Self-inflicted debt crises*

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February 26, 2021

*We thank Peter DeMarzo, Matthew Elliott (discussant), Michel Habib, Ali Lazrak, Dmitry Livdan, Simon Mayer (discussant), Erwan Morellec, Steven Ongena, Jean-Charles Rochet, Christoph Schiller (discussant), Harald Uhlig, seminar participants at the New York Fed, Università Cattolica del Sacro Cuore, ESSEC, Zurich, and participants at CRETE 2018, Cambridge Lausanne Conference 2018, Swiss Finance Institute Research Days 2019, CEPR ESSFM 2019, IFFR 2019, Paris-December Eurofidai Conference 2019, Word Finance Conference 2020, and the EFA 2020 for valuable comments. The authors gratefully acknowledge research support from the Swiss Finance Institute.

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Abstract

When present-biased borrowers have the option to default on their liabilities, we establish that they undervalue their default option by a U-shaped error. The biased beliefs gives rise to pseudo-wealth that explains why borrowers take excessive leverage, imperfectly smooth consumption, underinvest in good times, and risk shift in distress. Pseudo-wealth is an important consideration for the resolution of debt crises. When borrowers can extract concessions from lenders, the size and frequency of rescue packages depend on whether larger transfers exacerbate or alleviate the borrowers' belief bias. This result highlights that optimal crisis resolution requires to account for borrowers' time-inconsistency. We provide conditions for when borrowers procrastinate default and debt crises are protracted, and we show that distress can be cheaper to resolve when borrowers are more myopic. In turn, rational principals can benefit from myopic agents.

JEL: H63, G01, G4, D86

Keywords: borrower myopia, time-inconsistency, strategic default, debt crisis, bailout fund, real options

Debt crises are a recurrent phenomenon that affect a variety of borrowers, including sovereigns, banks, corporations, and consumers with devastating economic and social consequences. The Covid-19 pandemic is no exception even if its source is different from past crises. The G20 countries have provided over $10tn in short-term support to the public sector, businesses, and individuals since March 2020 (see Figure 1 for G-20 debt-to-GDP).[1] Much of this new debt are bailouts to avoid outright defaults and spillovers to other borrowers. In the last major debt crisis, ECB and IMF spent over €400bn on debt relief and set up new bailout agencies, including European Stability Mechanism (ESM) and Single Resolution Board (SRB).

When borrowers are fully rational the existing literature prescribes how bailout agencies can resolve debt crises (Mella-Barral and Perraudin, 1997; Mella-Barral, 1999; Bolton, 2016; Bolton and Oehmke, 2018; Lambrecht and Tse, 2019). However, the restructuring of debt and resolution of debt crises cannot holistically be addressed without considering borrower myopia. Myopia is an empirical regularity on intertemporal choice (Ericson and Laibson, 2019) and manifests itself in borrowing either as a behavioral trait of individuals (Laibson, 1997; DellaVigna and Malmendier, 2004) or as the result of a political system that sacrifices the future for short-term benefits (Aguiar and Amador, 2011). It is known that myopic behavior induces undersaving, overinvestment, and procrastination (Phelps and Pollak, 1968; Alesina and Tabellini, 1990; Grenadier and Wang, 2007; Laibson, Maxted, and Moll, 2020). Yet, an analysis is missing of how time-inconsistent borrower behavior affects the resolution of debt crises, including optimal bailout schemes and size of bailout funds. Any bailout agency, may it be the IMF, World Bank, Federal Reserve, ECB, ESM, or SRB, faces a policy tradeoff when faced with a distressed borrower: Generous financial transfers provide a quick fix to a looming debt crisis, but they are expensive and encourage repeated pleads for debt relief. How myopia alters this tradeoff is our focus.

This paper develops a continuous-time equilibrium model of debt crises that a myopic borrower and rational lenders, represented by a bailout agency, self-inflict through their strategic interaction. We develop this intuition in a dynamic Stackelberg game of repeated defaults and financial transfers by an agency that minimizes intertemporal bailout costs with non-commitment. The optimal bailout scheme takes the form of an ex-post incentive-compatible financial transfer contingent on the borrower's myopia and wealth.

[1] IMF staff, 2020, "G-20 report on strong, sustainable, balanced, and inclusive growth," retrieved at https://www.imf.org/external/np/g20/pdf/2020/110220.pdf.

The model provides a novel mechanism for how bailouts strengthen or weaken borrowers' incentives to strategically default and, ultimately, deepen debt crises and slow down recovery. We highlight two debt crisis dynamics: At one extreme, a debt spiral occurs in which expected wealth growth is procyclical and negative for low wealth, generating a point of no return beyond which default becomes inevitable. At another extreme, a trap occurs in which expected wealth growth is countercyclical and positive for low wealth, generating economic stagnation for long periods of time. The equilibrium dynamics depend on whether time-inconsistency alters the policy tradeoff in ways that amplify or ameliorate debt crises.

At the core of our mechanism is that myopic borrowers endogenously misprice their option to default. The pricing error has a U-shape and is negative at all times which makes a myopic borrower act as if it was richer than it actually is. The default mispricing distorts the borrower's timing of default and alters the optimal bailout scheme. Contrary to what one might expect, for a fixed bailout the borrower's wealth threshold that triggers default is invariant to myopia. In turn, given the equilibrium default threshold, there exists a target wealth level post-bailout, so that the curvature of the optimal bailout schedule is independent of myopia. Myopia affects default through the bailout agency's endogenous response, given its lack of commitment, to the borrower's myopia by shifting the intercept of the bailout schedule and, respectively, the equilibrium response of the myopic borrower to the bailout offered. In this sense, lenders and borrowers self-inflict debt crises through their strategic interaction.

The strategic interactions between borrower and lenders unfold with bilateral non-commitment.[2] Neither the borrower can commit on its post-bailout decisions, nor the agency can commit on its future bailout policy. This assumption is motivated by the observation that austerity measures are often hard to enforce and empirically the threat of spillovers results in prolonged episodes of debt relief.[3] We allow for two types of default. In a hard default, the borrower loses access to financial markets and enters autarky. Hard default yields immediate benefits for troubled borrowers at the expense of loss of access to borrowing and

[2]It is well known that commitment resolves distortions caused by time-inconsistent preferences (Ericson and Laibson, 2019).

[3]The European debt crisis is a prime example. Article 125 of the Lisbon Treaty states that the European Union (EU) *shall not be liable for the commitments of [...] any Member State. A Member State shall not be liable for the commitments of [...] another Member State.* From this rule it looks clear that all countries are responsible for their own debt, and no country has to rescue another. Still, the disclaimer has proven vacuous. The IMF, ECB, and Eurogroup granted bailouts to Ireland in 2010, Greece in 2010, 2012 and 2015, and other Eurozone countries between 2011 and 2016.

future bailouts.[4] In a soft default, the bailout agency provides debt relief through financial transfers, which guarantees the debt. Debt renegotiation frictions and the threat of a hard default keep the borrower from requesting financial aid excessively. The bailout agency designs an incentive-compatible financial transfer schedule in a stationary Markovian subgame perfect Nash equilibrium to minimize intertemporal debt crises costs. We focus on the dynamic Stackelberg game in which the borrower moves first by requesting financial aid and the bailout agency responds by a take-it-or-leave-it offer that is contingent on the borrower's wealth in default. The agency makes bailout offers, keeping in mind the potential for default recidivism. The expected time to next default depends on the borrower's time-inconsistent consumption, investment and borrowing decisions post-bailout.

The borrower exhibits short-term bias which emerges as the result of an intrapersonal game between "selves" of the borrower (Luttmer and Mariotti, 2003) that arrive over time at a fixed Poisson rate. The borrower discounts the consumption of future selves disproportionately compared to the current self. Decisions are taken with instantaneous gratification preferences (Harris and Laibson, 2012), which is the limit of quasi-hyperbolic preferences when tenure of selves becomes infinitesimally short. In this setup, a single parameter captures the borrower's degree of myopia, which facilitates analytical tractability while nesting time-consistency as a special case.

Myopia distorts the borrower's intertemporal consumption, saving, borrowing, and default decisions. In contrast to rational borrowers, myopic borrowers imperfectly smooth consumption by overconsuming more in good times than in bad times. Myopia thereby creates a retractive force that, like a rubber band, pulls time-inconsistent borrowers back into distress and inflicts serial default (Reinhart and Rogoff, 2009). This rubber-band effect is akin to distance-to-default-dependent discounting and only operates when *both* myopia and default options are present. Rational impatience, in contrast, exerts a constant retractive force to consumption growth independent of distance to default.

These distortions arise from how myopic borrowers value the option to default. Myopic borrowers overvalue the consumption claim compared to rational borrowers and undervalue the option to default. The pricing error is U-shaped in the distance to default, as it vanishes when wealth reaches the default

[4] Reinhart and Rogoff (2009) document that sovereign defaults are lengthy and cluster in time. We capture this fact by assuming that the bailout agency incurs spillover costs in a hard default.

threshold or diverges to infinity and stays negative in between. Simply put, myopic borrowers downplay during normal times the impact of default and behave as if they were richer than they actually are.

The borrower's mispricing of default plays an important role in debt renegotiation. Financial transfers are less effective at staving off default for myopic than rational borrowers, due to the retractive rubber-band force. In turn, future expected costs of default rise with myopia. The bailout agency balances, at the optimum, the marginal reduction in the risk-neutral default probability with the total cost of future defaults. Optimal transfers increase with myopia when non-myopia factors make future default more likely or costlier to the agency. Equilibrium transfers increase with myopia if spillovers are high, renegotiation frictions are high, risk aversion is high, growth opportunities are poor, and borrower default costs are low. Larger bailouts tend to resolve crises quickly and put the borrower back on a growth trajectory, but incentivize earlier default and worsen overspending. Smaller transfers tend to delay default and lead to frequent debt relief and prolong crisis, but discipline overspending.

The model yields closed-form expressions for the size of the required bailout fund and for credit spreads. The bailout fund depends on the cost of current bailouts, the frequency of future bailouts and the likelihood of negotiation failure, all of which depend on borrower myopia. A decomposition of bailout costs shows that myopic distress can be cheaper to resolve than rational distress. This occurs typically when myopia gets punished in equilibrium through a sequence of small transfers. It is not even clear that myopia always harms all borrowers. Rational agents (e.g., equityholders or citizens) can benefit from a borrower's myopia. This occurs when myopia gets rewarded and debt relief occurs frequently.

[Figure 1 About Here.]

Capital markets price risky bonds and credit-linked securities, such as credit default swaps, different when borrowers are myopic. Default-contingent claims have a mixed power-hypergeometric shape in our model, which nests the standard power shape of Merton (1974) and Black and Cox (1976) in the limiting case of borrower rationality. As a result, credit spread dynamics are more asymmetric than predicted by models with rational borrowers. Credit spreads rise more slowly than rational spreads, but then spike once the crisis starts to unfold. This behavior is in line with credit spread dynamics observed in practice. Figure 1 documents the credit markets during the European debt crisis when Greek debt-to-GDP rose

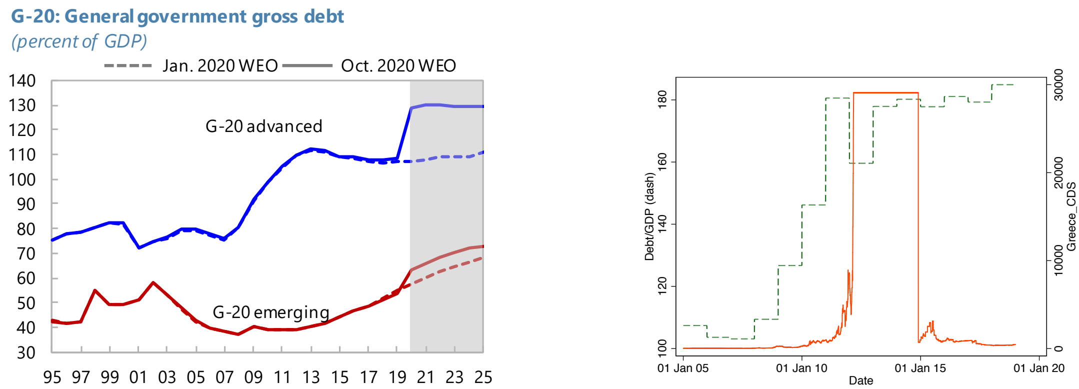

Figure 1: **Bailouts during Covid-19 and debt-to-GDP and CDS spreads during European debt crisis.** The figure documents IMF public-sector debt projections following the Covid-19 pandemic (left; source: IMF staff (2020)) and debt-to-GDP and CDS spread for Greece during the European debt crisis (right).

sharply before CDS spreads started to spike. In time series, there are two countervailing forces. Myopic borrowers travel the distance to default faster as they consume more. The distance to default is, however, larger when myopic borrowers default later. As a result, standard credit metrics based on the power shape are inadequate proxies for default risk under myopia. In the cross section, credit prices are *ceteris paribus* higher for more myopic borrowers since their recovery is slower.

Related Literature: Our model studies the implications of time-inconsistency for default and debt relief, both cross-sectionally and over time. Leland (1994) offers an early analysis on debt renegotiation. Strategic debt service and debt renegotiation have since been extensively studied in corporation finance (Anderson and Sundaresan, 1996; Mella-Barral and Perraudin, 1997; Mella-Barral, 1999; Fan and Sundaresan, 2000; Lambrecht and Tse, 2019). Bolton (2016) applies the corporate finance setting to sovereign debt based on the analogy between the fiat liability of sovereigns and corporate equity. We extend the debt renegotiation literature is several ways. Most importantly, we consider time-inconsistent borrowers.

Grenadier and Wang (2007) study the impact of myopia on investment under equity financing in a real options setting. Our goal is different. We solve for the joint intertemporal consumption, investment, and borrowing of a myopic borrower and, in particular, we consider the strategic interaction with a time-consistent bailout agency. In this regard, our modeling approach is similar to DellaVigna and Malmendier (2004) who assume that individuals are time-inconsistent while the firm with which they contract is time-consistent. While Grenadier and Wang (2007) show that call option exercise always get accelerated by time-inconsistency, which is a form of overinvestment, in our setting default option exercise gets accelerated or delayed depending on economic conditions.

Our paper also relates to the political economy literature that underlines the inclination of governmental borrowers to show preference for consumption while in office (Persson and Svensson, 1989; Alesina and Tabellini, 1990). Based on this premise, Aguiar and Amador (2011) map the inefficiency of the political system into a model of consumption with hyperbolic discounting (Laibson, 1997). Our model focuses on the design of debt relief in the presence of default spillovers. Acharya and Rajan (2013) model a myopic government that maximizes single-period public spending. They show that myopia can induce a government to default later, since it disregards the effects of borrowing on future taxes and output. In our model, myopia affects the default decision through its impact on bailout policy. When default spillovers are low (high), myopia leads to late (early) default.

Default in our model happens on the equilibrium path. In this manner, our paper is related to models of non-contingent sovereign debt. In the seminal contribution by Eaton and Gersovitz (1981) external debt financing can be sustained using exclusion from international debt markets as a deterrent for repudiation. Later contributions by Aguiar and Gopinath (2006) and Arellano (2008) indicate that the costs of financial exclusion are relatively small and insufficient to justify the levels of debt typically observed in the market. Higher levels of debt can be justified by richer models that control for maturity structure and output costs of default (Benjamin and Wright, 2009; Hatchondo and Martinez, 2009; Chatterjee and Eyigungor, 2012).[5]

The remainder of the paper is organized as follows. Section 1 sets up the model and derives the optimal investment and financing policy outside of default. Section 2 illustrates the endogenous mispricing of default. Section 3 explores the optimal bailout policy. Section 4 conducts policy analysis. Section 5 explores the implication of default options for growth. Section 6 explores the suitability of credit metrics under myopia. Section 7 considers the case of risky debt and Section 8 concludes. All proofs are relegated to an Appendix. The Internet Appendix discusses model extensions.

[5] Reinhart and Rogoff (2009) record the history of sovereign default since the 14th century and show that most countries have defaulted at least once. They further show that overborrowing is pronounced in good times; serial default on external debt is a phenomenon common to all regions of the world; and that new countries go through repeated debt restructuring before they "graduate" to an advanced rare-default status. Our model captures these empirical facts.

1 Model

We develop a dynamic model of debt crises featuring a myopic borrower and rational lenders that are represented by a bailout agency. We depart from the existing literature by assuming that the borrower seeks instantaneous gratification (Harris and Laibson, 2012). We first characterize the distortions in consumption, investment, and borrowing caused by myopia, before we lay out a dynamic Stackelberg game for debt renegotiation and characterize the Markovian subgame-perfect equilibrium.

1.1 Assumptions

Time is continuous and the horizon infinite. Financial markets for securities are complete and frictionless, except for debt renegotiation frictions. A borrower with net wealth W_t at time t decides on consumption, investment, borrowing, and default. Consumption is c_t with consumption share $\psi_t = \frac{c_t}{W_t}$. There exist riskless and risky savings technologies. To focus on the distortions arising from default and debt renegotiation frictions, we assume investment is frictionless. The investment share θ_t equals the fraction of wealth invested in the risky savings technology that follows a geometric Brownian motion with drift rate μ and volatility σ.[6] The risk-free rate is r and the price of risk is $\nu = \frac{\mu - r}{\sigma}$.

Funding gaps are financed by short-term debt that matures at $t+dt$ while funding surpluses are invested in a riskless bond. Under these assumptions, debt is riskless outside of default since it is short term and the path of wealth is continuous. In default, debt remains riskless since it is backed by a bailout fund. We assume the bailout agency (e.g., IMF, World Bank, ESM, SRB, Federal Reserve) has access to funding that suffices to reimburse creditors in full. Section 7 relaxes this assumption and allows for risky debt.

Wealth dynamics at any time t before default are given by

$$dW_t = [W_t(r + \theta_t(\mu - r)) - c_t]dt + \sigma W_t \theta_t dw_t. \tag{1}$$

In the remainder, we drop for notational convenience the dependence on t so long as there is no confusion.

[6]Throughout we consider interior optimal solutions for consumption and investment, explicitly ruling out doubling strategies and Ponzi schemes. See Merton (1969) for an analogous assumption in an optimal consumption-portfolio allocation problem.

In choosing its policies, the borrower faces self-control problems as it seeks instantaneous gratification (Harris and Laibson, 2012). Specifically, the borrower has CRRA preferences over consumption c with relative risk aversion γ, $u(c) = \frac{1}{1-\gamma}c^{1-\gamma}$. Utility streams are discounted at rate ρ until a Poisson-distributed random time τ with intensity λ at which a new self of the borrower arrives. The current self discounts utility streams after the new self arrives using the discount factor $\delta \times exp(-\rho t)$ where $\delta \in (0,1]$ and $\gamma + \delta > 1$. To ensure convergence of the value function, we impose a minimum bound on the subjective discount rate $\rho > \max\{(1-\gamma)(r+\frac{1}{\gamma}\frac{\nu^2}{2}), (1-\gamma)(\mu-\gamma\frac{\sigma^2}{2})\}$.[7] Instantaneous gratification assumes that future selves arrive instantaneously, that is, $\lambda \to \infty$. In this case, a single parameter δ captures the borrower's degree of myopia and the value functions become more tractable. The borrower is more time-inconsistent when δ is lower, with the two extremes $\delta \approx 0$ (myopic) and $\delta = 1$ (full rationality).

Hard vs. soft default: The borrower has the option to default on its debt and can possibly extract concessions from the lenders. Hard default acts as threat point for debt renegotiations in a soft default. Either of two default regimes, hard default and soft default can occur on the equilibrium path.

In a hard default, the borrower suffers from default costs and the lenders from the debt guarantee and the threat of spillovers. The borrower incurs direct cost of bankruptcy, as its wealth changes from W to $W^a(W) = \omega_0 + \omega_1 W$ with parameters ω_0 and ω_1 that depend on the enforcability of debt. The borrower faces indirect costs as, for simplicity, it stays in autarky forever and can no longer borrow or lend.[8] The agency seizes assets and picks up the net cost $W^a(W) - W$ from its debt guarantee. To capture the stylized facts that default episodes are lengthy and defaults cluster in time, we allow the agency to incur indirect costs in the form of expected spillover costs amounting to $\kappa \geq 0$.[9]

In a soft default, the borrower and the agency negotiate over a bailout transfer, with the agency holding all bargaining power. Hard default is a threat that the borrower can potentially use in these negotiations. To

[7]The restriction $\rho > (1-\gamma)(r + \frac{1}{\gamma}\frac{\nu^2}{2})$ is a standard transversality condition when a riskless savings technology exists (Merton, 1969), and $\rho > (1-\gamma)(\mu - \gamma\frac{\sigma^2}{2})$ is an alternative transversality condition when there is no riskless asset (Harris and Laibson, 2012).

[8]The autarky assumption is not crucial for any of the results, so long as default is costly.

[9]Reinhart and Rogoff (2009) document that during the last two centuries at least five waves of sovereign defaults occurred. The median duration of default episodes was 3–6 years, with a median duration of 6 (3) years pre (post) World War II. The literature has also documented spillover effects in the financial sector for banks, corporations, and consumers.

obtain an interior solution, we assume the debt renegotiation process is not frictionless. The assumption seems reasonable as negotiating relief deals requires in practice the approval from several committees and parties with sometimes conflicting incentives. Absent this assumption, the borrower would call for financial aid any time the promised aid is positive. We incorporate debt renegotiation frictions by assuming that the negotiations between borrower and agency fail with probability $p \in (0, 1)$. Once the borrower requests aid, it receives the financial transfer so long as the negotiations succeed and it continues operating with access to borrowing and lending markets. If negotiations fail, the borrower enters a hard default.

Debt renegotiation We model negotiations in a soft default by means of a dynamic Stackelberg game between the borrower and the agency in which the borrower moves first and the bailout agency responds optimally (Mella-Barral and Perraudin, 1997; Mella-Barral, 1999). The contracting space is incomplete: The bailout agency can incentivize the borrower through repeated financial transfers, but it can neither impose austerity nor enforce a first-best policy. The agency can also not credibly commit with respect to its future bailout policy.[10] The setup is equivalent to the agency posting an ex-post incentive-compatible transfer schedule $T(W)$ for any wealth W, and the borrower optimally responds by picking a soft default threshold $\underline{W}^-$ at which to request a financial transfer corresponding to the schedule $T(\underline{W}^-)$.

In this setting, the Markovian subgame perfect Nash equilibrium in threshold strategies can be summarized by the triplet

$$(\underline{W}^a, \underline{W}^-, \underline{W}^+). \tag{2}$$

When a hard default is optimal, the borrower enters autarky once wealth drops to $W \leq \underline{W}^a$. When a soft default is optimal, the borrower chooses the wealth threshold $\underline{W}^-$ at which point the borrower asks for financial concessions. The agency responds by offering the financial transfer $T(\underline{W}^-)$ that, if negotiations

[10] Non-commitment is a central assumption, as it puts the agency in a weak position to enforce borrower behavior in the lenders' interests. In an extension, we consider alternative settings in which either the agency moves first or both parties lack commitment. In our setting, the borrower moves first by declaring soft default. The agency then decides on the bailout T. The bailout policy is subgame perfect as it minimizes the cost to the guarantor given W. Optimal debt forgiveness, by contrast, requires commitment by the agency. The agency is the 'Stackelberg leader' in this alternative setting. The agency commits to a bailout T and the borrower decides when to plead for aid given T, that is, one solves for $W(T)$. When the agency can commit to the bailout, it can directly incentivize when the borrower defaults. The results are available upon request.

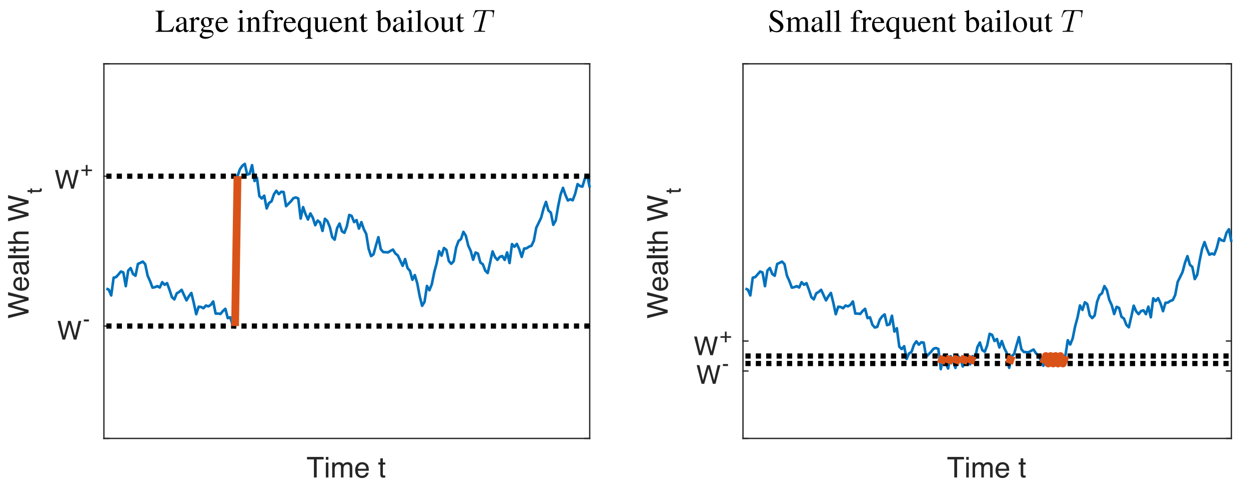

Figure 2: **Model illustration.** The figure plots wealth W_t over time, default threshold $\underline{W}^-$, and target wealth threshold $\underline{W}^+$ after a bailout. The vertical red lines indicate a bailout of size $T = \underline{W}^+ - \underline{W}^-$. The two figures assume different default and target thresholds $(\underline{W}^-, \underline{W}^+)$.

are successful, improves the borrower's financial situation to

$$\underline{W}^+ = \underline{W}^- + T(\underline{W}^-). \tag{3}$$

The bailout agency acts in the interests of the lenders by designing a transfer scheme $T(W)$ to minimize the size of the bailout fund which equals the intertemporal cost of rescuing the borrower, $I(c;T)$, by optimally balancing financial transfers with the frequency at which the borrower requests financial aid.

Figure 2 illustrates alternative bailout policies. In the left plot, infrequent large bailouts occur when $T(\underline{W}^-)$ is large and (in equilibrium) $\underline{W}^-$ is high. In the right plot, frequent small bailouts occur when $T(\underline{W}^-)$ is small and (in equilibrium) $\underline{W}^-$ is low. The latter resembles a tight-leash policy in which the lenders keep the borrower close to the default threshold in order to mitigate the incentive problem and thereby minimize policy distortions, which tends to procrastinate default in equilibrium.

[Figure 2 About Here.]

1.2 Value functions and optimality conditions

The value function $M(W)$ of the myopic borrower prior to its replacement differs from the corresponding value function $R(W)$ after the current borrower (or "self") has been replaced. The concept of equilibrium for the dynamic game played between the different selves of the borrower is Stationary Markov Perfect Equilibrium. Consumption and investment strategies away from the default threshold are contingent on the only payoff relevant variable, wealth W_t. Furthermore, every self maximizes its own value assuming that all future selves use the same policy functions.

Lemma 1 *The value functions $M(W)$ and, respectively, $R(W)$ at any wealth level W above the default threshold, $\underline{W}^-$ or $\underline{W}^a$, satisfy the system of Hamilton-Jacobi-Bellman (HJB) equations*

$$\rho M(W) - \lambda(\delta R(W) - M(W)) = \max_{c,\theta} \ u(c) + M'(W)[W(r + \theta(\mu - r)) - c] + \frac{\sigma^2}{2} W^2 \theta^2 M''(W), \quad (4)$$

$$\rho R(W) = u(c^*) + R'(W)[W(r + \theta^*(\mu - r)) - c^*] + \frac{\sigma^2}{2} W^2 (\theta^*)^2 R''(W), \quad (5)$$

where c^* are the consumption and θ^* the investment share that maximize $M(W)$.

The above formulation assumes that the borrower is sophisticated enough to account for the presence of the bias and correctly anticipates that its policies will continue to be myopic, even after its current self is replaced by a new one. This assumption is commonly termed "sophisticated" myopia (Phelps and Pollak, 1968; Laibson, 1997). The alternative is to assume that the borrower is naïve and not aware of its myopia.

The change in value from replacement by a new self, or "electoral effect," is captured in (4) by

$$E(W) = \lambda(\delta R(W) - M(W)). \quad (6)$$

Instantaneous gratification assumes that future selves arrive instantaneously, i.e., $\lambda \to \infty$. The following lemma describes the limiting relation between the value function of the current self $M(W)$ and its replacement value $R(W)$:

Lemma 2 *Fix a default threshold $\underline{W}$ and let the value of the borrower upon entering hard default be $\Omega(\underline{W}, \lambda)$. Furthermore, assume there exists a finite limit $\Omega(\underline{W})$ such that $\lim_{\lambda \to \infty} \Omega(\underline{W}, \lambda) = \Omega(\underline{W})$. Then $\lim_{\lambda \to \infty} M(W) - \delta R(W) = 0$ and $\lim_{\lambda \to \infty} E(W) + (1 - \delta) u(c^*(W)) = 0$.*

As new selves arrive instantaneously, the lemma shows that the value of the current self becomes a linear function of the replacement value. Therefore, the dynamic game between the different selves can be recast in terms of a single-person decision problem with value function $M(W) = \delta R(W)$. This simplification allows us to dispense with multiple equilibria arising from time-inconsistent consumption-savings problems and focus our analysis on the dynamic tradeoffs in the design of optimal bailout policy. Harris and Laibson (2012) prove a result similar to Lemma 2 when there is no default option, no riskless savings technology, and the utility of consumption is bounded from below. Here, there exists an endogenous lower bound on consumption for $W \geq \underline{W}^-$ (or $\underline{W}^a$) that is caused by the default option.

2 Allocation Distortions and (Mis)pricing of Default

This section shows that a myopic borrower undervalues the default claim compared to a rational borrower. The misconception creates a gap between the private and market valuations of the consumption claim that leads to allocation distortions. We provide the solution to the myopic consumption and investment policy and link it to the (mis)pricing of default in this section. We use the policy functions to characterize optimal bailout policy, equilibrium default thresholds, and bailout fund size in the next section.

2.1 Myopic consumption and investment policy

The first-order conditions for optimal consumption yield

$$u'(c^*) = M'(W) = \delta R'(W), \tag{7}$$

and, respectively, optimal investment satisfies

$$\theta^* = -\frac{R'(W)}{R''(W)W}\frac{\nu}{\sigma}. \tag{8}$$

Condition (7) illustrates the static effect of myopia. Consumption is higher when δ is lower, i.e., when the borrower is more time-inconsistent. The limiting consumption-to-wealth ratio as $W \to \infty$ equals

$$\psi = \frac{\rho - (1-\gamma)(r + \frac{1}{\gamma}\frac{\nu^2}{2})}{\gamma - (1-\delta)}. \tag{9}$$

It rises with myopia due to the term $1-\delta$ in the denominator of (9). Absent default options, there would be an isomorphism between myopia and impatience: For every value of the consumption-to wealth ratio that obtains under myopia, there exists a value of of the impatience parameter ρ that produces exactly the same consumption-to-wealth ratio assuming time-consistency. Once default options are present, this isomorphism ceases to exist because of the state-contingent nature of default.

Condition (8) shows that myopia does not change the nature of the investment policy directly but rather indirectly. Myopia affects investment to the extent that it changes the curvature of the value function.

To provide explicit solutions, we conjecture that the equilibrium consumption policy $c = c^*(W)$ is increasing in wealth and we denote by $Y(c)$ the borrower's subjective wealth given any level of equilibrium consumption, defined as $Y(c^*(W)) = W$. One can show that Y satisfies the following linear ODE (Karatzas et al., 1986):

$$rY(c) = (\gamma - (1-\delta))\frac{1}{\gamma}c + \left(r - \rho - (1-\frac{1}{\gamma})\frac{\nu^2}{2}\right)\frac{1}{\gamma}cY'(c) + \frac{\nu^2}{2}(\frac{1}{\gamma}c)^2 Y''(c), \tag{10}$$

subject to two boundary conditions. At the lower boundary $\underline{c}^- = c^*(\underline{W}^-) = M'(Y(\underline{c}^-))^{-\frac{1}{\gamma}}$, $Y(\underline{c}^-) = \underline{W}^-$, and $\lim_{c\to\infty} c/(Y(c)) = \psi$.

The solution to (10) has the form $Y(c) = \frac{c}{\psi} + Ac^{-\gamma\underline{h}} + Bc^{-\gamma h}$, where $\underline{h}$ and h are the negative and, respectively, positive solution to the characteristic equation $\frac{\nu^2}{2}h^2 - (r-\rho-\frac{\nu^2}{2})h - r = 0$ and independent of δ:

$$\begin{aligned} \underline{h} &= \tfrac{1}{\nu^2}\left(r - \rho - \tfrac{\nu^2}{2} - \sqrt{\left(r-\rho-\tfrac{\nu^2}{2}\right)^2 + 2r\nu^2}\right), \\ h &= \tfrac{1}{\nu^2}\left(r - \rho - \tfrac{\nu^2}{2} + \sqrt{\left(r-\rho-\tfrac{\nu^2}{2}\right)^2 + 2r\nu^2}\right). \end{aligned} \tag{11}$$

We can discard the rapidly growing term $c^{-\gamma\underline{h}}$ in the expression for Y by setting $A = 0$. The value-matching condition yields $B = -(\frac{\underline{c}^-}{\psi} - \underline{W}^-)(\underline{c}^-)^{\gamma h} < 0$, so that wealth in terms of consumption is

$$Y(c) = \frac{c}{\psi} - \underbrace{(\frac{\underline{c}^-}{\psi} - \underline{W}^-)\mathbf{MDef}(c)}_{\text{Default value} > 0}. \tag{12}$$

where the last term in expression (12) is the private valuation by a myopic borrower of a state-contingent claim that pays one unit when default occurs:

$$\mathbf{MDef}(c) = (\frac{c}{\underline{c}^-})^{-\gamma h}. \tag{13}$$

Expression (12) has an intuitive interpretation. The first term measures wealth without default, and the second term captures the impact of the option to default. Its impact on $Y(c)$ is negative since default is valuable and increases consumption c for any wealth level W. $\mathbf{MDef}(c)$ corresponds to the price of a default-contingent claim if the borrower is rational, but not if the borrower is myopic.

The myopic value function in terms of consumption, defined as $N(c^*(W)) = M(W)$, is given by

$$N(c) = \frac{1}{\rho}\left[(\gamma - (1-\delta))u(c) + u'(c)\left(rY(c) + \frac{1}{\gamma}\frac{\nu^2}{2}Y'(c)c\right)\right]. \tag{14}$$

Expression (14) highlights the impact of myopia on valuations.[11] The instantaneous utility of consumption, $u(c)$, gets scaled by $\gamma-(1-\delta)$ instead of γ, but more importantly, subjective wealth $Y(c)$ and its derivative are distorted due to the misvaluation of default claims by a myopic borrower that we explore now.

2.2 Pricing of default claim

To understand the nature of the distortions arising from borrower myopia, it will be useful to explore how a myopic borrower prices the default option. Myopic borrowers undervalue default-contingent claims which implies that they ignore in normal times the consequences of default. To see this, consider the shadow value of a claim on default for a myopic borrower, $\mathbf{MDef}(c)$ given in (13), compared to the market price of a default-contingent derivative claim, denoted $\mathbf{Def}(c)$. The price of a default-contingent derivative claim can be computed in closed form (all derivations are relegated to the Appendix), as follows:

Proposition 1 *The price of a default-contingent derivative claim equals*

$$\mathbf{Def}(c) = (\frac{c}{\underline{c}^-})^{-\gamma i}\frac{H(c)}{H(\underline{c}^-)}, \tag{15}$$

[11] From the envelope condition, $M''(W) = u''(c)/Y'(c)$ implying a concave value function in W for every $W > \underline{W}^-$.

where the coefficient i is the positive solution to the characteristic equation $\frac{\nu^2}{2}i^2 - (r - \rho - \frac{\nu^2}{2} - (1 - \delta)\psi)i - r = 0$ with solution

$$i = \frac{1}{\nu^2}\left(r - \rho - \frac{\nu^2}{2} - (1-\delta)\psi + \sqrt{\left(r - \rho - \frac{\nu^2}{2} - (1-\delta)\psi\right)^2 + 2r\nu^2}\right), \tag{16}$$

and H is the hypergeometric function $H(c) = {}_2F_1(\alpha_1, \alpha_2, \alpha_3; z(c))$ evaluated at the default-option wedge

$$z(c) = 1 - \psi Y'(c). \tag{17}$$

The coefficient i varies with δ and coincides with h in the rational case when $\delta = 1$. The function $H(c)$ in expression (15) is the main departure from the standard Merton (1974) and Black and Cox (1976) type solution. It denotes the hypergeometric function ${}_2F_1(\cdot)$ with coefficients $(\alpha_1, \alpha_2, \alpha_3)$ derived in the Appendix and evaluated at the default-option wedge $z(c)$. The wedge equals $z(\underline{c}^-) = 1$ at the default boundary and $\lim_{c\to\infty} z(c) = 0$ in good times.

The default claim can be used to illustrate how capital markets assess the borrower's default risk and to check how it differs from the borrower's distorted assessment. Comparing expression (13) to the market value of a default claim, **Def**(c), yields the following result.

Theorem 1 *A myopic borrower undervalues the default claim compared to a rational borrower. That is,* **MDef**$(c) \leq$ **Def**(c) *for $\delta < 1$. In the two limits $c = \underline{c}^-$ and $c \to \infty$,* **MDef**$(c) =$ **Def**(c).

Theorem 1 highlights that the (risk-neutral) probability of default under the borrower's subjective beliefs is lower than the one of a rational borrower, and it is lower the more myopic is the borrower. Only in the rational case do private and market values coincide.

Theorem 1 yields that myopic borrowers act as if they were richer than they actually are. This misconception creates a gap between the private valuation and the market value of the consumption claim that is absent under rationality. Myopic borrowers overvalue the consumption claim compared to the rational market. To see this, suppose the borrower valued the default option at its market value; wealth for a given level of consumption would be

$$\Psi(c) = \frac{c}{\psi} - (\frac{\underline{c}^-}{\psi} - \underline{W}^-)\mathbf{Def}(c). \tag{18}$$

We can write the subjective wealth $Y(c)$, which equals the private value of consumption, in terms of the $\Psi(c)$ claim and shadow wealth $G(c) \geq 0$ as

$$\begin{aligned}\text{(Subjective) wealth } Y(c) &= \frac{c}{\psi} - (\frac{\underline{c}^-}{\psi} - \underline{W}^-)(\frac{c}{\underline{c}^-})^{-\gamma h} \\ &= \Psi(c) + \underbrace{(\frac{\underline{c}^-}{\psi} - \underline{W}^-)[\mathbf{Def}(c) - \mathbf{MDef}(c)]}_{\text{Shadow wealth } G(c) \geq 0}. \end{aligned} \tag{19}$$

The comparison between the expressions for $\Psi(c)$ and $Y(c)$ illustrates that a myopic borrower acts richer than (s)he actually is: $\Psi(c) \leq Y(c)$ since a myopic borrower undervalues the default claim compared to a rational borrower, $\mathbf{MDef}(c) \leq \mathbf{Def}(c)$.[12]

The consequence of the misconception is that in good times myopic borrowers ignore the consequences of their actions on the likelihood of default. Only when default becomes imminent myopic borrowers can no longer ignore the impact on default. Close to default myopic borrowers start to behave more like rational borrowers. Default acts as a disciplining device for myopic borrowers. The marginal shadow wealth $G'(c) \lesseqgtr 0$ determines the sign and magnitude of the consumption and investment distortions.

[Figure 3 About Here.]

Figure 3 illustrates the shape of $G(c)$ (left) and its derivative $G'(c)$ (right) as a function of the probability of default. The blue line assumes that the borrower is time-consistent ($\delta = 1$), which represents our benchmark economy. The red line assumes that the borrower exhibits myopia ($\delta \approx 0$). The shadow

[12] Note that $\Psi(c)$ is not the market value of the consumption claim, since the borrower also misvalues the consumption stream. That is, the present value of future equilibrium consumption absent default is not equal to $\frac{c}{\psi}$. A more elaborate decomposition as (19) holds when $\Psi(c)$ is market value of equilibrium consumption until default and $p(c)$ denotes the market price of the consumption claim. We then have

$$Y(c) = \Psi(c) + \Phi(c) + G(c),$$

with

$$\begin{aligned}
\Psi(c) &= p(c) - (p(\underline{c}^-) - \underline{W}^-)\mathbf{Def}(c), && \text{(Market value of equilibrium } c^* \text{ until default)} \\
\Phi(c) &= \tfrac{c}{\psi} - p(c) - (\tfrac{\underline{c}^-}{\psi} - p(\underline{c}^-))\mathbf{Def}(c), && \text{(Mispricing of consumption until default)} \\
G(c) &= (\tfrac{\underline{c}^-}{\psi} - \underline{W}^-)[\mathbf{Def}(c) - \mathbf{MDef}(c)]. && \text{(Mispricing of default-contingent claim)}
\end{aligned}$$

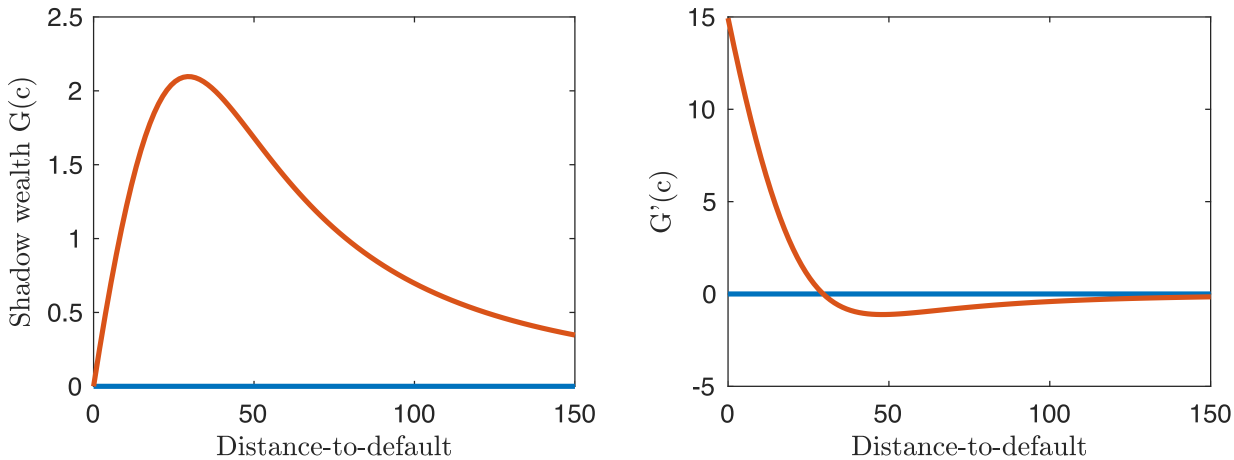

Figure 3: **Shadow wealth.** The figure plots the myopic wealth gap, $G(c)$ (left), and its marginal change with the consumption level, $G'(c)$ (right), as a function of the probability of default. The plot is constructed using the parametrization $(\gamma, \rho, r, \omega_0, \omega_1, p, \mu, \sigma, \kappa) = (6, 0.02, 0.02, 30, 0.7, 0.1, 0.04, 0.1, 150)$. The blue line assumes that the borrower is time-consistent ($\delta = 1$). The red line assumes the borrower is myopic ($\delta \approx 0$).

wealth G vanishes under rationality. It also disappears in the myopic case in the limit as $W \to \infty$ (that is, when Pr(Default)=0) and at the default threshold, that is, when Pr(Default)=1 for $W = \underline{W}^-$ or $\underline{W}^a$.[13]

2.3 Consumption and capital distortions from default options under myopia

2.3.1 Overconsumption and imperfect consumption smoothing

Myopia leads to overconsumption and imperfect consumption smoothing due to the undervaluation of the default option (Theorem 1). The consequence is a reduction and countercyclical variation in expected

[13]The following two thought experiments illustrate the nature of the misvaluation. Consider an otherwise identical rational borrower with distorted beliefs who makes two mistakes: The borrower wrongly believes the level of consumption is a fraction $1 - \frac{1-\delta}{\gamma}$ of its true value and, in addition, it assumes perfect consumption smoothing and therefore overestimates the growth rate in consumption by its time-varying component $(1-\delta)/Y'(c)$. An alternative interpretation is the following: Consider an otherwise identical borrower with distorted beliefs whose only mistake is to misjudge the consumption delta of the default option, **MDef**$'(c)$. So long as the borrower believes **MDef**$'(c) =$ **Def**$'(c)$, $\Psi'(c) = Y'(c)$ and market and private values of the consumption claim coincide, that is, $\Psi(c) = Y(c)$.

consumption growth. Consumption growth exhibits the equilibrium dynamics

$$\frac{dc}{c} = \left(r - \rho + (1 + \frac{1}{\gamma})\frac{\nu^2}{2} - \underbrace{\frac{1-\delta}{Y'(c)}}_{\text{Imperfect smoothing}} \right) \frac{1}{\gamma} dt + \frac{\nu}{\gamma} dw, \tag{20}$$

with $Y'(\underline{c}^-) = [1 + \gamma h(1 - \frac{\psi}{\underline{c}^-/(\underline{W}^-)})]\frac{1}{\psi} > \frac{1}{\psi} = \lim_{c\to\infty} Y'(c)$. The last term in parenthesis captures the imperfect smoothing through a reduction and countercyclical variation in consumption growth as $Y'(c) = \Psi'(c) + G'(c) > 0$ declines with c.

Discussion: Expression (20) shows that myopia produces a rubber-band effect: Expected consumption growth decreases with the distance to default, as myopic borrowers overconsume more in good times than in bad times. By contrast, rational borrowers have constant consumption growth and their tendency to overconsume stays constant at all times. This qualitative difference between myopia and impatience materializes because a default option is present. To understand its nature, first consider the electoral effect (6), i.e., the change in value upon the arrival of a new self. One can easily show that the imperfect smoothing term in (20) equals the marginal rate of substitution between present utility and the cost of a new self arrival: $-\frac{1-\delta}{Y'(c)} = \frac{E'(W)}{u'(c)}$. Expression (12) shows that there is a higher expected consumption growth when the marginal value of the electoral effect is higher. In this case, the borrower saves against a drop in value due to the arrival of a new self, which allows consumption to grow more over time.

The imperfect smoothing term $\frac{E'(W)}{u'(c)}$ is higher closer to default due to the wealth effect arising from the default option. Consumption is a constant fraction ψ of the sum of wealth and the value of the default option (see expression (12)). Since the value of the option increases close to default, consumption is convex in wealth. As the borrower relies more on the value of the default option in bad times to finance its consumption, it becomes more concerned with the possibility of a new self arriving, which implies that the marginal rate of substitution between present utility and the electoral effect increases.

[Figure 4 About Here.]

Figure 4 illustrates the consumption distortions arising from borrower myopia. The figures plot con-

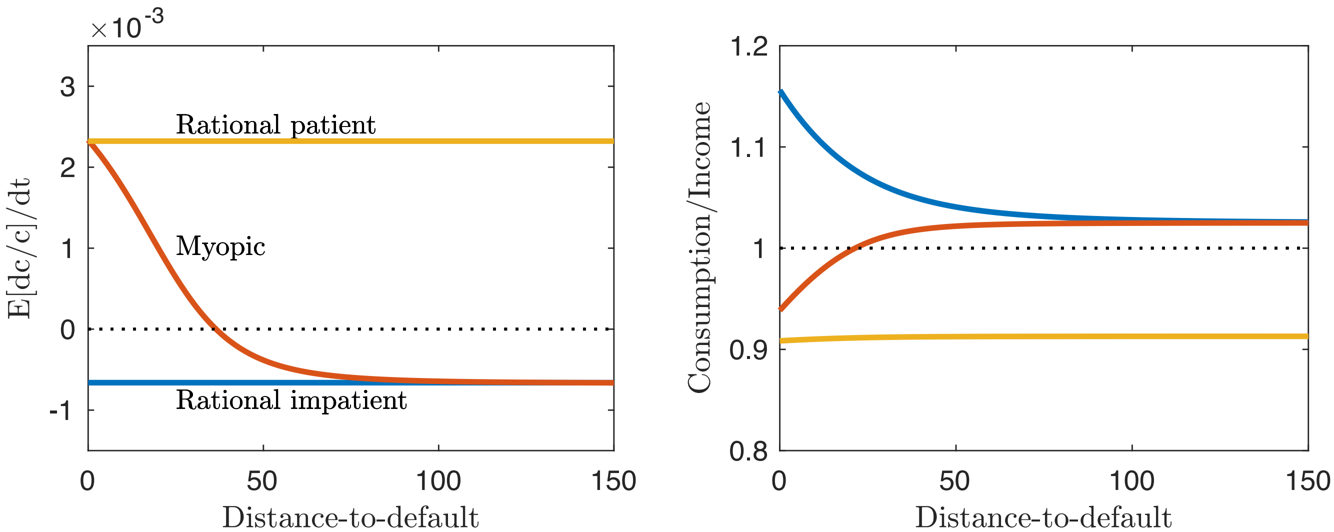

Figure 4: **Imperfect consumption smoothing.** The left figure plots expected consumption growth. The right figure plots consumption normalized by expected income $W(r + \theta(\mu - r))$. The red line assumes the borrower is myopic ($\delta \approx 0$). The yellow line corresponds to a patient rational borrower ($\delta = 1$) with same impatience parameter ρ as the myopic borrower. The blue line corresponds to an impatient rational borrower ($\delta = 1$) with higher impatience parameter ρ than the myopic borrower, calibrated so that the consumption-to-wealth ratios in normal times are equalized.

sumption growth (left) and consumption over expected income $W(r + \theta(\mu - r))$ (right) as functions of the endogenous distance to default. The yellow line assumes that the borrower is time-consistent ($\delta = 1$) and patient, which is our benchmark. The blue line captures a time-consistent borrower that is less patient. The red line captures a myopic borrower ($\delta \approx 0$). The impatience parameter ρ is calibrated so that the consumption-to-wealth ratio in normal times is equal to the case of myopia. The right figure shows that myopic borrowers overconsume relative to the rational case. The left figure plots the expected consumption growth rate. The figure shows that myopic borrowers smooth consumption imperfectly. More importantly, defaults acts as a disciplining device: As the borrower approaches default, consumption declines and the expected growth rate in consumption approaches the rational benchmark.

2.3.2 Underinvestment in normal times and risk shifting in crisis times

Myopia leads also to investment distortions. The myopic investment share equals

$$\theta^* = \frac{Y'(c^*)c^*}{Y(c^*)}\theta^\star = \left(\frac{c^*}{W} + \gamma h(\frac{\underline{c}^-}{\underline{W}^-} - \psi)\frac{W^-}{W}\mathbf{MDef}(c^*)\right)\frac{1}{\psi}\theta^\star, \tag{21}$$

where $\theta^\star = \frac{1}{\gamma}\frac{\nu}{\sigma}$ is Merton's fixed investment share. Total investment relative to consumption at time t can be decomposed as

$$\frac{\theta^* W}{c^*} = (\underbrace{\Psi'(c^*)}_{\text{Effect of default}} + \underbrace{G'(c^*)}_{\text{Effect of myopia: marginal shadow wealth}}) \times \theta^\star. \tag{22}$$

The first term, Ψ', leads to risk shifting in crisis times (gambling for resurrection). The second term in expression (22) shows that myopia distorts investment further. Combined with expression (19) and the right panel of Figure 3, it becomes clear that myopia leads to underinvestment in normal times times when wealth is high and to overinvestment relative to the rational benchmark in crisis times when wealth is low, since $G'(c)$ is negative for high c and $G'(c)$ is positive for low c.

3 Optimal Bailout Fund

This section derives the optimal incentive-compatible bailout policy that minimizes intertemporal costs. We start by characterizing the hard-default threshold $\underline{W}^a$ and soft-default threshold $\underline{W}^-$ and then determine bailout fund size and equilibrium bailout scheme, $T(W)$.

3.1 Hard vs. soft default

As discussed in Section 1, we assume for simplicity that in a hard default the borrower becomes autark and the bailout agency seizes assets so that wealth changes from W to $W^a(W) = \omega_0 + \omega_1 W$, with parameters ω_0, ω_1 determined by the enforcability of debt. Optimal consumption in autarky is a constant fraction of wealth, $c^a(W) = \psi^a W$, with autark consumption-wealth ratio

$$\psi^a = \frac{\rho - (1-\gamma)(\mu - \gamma\frac{\sigma^2}{2})}{\gamma - (1-\delta)}. \tag{23}$$

The borrower's value function in autarky equals

$$\Omega(W) = \frac{1}{\psi^a}\frac{c^a(W)^{1-\gamma}}{1-\gamma} = (\psi^a)^{-\gamma}\frac{W^a(W)^{1-\gamma}}{1-\gamma}. \tag{24}$$

If a hard default (HD) is optimal, it occurs when wealth drops to the threshold $\underline{W}^a$ that satisfies the value-matching and smooth-pasting conditions (see Grenadier and Wang (2007))

$$VMC\text{ HD}: \qquad M(\underline{W}^a) = \Omega(\underline{W}^a), \tag{25}$$

$$SPC\text{ HD}: \qquad M'(\underline{W}^a) = \Omega'(\underline{W}^a). \tag{26}$$

These conditions determine the hard-default threshold and the consumption policy. Appendix B shows that the hard-default threshold depends on the ratio of the propensities to consume in autarky and good times, ψ^a/ψ. Myopia increases both propensities, yet in a symmetric fashion, so that their ratio remains independent of myopia.

Lemma 3 shows that myopia does not distort a myopic borrower's default decision in the absence of a bailout agency that provides optimal incentives. This benchmark result follows from the assumption of instantaneous gratification with constant relative risk aversion. In this case, the marginal cost and marginal benefit of default vary symmetrically with δ. Any distortions in default decisions thus arise from the strategic interaction between borrower and bailout agency.

Lemma 3 *The hard-default threshold $\underline{W}^a$ is invariant to the myopia parameter δ, with*

$$\underline{W}^a = \frac{\left(\mu - \gamma\frac{\sigma^2}{2} - \frac{\nu^2}{2}\frac{\psi^a}{\psi}\left(\frac{1}{\gamma}+h\right) - \rho\frac{1-(\omega_1)^{\frac{1}{\gamma}-1}}{1-\gamma}\right)(\omega_1)^{-\frac{1}{\gamma}}\omega_0}{r - \frac{\nu^2}{2}h - \left(\mu - \gamma\frac{\sigma^2}{2} - \frac{\nu^2}{2}\frac{\psi^a}{\psi}\left(\frac{1}{\gamma}+h\right) - \rho\frac{1-(\omega_1)^{\frac{1}{\gamma}-1}}{1-\gamma}\right)(\omega_1)^{1-\frac{1}{\gamma}}}. \tag{27}$$

Consumption at the hard-default threshold equals $\underline{c}^a = (\omega_1)^{-\frac{1}{\gamma}}\psi^a(\omega_0 + \omega_1\underline{W}^a)$.

In a soft default (SD), borrower and bailout agency negotiate over a financial transfer to keep the borrower from outright default. Following Mella-Barral (1999), we model the interaction as a dynamic game in which the borrower acts as the Stackelberg leader and the agency acts as the follower. The borrower pleads for financial aid at the soft-default threshold $\underline{W}^-$. The agency responds by offering an incentive-compatible transfer that changes wealth to $\underline{W}^+ = \underline{W}^- + T(\underline{W}^-)$. The agency offers $T(\underline{W}^-)$ to keep the borrower afloat, which requires an individual rationality constraint for the borrower:

$$IR \text{ borrower SD}: \quad M(\underline{W}^- + T(\underline{W}^-)) \geq \Omega(\underline{W}^- + T(\underline{W}^-)), \tag{28}$$

with $M(W)$ from (14) and $\Omega(W)$ from (24). The threshold $\underline{W}^-$ is a best response given the transfer policy T and the history of past transfers and default thresholds. In turn, the transfer policy is a best response to any default threshold W chosen by the borrower, given the history of past transfers and default thresholds. We analyze a stationary subgame-perfect Nash equilibrium in which $\underline{W}^-$ and $T(\underline{W}^-)$ are time invariant.

In a soft default, the value-matching condition requires that the borrower's value functions are probabilistically matched before and after default. The borrower optimally decides when to call on the agency's help. The optimality condition requires that marginal benefit and cost are equalized. With probability $0 \leq p < 1$ that negotiations fail and value $\Omega(W)$ after a hard default, the value-matching and smooth-pasting conditions for the soft-default threshold $\underline{W}^-$ read

$$VMC \text{ SD}: \quad M(\underline{W}^-) = (1-p)M(\underline{W}^+) + p\Omega(\underline{W}^-), \tag{29}$$

$$SPC \text{ SD}: \quad M'(\underline{W}^-) = (1-p)M'(\underline{W}^+)(1 + T'(\underline{W}^-)) + p\Omega'(\underline{W}^-). \tag{30}$$

The term $T'(\underline{W}^-)$ in condition (30) arises because the borrower is strategic in default about how transfer $T(\underline{W}^-)$ depends on the wealth level $\underline{W}^-$ at which the borrower defaults.

The borrower's default incentive thus depends on the steepness T' of the bailout scheme. The agency, in turn, can design the transfer to influence the borrower's default decision. A steeper contract with larger T' induces the borrower to default later, potentially reducing the cost to the agency.

A solution to (29) and (30) does not exist for all parameter values.[14] If it exists, we can verify that the solution is an equilibrium. In turn, a hard default is an equilibrium if no threshold satisfying (29) and (30) exists, or if bailout costs exceed the cost of hard default. Positive transfers require therefore the agency's

[14]See Tan et al. (2019) for generic conditions guaranteeing that the solution to the SPC solves the Bellman system and, hence, is an equilibrium. The authors also show that there exists no equilibrium that cannot be obtained by the SPC.

individual rationality constraint

$$IR \text{ agency SD}: \quad I(c(\underline{W}^-)) \leq \kappa + W^a(\underline{W}^-) - \underline{W}^- \tag{31}$$

where $I(c)$ are the intertemporal bailout costs under SD.

3.2 Bailout fund

To determine the optimal bailout policy and bailout fund size, we need to value the agency's payment obligations given an incentive-compatible default threshold $\underline{W}^-$ and an equilibrium bailout schedule $T = T(\underline{W}^-)$. The bailout agency minimizes the intertemporal costs, trading off infrequent large against frequent small transfers, or not bailing out at all. Expected bailout costs $I(c)$ are the present value of costs conditional on default:

$$I(c) = I(\underline{c}^-) \times \textbf{Def}(c). \tag{32}$$

Upon reaching the soft-default threshold, the agency's cost incorporate the current transfers, the cost of future bailouts, and the possibility of a hard default:

$$I(\underline{c}^-) = (1-p)(\underbrace{T}_{\text{Current transfer}} + \underbrace{I(\underline{c}^+)}_{\text{Future bailouts}}) + p(\underbrace{\kappa + W^a(\underline{W}^-) - \underline{W}^-}_{\text{Hard default}}). \tag{33}$$

Evaluating (32) at the post-transfer equilibrium consumption $\underline{c}^+ = c(\underline{W}^- + T)$ and substituting in (33) yields the annuity value of transfers and hard default costs. Solving forward, one obtains

$$I(c) = \frac{(1-p)T + p(\kappa + W^a(\underline{W}^-) - \underline{W}^-)}{1 - (1-p)\textbf{Def}(\underline{c}^+)} \times \textbf{Def}(c). \tag{34}$$

Expression (34) expresses the required bailout funds in terms of the agency's transfers, which is crucial for characterizing the optimal bailout policy.

3.3 Equilibrium bailout scheme

The optimal incentive-compatible bailout policy minimizes the intertemporal costs to the bailout agency in (32), given that the borrower follows a threshold strategy and asks for financial aid whenever wealth drops below the endogenous threshold $\underline{W}^-$ determined by (29) and (30). A hard default is optimal if no such threshold exists or the bailout costs exceed the cost of outright default.

We now characterize the optimal bailout, given the agency's and the borrower's objectives and sequence of moves. By the one-shot deviation principle, the optimal bailout is pinned down by deviations from the subgame-perfect Nash equilibrium in a single round of bailout negotiations. The agency when deciding its transfer policy $T(W)$ takes into account that the borrower changes its default policy with the size of the offered bailout. The optimality condition for $T(W)$ equates the marginal benefit of the transfer to its marginal cost. As a result, the optimal bailout policy depends on borrower myopia.

Lemma 4 *Fix a soft default threshold $\underline{W}^-$. It is optimal for the bailout agency to offer a bailout $T > 0$ if and only if default spillovers exceed $\underline{\kappa} = -\frac{Y'(\underline{c}^-)}{\mathbf{Def}'(\underline{c}^-)} - (\omega_0 - (1-\omega_1)\underline{W}^-)$.*

The size of the optimal bailout, when positive, is given by the following proposition:

Proposition 2 *If $\kappa > \underline{\kappa}$, the optimal bailout is determined by setting*

$$\underbrace{-\frac{d\mathbf{Def}(\underline{c}^+)}{dT}}_{\substack{\textit{Marginal decline in}\\ \textit{price of default}\\ (\searrow \textit{ under myopia})}} \times \underbrace{I(\underline{c}^-)}_{\substack{\textit{Discounted cost}\\ \textit{of future defaults}\\ (\nearrow \textit{ under myopia})}} = \underbrace{1}_{\substack{\textit{Marginal cost}\\ \textit{of bailout}}} . \tag{35}$$

Condition (35) illustrates how the optimal bailout policy depends on myopia. The marginal benefit of the transfer is the reduction in the cost of future defaults through a reduction in the risk-neutral default probability (first term in expression (35)) times the discounted losses given default to the bailout agency (second term in expression (35)). The marginal cost of the bailout on the right-hand side of (35) is 1.

Equilibrium consumption dynamics (20) with subsistence level $\underline{c}^-$ and jump size $\underline{c}^+ - \underline{c}^-$, and wealth dynamics with soft-default threshold $\underline{W}^-$ and financial transfer $T(\underline{W}^-) = Y(\underline{c}^+) - \underline{W}^-$ are pinned down by the conditions SPC: $\underline{c}^- = (p\omega_1)^{-\frac{1}{\gamma}}\psi^a(\omega_0 + \omega_1\underline{W}^-)$, VMC: $N(\underline{c}^-) = (1-p)N(\underline{c}^+) + p\Omega(\underline{W}^-)$, $\underline{c}^+$: $Y'(\underline{c}^+) = -\frac{\partial \mathbf{Def}(\underline{c}^+)}{\partial \underline{c}^+} I(\underline{c}^-)$, with the expressions for ψ^a from (23), $N(c)$ from (14), $\Omega(W)$ from (24), $Y(c)$ from (12), **Def**(c) from (15), and $I(c)$ from (34).

The impact of myopia on the optimal bailout depends on the sensitivity to wealth of the price of default and on the magnitude of bailout costs. Financial transfers are less effective under myopia than rationality at reducing the risk-neutral default probability. Therefore, the risk-neutral default probability is not only higher but also less sensitive to wealth under myopia than rationality, and the cost of default is higher. These forces imply that optimal financial transfers can be higher or lower under myopia than rationality. In case financial transfers are less effective at staving off default and reducing the price of the default claim, the claim is less sensitive to wealth under myopia than rationality. As a result, optimal transfers are lower under myopia. On the other hand, the need for larger bailouts is higher when future bailouts arrive sooner. Transfers are therefore higher under myopia if default is sufficiently more costly.

3.4 Tradeoff in myopia-optimal bailout

To illustrate the tradeoff inherent in the optimal bailout and its interaction with the borrower's incentive to default, we consider two counterfactuals. First, we fix the policy of the bailout agency and ask how does a borrower's default policy change with myopia? Second, we fix the default policy and ask how does the bailout policy change with myopia?

Consider first how a borrower's default policy changes with myopia, given the policy of the agency. In deciding whether to default or not, the borrower balances the benefits of default (immediate increase in wealth) against its future costs (loss of access to the risk free asset and eligibility for future bailouts). Myopia diminishes the importance the borrower attaches to the future, either after a bailout or after a hard default. The following result shows that myopia leaves unaffected the default policy of the borrower, holding fixed the policy of the bailout agency. This myopia irrelevance result follows from the same tradeoff as described in Lemma 3.

Theorem 2 *(Myopia irrelevance) Fix a transfer T that is offered by the bailout agency once the borrower reaches a soft-default threshold $\underline{W}^-$. Then $\underline{W}^-$ is invariant to the myopia parameter δ.*

Consider next how the bailout policy changes with myopia, given the default policy of the borrower. Holding fixed the default policy, the equilibrium bailout scheme has the following important property.

Proposition 3 *(Target leverage) Given an equilibrium default threshold $\underline{W}^-$, there exists a target wealth $\underline{W}^+$, so that the optimal bailout scheme has the property $T'(W) = -1$ for all W, which is independent of the myopia parameter δ.*

The result follows from the first-order condition that is independent of W:

$$\frac{\partial I(c(W);T)}{\partial T} = (1-p)(1 + I(\underline{c}^-)\frac{\partial \mathbf{Def}(c^+)}{\partial c}\frac{\partial c^+}{\partial T}) = 0. \tag{36}$$

In a soft default, if the borrower receives a bailout, one-shot deviations thus have the property that $\underline{W}^+$ and $\underline{c}^+ = c(\underline{W}^+)$ are independent of when the borrower defaults. Alternative bailout schemes differ, as a result, in the intercept but not the slope of the bailout schedule.

4 Default Procrastination and the Impunity of Myopia

4.1 Comparing bailouts: When does myopia get rewarded?

We now return to the question of how a bailout agency optimally adapts its policy and chooses its bailout fund size depending on a borrower's myopia. Below we concentrate on bailout policy, leaving aside any structural policy aimed at strengthening the time-consistency of a borrower.

Whether myopic borrowers are treated more strictly or more leniently by bailout agencies ultimately depends on economic determinants that affect the growth rate of the economy, the costs of a hard default, and the risk of negotiation failure. To demonstrate the economic intuition, we show how transfers differ between myopic ($\delta = 0$) and rational borrowers ($\delta = 1$) as we vary the parameters of the model. Let Θ denote the domain of the model parameters $\{\gamma, \rho, r, \omega_0, \omega_1, p, \mu, \sigma, \kappa\}$.

Lemma 5 *Myopia is punished (rewarded) when a myopic borrower ($0 \leq \delta < 1$) receives a smaller (larger) financial transfer than a rational borrower ($\delta = 1$). For $\Delta \subseteq \Theta$, punishment and reward depend on the following conditions:*

1. *Myopia is punished for low levels of negotiation risk p, and rewarded otherwise.*

2. *Myopia is punished when spillover costs κ are low, and rewarded otherwise.*

3. *Myopia is punished for low values of risk aversion γ, and rewarded otherwise.*

4. *Myopia is punished for low values of impatience ρ, and rewarded otherwise.*

5. *Myopia is punished when the market price of risk ν is high, and rewarded otherwise.*

6. *Myopia is punished when default cost ω_0 is high and ω_1 is low, and rewarded otherwise.*

Figure 5 illustrates when myopia gets punished in equilibrium (blue region) and when it gets rewarded (red region). The white and grey regions indicate no bailout in equilibrium. Myopia gets punished when the equilibrium transfer is smaller under myopia than under rationality, $T(\underline{W}^-; \delta \approx 0) > T(\underline{W}^-; \delta = 1)$. Myopia gets rewarded when the equilibrium transfer is larger under myopia than under rationality, $T(\underline{W}^-; \delta \approx 0) < T(\underline{W}^-; \delta = 1)$. We start with the base case $(\gamma, \rho, r, \omega_0, \omega_1, p, \mu, \sigma, \kappa) = (5, 0.02, 0.015, 1, 0.5, 0.2, 0.025, 0.05, 10)$ and in each plot we vary two parameters. The top left figure varies spillover cost κ and renegotiation friction p. The top right figure varies impatience ρ and risk aversion γ. The bottom left figure varies volatility σ and growth rate μ. The bottom right figure varies the recovery parameters ω_0 and ω_1. The plots confirm Lemma 5. Equilibrium transfers increase under myopia relative to rationality when spillover cost κ, renegotiation frictions p, risk aversion γ, volatility σ, and variable recovery ω_1 are higher, and when impatience ρ, growth rate μ, and fixed recovery ω_0 are lower.

[Figure 5 About Here.]

4.2 Procrastinated default

Default procrastination is a recurrent issue for distressed borrowers. Under what conditions does myopia lead to early or late default, and why do myopic borrowers procrastinate default?

Figure 6 illustrates when myopia leads to early default (red region) or late default (blue region). The white and grey region indicate no bailout in equilibrium and, hence, a hard default occurs at $\underline{W}^a$. We consider soft default to occur early under myopia when it occurs earlier than hard default ($\underline{W}^- > \underline{W}^a$). Correspondingly, soft default occurs late under myopia when it occurs later than hard default ($\underline{W}^- < \underline{W}^a$). The figure provides a simple intuition: Larger bailouts incentivize earlier default.

The question that is pertinent for policy makers concerns the interaction between bailout and default incentive. Figure 7 illustrates how the parameters affect whether myopia leads to procrastinated soft default (red region) or accelerated soft default (blue region). The white and grey regions indicate no bailout in equilibrium and, hence, a hard default occurs at $\underline{W}^a$. We consider a default as procrastinated under myopia when it occurs later than rational default ($\underline{W}^-(\delta \approx 0) < \underline{W}^-(\delta = 1)$). Correspondingly, default is accelerated under myopia when it occurs earlier than rational default ($\underline{W}^-(\delta \approx 0) > \underline{W}^-(\delta = 1)$). The figure shows that default procrastination and borrower myopia are intricately related. Myopic borrowers tend to procrastinate default when myopia is punished. This occurs for low spillovers, low negotiation frictions, high impatience, high growth, low volatility, high fixed and low variable default costs.

[Figures 6 and 7 About Here.]

4.3 Is rational or myopic default cheaper to resolve?

For the size of the bailout fund, it is important to know if rational or myopic default is cheaper to resolve.

Proposition 4 *The bailout costs upon default, $I(\underline{c}^-)$, monotonically fall with δ. The size of the bailout fund, given by expected bailout costs $I(c(W))$, falls or rises with δ.*

Expected bailout costs given W are $I(c(W)) = I(\underline{c}^-)\mathbf{Def}(c(W))$. Bailout costs upon default are equal to $I(\underline{c}^-) = \frac{(1-p)T + p(\kappa + W^a(\underline{W}^-) - \underline{W}^-)}{1-(1-p)\mathbf{Def}(\underline{c}^+)}$. Hence,

$$\frac{d}{d\delta} I(c(W)) = [\underbrace{\frac{\partial I(\underline{c}^-)}{\partial \delta}}_{<0} + \underbrace{I'(\underline{c}^-)\frac{\partial \underline{c}^-}{\partial \delta}}_{>0}]\mathbf{Def} + I(\underline{c}^-)[\underbrace{\frac{\partial \mathbf{Def}(c)}{\partial \delta}}_{>0} + \underbrace{\mathbf{Def}'(c)\frac{\partial c(W)}{\partial \delta}}_{<0}]. \tag{37}$$

Expression (37) illustrates that competing forces determine if the expected bailout costs $I(c(W))$ fall or rise with δ. The first and last terms in the two brackets are negative, while the second and third terms are positive. Depending on the parameters, the negative or the positive forces dominate.

Figure 8 illustrates when myopia is more expensive to resolve than rational default and when it is cheaper to resolve. The red area indicates regions where myopic default is more expensive to resolve than rational default, such that $I(c(\underline{W}^-); \delta \approx 0) > I(c(\underline{W}^-); \delta = 1)$. The blue area indicates regions where myopic default is cheaper to resolve than rational default, such that $I(c(\underline{W}^-); \delta \approx 0) < I(c(\underline{W}^-); \delta = 1)$. The figure reveals an important intuition. Myopic default tends to be cheaper to resolve when myopia is punished, but sometimes also when myopia is rewarded.

[Figure 8 About Here.]

4.4 Does myopia harm all or benefit some?

Myopia harms a borrower through imperfect consumption smoothing. But can myopia, in fact, benefit rational stakeholders when only the decision maker is myopic, such as citizens under a populist government or equityholders under a myopic management? The size and frequency of bailouts depend on the myopia of the borrower. The following result shows that the strategic impact on the bailout policy and the endogenous timing of soft default can dominate the consumption-savings distortions.

Proposition 5 *The value function $R(W)$ of a rational agent under a myopic borrower rises or falls with δ. In particular, $\frac{d}{d\delta} R(\underline{W}^-; \delta = 1) \lesseqgtr 0$.*

The value function of a rational agent under a myopic borrower equals the value function $R(W)$ after the current self has been replaced. From Section 2.1, it is given by

$$R(W) = \frac{1}{\delta\rho}\left[(\gamma - (1-\delta))u(c^*) + u'(c^*)\left((r - \frac{\nu^2}{2}h)W + \frac{\nu^2}{2}\frac{c^*}{\psi}(\frac{1}{\gamma} + h)\right)\right]. \tag{38}$$

Its total derivative with respect to δ, $\frac{d}{d\delta}R(W) = \frac{\partial}{\partial\delta}R(W) + \frac{\partial}{\partial c^*}R(W)\frac{\partial}{\partial\delta}c(W)$, can be either positive or negative depending on the parameters. As a result, a rational agent may benefit from myopia due to the bailout concessions and, more importantly, the change in the endogenous timing of default.

5 Debt Spirals and Growth Traps

The distortions that myopia imposes on consumption and investment reflect on the dynamics of wealth in ways that have been documented empirically. As shown in the Appendix, Itô's lemma applied on the inverse consumption function (12) yields wealth dynamics $dW = m(c)dt + \frac{\nu}{\gamma}Y'(c)cdw$, with expected wealth growth $m(c) = rY(c) - c + \frac{\nu^2}{\gamma}Y'(c)c$.

Wealth dynamics nest several distinct patterns of growth, depending on growth opportunities and borrower characteristics. To characterize them, we use the following definition:

Definition 1 *Wealth dynamics exhibit*

i. Global growth if $m(c) \geq 0$ *for every* $c \geq \underline{c}^-$.

ii. A global spiral if $m(c) \leq 0$ *for every* $c \geq \underline{c}^-$.

iii. A local spiral if there exists $c^\dagger > \underline{c}^-$ *such that* $m(c) \leq 0$ *for* $c \in [\underline{c}^-, c^\dagger)$ *and* $m(c) \geq 0$ *for* $c \in (c^\dagger, \infty)$.

iv. A trap if there exists $c^\ddagger > \underline{c}^-$ *such that* $m(c) \geq 0$ *for* $c \in [\underline{c}^-, c^\ddagger)$ *and* $m(c) \leq 0$ *for* $c \in (c^\ddagger, \infty)$.

Global growth corresponds to the case in which the trend in wealth is positive for any wealth level above the default threshold. In such an environment, default occurs due only to negative economic shocks. Instead, default seems rather unavoidable in a global spiral, as the trend in wealth is always negative. The special cases are local spiral and trap. A local spiral occurs when the expected growth in wealth is positive in normal times but turns negative once wealth drops below the critical point $c^\dagger$, rendering default imminent. In a trap, the expected growth pattern is reversed. Wealth growth is positive below and negative above the critical point $c^\ddagger$, such that the borrower is trapped in the neighborhood of the critical point.

Lemma 6 *Let* $m(c^*)$ *be defined on the* $(0, \infty)$ *domain. Then* $m(\cdot)$ *is concave in equilibrium consumption* c^* *whenever* $r > \nu^2 h$ *and convex in* c^* *whenever* $r < \nu^2 h$*. Furthermore, let* $K \equiv \frac{r-\nu^2 h}{\psi-(r+\frac{\nu^2}{\gamma})}$*. Then* $m(\cdot)$ *has at most one root in* c^* *which exists if and only if* $K > 0$ *and* $\underline{W}^- \leq \underline{W}_0$*, where* $\underline{W}_0 = \frac{K-1}{\psi K}\underline{c}^-$*. In this case, the zero-trend equilibrium consumption equals* $c_0 = [\psi K(\frac{\underline{c}^-}{\psi} - \underline{W}^-)]^{\frac{1}{1+\gamma h}} (\underline{c}^-)^{\frac{\gamma h}{1+\gamma h}}$*. In addition,* $m'(\cdot)$ *has at most one root in equilibrium consumption which exists if and only if* $K > 0$ *and* $\underline{W}^- \leq \underline{W}_1$*, where* $\underline{W}_1 = \frac{\gamma h K \underline{c}^- - (\underline{c}^-)^{1+\gamma h}}{\gamma h K \psi}$*, which equals* $c_1 = [\psi K(\frac{\underline{c}^-}{\psi} - \underline{W}^-)]^{\frac{1}{1+\gamma h}} (\gamma h)^{\frac{1}{1+\gamma h}}$.

In the above lemma, the zero-trend consumption, c_0, and the zero-marginal trend consumption, c_1, are obviously infeasible when they lie below the soft-default boundary $\underline{c}^-$. Their relative ranking is instrumental in ordering the wealth thresholds $\underline{W}_0$ and $\underline{W}_1$ that characterize the growth patterns, as shown in the next proposition.

Proposition 6 *Assume $K > 0$. The dynamics of wealth exhibit:*

i. Local spiral if $r > \nu^2 h$ and $\underline{W} < \underline{W}_1 < \underline{W}_0$, or $r < \nu^2 h$ and $\underline{W} < \underline{W}_0 < \underline{W}_1$.

ii. Trap if $r > \nu^2 h$ and $\underline{W} < \underline{W}_0 < \underline{W}_1$, or $r < \nu^2 h$ and $\underline{W} < \underline{W}_1 < \underline{W}_0$.

iii. Global growth if $r > \nu^2 h$ and $\underline{W}_1 < \underline{W}_0 < \underline{W}$, or $r < \nu^2 h$ and $\underline{W}_0 < \underline{W}_1 < \underline{W}$.

iv. Global spiral if $r > \nu^2 h$ and $\underline{W}_0 < \underline{W}_1 < \underline{W}$, or $r < \nu^2 h$ and $\underline{W}_1 < \underline{W}_0 < \underline{W}$.

Assume $K < 0$. The dynamics of wealth exhibit global spiral if $r > \nu^2 h$ and global growth if $r < \nu^2 h$.

Depending on parameter values, all of the four cases are possible. We concentrate below on the two interesting cases from an economic perspective, local spiral and trap.

Case *i.* In a local debt spiral, the threat of default does not discipline the borrower. Expected wealth growth is positive in normal times, but when the borrower crosses the critical point $Y(c^\dagger)$, expected wealth growth turns negative. This is illustrated in Figure 9 a)-c) which demonstrates wealth dynamics. The early years are characterized by repeated default episodes. Default is followed by debt relief, but soon enough the negative trend in wealth pushes the borrower back to the default boundary. The density of the distance-to-default $W_t - \underline{W}^-$ in Figure 9 c) peaks at low levels, indicating prolonged crisis. The dotted black line in Figure 9 a) shows the distance to default for which the trend in wealth becomes zero. In the vicinity of that distance, expected growth fluctuates between positive and negative levels according to random shock realizations. The red bar marks the last time in the graph for which the borrower crosses the zero-trend threshold. Ultimately, the borrower enters a prolonged period of positive expected growth.[15]

[15]In the spirit of Reinhart and Rogoff (2009), these growth dynamics can be interpreted as "graduation" to an advanced level, away from the default region.

Case *ii.* A trap reflects an economic environment in which wealth exhibits positive expected growth in the vicinity of default (Figure 9 d)-f)). However, as wealth grows beyond the critical point $Y(c^{\ddagger})$, the trend in wealth turns negative so that wealth returns to the positive trend territory, thus producing economic stagnation. Early growth is followed by a prolonged period in which wealth fluctuates around a constant steady-state level. The borrower stays in the vicinity of default. This implies that even if there is a prolonged period for which no default happens, the value of the default option remains an important determinant in the consumption and investment decisions of the borrower.

Myopia renders the trap more likely. The value of the default option reduces both the marginal utility of continuation and risk aversion, thus inducing both higher consumption and investment. In a local spiral, the consumption effect dominates close to default, whereas in a trap the investment effect dominates and more so under myopia. The result are vastly different patterns in wealth growth and crisis duration.

[Figure 9 About Here.]

6 Credit Spread Dynamics

The time-inconsistency of consumption and risk-taking policies under myopia renders standard market-based credit metrics unsuitable proxies for default risk. For sovereigns, Tomz and Wright (2013) document that credit spreads are highly nonlinear and excessively sensitive to fundamentals. These features arise in our setting as natural characteristics of credit risk dynamics.

In the standard Merton (1974) and Black and Cox (1976) settings, the price of a derivative claim has a power shape, exactly like **MDef**(c), such that on a log-log scale it increases linearly as the distance to default declines. The market price of a default-contingent derivative claim, **Def**(c), has, by contrast, a mixed power-hypergeometric shape in our setting:

$$\text{Default claim price with rational borrower: } \mathbf{MDef}(c) = \left(\frac{c}{\underline{c}^{-}}\right)^{-\gamma h}, \tag{39}$$

$$\text{Default claim price with myopic borrower: } \mathbf{Def}(c) = \left(\frac{c}{\underline{c}^{-}}\right)^{-\gamma i} \frac{H(c)}{H(\underline{c}^{-})}. \tag{40}$$

The mixed power-hypergeometric function has a distinctive shape. For low default probability, it is less convex than the power function and plots concave in the distance to default on a log-log scale. Hence, the default claim is very sensitive to small changes in the likelihood of default for high distance to default. There exists a critical point, however, where the relation switches. For high default probability, the mixed power-hypergeometric function is more convex than the power function and plots convex in the distance to default on a log-log scale. This means the price of the derivative claim rises at an accelerating rate as the distance to default declines.

Figure 10 illustrates the relation between default probability and the price of a default-contingent claim when the borrower is myopic. The left figure plots the price of a default-contingent claim, $\mathbf{Def}(c)$, as a function of the distance-to-default. The blue line assumes that the borrower is rational ($\delta = 1$). The red line assumes that the borrower is myopic ($\delta \approx 0$). The right figure plots the price of a default-contingent claim, $\mathbf{Def}(c)$, against the private valuation by a myopic borrower of a state-contingent claim that pays one unit when default occurs, $\mathbf{MDef}(c)$. The blue line assumes that the borrower is rational ($\delta = 1$). The red line assumes that the borrower is myopic ($\delta \approx 0$).

[Figure 10 About Here.]

7 Extension: Inadequate Bailout Funds and Risky Debt

The analysis so far assumes that the agency's bailout fund suffices to reimburse creditors in full, which requires a fund size $W^{\alpha}(\underline{W}^{-}) - \underline{W}^{-}$ in case of a hard default. We now relax this assumption by allowing creditors to suffer losses upon default. Below we provide the economic setup and discuss the main economic conclusions, while relegating all proofs to the Internet Appendix.

To model debt dynamics as the borrower approaches default, we will retain the assumption of continuously adjusted debt. However, we modify the debt issuance technology as follows: Each bond issued pays one unit upon maturity and zero otherwise, provided the borrower does not default on its obligations. Each of the bonds has an exogenous random maturity which arrives at Poisson rate m. Maturity events are independently distributed across bonds so that, at each instant, fraction m of existing bonds matures.

Newly issued bonds that have not matured receive a take-it-or-leave-it repurchase offer by the borrower. The offer equals the market price of the bond since investors are competitive. Non-tendering bondholders do not receive a further offer in the future, which implies that they bear the credit risk of the bond.

The borrower retains a fraction ω_1 of the risky investment and repays an equal fraction ω_1 of the debt due in a hard default at wealth threshold $\underline{W}^- < 0$.[16] In addition, creditors seize fraction ω_0 of the risky investment lost to the borrower. The recovered value is distributed among creditors in a *pari-passu* fashion. The bailout agency suffers spillover cost that are affine in the change in gross wealth of the borrower upon hard default with intercept κ_0 and slope parameter κ_1. Finally, the borrower enjoys constant endowment flow e in normal times which is reduced to $e \times (1 - \phi)$ after a hard default, for $\phi \in (0, 1)$.

In such an event, the creditors' total recovery equals to

$$\mathcal{R} = -\omega_1(1 - \theta(\underline{c}^-))(\underline{W}^- + e) + \omega_0(1 - \omega_1)\theta(\underline{c}^-)(\underline{W}^- + e). \tag{41}$$

The first term in $\mathcal{R}$ is the part of recovery related to the amount of debt outstanding and the second is the recovery from the risky investment. Given this economic environment, we can characterize bond prices as a function of equilibrium consumption.

Proposition 7 *Bond prices are*

$$D(c) = \frac{m}{m+r} - \left(\frac{m}{m+r} - D(\underline{c}^-)\right) P_m(c), \tag{42}$$

where $D(\underline{c}^-)$ and $P_m(c)$ are respectively the bond value at the default boundary and the price of a claim that pays 1 upon default and zero otherwise, both given in the Internet Appendix.

Investment in the risky asset incorporates a hedging demand against bond price movements, whenever there is debt outstanding:

$$\theta^*(c) = -\frac{R'(W)}{R''(W)(W+e)} \times \frac{\mu - \mu_D(c)}{(\sigma - \sigma_D(c))^2} - \frac{\sigma_D(c)}{\sigma - \sigma_D(c)}. \tag{43}$$

[16] The equality between the fraction of risky assets retained and the fraction of debt repaid simplifies the autarky value for the borrower. Other than that, this assumption plays no economic role in the model.

where $\mu_D(c)$ and $\sigma_D(c)$ are respectively the expected return and volatility of the bond given in the Internet Appendix. At the same time, the inverse consumption is modified to account for the endowment, $Y(c) + e = \frac{c}{\psi} - (\frac{\underline{c}^-}{\psi} - \underline{W}^- - e)(\frac{c}{\underline{c}^-})^{-\gamma h}$.

Figure 11 compares the bond value for a myopic versus a time-consistent borrower as a function of wealth. In both cases bond values converge to their risk free value as wealth tends to infinity which lies above the value at the soft-default boundary. In the example shown, myopia leads to procrastination of default which implies that the bond values of the myopic borrower lie above that of a rational borrower. The volatility of bond prices increases closer to default, since bond prices become more sensitive to wealth shocks. As a result, the hedging demand component in the risky investment share ratio increases. Intuitively, the borrower hedges against adverse bond price movements by increasing investment in the risky asset. This gives rise to the possibility of both debt values and investment rising closer to default. This behavior resembles a debt-financed investment spree prior to default.

[Figure 11 About Here.]

To assess the implications of a hard default, on wealth notice first that the investment share θ and consumption c have continuous time paths for any $W > \underline{W}^-$. The value of the risky investment at the default threshold is $\lim_{c \downarrow \underline{c}^-} \theta(c)(W+e) = \theta(\underline{c}^-)(\underline{W}^- + e)$, and debt issued stands at $\lim_{c \downarrow \underline{c}^-} -(1-\theta(c))(W+e) = -(1-\theta(\underline{c}^-))(\underline{W}^- + e)$. The borrower repays a fraction ω_1 of the debt, loses a fraction $(1-\omega_1)$ of the risky asset investment, and receives an endowment reduced by bankruptcy costs, $e(1-\phi)$, so that gross wealth equals

$$W^a(\underline{W}^-) = \underbrace{\underline{W}^- + e}_{\text{value after last investment}} + \underbrace{(1-\omega_1)(\theta(\underline{c}^-) - 1)(\underline{W}^- + e)}_{\text{debt relief}} - \underbrace{(1-\omega_1)\theta(\underline{c}^-)(\underline{W}^- + e)}_{\text{risky assets lost}} + \underbrace{(1-\phi)e}_{\text{post-default endowment}}, \quad (44)$$

which rewrites compactly as $W^a(\underline{W}^-) = \omega_1(\underline{W}^- + e) + (1-\phi)e$.

The cost of the agency at the default boundary is still affine in $\underline{W}^-$. Upon reaching the soft-default threshold, the agency's cost incorporate current transfers, the cost of future bailouts, and the possibility of a hard default:

$$I(\underline{c}^-) = (1-p)(T + I(\underline{c}^+)) + p(\kappa_0 + \kappa_1(W^a(\underline{W}^-) - (\underline{W}^- + e))) \tag{45}$$

The pricing of an infinite maturity default claim that pays one unit upon soft default continues to be given by (15), consumption dynamics still follows equation (20), and the intertemporal cost of the bailout agency away from default is still characterized by (32). As a result, any positive transfer still obeys (35), and the contract is given by the same conditions with consumption at the soft boundary modified to account for the endowment: $\underline{c}^- = (p\omega_1)^{-\frac{1}{\gamma}} \psi^a((1-\phi+\omega_1)e + \omega_1 \underline{W}^-)$.

As a result, the implications of myopia for optimal bailout design of Section 4 extend in an analogous fashion to the case of risky debt.

8 Conclusion

Since debt crises have become a recurrent phenomenon, notwithstanding the fact that the recent surge in global debt has been due to a pandemic, this paper explores the impact of time-inconsistent borrower preferences on debt crisis dynamics and optimal crisis resolution in a continuous-time model of strategic defaults and repeated bailouts.

The threat of default creates a retractive force that, like a rubber band, pulls time-inconsistent borrowers back into distress and inflicts serial default. Myopic borrowers, as a result, imperfectly smooth consumption by overconsuming more in good times than in bad times. At the core of this effect is that myopic borrowers endogenously misprice their option to default by a U-shaped negative pricing error. The default mispricing distorts the borrower's timing of default and affects the lenders' optimal bailout policy.

Bailout agencies face a tradeoff to either punish or reward borrower myopia through smaller or, respectively, larger financial transfers. Smaller transfers lead to more procrastinated default and protracted crises. We show that the equilibrium transfer schedule differs across more or less myopic borrowers in its intercept, but myopia is irrelevant for its curvature. Optimal bailouts punish or reward myopia depending on whether transfers exacerbate or alleviate the borrowers' misperception of default risk.

The model captures the sequence of events during the European debt crisis in which the ECB, EFSF, ESM, and IMF were involved in the bailout of Ireland, Portugal, Spain, Greece, Cyprus, Latvia, Romania, and Hungary. The varying degree of duration, size, and success of these programs have stirred policy debates regarding their appropriate design. Our model shows that some borrowers optimally receive larger bailouts to resolve a debt crisis quickly ("southern view") while others receive a series of smaller bailouts to discipline borrower behavior ("northern view"). We also show to what extent bailing out myopic borrowers can be cheaper than bailing out rational borrowers, and why populist behavior can be a bargaining chip to extract higher financial transfers.

The restrictions on economic activity due to the Covid-19 pandemic have inevitably lead to a surge in debt, parts of which will need to be renegotiated. Bailout agencies can use our analysis to guide their considerations towards the design of Covid-19 related bailout schemes.

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Appendix A Derivations in Sections 1 and 2

Value functions $N(c)$ and $M(W)$: After plugging in optimal consumption c^* and investment θ^* that maximize (4),

$$\begin{aligned}\rho M(W) &= \delta u(c^*) + M'(W)[W(r+\theta^*(\mu-r)) - c^*] + \frac{\sigma^2}{2}W^2(\theta^*)^2 M''(W) \\ &= (\frac{\delta}{1-\gamma} - 1)M'(W)^{1-\frac{1}{\gamma}} + M'(W)(Wr - \frac{\nu^2}{2}\frac{M'(W)}{M''(W)}),\end{aligned} \tag{A1}$$

The ODE for $Y(c)$ obtains as follows. We have $Y'(c^*(W))\frac{\partial c^*(W)}{\partial W} = 1$ and $Y''(c^*(W))(\frac{\partial c^*(W)}{\partial W})^2 + Y'(c^*(W))\frac{\partial^2 c^*(W)}{(\partial W)^2} = 0$. Hence, $\frac{\partial c^*(W)}{\partial W} = \frac{1}{Y'(c^*(W))}$ and $\frac{\partial^2 c^*(W)}{(\partial W)^2} = -\frac{Y''(c^*(W))}{Y'(c^*(W))^3}$. Differentiating both sides of (7) with respect to W yields

$$M''(W) = u''(c^*(W))\frac{\partial c^*(W)}{\partial W} = u''(c^*(W))\frac{1}{Y'(c^*(W))},$$

and

$$\theta^* = -\frac{u'(c^*)}{u''(c^*)}\frac{Y'(c^*)}{Y(c^*)}\frac{\nu}{\sigma}. \tag{A2}$$

Substituting in (A1),

$$\rho M(Y(c^*)) = (\frac{\delta}{1-\gamma} - 1)u'(c^*)^{1-\frac{1}{\gamma}} + u'(c^*)(Y(c^*)r - \frac{\nu^2}{2}\frac{u'(c^*)}{u''(c^*)}Y'(c^*)). \tag{A3}$$

Differentiating both sides and rearranging yields

$$rY(c^*) = (\frac{\delta}{1-\gamma} - 1)(\frac{1}{\gamma} - 1)c^* + [r - \rho + (\frac{1}{\gamma} - 1)\frac{\nu^2}{2}]\frac{1}{\gamma}c^* Y'(c^*) + \frac{\nu^2}{2}(\frac{1}{\gamma}c^*)^2 Y''(c^*). \tag{A4}$$

The ODE for $M(W) = N(c^*(W))$ in (A1) yields the following ODE for $N(c)$:

$$\rho N(c) = (\frac{\delta}{1-\gamma} - 1)\left(\frac{N'(c)}{Y'(c)}\right)^{1-\frac{1}{\gamma}} + N'(c)\left(\frac{Y(c)}{Y'(c)}r - \frac{\nu^2}{2}\frac{Y'(c)N'(c)}{Y'(c)N''(c) - Y''(c)N'(c)}\right). \tag{A5}$$

The value functions $N(c)$ and $M(W)$ can now be solved for, where we use that $cY'(c) = \frac{c}{\psi} - \gamma h(Y(c) - \frac{c}{\psi})$.

Proof of Lemma 2: Let $J(W) = M(W) - \delta R(W)$. Index the optimal consumption policy and the investment share by λ, to read as $c^*_\lambda(W)$ and $\theta^*_\lambda(W)$, respectively. Multiply both sides of (5) by δ and subtract from (4) to obtain

$$(\rho+\lambda)J(W) = (1-\delta)u(c^*_\lambda(W)) + J'(W)\left[W(r+\theta^*_\lambda(W)(\mu-r)) - c^*_\lambda(W)\right] + \frac{1}{2}J''(W)W^2(\theta^*_\lambda(W)\sigma)^2$$

From the Feynman-Kac formula:

$$J(W) = (1-\delta)E\left[\int_0^\infty e^{-(\rho+\lambda)t}u(c^*_\lambda(W_t))dt\right],$$

where $W_0 = W$ and the budget dynamics (1) hold with $c = c^*_\lambda(W)$ and $\theta = \theta^*_\lambda(W)$.

If $\gamma \in (1/2, 1)$:
Select $q \in (1, \frac{1}{1-\gamma})$ and let $\underline{q}$ satisfy $\frac{1}{q} + \frac{1}{\underline{q}} = 1$. From Hölder's inequality:

$$\int_0^\infty |e^{-(\rho+\lambda)t}(u(c^*_\lambda(W_t)))|dt \le \left(\int_0^\infty e^{-\underline{q}\lambda t}dt\right)^{1/\underline{q}}\left(\int_0^\infty e^{-q\rho t}u^q(c^*_\lambda(W_t))dt\right)^{1/q} = \frac{(1-\widetilde{\gamma})^{1/q}}{(\underline{q}\lambda)^{1/\underline{q}}(1-\gamma)}\left(\int_0^\infty e^{-\widetilde{\rho}t}\widetilde{u}(c^*_\lambda(W_t))dt\right)^{1/q}$$

where $\widetilde{\rho} = q\rho, \widetilde{\gamma} = 1 - q(1-\gamma) \in (0,1)$ and $\widetilde{u}(c) = \frac{c^{q-q\gamma}}{q-q\gamma} = \frac{1}{1-\widetilde{\gamma}} c^{1-\widetilde{\gamma}} > 0$. Therefore,

$$0 \leq E \int_0^\infty e^{-(\rho+\lambda)t} u(c_\lambda^*(W_t))dt \leq \frac{(1-\widetilde{\gamma})^{1/q}}{(\underline{q}\lambda)^{1/\underline{q}}(1-\gamma)} E\left[\int_0^\infty e^{-\widetilde{\rho}t}\widetilde{u}(c_\lambda^*(W_t))dt\right]^{1/q}.$$

From Jensen's inequality,

$$E\left[\int_0^\infty e^{-\widetilde{\rho}t}\widetilde{u}(c_\lambda^*(W_t))dt\right]^{1/q} \leq \left[\int_0^\infty e^{-\widetilde{\rho}t}E(\widetilde{u}(c_\lambda^*(W_t)))dt\right]^{1/q}.$$

Let $\widetilde{R}(W,\widetilde{\gamma}) \equiv \sup_{c,\theta} \int_0^\infty e^{-\widetilde{\rho}t}E(\widetilde{u}(c_t))dt$ subject to the constraint that the budget dynamics of Lemma 1 hold.
Since $\int_0^\infty e^{-\widetilde{\rho}t}E(\widetilde{u}(c_\lambda^*(W_t)))dt \leq \widetilde{R}(W,\widetilde{\gamma})$:

$$\begin{aligned} & 0 \leq E\left[\int_0^\infty e^{-(\rho+\lambda)t}u(c_\lambda^*(W_t))dt\right] \leq \frac{(1-\widetilde{\gamma})^{1/q}}{(\underline{q}\lambda)^{1/\underline{q}}(1-\gamma)}\widetilde{R}^{1/q}(W,\widetilde{\gamma}) \\ \Rightarrow \quad & 0 \leq J(W) \leq (1-\delta)\frac{(1-\widetilde{\gamma})^{1/q}}{(\underline{q}\lambda)^{1/\underline{q}}(1-\gamma)}\widetilde{R}^{1/q}(W,\widetilde{\gamma}). \end{aligned}$$

Taking limits on both sides as $\lambda \to \infty$ and noticing that $\widetilde{R}(W,\widetilde{\gamma})$ is invariant to λ, $\lim_{\lambda\to\infty} J(W) = 0$.

If $\gamma > 1$:
First notice that $J(W) \leq 0$. Defining as $\mathcal{T}_\lambda$ the stopping time of hard default, we have

$$J(W)/(1-\delta) = E\left[\int_0^{\mathcal{T}_\lambda} e^{-(\rho+\lambda)t}u(c_\lambda^*(W_t)dt\right] + E(e^{-(\rho+\lambda)\mathcal{T}_\lambda})\Omega(\underline{W},\lambda)$$

Since $\lim_{c\downarrow 0} u'(c) = -\infty$ for $\gamma > 1$, there exists a strictly positive lower bound for consumption, i.e. $c_\lambda^*(W) \geq \underline{c}_\lambda^* > 0$ for every $W \geq \underline{W}$ and $\lim_{\lambda\to\infty} \underline{c}_\lambda^* > 0$. As a consequence $0 \geq J(W)/(1-\delta) \geq E\left(\frac{1-e^{-(\rho+\lambda)\mathcal{T}_\lambda}}{\rho+\lambda}\right)u(\underline{c}_\lambda^*) + E\left(e^{-(\rho+\lambda)\mathcal{T}_\lambda}\right)\Omega(\underline{W},\lambda)$.
Taking limits on both sides as $\lambda \to \infty$, $\lim_{\lambda\to\infty} J(W) = 0$.

If $\gamma = 1$:
Then $u(c) = log(c) \leq \frac{c^{1-\underline{\gamma}}}{1-\underline{\gamma}}$ for $\underline{\gamma} \in (0,1)$. For this reason, $E\int_0^\infty e^{-(\rho+\lambda)t}u(c_\lambda^*(W))dt \leq E\int_0^\infty e^{-(\rho+\lambda)t}\frac{c_\lambda^*(W)^{1-\underline{\gamma}}}{1-\underline{\gamma}}dt \leq \frac{(1-\overline{\gamma})^{1/\widehat{q}}}{(\underline{\widehat{q}}\lambda)^{1/\underline{\widehat{q}}}(1-\underline{\gamma})}\widetilde{R}^{1/\widehat{q}}(W,\overline{\gamma})$, or

$$J(W) \leq (1-\delta)\frac{(1-\overline{\gamma})^{1/\widehat{q}}}{(\underline{\widehat{q}}\lambda)^{1/\underline{\widehat{q}}}(1-\underline{\gamma})}\widetilde{R}^{1/\widehat{q}}(W,\overline{\gamma})$$

where $\widehat{q} \in (1, \frac{1}{1-\underline{\gamma}})$, $\frac{1}{\widehat{q}} + \frac{1}{\underline{\widehat{q}}} = 1$ and $\overline{\gamma} = 1 - \widehat{q}(1-\underline{\gamma})$
$\overline{\gamma} = 2\underline{\gamma} - 1$.
Following the analysis of the case $\gamma > 1$ above,

$$J(W)/(1-\delta) \geq E\left(\frac{1-e^{-(\rho+\lambda)\mathcal{T}_\lambda}}{\rho+\lambda}\right)u(\underline{c}_\lambda^*) + E\left(e^{-(\rho+\lambda)\mathcal{T}_\lambda}\right)\Omega(\underline{W},\lambda)$$

As before, we conclude $\lim_{\lambda\to\infty} J(W) = 0$.
Finally, multiplying (5) by δ, subtracting from (4) and taking the limit as $\lambda \to \infty$ we have

$$\lim_{\lambda\to\infty} E(W) + (1-\delta)u(c_\lambda^*(W)) = 0.$$

Shadow wealth: Consider how a rational borrower would value a claim on equilibrium consumption up until default, denoted by $\Psi(c)$. The no-arbitrage condition

$$r\Psi(c) = c + \left(r - \rho - (1 - \frac{1}{\gamma})\frac{\nu^2}{2} - \frac{1-\delta}{Y'(c)}\right)\frac{1}{\gamma}c\Psi'(c) + \frac{\nu^2}{2}(\frac{1}{\gamma}c)^2\Psi''(c), \tag{A6}$$

with boundary conditions $\Psi(\underline{c}^-) = \underline{W}^-$ and $\lim_{c\to\infty}\Psi(c)/c = 1/\psi$ has two solutions. The solution that imposes $\Psi'(c) = Y'(c)$ for all c yields the subjective wealth equation (12). This can be seen from a comparison between (10) and (A6). The ODE (10) is a special case of (A6) when $\Psi'(c) = Y'(c)$. The general solution that does not impose this restrictive, myopic condition yields the market value of a consumption claim.

Appendix B Hard Default

After a hard default, there is autarky with no risk-free lending and savings technology. Then the value function in autarky, $\Omega(W)$, satisfies

$$\rho\Omega(W) = \delta u(c^a) + \Omega'(W)(W\mu - c^a) + \frac{\sigma^2}{2}W^2\Omega''(W), \tag{B1}$$

with $u'(c^a) = \Omega'(W)$ and $c^a = c^a(W)$ is consumption in autarky. The ODE becomes

$$\rho\Omega(W) = (\gamma - (1-\delta))\frac{1}{1-\gamma}\Omega'(W)^{1-\frac{1}{\gamma}} + \Omega'(W)W\mu + \frac{\sigma^2}{2}W^2\Omega''(W), \tag{B2}$$

which yields

$$\Omega(W) = (\psi^a)^{-\gamma}\frac{W^{1-\gamma}}{1-\gamma}, \tag{B3}$$

with the autarky consumption-wealth ratio ψ^a is equal to

$$\psi^a = \frac{\rho - (1-\gamma)(\mu - \gamma\frac{\sigma^2}{2})}{\gamma - (1-\delta)}. \tag{B4}$$

The smooth-pasting condition (26) can be written as $u'(\underline{c}^{-a}) = \Omega'(\underline{W}^{-a})$, or

$$\underline{c}^{-a} = (\omega_1)^{-\frac{1}{\gamma}}\psi^a(\omega_0 + \omega_1\underline{W}^{-a}). \tag{B5}$$

The left-hand side of the value-matching condition (25) rewrites as

$$M(\underline{W}^{-a}) \quad = \quad \frac{1}{\rho}\left[(\gamma - (1-\delta))u(\underline{c}^{-a}) + u'(\underline{c}^{-a})\left((r - \frac{\nu^2}{2}h)(\underline{W}^{-a}) + \frac{\nu^2}{2}\frac{\underline{c}^{-a}}{\psi}(\frac{1}{\gamma} + h)\right)\right]. \tag{B6}$$

From (24) the right-hand side of the same condition rewrites as

$$\Omega(\underline{W}^{-a}) = (\psi^a)^{-\gamma}\frac{(\omega_0 + \omega_1\underline{W}^{-a})^{1-\gamma}}{1-\gamma}. \tag{B7}$$

Substituting (B5) in (B6) and solving the value-matching condition (25) for the hard default threshold we obtain expression (27) for $\underline{W}^{-a}$ which is independent of δ and proves Lemma 3.

Appendix C Valuing Default Claim and Cost of Bailout $I(c)$

The price $P(c)$ of any default-contingent claim with payoff linked to consumption c and $\lim_{c\to\infty} P(c) = 0$, assuming complete markets, satisfies the no-arbitrage condition

$$rP(c) = \left(r - \rho - (1 - \frac{1}{\gamma})\frac{\nu^2}{2} - \frac{1-\delta}{Y'(c)}\right)\frac{1}{\gamma}cP'(c) + \frac{\nu^2}{2}(\frac{1}{\gamma}c)^2 P''(c). \tag{C1}$$

We conjecture that the general solution to equation (C1) is additively linear in compound functions of the form

$$P(c) = Ac^{-\gamma i}H(c), \tag{C2}$$

with constant A and coefficient i that satisfies the quadratic equation

$$\frac{\nu^2}{2}i^2 - \left(r - \rho - \frac{\nu^2}{2} - (1-\delta)\psi\right)i - r = 0, \tag{C3}$$

and $H(c)$ is a hypergeometric function. We have

$$\begin{aligned} P'(c) &= -\gamma i A c^{-\gamma i}H(c)/c + Ac^{-\gamma i}H'(c), \\ P''(c) &= -\gamma i(-\gamma i - 1)Ac^{-\gamma i}H(c)/c^2 - 2\gamma i A c^{-\gamma i}H'(c)/c + Ac^{-\gamma i}H''(c). \end{aligned}$$

This yields

$$\begin{aligned} &rH(c) = \left(r - \rho - (1-\tfrac{1}{\gamma})\tfrac{\nu^2}{2} - \tfrac{1-\delta}{Y'(c)}\right)\left(-iH(c) + \tfrac{1}{\gamma}H'(c)c\right) + \tfrac{\nu^2}{2}(\tfrac{1}{\gamma})^2[\gamma i(1+\gamma i)H(c) - 2\gamma i H'(c)c + H''(c)c^2], \\ \Leftrightarrow\ &\left[r + \left(r - \rho - (1+i)\tfrac{\nu^2}{2} - \tfrac{1-\delta}{Y'(c)}\right)i\right]H(c) = \left(r - \rho - (1 - \tfrac{1}{\gamma} + 2i)\tfrac{\nu^2}{2} - \tfrac{1-\delta}{Y'(c)}\right)\tfrac{1}{\gamma}cH'(c) + \tfrac{\nu^2}{2}(\tfrac{1}{\gamma}c)^2H''(c). \end{aligned}$$

We make the change of variables

$$z = 1 - \psi Y'(c) = -\frac{\gamma h(\frac{\underline{c}^-}{\psi} - \underline{W}^-)(\frac{c}{\underline{c}^-})^{-\gamma h}}{\frac{c}{\psi}} < 0, \tag{C4}$$

and let $H(c) = G(z)$, $H'(c) = G'(z)z'(c)$, $H''(c) = G''(z)z'(c)^2 + G'(z)z''(c)$, with $z(c) = 1 - \psi Y'(c) = -\psi Bc^{-\gamma h}/c$ and $B = \gamma h(\frac{\underline{c}^-}{\psi} - \underline{W}^-)(\frac{1}{\underline{c}^-})^{-\gamma h}$.

We can transform the ODE for $G(z)$ into a hypergeometric differential equation, where condition (C3) helps to simplify, as follows:

$$\begin{aligned} &\left[r + \left(r - \rho - (1+i)\tfrac{\nu^2}{2} - \tfrac{(1-\delta)\psi}{1-z}\right)i\right]G(z) = \\ &-\left(r - \rho - (1 + 2i + h + \tfrac{1}{\gamma})\tfrac{\nu^2}{2} - \tfrac{(1-\delta)\psi}{1-z}\right)(h + \tfrac{1}{\gamma})G'(z)z + \tfrac{\nu^2}{2}(h+\tfrac{1}{\gamma})^2G''(z)z^2 \\ \Leftrightarrow\ &\left[\underbrace{\left\{r + \left(r - \rho - (1+i)\frac{\nu^2}{2} - (1-\delta)\psi\right)i\right\}}_{0}\tfrac{1}{z} - \underbrace{\left\{r + \left(r - \rho - (1+i)\frac{\nu^2}{2}\right)i\right\}}_{(1-\delta)\psi i}\right]G(z) = \\ &-\left[\left(r - \rho - (1 + 2i + h + \tfrac{1}{\gamma})\tfrac{\nu^2}{2}\right)(1-z) - (1-\delta)\psi\right](h + \tfrac{1}{\gamma})G'(z) + \tfrac{\nu^2}{2}(h + \tfrac{1}{\gamma})^2G''(z)z(1-z) \text{ (by (C3))} \\ \Leftrightarrow\ &(1-\delta)\psi i G(z) - \left[r - \rho - (1 + 2i + h + \tfrac{1}{\gamma})\tfrac{\nu^2}{2} - (1-\delta)\psi - \left(r - \rho - (1 + 2i + h + \tfrac{1}{\gamma})\tfrac{\nu^2}{2}\right)z\right](h + \tfrac{1}{\gamma})G'(z) \\ &+ \tfrac{\nu^2}{2}(h + \tfrac{1}{\gamma})^2G''(z)z(1-z) = 0 \\ \Leftrightarrow\ &\frac{(1-\delta)\psi i}{\frac{\nu^2}{2}(h+\frac{1}{\gamma})^2}G(z) + \left[\frac{\frac{r}{i} + (i + h + \frac{1}{\gamma})\frac{\nu^2}{2}}{\frac{\nu^2}{2}(h + \frac{1}{\gamma})} + \frac{r - \rho - (1 + 2i + h + \frac{1}{\gamma})\frac{\nu^2}{2}}{\frac{\nu^2}{2}(h+\frac{1}{\gamma})}z\right]G'(z) + G''(z)z(1-z) = 0. \end{aligned}$$

We obtain the hypergeometric differential equation for $G(z)$:

$$-\alpha_1\alpha_2 G(z) + [\alpha_3 - (\alpha_1+\alpha_2+1)\, z]\, G'(z) + G''(z)z(1-z) = 0. \tag{C5}$$

The differential equation for $G(z)$ is of hypergeometric form and has the solution $G(z) = {}_2F_1(\alpha_1,\alpha_2;\alpha_3;z) = (1-z)^{-\alpha_1}$ ${}_2F_1(\alpha_1,\alpha_3-\alpha_2;\alpha_3;\frac{z}{z-1})$. The second equality follows from the Pfaff transformation of the hypergeometric function and guarantees that $\frac{z}{z-1}\in[0,1]$. The coefficients α_1,α_2,α_3 are the roots to the characteristic equations

$$\begin{aligned}
\alpha_3 &= \frac{\frac{r}{i} + (i+h+\frac{1}{\gamma})\frac{\nu^2}{2}}{\frac{\nu^2}{2}(h+\frac{1}{\gamma})},\\
\alpha_1\alpha_2 &= -\frac{(1-\delta)\psi i}{\frac{\nu^2}{2}(h+\frac{1}{\gamma})^2},\\
\alpha_1+\alpha_2 &= -\frac{r-\rho-(1+2i)\frac{\nu^2}{2}}{\frac{\nu^2}{2}(h+\frac{1}{\gamma})},
\end{aligned}$$

and $0 = \frac{\nu^2}{2}i^2 - \Big(r-\rho-\frac{\nu^2}{2}-(1-\delta)\psi\Big)\, i - r$. The coefficients of the hypergeometric functions are

$$\begin{aligned}
\alpha_1 &= \tfrac{i-h}{h+\frac{1}{\gamma}} = \tfrac{1}{2\frac{\nu^2}{2}(h+\frac{1}{\gamma})}\left(\sqrt{\Big(r-\rho-\tfrac{\nu^2}{2}-(1-\delta)\psi\Big)^2+2r\nu^2} - (1-\delta)\psi - \sqrt{\Big(r-\rho-\tfrac{\nu^2}{2}\Big)^2+2r\nu^2}\right),\\
\alpha_2 &= \tfrac{i-h}{h+\frac{1}{\gamma}} = \tfrac{1}{2\frac{\nu^2}{2}(h+\frac{1}{\gamma})}\left(\sqrt{\Big(r-\rho-\tfrac{\nu^2}{2}-(1-\delta)\psi\Big)^2+2r\nu^2} - (1-\delta)\psi + \sqrt{\Big(r-\rho-\tfrac{\nu^2}{2}\Big)^2+2r\nu^2}\right),\\
\alpha_3 &= \tfrac{i-\underline{i}}{h+\frac{1}{\gamma}}+1 = 1+\tfrac{1}{\frac{\nu^2}{2}(h+\frac{1}{\gamma})}\sqrt{\Big(r-\rho-\tfrac{\nu^2}{2}-(1-\delta)\psi\Big)^2+2r\nu^2}.
\end{aligned} \tag{C6}$$

To obtain the constant A for the default cost $I(c)$, we use that

$$\begin{aligned}
I(\underline{c}^-) &= A(\underline{c}^-)^{-\gamma i}H(\underline{c}^-),\\
I(\underline{c}^+) &= A(\underline{c}^+)^{-\gamma i}H(\underline{c}^+),\\
I(\underline{c}^-) &= (1-p)(T(\underline{W}^-)+I(\underline{c}^+)) + p(\kappa + W^a(\underline{W}^-) - \underline{W}^-).
\end{aligned}$$

Plugging in, we obtain

$$A = \frac{(1-p)T(\underline{W}^-) + p(\kappa + W^a(\underline{W}^-) - \underline{W}^-)}{(\underline{c}^-)^{-\gamma i}H(\underline{c}^-) - (1-p)(\underline{c}^+)^{-\gamma i}H(\underline{c}^+)}. \tag{C7}$$

Appendix D Proof of Proposition 4

Assume that $\kappa > \underline{\kappa}$ and that is optimal for the bailout agency to offer $T = 0$. We will show that there is a one-shot deviation that reduces the agency's expenses. If the agency offers a positive bailout once, and commits to no bailout in the future, its objective reads as

$$\min_{\underline{c}^+\geq\underline{c}^-} J(\underline{c}^+) \equiv ((1-p)\left[Y(\underline{c}^+) - \underline{W}^- + \mathbf{Def}(\underline{c}^+)\left(\kappa+\omega_0-(1-\omega_1)\underline{W}^-\right)\right] + p\left(\kappa+\omega_0-(1-\omega_1)\underline{W}^-\right) \tag{D1}$$

The first term is the discounted value of the agency's costs, when it extends an offer $T = Y(\underline{c}^+) - \underline{W}^-$ once and commits to $T = 0$ in future pleads by the borrower, which leads to a hard default. The second term shows the cost of hard default in case the offer by the agency is not successful, which happens with probability p.

Fixing $\underline{W}^-, \underline{c}^-$,

$$\lim_{\underline{c}^+\downarrow\underline{c}^-} J'(\underline{c}^+) = (1-p)\left[Y'(\underline{c}^-) + \mathbf{Def}'(\underline{c}^-)\left(\kappa+\omega_0-(1-\omega_1)\underline{W}^-\right)\right] \leq (1-p)\left[Y'(\underline{c}^-) + \mathbf{Def}'(\underline{c}^-)\left(\underline{\kappa}+\omega_0-(1-\omega_1)\underline{W}^-\right)\right] = 0 \tag{D2}$$

which contradicts the fact that $T = 0$ is optimal.

Instead, assume now that $\kappa \leq \underline{\kappa}$ and that the agency's optimal offer is $T > 0$. The present value of the agency's costs at the optimal transfer satisfies

$$
\begin{aligned}
& I(\underline{c}^-) = (1-p)\left[T + \mathbf{Def}(\underline{c}^+) I(\underline{c}^-)\right] + p\left(\kappa + \omega_0 - (1-\omega_1)\underline{W}^-\right) \\
\Leftrightarrow \qquad & I(\underline{c}^-) = \frac{(1-p)T + p\left(\kappa + \omega_0 - (1-\omega_1)\underline{W}^-\right)}{1-(1-p)\mathbf{Def}(\underline{c}^+)}.
\end{aligned} \tag{D3}
$$

For T to be optimal, the agency must prefer to offer T than make no offer at all:

$$
\begin{aligned}
& I(\underline{c}^-) - \left(\kappa + \omega_0 - (1-\omega_1)\underline{W}^-\right) \leq 0 \\
\Leftrightarrow \qquad & Y(\underline{c}^+) - \underline{W}^- + \mathbf{Def}(\underline{c}^+)\left(\kappa + \omega_0 - (1-\omega_1)\underline{W}^-\right) \leq \left(\kappa + \omega_0 - (1-\omega_1)\underline{W}^-\right).
\end{aligned} \tag{D4}
$$

That means the agency must prefer to offer $T > 0$ once than to offer zero transfer always. However, for $\kappa < \underline{\kappa}$ the agency prefers to offer zero than to offer $T > 0$ once, therefore $T > 0$ is not optimal.

Appendix E Optimal Bailout Scheme

First, any bailout offer $T > 0$ that makes the borrower indifferent between defaulting and not, i.e. $M(W+T) = \Omega(W+T)$ is dominated by an offer $T + \varepsilon$ for some $\varepsilon > 0$, so that the borrower does not ask for a new transfer immediately upon receiving T. Therefore the restriction $M(W+T) \geq \Omega(W+T)$ holds as a strict inequality and the associated Lagrange multiplier in the agency's cost minimization problem is zero. Second, by the one-shot deviation principle, the optimum in the bailout agency's problem is pinned down by considering deviations from the Subgame Perfect Nash Equilibrium in a single round of bailout negotiations, reverting to the equilibrium afterwards. In particular, assume that the borrower pleads for aid when its wealth reaches W, retaining threshold $\underline{W}$ for all future soft default events.

The bailout agency minimizes

$$
I(c(W)) = (1-p)(T + I(c(W+T)) + p(\kappa + W^a(W) - W) \tag{E1}
$$

Since in the future both the borrower and the agency will stick to the equilibrium,

$$
I(c(W+T) = \mathbf{Def}(c(W+T)) I(\underline{c}^-). \tag{E2}
$$

Writing $c^+ = c(W+T)$, the first-order condition of the agency reads as:

$$
\frac{\partial I(c(W);T)}{\partial T} = (1-p)\left[1 + \frac{\partial \mathbf{Def}(c^+)}{\partial c^+}\frac{\partial c^+}{\partial T} I(\underline{c}^-)\right] = 0, \tag{E3}
$$

or, equivalently,

$$
1 + \frac{\partial \mathbf{Def}(c^+)}{\partial c^+}\frac{1}{Y'(c^+)} I(\underline{c}^-) = 0 \tag{E4}
$$

The above condition shows that the target consumption level c^+ and, hence, the target wealth level $W^+ = W + T$ are independent of W. In equilibrium, $c^+ = \underline{c}^+$ and $W^+ = \underline{W}^+$, so that the contract curve offered by the bailout agency has the property $T'(W) = -1$ for all W.

From the SPC, $M'(\underline{W}^-) = u'(\underline{c}^-) = p\Omega'(\underline{W}^-) = p\omega_1(\psi^a)^{-\gamma}(\omega_0 + \omega_1 \underline{W}^-)^{-\gamma}$, the lower consumption threshold is

$$
\underline{c}^- = (p\omega_1)^{-\frac{1}{\gamma}} c^a(\underline{W}^-) = (p\omega_1)^{-\frac{1}{\gamma}} \psi^a(\omega_0 + \omega_1 \underline{W}^-), \tag{E5}
$$

or, equivalently, $\underline{W}^- = \frac{1}{\omega_1}((p\omega_1)^{\frac{1}{\gamma}} \frac{\underline{c}^-}{\psi^a} - \omega_0)$. Therefore, whenever a positive bailout is offered by the agency, it satisfies the agency's optimality condition, the borrower's smooth-pasting condition, and the value-matching condition.

Appendix F Proof of Theorem 2

Totally differentiate the value-matching condition (29) with respect to δ:

$$\begin{aligned}
&\frac{dM(\underline{W}^-)}{d\delta} = (1-p)\frac{dM(\underline{W}^- + T)}{d\delta} + p\frac{d\Omega(\underline{W}^-)}{d\delta} \\
\Leftrightarrow &\frac{\partial M(\underline{W}^-)}{\partial\delta} + M'(\underline{W}^-)\frac{d\underline{W}^-}{d\delta} = (1-p)\left[\frac{\partial M(\underline{W}^- + T)}{\partial\delta} + M'(\underline{W}^- + T)\frac{d\underline{W}^-}{d\delta}\right] + p\frac{d\Omega(\underline{W}^-)}{d\delta} \\
\Leftrightarrow &\frac{1}{\rho}u(\underline{c}^-) + M'(\underline{W}^-)\frac{d\underline{W}}{d\delta} = (1-p)\left[\frac{1}{\rho}u(\underline{c}^+) + M'(\underline{W}^- + T)\frac{d\underline{W}^-}{d\delta}\right] + p\left[\frac{\partial\Omega(\underline{W}^-)}{\partial\delta} + \Omega'(\underline{W}^-)\frac{d\underline{W}^-}{d\delta}\right].
\end{aligned} \tag{F1}$$

Substituting in (F1) the smooth-pasting condition (30),

$$\frac{1}{\rho}u(\underline{c}^-) = (1-p)\frac{1}{\rho}u(\underline{c}^+) + p\frac{\partial\Omega(\underline{W}^-)}{\partial\delta}. \tag{F2}$$

From (E5) and (24)

$$(\gamma-(1-\delta))u(\underline{c}^-) = \gamma-(1-\delta)\psi^\alpha(p\omega_1)^{\frac{\gamma-1}{\gamma}}\Omega(\underline{W}^-) = (\gamma-(1-\delta))^\gamma\,\Lambda u(W^\alpha(\underline{W}^-)), \tag{F3}$$

where $\Lambda = (\rho-(1-\gamma)(\mu-\gamma\frac{\sigma^2}{2}))^{1-\gamma}(p\omega_1)^{\frac{\gamma-1}{\gamma}}$.
Solving (F2) for $\underline{c}^-$,

$$u'(\underline{c}^-) = ((1-\gamma)u(\underline{c}^-))^{-\frac{\gamma}{1-\gamma}} = (\gamma-(1-\delta))^\gamma\left((1-\gamma)\Lambda u(W^\alpha(\underline{W}^-))\right)^{-\frac{\gamma}{1-\gamma}} \tag{F4}$$

Substituting (F3) in (F2) and rearranging,

$$\begin{aligned}
&(1-p)u(\underline{c}^+) = \psi^\alpha(p\omega_1)^{\frac{\gamma-1}{\gamma}}\Omega(\underline{W}^-) - p\rho\frac{\partial\Omega(\underline{W}^-)}{d\delta} = \left[\psi^\alpha(p\omega_1)^{\frac{\gamma-1}{\gamma}} + p\rho\gamma\frac{1}{\psi^\alpha}\frac{\partial\psi^\alpha}{\partial\delta}\right]\Omega(\underline{W}^-) = \\
&\frac{1}{\gamma-(1-\delta)}\left[(\rho-(1-\gamma)(\mu-\gamma\frac{\sigma^2}{2}))(p\omega_1)^{\frac{\gamma-1}{\gamma}} - p\rho\gamma\right]\Omega(\underline{W}^-) \\
\Rightarrow &(1-p)u(\underline{c}^+) = \left(\frac{1}{\gamma-(1-\delta)}\right)^{1-\gamma} Ku(W^\alpha(\underline{W}^-)) \\
\Rightarrow &(1-p)(\gamma-(1-\delta))u(\underline{c}^+) = (\gamma-(1-\delta))^\gamma\,Ku(W^\alpha(\underline{W}^-)),
\end{aligned} \tag{F5}$$

where $K = \left[(\rho-(1-\gamma)(\mu-\gamma\frac{\sigma^2}{2}))(p\omega_1)^{\frac{\gamma-1}{\gamma}} - p\rho\gamma\right](\rho-(1-\gamma)(\mu-\gamma\frac{\sigma^2}{2}))^{-\gamma}$.
Solving (F5) for $\underline{c}^+$,

$$u'(\underline{c}^+) = ((1-\gamma)u(\underline{c}^+))^{-\frac{\gamma}{1-\gamma}} = (\gamma-(1-\delta))^\gamma\left(\frac{1-\gamma}{1-p}Ku(W^\alpha(\underline{W}^-))\right)^{-\frac{\gamma}{1-\gamma}}. \tag{F6}$$

From (E5) and (9),

$$\frac{\underline{c}^-}{\psi} = (p\omega_1)^{-1/\gamma}\frac{\psi^\alpha}{\psi}W^\alpha(\underline{W}^-) = (p\omega_1)^{-1/\gamma}\frac{\rho-(1-\gamma)(\mu-\gamma\frac{\sigma^2}{2})}{\rho-(1-\gamma)(r+\frac{1}{\gamma}\frac{\sigma^2}{2})}W^\alpha(\underline{W}^-) \tag{F7}$$

Solving (F5) for $\underline{c}^+$ and dividing by ψ,

$$\frac{\underline{c}^+}{\psi} = \frac{1}{\psi}\left(\frac{1}{\gamma-(1-\delta)}\right)\left(\frac{(1-\gamma)Ku(W^\alpha(\underline{W}^-))}{(1-p)}\right)^{1/(1-\gamma)} = \left(\rho-(1-\gamma)(r+\frac{1}{\gamma}\frac{\sigma^2}{2})\right)\left(\frac{(1-\gamma)Ku(W^\alpha(\underline{W}^-))}{(1-p)}\right)^{1/(1-\gamma)} \tag{F8}$$

Evaluating the myopic value function at $W = \underline{W}^-$, substituting (F3) and (F4), and factoring out $(\gamma - (1-\delta))^\gamma$,

$$\rho M(\underline{W}^-) = (\gamma - (1-\delta))^\gamma \left[\Lambda u(W^\alpha(\underline{W}^-)) + ((1-\gamma)\Lambda u(W^\alpha(\underline{W}^-)))^{-\frac{\gamma}{1-\gamma}} \left(\left(r - \frac{\nu^2}{2}h\right)\underline{W}^- + \frac{\nu^2}{2}\frac{\underline{c}^-}{\psi}\left(\frac{1}{\gamma}+h\right)\right)\right] \tag{F9}$$

Evaluating the myopic value function at $W = \underline{W}^- + T$,

$$\rho(1-p)M(\underline{W}^- + T) =$$
$$(\gamma-(1-\delta))^\gamma \left[Ku(W^\alpha(\underline{W}^- + T)) + (1-p)\left(\frac{1-\gamma}{1-p}Ku(W^\alpha(\underline{W}^-))\right)^{-\frac{\gamma}{1-\gamma}} \left(\left(r - \frac{\nu^2}{2}h\right)(\underline{W}^- + T) + \frac{\nu^2}{2}\frac{\underline{c}^+}{\psi}\left(\frac{1}{\gamma}+h\right)\right)\right] \tag{F10}$$

Rewrite the borrower's value in autarky as

$$\Omega(\underline{W}^-) = (\gamma - (1-\delta))^\gamma((\rho - (1-\gamma)(\mu - \gamma\frac{\sigma^2}{2})))^{-\gamma}u(W^\alpha(\underline{W}^-)). \tag{F11}$$

Divide the value-matching condition (29) by $(\gamma - (1-\delta))^\gamma$ and substitute (F3), (F4), (F5) and (F6):

$$\Lambda u(W^\alpha(\underline{W}^-)) + ((1-\gamma)\Lambda u(W^\alpha(\underline{W}^-)))^{-\frac{\gamma}{1-\gamma}} \left(\left(r - \frac{\nu^2}{2}h\right)\underline{W}^- + \frac{\nu^2}{2}\frac{\underline{c}^-}{\psi}\left(\frac{1}{\gamma}+h\right)\right) =$$
$$Ku(W^\alpha(\underline{W}^- + T)) + (1-p)\left(\frac{1-\gamma}{1-p}Ku(W^\alpha(\underline{W}^-))\right)^{-\frac{\gamma}{1-\gamma}} \left(\left(r - \frac{\nu^2}{2}h\right)(\underline{W}^- + T) + \frac{\nu^2}{2}\frac{\underline{c}^+}{\psi}\left(\frac{1}{\gamma}+h\right)\right) +$$
$$\rho((\rho - (1-\gamma)(\mu - \gamma\frac{\sigma^2}{2})))^{-\gamma}u(W^\alpha(\underline{W}^-)). \tag{F12}$$

From (F7) and (F8) $\frac{\underline{c}^-}{\psi}$ and $\frac{\underline{c}^+}{\psi}$ are both invariant to δ. Therefore (F12) is an expression invariant to δ which shows that $\underline{W}^-$ does not change with δ given T.

Appendix G Characterizing wealth dynamics

Wealth dynamics are given by

$$\begin{aligned} dW_t &= [W_t(r + \theta_t(\mu - r)) - c_t]dt + \sigma W_t \theta_t dw_t \\ &= [rY(c_t) - (1 - \frac{\nu^2}{\gamma}Y'(c_t))c_t]dt + \frac{\nu}{\gamma}Y'(c_t)c_t dw_t, \end{aligned} \tag{G1}$$

since

$$\theta_t = -\frac{u'(c_t)}{u''(c_t)}\frac{Y'(c_t)}{Y(c_t)}\frac{\nu}{\sigma}. \tag{G2}$$

Proof of Lemma 6: From $m(c) = rY(c) - (1 - \frac{\nu^2}{\gamma}Y'(c))c$ we have $m'(c) = (r + \frac{\nu^2}{\gamma})Y'(c) - 1 + \frac{\nu^2}{\gamma}Y''(c)c = (r - \nu^2 h)\gamma h\left(\frac{\underline{c}^-}{\psi} - \underline{W}^-\right)\left(\frac{c^{-\gamma h - 1}}{\underline{c}^{-\gamma h}}\right) - \frac{\psi - (r + \frac{\nu^2}{\gamma})}{\psi}$ and $m''(c) = (r + \frac{2\nu^2}{\gamma})Y''(c) + \frac{\nu^2}{\gamma}Y'''(c)c = -(r - \nu^2 h)\gamma h(\gamma h + 1)\left(\frac{\underline{c}^-}{\psi} - \underline{W}^-\right)\left(\frac{c^{-\gamma h - 2}}{\underline{c}^{-\gamma h}}\right)$. Therefore the sign of $m''(\cdot)$ is that of $\nu^2 h - r$. Since $m(\cdot)$ is globally concave or convex, $m'(\cdot)$ admits at most one root. Furthermore, $m(c) = r(\frac{c}{\psi} - \left(\frac{\underline{c}^-}{\psi} - \underline{W}^-\right)\left(\frac{c}{\underline{c}^-}\right)^{-\gamma h}) - c + \frac{\nu^2}{\gamma}(\frac{c}{\psi} + \gamma h\left(\frac{\underline{c}^-}{\psi} - \underline{W}^-\right)\left(\frac{c}{\underline{c}^-}\right)^{-\gamma h}) = (r - \psi + \frac{\nu^2}{\gamma})\frac{c}{\psi} - (r - \nu^2 h)\left(\frac{\underline{c}^-}{\psi} - \underline{W}^-\right)\left(\frac{c}{\underline{c}^-}\right)^{-\gamma h}$. Therefore if $K < 0$, $m(c) < 0$, $m'(c) > 0$ whenever $r > \nu^2 h$ and $m(c) > 0$, $m'(c) < 0$ whenever $r > \nu^2 h$. If $K > 0$ then $m(c) = 0$ yields c_0 as the unique root and $m'(c) = 0$ yields c_1 as the unique root.

Proof of Proposition 6: The cases corresponding to $K < 0$ and are shown in the proof of Lemma 6. If $K > 0$, we can easily show that the following equivalences hold:

i. $c_0 > c_1 \Leftrightarrow \underline{c}^- < (\gamma h)^{\frac{1}{\gamma h}}$

ii. $c_0 > \underline{c}^- \Leftrightarrow \underline{W}^- < \frac{K-1}{\psi K}\underline{c}^- = \underline{W}_0$

iii. $c_1 > \underline{c}^- \Leftrightarrow \underline{W}^- < \frac{\gamma h K \underline{c}^- - (\underline{c}^-)^{1+\gamma h}}{\gamma h K \psi} = \underline{W}_1$

iv. $c_0 > c_1 \Leftrightarrow \underline{W}_0 < \underline{W}_1$

Consider the ordering of $\underline{W}_0, \underline{W}_1, \underline{W}^-$. The case $\underline{W}_1 < \underline{W}^- < \underline{W}_0$ is impossible since it implies $\underline{c}^- < c_0 < c_1 < \underline{c}^-$. The case $\underline{W}_0 < \underline{W}^- < \underline{W}_1$ is also impossible since it implies $\underline{c}^- < c_1 < c_0 < \underline{c}^-$.

With respect to the remaining possibilities

a) if $\underline{W}^- < \underline{W}_0 < \underline{W}_1$ then $\underline{c}^- < c_1 < c_0$. If $r > \nu^2 h$, $m'(c) \geq 0$ iff $c \leq c_1$, which implies $m'(c_0) < 0$. Since c_0 is the unique root of $m(.)$, $m(c) \geq 0$ iff $c \leq c_0$ which implies a trap. Symmetrically, if $r < \nu^2 h$, $m'(c) \geq 0$ iff $c \geq c_1$, which implies $m'(c_0) > 0$ and $m(c) \geq 0$ iff $c \geq c_0$ which implies a local spiral.

b) if $\underline{W}_0 < \underline{W}_1 < \underline{W}^-$ then $c_1 < c_0 < \underline{c}^-$. If $r > \nu^2 h$ then $m'(c) < 0$ and $m(c) \leq m(\underline{c}^-) \leq m(c_0) = 0$ for every $c \geq \underline{c}^-$ which implies a global spiral. Symmetrically, if $r < \nu^2 h$ then $m'(c) > 0$ and $m(c) \geq m(\underline{c}^-) \geq m(c_0) = 0$ for every $c \geq \underline{c}^-$ which implies global growth.

c) if $\underline{W}^- < \underline{W}_1 < \underline{W}_0$ then $\underline{c}^- < c_0 < c_1$. If $r > \nu^2 h$ then $m'(c) \geq 0$ iff $c \leq c_1$,which implies $m'(c_0) > 0$. Since $c_0$is the unique root of $m(.)$, $m(c_0) \geq 0$ iff $c \geq c_0$ which implies a local spiral. If $r < \nu^2 h$ then $m'(c) \geq 0$ iff $c \geq c_1$,which implies $m'(c_0) < 0$. Since $c_0$is the unique root of $m(.)$, $m(c_0) \geq 0$ iff $c \leq c_0$ which implies a trap.

d) if $\underline{W}_1 < \underline{W}_0 < \underline{W}^-$ then $c_0 < c_1 < \underline{c}^-$. If $r > \nu^2 h$ then $m'(c_0) \leq m'(c_1) = 0$ which means $m(.)$ is positive in the right neighborhood of c_0. Since c_0 is the unique root of $m(.)$, $m(c) \geq m(c_0) = 0$ for every $c \geq c_0$, which implies global growth in the domain $(\underline{c}^-, \infty)$.

Collecting cases a)-d) we have the results i)-iv) of the proposition.

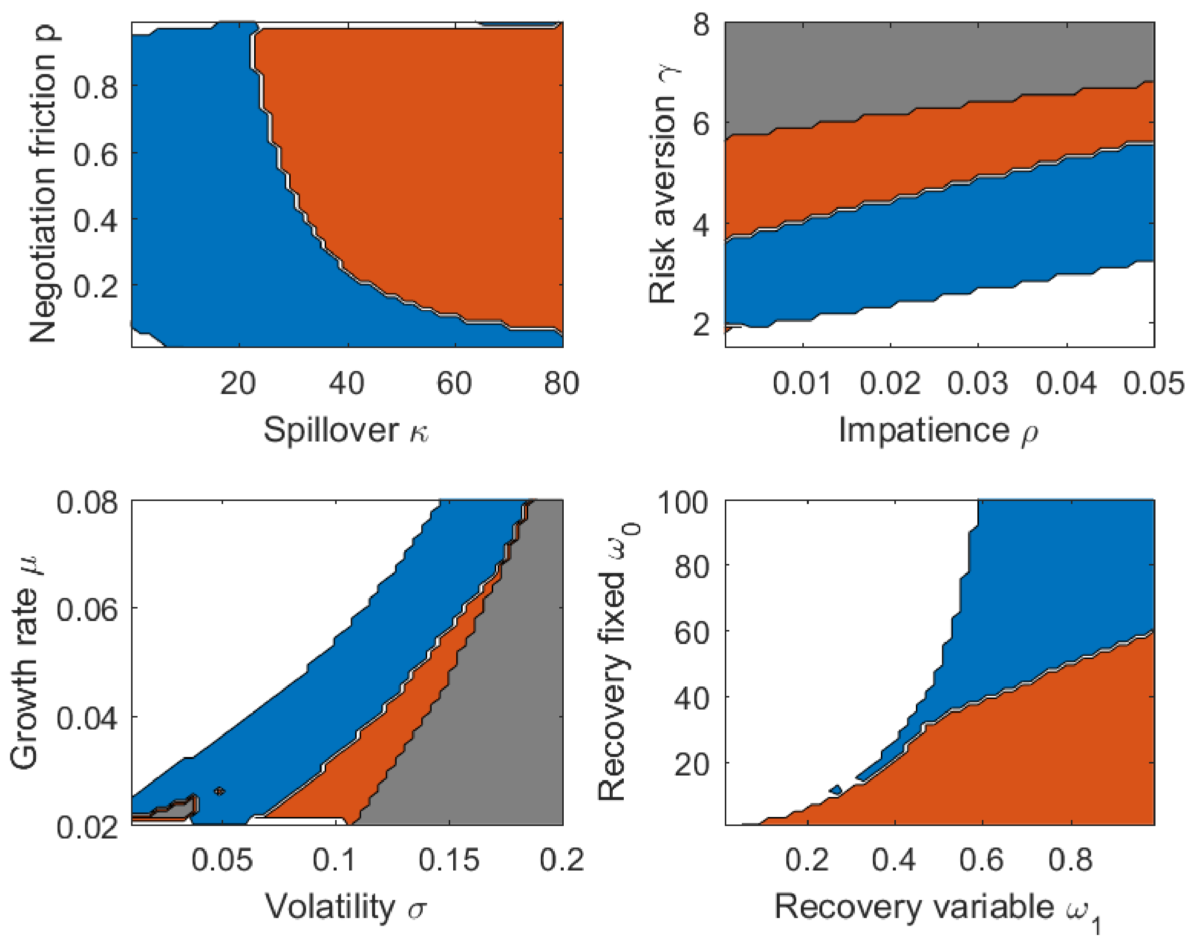

Figure 5: **Is myopia punished or rewarded?** The figure documents when myopia gets punished in equilibrium (blue region) and when it gets rewarded (red region). The white and grey region indicate no bailout in equilibrium. Myopia gets punished when the equilibrium transfer is smaller under myopia than under rationality, $T(\underline{W}^{-};\delta \approx 0) < T(\underline{W}^{-};\delta = 1)$. Myopia gets rewarded when the equilibrium transfer is larger under myopia than under rationality, $T(\underline{W}^{-};\delta \approx 0) > T(\underline{W}^{-};\delta = 1)$. We start with the base case $(\gamma, \rho, r, \omega_0, \omega_1, p, \mu, \sigma, \kappa) = (5, 0.02, 0.015, 1, 0.5, 0.2, 0.025, 0.05, 10)$, and in each plot we vary two parameters. The top left figure varies spillover cost κ and renegotiation friction p. The top right figure varies impatience ρ and risk aversion γ. The bottom left figure varies volatility σ and growth rate μ. The bottom right figure varies the recovery parameters ω_0 and ω_1.

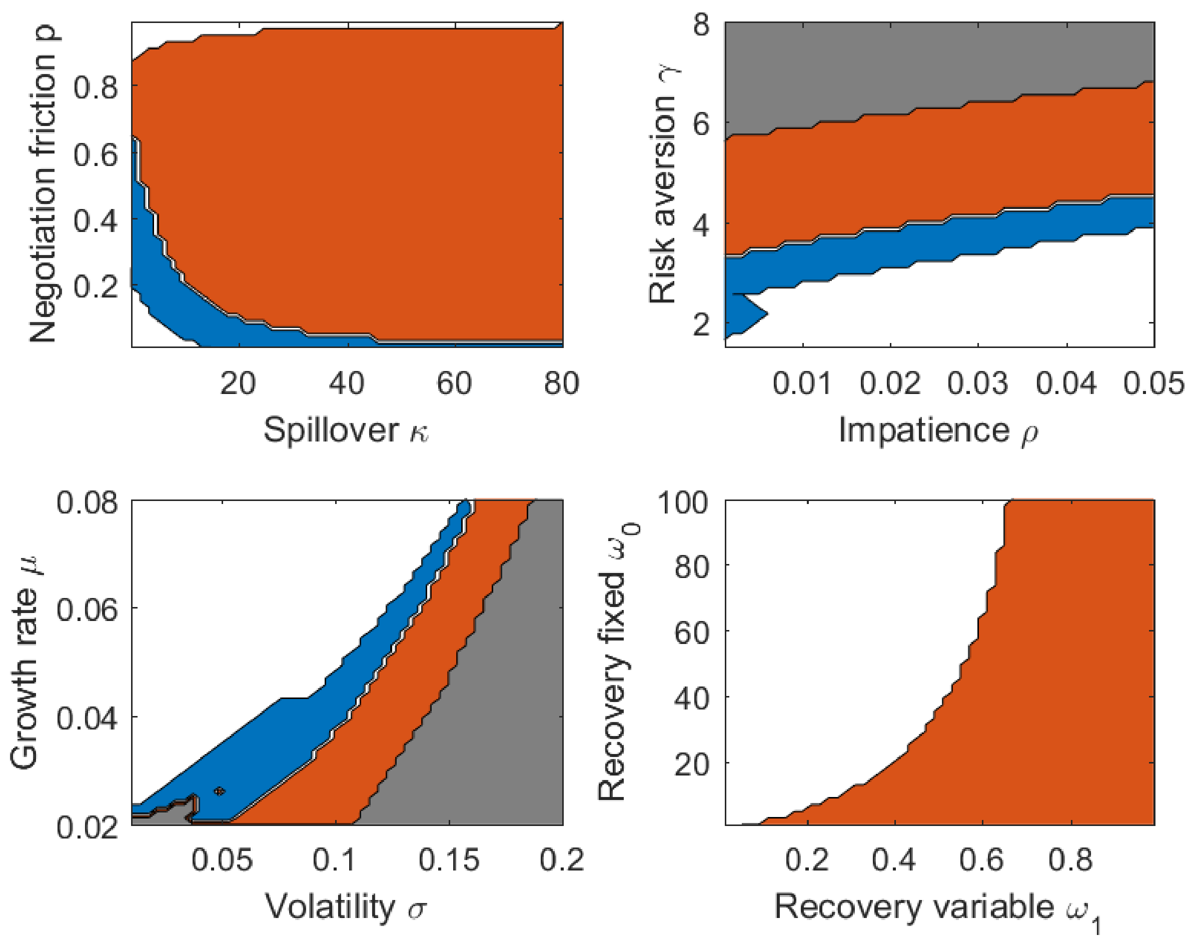

Figure 6: **Does myopia lead to early or late default?** The figure documents when myopia leads to early default (red region) and when it leads to late default (blue region). The white and grey region indicate no bailout in equilibrium and, hence, a hard default occurs at $\underline{W}^a$. Soft default occurs early under myopia when it occurs earlier than hard default ($\underline{W}^- > \underline{W}^a$). Soft default occurs late under myopia when it occurs later than hard default ($\underline{W}^- < \underline{W}^a$). We start with the base case $(\gamma, \rho, r, \omega_0, \omega_1, p, \mu, \sigma, \kappa) = (5, 0.02, 0.015, 1, 0.5, 0.2, 0.025, 0.05, 10)$, and in each plot we vary two parameters. The top left figure varies spillover cost κ and renegotiation friction p. The top right figure varies impatience ρ and risk aversion γ. The bottom left figure varies volatility σ and growth rate μ. The bottom right figure varies the recovery parameters ω_0 and ω_1.

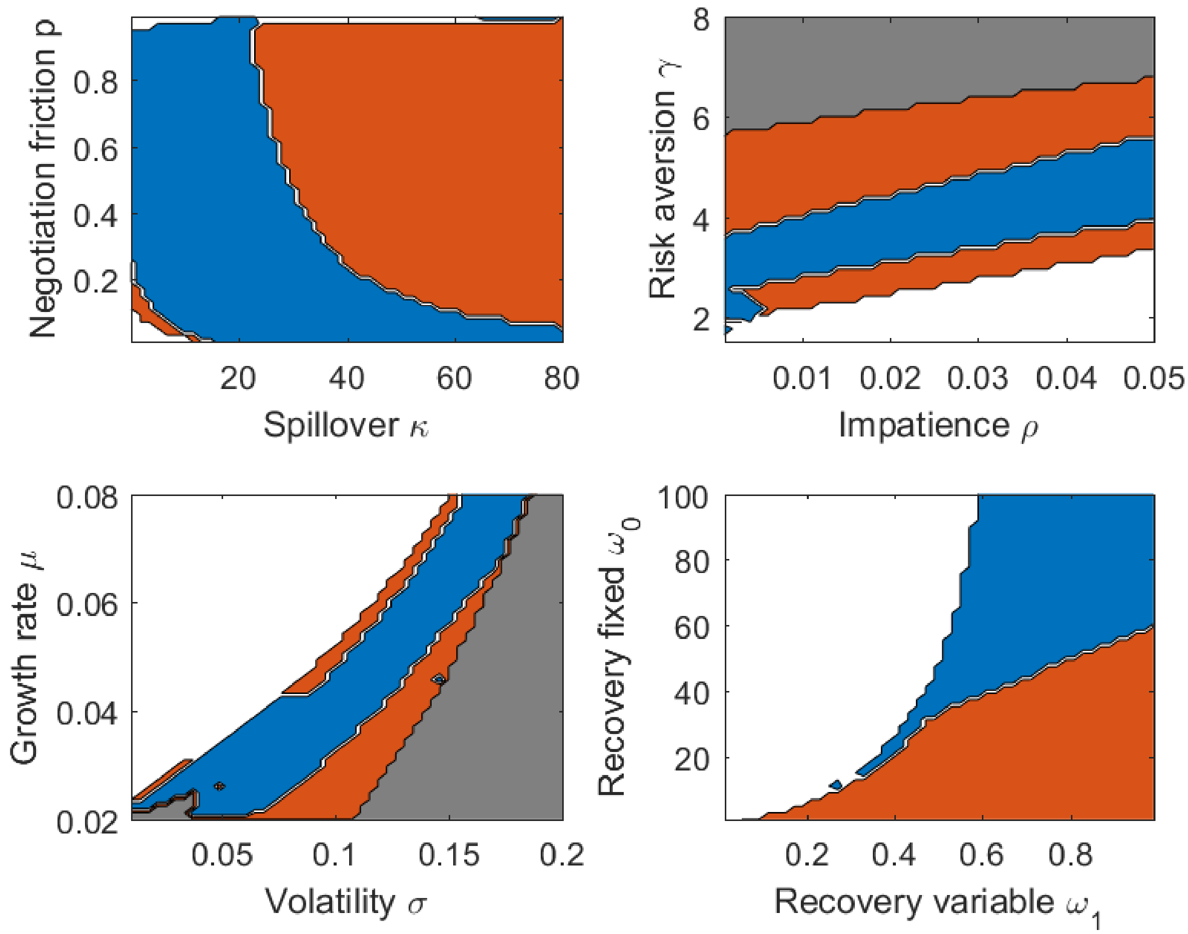

Figure 7: **When does myopia lead to procrastinated default?** The figure documents when myopia leads to procrastinated soft default (blue region) and when it leads to accelerated soft default (red region). The white and grey region indicate no bailout in equilibrium and, hence, a hard default occurs at $\underline{W}^a$. Default is procrastinated under myopia when it occurs later than rational default ($\underline{W}^-(\delta \approx 0) < \underline{W}^-(\delta = 1)$). Default is accelerated under myopia when it occurs earlier than rational default ($\underline{W}^-(\delta \approx 0) > \underline{W}^-(\delta = 1)$). We start with the base case $(\gamma, \rho, r, \omega_0, \omega_1, p, \mu, \sigma, \kappa) = (5, 0.02, 0.015, 1, 0.5, 0.2, 0.025, 0.05, 10)$, and in each plot we vary two parameters. The top left figure varies spillover cost κ and renegotiation friction p. The top right figure varies impatience ρ and risk aversion γ. The bottom left figure varies volatility σ and growth rate μ. The bottom right figure varies the recovery parameters ω_0 and ω_1.

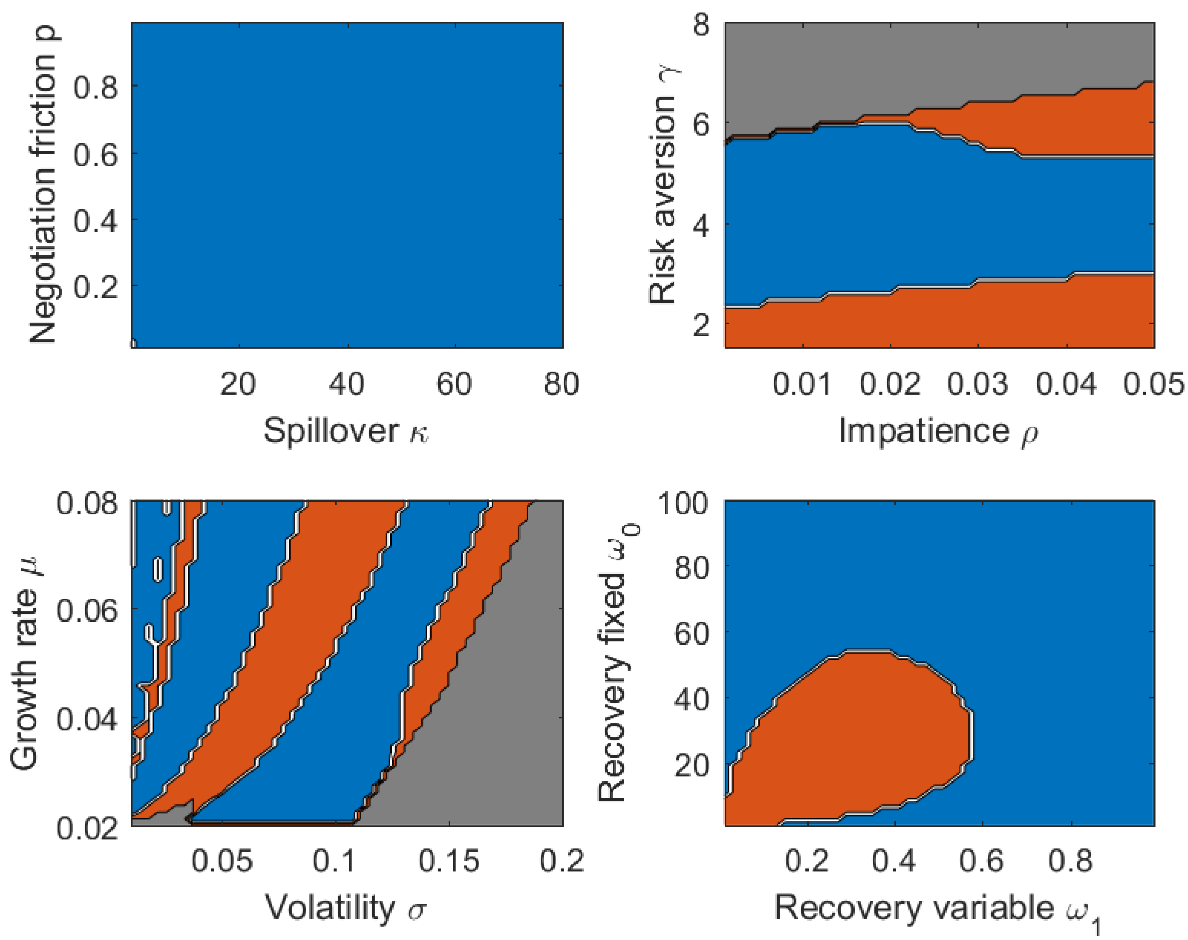

Figure 8: **Is rational or myopic default cheaper to resolve?** The figure documents when myopia is more expensive to resolve than rational default (red region) and when it is cheaper to resolve (blue region). Myopia is more expensive to resolve than rational default when $I(c(\underline{W}^-); \delta \approx 0) > I(c(\underline{W}^-); \delta = 1)$. Myopia is cheaper to resolve than rational default when $I(c(\underline{W}^-); \delta \approx 0) < I(c(\underline{W}^-); \delta = 1)$. We start with the base case $(\gamma, \rho, r, \omega_0, \omega_1, p, \mu, \sigma, \kappa) = (5, 0.02, 0.015, 1, 0.5, 0.2, 0.025, 0.05, 10)$, and in each plot we vary two parameters. The top left figure varies spillover cost κ and renegotiation friction p. The top right figure varies impatience ρ and risk aversion γ. The bottom left figure varies volatility σ and growth rate μ. The bottom right figure varies the recovery parameters ω_0 and ω_1.

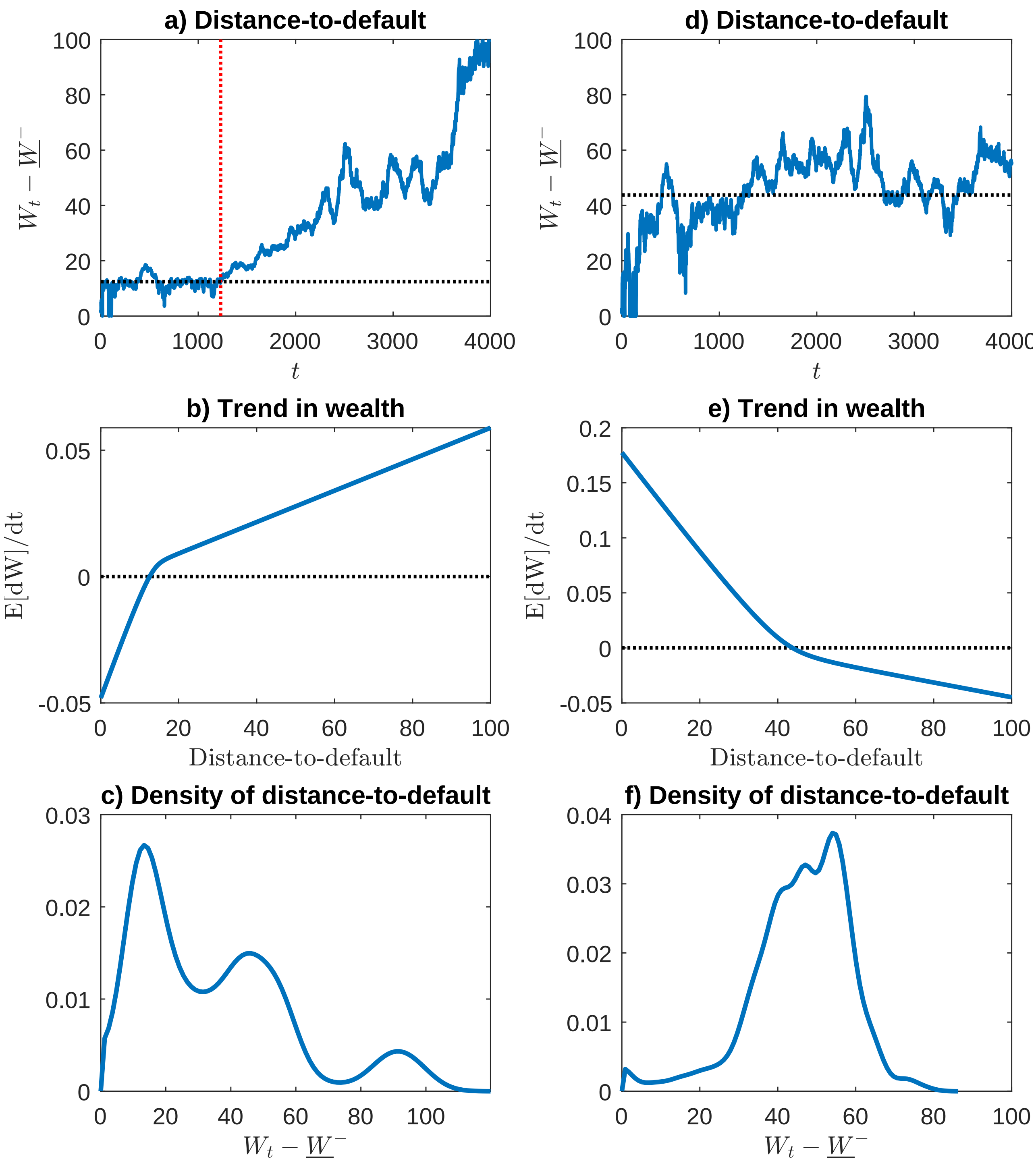

Figure 9: **Local spiral vs. trap.** Plots a)-c) in the left column correspond to the case of a local spiral. Plot a) shows a sample path of the distance-to-default $W_t - \underline{W}^-$. The dotted black line is the distance to default for which the trend in wealth is zero. The dotted red line shows the time of graduation, i.e. the last time in which wealth crosses the level for which the trend in wealth is zero. Plot b) shows the trend in wealth as a function of the distance-to-default. Plot c) shows the density of the distance to default corresponding to the path shown in plot a). Plots d)-f) show the corresponding plots for the case of a global trap. Paths in a)-d) are conditional on no hard default taking place.

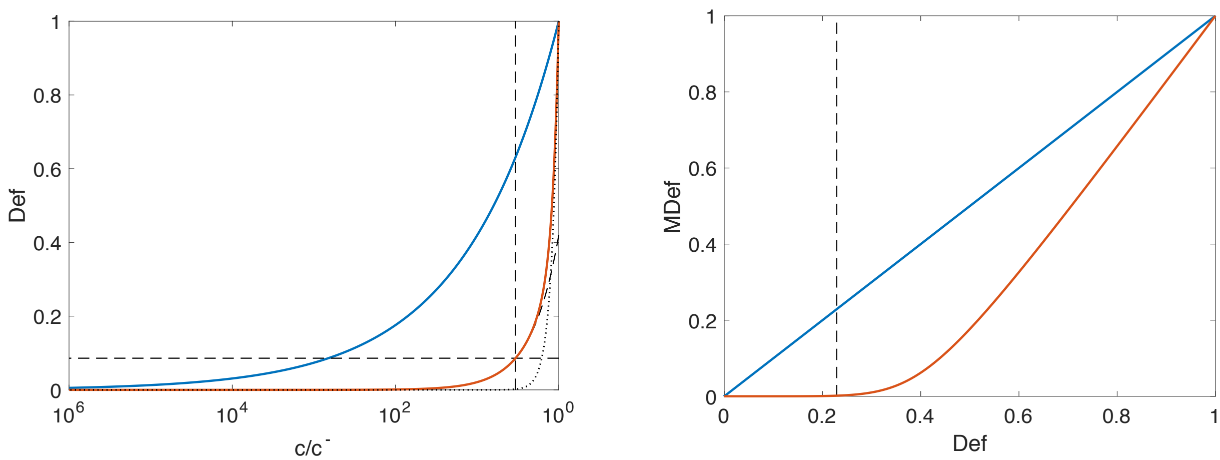

Figure 10: **Credit spread dynamics and credit risk metrics.** The left figure shows the price of a default-contingent claim, **Def**(c), as a function of the distance-to-default. The blue line assumes that the borrower is rational ($\delta = 1$). The red line assumes that the borrower is myopic ($\delta \approx 0$). The right figure plots the price of a default-contingent claim, **Def**(c), against the private valuation by a myopic borrower of a state-contingent claim that pays one unit when default occurs, **MDef**(c). The blue line assumes that the borrower is rational ($\delta = 1$). The red line assumes that the borrower is myopic ($\delta \approx 0$).

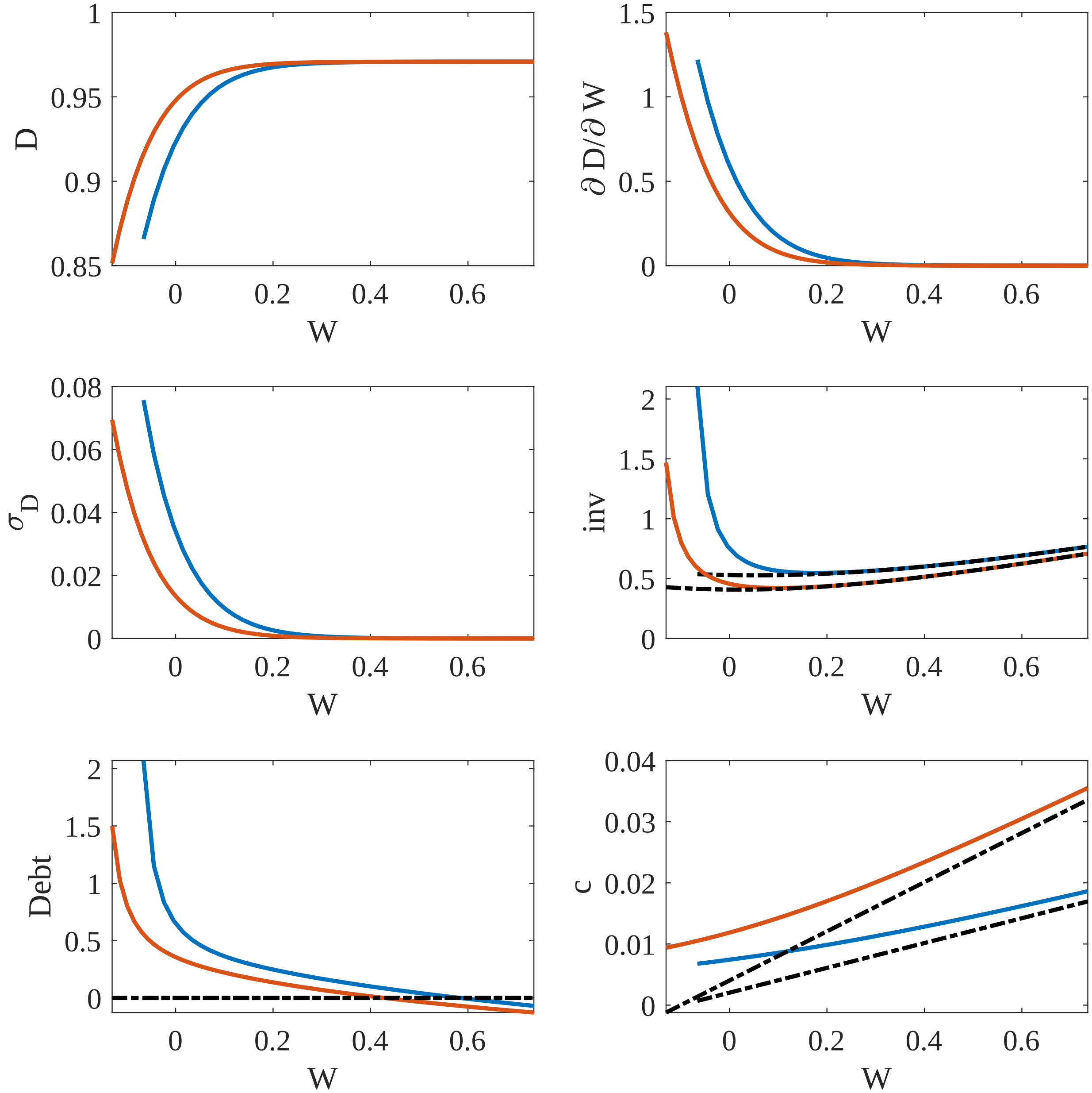

Figure 11: **Risky debt.** This figure shows bond prices $D(W)$, the sensitivity of bond prices to wealth $D'(W)$, the volatility of bond returns σ_D, the investment in the risky asset $\theta(W+e)$, debt issuance $-(1-\theta)(W+e)$ and the level of consumption c as a function of wealth. The blue line corresponds to a patient rational borrower ($\delta = 1$). The red line corresponds to a myopic borrower ($\delta \approx 0$). The dotted black lines in the investment figure ignore the hedging demand for the risky asset, setting $\sigma_D = 0$. The dotted black lines in the consumption figure ignore the value of default option and set $c = \psi(W+e)$.

Internet Appendix

This Internet Appendix extends the model with risky long-term debt.

Assumptions in the model with risky debt: Let the Itô process for the value of the bond be

$$\frac{dD}{D} = \overline{\mu}_D dt + \sigma_D dt, \tag{IA1}$$

where $\overline{\mu}_D$ and σ_D are the expected capital gains and volatility of the bond contingent on the wealth of the borrower. When there is debt outstanding, we can characterize the bond pricing equation in terms of equilibrium consumption $c = c^*(W)$ by:

$$rD(c) = D'(c)c(\mu_c(c) - \nu\sigma_c(c)) + D''(c)\frac{\sigma_c(c)^2 c^2}{2} + m(1 - D(c)), \tag{IA2}$$

where $\mu_c(c)$ and $\sigma_c(c)$ are, respectively, the expected value and volatility of equilibrium consumption growth. The HJB equation (IA2) for bond prices is subject to two boundary conditions:

$$\lim_{c\to\infty} D(c) = \frac{m}{m+r}, \tag{IA3}$$

$$D(\underline{c}^-) = (1-p)D(\underline{c}^+) + p\frac{\mathcal{R}}{\Pi(\underline{c}^-)}, \tag{IA4}$$

The first boundary condition regulates the value of the bond as default becomes extremely unlikely. Recall that the borrower repurchases non-maturing bonds at market price. Non-tendering bondholders realize that the market price of the bond will stay below the riskless value $\frac{m}{m+r}$ even in future states when the borrower will become a saver ($\theta < 1$). The reason is that the bond matures at a random future instant and default arrives before maturity with positive probability. The second boundary condition specifies the bond value at the soft-default boundary, $\underline{c}^- = c^*(\underline{W}^-)$. In that case, if the borrower ends up in autarky, the creditors' total recovery is $\mathcal{R}$, which is distributed equally among each of the bonds outstanding, whose number totals to

$$\Pi(\underline{c}^-) = -\frac{(1-\theta(\underline{c}^-))(\underline{W}^- + e)}{D(\underline{c}^-)}. \tag{IA5}$$

The value of the bond at the soft-default boundary reflects the possibility of a bailout, in which case consumption increases to $\underline{c}^+ = c^*(\underline{W}^- + T(\underline{W}^-))$.

Budget dynamics: The borrower's wealth equals the sum of the risky investment and the bond, with negative positions in the bond representing debt issuance. Two cases arise: The first is when the borrower has savings in the risk-free bond, and the second captures the case with debt issuance.

Lemma 7 *For any $W > \underline{W}^-$, the wealth dynamics of the borrower are given by*

$$dW = (W+e)(r + \theta(\mu - r))dt + (W+e)\theta\sigma dw - cdt, \qquad \textit{if } (W+e)(1-\theta) > 0, \tag{IA6}$$

$$dW = (W+e)(\mu_D + \theta(\mu - \mu_D))dt + (W+e)(\sigma_D + \theta(\sigma - \sigma_D))dw - cdt, \quad \textit{otherwise,} \tag{IA7}$$

where $\mu_D = m(1/D_t - 1) + \overline{\mu}_D$ is the expected return on the bond.

Proof. Wealth dynamics in case $(W_t + e)(1 - \theta_t) > 0$ are standard. In case $(W_t + e)(1 - \theta_t) < 0$ there is borrowing with risky debt. We derive the wealth dynamics by first considering the discrete time case and then taking the limit as the length of the discrete time period h shrinks to zero. The derivation proceeds in similar steps as for the case when borrowing is not risky. Let A be the value of investment in the risky asset, Π the number of bonds outstanding at the start of the period, D be the price of new bonds issued, B the value of all bonds outstanding and $\underline{\Pi}_t$ be the number of bonds outstanding at the end of period t before the purchase of new bonds.

Since bonds mature at rate m,

$$\underline{\Pi}_t = \Pi_{t-h} exp(-mh). \tag{IA8}$$

Wealth and endowment are spent on investment in the risky asset and bonds

$$W_t + e = A_t + B_t = A_t + \Pi_t D_t. \tag{IA9}$$

If the sovereign consumes c_t per unit of time, then

$$c_t h = \frac{A_{t-h}}{X_{t-h}} X_t - A_t + (\Pi_{t-h} - \underline{\Pi}_t) + (\underline{\Pi}_t - \Pi_t) D_t. \tag{IA10}$$

On the right-hand side, the first term is the value of the investment in the risky asset, as of date t. The second term is the new investment in the risky asset. The term in the parenthesis that follows is the payoff from maturing bonds. The last term is the amount of new investment in bonds. The same equation rewrites as:

$$\begin{aligned} c_t h &= A_{t-h} \frac{\Delta X_t}{X_{t-h}} + A_{t-h} - A_t + \Pi_{t-h} (1 - exp(-mh)) + (\Pi_{t-h} exp(-mh) D_t - B_t \Rightarrow \\ &= A_{t-h} \frac{\Delta X_t}{X_{t-h}} + W_{t-h} - W_t + \Pi_{t-h} (1 - exp(-mh)) + B_{t-h} exp(-mh) \frac{\Delta D_t}{D_{t-h}} + B_{t-h}(exp(-mh) - 1). \end{aligned}$$

Since $\lim_{h \to 0} \frac{1 - exp(-mh)}{h} = m$, the continuous time limit is:

$$\begin{aligned} c_t dt &= A_t \frac{dX_t}{X_t} - dW_t + m\Pi_t dt + B_t \frac{dD_t}{D_t} - mB_t dt \Rightarrow \\ dW_t &= A_t \frac{dX_t}{X_t} + B_t \frac{dD_t}{D_t} + m(\Pi_t - B_t) dt - c_t dt \Rightarrow \\ &= (W_t + e_t)(\mu_D + \theta_t(\mu - \mu_D)) dt + (W_t + e_t)(\sigma_D + \theta(\sigma - \sigma_D)) dw - c_t dt. \end{aligned}$$

■

Instantaneous gratification with risky debt: The Hamilton-Jacobi-Bellman (HJB) equations in case of debt issuance ($\theta^* > 1$) for $M(W)$ and, respectively, $R(W)$ at any wealth level W above the default threshold $\underline{W}^-$ are given by[17]:

$$\rho M(W) - E(W) = \max_{c,\theta} u(c) + M'(W)[(W+e)(\mu_D(c) + \theta(\mu - \mu_D(c))) - c] + \frac{1}{2}M''(W)(W+e)^2(\sigma_D(c) + \theta(\sigma - \sigma_D(c)))^2, \quad \text{(IA11)}$$

$$\rho R(W) = u(c^*) + R'(W)[(W+e)(\mu_D(c^*) + \theta^*(\mu - \mu_D(c^*)) - c^*] + \frac{1}{2}R''(W)(W+e)^2(\sigma_D(c^*) + \theta^*(\sigma - \sigma_D(c^*)))^2. \quad \text{(IA12)}$$

Lemma 8 *Fix a default threshold* $\underline{W}$ *and let the value of the borrower upon entering hard default be* $\Omega(\underline{W}, \lambda)$. *Furthermore, assume there exists a finite limit* $\Omega(\underline{W})$ *such that* $\lim_{\lambda \to \infty} \Omega(\underline{W}, \lambda) = \Omega(\underline{W})$. *Then* $\lim_{\lambda \to \infty} M(W) - \delta R(W) = 0$ *and* $\lim_{\lambda \to \infty} E(W) + (1-\delta)u(c^*(W)) = 0$.

Proof.

Let $J(W) = M(W) - \delta R(W)$. Index the optimal consumption policy and the investment share by λ, to read as $c^*_\lambda(W)$ and $\theta^*_\lambda(W)$ respectively. Multiply both sides of (IA12) by δ and subtract from (IA11) to obtain

$$(\rho + \lambda)J(W) = (1-\delta)u(c^*_\lambda(W)) + J'(W)\left[(W+e)(\mu_D + \theta^*_\lambda(W)(\mu - \mu_D)) - c^*_\lambda(W)\right] + \frac{1}{2}J''(W)(W+e)^2(\sigma_D + \theta^*_\lambda(W)(\sigma - \sigma_D))^2.$$

From the Feynman-Kac formula:

$$J(W) = (1-\delta)E\left[\int_0^\infty e^{-(\rho+\lambda)t}u(c^*_\lambda(W_t))dt\right],$$

where $W_0 = W$ and the budget dynamics of Lemma (10) hold with $c = c^*_\lambda(W)$ and $\theta = \theta^*_\lambda(W)$.

If $\gamma \in (1/2, 1)$*:*

Select $q \in (1, \frac{1}{1-\gamma})$ and let $\underline{q}$ satisfy $\frac{1}{q} + \frac{1}{\underline{q}} = 1$. From Hölder's inequality:

$$\int_0^\infty |e^{-(\rho+\lambda)t}(u(c^*_\lambda(W_t)))|dt \le \left(\int_0^\infty e^{-\underline{q}\lambda t}dt\right)^{1/\underline{q}} \left(\int_0^\infty e^{-q\rho t}u^q(c^*_\lambda(W_t))dt\right)^{1/q} = \frac{(1-\widetilde{\gamma})^{1/q}}{(\underline{q}\lambda)^{1/\underline{q}}(1-\gamma)} \left(\int_0^\infty e^{-\widetilde{\rho}t}\widetilde{u}(c^*_\lambda(W_t))dt\right)^{1/q},$$

where $\widetilde{\rho} = q\rho, \widetilde{\gamma} = 1 - q(1-\gamma) \in (0,1)$ and $\widetilde{u}(c) = \frac{c^{q-q\gamma}}{q-q\gamma} = \frac{1}{1-\widetilde{\gamma}}c^{1-\widetilde{\gamma}} > 0$.

[17] In case there is saving in the risk free asset rather than bond issuance, then can we set $\mu_D(c) = r$ and $\sigma_D(c) = 0$.

Therefore

$$0 \leq E\int_0^{\infty} e^{-(\rho+\lambda)t}u(c^*_\lambda(W_t))dt \leq \frac{(1-\widetilde{\gamma})^{1/q}}{\left(\underline{q}\lambda\right)^{1/\underline{q}}(1-\gamma)}E\left[\int_0^{\infty} e^{-\widetilde{\rho}t}\widetilde{u}(c^*_\lambda(W_t))dt\right]^{1/q}.$$

From Jensen's inequality

$$E\left[\int_0^{\infty} e^{-\widetilde{\rho}t}\widetilde{u}(c^*_\lambda(W_t))dt\right]^{1/q} \leq \left[\int_0^{\infty} e^{-\widetilde{\rho}t}E(\widetilde{u}(c^*_\lambda(W_t)))dt\right]^{1/q}.$$

Let $\widetilde{R}(W,\widetilde{\gamma}) \equiv \sup\limits_{c,\theta}\int_0^\infty e^{-\widetilde{\rho}t}E(\widetilde{u}(c_t))dt$ subject to the constraint that the budget dynamics of Lemma (7) hold. Since $\int_0^\infty e^{-\widetilde{\rho}t}E(\widetilde{u}(c^*_\lambda(W_t)))dt \leq \widetilde{R}(W,\widetilde{\gamma})$:

$$\begin{aligned} 0 \;&\leq E\left[\int_0^{\infty} e^{-(\rho+\lambda)t}u(c^*_\lambda(W_t))dt\right] \leq \frac{(1-\widetilde{\gamma})^{1/q}}{\left(\underline{q}\lambda\right)^{1/\underline{q}}(1-\gamma)}\widetilde{R}^{1/q}(W,\widetilde{\gamma}) \Rightarrow \\ 0 \;&\leq J(W) \leq (1-\delta)\frac{(1-\widetilde{\gamma})^{1/q}}{\left(\underline{q}\lambda\right)^{1/\underline{q}}(1-\gamma)}\widetilde{R}^{1/q}(W,\widetilde{\gamma}). \end{aligned}$$

Taking limits on both sides as $\lambda \to \infty$ and noticing that $\widetilde{R}(W,\widetilde{\gamma})$ is invariant to λ, $\lim\limits_{\lambda\to\infty} J(W) = 0$.

If $\gamma > 1$*:*

First notice that $J(W) \leq 0$.

Defining as $\mathcal{T}_\lambda$ the stopping time of hard default, we have

$$J(W)/(1-\delta) = E\left[\int_0^{\mathcal{T}_\lambda} e^{-(\rho+\lambda)t}u(c^*_\lambda(W_t)dt\right] + E(e^{-(\rho+\lambda)\mathcal{T}_\lambda})\Omega(\underline{W},\lambda).$$

Since $\lim\limits_{c\downarrow 0} u'(c) = -\infty$ for $\gamma > 1$, there exists a strictly positive lower bound for consumption, i.e. $c^*_\lambda(W) \geq \underline{c}^*_\lambda > 0$ for every $W \geq \underline{W}$ and $\lim\limits_{\lambda\to\infty}\underline{c}^*_\lambda > 0$.

As a consequence $0 \geq J(W)/(1-\delta) \geq E\left(\frac{1-e^{-(\rho+\lambda)\mathcal{T}_\lambda}}{\rho+\lambda}\right)u(\underline{c}^*_\lambda) + E\left(e^{-(\rho+\lambda)\mathcal{T}_\lambda}\right)\Omega(\underline{W},\lambda)$

Taking limits on both sides as $\lambda \to \infty$, $\lim\limits_{\lambda\to\infty} J(W) = 0$.

If $\gamma = 1$*:*

Then $u(c) = log(c) \leq \frac{c^{1-\underline{\gamma}}}{1-\underline{\gamma}}$ for $\underline{\gamma} \in (0,1)$.

For this reason, $E\int_0^\infty e^{-(\rho+\lambda)t}u(c^*_\lambda(W))dt \leq E\int_0^\infty e^{-(\rho+\lambda)t}\frac{c^*_\lambda(W)^{1-\underline{\gamma}}}{1-\underline{\gamma}}dt \leq \frac{(1-\overline{\gamma})^{1/\widehat{q}}}{\left(\underline{\widehat{q}}\lambda\right)^{1/\underline{\widehat{q}}}(1-\underline{\gamma})}\widetilde{R}^{1/\widehat{q}}(W,\overline{\gamma})$, or

$$J(W) \leq (1-\delta)\frac{(1-\overline{\gamma})^{1/\widehat{q}}}{\left(\underline{\widehat{q}}\lambda\right)^{1/\underline{\widehat{q}}}(1-\underline{\gamma})}\widetilde{R}^{1/\widehat{q}}(W,\overline{\gamma}),$$

where $\widehat{q} \in (1, \frac{1}{1-\underline{\gamma}})$, $\frac{1}{\widehat{q}} + \frac{1}{\underline{q}} = 1$ and $\overline{\gamma} = 1 - \widehat{q}(1 - \underline{\gamma})$.

$\overline{\gamma} = 2\underline{\gamma} - 1$.

Following the analysis of the case $\gamma > 1$ above,

$$J(W)/(1-\delta) \geq E\left(\frac{1 - e^{-(\rho+\lambda)\mathcal{T}_\lambda}}{\rho + \lambda}\right) u(\underline{c}^*_\lambda) + E\left(e^{-(\rho+\lambda)\mathcal{T}_\lambda}\right) \Omega(\underline{W}, \lambda).$$

As before, we conclude $\lim_{\lambda\to\infty} J(W) = 0$.

Finally, multiplying (IA12) by δ, subtracting from (IA11) and taking the limit as $\lambda \to \infty$ we have

$$\lim_{\lambda\to\infty} E(W) + (1-\delta)u(c^*_\lambda(W)) = 0.$$

■

Wealth and consumption dynamics:

Lemma 9 *Consumption and wealth dynamics, when there is risky debt outstanding, are given respectively by:*

$$\frac{dW}{W+e} = \left(r - \frac{c}{Y(c)+e} + \frac{\nu^2}{\gamma}\frac{Y'(c)c}{Y(c)+e}\right)dt + \frac{\nu}{\gamma}\frac{Y'(c)c}{Y(c)+e}dw, \quad \text{(IA13)}$$

$$\frac{dc_t}{c_t} = \left(r - \rho + (1+\frac{1}{\gamma})\frac{\nu^2}{2} - \frac{1-\delta}{Y'(c_t)}\right)\frac{1}{\gamma}dt + \frac{\nu}{\gamma}dw_t. \quad \text{(IA14)}$$

Proof. Wealth dynamics with $\theta = \theta^*(W)$ and $c = c^*(W)$ are given by (IA13) since

$$(W_t + e)\theta = -\frac{u'(c_t)}{u''(c_t)}Y'(c_t)\frac{\nu}{\sigma - \sigma_D} - \frac{\sigma_D}{\sigma - \sigma_D}(Y(c_t) + e).$$

■

Expected return and volatility of risky bonds: The expected return and volatility of the bond are given by a no-arbitrage argument.

Lemma 10 *No arbitrage implies $\mu_D(c) = r + \nu\sigma_D(c)$ and $\sigma_D(c) = \frac{D'(c)c}{D(c)} \times \sigma_c(c)$.*

Proof. From Itô's lemma

$$dD = \left(D'(c)c\mu_c(c) + \frac{1}{2}D''(c)c^2\sigma_c^2(c)\right)dt + D'(c)c\sigma_c(c)dw.$$

Matching dt and dw terms with equation (IA1):

$$\overline{\mu}_D(c) = \frac{D'(c)c\mu_c(c) + \frac{1}{2}D''(c)c^2\sigma_c^2(c)}{D(c)} = r + m - m(\frac{1}{D(c)}) + \frac{\nu\sigma_c(c)}{D(c)}D'(c)c,$$

and

$$\sigma_D(c) = \frac{D'(c)c\sigma_c(c)}{D(c)}.$$

Therefore

$$\mu_D = \overline{\mu}_D + m(\frac{1}{D(c)} - 1) = r + \nu\sigma_c(c)\frac{D'(c)c}{D(c)} = r + \nu\sigma_D(c).$$

■

Debt pricing solution: The pricing equation for a default claim with random expiration at Poisson rate m is

$$(r+m)P_m(c) = \left(r - \rho - (1-\frac{1}{\gamma})\frac{\nu^2}{2} - \frac{1-\delta}{Y'(c)}\right)\frac{c}{\gamma}P'_m(c) + \frac{\nu^2}{2}(\frac{c}{\gamma})^2P''_m(c), \quad \text{(IA15)}$$

which is solved subject to the boundary conditions $P_m(\underline{c}^-) = 1$ and $\lim_{c\to\infty} P_m(c) = 0$.

This is the same problem as the default claim pricing problem of section 2, with the exception that the expiration rate m appears on the LHS of the pricing equation. We can show that

$$P_m(c) = \left(\frac{c}{\underline{c}^-}\right)^{-\gamma j}\frac{H_m(c)}{H_m(\underline{c}^-)},$$

where j is the positive root of the polynomial of x:

$$0 = \frac{\nu^2}{2}x^2 - \left(r - \rho - \frac{\nu^2}{2} - (1-\delta)\psi\right)x - r - m, \quad \text{(IA16)}$$

and

$$H_m(c) =_2 F_1(\beta_1, \beta_2, \beta_3; z(c)),$$

with

$$\beta_1 = \frac{j-h}{h+\frac{1}{\gamma}}. \quad \text{(IA17)}$$

$$\beta_1 + \beta_2 = -\frac{r - \rho - (1+2j)\frac{\nu^2}{2}}{\frac{\nu^2}{2}(h+\frac{1}{\gamma})}. \quad \text{(IA18)}$$

$$\beta_3 = \frac{\frac{r}{j} + (j+h+\frac{1}{\gamma})\frac{\nu^2}{2}}{\frac{\nu^2}{2}(h+\frac{1}{\gamma})}. \quad \text{(IA19)}$$

The negative solution $\underline{j}$ of (IA16) corresponds to the conjugate solution $P_m(c) = \left(\frac{c}{\underline{c}^-}\right)^{-\gamma\underline{j}}\frac{\underline{H}_m(c)}{\underline{H}_m(\underline{c}^-)}$ where $\underline{H}_m$ is the same ${}_2F_1$ function as H_m with β's evaluated at $\underline{j}$. This conjugate solution diverges and does not satisfy the boundary condition $\lim_{c\to\infty} P_m(c) = 0$.

To solve the debt boundary value problem first notice that the function $D(c) - \frac{m}{m+r}$ satisfies the default pricing equation (IA15). By the superposition principle, there exists coefficients κ_1, κ_2 such that

$$D(c) - \frac{m}{m+r} = \kappa_1 P_m(c) + \kappa_2 \underline{P}_m(c).$$

Since $\lim_{c\to\infty} D(c) - \frac{m}{m+r} = 0$ and $\underline{P}_m(c)$ diverges, we have $\kappa_2 = 0$. Solving for the boundary condition at the soft-default threshold we have

$$\kappa_1 = D(\underline{c}^-) - \frac{m}{m+r}.$$

The value of debt at the default threshold obtains by combining equations (IA5), (IA4) and (42) evaluated at $\underline{c}^+$:

$$D(\underline{c}^-) = \frac{(1-p)(1-P_m(\underline{c}^+))}{1-(1-p)P_m(\underline{c}^+) + \frac{pR}{(1-\theta(\underline{c}^-))(\underline{W}^- + e)}} \frac{m}{m+r}. \tag{IA20}$$

To find the investment share at the soft-default threshold, first notice that

$$\begin{aligned}
\sigma_D(\underline{c}^-) &= \frac{D'(\underline{c}^-)}{D(\underline{c}^-)}\underline{c}^- \frac{\nu}{\gamma} = -\left(\frac{m}{m+r} - D(\underline{c}^-)\right)\frac{P_m'(\underline{c}^-)}{D(\underline{c}^-)}\underline{c}^- \frac{\nu}{\gamma} \\
&= \frac{-p(1-\omega_1)(1-\theta(\underline{c}^-)) - p\omega_0(1-\omega_1)\theta(\underline{c}^-)}{(1-p)(1-P_m(\underline{c}^+))(1-\theta(\underline{c}^-))} P_m'(\underline{c}^-)\underline{c}^- \frac{\nu}{\gamma} \\
&= \underline{\sigma}_1 + \underline{\sigma}_2 \frac{\theta(\underline{c}^-)}{(1-\theta(\underline{c}^-))},
\end{aligned}$$

where $\underline{\sigma}_1 = -\frac{p(1-\omega_1)P_m'(\underline{c}^-)}{(1-p)(1-P_m(\underline{c}^+))}\underline{c}^- \frac{\nu}{\gamma}$ and $\underline{\sigma}_2 = \omega_0 \underline{\sigma}_1$. Furthermore,

$$\begin{aligned}
&\theta(\underline{c}^-) = \frac{Y'(\underline{c}^-)}{Y(\underline{c}^-)+e}\frac{\underline{c}^-}{\gamma}\frac{\nu}{\sigma - \sigma_D(\underline{c}^-)} - \frac{\sigma_D(\underline{c}^-)}{\sigma - \sigma_D(\underline{c}^-)} \Rightarrow \\
&\left(\sigma - \underline{\sigma}_1 - \underline{\sigma}_2 \frac{\theta(\underline{c}^-)}{(1-\theta(\underline{c}^-))}\right)(1-\theta(\underline{c}^-)) = \sigma - \frac{Y'(\underline{c}^-)\underline{c}^-}{Y(\underline{c}^-)+e}\frac{\nu}{\gamma} \Rightarrow \\
&\theta(\underline{c}^-) = \frac{\frac{Y'(\underline{c}^-)}{\underline{W}+e} + p(1-\omega_1)P_m'(\underline{c}^-)}{\sigma(1-p)(1-P_m(\underline{c}^+)) + p(1-\omega_1)(1-\omega_0)P_m'(\underline{c}^-)\underline{c}^- \frac{\nu}{\gamma}} \underline{c}^- \frac{\nu}{\gamma}.
\end{aligned}$$