Passive Investing and Price Efficiency

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Abstract

This paper studies how falling fees for delegated investments affect price efficiency in a theoretical framework, in which the investors' allocations, management fees, and asset prices are all determined in a general equilibrium. Importantly, investors optimally decide whether to participate in the financial market or simply hold the safe asset, and active managers trade strategically, adjusting the traded quantities according to market liquidity. Perhaps surprisingly, and in contrast to the broad theoretical literature, prices of the index fund become more efficient as passive fees decrease and more investors choose the uninformed index fund. Prices can become more efficient even when the inflow to passive funds comes at the expense of the outflow from active funds, as has been the case more recently since 2007. Combined with the observed downward trend in fees, the finding is consistent with recent empirical evidence that prices, especially those of S&P 500, have become more informative over time.

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Delegated investments in the United States have been growing substantially over the past few decades. Individuals have been moving away from direct ownership towards institutional investments, with the fraction of individual ownership falling from 48% of the equity market in 1980 to 20% in 2010 (French (2008); Stambaugh (2014); Garleanu and Pedersen (2019)). Especially pronounced in this growth is the rise of passive management. Passive mutual funds and ETFs such as index funds grew from 2% of the U.S. equity market capitalization in 1998 to about 14% in March 2020 (Appel et al. (2016); Anadu et al. (2019)). Fees and expenses have continually declined, especially for passive funds. Average fees and expenses of mutual funds fell from about 2% in 1980 to 1% in 2006, and average passive fees fell from 30 basis points in 1990 to 13 basis points in 2019 (French (2008); Morningstar (2020)). While low-cost index funds allows more investors to participate in the financial market cheaply, without having to pay for specific market timing or stock picking skills, it raises concerns that the surge of passive investing, and thus the increased importance of uninformed capital, may harm price efficiency of the financial market, hampering its key role of aggregating and transmitting information for the broader economy (Hayek (1945); Bond et al. (2012)).

This paper studies how falling fees for delegated investments affect price efficiency in a theoretical framework, in which the investors' allocations, management fees, and asset prices are all determined in a general equilibrium. Importantly, investors optimally decide whether to participate in the financial market or simply hold the safe asset, and informed (active) managers trade strategically, adjusting the traded quantities according to market liquidity. Perhaps surprisingly, and in contrast to the broad theoretical literature, prices of the index fund become more efficient as passive fees decrease and more investors choose the uninformed index fund. Prices can become more efficient even when the inflow to passive funds comes at the expense of the outflow from active funds, as has been the case more recently since 2007. Combined with the observed downward trend in fees, the finding is consistent with recent empirical evidence that prices, especially those of S&P 500, have become more informative over time (Bai et al. (2016); Farboodi et al. (2020); Davila and Parlatore (2020)).[1]

In a nutshell, price efficiency is mainly governed by the number of active investors and the liquidity with which their managers trade on their information, both of which are determined in equilibrium. Participation of passive investors affect price efficiency by improving liquidity for active managers, which in turn raises the value of investing with those managers and thus increases the number of active investors. While the equilibrium effects on liquidity and the number of active investors depend on specific comparative statics since the increase of the latter deteriorates the former, overall price efficiency, which is determined by the product

[1] Bennett et al. (2020) on the other hand show evidence that individual stock's prices have become less efficient. See also Sammon (2020).

of the two, tends to increase. These effects of participation are absent in the competitive benchmark, in which managers take prices as given since they trade as if the market is always perfectly liquid and thus do not respond to liquidity provided by passive investors. Hence, unlike in the competitive benchmark, in which price efficiency essentially is unaffected by passive investors and must decrease as fewer investors choose active managers, participation of passive investors in fact play an active role in determining price efficiency and can improve rather than hamper price efficiency.

In the model, investors choose mainly among three options: they can hold the safe asset, invest with passive managers, or invest with active managers. Focus for now on active managers who trade the index fund using their private information to time the market, while the analysis easily extends to allowing other types of active managers who specialize in stock picking with or without private information. To invest with either passive or active managers, investors first pay a cost to search and vet them and then negotiate a management fee. Then all managers trade, incurring per-investor trading costs, with exogenous noise traders, which may represent direct trading by individual investors. While passive managers trade competitively, consistent with small uninformed investors frequently buying and selling the index funds without price impact, especially for the ETFs, active managers attempt to time the market accordingly, taking into account the price impact of trade for all of their investors. In equilibrium, fees for passive and active investments, determined so that investors are indifferent among saving and passive and active managements, reflect the search cost for investors and the trading cost for managers.

The main results concern the comparative statics of price efficiency as trading and search costs. First, as trading costs decline, both passive and active fees decline, while the fee spread between the two remains constant, and prices become more efficient. As fees decline, it becomes cheaper to participate in the financial market, and naturally, more investors choose passive managements rather than saving. The increased participation improves liquidity for active managers, since the aggregate demand from passive managers can better absorb the trade by active managers, who then face an increasingly flatter supply curve. The improved liquidity, in turn, makes investing with active managers more attractive, since active managers can trade a larger quantity and better take advantage of their private information. When choosing between passive and active managements, investors trade off the additional value created by active managers relative to passive managers (i.e. the value of information), with spread between active and passive fees. Since the falling trading costs increase the value of information through improved liquidity but do not affect the spread, the number of active investors as well as that of passive investors increase. While the equilibrium liquidity tends to decrease since active managers now have to trade for more investors, which deteriorates liquidity, the effect of increasing active investors outweighs, and prices of the index fund become more efficient rather than less efficient.

Second, as search costs for uninformed managers decline, passive fees also decline while active fees are unaffected so that the fee spread between active and passive managements increases. Prices can still become more efficient, although the increasing spread discourages investors from choosing active managements and the number of active investors thus decreases. Like above, decreasing passive fees increases participation and improves liquidity for active managers, which again increases the value of information. The increasing fee spread, however, makes active managements relatively more expensive, and the equilibrium number of active investors decreases. Both increasing passive investors and decreasing active investors improve liquidity, and active managers trade on their information much more aggressively. Provided that active fees are sufficiently low, the effect of improved liquidity outweighs that of decreasing active investors, and prices become more efficient.

Third, as search costs for informed managers decline, active fees decline while passive fees are unaffected so that the fee spread decreases. Prices become more efficient since the decreasing spread encourages investors to choose active managements, and the number of active investors increases. While this tends to deteriorate liquidity, the direct effect of the increasing active investors outweighs, and prices become more efficient, albeit to a much less degree than would be in the hypothetical competitive benchmark.

Fourth, price efficiency is independent of the amount of noise trading. In the partial equilibrium, noise trading makes prices less informative and presents profitable opportunities for both informed and uninformed investors. As investors optimally respond to the change in noise trading in a general equilibrium, noise trading has no effect on price efficiency. Although this result is not new to this model and also holds in the competitive benchmark, it stands in contrast to Stambaugh (2014), who shows that the decrease in direct investments by individuals, which reduces noise trading, makes prices less informative.

Finally, the model has implications for impossibility of fully efficient market. The Grossman-Stiglitz paradox starts with a premise that when there is no noise but some investors are informed, prices would be fully informative. This does not happen with strategic trading, and there is no paradox. Garleanu and Pedersen (2018) show that prices become fully efficient as the search cost vanishes, and the industry of informed managers concentrates. Again, this does not happen with strategic trading. Although the number of informed investors goes to infinity, liquidity evaporates, and together price remains partially informative.

This paper is related to the literature on delegated asset management.[2] This paper in

[2]The classic papers in this literature on the role of the asset management industry include Admati

particular focuses on the effect of passive investing, or indexing, on price efficiency. The paper builds on the model of Garleanu and Pedersen (2018, 2019), who introduce asset managers into the rational expectations equilibrium framework. Differently from theirs, informed managers trade strategically and investor participation is endogenous. These changes produce opposite implications for price implications of declining passive fees.

The literature models passive investing largely in three ways: as an index portfolio, whose relative weights are determined by the exogenous supply (e.g. Bond and Garcia (2017)), as an optimal portfolio constructed by uninformed investors or managers (e.g. Garleanu and Pedersen (2019)), and as a financial innovation that relaxes various frictions in the financial market (e.g. Subrahmanyam (1991); Gorton and Pennacchi (1993); Cong and Xu (2016)). The paper is more closely related to the first two, as the share of passive and active investors are determined in equilibrium, while the third literature takes the investor allocation exogenously. This paper takes the first approach and adopts the idea of Bond and Garcia (2017) in re-spanning the asset space into the index portfolio and spread (or long-short) portfolios.

Investor participation is also studied by Bond and Garcia (2017), in whose model, investors who are endowed with information and hedging needs endogenously choose to participate in the financial market. They show that lowering the cost to participate reduces the price efficiency of the index. Malikov (2019) extend the framework and allow information acquisition. He shows that lowering the information cost for spread portfolios increases their price efficiency, and it discourages investor participation for the spread portfolios. Lowering the participation cost, in his framework, still makes the index portfolio less efficient. Buss and Sundaresan (2019) is the only other paper to my knowledge that increasing passive investments can make prices more efficient. They emphasize the cross-sectional implications that stocks with more passive investors have higher price efficiency. The shares of investors in their model are exogenous, and the time-series implications depend on the assumptions of exogenous shares of uninformed and informed investors.

The paper also contributes to the relatively small literature on the role of market power in delegated management. Kyle et al. (2011) focus on contracting features and heterogeneous risk aversion. Kacperczyk et al. (2018) allow informed traders to exercise market power, but in their model, more uninformed investments make price less informative, and their results are based on numerical rather than analytical. Incorporating market power makes problems in endogenous information acquisition and delegated management complicated very quickly. On technical front, this paper provides an intuitive expression of the value of

and Pfleiderer (1986), Berk and Green (2004), Garcia and Vanden (2009), Basak and Pavlova (2013), and Kacperczyk et al. (2016).

information, when managers compete in demand schedules, as in the classic paper of Kyle (1989), who also shows that increasing uninformed investors reduces price impact and can increase price efficiency, while the shares of investors are exogenous. Corum et al. (2020) also have endogenous participation and strategic trading, albeit with risk-neutral investors. Their focus is on corporate governance.

The paper is organized as follows. Section 1 describes the setup of the model. Section 2 solves and characterizes equilibrium. Section 3 analyzes implications of declining fees for price efficiency. Section 4 discuss empirical implications. Section 5 provides further discussions. Section 6 concludes.

1 Model Setup

The framework is motivated by Garleanu and Pedersen (2018, 2019), who introduce asset managers into the noisy rational expectations equilibrium framework of Grossman and Stiglitz (1980). The key differences are that (1) informed managers trade strategically, taking into account their price impact, à la Kyle (1989); and (2) investors optimally decide whether to participate in the financial market, as well as whether to invest with informed or uninformed managers.

Funds There are one safe asset and J risky assets, indexed by $j = 1, \ldots, J$. The safe asset is a numeraire, whose return is normalized to one and net supply is zero. The first risky asset is an index fund. The rest of $J-1$ risky assets include long-short portfolios whose payoffs are orthogonal to that of the index. They can also include other risky assets that are excluded from the index, provided that the payoff of excluded assets are orthogonal to that of the index. Bond and Garcia (2017) introduce ways to re-span the asset space with the index fund and long-short portfolios that are orthogonal to the index fund.In this paper, I assume that For simplicity, assume $J = 2$, and let the first risky asset ($j = 1$) represent the index fund, denoted x, and the second risky asset ($j = 2$) represent any non-index assets/funds, denoted y. Let v_j denote the payoff of fund j, which is distributed normally with mean $\bar{v}_j$ and variance $\sigma^2_{v_j}$ and $\bar{z}_j$ denote the net supply.

There are three types of agents: managers, investors, and noise traders.

Managers There are four sectors of asset/fund managers: uninformed managers of the index, informed managers of the index, uninformed managers of the non-index funds, and informed managers of non-index funds, described in Table 1. Uninformed managers choose optimal strategies without private information, while informed managers acquire and use

	Uninformed	Informed
Index	Passive	Active
Non-index	Active	Active

Table 1: Passive and active managers

private information. Managers of the index fund restrict themselves to trading the index fund and the safe asset, while managers of the non-index fund trade non-index assets (long-short portfolios and assets excluded from the index). Uninformed managers of the index fund are passive managers, and the rest of three types of managers are active managers. Informed managers of the index fund actively time the market using their private information. Managers of the non-index fund actively pick stocks with and without private information.

All managers are risk-neutral. Managers (informed and uninformed) incur a marginal cost k_j per investor, and the cost is cheaper for the index fund than for the non-index fund (i.e. $k_x \leq k_y$). Informed managers of the fund j also incur a fixed cost K_j to acquire private signal s_j, given by

$$s_j = v_j + \epsilon_j, \qquad \text{where} \qquad \epsilon_j \sim \mathrm{N}\left(0, \sigma_{v_j}^2 \tau_j^{-1}\right). \tag{1}$$

The precision parameter τ_j is the ratio of the variance of payoff to that of the error, representing the signal-to-noise ratio.

Informed managers trade strategically, taking into account their price impact. They submit demand schedules, specifying quantities to be traded contingent on realized prices, to maximize the expected utilities of their investors' payoffs. There are M_x^I informed managers of the index fund and M_y^I informed managers of the non-index fund. For now, assume $M_x^I = M_y^I = 1$. For simplicity, also assume that uninformed managers trade competitively, taking prices as given.

Investors There is a mass-N continuum of investors, who have constant absolute risk aversion (CARA) preferences with risk aversion γ. Investors can choose between saving and investing with four different types of managers. Saving is free and can be considered as non-participation in the stock/financial market. If investors choose to participate in the financial market, they can choose whether to invest with informed or uninformed managers and whether to invest in the index fund or the non-index fund or both.

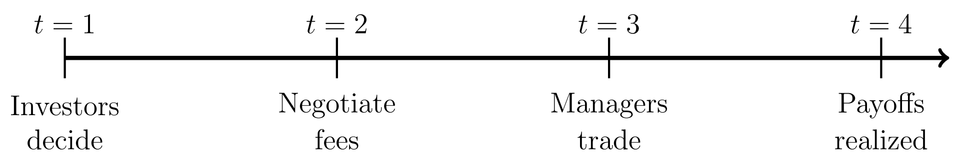


Figure 1: **Time Line**

Investing with managers requires a search, which reflects time and resources that investors must incur to find and vet a manager (see Garleanu and Pedersen (2018) for description of the process in practice). To invest with a manager of fund j, investors first pay a search cost c_j^I for informed and c_j^U for uninformed managers respectively, after which they meet with one and negotiate a management fee f_j^I for informed and f_j^U for uninformed managers respectively, via Nash bargaining with, for simplicity, equal bargaining power. It is cheaper to find and vet an uninformed manager than an informed manager (i.e. $c_j^U \leq c_j^I$) for a given fund j, and an uninformed manager of the index fund is cheaper to find than an uninformed manager of the non-index fund (i.e. $c_x^U \leq c_y^U$).

Let I_j and U_j denote the mass of investors who invest with an informed manager (called "informed investors" for brevity) and the mass of investors who invest with an uninformed manager (called "uninformed investors") of the fund j, respectively. The rest of $N-(I_j+U_j)$ investors choose not to participate in the fund j, and those who do not participate in either fund only save in the safe asset.

Finally, there is a noise trader who trades an exogenous random quantity $-z_j$. Alternatively, z_j can be interpreted as the random supply of the risky asset. The quantity z_j is distributed normally with mean zero and variance ${\sigma_{zj}}^2$. The payoffs v_j, the errors ϵ_j, and z_j are jointly normally and independently distributed.

Timeline To summarize, the model is static with four stages, as illustrated in Figure 1. At $t = 1$, investors decide whether to participate in the market for the risky asset or not. If investors choose to participate, they choose whether to invest in the index fund or the non-index fund or both and whether to search for an informed manager or for an uninformed manager. At $t = 2$, investors meet and negotiate with the managers to determine fees. At $t = 3$, informed managers and uninformed managers trade both funds. At $t = 4$, the payoff of the funds are realized.

Equilibrium definition An equilibrium is found when (1) investors optimally choose among saving and four types of asset managers; (2) management fees are set via Nash bargaining; (3) informed and uninformed managers choose demand schedules to maximize their investors' utility; and (4) the market for the two funds clears. For now, I focus on an interior equilibrium, in which investors are indifferent among saving and uninformed and informed investments of both funds. In the trading stage, I restrict to a linear equilibrium, in which demand schedules are linear in traders' prices and signals.

2 Equilibrium

The model is solved by backward induction, starting with the trading stage.

2.1 Trading Stage Equilibrium

Since the index and non-index funds are orthogonal to each other, the equilibrium for each fund can be determined separately and in the same way.

A linear equilibrium in the market for fund j is found, taking as given the masses of informed and uninformed investors I_j and U_j, which are later determined by the investors' choice. The partial equilibrium characterized in this section is similar to that of Kyle (1989), in which both informed and uninformed traders are strategic, but there are no asset/fund managers, and the number of investors who invest with informed and uninformed managers are exogenous.[3] For completeness as well as to introduce useful notations, I describe the process of solving for an equilibrium briefly below.

The informed manager trades on behalf of their I_j investors. He finds the optimal demand schedule for each investor $q_j^I(p_j)$, taking into account price impact of all his investors, solving

$$\max_{q_j^I} \left\{ \mathrm{E}\left[\left(v_j - p_j^I - \lambda_j I_j q_j^I \right) q_j^I \mid s_j \right] - \frac{\gamma}{2} \operatorname{var}\left[v_j \mid s_j \right] \left(q_j^I \right)^2 \right\}, \tag{2}$$

where $q_j^I = q_j^I(p_j)$ is the quantity that the manager trades, p_j^I is the price that would prevail if the manager were not to trade, and λ_j is price impact à la Kyle (1985, 1989), the marginal effect of trading one additional share on the per-share price of risky asset.[4] Since his private signal s_j encompasses information in prices, he does not need to learn from prices.

Uninformed managers find their optimal demand schedules, taking prices as given and learning rationally from prices. Let $q_j^U(p_j)$ denote the demand schedule of each uninformed investor. Lastly, the market clears: $I_j q_j^I(p_j) + U_j q_j^U(p_j) = z_j$.

[3] There are two differences: In this model, (i) there is only one informed manager; and (ii) uninformed managers competitively. These are simplifying assumptions useful for later analysis and more explicit expressions for intuition, but they do not alter the main results.

[4] This formulation is equivalent to the informed manager aggregating his investors preferences and solving the optimal demand schedule; I thank Mariana Khapko for the suggestion.

To characterize equilibrium, it is useful to use the two variables below. First, following Lee and Kyle (2018) with slight modification, define

$$\chi_j := \frac{\gamma \mathrm{var}\left[v_j \mid s_j\right]}{\lambda I_j + \gamma \mathrm{var}\left[v_j \mid s_j\right]}. \tag{3}$$

It decreases in price impact λ and increases in the riskiness of the risky asset, multiplied by risk aversion. χ_j computes the relative importance between risk and price impact. Using χ_j, the informed manager's demand schedule $q_j^I(p_j)$ can be expressed as

$$q_j^I(p_j) = \chi_j \left(\frac{\mathrm{E}\left[v_j \mid s_j\right] - p_j}{\gamma \mathrm{var}\left[v_j \mid s_j\right]} \right). \tag{4}$$

In the parentheses on the right hand side is the competitive demand schedule. Hence, χ_j measures how unconstrained the informed manager is by his price impact. It lies between zero and one. As χ_j increases to one, the manager trades as if there is no price impact. I use χ_j as a proxy for *liquidity* of the market as experienced by the informed manager.

Second, following Kyle (1989), define

$$\phi_j := \frac{\mathrm{var}^{-1}\left[v_j \mid p_j\right] - \mathrm{var}^{-1}\left[v_j\right]}{\mathrm{var}^{-1}\left[v_j \mid s_j\right] - \mathrm{var}^{-1}\left[v_j\right]}. \tag{5}$$

It is the ratio of the precision of new information from prices to the precision of new information from the signal, and it lies between zero and one. As ϕ_j approaches one, uninformed managers extract from prices all information contained in the private signal. As ϕ_j approaches zero, they extract no information from prices. I use ϕ_j as a proxy for *price efficiency*. It is broadly consistent with other measures of price efficiency in the literature; see Section 4 for description of the relationship among various measures and the empirical implications.

The next proposition fully characterize the equilibrium. All proofs are in the Appendix.

Proposition 1 (Partial Equilibrium). *Assume γ, σ_{v_j}, σ_{z_j}, τ_j, U_j, and I_j are all strictly positive for all j. Then there exists a unique linear equilibrium with trade, in which the price of fund j is*

$$p_j = \left(1 + \frac{\chi_j I_j + U_j}{\left(\chi_j I_j + \phi_j U_j\right)\tau_j}\right)^{-1} \left(s_j - \frac{\gamma \sigma_{v_j}^2}{\tau_j \chi_j I_j} z_j\right), \tag{6}$$

where liquidity $\chi_j \in (0,1)$ is the unique solution to the cubic equation:

$$\left(\gamma \sigma_{v_j} \sigma_{z_j}\right)^2 U_j = \left(\gamma \sigma_{v_j} \sigma_{z_j}\right)^2 \left(U_j + \left(1+\tau_j\right) I_j\right) \chi_j + \tau_j \left(1+\tau_j\right) U_j I_j^2 \chi_j^2 + \tau_j \left(1+\tau_j\right) I_j^3 \chi_j^3, \tag{7}$$

and price efficiency $\phi_j \in (0,1)$ is given by

$$\phi_j = \left(1 + \frac{\left(\gamma \sigma_{v_j} \sigma_{z_j}\right)^2}{\tau_j \left(\chi_j I_j\right)^2}\right)^{-1}. \tag{8}$$

Aside from endogenous liquidity χ_j, for now, price efficiency is unaffected by the number of uninformed investors U_j. It is determined by the number of informed investors I_j, the precision of the private signal τ_j, and the amount of noise trading $\gamma \sigma_{vj} \sigma_{zj}$ in units of util (σ_{zj} is in the number of shares demanded by noise traders, σ_{vj} is the risk in each share, and γ is in the number of utils per dollar). Intuitively, prices become more informative, as there are more informed investors, the private signal becomes more precise, and the amount of noise trading decreases.

Note that plugging $\chi_j = 1$ into (6) and (8) above characterizes the equilibrium in the competitive benchmark, in which all traders take prices as given, regardless of their impact on prices. Hence, the key difference between competitive and strategic informed trading for price efficiency is captured by the endogenous liquidity χ_j. Importantly, the number of uninformed investors U_j has no effect on price efficiency in the competitive benchmark, while it has an indirect effect through liquidity with strategic informed trading.

2.2 Fee Determination

The asset management fees for informed and uninformed investments, f_j^I and f_j^U, for both funds are determined by an appropriate manager manager and an investor who meets with him via Nash bargaining.

Let u_j^I and u_j^U denote the certainty equivalent of an investor investing with an informed manager and that of investing with an uninformed manager, respectively:

$$u_j^I := -\frac{1}{\gamma} \log \left(\mathrm{E}\left[-\exp\left(-\gamma \left(v_j - p_j\right) q_j^I(p_j)\right)\right]\right); \tag{9}$$

$$u_j^U := -\frac{1}{\gamma} \log \left(\mathrm{E}\left[-\exp\left(-\gamma \left(v_j - p_j\right) q_j^U(p_j)\right)\right]\right). \tag{10}$$

Note that an investor may choose to invest in both funds. For example, the investor may invest with an uninformed manager of the index fund and with an informed manager of the non-index fund. From the orthogonality assumption, the certainty equivalent of investing in both funds can be simply summed up. Thus, the investor in the example obtains the certainty equivalent of $u_x^U + u_y^I$.

Using the equilibrium price in Proposition 1 and the usual properties of the momentgeneration function, the lemma below characterizes the certainty equivalent of uninformed investments, which represents the dollar value of participating in the market for fund j, and the difference in the certainty equivalents of informed and uninformed investments, which represents the incremental dollar value of private information:

Lemma 1 (Values of Participation and Information)**.** *With ϕ_j and χ_j given by Proposition 1, the dollar value of participation is*

$$u_j^U = \frac{1}{2\gamma}\log\left(\frac{\mathrm{var}\left[v_j - p_j\right]}{\mathrm{var}\left[v_j \mid p_j\right]}\right) \tag{11}$$

and the incremental dollar value of information is

$$u_j^I - u_j^U = \frac{1}{2\gamma}\log\left(\frac{1-\chi_j}{e^{2\gamma u_j^U}} + \left(\frac{1+\tau_j}{1+\phi_j\tau_j}\right)\chi_j\right), \tag{12}$$

where

$$\frac{\mathrm{var}\left[v_j - p_j\right]}{\mathrm{var}\left[v_j \mid p_j\right]} = 1 + \frac{\tau_j\left(1-\phi_j\right)^2}{\phi_j\left(1+\tau_j+\left(1+\phi_j\tau_j\right)U_j\left(\chi_j I_j\right)^{-1}\right)^2}. \tag{13}$$

After paying the search cost c_j^I and meeting with the informed manager, the investor gains $u_j^I - f_j^I$ by agreeing to pay the fee. If the negotiation fails, the investor gains zero, if he chooses to save. Since investors are indifferent among saving and four types of managements in an interior equilibrium, the investor's second-best outside option yields zero utility gain. Assuming, for simplicity, the equal bargaining power between an investor and an informed manager, f_j^I is determined to maximize the product of utility gains $(u_j^I - f_j^I)(f_j^I - k_j)$. The fee for uninformed manager is found in the same way. In summary, the equilibrium management fees are

$$f_j^I = \frac{u_j^I + k_j}{2} \qquad \text{and} \qquad f_j^U = \frac{u_j^U + k_j}{2}. \tag{14}$$

2.3 Investors Choice

Investors optimally choose among saving, uninformed and informed investments, rationally anticipating the fees and the asset market equilibrium in Proposition 1.

Assuming that the total number of investors N is sufficiently large to ensure an interior equilibrium, for now, investors are indifferent among saving and searching for any of the four types of managers:

$$u_j^I - (c_j^I + f_j^I) = u_j^U - (c_j^U + f_j^U) = 0, \qquad \forall j. \tag{15}$$

Substituting the fees (14) into above, the equilibrium masses of informed and uninformed investors I_j and U_j are determined such that

$$u_j^I = k_j + 2c_j^I \qquad \text{and} \qquad u_j^U = k_j + 2c_j^U . \tag{16}$$

Finally, combining this with the certainty equivalent in Lemma 1 determines a general equilibrium. The following result summarizes the solution.

Proposition 2 (General Equilibrium)**.** *Assume that γ, σ_{v_j}, σ_{z_j}, τ_j, c_j^I, and k_j $\forall j$ are all strictly positive and that N is sufficiently large. There exists a unique interior equilibrium, in which (1) the prices of funds are given by (6), (2) asset management fees are given by*

$$f_j^I = k_j + c_j^I \qquad \text{and} \qquad f_j^U = k_j + c_j^U , \qquad \forall j \tag{17}$$

and (3) the masses of informed and uninformed investors for each fund are

$$I_j = \frac{\gamma \sigma_{v_j} \sigma_{z_j}}{\chi_j} \sqrt{\frac{1}{\tau_j} \left(\frac{\phi_j}{1-\phi_j} \right)}, \tag{18}$$

$$U_j = \frac{\gamma \sigma_{v_j} \sigma_{z_j} (1+\tau_j)}{(1-\chi_j)(1-\phi_j) - (1+\tau_j)\phi_j} \sqrt{\frac{1}{\tau_j} \left(\frac{\phi_j}{1-\phi_j} \right)}, \tag{19}$$

where liquidity χ_j is

$$\chi_j = \frac{(1+\phi_j \tau_j)\left(e^{2\gamma(k_j+2c_j^I)} - 1\right)}{(1+\tau_j)e^{2\gamma(k_j+2c_j^U)} - (1+\phi_j\tau_j)}, \tag{20}$$

and price efficiency $\phi_j \in (0,1)$ is the unique solution that satisfies

$$1 = \left(\frac{2}{1+\phi_j\tau_j} - \frac{e^{2\gamma(k_j+2c_j^I)} - 1}{(1+\tau_j)e^{2\gamma(k_j+2c_j^U)} - (1+\phi_j\tau_j)} \right) \left(1 - \phi_j - (1+\tau_j)\left(\tau_j^{-1} \left(e^{2\gamma(k_j+2c_j^U)} - 1 \right) \phi_j \right)^{1/2} \right). \tag{21}$$

3 Declining Fees and Price Efficiency

Asset management fees, especially those for passive index funds, have been steadily decreasing in the past few decades. In this section, I analyze comparative statics, in which parameters that determine management fees change, and study the impact on the general equilibrium price efficiency. These results are then connected to empirical facts on fees, liquidity, and price efficiency in Section 4.

Recall that in Proposition 2, the management fees (f_j^I and f_j^U) are determined by and increase in the marginal trading cost that both informed and uninformed managers face per investor (k_j) and the search costs that investors pay to meet with informed and uninformed managers (c_j^I and c_j^U). In the following subsections, I study the general equilibrium effect of decreasing management fees through three separate channels: (i) the decrease in the trading cost; (ii) the decrease in the search cost for an uninformed manager; and (iii) the decrease in the search cost for an informed manager.

3.1 Decrease in the trading cost

The proposition below provides a comparative static of price efficiency in the general equilibrium characterized in Proposition 2 as the trading cost that affects the management fees of both the informed and uninformed managers varies.

Proposition 3 (Declining trading cost)**.** *As the trading cost of a given portfolio k_j decreases, general equilibrium price efficiency of the fund ϕ_j increases.*

Hence, prices become more efficient as the fall of the trading cost makes both informed and uninformed investments cheaper, while the spread between two fees remain the same. To provide economic intuition for this result, let me illustrate the mechanism in each step, although in a general equilibrium everything is affected by everything else.

Value of Participation First, declining k_j makes it cheaper to participate in the market for fund j without any private information, while the spread between informed and uninformed funds remains constant. Intuitively, as it becomes cheaper to participate in the market, the mass of investors who invest with uninformed managers U increases. This intuition also can be seen in the representation of the certainty equivalent that investors obtain by investing with an uninformed manager u_j^I, which decreases as k_j decreases in equilibrium (i.e. $u_j^U = k_j + 2c_j^U$), from (16). Furthermore, from Lemma 1, the certainty equivalent u_j^I tends to decrease in the mass of uninformed investors U_j. Hence, as it becomes cheaper to participate in the market, more uninformed investors enter so that each investor's gain from participation decreases, corresponding to the lower management fees they pay.

Liquidity Next, the increase in uninformed investors makes the market more liquid for the informed manager, who then can trade more aggressively on their private information. The next result directly follows the partial equilibrium characterization:

Corollary 1 (Liquidity)**.** *In the partial equilibrium characterized in Proposition 1, liquidity χ_j increases in the number of uninformed investors U_j and the amount of noise trading*

$\gamma\sigma_{vj}\sigma_{zj}$ and decreases in the number of informed investors I_j and the precision of private information τ_j.

More noise trading and more uninformed investors improve liquidity for the informed manager, while more precise information and more investors to trade for makes the market less liquid for the informed manager. To see why increasing uninformed investors improve liquidity, recall that each uninformed investor has a downward-slopping demand schedule in equilibrium. Since they buy more when prices are low and sell more when prices are high, uninformed investors dampen the effect of informed investors on prices. As there are more uninformed investors, aggregate demand schedule from uninformed investors becomes steeper. This in turn implies that the residual supply schedule of informed investors becomes flatter, which means lower price impact and higher liquidity.

Note that there is a mitigating effect through price efficiency. Since liquidity allows informed investors to trade aggressively, prices become more informative. More informative prices imply that higher prices are now more likely to come from better fundamental. This discourages each uninformed investor from providing depth to the market and taking the other sides of informed investors. Hence the demand schedule of each uninformed investor becomes flatter as there are more informed investors. The overall effect on the aggregate demand schedule, however, is dominated by the increase in uninformed investors (extensive margin) rather than by the decrease in the willingness of each uninformed investor to provide liquidity (intensive margin).

Value of Information More liquid the market is for the informed manager, more valuable a given private information is for his investors. Recall that liquidity χ is defined as the extent to which the informed manager can trade without being constrained by his price impact. Hence, as liquidity improves, the informed manager can trade a larger quantity and arrive closer to his target inventory. This then translates into the value he can generate with a given information. From Lemma 1, the value of information, captured by the spread in the certainty equivalent, is

$$\mathbf{VI} := u_j^I - u_j^U = \frac{1}{2\gamma}\log\left(\frac{1-\chi_j}{e^{2\gamma u_j^U}} + \left(\frac{1+\tau_j}{1+\phi_j\tau_j}\right)\chi_j\right). \tag{22}$$

The first term in the parenthesis on the right hand side represents the opportunity cost of being informed. Due to limited liquidity, the manager cannot trade as much as he would have liked to, and for this amount, investors lose the value they could have obtained by investing with an uninformed manager.

The second term represents the gains from private information, and the extent to which the manager could take advantage of their private information depends not only on price informativeness (ϕ), with which uninformed managers learn about private information from prices, but also on liquidity χ. Note that if the informed manager were to take prices as given, and thus he perceives liquidity χ as exogenous and equal to one, then the right hand side of (22) will simplify into

$$\mathbf{VI}^{\text{com}} = \frac{1}{2\gamma} \log\left(\frac{1+\tau_j}{1+\phi_j\tau_j}\right), \tag{23}$$

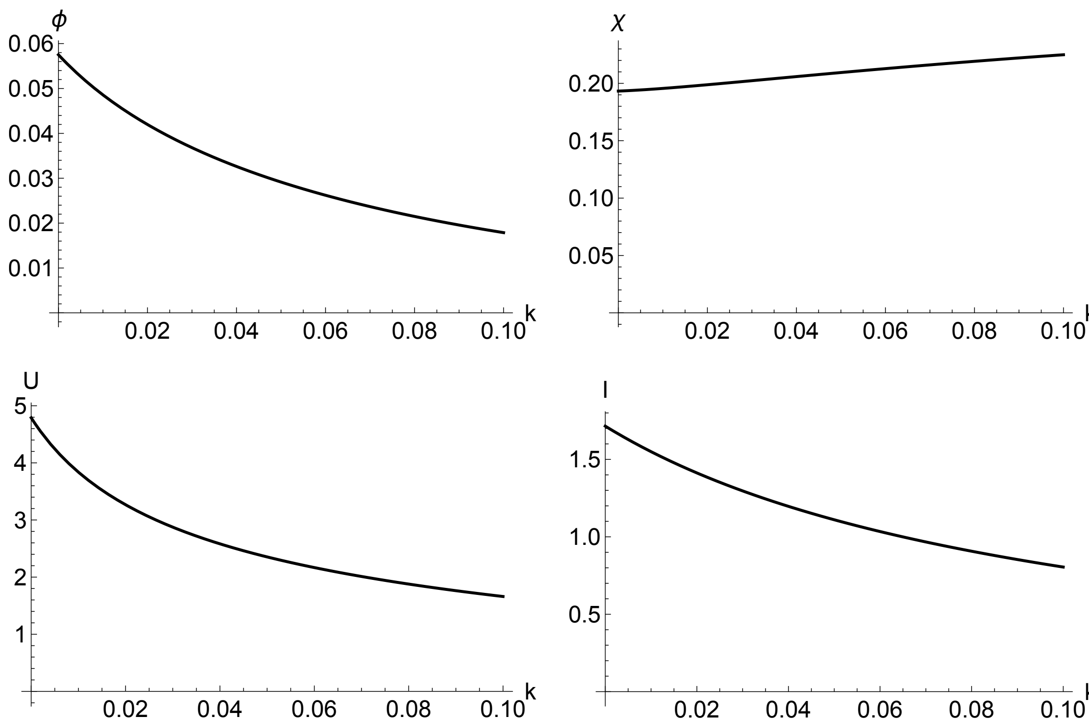
which corresponds to the value of information in the competitive benchmark of Grossman and Stiglitz (1980). Hence, while the improved liquidity provided by the increased participation of uninformed investors tends to make the information more valuable for the strategic informed manager, it has no effect for those who trade competitively.

Recall that as the trading cost k_j decreases, the spread between the fee for the informed manager and that for the uninformed manager remains constant: $f_j^I - f_j^U = c_j^I - c_j^U$ from (17) in Proposition 2 does not depend on the trading cost k_j. As k_j decreases, the incremental value of investing with an informed manager relative to an uninformed manager increases, while the differences in their fees remain the same. Intuitively, more investors choose to invest with an informed manager (i.e. the mass of informed investors I increases), which then decreases the value of information for each informed investor in equilibrium corresponding to the constant fee spread.

Price efficiency Finally, the increase in the mass of informed investors makes prices more informative. Recall in the partial equilibrium characterized by Proposition 1, price efficiency is determined by the mass of informed investors and liquidity, along with the precision of private information, the amount of noise trading, given by (8). With the precision and noise trading fixed, the effect of trading cost (k_j) on the general equilibrium price efficiency is then determined by how the change in k_j affects the product between mass of informed investors (I_j) and liquidity (χ_j) in equilibrium. While improved liquidity provided by uninformed investors leads more investors to choose the informed manager, the increase in informed investors also deteriorates liquidity, and thus the overall equilibrium effect of the trading cost on liquidity is ambiguous. The overall effect on price efficiency, however, turns out always positive, as the effect of the number of informed investors dominates: prices become more efficient as the trading costs decline.

To summarize, the intuition for the result that general equilibrium price efficiency increases as the trading costs declines is that (i) more uninformed investors enter as it becomes cheaper to participate in the financial market; (ii) they provide liquidity and make the mar-

Figure 2: Effects of declining trading costs



Notes: Figure shows the comparative statics with respect to the trading cost k_j. The dependent variables are price efficiency ϕ_j, liquidity χ_j, the number of investors who invest with an uninformed manager U_j, and the number of investors who invest with an informed manager I_j. The parameter values used are $\sigma_v = \sigma_{zj} = 1$, $\gamma = 3$, $\tau_j = 5$, $c_j^U = 1\%$, and $c_j^I = 5\%$.

ket more liquid for the informed manager; (iii) liquidity makes information more valuable, and more investors choose to invest with an informed manager; and thus (iv) prices become more efficient as there are more informed investors; see also Figure 2.

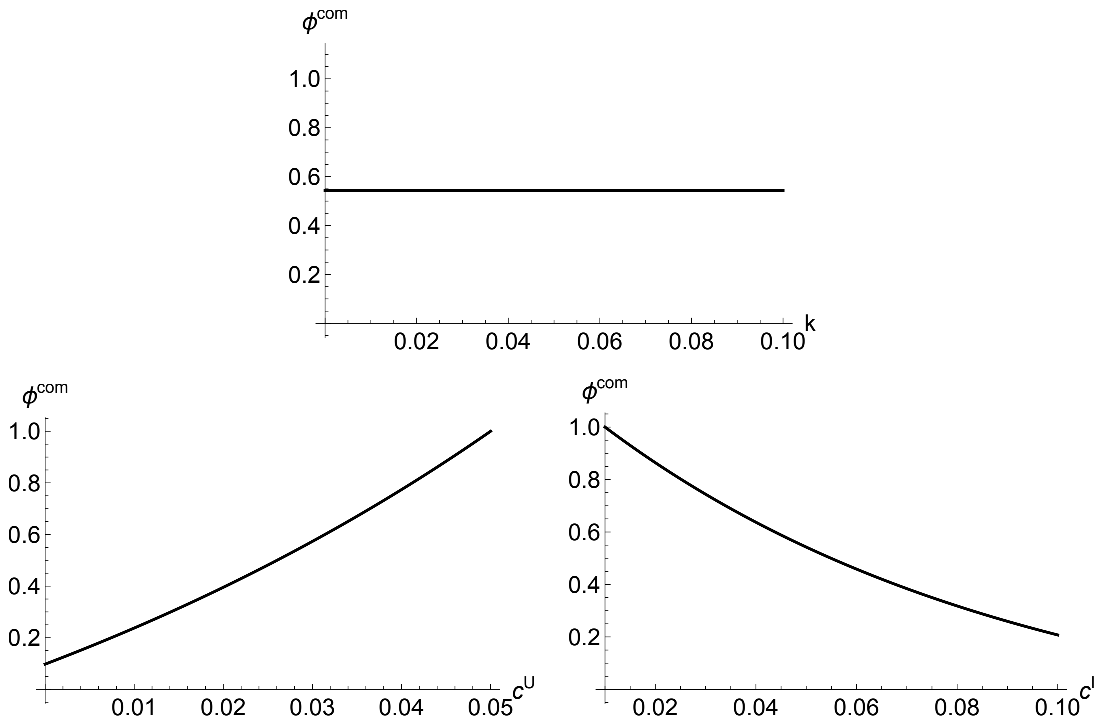
Role of Endogenous Participation Two key elements in this result are strategic informed trading and endogenous participation. To clearly explain the role of each element, I start with endogenous participation to the financial market. , which contrasts to a common assumption in the literature that starts with an exogenous number of participants, who must choose whether to invest with or without information (e.g. Grossman and Stiglitz (1980) normalize the mass of total participants is normalized to one).

In this model, the total number of participants for fund j is endogenous: Among N total investors, I_j investors choose to invest with an informed manager, U_j investors choose to invest with an uninformed manager, while $N - I_j - U_j$ investors choose to not participate in the financial market for fund j. This is crucial for the result of Proposition 3 since the key mechanism through which lower trading costs increase price efficiency is that more investors choose to invest with an informed manager, as more investors also choose to invest with an uninformed manager, i.e. both types of investors increase simultaneously in equilibrium.

Bond and Garcia (2017) examine the endogenous decision to participate in the financial market in the context of the rational expectations framework. In their model, as the index portfolio becomes cheaper to invest and thus it becomes less expensive to participate, more investors participate and prices become less informative. In their model, investors are endowed with private information and endowment shocks. They show that the participation decision exhibits strategic complementarity. Similar to the so-called "Hirshleifer effect", less informative prices makes participating more valuable. Malikov (2019) also study endogenous participation in the multiple asset extension of Grossman and Stiglitz (1980), where investors optimally choose whether to acquire information or not in addition to whether to participate or not. Both Bond and Garcia (2017) and Malikov (2019) are based on the competitive rational expectations equilibrium, in which all traders take prices as given. To discuss how their results differ from mine and highlight the role of strategic informed trading in this model, below I consider the competitive benchmark, in which the informed manager takes prices as given in an otherwise identical setting.

Comparison with the Competitive Benchmark Recall that in the partial equilibrium characterized by Proposition 1, simply assuming exogenous, perfect liquidity yields the

Figure 3: Price efficiency in the competitive benchmark



Notes: Figures show the comparative statics of the competitive benchmark price efficiency with respect to the trading cost (top), the search cost for an uninformed manager (left), and the search cost for an informed manager (right). The parameter values used are $\sigma_v = \sigma_{zj} = 1$, $\gamma = 3$, $\tau_j = 5$, $k_j = 1\%$, $c_j^U = 1\%$and $c_j^I = 5\%$.

competitive counterpart. From plugging $\chi = 1$ into (8), price efficiency is

$$\phi_j^{\text{com}} = \left(1 + \frac{\left(\gamma \sigma_{v_j} \sigma_{z_j}\right)^2}{\tau_j I_j^2}\right)^{-1}. \tag{24}$$

Hence, other than the exogenous parameters such as the precision and noise trading, price efficiency depends only on the mass of informed investors. Absent the endogenous liquidity channel, the number of uninformed investors has no effect on price efficiency. Furthermore, the value of information in this benchmark ($\mathbf{VI}^{\text{com}}$ in (23)) depends only on price efficiency and the precision parameter. Below is the characterization of the general equilibrium in the competitive benchmark:

Proposition 4 (Competitive Benchmark)**.** *Assume that the informed manager takes prices as given. The price efficiency in an interior general equilibrium is given by*

$$\phi_j^{com} = \tau_j^{-1}\left((1+\tau_j)e^{-4\gamma(c_j^I - c_j^U)} - 1\right). \tag{25}$$

Hence, (i) as the marginal cost for managing uninformed investments k_j declines, price efficiency is unaffected; (ii) as the search cost for an uninformed manager c_j^U declines, prices become less informative; and (iii) as the search cost for an informed manager c_j^I declines, prices become more informative.

Since the partial equilibrium price efficiency in the competitive benchmark essentially depends on the number of informed investors, which in turn is determined by the cost of information, or the fee spread between informed and uninformed investments, the general equilibrium price efficiency also depends on the fee spread (i.e. $2(c_j^I - c_j^U)$) and is independent of the trading cost k_j. Intuitively, prices become more informative as it becomes more expensive to invest with an informed manager relative to investing with an uninformed manager.

As the trading cost decreases and thus it becomes cheaper to participate in the financial market, more investors do enter. But, the entry of uninformed investors does not affect price efficiency, since the price-taking informed manager, who behaves as if he can trade with perfect liquidity, does not respond to the liquidity provided by uninformed investors. Thus, the value of information, as well as investors' decision to invest with an informed manager are unaffected. Hence, price efficiency remains constant as the trading cost declines in the competitive benchmark.

3.2 Decrease in the search cost for an uninformed manager

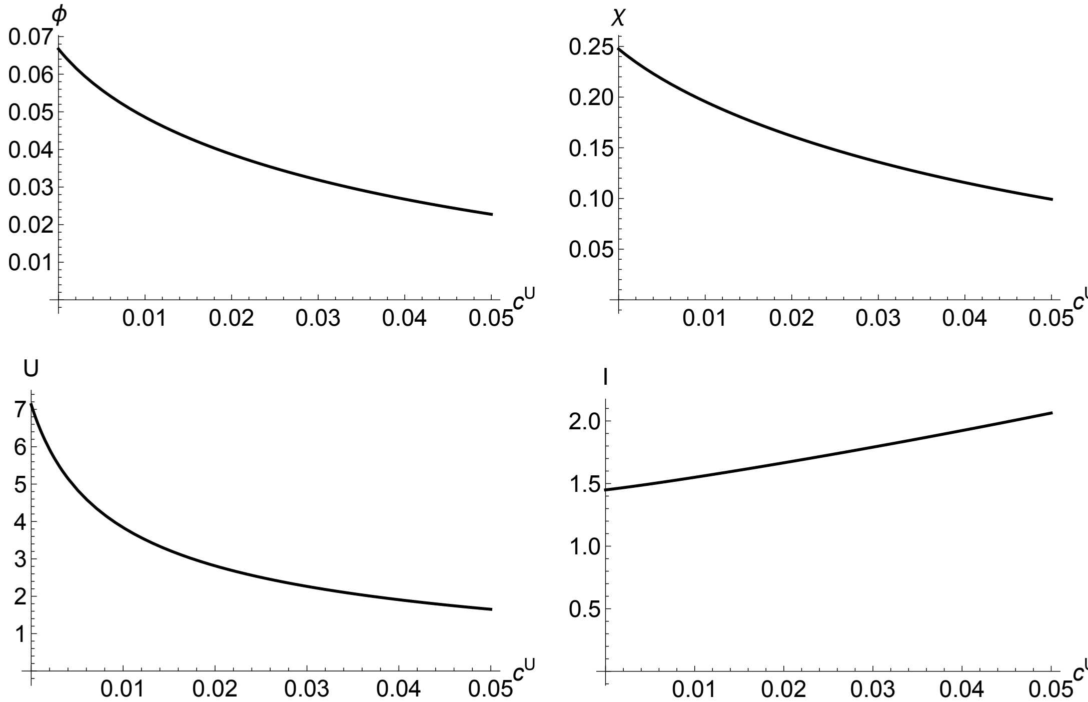
The next proposition is a comparative static of price efficiency with respect to the search cost for an uninformed manager, which affects the management fee for an uninformed manager but has no effect on that for an informed manager.

Proposition 5 (Declining search costs for uninformed managers)**.** *As the search cost for the uninformed manager of fund j (c_j^U) decreases, price efficiency ϕ_j increases provided that c_j^I is sufficiently low.*

Hence, prices can become more efficient, as the fee for an uninformed manager falls, even though it makes informed investments relatively more expensive. This result directly contrasts with that of the competitive benchmark in Proposition 4: Price efficiency always decreases as the search cost for an uninformed manager decreases, since it increases the fee spread between informed and uninformed investments, and thus it discourages investors from choosing the informed manager.

To provide intuition for this result, let me illustrate each step of the mechanism. Like in the comparative static with respect to the trading cost of Proposition 3, the decrease

Figure 4: Effects of declining search costs for an uninformed manager



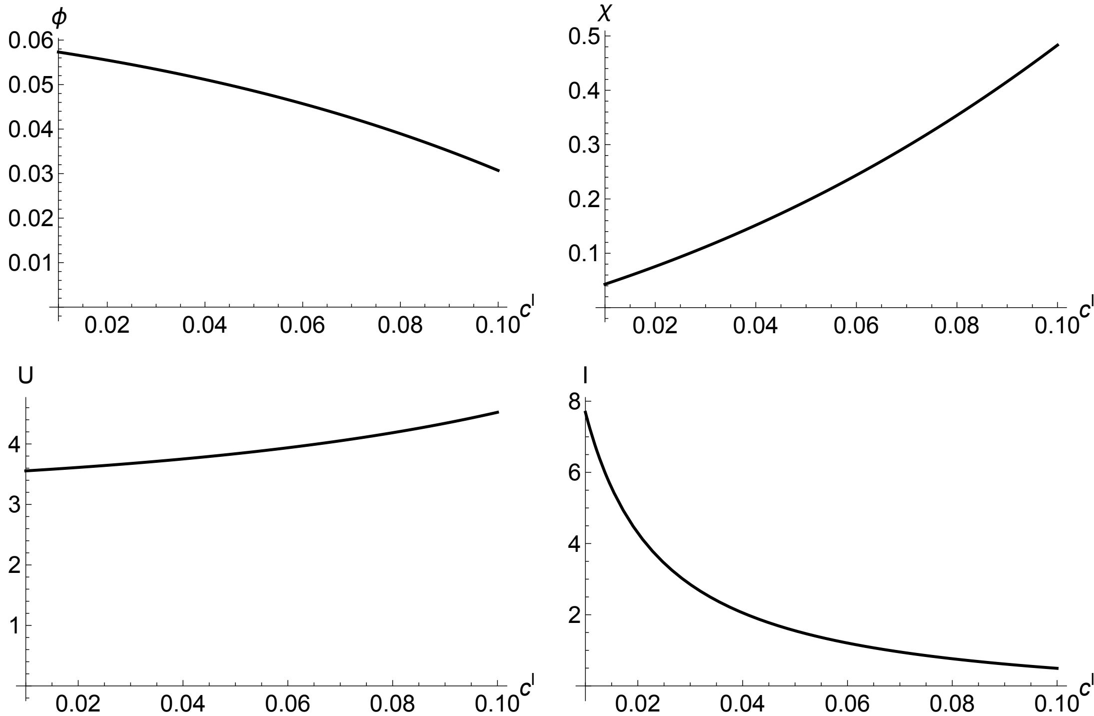
Notes: The parameter values used are $\sigma_v = \sigma_{zj} = 1$, $\gamma = 3$, $\tau_j = 5$, $k_j = 1\%$, and $c_j^I = 5\%$.

in the search cost for an uninformed manager lowers the equilibrium fee for an uninformed manager and thus makes it cheaper to participate in the financial market. Hence, the value of participation must also decrease correspondingly, by more investors entering the financial market. This tends to improve liquidity for the informed manager, and it increases the value of given information since the manager can trade more aggressively on that information.

Unlike the decrease in the trading cost, which affects both the fees for uninformed and informed managers the same way, the decrease in c_j^U does not lower the equilibrium fee for an informed manager ((17) in Proposition 2). Thus, as c_j^U decreases, the spread between the informed and uninformed managements increases. This implies that it not only becomes cheaper to participate in the financial market, but also becomes relatively more expensive to invest with an informed manager as the incremental fees for the informed manager increases. Intuitively, the higher cost would decrease investors from investing with an informed manager.

While this effect would always make prices less informative in the competitive benchmark, it faces countervailing forces with strategic informed trading. First, as mentioned above, improved liquidity from the increased participation of uninformed investors makes investing with an informed manager more valuable. Second, the decrease in informed investors due

Figure 5: Effects of declining search costs for an informed manager



Notes: The parameter values used are $\sigma_v = \sigma_{zj} = 1$, $\gamma = 3$, $\tau_j = 5$, and $c_j^U = k_j = 1\%$.

to the high spread further improves liquidity: recall in Corollary 1, the partial equilibrium liquidity decreases in the number of informed investors as well as increases in the number of uninformed investors. Hence, even as it becomes relatively more expensive to invest with an informed manager, prices can be actually more informative, since liquidity, with which the informed manager trades on their information, improves faster so that the product between liquidity and the mass of informed investors increases.

The result does not always hold and requires the search cost for an informed manager to be sufficiently low so that the fee spread does not become too high as the search cost for an uninformed manager decreases. When the search cost for an informed manager is too high, the number of informed investors can decrease so fast that price efficiency becomes non-monotonic and can decrease, even to zero. In Section 4, I examine whether the condition is likely to hold in the real world using a range of reasonable parameter values.

3.3 Decrease in the search cost for an informed manager

The third, and final, comparative static studies how price efficiency is affected by the search cost for an informed manager:

Proposition 6 (Declining search costs for informed managers)**.** *As the search cost for the informed manager c_j^I decreases, price efficiency ϕ_j increases.*

As the search cost for an informed manager declines, the management fee for informed investments also declines, while the fee for uninformed investments remains constant. This implies that while the cost to participate in the market is unaffected, investing with an informed manager becomes relatively cheaper. It leads to more investors choosing to invest with the informed manager and more informative prices, like in the competitive benchmark (Proposition 4), where the decrease in the fee spread makes prices more informative. Since more informed investors deteriorate liquidity with which the informed manager can trade, both the level of price efficiency and the effect of the lower fee on price efficiency are dampened with strategic informed trading relative to the competitive benchmark.

3.4 Effect of noise trading

The growth in the professional asset management industry (including both passive and active) in the last decades is accompanied with the decline in direct investing of individual investors. At least some part of direct investing may be close to noise trading. Although noise trading is modeled exogenous for simplicity, it may capture trading based on hedging needs, private values, etc. Below I study the implication of noise trading for price efficiency. The result directly follows from Propositions 2 and 4.

Corollary 2 (Noise Trading)**.** *Price efficiency in a general equilibrium is independent of the amount of noise trading, regardless of whether informed managers trade strategically.*

Noise trading tends to make prices less informative in the asset market equilibrium, with the numbers of informed and uninformed investors exogenously given. In a general equilibrium, it has no effect. Intuitively, noise trading provides profitable opportunities to both informed and uninformed investors, and they optimally respond so that price efficiency is unchanged.

This result is in contrast to Stambaugh (2014), who shows the decrease in noise trading (representing direct investing of individuals) makes prices more informative, or equivalently, reduce mispricing. In his model, active managers, who trade against noise traders, face exogenous, quadratic trading costs. Although price impact also causes quadratic trading costs, the difference is that trading costs are endogenously determined in my model. It is also interesting to note that as far as the implication of noise trading is concerned, the competitive equilibrium yields the same result.

Finally, recall that in characterizing equilibrium, the standard deviation of noise trading σ_{zj} always appears in the product $\gamma\sigma_v\sigma_{z_j}$, which together I interpret as the noise trading in utils. This also implies that risk aversion γ and the standard deviation of the fundamental σ_v have exactly the same comparative statics as that of σ_{zj}, except that γ affects price efficiency through its effect on the dollar costs of k and c. Hence, with the trading and search costs fixed in units of utils, changing the risk aversion of the prior variance would have no effect on price efficiency.

4 Empirical Implications

The theoretical results from previous sections are now used to draw empirical implications. To ensure relevancy of the implications, I start by discussing key trends in the asset management industry documented in the existing literature. I then connect the trends with the theoretical framework to provide implications for price efficiency and relate them to the empirical evidence on time-series and cross-sectional price efficiency in the literature.

4.1 Key trends in the asset management industry

Growth in delegated investments Delegated investments in the United States have been growing tremendously over the past few decades. Individuals have been moving away from direct equity ownership towards institutional investments, with the fraction of individual ownership falling from 48% in 1980 to 20% in 2010 (French (2008); Stambaugh (2014)). And the trend of falling direct ownership has continued since the end of the Second World War (Garleanu and Pedersen (2019)). Combining both direct and institutional ownership, the fraction of American households with any stock has also increased. The rate grew from 32% in 1989 to 49% in 1998 (Bertaut and Starr (2000); Hong et al. (2004)). The growth in various retirement vehicles such as 401 (k) plans and IRAs further allow individuals to participate in the stock market (Poterba et al. (1998)).

Following the interpretation that individual ownership likely resembles noise trading (Stambaugh (2014); Black (1986)), both trends of the growing stock market participation and the growing delegated investments can be mapped into the increase in participation in the theoretical model.

Rise of passive investing Especially pronounced in the growth of delegated investments is the rise of passive investing. Since the first index mutual fund was introduced in 1975, the industry of low-cost index investing has grown tremendously.[5] Appel et al. (2016) document

[5]For those who are interested, this podcast episode features Jack Bogle and AQR's co-founder Cliff Asness who discuss merits of passive and active investing and the beginning of index funds: https://www.aqr.com/Insights/Podcasts/The-Curious-Investor/Season-One/Active-versus-Passive

that between 1998 and 2014, passively managed funds grew from under 10% to over 30% of all equity mutual funds and from under 2% to close to 8% of the total market capitalization. Blackrock (2017) estimates that passive investors, including those outside mutual funds and ETFs, own 18% of all global equity in 2016. Anadu et al. (2019) report that passive mutual funds and ETFs grew from less than 4% of the U.S. equity market in 2005 to about 14% in March 2020. On the other hand, Easley et al. (2018) emphasize that determining "activeness" in the ETF can be complex and argue that the market has not necessarily become more passive.

The rise of passive investing is accompanied by the steady decline in management fees. French (2008) documents that the average fees and expenses for mutual funds fell from 2.08% of assets under management to 0.95% between 1980 and 2006, while the average spread between active and passive funds remained stable at about 70 basis points during the same period. The decline in average fees is due to the falling fees of funds as well as the movement towards the cheaper passive funds. Morningstar (2020) reports that asset-weighted passive fund fees declined from 30 basis points to 13 basis points between 1990 and 2019, while active fees fell from 85 basis points to 67 basis points during the same period, which also shows that the spread has remained almost constant, changing only from 55 to 54 basis points. Cremers et al. (2016) provide evidence that the decline in active fees may in fact be caused by competitive pressure from low-cost index funds.

In the model, I interpret the uninformed managers of the exogenous index portfolio as passive managers, while there are three types of active funds: informed managers of the index portfolio, and uninformed and informed managers of the non-index portfolios such as spread portfolios or portfolios that consist of assets not included in the index portfolio. The first type of the active manager employs a market-timing strategy, deciding when to buy or sell the index fund using private information, while the other two types implement a stock-picking strategy, determining assets that are mispriced relative to the index portfolio with or without private information. The falling management fees in the real world, especially for passive investments, can be mapped into the decline in determinants of the equilibrium fees, both trading and search costs, especially for the uninformed managers, while the stable spread between active and passive funds can represent that the fall in fees is mainly caused by that of the trading cost especially for the index fund.

4.2 Time-series and cross-sectional implications for price efficiency

Proxies for Price Efficiency This paper uses ϕ_j, the fraction of private information that is revealed in prices, as defined by (5), as a proxy for price efficiency. It is related to and broadly consistent with a variety of measures used in the literature.

First, Grossman and Stiglitz (1980) use the ratio between the total precision with and without private information, where the difference from ϕ_j is that it includes that from the prior variance, and hence the term total, i.e.

$$\frac{\operatorname{var}^{-1}[v_j \mid p_j]}{\operatorname{var}^{-1}[v_j \mid s_j]} = \frac{1+\phi_j\tau_j}{1+\tau_j}, \tag{26}$$

where one in both the numerator and the denominator on the left hand side represent the importance of the prior variance (recall that the precision τ_j in (1) is defined as the precision of the error normalized by that of the fundamental). With a given precision of the private signal τ_j, prices become more efficient as ϕ_j increases and is fully efficient if and only if ϕ_j approaches one.

Closely related to above, Garleanu and Pedersen (2018, 2019) use

$$-\frac{1}{2}\log\left(\frac{1+\phi_j\tau_j}{1+\tau_j}\right) \tag{27}$$

as a measure of price inefficiency. A nice property is that in the competitive benchmark such as theirs, it corresponds to the value of information, as discussed in (23), and thus related to the difference in performance between informed and active funds.

From the perspective of an econometrician, Bai et al. (2016) measure price efficiency using the variance of the predictable component in the fundamental, which represents the R^2 of a regression of prices on future asset values. Using the notations this model, the measure is expressed as

$$\operatorname{var}[\mathrm{E}[v_j \mid p_j]] = \operatorname{var}[v_j] - \operatorname{var}[v_j \mid p_j] = \left(1 - \frac{1}{1+\phi_j\tau_j}\right)\sigma_{v_j}^2. \tag{28}$$

The measure is related to Bond et al. (2012), who articulate the difference between forecasting price efficiency and revelatory price efficiency, and captures the former, which includes information already known to insiders of the firm as well as information insiders learn from the market.

Furthermore, Farboodi et al. (2020) use the measure above as well as develop a structural framework, in which price efficiency is given by the covariance between price and future cash flows, normalized by the standard deviation of prices. **?** use relative price informativeness, which corresponds to (28) normalized by the conditional variance of cash flows given prices, i.e. $\operatorname{var}[\mathrm{E}[v_j \mid p_j]] / \operatorname{var}[v_j \mid p_j]$. Huang et al. (2020) use the inverse of the conditional variance of the fundamental given prices, i.e. $\operatorname{var}^{-1}[v_j \mid p_j]$. Using notations of this paper, the last two measures simplify to $\phi_j \tau_j$ and $(1 + \phi_j \tau_j)\, \sigma_{v_j}^{-2}$, respectively.

Although the list is certainly not exhaustive of all proxies of price efficiency in the literature, it demonstrates the common theme that more informative prices should lower the conditional variance of the fundamental given the price (var $[v_j \mid p_j]$), while individual measures differ in the way that the reduction in the conditional variance is normalized. Therefore, price efficiency generally increases in ϕ_j, which is thus used to connect to empirical predictions and evidence below. Note that the relationship is exactly monotonic if the the prior variance of the fundamental σ_v and the precision τ_j remain constant, which is the case in main comparative statics analysis of the trading and search costs, and thus the management fees of active and passive funds.

Model predictions for price efficiency and empirical evidence Based on the evidence of falling management fees as discussed above, the theoretical framework of this paper implies that price efficiency of the index portfolio would generally increase over time. From Propositions 3, 5, and 6, price efficiency of a portfolio increases as the trading and search costs for that portfolio decrease. With fees for passive investments being even lower than those of active investments, price efficiency of the index portfolio is likely to be higher and increase more than that of the non-index portfolios such as spread portfolios and portfolios of assets excluded from the index portfolio.

These results on portfolio price efficiency can be translated into price efficiency of individual stocks, in spirit of the single-index model of Sharpe (1963), i.e.

$$v_i = \overline{v_i} + \beta_i v_x + e_i, \tag{29}$$

where v_i is the value of the asset i, β_i is the factor loading on the index portfolio v_x, and e_i is the idiosyncratic component. The error e_i in turn can be written as $w_i v_y$, where w_i is the factor loading on the non-index portfolio, independent of the index portfolio, v_y (or the weighted sum of several such portfolios). Garleanu and Pedersen (2019) assume the factor structure for fundamentals and determine factor and idiosyncratic portfolios endogenously. They provide conditions in which the factor portfolio is indeed proportional to the index portfolio.

Under the assumption of the factor structure (29), price efficiency of individual stocks in the context of Bai et al. (2016) as in (28) can be decomposed into the systematic and idiosyncratic parts:

$$\text{var}[\text{E}[v_i \mid p_x, p_y]] = \left(1 - \frac{1}{1 + \phi_x \tau_x}\right) \beta_i^2 \sigma_{v_x}^2 + \left(1 - \frac{1}{1 + \phi_y \tau_y}\right) w_i^2 \sigma_{v_y}^2. \tag{30}$$

In this context, the increased price efficiency of the index fund as a result of the decrease in passive fees and the unaffected or moderately increased price efficiency of the non-index fund can be combined to provide cross-sectional implications. This suggests that stocks included in the index fund are more likely to experience increased price efficiency than those excluded in the index fund and that stocks that comprise an important part of the index fund are more likely to have higher price efficiency than others. The result is consistent with the recent empirical evidence that financial prices have become more informative, especially those of the large stocks included in the S&P 500 index (Bai et al. (2016); Farboodi et al. (2018); Davila and Parlatore (2020)).

5 Discussion

The results in this model shed light on the well-known Grossman-Stiglitz paradox.[6] They show that prices cannot fully reveal costly private information. In their discussion of Theorem 5 in their paper, they write "If there is no noise and some traders become informed, then all their information is transmitted to the uninformed by the price system. Hence each informed trader acting as a price taker thinks the efficiency of the price system will be unchanged if he becomes uninformed... On the other hand, if no traders are informed, then each uninformed trader learns nothing from the price system, and thus he has a desire to become informed"

Recall that in a general equilibrium, noise trading has no effect on price efficiency (Corollary 2). Although vanishing noise trading would make prices fully informative in the asset market equilibrium, it is not the case in a general equilibrium since the number of informed and uninformed investors optimally respond to the change in noise trading. In particular, the number of informed investors as well as the number of uninformed investors approach zero. This applies to both my model and the Grossman-Stiglitz framework.

Now consider their thought experiment. What happens if there are some informed investors even as noise trading vanishes? Do prices become fully efficient? Plugging $\sigma_{zj} \to 0$ into the asset market equilibrium of Proposition 1 shows that liquidity evaporates in this limit ($\chi_j \to 0$). Hence, although noise trading vanishes, price efficiency remains strictly below one: $\phi_j \to 1/(2+\tau_j)$. This, however, does not mean that investors would have incentives to become informed. Recall that the value of information depends on not only price efficiency but also liquidity (22). If there is no liquidity, information is worthless, since one cannot trade on their information. Hence, there is no paradox: regardless of fully revealing prices, the number of informed investors must approach zero, as noise vanishes.

[6]See also Muendler (2007) who discusses the possibility fully revealing equilibrium with finitely many traders as well as Ou-Yang and Wu (2017) who show the error in the GS.

Garleanu and Pedersen (2018) propose a different possibility of fully efficient asset market. They show that as search costs decrease, the industry of informed managers become more concentrated and the asset market becomes more efficient. In their model, as the search cost approaches zero, the market becomes fully efficient. They explain this as a result of the economy of scale in the asset-management industry: each informed manager has more informed investors, who can share the cost of information.

The limiting efficient market results from the price-taking assumption of informed managers, which becomes less plausible as the industry becomes more concentrated, as a result of declining search cost. In this model with strategic informed trading, prices do not become fully revealing in the limit as search costs vanish. The declining search cost, and thus declining fee for informed investments do increase the number of informed investors. Although the number of informed investors goes to infinity, liquidity of the informed manager evaporates, and thus together, prices remain not fully efficient in the limit.

Corollary 3. *As both a search cost c a trading cost k decrease to zero, liquidity approaches zero, and price efficiency ϕ_j approaches $1/(2+\tau_j)$ and remains below one half.*

6 Conclusion

Investigating how uninformed investments affect price efficiency, I find that a minor change in the assumption that informed investors trade strategically drastically change the implication. In the competitive benchmark of Grossman and Stiglitz (1980), uninformed investors do not play an active role. Informed trading does not respond to uninformed trading. Hence, price efficiency is generally independent of uninformed investments and decreases as the share of uninformed investments relative to informed investments increases. However, when informed investors trade strategically, uninformed investors play a role of liquidity provider. By making informed investors trade more aggressively, they make prices more informative. Even as the share of uninformed investments relative to informed investments increases, prices become more informative. The result is consistent with the finding that prices have become more informative over the past couple decades.

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Appendix: Proofs

Proof of Propositions 1. Recall the optimal demand schedule of the manager (4), into which the conditional expectation and variance given the signal ($\mathrm{E}\left[v_j \mid s_j\right] = \frac{\tau_j}{1+\tau_j} s_j$; $\mathrm{var}\left(v_j \mid s_j\right) = \frac{\sigma_{v_j}^2}{1+\tau_j}$) can be substituted into to obtain

$$q_j^I\left(p_j\right) = \chi_j \left(\frac{\tau_j s_j - \left(1+\tau_j\right) p_j}{\gamma \sigma_{v_j}^2}\right). \tag{A1}$$

For an uninformed manager's demand schedule, start with a linear price conjecture:

$$p_j = \theta_{s_j}\left(s_j - \theta_{z_j} z_j\right), \tag{A2}$$

from which $\frac{p_j}{\theta_{s_j}}$ is extracted with the error variance:

$$\mathrm{var}\left[\epsilon_j - \theta_{z_j} z_j\right] = \sigma_{v_j}^2 \tau_j^{-1} + \theta_{z_j}^2 \sigma_{z_j}^2 = \sigma_{v_j}^2 \tau_j^{-1}\left(1 + \frac{\theta_{z_j}^2 \sigma_{z_j}^2}{\sigma_{v_j}^2 \tau_j^{-1}}\right). \tag{A3}$$

Since ϕ_j, defined by (5), is the fraction of precision revealed by prices relative to the total precision,

$$\phi_j = \left(1 + \frac{\theta_{z_j}^2 \sigma_{z_j}^2}{\sigma_{v_j}^2 \tau_j^{-1}}\right)^{-1}, \tag{A4}$$

which, together with Bayes' law and normality, implies $\mathrm{E}\left[v_j \mid p_j\right] = \frac{\phi_j \tau_j}{1+\phi_j \tau_j} \frac{p_j}{\theta_{s_j}}$ and $\mathrm{var}\left[v_j \mid p_j\right] = \frac{\sigma_{v_j}^2}{1+\phi_j \tau_j}$. Then the uninformed manager's optimal demand schedule is

$$q_j^U\left(p_j\right) = \frac{\mathrm{E}\left[v_j \mid p_j\right] - p_j}{\gamma \mathrm{var}\left[v_j \mid p_j\right]} = -\frac{\left(1 + \left(1 - \theta_{s_j}^{-1}\right)\phi_j \tau_j\right)}{\gamma \sigma_{v_j}^2} p_j. \tag{A5}$$

Clearing the market with (A1) and (A5) (i.e. $I_j q_j^I(p_j) + U_j q_j^U(p_j) = z_j$), the price is

$$p_j = \frac{1}{\left(1 + \tau_j^{-1}\left(1 + \frac{U_j}{\chi_j I_j}\left(1 + \left(1 - \theta_{s_j}^{-1}\right)\phi_j \tau_j\right)\right)\right)}\left(s_j - \frac{\gamma \sigma_{v_j}^2}{\chi_j I_j \tau_j} z_j\right). \tag{A6}$$

Now, we make sure the market clearing price is consistent with the conjectures. First, solve the coefficient on s_j to obtain

$$\theta_{s_j} = \left(1 + \frac{\chi_j I_j + U_j}{\tau_j\left(\chi_j I_j + \phi_j U_j\right)}\right)^{-1}. \tag{A7}$$

Second, substituting $\theta_{z_j} = \frac{\gamma \sigma_{v_j}^2}{\chi_j I_j \tau_j}$ into (A4) implies

$$\phi_j = \left(1 + \frac{\left(\gamma \sigma_{v_j} \sigma_{z_j}\right)^2}{\tau_j\left(\chi_j I_j\right)^2}\right)^{-1}. \tag{A8}$$

Lastly, to find χ_j, rewrite market clearing condition as

$$p_j = \left(\frac{\gamma \sigma_{v_j}^2}{U_j\left(1 + \left(1 - \theta_{s_j}^{-1}\right)\phi_j \tau_j\right)}\right)\left(q_j^I - z_j\right) = \left(\frac{\gamma \sigma_{v_j}^2\left(\chi_j I_j + \phi_j U_j\right)}{(1-\phi_j)\chi_j I_j U_j}\right)\left(q_j^I - z_j\right), \tag{A9}$$

to obtain λ, which can be substituted into the definition of χ_j (3) to solve for ϕ_j as

$$\phi_j = \frac{\left(1-\chi_j\right)U_j - \left(1+\tau_j\right)\chi_j I_j}{\left(2 - \chi_j + \tau_j\right)U_j}. \tag{A10}$$

Combining this with (A8) yields the cubic equation of χ_j:

$$\left(\gamma \sigma_{v_j} \sigma_{z_j}\right)^2\left(U_j - \left(U_j + \left(1+\tau_j\right)I_j\right)\chi_j\right) = \tau_j\left(1+\tau_j\right)\left(U_j + \chi_j I_j\right)\left(\chi_j I_j\right)^2. \tag{A11}$$

Note that for $\chi_j \in [0,1]$, the LHS is monotonically decreasing from $\left(\gamma \sigma_{v_j} \sigma_{z_j}\right)^2 U_j$ to a negative constant, while the RHS is monotonically increasing from 0 to $\tau_j\left(1+\tau_j\right)\left(U_j + I_j\right)I_j^2$. Hence, there exists a unique $\chi_j \in (0,1)$ that solves the cubic equation (A11). Also, notice that $\chi_j > 0$ ensures that the second order condition for the informed manager holds. □

Proof of Lemma 1. Use the well-known property of the chi-squared distribution: For a standardized normal random variable $z \sim \mathrm{N}(0,1)$, provided that $1 + 2\gamma K > 0$,

$$\mathrm{E}\left[\exp\left(-\gamma\left(Kz^2 + Lz + M\right)\right)\right] = \frac{1}{\sqrt{1+2\gamma K}}\exp\left(-\gamma\left(-\frac{\gamma L^2}{2\left(1+2\gamma K\right)} + M\right)\right). \tag{A12}$$

First, start with uninformed managers. Define $\epsilon_j^{'} := \frac{v_j - \mathrm{E}[v_j|p_j]}{\sqrt{\mathrm{var}[v_j|p_j]}}$. Using $\epsilon_j^{'}$ and applying

(A12) imply

$$\mathrm{E}\left[\exp\left(-\gamma\left(v_j-p_j\right)q_j^U\right)\mid p_j\right]=\exp\left(-\gamma\left(\left(\mathrm{E}\left[v_j\mid p_j\right]-p_j\right)q_j^U-\frac{\gamma\mathrm{var}\left[v_j\mid p_j\right]}{2}q_j^{U2}\right)\right). \tag{A13}$$

Substituting the uninformed manager's demand schedule into above and again applying (A12) produce

$$\mathrm{E}\left[\exp\left(-\gamma\frac{\left(\mathrm{E}\left[v_j\mid p_j\right]-p_j\right)^2}{2\gamma\mathrm{var}\left[v_j\mid p_j\right]}\right)\right]=\sqrt{\frac{\mathrm{var}\left[v_j\mid p_j\right]}{\mathrm{var}\left[v_j\mid p_j\right]+\mathrm{var}\left[\mathrm{E}\left[v_j-p_j\mid p_j\right]\right]}}=\left(\frac{\mathrm{var}\left[v_j-p_j\right]}{\mathrm{var}\left[v_j\mid p_j\right]}\right)^{-\frac{1}{2}}, \tag{A14}$$

where the second equality follows from the law of total variance.

Next, consider the informed manager. With a slight abuse of notation, redefine $\epsilon_j' := \frac{v_j-\mathrm{E}[v_j|s_j]}{\sqrt{\mathrm{var}[v_j|s_j]}}$. Similarly to before, use the informed manager's demand schedule and apply (A12) to obtain

$$\mathrm{E}\left[\exp\left(-\gamma\left(v_j-p_j\right)q_j^I/I_j\right)\mid s_j,p_j\right]=\exp\left(-\gamma\frac{\chi_j\left(\mathrm{E}\left[v_j\mid s_j\right]-p_j\right)^2}{2\gamma\mathrm{var}\left[v_j\mid s_j\right]}\right). \tag{A15}$$

To get the conditional expectation given p_j, define α as a regression coefficient of $\mathrm{E}\left[v_j-p_j\mid s_j\right]$ onto p_j (i.e. $\mathrm{E}\left[v_j-p_j\mid s_j\right]=\alpha p_j+\beta\epsilon_j''$) so that $\epsilon_j''\sim\mathrm{N}\left(0,1\right)$ is independent of s_j and p_j. Using α and β,

$$\mathrm{E}\left[\exp\left(-\gamma\left(v_j-p_j\right)q_j^I/I_j\right)\mid s_j,p_j\right]=\exp\left(-\gamma\frac{\chi_j}{2\gamma\mathrm{var}\left[v_j\mid s_j\right]}\left(\alpha^2p_j^2+2\alpha\beta p_j\epsilon_j''+\beta^2\left(\epsilon_j''\right)^2\right)\right). \tag{A16}$$

Applying (A12) to above implies

$$\mathrm{E}\left[\exp\left(-\gamma\left(v_j-p_j\right)q_j^I/I_j\right)\mid p_j\right]=\frac{1}{\sqrt{1+\frac{\beta^2\chi_j}{\mathrm{var}[v_j|s_j]}}}\exp\left(\frac{\chi_j\alpha^2p_j^2}{2\gamma\left(\mathrm{var}\left[v_j\mid s_j\right]+\chi_j\beta^2\right)}\right). \tag{A17}$$

With $p_j\sim N(0,\mathrm{var}[p_j])$, again applying (A12) to above gives

$$\mathrm{E}\left[\exp\left(-\gamma\left(v_j-p_j\right)q_j^I/I_j\right)\right]=\left(1+\chi_j\frac{\left(\beta^2+\alpha^2\mathrm{var}[p_j]\right)}{\mathrm{var}\left[v_j\mid s_j\right]}\right)^{-\frac{1}{2}}=\left(1+\chi_j\frac{\mathrm{var}\left[\mathrm{E}\left[v_j-p_j\mid s_j\right]\right]}{\mathrm{var}\left[v_j\mid s_j\right]}\right)^{-\frac{1}{2}}. \tag{A18}$$

From the law of total variance, the RHS can be further simplified to

$$\left(1-\chi_j+\chi_j\frac{\mathrm{var}\left[v_j-p_j\right]}{\mathrm{var}\left[v_j\mid s_j\right]}\right)^{-\frac{1}{2}}=\left(1-\chi_j+\chi_j\left(\frac{1+\tau_j}{1+\phi_j\tau_j}\right)\left(\frac{\mathrm{var}\left[v_j-p_j\right]}{\mathrm{var}\left[v_j\mid p_j\right]}\right)\right)^{-\frac{1}{2}}. \tag{A19}$$

Finally, to determine $\text{var}\left[v_j - p_j\right]$, use the equilibrium price (6) to express

$$v_j - p_j = \frac{(\chi_j I_j + U_j)v_j - (\chi_j I_j \tau_j + U_j \phi_j \tau_j)\left(\epsilon_j - \frac{\gamma \sigma_{v_j}^2}{\tau_j \chi_j I_j} z_j\right)}{\chi_j I_j \left(1 + \tau_j\right) + U_j \left(1 + \phi_j \tau_j\right)} \tag{A20}$$

Hence,

$$\text{var}\left[v_j - p_j\right] = \frac{\left(\chi_j I_j + U_j\right)^2 + \left(\chi_j I_j + U_j \phi_j\right)^2 \frac{\tau_j}{\phi_j}}{\left(\chi_j I_j \left(1 + \tau_j\right) + U_j \left(1 + \phi_j \tau_j\right)\right)^2} \sigma_{v_j}^2, \tag{A21}$$

and combining this with $\text{var}[v_j \mid p_j] = \frac{\sigma_{v_j}^2}{1+\phi_j \tau_j}$ and rearranging give the ratio:

$$\frac{\text{var}\left[v_j - p_j\right]}{\text{var}\left[v_j \mid p_j\right]} = 1 + \frac{\tau_j \left(1 - \phi_j\right)^2}{\phi_j \left(1 + \tau_j + \left(1 + \phi_j \tau_j\right) U_j \left(\chi_j I_j\right)^{-1}\right)^2}. \tag{A22}$$

$\square$

Proof of Propositions 2. First, for given ϕ_j and χ_j in Proposition 1, find the masses of informed investors I_j and U_j. From the equilibrium price efficiency (8),

$$I_j = \frac{\gamma \sigma_{v_j} \sigma_{z_j}}{\chi_j} \sqrt{\frac{\phi_j}{\tau_j \left(1 - \phi_j\right)}}. \tag{A23}$$

From the equilibrium liquidity (7),

$$U_j = \frac{\left(1 + \tau_j\right)\left(\chi_j I_j\right)\left(\left(\gamma \sigma_{v_j} \sigma_{z_j}\right)^2 + \tau_j \left(\chi_j I_j\right)^2\right)}{\left(\gamma \sigma_{v_j} \sigma_{z_j}\right)^2 \left(1 - \chi_j\right) - \tau_j \left(1 + \tau_j\right)\left(\chi_j I_j\right)^2} = \frac{\left(1 + \tau_j\right)\left(\chi_j I_j\right)}{\left(1 - \chi_j\right)\left(1 - \phi_j\right) - \left(1 + \tau_j\right)\phi_j}, \tag{A24}$$

where the second equality follows from (A23).

In an interior equilibrium, the certainty equivalents u_j^I and u_j^U satisfy the indifference condition (16). From Lemma 1,

$$e^{2\gamma(k_j + 2c_j^I)} = 1 - \chi_j + e^{2\gamma(k_j + 2c_j^U)} \left(\frac{1 + \tau_j}{1 + \phi_j \tau_j}\right) \chi_j, \tag{A25}$$

which can be solved for χ_j as

$$\chi_j = \frac{\left(1 + \phi_j \tau_j\right)\left(e^{2\gamma(k_j + 2c_j^I)} - 1\right)}{(1 + \tau_j)e^{2\gamma(k_j + 2c_j^U)} - \left(1 + \phi_j \tau_j\right)}. \tag{A26}$$

Finally, we need to solve for ϕ_j. Again from Lemma 1,

$$e^{2\gamma(k_j+2c_j^U)} = 1 + \frac{\tau_j \left(1-\phi_j\right)^2}{\phi_j \left(1+\tau_j+\left(1+\phi_j\tau_j\right) U_j \left(\chi_j I_j\right)^{-1}\right)^2}, \tag{A27}$$

which can be simplified into

$$\sqrt{\phi_j \tau_j^{-1}\left(e^{2\gamma(k_j+2c_j^U)}-1\right)} = \frac{1-\phi_j}{1+\tau_j+U_j(\chi_j I_j)^{-1}(1+\phi_j\tau_j)} \tag{A28}$$

Substituting (A24) into the R.H.S and rearranging yield

$$\frac{1+\phi_j\tau_j}{2-\chi_j} = 1-\phi_j-(1+\tau_j)\sqrt{\phi_j\tau_j^{-1}\left(e^{2\gamma(k_j+2c_j^U)}-1\right)}, \tag{A29}$$

and finally substituting (A26) into above yields

$$1 = \left(\frac{2}{1+\phi_j\tau_j} - \frac{e^{2\gamma(k_j+2c_j^I)}-1}{(1+\tau_j)e^{2\gamma(k_j+2c_j^U)}-(1+\phi_j\tau_j)}\right)\left(1-\phi_j-(1+\tau_j)\left(\tau_j^{-1}\left(e^{2\gamma(k_j+2c_j^U)}-1\right)\phi_j\right)^{1/2}\right). \tag{A30}$$

To show the existence and uniqueness, note that in (A29), the RHS is decreasing in ϕ_j, while the LHS is increasing in ϕ_j since χ_j increases in ϕ_j from (A26). Hence, the solution is unique if it exists. Further, for $\phi_j \to 0$, the LHS approaches $\frac{1}{2-\chi_j} < 1$, while the RHS approaches 1. For $\phi_j \to 1$, the RHS goes negative. Therefore, there is a unique solution ϕ_j that is between zero and one. □

Proof of Corollary 1. To determine $\frac{d\chi_j}{dU}$, rewrite (7) as

$$\left(\gamma\sigma_{v_j}\sigma_{z_j}\right)^2\left(1-\chi_j\right)U_j - \tau_j\left(1+\tau_j\right)I_j^2\chi_j^2U_j = \left(\gamma\sigma_{v_j}\sigma_{z_j}\right)^2\left(1+\tau_j\right)I_j\chi_j + \tau_j\left(1+\tau_j\right)I_j^3\chi_j^3, \tag{A31}$$

where the left hand side is decreasing in χ_j and the right hand side is increasing in χ_j. Since increasing U_j would increase the left hand side while the right hand side is unaffected, χ_j increases (i.e. $d\chi_j/dU > 0$). □

Proof of Proposition **??**. From (8) in Proposition 1, we have

$$\frac{d\phi_j}{dU} = \frac{\partial\phi_j}{\partial\chi_j}\frac{d\chi_j}{dU}, \tag{A32}$$

where $\frac{\partial\phi_j}{\partial\chi_j} > 0$. Since χ_j increases in U_j from Corollary 1, ϕ_j increases in U_j.

If informed investors are assumed to take prices as given with $\chi_j = 1$, ϕ_j (given by (**??**)) is independent of U_j (i.e. $d\phi_j/dU = 0$). If we additionally assume $I_j + U_j$ is a given constant, ϕ_j decreases in U_j, since ϕ_j increases in I_j and I_j decreases in U_j. $\square$

Proof of Proposition **??**. Begin with (i) the comparative statics with respect to k. Recall (A29):

$$\frac{1+\phi_j\tau_j}{2-\chi_j} = 1 - \phi_j - (1+\tau_j)\sqrt{\phi_j\tau_j^{-1}\left(e^{2\gamma k}-1\right)}, \tag{A33}$$

where the LHS is increasing in ϕ_j and the RHS is decreasing in ϕ_j. Again recall (A26):

$$\chi_j = \frac{e^{4\gamma c} - e^{-2\gamma k}}{\frac{1+\tau_j}{1+\phi_j\tau_j} - e^{-2\gamma k}} = 1 - \frac{\frac{1+\tau_j}{1+\phi_j\tau_j} - e^{4\gamma c}}{\frac{1+\tau_j}{1+\phi_j\tau_j} - e^{-2\gamma k}}, \tag{A34}$$

which increases in k since χ_j is less than one. Hence, keeping ϕ_j fixed, the LHS of (A33) increases in k and the RHS decreases in k. Together, as k increases, the LHS and the RHS meet at ϕ_j that is lower, and ϕ_j decreases in k (i.e. $d\phi_j/dk < 0$).

For (ii) the comparative statics with respect to c, note that the RHS in (A33) does not directly depend on c, while the LHS increases in c as so does χ_j. Hence, increasing c will decrease the equilibrium ϕ_j, at which the LHS and the RHS meet (i.e. $d\phi_j/dc < 0$). $\square$

Proof of Corollary 2. In (21) in Proposition 2, noise trading σ_{zj} has no effect on ϕ_j. More generally, the product $\gamma\sigma_v\sigma_{zj}$ does not affect ϕ_j, except for γ being used to transform the dollars such as k and c into utility. $\square$

Proof of Corollary 3. Take the limit as both c and k approach zero in (20) in Proposition 2 to obtain that χ_j approaches zero. Again taking the same limit in (21) yields that ϕ_j approaches $1/(2+\tau_j)$. $\square$