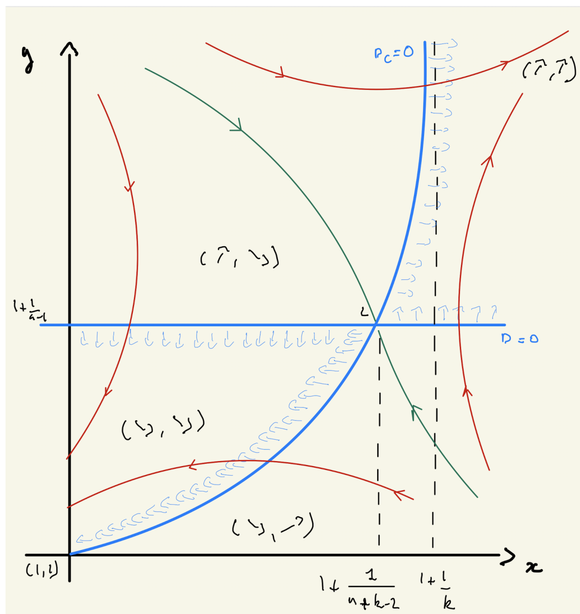


Figure 11: *Asymptotic phase diagram:* the small blue and black arrows indicates the variations of $x(\cdot)$ and $y(\cdot)$ from (44) and (45) setting $\varepsilon(\cdot)$ and $\varepsilon_c(\cdot)$ to 0. The blue lines are $D(x, y|0) = 0$ and $D_c(x, y|0) = 0$, at which $x^{(1)}(\cdot)$ and $y^{(1)}(\cdot)$ change sign. The two paths compatible with a first-price auction are in green, the incompatible ones in red.

Figure 11, which assumes $\varepsilon(\cdot) = 0$ and $\varepsilon_c(\cdot) = 0$, suggests that the growth rates can diverge, converge to limits on $x = 1$ or $y = 1$, or to the candidate limit $\left(\frac{n}{n-1}, \frac{n+k-1}{n+k-2}\right)$. The rest of the proof shows that the two first cases are not possible for solutions of (42)-(43), even in the presence of non zero linearization terms, so that proper auction inverse strategies must have derivatives $\left(\frac{n}{n-1}, \frac{n+k-1}{n+k-2}\right)$ at $\underline{b}$.

B.2 Intermediary claims

Define

$$(\nearrow,\searrow)_t = \left\{ (x,y) \in (1,\infty)^2 \,\middle|\, \begin{array}{c} D(x,y|\varepsilon(t)) > 0 \\ \text{and } D_c(x,y|\varepsilon_c(t)) < 0 \end{array} \right\},$$

$$(\nearrow,\nearrow)_t = \left\{ (x,y) \in (1,\infty)^2 \,\middle|\, \begin{array}{c} D(x,y|\varepsilon(t)) > 0 \\ \text{and } D_c(x,y|\varepsilon_c(t)) > 0 \end{array} \right\},$$

$$(\searrow,\nearrow)_t = \left\{ (x,y) \in (1,\infty)^2 \,\middle|\, \begin{array}{c} D(x,y|\varepsilon(t)) > 0 \\ \text{and } D_c(x,y|\varepsilon_c(t)) < 0 \end{array} \right\},$$

$$(\searrow,\searrow)_t = \left\{ (x,y) \in (1,\infty)^2 \,\middle|\, \begin{array}{c} D(x,y|\varepsilon(t)) < 0 \\ \text{and } D_c(x,y|\varepsilon_c(t)) < 0 \end{array} \right\},$$

abbreviating $(\nearrow,\searrow)_\infty$ into $(\nearrow,\searrow)$, so that

$$(\nearrow,\searrow) = \left\{ (x,y) \in (1,\infty)^2 \,|D(x,y|0) > 0 \text{ and } D_c(x,y|0) < 0 \right\},$$

$(\nearrow,\nearrow)$, $(\searrow,\nearrow)$ and $(\searrow,\searrow)$ being similarly defined. The blue curves of Figure 11 define the boundaries of these four sets. The sets $(\nearrow,\searrow)_t$, $(\nearrow,\nearrow)_t$, $(\searrow,\nearrow)_t$ and $(\searrow,\searrow)_t$ correspond to the four possible variations at t of $(x(t),y(t))$ solving the system (44,45). For instance, $x^{(1)}(t) > 0$ and $y^{(1)}(t) < 0$ if $(x(t),y(t))$ belongs to $(\nearrow,\searrow)_t$. The curves of Figure 11 give a rough qualitative description of the solutions of (44)-(45). It will be argued that only the ones in green can be generated by Bayesian Nash Equilibrium bidding strategies.

The first claim shows that the only possible limit in $(1,\infty)^2$ for solutions of (44) and (45) is $\left(\frac{n}{n-1}, \frac{n+k-1}{n+k-2}\right)$.

Claim B.1 *Suppose $(x(\cdot),y(\cdot))$ solves (44) and (45), and that $\lim_{t\uparrow\infty} x(t) = \ell$ and $\lim_{t\uparrow\infty} y(t) = \ell_c$ with ℓ and ℓ_c in $(1,\infty)$. Then it must hold*

$$\lim_{t\uparrow\infty} x(t) = \frac{n+k-1}{n+k-2} \text{ and } \lim_{t\uparrow\infty} y(t) = \frac{n}{n-1}.$$

Proof of Claim B.1. Note that (44) shows

$$x^{(1)}(t) = \frac{1+o(1)}{t}\frac{\ell}{\ell_c - 1}\left(\ell_c - \frac{n}{n-1} + o(1)\right)$$

when t grows. Then

$$x(t) = (1+o(1))\,\frac{\ell}{\ell_c - 1}\left(\ell_c - \frac{n}{n-1}\right)\ln t$$

and diverges if $\ell_c \neq \frac{n}{n-1}$, a contradiction. Then $\ell_c = \frac{n}{n-1}$. Now (45) yields similarly

$$y(t) = (1+o(1))\,\frac{(n+k-2)\ell_c}{k(\ell-1)}\left(\ell - \frac{n+k-1}{n+k-2}\right)\ln t$$

which implies again that it must hold $\ell = \frac{n+k-1}{n+k-2}$. $\square$

The two next Claims shows that growth rate paths of admissible inverse strategies cannot reach some interior parts of $(\nearrow, \nearrow)$ and $(\searrow, \searrow)$, hatched in red in Figure 12 below. From now on, we will use of the monotonous modifications of the linearization terms $\varepsilon(\cdot)$ and $\varepsilon_c(\cdot)$,

$$\overline{\varepsilon}(t) = \sup_{s\geq t}\left\{|\varepsilon(s)| + |\varepsilon_c(s)|\right\} \geq 0,\ \underline{\varepsilon}(t) = -\overline{\varepsilon}(t), \tag{46}$$

which are such that, for all $x, y > 1$ and $t > 0$,

$$D(x,y|\overline{\varepsilon}(t)) \leq D(x,y|\varepsilon(t)) \leq D(x,y|\underline{\varepsilon}(t)),$$
$$D_c(x,y|\overline{\varepsilon}(t)) \leq D_c(x,y|\varepsilon_c(t)) \leq D_c(x,y|\underline{\varepsilon}(t)).$$

Claim B.2 *Suppose $(\bar{x}(\cdot), \bar{y}(\cdot))$ solves (44) and (45), and that, for $\kappa > 1$, there exists a $t^\star > 0$ with $\overline{\varepsilon}(t^\star) \leq \frac{1}{2}$ such that*

$$D\left(\bar{x}(t^\star), \bar{y}(t^\star)\right|\kappa\overline{\varepsilon}(t^\star)) > 0 \text{ and } D_c\left(\bar{x}(t^\star), \bar{y}(t^\star)\right|\overline{\varepsilon}(t^\star)) > 0,$$

implying that $(\bar{x}(t^\star), \bar{y}(t^\star))$ *belongs to the interior of* $(\nearrow, \nearrow)$.

Then there is a large enough κ*, independent of* $(\bar{x}(\cdot), \bar{y}(\cdot))$*, such that* $(\bar{x}(t), \bar{y}(t))$ *belongs to* $(\nearrow, \nearrow)_t$ *for all* $t \geq t^\star$*. Moreover* $(\bar{x}(\cdot), \bar{y}(\cdot))$ *cannot be generated by an inverse strategy solving (42) and (43).*

Claim B.3 *Suppose* $(\bar{x}(\cdot), \bar{y}(\cdot))$ *solves (44) and (45), and that, for* $\kappa > 1$*, there exists a* $t^\star > 0$ *with* $\overline{\varepsilon}(t^\star) \leq \frac{1}{2}$ *such that*

$$D\left(\bar{x}(t^\star), \bar{y}(t^\star)\right| \kappa \underline{\varepsilon}(t^\star)) < 0 \text{ and } D_c\left(\bar{x}(t^\star), \bar{y}(t^\star)\right| \underline{\varepsilon}(t^\star)) < 0,$$

implying that $(\bar{x}(t^\star), \bar{y}(t^\star))$ *belongs to the interior of* $(\searrow, \searrow)$.

Then there is a large enough κ*, independent of* $(\bar{x}(\cdot), \bar{y}(\cdot))$*, such that* $(\bar{x}(t), \bar{y}(t))$ *belongs to* $(\searrow, \searrow)_t$ *for all* $t \geq t^\star$*. Moreover* $(\bar{x}(\cdot), \bar{y}(\cdot))$ *cannot be generated by an inverse strategy solving (42) and (43).*

Proof of Claim B.2. The first step of the proof checks that $(\bar{x}(t), \bar{y}(t))$ belongs to $(\nearrow, \nearrow)_t$ for all $t \geq t^\star$.[14] Note that, by definition of $\overline{\varepsilon}(t^\star)$ and $\varepsilon_c^M(t^\star)$,

$$D\left(\bar{x}(t^\star), \bar{y}(t^\star)\right| \varepsilon(t^\star)) > 0 \text{ and } D_c\left(\bar{x}(t^\star), \bar{y}(t^\star)\right| \varepsilon_c(t^\star)) > 0.$$

Observe that the slope of the tangent of $t \mapsto (\bar{x}(t), \bar{y}(t))$ at t is, by (44) and (45),

$$sl(t) = \frac{\bar{y}^{(1)}(t)}{\bar{x}^{(1)}(t)} = \frac{\bar{y}(t) D_c\left(\bar{x}(t), \bar{y}(t)\right| \varepsilon_c(t))}{k\bar{x}(t) D\left(\bar{x}(t), \bar{y}(t)\right| \varepsilon(t))}.$$

Let

$$t_0 = \inf\left\{t \geq t^\star \right| D\left(\bar{x}(t), \bar{y}(t)|\varepsilon(t)\right) \leq 0\}, \text{ so that, if } t_0 < \infty,\ D\left(\bar{x}(t_0), \bar{y}(t_0)|\varepsilon(t_0)\right) = 0,$$

$$t_c = \inf\left\{t \geq t^\star \right| D_c\left(\bar{x}(t), \bar{y}(t)|\varepsilon_c(t)\right) \leq 0\}, \text{ so that, if } t_c < \infty,\ D_c\left(\bar{x}(t_c), \bar{y}(t_c)|\varepsilon_c(t_c)\right) = 0.$$

[14] This is straightforward to show in the absence of the linearization terms $(\varepsilon(\cdot), \varepsilon_c(\cdot))$ as $(\bar{x}(\cdot), \bar{y}(\cdot)$ cannot cross the frontiers $D(x, y|0) = 0$ downward and $D_c(x, y|0)$ upward. In the presence of $(\varepsilon(\cdot), \varepsilon_c(\cdot))$, $(\bar{x}(s), \bar{y}(s)$ can be above the curve $D_c(x, y|\varepsilon_c(t)) = 0$ for all $s \leq t$ close enough to t, so downward crossings could not be excluded, and could even be necessary for the result to hold. This complicates the proof, which considers crossings of the curve $D_c(x, y|\overline{\varepsilon}(t^\star)) = 0$.

Hence, if finite, t_0 satisfies

$$\bar{y}(t_0) = \frac{n+\varepsilon(t_0)}{n-1} \leq \frac{n+\overline{\varepsilon}(t^\star)}{n-1} < \bar{y}(t^\star),$$

with $\bar{y}^{(1)}(t^\star) > 0$ by (44). It follows that $\bar{y}(\cdot)$ must decrease for some t between $t^\star$ and t_0, yielding $D_c\left(\bar{x}(t), \bar{y}(t)|\varepsilon(t)\right) < 0$, so that it must be that $t_c < t_0$ if $t_0 < \infty$. Now, if t_c is finite, it holds since $\varepsilon_c(t_c) \leq \overline{\varepsilon}(t^\star)$,

$$0 = D_c(x, y|\varepsilon_c(t_c)) = \frac{k(n-1)}{1+\varepsilon_c(t_c)} + \frac{n-2}{y-1} - \frac{n-1}{x-1} \geq D_c(x, y|\overline{\varepsilon}(t^\star)),$$

implying that there is a $t_c^\star$

$$t_c^\star = \inf\left\{t \in [t^\star, t_c] \middle| \, D_c(\bar{x}(t), \bar{y}(t)|\overline{\varepsilon}(t^\star)) \leq 0\right\}$$

at which $(\bar{x}(\cdot), \bar{y}(\cdot))$ crosses the curve $y_c^\star(\cdot)$ with $D_c(x, y_c^\star(x)|\overline{\varepsilon}(t^\star)) = 0$ from below,

$$\bar{y}(t_c^\star) = y_c^\star\left(\bar{x}(t_c^\star)\right) \text{ and } sl(t_c^\star) \geq \frac{dy_c^\star\left(\bar{x}(t_c^\star)\right)}{dx}.$$

We now work out the implications of the inequality above. Note that $D_c(x, y_c^\star(x)|\overline{\varepsilon}(t^\star)) = 0$ gives

$$\begin{aligned} x &= 1 + \frac{1}{\frac{k}{1+\overline{\varepsilon}(t^\star)} + \frac{n-2}{(n-1)(y_c^\star(x)-1)}} = 1 + \frac{(n-1)(y_c^\star(x)-1)}{\frac{k(n-1)}{1+\overline{\varepsilon}(t^\star)}\left(y_c^\star(x)-1\right)+n-2}, \\ \frac{dy_c^\star(x)}{dx} &= \frac{n-1}{n-2}\left(\frac{y_c^\star(x)-1}{x-1}\right)^2 = \frac{1}{(n-2)(n-1)}\left(\frac{k(n-1)}{1+\overline{\varepsilon}(t^\star)}\left(y_c^\star(x)-1\right)+n-2\right)^2 \\ &\geq \frac{4}{9}k\left(y_c^\star(x)-1\right)^2, \end{aligned}$$

while $D_c(\bar{x}(t_c^\star), \bar{y}(t_c^\star)|\overline{\varepsilon}(t^\star)) = 0$ and $\bar{y}(t_c^\star) - \frac{n+\varepsilon(t_c^\star)}{n-1} \geq \bar{y}(t^\star) - \frac{n+\varepsilon(t_c^\star)}{n-1} \geq \frac{(\kappa-1)\overline{\varepsilon}(t^\star)}{n-1}$ imply

$$sl(t_c^\star) = \frac{\bar{y}(t_c^\star)k(n-1)\frac{\overline{\varepsilon}(t^\star)-\varepsilon_c(t_c^\star)}{(1+\overline{\varepsilon}(t^\star))(1+\varepsilon_c(t_c^\star))}}{\frac{k\bar{x}(t_c^\star)}{\bar{y}(t_c^\star)-1}\left(\bar{y}(t_c^\star) - \frac{n+\varepsilon(t_c^\star)}{n-1}\right)} \leq \frac{4(n-1)^2}{\kappa-1}\bar{y}(t_c^\star)\left(\bar{y}(t_c^\star)-1\right).$$

Hence the inequality $sl(t_c^\star) \geq \frac{dy_c^\star(\bar{x}(t_c^\star))}{dx}$ implies

$$\frac{4(n-1)^2}{\kappa-1}\bar{y}(t_c^\star) \geq \frac{4}{9}k\left(\bar{y}(t_c^\star)-1\right)$$

which is impossible if κ is taken large enough as $\bar{y}(t_c^\star) > \bar{y}(t^\star) \geq \frac{n}{n-1} > 1$. It follows that $t_c = t_0 = +\infty$, so that $D(\bar{x}(t), \bar{y}(t)|\varepsilon(t)), D_c(\bar{x}(t), \bar{y}(t)|\varepsilon_c(t)) > 0$ for all $t \geq t^\star$, so that $\bar{x}(\cdot)$ and $\bar{y}(\cdot)$ are both increasing over $[t^\star, \infty)$, see also Figure 11.

The next step shows that $(\bar{x}(t), \bar{y}(t))$ cannot be generated by an inverse strategy. From now on, write $(\bar{x}(t), \bar{y}(t))$ as $\left(t\bar{\xi}(t), t\bar{\xi}_c(t)\right)$. We show that assuming that (43) holds, as requested for inverse strategies, gives a contradiction. As $t\bar{\xi}(t)$ and $t\bar{\xi}_c(t)$ are both strictly increasing for $t \geq t^\star$, Claim B.1 then implies that at least one of these functions must diverge when t grows.

Suppose first that $t\bar{\xi}_c(t)$ diverges. Then (43) implies

$$\bar{\xi}_c^{(1)}(t) = \frac{n-2}{k(n-1)} \frac{1+o(1)}{t^2} - \frac{1+o(1)}{k} \frac{t\bar{\xi}_c(t)}{t^2\left(t\bar{\xi}(t)-1\right)}$$

and then as $\int_t^\infty \frac{u\bar{\xi}_c(u)}{u^2\left(u\bar{\xi}(u)-1\right)} du \geq 0$, it holds

$$\begin{aligned} t\bar{\xi}_c(t) &= t\left(\frac{n-2}{k(n-1)} \frac{1+o(1)}{t} - \frac{1+o(1)}{k} \int_t^\infty \frac{u\bar{\xi}_c(u)}{u^2\left(u\bar{\xi}(u)-1\right)} du\right) \\ &\leq \frac{n-2}{k(n-1)} + o(1) < 1 \text{ for } t \text{ large enough,} \end{aligned}$$

which contradicts $t\bar{\xi}_c(t) > 1$ from (41).

Suppose now that $t\bar{\xi}(t)$ diverges and that the increasing $t\bar{\xi}_c(t) > 1$ stays bounded from infinity when t grows. Then (45) gives for t large enough

$$\begin{aligned} \frac{d}{dt}\left[\ln\left(t\bar{\xi}_c(t)\right)\right] &= \frac{1}{k(n-1)t}\left(k(n-1) + \frac{(1+\varepsilon_c(t))\,(n-2)}{t\bar{\xi}_c(t)-1} - \frac{(1+\varepsilon_c(t))\,(n-1)}{t\bar{\xi}(t)-1}\right) \\ &\geq \frac{C}{t} \end{aligned}$$

so that $\ln\left(t\bar{\xi}_c(t)\right) \geq C\ln t$ and $t\bar{\xi}_c(t)$ must diverge, a contradiction. □

Proof of Claim B.3. Showing that $(\bar{x}(t), \bar{y}(t))$ belongs to $(\searrow, \searrow)_t$ for all $t \geq t^\star$ is similar to the first step of proof of Claim B.2 and will not be repeated. From now on, write $(\bar{x}(t), \bar{y}(t))$ as $\left(t\bar{\xi}(t), t\bar{\xi}_c(t)\right)$. Under the conditions of the Claim, $t\bar{\xi}(t) > 1$ and $t\bar{\xi}_c(t) > 1$ both strictly decrease with t, so that these functions should converge to some limits $1 \leq \ell < \frac{n+k-1}{n+k-2}$ and $1 \leq \ell_c < \frac{n}{n-1}$, respectively. By Claim B.1, one of these limit should be 1. The proof now works by showing that some of these limits are reached for a finite t.

Suppose first that $\lim_{t\uparrow\infty} t\bar{\xi}_c(t) = 1$. Assume that $t\bar{\xi}_c(t) > 1$ for all t, so that the LHS of (42) exists. Then the differential equation (42) gives that for some $\eta(t) = \frac{(n-1)(t\bar{\xi}_c(t)-1)}{1+\varepsilon(t)} > 0$ for t large enough with $\eta(t) = o(1)$, $\frac{\bar{\xi}^{(1)}(t)}{\bar{\xi}(t)} = -\frac{1}{t\eta(t)}$ implying by integration between $t/2$ and t,

$$\ln \frac{\bar{\xi}(t)}{\bar{\xi}(t/2)} = -\int_{\frac{t}{2}}^{t} \frac{du}{u\eta(u)} \leq \max_{u\in\left[\frac{t}{2},t\right]} \left\{-\frac{1}{\eta(u)}\right\} \ln 2 \to -\infty,$$

while

$$\ln \frac{\bar{\xi}(t)}{\bar{\xi}(t/2)} = \ln \frac{\frac{\ell(1+o(1))}{t}}{\frac{\ell(1+o(1))}{t/2}} \to -\ln 2$$

a contradiction. Then it must hold that $t\bar{\xi}_c(t) = 1$ for t large enough, and (41) cannot hold, so that such $\left(\bar{\xi}(\cdot), \bar{\xi}_c(\cdot)\right)$ are not Bayesian Nash Equilibrium inverse strategies.

Suppose now that $\lim_{t\uparrow\infty} t\bar{\xi}(t) = 1$ with $\lim_{t\uparrow\infty} t\bar{\xi}_c(t) > 1$. Assume that $t\bar{\xi}(t) > 1$ for all t, so that the LHS of (43) is well-defined. Then arguing as above with the differential equation (43) gives a similar contradiction unless $t\bar{\xi}(t) = 1$ for t large enough, so that the pair $\left(\bar{\xi}(\cdot), \bar{\xi}_c(\cdot)\right)$ cannot be inverse strategies. □

The four next Claims show that growth rates of admissible inverse strategies cannot enter some vicinities of $D(x, y|0) = 0$ and $D_c(x, y|0) = 0$, see the orange and purple areas of Figure 12 below.

Claim B.4 *Let $\kappa > 1$ be as in Claim B.2, and consider a large enough $\kappa_1 > \kappa$. Suppose $(\bar{x}(\cdot), \bar{y}(\cdot))$ solves (44) and (45) with $\bar{x}(t), \bar{y}(t) > 1$ for all t. Then if there exists a large enough $t^\dagger$ with in particular $\overline{\varepsilon}(t^\dagger) < \frac{1}{2}$,*

$$D\left(\bar{x}(t^\dagger), \bar{y}(t^\dagger)\middle|\, \kappa_1\overline{\varepsilon}(t^\dagger)\right) > 0 \text{ and } D_c\left(\bar{x}(t^\dagger), \bar{y}(t^\dagger)\middle|\, \underline{\varepsilon}(t^\dagger)\right) \geq 0,$$

then there exists a $t^\star \geq t^\dagger$ *such that* $\overline{\varepsilon}(t^\star) < \frac{1}{2}$,

$$D\left(\bar{x}(t^\star), \bar{y}(t^\star)\middle|\, \kappa\overline{\varepsilon}(t^\star)\right) > 0 \text{ and } D_c\left(\bar{x}(t^\star), \bar{y}(t^\star)\middle|\, \overline{\varepsilon}(t^\star)\right) > 0.$$

Moreover $(\bar{x}(\cdot), \bar{y}(\cdot))$ *cannot be generated by inverse strategies solving (42) and (43).*

Claim B.5 *Let* $\kappa > 1$ *be as in Claim B.3, and consider a large enough* $\kappa_1 > \kappa$. *Suppose* $(\bar{x}(\cdot), \bar{y}(\cdot))$ *solves (44) and (45) with* $\bar{x}(t), \bar{y}(t) > 1$ *for all* t. *Then if there exists a large enough* $t^\dagger$ *with in particular* $\overline{\varepsilon}(t^\dagger) < \frac{1}{2}$,

$$D\left(\bar{x}(t^\dagger), \bar{y}(t^\dagger)\middle|\, \kappa_1\overline{\varepsilon}(t^\dagger)\right) < 0 \text{ and } D_c\left(\bar{x}(t^\dagger), \bar{y}(t^\dagger)\middle|\, \overline{\varepsilon}(t^\dagger)\right) \leq 0,$$

then there exists a $t^\star \geq t^\dagger$ *such that* $\overline{\varepsilon}(t^\star) < \frac{1}{2}$,

$$D\left(\bar{x}(t^\star), \bar{y}(t^\star)\middle|\, \kappa\overline{\varepsilon}(t^\star)\right) < 0 \text{ and } D_c\left(\bar{x}(t^\star), \bar{y}(t^\star)\middle|\, \underline{\varepsilon}(t^\star)\right) < 0.$$

Moreover $(\bar{x}(\cdot), \bar{y}(\cdot))$ *cannot be generated by inverse strategies solving (42) and (43).*

Claim B.4 improves the constraint $D_c\left(\bar{x}(t^\star), \bar{y}(t^\star)\middle|\, \overline{\varepsilon}(t^\star)\right) > 0$ of Claim B.2 to $D_c\left(\bar{x}(t^\dagger), \bar{y}(t^\dagger)\middle|\, \underline{\varepsilon}(t^\dagger)\right) \geq 0$, observing that $-\frac{1}{2} \leq \underline{\varepsilon}(t^\dagger) \leq 0 \leq \overline{\varepsilon}(t^\star)$ yields that $D_c\left(x, y\middle|\, \overline{\varepsilon}(t^\star)\right) \leq D_c\left(x, y\middle|\, \underline{\varepsilon}(t^\dagger)\right)$. In particular, $(\bar{x}(t^\dagger), \bar{y}(t^\dagger))$ can now belong to the closure of $(\nearrow, \searrow)_{t^\dagger}$, as $\varepsilon(t^\dagger) \geq \underline{\varepsilon}(t^\dagger)$ yields $D_c\left(x, y\middle|\, \varepsilon(t^\dagger)\right) \leq D_c\left(x, y\middle|\, \underline{\varepsilon}(t^\dagger)\right)$, so that $D_c\left(x, y\middle|\, \varepsilon(t^\dagger)\right)$ can be negative. See the orange hatched area of Figure 12 below. The existence of $t^\star$ implies that $(\bar{x}(\cdot), \bar{y}(\cdot))$ enters the interior $(\nearrow, \nearrow)_{t^\star}$ and satisfies the conditions of Claim B.2, so that it cannot be rationalized by a first-price auction game. Claim B.5 has a similar interpretation.

Proof of Claims B.4 and B.5. We focus on Claim B.4 as Claim B.5 can be proven similarly. Note that for any $t^\star \geq t^\dagger$ it holds $0 \leq \overline{\varepsilon}(t^\star) \leq \overline{\varepsilon}(t^\dagger) \leq \frac{1}{2}$. Assume that $D_c(\bar{x}(t^\dagger), \bar{y}(t^\dagger)|\overline{\varepsilon}(t^\dagger)) \leq 0$, otherwise taking $t^\star = t^\dagger$ ends the proof of the Claim.

Let $\mathcal{D}$ be the domain having the curve $D_c(x, y|\underline{\varepsilon}(t^\dagger)) = 0$ as its left boundary and $D_c(x, y|\overline{\varepsilon}(t^\dagger)) = 0$ as its right one, with $x, y > 1$. Hence, for all (x, y) of $\mathcal{D}$,

$$D_c(x, y|\underline{\varepsilon}(t^\dagger)) \geq 0 \text{ and } D_c(x, y|\overline{\varepsilon}(t^\dagger)) \leq 0.$$

Under the restriction above $(\bar{x}(t^\dagger), \bar{y}(t^\dagger))$ is in $\mathcal{D}$.

Suppose that, for some $t \geq t^\dagger$, $(\bar{x}(t), \bar{y}(t))$ is in the domain $\mathcal{D}$. Then

$$D_c(\bar{x}(t), \bar{y}(t)|\overline{\varepsilon}(t^\dagger)) \leq 0 \leq D_c(\bar{x}(t), \bar{y}(t)|\underline{\varepsilon}(t^\dagger)),$$

so that there is a ε in $[\underline{\varepsilon}(t^\dagger), \overline{\varepsilon}(t^\dagger)]$ such that $D_c(\bar{x}(t), \bar{y}(t)|\varepsilon) = 0$ by the Intermediate Value Theorem. As $\varepsilon_c(t)$ is also in $[\underline{\varepsilon}(t^\dagger), \overline{\varepsilon}(t^\dagger)]$, it holds

$$\begin{aligned} D_c(\bar{x}(t), \bar{y}(t)|\varepsilon_c(t)) &= D_c(\bar{x}(t), \bar{y}(t)|\varepsilon_c(t)) - D_c(\bar{x}(t), \bar{y}(t)|\varepsilon) \\ &= k(n-1)\left(\frac{1}{1+\varepsilon_c(t)} - \frac{1}{1+\varepsilon}\right) = k(n-1)\frac{\varepsilon - \varepsilon_c(t)}{(1+\varepsilon)(1+\varepsilon_c(t))} \\ &\in 8k(n-1)\left[-\overline{\varepsilon}(t^\dagger), \overline{\varepsilon}(t^\dagger)\right]. \end{aligned}$$

The proof works by contradiction. Suppose $(\bar{x}(\cdot), \bar{y}(\cdot))$ never exits $\mathcal{D}$ by crossing the RHS boundary $D_c(x, y|\overline{\varepsilon}(t^\dagger)) = 0$. As the lower bound above implies

$$\bar{y}^{(1)}(t) = \bar{y}(t)\left(1 + \varepsilon_c(t)\right)\frac{D_c(\bar{x}(t), \bar{y}(t)|\varepsilon_c(t))}{kt} \geq -\bar{y}(t)\frac{8(n-1)\overline{\varepsilon}(t^\dagger)}{t},$$

so that, for those $t \leq \exp\left(\frac{\lambda}{8(n-1)n}\right)t^\dagger$ and since $\bar{y}(t^\dagger) \geq \frac{n+\kappa_1\overline{\varepsilon}(t^\dagger)}{n-1}$,

$$\begin{aligned} \bar{y}(t) \geq \bar{y}(t^\dagger)\left(\frac{t}{t^\dagger}\right)^{-8(n-1)\overline{\varepsilon}(t^\dagger)} &\geq \frac{n+\kappa_1\overline{\varepsilon}(t^\dagger)}{n-1}\left(\frac{t}{t^\dagger}\right)^{-8(n-1)\overline{\varepsilon}(t^\dagger)} \\ \geq \frac{n+\kappa_1\overline{\varepsilon}(t^\dagger)}{n-1}\left(1 - \frac{\lambda}{n}\overline{\varepsilon}(t^\dagger) + O\left(\overline{\varepsilon}^2(t^\dagger)\right)\right) &= \frac{n+(\kappa_1-\lambda)\overline{\varepsilon}(t^\dagger)}{n-1} + O\left(\overline{\varepsilon}^2(t^\dagger)\right) \end{aligned}$$

when $\overline{\varepsilon}(t^\dagger)$ goes to 0, that is if $t^\dagger$ diverges. Hence if $\kappa_1 - \lambda$ is large enough, $\bar{y}(t) \geq \frac{n+\kappa_1\overline{\varepsilon}(t^\dagger)}{n-1}$ and $\bar{x}^{(1)}(t) > 0$ for all $t \geq t^\dagger$ less than $\exp\left(\frac{\lambda}{8(n-1)n}\right)t^\dagger$.

As $(\bar{x}(t),\bar{y}(t))$ lies in $\mathcal{D}$ for all $t \geq t^{\dagger}$, it then holds

$$\bar{y}(t) \geq \frac{n}{n-1} + O\left(\overline{\varepsilon}(t^{\dagger})\right),$$
$$\bar{x}(t) \leq \frac{k+1}{k} + O\left(\overline{\varepsilon}(t^{\dagger})\right).$$

It follows, for some $C_x, C_y > 0$ which do not depend upon κ, κ_1 and λ,

$$\begin{aligned}\frac{d}{dt}\left[\frac{n-2}{\bar{y}(t)-1} - \frac{n-1}{\bar{x}(t)-1}\right] &= -\frac{n-2}{\left(\bar{y}(t)-1\right)^2}\bar{y}^{(1)}(t) + \frac{n-1}{\left(\bar{x}(t)-1\right)^2}\bar{x}^{(1)}(t) \\ &= -\frac{(n-2)\bar{y}(t)D_c\left(\bar{x}(t),\bar{y}(t)|\,\varepsilon_c(t)\right)}{\left(\bar{y}(t)-1\right)^2 t} + \frac{(n-1)\bar{x}(t)D\left(\bar{x}(t),\bar{y}(t)|\,\varepsilon(t)\right)}{\left(\bar{x}(t)-1\right)^2 t} \\ &\geq \frac{-C_y\overline{\varepsilon}(t^{\dagger}) + C_x(\kappa_1-\lambda-1)\overline{\varepsilon}(t^{\dagger}) + O\left(\overline{\varepsilon}^2(t^{\dagger})\right)}{t}\end{aligned}$$

for all t in $\left[t^{\dagger}, \exp\left(\frac{\lambda}{8(n-1)n}\right)t^{\dagger}\right]$. Hence, since

$$\begin{aligned}&\frac{n-2}{\bar{y}\left(\exp\left(\frac{\lambda}{8(n-1)n}\right)t^{\dagger}\right)-1} - \frac{n-1}{\bar{x}\left(\exp\left(\frac{\lambda}{8(n-1)n}\right)t^{\dagger}\right)-1} - \left(\frac{n-2}{\bar{y}(t^{\dagger})-1} - \frac{n-1}{\bar{x}(t^{\dagger})-1}\right) \\ &\quad\leq \frac{k(n-1)}{1-\overline{\varepsilon}(t^{\dagger})} - \frac{k(n-1)}{1+\overline{\varepsilon}(t^{\dagger})} \leq 8k(n-1)\overline{\varepsilon}(t^{\dagger}),\end{aligned}$$

it must holds

$$\begin{aligned}8k(n-1)\overline{\varepsilon}(t^{\dagger}) &\geq \int_{t^{\dagger}}^{\exp\left(\frac{\lambda}{8(n-1)n}\right)t^{\dagger}} \frac{d}{dt}\left[\frac{n-2}{\bar{y}(t)-1} - \frac{n-1}{\bar{x}(t)-1}\right]dt \\ &\geq \int_{t^{\dagger}}^{\exp\left(\frac{\lambda}{8(n-1)n}\right)t^{\dagger}} \frac{\overline{\varepsilon}(t^{\dagger})\left(C_x(\kappa_1-\lambda-1)-C_y\right) + O\left(\overline{\varepsilon}^2(t^{\dagger})\right)}{t}dt \\ &= \left[\left(C_x(\kappa_1-\lambda-1)-C_y\right)\overline{\varepsilon}(t^{\dagger}) + O\left(\overline{\varepsilon}^2(t^{\dagger})\right)\right]\frac{\lambda}{8(n-1)n},\end{aligned}$$

which cannot hold for $\lambda = 1$ provided κ_1 is large enough, if $t^{\dagger}$ is large enough. Hence $(\bar{x}(\cdot),\bar{y}(\cdot))$ must exit $\mathcal{D}$ by crossing the RHS boundary $D_c(x,y|\overline{\varepsilon}(t^{\dagger})) = 0$. After this crossing $(\bar{x}(t),\bar{y}(t))$ is in $(\nearrow,\nearrow)_t$, which implies the conclusion of the Claim. □

Claim B.6 *Let $\kappa > 1$ be as in Claim B.2 and consider a large enough $\kappa_2 > 1$. Suppose $(\bar{x}(\cdot),\bar{y}(\cdot))$ solves (44) and (45) with $\bar{x}(t),\bar{y}(t) > 1$ for all t. Then if there* *exists a large enough* $t^\dagger$ *with in particular* $\overline{\varepsilon}(t^\dagger) < \frac{1}{2\kappa}$,

$$D\left(\bar{x}(t^\dagger), \bar{y}(t^\dagger)\middle|\, \kappa\underline{\varepsilon}(t^\dagger)\right) > 0 \text{ and } D_c\left(\bar{x}(t^\dagger), \bar{y}(t^\dagger)\middle|\, \kappa_2\overline{\varepsilon}(t^\dagger)\right) \geq 0,$$

then there exists a $t^\star \geq t^\dagger$ *such that* $\overline{\varepsilon}(t^\star) < \frac{1}{2\kappa}$,

$$D\left(\bar{x}(t^\star), \bar{y}(t^\star)\middle|\, \kappa\overline{\varepsilon}(t^\star)\right) > 0 \text{ and } D_c\left(\bar{x}(t^\star), \bar{y}(t^\star)\middle|\, \overline{\varepsilon}(t^\star)\right) > 0.$$

Moreover $(\bar{x}(\cdot), \bar{y}(\cdot))$ *cannot be generated by inverse strategies solving (42) and (43).*

Claim B.7 *Let* $\kappa > 1$ *be as in Claim B.3, and consider a large enough* $\kappa_2 > 1$. *Suppose* $(\bar{x}(\cdot), \bar{y}(\cdot))$ *solves (44) and (45) with* $\bar{x}(t), \bar{y}(t) > 1$ *for all* t. *Then if there exists a large enough* $t^\dagger$ *with in particular* $\overline{\varepsilon}(t^\dagger) < \frac{1}{2\kappa}$,

$$D\left(\bar{x}(t^\dagger), \bar{y}(t^\dagger)\middle|\, \kappa\overline{\varepsilon}(t^\dagger)\right) < 0 \text{ and } D_c\left(\bar{x}(t^\dagger), \bar{y}(t^\dagger)\middle|\, \kappa_2\underline{\varepsilon}(t^\dagger)\right) \leq 0,$$

then there exists a $t^\star \geq t^\dagger$ *such that* $\overline{\varepsilon}(t^\star) < \frac{1}{2\kappa}$,

$$D\left(\bar{x}(t^\star), \bar{y}(t^\star)\middle|\, \kappa\underline{\varepsilon}(t^\star)\right) < 0 \text{ and } D_c\left(\bar{x}(t^\star), \bar{y}(t^\star)\middle|\, \underline{\varepsilon}(t^\star)\right) < 0.$$

Moreover $(\bar{x}(\cdot), \bar{y}(\cdot))$ *cannot be generated by inverse strategies solving (42) and (43).*

Claims B.6 and B.7 play a role similar to Claims B.4 and B.5. For instance, Claim B.7 improves the constraint $D\left(\bar{x}(t^\star), \bar{y}(t^\star)\middle|\, \kappa\underline{\varepsilon}(t^\star)\right) < 0$ of Claim B.3, which is equivalent to $\bar{y}(t^\star) \leq \frac{n-\kappa\overline{\varepsilon}(t^\star)}{n-1}$, to $\bar{y}(t^\dagger) \leq \frac{n+\kappa\overline{\varepsilon}(t^\dagger)}{n-1}$.

Proof of Claims B.6 and B.7. The parts on non admissibility of $(\bar{x}(\cdot), \bar{y}(\cdot))$ as auction strategy follows from Claims B.2 and B.3. We now establish Claim B.7, the proof of Claim B.6 being similar. As the result follows from Claim B.3 if $D\left(\bar{x}(t^\dagger), \bar{y}(t^\dagger)\middle|\, \kappa\underline{\varepsilon}(t^\dagger)\right) < 0$, we shall assume that $D\left(\bar{x}(t^\dagger), \bar{y}(t^\dagger)\middle|\, \kappa\underline{\varepsilon}(t^\dagger)\right) \geq 0$, that is

$$\frac{n-\kappa\overline{\varepsilon}(t^\dagger)}{n-1} \leq \bar{y}(t^\dagger) \leq \frac{n+\kappa\overline{\varepsilon}(t^\dagger)}{n-1}.$$

Let $\mathcal{D}$ be the corresponding domain $\frac{n-\kappa\overline{\varepsilon}(t^\dagger)}{n-1} \leq y \leq \frac{n+\kappa\overline{\varepsilon}(t^\dagger)}{n-1}$. Suppose that, for some $t \geq t^\dagger$ implying $\overline{\varepsilon}(t) \leq \frac{1}{2}$, $(\bar{x}(t), \bar{y}(t))$ is in $\mathcal{D}$. Then as $D(x, y|\varepsilon(t))$ increases with y and $-\overline{\varepsilon}(t) \leq \varepsilon(t) \leq \overline{\varepsilon}(t)$, it holds

$$\begin{aligned}
D(\bar{x}(t), \bar{y}(t)|\varepsilon(t)) &\leq \frac{1}{\frac{n-\kappa\overline{\varepsilon}(t^\dagger)}{n-1} - 1}\left(\frac{n+\kappa\overline{\varepsilon}(t^\dagger)}{n-1} - \frac{n+\kappa\varepsilon(t)}{n-1}\right) \leq 2(n-1)\frac{2\kappa\overline{\varepsilon}(t^\dagger)}{n-1} \\
&= 4\kappa\overline{\varepsilon}(t^\dagger), \\
D(\bar{x}(t), \bar{y}(t)|\varepsilon(t)) &\geq \frac{1}{\frac{n-\kappa\overline{\varepsilon}(t^\dagger)}{n-1} - 1}\left(\frac{n-\kappa\overline{\varepsilon}(t^\dagger)}{n-1} - \frac{n+\kappa\varepsilon(t)}{n-1}\right) \geq -4\kappa\overline{\varepsilon}(t^\dagger),
\end{aligned}$$

so that

$$-4\kappa\overline{\varepsilon}(t^\dagger) \leq D(\bar{x}(t), \bar{y}(t)|\varepsilon(t)) \leq 4\kappa\overline{\varepsilon}(t^\dagger).$$

Because $(\bar{x}(t^\dagger), \bar{y}(t^\dagger))$ lies in $\mathcal{D}$ and $D_c\left(\bar{x}(t^\dagger), \bar{y}(t^\dagger)\middle|\, \kappa_2\underline{\varepsilon}(t^\dagger)\right) \leq 0$, $\bar{x}(t^\dagger)$ satisfies

$$\begin{aligned}
1 \leq \bar{x}(t^\dagger) &\leq 1 + \frac{1}{\frac{k}{1+\kappa_2\underline{\varepsilon}(t^\dagger)} + \frac{n-2}{(n-1)(\bar{y}(t^\dagger)-1)}} \leq 1 + \frac{1}{\frac{k}{1-\kappa_2\overline{\varepsilon}(t^\dagger)} + \frac{n-2}{1+\kappa\overline{\varepsilon}(t^\dagger)}} \\
&\leq 1 + \frac{1 + (\kappa - \kappa_2)\overline{\varepsilon}(t^\dagger)}{n+k-2} + O\left(\overline{\varepsilon}^2(t^\dagger)\right)
\end{aligned}$$

when $\overline{\varepsilon}^2(t^\dagger)$ goes to 0.

The proof works by contradiction. Suppose $(\bar{x}(\cdot), \bar{y}(\cdot))$ never exits $\mathcal{D}$ by crossing the lower boundary $y = \frac{n-\kappa\overline{\varepsilon}(t^\dagger)}{n-1}$. Consider $\lambda > 0$. As

$$\bar{x}^{(1)}(t) \leq \frac{\bar{x}(t)4\kappa\overline{\varepsilon}(t^\dagger)}{t}$$

it holds for all t in $\left[t^\dagger, \exp\left(\frac{\lambda}{4\kappa}\right)t^\dagger\right]$

$$\begin{aligned}
\bar{x}(t) &\leq \bar{x}(t^\dagger)\exp\left(\lambda\overline{\varepsilon}(t^\dagger)\right) \leq \bar{x}(t^\dagger)\left(1 + \lambda\overline{\varepsilon}(t^\dagger) + O\left((\lambda\overline{\varepsilon}(t^\dagger))^2\right)\right) \\
&\leq 1 + \frac{1 + \left(\kappa + (n+k-2)\lambda - \kappa_2\right)\overline{\varepsilon}(t^\dagger)}{n+k-2} + O\left((\lambda^2+1)\overline{\varepsilon}^2(t^\dagger)\right).
\end{aligned}$$

This gives

$$D_c\left(\bar{x}(t),\bar{y}(t)\right|\varepsilon_c(t)) = \frac{k(n-1)}{1+\varepsilon_c(t)} + \frac{n-2}{\bar{y}(t)-1} - \frac{n-1}{\bar{x}(t)-1}$$
$$\leq \frac{k(n-1)}{1-\bar{\varepsilon}(t^\dagger)} + \frac{n-2}{\frac{1-\kappa\bar{\varepsilon}(t^\dagger)}{n-1}} - \frac{n-1}{\frac{1+(\kappa+(n+k-2)\lambda-\kappa_2)\bar{\varepsilon}(t^\dagger)}{n+k-2}} + O\left((\lambda^2+1)\bar{\varepsilon}^2(t^\dagger)\right)$$
$$= \left[(n-1)(n+k-2)\left(\kappa-(\kappa_2-(n+k-2)\lambda)\right)+(n-1)(n-2)\kappa+k(n-1)\right]\bar{\varepsilon}(t^\dagger)$$
$$+ O\left((\lambda^2+1)\bar{\varepsilon}^2(t^\dagger)\right) = -C(1+o(1))\bar{\varepsilon}(t^\dagger) < 0$$

provided $\kappa_2-(n+k-2)\lambda$ and $t^\dagger$ are taken large enough. It follows that $\bar{y}(\cdot)$ is strictly decreasing over $\left[t^\dagger, \exp\left(\frac{\lambda}{4\kappa}\right)t^\dagger\right]$. This gives

$$-2\frac{\kappa\bar{\varepsilon}(t^\dagger)}{n+1} \leq \int_{t^\dagger}^{\exp\left(\frac{\lambda}{4\kappa}\right)t^\dagger} \bar{y}^{(1)}(t)dt \leq \int_{t^\dagger}^{\exp\left(\frac{\lambda}{4\kappa}\right)t^\dagger} \frac{-C(1+o(1))\bar{\varepsilon}(t^\dagger)}{kt}dt = -\frac{C(1+o(1))\lambda\bar{\varepsilon}(t^\dagger)}{kn}$$

a contradiction if λ is taken large enough. Then for this choice of κ_2, λ and $t^\dagger$, $(\bar{x}(\cdot),\bar{y}(\cdot))$ must cross $y=\frac{n-\kappa\bar{\varepsilon}(t^\dagger)}{n+1}$. □

B.3 Proof of the proposition

Figure 12 summarizes the admissible region for growth rates $(x(t), y(t))$ generated by a first-price auction strategy for large enough t as described by Claims B.2-B.7. It consists into three parts defined by interior subsets of $(\searrow,\nearrow)_t$ and $(\nearrow,\searrow)_t$ and a central area. The interior regions are

$$(\overset{\circ}{\nearrow,\searrow})_t = \left\{(x,y)\in(1,\infty)^2 \,\middle|\, y \geq \frac{n+\kappa\bar{\varepsilon}(t)}{n-1} \text{ and } D_c(x,y|\underline{\varepsilon}(t)) \leq 0\right\},$$
$$(\overset{\circ}{\searrow,\nearrow})_t = \left\{(x,y)\in(1,\infty)^2 \,\middle|\, y \leq \frac{n-\kappa\bar{\varepsilon}(t)}{n-1} \text{ and } D_c(x,y|\bar{\varepsilon}(t)) \geq 0\right\}.$$

The boundaries $\partial\mathcal{C}_t$ of the central part $\mathcal{C}_t$ are defined using the $A_{j,t}$'s, $j=1,\ldots,12$, reported in Figure 12. $\mathcal{C}_t$ is the union of $\mathcal{C}_{1t}$ and $\mathcal{C}_{2t}$ with, κ, κ_1 and κ_2 being the

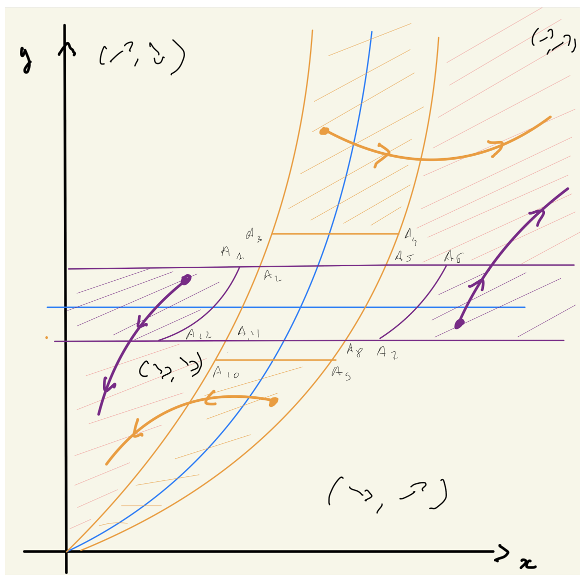


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Figure 12: *(Non) admissible regions for inverse strategies.* Hatched areas are excluded by Claims B.2 and B.3 (red), Claims B.4 and B.5 (Orange), Claims B.6 and B.7 (Purple). Relevant boundaries are defined below. Blues lines are $D(x, y|\varepsilon(t)) = 0$ and $D_c(x, y|\varepsilon_c(t)) = 0$.

constants introduced in Claims B.2-B.7:

$$\mathcal{C}_{1t} = \left\{ (x,y) \in (1,\infty)^2 \,\middle|\, \begin{array}{l} \frac{n-\kappa_1\overline{\varepsilon}(t)}{n-1} \leq y \leq \frac{n+\kappa_1\overline{\varepsilon}(t)}{n-1} \text{ and} \\ D_c(x,y|\overline{\varepsilon}(t)) \leq 0 \leq D_c(x,y|\underline{\varepsilon}(t)) \end{array} \right\},$$

$$\mathcal{C}_{2t} = \left\{ (x,y) \in (1,\infty)^2 \,\middle|\, \begin{array}{l} \frac{n-\kappa\overline{\varepsilon}(t)}{n-1} \leq y \leq \frac{n+\kappa\overline{\varepsilon}(t)}{n-1} \text{ and} \\ D_c(x,y|\kappa_2\overline{\varepsilon}(t)) \leq 0 \leq D_c(x,y|\kappa_2\underline{\varepsilon}(t)) \end{array} \right\}.$$

For instance, the boundary $\partial_{A_1A_2}\mathcal{C}_t$ of $\mathcal{C}_t$ between $A_{1,t}$ and $A_{2,t}$ is $y = \frac{n+\kappa\overline{\varepsilon}(t)}{n-1}$, these two points being at the intersection of this straight line with $D_c(x,y|\kappa_2\underline{\varepsilon}(t)) = 0$ and $D_c(x,y|\underline{\varepsilon}(t)) = 0$, respectively.

Limit of the growth rates. Suppose from now on that $(\bar{x}(\cdot), \bar{y}(\cdot))$ is generated by an auction strategy, so that

$$(\bar{x}(t), \bar{y}(t)) \text{ lies in } (\nearrow \overset{\circ}{,} \searrow)_t \cup \mathcal{C}_t \cup (\searrow \overset{\circ}{,} \nearrow)_t$$

for all t large enough by Claims B.2-B.7. Consider a sequence t_n such that $\overline{\varepsilon}(t_n)$ is strictly decreasing, with $\overline{\varepsilon}(t) < \overline{\varepsilon}(t_n)$ if $t > t_n$. Let $N_{\mathcal{C}}$ be the number of n such that there is t in $[t_n, t_{n+1})$ such such that $(x(\cdot), y(\cdot))$ enters $\mathcal{C}_t \subset \mathcal{C}_{t_n}$ at t, that is in the interior of $\mathcal{C}_t$ over a small interval $(t, t+\epsilon)$, for instance crossing either $\partial_{A_1A_3}\mathcal{C}_t$ from $(\nearrow \overset{\circ}{,} \searrow)_t$ or $\partial_{A_7A_9}\mathcal{C}_t$ from $(\searrow \overset{\circ}{,} \nearrow)_t$.

If $N_{\mathcal{C}}$ is finite, then the signs of $D(x(t), y(t)|\varepsilon(t))$ and $D_c(x(t), y(t)|\varepsilon_c(t))$ are constant for all $t \geq T$ large enough, implying that $(x(t), y(t)))$ belongs to $(\nearrow \overset{\circ}{,} \searrow)_t$ for all $t \geq T$, or to $(\searrow \overset{\circ}{,} \nearrow)_t$ for all $t \geq T$. If $(x(t), y(t)))$ belongs to $(\nearrow \overset{\circ}{,} \searrow)_t$ for all $t \geq T$, then $x(t)$ is strictly increasing and bounded from above, $y(t)$ is strictly decreasing and bounded from below so that Claim B.1 implies

$$\lim_{t\uparrow\infty} x(t) = \frac{n+k-1}{n+k-2} \text{ and } \lim_{t\uparrow\infty} y(t) = \frac{n}{n-1}$$

as stated in the Proposition. The case where $(x(t), y(t)))$ belongs to $(\searrow \overset{\circ}{,} \nearrow)_t$ for all $t \geq T$ is similar.

Suppose that $N_{\mathcal{C}}$ is infinite. Then we can extract from $\{t_n\}$ a sequence $\{\tau_n\}$ such that, for each n, there is a $t_n(\mathcal{C})$ in $[\tau_n, \tau_{n+1})$ with $(x(t_n(\mathcal{C})), y(t_n(\mathcal{C})))$ in the interior of $\mathcal{C}_{\tau_n}$. We now show by contradiction that $(x(t), y(t))$ belongs to $\mathcal{C}_{\tau_n}$ for all $t \geq t_n(\mathcal{C})$. Suppose that there is a $t_\star > t_n(\mathcal{C}) \geq \tau_n$ so that $(x(\cdot), y(\cdot)$ exits $\mathcal{C}_{\tau_n}$ at $t_\star$, ie crosses $\partial_{A_1A_3}\mathcal{C}_{\tau_n}$ or $\partial_{A_7A_9}\mathcal{C}_{\tau_n}$. Consider the case of $\partial_{A_1A_3}\mathcal{C}_{\tau_n}$, the other one being similar. As $\overline{\varepsilon}(t_\star) < \overline{\varepsilon}(\tau_n)$ by construction of the sequence $\{t_n\}$, $D(x, y|\varepsilon(t_\star)) = 0$ and $D_c(x, y|\varepsilon_c(t_\star)) = 0$ (the blue lines of Figure 12) are, respectively, strictly below $D(x, y|\kappa\overline{\varepsilon}(\tau_n)) = 0$ (the upper purple straight line) and $D_c(x, y|\underline{\varepsilon}(\tau_n)) = 0$ (the RHS orange curve of Figure 12). Hence (44) and (45) imply that

$$x^{(1)}(t_\star) > 0 \text{ and } y^{(1)}(t_\star) < 0$$

showing that $(x(\cdot), y(\cdot))$ cannot cross $\partial_{A_1A_3}\mathcal{C}_{\tau_n}$.

As a consequence, there is a diverging sequence $\{t_n(\mathcal{C})\}$ such that $(x(t), y(t))$ lies in $\mathcal{C}_{\tau_n}$ for all $t \geq \mathcal{C}_{\tau_n}$. As τ_n diverges, $\mathcal{C}_{\tau_n}$ shrinks to $\left(\frac{n+k-1}{n+k-2}, \frac{n}{n-1}\right)$, showing that $(x(t), y(t))$ converges to the desired limit when t grows.

Continuity of strategy derivatives. Let

$$(x(t), y(t)) = t\left(\xi\left(\underline{b} + \frac{1}{t}\right) - \xi\left(\underline{b}\right), \xi_c\left(\underline{b} + \frac{1}{t}\right) - \xi_c\left(\underline{b}\right)\right)$$

be generated by a proprer first-price auction strategy. Observe that

$$x^{(1)}(t) = \xi\left(\underline{b} + \frac{1}{t}\right) - \xi\left(\underline{b}\right) - \frac{1}{t}\xi^{(1)}\left(\underline{b} + \frac{1}{t}\right),$$
$$y^{(1)}(t) = \xi_c\left(\underline{b} + \frac{1}{t}\right) - \xi_c\left(\underline{b}\right) - \frac{1}{t}\xi_c^{(1)}\left(\underline{b} + \frac{1}{t}\right).$$

Hence the differential system (44)-(45) shows

$$t\left(\xi\left(\underline{b} + \frac{1}{t}\right) - \xi\left(\underline{b}\right)\right) - \xi^{(1)}\left(\underline{b} + \frac{1}{t}\right) = D\left(x(t), y(t)\right|\varepsilon(t)),$$
$$t\left(\xi_c\left(\underline{b} + \frac{1}{t}\right) - \xi_c\left(\underline{b}\right)\right) - \xi_c^{(1)}\left(\underline{b} + \frac{1}{t}\right) = \frac{D_c\left(x(t), y(t)\right|\varepsilon_c(t))}{k}.$$

with $\lim_{t\uparrow\infty} D\left(x(t), y(t)\right| \varepsilon(t)) = \lim_{t\uparrow\infty} D_c\left(x(t), y(t)\right| \varepsilon_c(t)) = 0$. Hence

$$\lim_{t\uparrow\infty} \xi^{(1)}\left(\underline{b}+\frac{1}{t}\right) = \lim_{t\uparrow\infty} t\left(\xi\left(\underline{b}+\frac{1}{t}\right) - \xi\left(\underline{b}\right)\right) = \frac{n+k-1}{n+k-2},$$
$$\lim_{t\uparrow\infty} \xi^{(1)}\left(\underline{b}+\frac{1}{t}\right) = \lim_{t\uparrow\infty} t\left(\xi_c\left(\underline{b}+\frac{1}{t}\right) - \xi_c\left(\underline{b}\right)\right) = \frac{n}{n-1},$$

implying that $\xi^{(1)}(\cdot)$ and $\xi_c^{(1)}(\cdot)$ are continuous at $\underline{b}$. Continuity of these derivatives over $(\underline{b}, \overline{b}]$ follows from Lebrun (1997), see also the differential system (22)-(23). This latter system also shows that $\xi^{(1)}(\cdot), \xi_c^{(1)}(\cdot) > 0$ over $(\underline{b}, \overline{b}]$ as $\xi(b), \xi_c(b) > b$ over this set. $\square$

B.4 Proof of Theorem B.1

Rewrite the differential system (22)-(23) as

$$\xi^{(1)}(b) = \Delta\left(\xi(b), \xi_c(b), b\right) \text{ and } \xi_c^{(1)}(b) = \Delta\left(\xi(b), \xi_c(b), b\right)$$

where

$$\Delta\left(\xi, \xi_c, b\right) = \frac{F(\xi)}{f(\xi)} \frac{1}{n-1} \frac{1}{\xi_c - b},$$
$$\Delta_c\left(\xi, \xi_c, b\right) = \frac{F(\xi_c)}{f(\xi_c)} \frac{1}{k(n-1)} \left(\frac{n-1}{\xi - b} - \frac{n-2}{\xi_c - b}\right).$$

Suppose now that there is two Bayesian Nash Equilibrium inverse strategies $(\xi(\cdot), \xi_c(\cdot))$ and $\left(\widetilde{\xi}(\cdot), \widetilde{\xi}_c(\cdot)\right)$, which must solve this differential system. Consider $b^\star \leq \max\left(\xi^{-1}(\overline{v}), \widetilde{\xi}^{-1}(\overline{v})\right)$ with $b^\star > \underline{b}$ By Proposition B.1, for any $\epsilon > 0$, there exists $\eta > 0$ with $\underline{b} + \eta < b^\star$ such that, for all b in $[\underline{b}, \underline{b} + \eta]$,

$$\left|\widetilde{\xi}(b) - \xi(b)\right|, \quad \left|\widetilde{\xi}_c(b) - \xi_c(b)\right| \leq \epsilon\left(b - \underline{b}\right),$$
$$\left|\Delta\left(\widetilde{\xi}(b), \widetilde{\xi}_c(b), b\right) - \Delta\left(\xi(b), \xi_c(b), b\right)\right|, \quad \left|\Delta_c\left(\widetilde{\xi}(b), \widetilde{\xi}_c(b), b\right) - \Delta_c\left(\xi(b), \xi_c(b), b\right)\right| \leq \epsilon.$$

For all b in $[\underline{b}+\eta, b^\star]$, $\xi(b)-b, \xi_c(b)-b, \geq C>0$. As $\Delta(\xi,\xi_c,b)$ and $\Delta_c(\xi,\xi_c,b)$ are Lipshitz in (ξ,ξ_c) over any compact domain such that $\xi-b, \xi_c-b\geq C$, there is a $\kappa>0$ such that for all b in $[\underline{b}+\eta, b^\star]$,

$$\left|\Delta\left(\widetilde{\xi}(b),\widetilde{\xi}_c(b),b\right)-\Delta\left(\xi(b),\xi_c(b),b\right)\right|\leq\kappa\left\{\left|\widetilde{\xi}(b)-\xi(b)\right|+\left|\widetilde{\xi}_c(b)-\xi_c(b)\right|\right\},$$
$$\left|\Delta_c\left(\widetilde{\xi}(b),\widetilde{\xi}_c(b),b\right)-\Delta_c\left(\xi(b),\xi_c(b),b\right)\right|\leq\kappa\left\{\left|\widetilde{\xi}(b)-\xi(b)\right|+\left|\widetilde{\xi}_c(b)-\xi_c(b)\right|\right\},$$
$$\left|\widetilde{\xi}(b)-\xi(b)\right|,\ \left|\widetilde{\xi}_c(b)-\xi_c(b)\right|\leq\kappa b$$

where the last inequality follows from Proposition B.1. Hence it holds over $[\underline{b}, b^\star]$

$$\left|\widetilde{\xi}(b)-\xi(b)\right|,\ \left|\widetilde{\xi}_c(b)-\xi_c(b)\right|\leq\delta_1(b)=\epsilon b\mathbb{I}\left(b\in[\underline{b},\underline{b}+\eta]\right)+\kappa b\mathbb{I}\left(b\in[\underline{b}+\eta,b^\star]\right),$$

assuming $\epsilon\leq\kappa$ without loss of generality. This gives, for any b in $[\underline{b}, b^\star]$

$$\begin{aligned}\left|\widetilde{\xi}(b)-\xi(b)\right|&=\left|\int_{\underline{b}}^{b}\left(\Delta\left(\widetilde{\xi}(t),\widetilde{\xi}_c(t),t\right)-\Delta\left(\xi(t),\xi_c(t),t\right)\right)dt\right|\\&\leq\mathbb{I}\left(b\in[\underline{b},\underline{b}+\eta]\right)\int_{\underline{b}}^{b}\epsilon dt+\mathbb{I}\left(b\in[\underline{b}+\eta,b^\star]\right)\int_{\underline{b}}^{b}2\kappa^2tdt\\&\leq\mathbb{I}\left(b\in[\underline{b},\underline{b}+\eta]\right)\epsilon(b-\underline{b})+\mathbb{I}\left(b\in[\underline{b}+\eta,b^\star]\right)\frac{(2\kappa)^2}{2!}b^2=\delta_2(b).\end{aligned}$$

Proceeding similarly gives $\left|\widetilde{\xi}_c(b)-\xi_c(b)\right|\leq\delta_2(b)$ over $[\underline{b}, b^\star]$. Iterating gives

$$\begin{aligned}\left|\widetilde{\xi}(b)-\xi(b)\right|&\leq\mathbb{I}\left(b\in[\underline{b},\underline{b}+\eta]\right)\int_{\underline{b}}^{b}\epsilon dt+\mathbb{I}\left(b\in[\underline{b}+\eta,b^\star]\right)2\kappa\int_{\underline{b}}^{b}(2\kappa)^2\frac{t^2}{2}dt\\&\leq\mathbb{I}\left(b\in[\underline{b},\underline{b}+\eta]\right)\epsilon(b-\underline{b})+\mathbb{I}\left(b\in[\underline{b}+\eta,b^\star]\right)\frac{(2\kappa)^3}{3!}b^3=\delta_3(b)\end{aligned}$$

and $\left|\widetilde{\xi}_c(b)-\xi_c(b)\right|\leq\delta_3(b)$ over $[\underline{b}, b^\star]$. Further iterations give, for any integer number p,

$$\left|\widetilde{\xi}(b)-\xi(b)\right|,\ \left|\widetilde{\xi}_c(b)-\xi_c(b)\right|\leq\delta_p(b)=\epsilon b\mathbb{I}\left(b\in[\underline{b},\underline{b}+\eta]\right)+\frac{(2\kappa b)^p}{p!}\mathbb{I}\left(b\in[\underline{b}+\eta,b^\star]\right)$$

over $[\underline{b}, b^\star]$. Hence

$$\left|\widetilde{\xi}(b)-\xi(b)\right|,\ \left|\widetilde{\xi}_c(b)-\xi_c(b)\right|\leq\lim_{p\uparrow\infty}\delta_p(b)=\epsilon b\mathbb{I}\left(b\in[\underline{b},\underline{b}+\eta]\right)$$

over $[\underline{b}, b^\star]$, and then, since ϵ can be taken arbitrarily small,

$$\widetilde{\xi}(b) = \xi(b) \text{ and } \widetilde{\xi}_c(b) = \xi_c(b) \text{ over } [\underline{b}, b^\star].$$

It is easily seen that this implies $\xi^{-1}(\overline{v}) = \widetilde{\xi}^{-1}(\overline{v})$. Setting $b^\star$ to this value shows that the two inverse strategies are identical. Hence the Theorem is proved. $\square$

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