Screening for breakthroughs*

Gregorio Curello
University of Bonn

Ludvig Sinander
Northwestern University

13 July 2021

Abstract

We identify a new dynamic agency problem: that of incentivising the *prompt* disclosure of productive information. To study it, we introduce a general model in which a technological breakthrough occurs at an uncertain time and is privately observed by an agent, and a principal must incentivise disclosure via her control of the agent's utility. We uncover a deadline structure of optimal mechanisms: they have a simple deadline form in an important special case, and a graduated deadline structure in general. We apply our results to the design of unemployment insurance schemes.

1 Introduction

Society advances by finding better ways of doing things. When such a technological breakthrough occurs, it frequently becomes known only to certain individuals with particular expertise. Only if such individuals share their knowledge promptly can the promise of progress be unlocked.

The resulting need to incentivise prompt disclosure engenders a new type of screening problem: one in which the agent's private information is about *when,* rather than about *what.* We call this *screening for breakthroughs.*

*We are deeply grateful to Eddie Dekel for many detailed comments. Sinander thanks him, Alessandro Pavan and Bruno Strulovici for their guidance and support. We have profited from comments and suggestions provided by Miguel Ballester, Lori Beaman, Iván Canay, Sylvain Chassang, Janet Currie, Théo Durandard, Piotr Dworczak, Andrew Ellis, Jeff Ely, Alex Frug, George Georgiadis, Brett Green, Faruk Gül, Yingni Guo, Marina Halac, Oliver Hart, Ian Jewitt, Alessandro Lizzeri, Guido Lorenzoni, Eric Maskin, Matt Notowidigdo, Wojciech Olszewski, Harry Pei, Dan Quigley, Wolfgang Pesendorfer, Ronny Razin, Wolfgang Ridinger, Marciano Siniscalchi, Can Ürgün, Chris Wallace, Asher Wolinsky, Leeat Yariv and audiences at Bonn, Harvard, LSE, Lund, Northwestern, Oxford, Princeton, *Seminars in Economic Theory* and the 2021 *REStud* Tour.

The need to screen for breakthroughs is pervasive. One example is the much-discussed problem of *talent-hoarding* in organisations (e.g. Lublin, 2017). The manager of a team is well-placed to know when one of her subordinates acquires a skill. When this happens, headquarters may wish to re-assign the worker to a new role better-suited to her abilities. Managers, however, have a documented tendency to want to hold on to their workers. Careful design is thus needed to incentivise prompt disclosure.

Another example is unemployment insurance: since unemployed workers are typically privately informed about when they receive a job offer, benefits must be designed with a view to incentivising them to accept employment. A third example concerns technical innovations that reduce firms' greenhouse-gas emissions, at the price of raising production costs.[1] Only with suitable regulation will firms which discover such innovations choose to adopt them.

In this paper, we study the general problem of screening for breakthroughs. We introduce a model in which an agent privately observes when a new productive technology arrives. This breakthrough expands utility possibilities for the agent and principal, but generates a conflict of interest between them. The agent decides whether and when to disclose the breakthrough, and the principal controls a payoff-relevant physical allocation over time.

We ask how the principal can best incentivise prompt disclosure of the breakthrough. Our answer uncovers a striking deadline structure of optimal mechanisms: only simple *deadline mechanisms* are optimal in an important special case, while a graduated deadline structure characterises optimal incentives in general. We apply these insights to the design of unemployment insurance schemes.

Our contribution is threefold. First, we identify a new dynamic agency problem that is pervasive in practice: that of incentivising the *prompt* disclosure of productive information. Secondly, we introduce a general model in which this agency problem may be studied in isolation from other complicating factors. In that model, we fully characterise optimal mechanisms, showing them to have a deadline structure. Finally, we develop a novel set of techniques suitable for the study of our new agency problem. We expect these tools to prove useful also for richer environments in which our dynamic agency problem interacts with other incentive issues.

[1] Such innovations are expected to account for the bulk of abatement in the cement industry, currently the source of about 7% of all CO_2 emissions (Czigler et al., 2020).

1.1 Overview of model and results

A breakthrough occurs at a random time, making available a new technology that expands utility possibilities for an agent and a principal. There is a conflict of interest: were the principal to operate the old and new technologies in her own interest, the agent would be better off under the old one. The agent privately observes when the breakthrough occurs, and (verifiably) discloses it at a time of her choosing. The principal controls a physical allocation that determines the agent's utility over time. (The description of a physical allocation may include a specification of monetary payments to the agent.)

To focus on the robust qualitative features of optimal screening, we allow for general technologies, and study *undominated* mechanisms, meaning those such that no alternative mechanism is weakly better for the principal under any arrival distribution of the breakthrough and strictly better under some. We further describe, for any given breakthrough distribution, the principal's optimal choice among undominated mechanisms.

Toward our deadline characterisation, we first study how undominated mechanisms incentivise the agent. We show that the agent should be indifferent at all times between prompt and delayed disclosure (Theorem 1). This is despite the fact that the standard argument fails: were the agent strictly to prefer prompt to delayed disclosure, then lowering the agent's post-disclosure utility would *not* necessarily benefit the principal.

We then elucidate the deadline structure of undominated mechanisms when the pre-breakthrough technology's utility possibilities have an affine shape. Theorem 2 asserts that in this case, all undominated mechanisms belong to a small class of simple *deadline mechanisms*. Absent disclosure, these mechanisms give the agent a Pareto-efficient utility u^0 before a deadline, and an inefficiently low utility $u^\star$ afterwards.[2] The proof of Theorem 2 argues (loosely) that any mechanism may be improved by *front-loading* the agent's pre-disclosure utility, making it higher early and lower late while preserving its total discounted value. We further characterise the principal's optimal choice of deadline as a function of the breakthrough distribution (Proposition 2).

Outside of the affine case, optimal mechanisms exhibit a graduated deadline structure (Theorem 3): absent disclosure, the agent's utility still starts at the efficient level u^0 and declines monotonically toward the inefficiently low level $u^\star$, but the transition may be gradual. For any given breakthrough distribution, we describe the optimal transition (Proposition 3).

We conclude by applying our results to the design of unemployment insurance schemes. An unemployed worker (agent) receives a job offer at a

[2] u^0 and $u^\star$ are functions of the technologies, so the deadline is the only free parameter.

random time, and chooses whether and when to accept. Offers are private, but the state (principal) observes when the worker accepts a job. The state controls the worker's consumption and labour supply through taxes and benefits, and cares both about the worker's welfare and net tax revenue.

Many countries, such as Germany and France, pay a generous unemployment benefit until a deadline, and provide only a low benefit to those remaining unemployed beyond this deadline. Our results provide a potential rationale for such deadline schemes: they are approximately optimal provided that either (a) the worker's consumption utility has limited curvature, or (b) tax revenue is comparatively unimportant for social welfare. Conversely, our analysis suggests that where neither (a) nor (b) is satisfied, substantial welfare gains could be achieved by tapering benefits gradually, as in Italy.

1.2 Related literature

This paper belongs to the literature on incentive design for a proposing agent, initiated by Armstrong and Vickers (2010).[3] In their (static) model, the agent privately observes which physical allocations are available, then proposes one (or several). The key assumptions are that

(a) the agent can propose only available allocations, and that

(b) the principal can implement only proposed allocations.

Our dynamic problem shares these key features: the new technology (a) can only be disclosed (proposed) once available, and (b) can be utilised by the principal only once disclosed.

Bird and Frug (2019) study a different dynamic environment with features (a) and (b). Payoffs are simple: there is an allocation α preferred by the principal and a default allocation favoured by the agent,[4] and the principal can furthermore reward the agent at a linear cost. In each period, the agent privately observes whether α is available; it can (a) be disclosed only if available, and (b) be implemented only if disclosed. Were rewards unrestricted, α could be implemented whenever available by rewarding the agent just enough to induce disclosure. (And this is optimal; thus there is no conflict of interest in our sense.) The authors instead subject promised rewards to a

[3] See also Nocke and Whinston (2013) and Guo and Shmaya (2021). Our account of the literature follows the latter authors' insightful discussion. The literature has precedents in applied work on corporate finance (Berkovitch & Israel, 2004) and antitrust (Lyons, 2003).

[4] There is an extension to multiple allocations α; little changes.

dynamic budget constraint,[5] and study how the budget should be spent over time. By comparison, we allow for general payoffs (technologies) and impose no dynamic constraints, focussing instead on a conflict of interest.

Feature (a) means that the agent's disclosures are verifiable, a possibility first studied by Grossman and Hart (1980), Milgrom (1981) and Grossman (1981). A strand of the subsequent literature examines the role of commitment in static models,[6] while another studies the timing of disclosure absent commitment;[7] our environment features both commitment and dynamics.[8] These models lack property (b): the agent cannot constrain the principal.

More distantly related is the large literature on dynamic adverse-selection models with cheap-talk communication (contrast with (a)) and no scope for the agent to constrain the principal's choice of allocation (contrast with (b)). The strand on dynamic 'delegation' allows for non-transferable utility, as we do;[9] otherwise the literature tends to focus on monetary transfers.[10] Green and Taylor (2016) show how moral hazard may be mitigated by conditioning pay and termination on self-reported progress,[11] while Madsen (2020) studies how monitoring can help to elicit progress reports.[12]

[5] They assume in particular that the agent can be rewarded only using exogenous reward 'opportunities', which arrive randomly over time; but nothing changes if rewards take other forms, e.g. (flow) monetary payments subject to a per-period cap.

[6] Particularly Glazer and Rubinstein (2004, 2006), Sher (2011), Hart, Kremer and Perry (2017) and Ben-Porath, Dekel and Lipman (2019).

[7] See Dye and Sridhar (1995), Acharya, DeMarzo and Kremer (2011), Guttman, Kremer and Skrzypacz (2014), Campbell, Ederer and Spinnewijn (2014) and Curello (2021). The last two papers feature 'breakthroughs', but these engender no conflict of interest in our sense; the incentive problem is instead that of deterring shirking.

[8] So does recent work on revenue management, where a firm contracts with customers who arrive unobservably over time and choose when verifiably to reveal themselves; see Pai and Vohra (2013), Board and Skrzypacz (2016), Mierendorff (2016), Garrett (2016, 2017), Gershkov, Moldovanu and Strack (2018) and Dilmé and Li (2019).

[9] See Jackson and Sonnenschein (2007), Frankel (2016), Guo (2016), Grenadier, Malenko and Malenko (2016), Li, Matouschek and Powell (2017), Lipnowski and Ramos (2020), Guo and Hörner (2020) and de Clippel, Eliaz, Fershtman and Rozen (2021).

[10] E.g. Roberts (1983), Baron and Besanko (1984), Courty and Li (2000), Battaglini (2005), Eső and Szentes (2007a, 2007b), Board (2007) and Pavan, Segal and Toikka (2014).

[11] In their model, the agent privately observes a signal indicating that a valuable (and observable) breakthrough is within reach. There is no conflict of interest in our sense; the difficulty is instead that of incentivising unobservable (breakthrough-hastening) effort. Although there is adverse selection, moral hazard is the focus: without it, the principal has no reason to elicit the signal. See also Feng, Taylor, Westerfield and Zhang (2021).

[12] In this paper, an agent privately observes when a project 'expires', and a principal chooses when to terminate the project. The agent prefers late termination, while the principal wishes to terminate at expiry. The focus is on how best to condition pay and termination on a noisy signal of expiry. (Were there no signal, non-trivial screening would

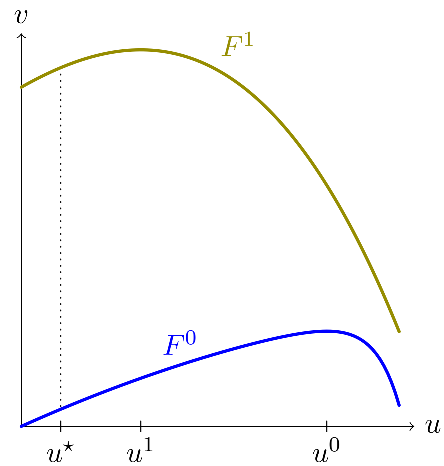


Figure 1: Utility possibility frontiers. The new technology expands utility possibilities ($F^1 \geq F^0$), but creates a conflict of interest ($u^1 < u^0$). $u^\star$ denotes the rightmost point to the left of u^0 at which F^0, F^1 have equal slopes.

1.3 Roadmap

We introduce the model in the next section, then formulate the principal's problem in §3. In §4, we show that undominated mechanisms incentivise the agent by keeping her always indifferent. We then describe the deadline structure of optimal mechanisms (§5 and §6). We conclude in §7 by applying our results to the design of unemployment insurance schemes.

2 Model

There is an agent and a principal, whose utilities are denoted by $u \in [0, \infty)$ and $v \in [-\infty, \infty)$, respectively. A frontier $F^0 : [0, \infty) \to [-\infty, \infty)$ describes utility possibilities: $F^0(u)$ is the highest utility that the principal can attain subject to giving the agent utility u. We assume that F^0 is concave and upper semi-continuous, that it has a unique peak $u^0 > 0$ (namely, $F^0(u^0) > F^0(u)$ for every $u \neq u^0$), and that it is finite on $(0, u^0]$. Such a frontier is depicted in Figure 1.

Time $t \in \mathbf{R}_+$ is continuous. The principal controls the agent's (flow) utility u (and thus her own utility $F^0(u)$) over time, and is able to commit.

We interpret this abstract description of utility possibilities in the standard fashion: there is an (unmodelled) set of feasible physical allocations over which the agent and principal have preferences.[13] This allows for a

be impossible since the agent's preferences are the same whatever her type [expiry date].)

[13]Formally, there is a set $\mathcal{A}^0$ of allocations, and the agent and principal have utility functions $U, V : \mathcal{A}^0 \to \mathbf{R}$. The frontier is defined by $F^0(u) := \sup_{a \in \mathcal{A}^0} \{V(a) : U(a) = u\}$,

broad range of applications. Allocations may be multi-dimensional, and some dimensions may correspond to observable actions taken by the agent. (The principal controls these by issuing action recommendations, backed by the threat of giving the agent zero utility forever unless she complies.) One dimension of the allocation may describe monetary payments to the agent; we discuss this possibility in the next section.

At a random time τ, a *breakthrough* occurs: a new technology becomes available which expands the utility possibility frontier to $F^1 \geq F^0$.[14] The new frontier is likewise concave and upper semi-continuous, with a unique peak denoted by u^1. The breakthrough engenders a conflict of interest: the new frontier peaks at a strictly lower agent utility ($u^1 < u^0$), so that the breakthrough would hurt the agent were the principal to operate both technologies in her own interest. This is illustrated in Figure 1.

The breakthrough is observed only by the agent. At any time $t \geq \tau$ after the breakthrough, she can verifiably disclose to the principal that it has occurred. (That is, she can *prove* that the new technology is available.)

The agent and principal discount their flow payoffs at rate $r > 0$ and have expected-utility preferences, so that their respective payoffs from random flow utilities $t \mapsto x_t$ and $t \mapsto y_t$ are

$$\mathbf{E}\left(r \int_0^\infty e^{-rt} x_t \mathrm{d}t \right) \quad \text{and} \quad \mathbf{E}\left(r \int_0^\infty e^{-rt} y_t \mathrm{d}t \right).$$

The random time τ at which the breakthrough occurs is distributed according to an arbitrary CDF G.

We write $u^\star$ for the rightmost $u \in [0, u^0]$ at which the old and new frontiers F^0, F^1 have equal slopes.[15] This utility level will feature prominently in our analysis. To avoid trivialities, we impose the weak genericity assumption that $u^\star$ is a strict local maximum of $F^1 - F^0$, rather than a saddle point.

2.1 A simple illustration

In the simplest applications, there are finitely many (old) allocations, and the agent privately observes when a new allocation becomes available. For example, a manager may observe when a member of her team acquires a skill, or a firm may discover an emissions-reducing innovation.

---

with $F^0(u) \coloneqq -\infty$ if there is no allocation $a \in \mathcal{A}^0$ such that $U(a) = u$.

[14]In the language of allocations (footnote 13), the breakthrough enlarges the set of available allocations to some $\mathcal{A}^1 \supset \mathcal{A}^0$, and thus increases the frontier pointwise.

[15]'Equal slopes' formally means that F^0, F^1 share a supergradient (see Rockafellar, 1970, part V). $u^\star$ is well-defined because at $u = 0$, both F^0 and F^1 admit ∞ as a supergradient.

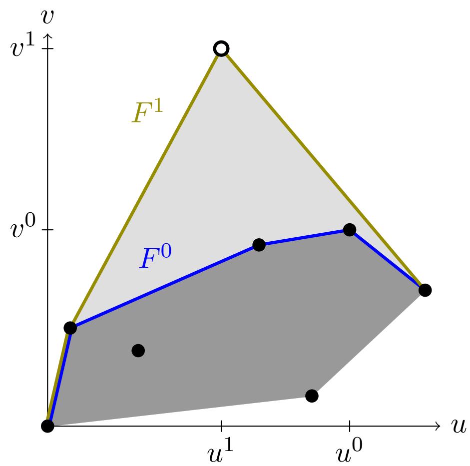


Figure 2: Finitely many allocations: the old (●), the new (○), and utility possibilities (grey).

Each allocation provides some utilities (u, v) to the agent and principal, which may be plotted as in Figure 2. The utility possibility set is the convex hull of these profiles,[16] and the frontier F^0 is its upper boundary.

The agent privately observes when a new allocation (u^1, v^1) becomes available. The principal likes the new allocation better than any other, whereas the agent prefers the principal's favourite old allocation (u^0, v^0). Thus utility possibilities expand, but there is a conflict of interest.

Richer applications feature (infinitely) many allocations. In our application to unemployment insurance (§7), for example, an allocation specifies the worker's consumption and (if she is employed) her labour supply.

2.2 Discussion of the assumptions

Two of our assumptions are economically substantive. First, the agent privately observes a technological breakthrough, but cannot utilise the new technology without the principal's knowledge. Many economic environments have this feature: in unemployment insurance, for instance, the state observes the worker's employment status (from e.g. tax records).

Secondly, there is a conflict of interest, captured by $u^1 < u^0$. Such conflicts arise naturally in applications: in unemployment insurance, for example, the state (principal) would like an employed worker (agent) to work and pay taxes, but the worker would rather not.[17]

[16] In-between profiles are achieved by rapidly switching back and forth (or randomising).

[17] Absent a conflict of interest, the principal can attain first-best (see Remark 1 below).

The remaining model assumptions are innocuous, as we next briefly relate. Further details are provided in supplemental appendix I.

Utility possibilities. The assumption that the frontiers are concave is without loss of generality: if one of them were not, then the principal could get arbitrarily close to any point on its concave upper envelope by rapidly switching back and forth between agent utility levels. Upper semi-continuity is similarly innocuous. The stipulation that $u^\star$ is a strict local maximum of $F^1 - F^0$ essentially just rules out a saddle point, and is anyway dispensable.

Not every agent utility $u \in [0, \infty)$ need be feasible: if no physical allocation provides utility u, then we let $F^j(u) := -\infty$, ensuring that u is never chosen by the principal. Our assumption that F^0 is finite on $\left(0, u^0\right]$ is without loss.

We have required the agent's flow utility u to be non-negative, meaning that there is a bound (normalised to zero) on how much misery the principal can inflict on the agent. This assumption may be replaced with a participation constraint without affecting our results.

Distribution. The distribution G of the breakthrough time is completely unrestricted: it can have atoms, for example, and need not have full support.

Monetary transfers. As mentioned, our formalism allows for monetary transfers. The conflict-of-interest assumption rules out unrestricted transfers from the agent to the principal,[18] but is consistent with arbitrary payments *to* the agent. Our analysis thus applies whenever the agent is protected by limited liability, a common assumption in contract theory.

Uncertain technology. Our analysis applies unchanged if the new frontier F^1 is random, provided the agent does not have private information about its realisation.

Cheap talk. Nothing changes if the agent's disclosures are non-verifiable, provided the principal observes her own payoffs in real time, since she can then verify cheap-talk reports at negligible cost.[19]

2.3 Mechanisms and incentive-compatibility

A *mechanism* specifies, for each period t, the flow utility x_t^0 that the agent enjoys at t if she has not yet disclosed, as well as the continuation utility X_t^1 that she earns by disclosing at t. Formally, a mechanism is a pair $\left(x^0, X^1\right)$,

[18] That would make both frontiers downward-sloping, with peaks $u^0 = u^1 = 0$.

[19] Following a report, the principal can provide utility u^1 for a short time, earning $F^1(u^1)$ if the breakthrough really did occur and $F^0(u^1) < F^1(u^1)$ if not.

where $x^0 : \mathbf{R}_+ \to \mathbf{R}_+$ is Lebesgue-measurable and X^1 is a function $\mathbf{R}_+ \to [0, \infty]$. We call x^0 the *pre-disclosure flow*, and X^1 the *disclosure reward.*

Note that the description of a mechanism does not specify what utility flow $s \mapsto x_s^{1,t}$ the agent enjoys after disclosing at t, only its present value

$$X_t^1 = r \int_t^{\infty} e^{-r(s-t)} x_s^{1,t} \mathrm{d}s.$$

Nor does the definition specify which technology is used when both are available. These omissions do not matter for the agent's incentives, so we shall address them when we formulate the principal's problem (next section).

A mechanism is *incentive-compatible ('IC')* iff the agent prefers disclosing promptly to (a) disclosing with a delay or (b) never disclosing. Formally:

Definition 1. A mechanism (x^0, X^1) is *incentive-compatible ('IC')* iff for every period $t \in \mathbf{R}_+$,

(a) $X_t^1 \geq r \int_t^{t+d} e^{-r(s-t)} x_s^0 \mathrm{d}s + e^{-rd} X_{t+d}^1$ for every $d > 0$, and

(b) $X_t^1 \geq r \int_t^{\infty} e^{-r(s-t)} x_s^0 \mathrm{d}s$.

By a revelation principle, we may restrict attention to incentive-compatible mechanisms. (See supplemental appendix J for details.)

Remark 1. Although we have not yet stated the principal's problem, it is clear that her first-best is the mechanism $(x^0, X^1) \equiv (u^0, u^1)$, which fails to be incentive-compatible due to the conflict of interest ($u^1 < u^0$). If there were no conflict of interest ($u^1 \geq u^0$), then the first-best would be IC.

In the sequel, we equip the set $\mathbf{R}_+$ of times with the Lebesgue measure, so that a 'null set of times' means a set of Lebesgue measure zero, and 'almost everywhere (a.e.)' means 'except possibly on a null set of times'.

Observe that two IC mechanisms (x^0, X^1) and $(x^{0\dagger}, X^1)$ which differ only in that $x^0 \neq x^{0\dagger}$ on a null set are payoff-equivalent.[20] For this reason, we shall not distinguish between such mechanisms in the sequel, instead treating them as identical.[21]

[20] x^0 enters payoffs as $\mathbf{E}_G\left(\int_0^{\tau} e^{-rt} x_t^0 \mathrm{d}t\right)$ and $\mathbf{E}_G\left(\int_0^{\tau} e^{-rt} F^0\left(x_t^0\right) \mathrm{d}t\right)$, respectively. Modifying x^0 on a null set has no effect on the integrals, and thus leaves both players' payoffs unchanged, no matter what the breakthrough distribution G.

[21] We term such (x^0, X^1) and $(x^{0\dagger}, X^1)$ *versions* of each other. A mechanism is really an equivalence class: a maximal set whose every element is a version of every other.

3 The principal's problem

In this section, we formulate the principal's problem, and define undominated and optimal mechanisms. We then derive an upper bound on the agent's utility in undominated mechanisms.

3.1 After disclosure

To determine the principal's payoff, we must fill in the gaps in the definition of a mechanism. So fix a mechanism (x^0, X^1), and suppose that the agent discloses at time t. For each of the remaining periods $s \in [t, \infty)$, the principal must determine

(1) which technology (F^0 or F^1) will be used, and

(2) what flow utility $x_s^{1,t}$ the agent will enjoy.

Part (1) is straightforward: the principal is always (weakly) better off using the new technology.

For (2), the principal must choose a (measurable) utility flow $x^{1,t} : [t, \infty) \to [0, \infty)$ subject to providing the agent with the continuation utility specified by the mechanism:

$$r \int_t^\infty e^{-r(s-t)} x_s^{1,t} \mathrm{d}s = X_t^1.$$

She chooses so as to maximise her post-disclosure payoff

$$r \int_t^\infty e^{-r(s-t)} F^1\left(x_s^{1,t}\right) \mathrm{d}s.$$

Since the frontier F^1 is concave, the constant flow $x^{1,t} \equiv X_t^1$ is optimal.

Parts (1) and (2) together imply that the principal earns a flow payoff of $F^1(X_t^1)$ forever following a time-t disclosure in a mechanism (x^0, X^1).

3.2 Undominated and optimal mechanisms

The principal's payoff from an incentive-compatible mechanism (x^0, X^1) is

$$\Pi_G\left(x^0, X^1\right) := \mathbf{E}_G\left(r \int_0^\tau e^{-rt} F^0\left(x_t^0\right) \mathrm{d}t + e^{-r\tau} F^1\left(X_\tau^1\right)\right),$$

where the expectation is over the random breakthrough time $\tau \sim G$.[22] Her problem is to maximise her payoff by choosing among IC mechanisms.

[22] To allow for $X_\tau^1 = \infty$, extend F^1 upper semi-continuously to $[0, \infty]$ (so $F^1(\infty) = -\infty$).

A basic adequacy criterion for a mechanism is that it not be *dominated* by another mechanism, by which we mean that the alternative mechanism is weakly better under every distribution and strictly better under at least one distribution. Formally:

Definition 2. Let (x^0, X^1) and $(x^{0\dagger}, X^{1\dagger})$ be incentive-compatible mechanisms. The former *dominates* the latter iff

$$\Pi_G\left(x^0, X^1\right) \geq (>)\, \Pi_G\left(x^{0\dagger}, X^{1\dagger}\right) \quad \text{for every (some) distribution } G.$$

An IC mechanism is *undominated* iff no IC mechanism dominates it.

Domination is a distribution-free concept: the principal (weakly) prefers a dominating mechanism no matter what her belief G about the likely time of the breakthrough.

Definition 3. An incentive-compatible mechanism is *optimal* for a distribution G iff it maximises Π_G and is undominated.

We show in supplemental appendix K that undominated and optimal mechanisms exist.

3.3 An upper bound on the agent's utility

Absent incentive concerns, the principal never wishes to give the agent utility strictly exceeding u^0, since both frontiers are downward-sloping to the right of u^0. The principal could use utility promises in excess of u^0 as an incentive tool, however. This is never worthwhile:

Lemma 0. Any undominated incentive-compatible mechanism (x^0, X^1) satisfies $x^0 \leq u^0$ almost everywhere.

Proof. Let (x^0, X^1) be an IC mechanism in which $x^0 > u^0$ on a non-null set of times. Consider the alternative mechanism $(\min\{x^0, u^0\}, X^1)$ in which the agent's pre-disclosure flow is capped at u^0. This mechanism dominates the original one: its pre-disclosure flow is lower, strictly on a non-null set, and the frontier F^0 is strictly decreasing on $[u^0, \infty)$. And it is incentive-compatible: prompt disclosure is as attractive as in the original (IC) mechanism, and disclosing with delay (or never disclosing) is weakly less attractive since the agent earns a lower flow payoff $\min\{x^0, u^0\} \leq x^0$ while delaying. ■

4 Keeping the agent indifferent

In this section, we describe how undominated mechanisms incentivise the agent. This result is a stepping stone to the deadline characterisation of undominated mechanisms that we develop in next two sections.

To formulate the agent's problem in a mechanism (x^0, X^1), let X^0_t denote the period-t present value of the remainder of the pre-disclosure flow x^0:

$$X^0_t \coloneqq r \int_t^\infty e^{-r(s-t)} x^0_s \mathrm{d}s.$$

In a period t in which the agent has observed but not yet disclosed the breakthrough, she chooses between

- disclosing promptly (payoff X^1_t),
- disclosing with any delay $d > 0$ (payoff $X^0_t + e^{-rd}(X^1_{t+d} - X^0_{t+d})$), and
- never disclosing (payoff X^0_t).

Incentive-compatibility demands precisely that the agent weakly prefer the first option. Our first theorem asserts that in an undominated mechanism, she must in fact be indifferent between all three options:

Theorem 1. Any undominated incentive-compatible mechanism (x^0, X^1) satisfies $X^0 = X^1$.

That is, the reward X^1_t for disclosure must equal the present value $X^0_t = r \int_t^\infty e^{-r(s-t)} x^0_s \mathrm{d}s$ of the remainder of the pre-disclosure flow x^0.

A naïve intuition for Theorem 1 is that, were the agent strictly to prefer prompt disclosure in some period t, the principal could reduce her disclosure reward X^1_t without violating IC. The trouble with this idea is that if $X^1_t \leq u^1$, then lowering X^1_t would *hurt* the principal (refer to Figure 1 on p. 6). This is no mere quibble, for (as we shall see) undominated mechanisms will spend time in $[0, u^1]$. More broadly, in a general dynamic environment, it is not clear that IC ought to bind everywhere.

The proof is in appendix B. Below, we outline the main idea in discrete time, then highlight the additional difficulties posed by continuous time.

Sketch proof. Let time $t \in \{0, 1, 2, \dots\}$ be discrete, and write $\beta \coloneqq e^{-r}$ for the discount factor. A mechanism (x^0, X^1) is incentive-compatible iff in each period s, the agent prefers prompt disclosure to delaying by one period and to never disclosing:

$$X_s^1 \geq (1-\beta)x_s^0 + \beta X_{s+1}^1 \qquad \text{(delay IC)}$$
$$X_s^1 \geq X_s^0. \qquad \text{(non-disclosure IC)}$$

(Delay IC also deters delay by two or more periods.) We shall show that undominatedness requires that the delay IC inequalities be equalities; we omit the argument that non-disclosure IC must also hold with equality.

So let (x^0, X^1) be an IC mechanism with delay IC slack in some period t:

$$X_t^1 > (1-\beta)x_t^0 + \beta X_{t+1}^1.$$

Observe that if the terms x_t^0 and X_{t+1}^1 on the right-hand side are $\geq u^1$, then the left-hand side X_t^1 must strictly exceed u^1. Equivalently, it must be that either

$$\text{(i) } X_t^1 > u^1, \quad \text{(ii) } x_t^0 < u^1, \quad \text{or} \quad \text{(iii) } X_{t+1}^1 < u^1.$$

In each of these cases, we shall find a mechanism that dominates (x^0, X^1).

In case (i), the naïve intuition is vindicated: lowering X_t^1 toward u^1 really does improve the principal's payoff (strictly in case of a breakthrough in period t). And this preserves IC: the (slack) period-t delay IC holds for a small enough decrease, while delay IC *slackens* in period $t-1$ and is unaffected in all other periods. Non-disclosure IC is easily shown also to hold.

In case (ii), increase x_t^0 toward u^1, by an amount small enough to preserve period-t delay IC. Other periods' delay IC is undisturbed, and non-disclosure IC in fact continues to hold. Since F^0 increases strictly to the left of $u^1 < u^0$, the principal's payoff improves (strictly in case of a breakthrough after t).

Finally, in case (iii), *increase* X_{t+1}^1 toward u^1. (The opposite of the naïve intuition.) The principal is better off (strictly in case of a period-$(t+1)$ breakthrough). Period-t delay IC abides provided the modification is small, while delay IC is loosened in period $t+1$ and unaffected in other periods. Non-disclosure IC is clearly preserved. ■

The proof in appendix B is based on the logic of the sketch above, but must handle two issues that arise in continuous time. First, in case (ii), x^0 must be increased on a *non-null* set of times if the principal's payoff is to increase strictly under some distribution. Secondly, in cases (i) and (iii), it is typically not possible to modify X^1 in a single period while preserving IC.

In light of Theorem 1, an undominated incentive-compatible mechanism (x^0, X^1) is pinned down by the pre-disclosure flow x^0, since the disclosure

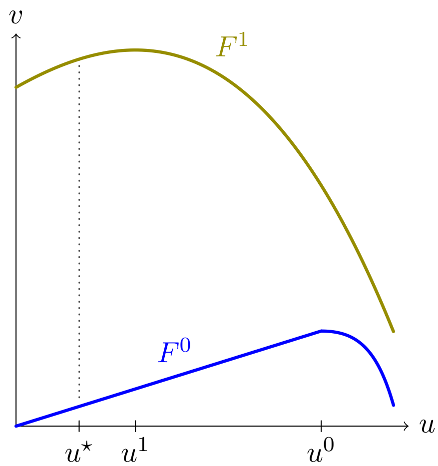


Figure 3: Utility possibility frontiers in the affine case. $u^\star$ is where the frontiers are furthest apart.

reward X^1 must always equal the present value of the remainder of x^0:

$$X_t^1 = X_t^0 = r \int_t^\infty e^{-r(s-t)} x_s^0 \mathrm{d}s \quad \text{for each } t \in \mathbf{R}_+.$$

We therefore drop superscripts in the sequel, writing an IC mechanism simply as (x, X), where $X_t \coloneqq r \int_t^\infty e^{-r(s-t)} x_s \mathrm{d}s$. (Since mechanisms of this form are automatically IC, we refer to them simply as a 'mechanisms'.) By Lemma 0, we need only consider mechanisms (x, X) that satisfy $x \leq u^0$ a.e.

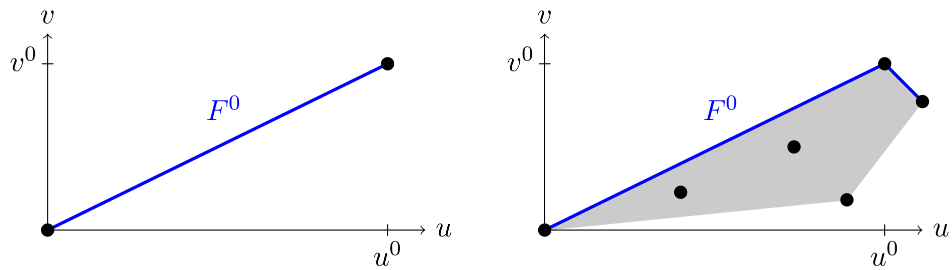
5 Deadline mechanisms

In this section, we uncover a striking deadline structure of undominated mechanisms when the old utility possibility frontier F^0 is affine on $[0, u^0]$, as in Figure 3. We further characterise the optimal choice of deadline, given the breakthrough distribution.

The affine case is important because (approximate) affineness frequently arises in applications, for two basic reasons. The first is that in policy applications, the principal cares directly about the agent's welfare, so that lowering the agent's utility reduces the principal's at a constant rate. This force yields approximate affineness in unemployment insurance (§7).

The second reason is concavification. In the simplest case, with just two allocations, the utility possibility frontier is the straight line connecting the two feasible utility profiles (Figure 4a).[23] More generally (Figure 4b), the

[23]In-between profiles are attained by rapidly switching back and forth (or randomising).



(a) Two allocations providing utilities $(0,0)$ and $\left(u^0, v^0\right)$, and the frontier F^0.

(b) Many allocations: utility possibility set (grey) and its upper boundary F^0.

Figure 4: Affineness on $\left[0, u^0\right]$ arising from concavification.

utility possibility set is the convex hull of all feasible utility profiles, and its upper boundary F^0 is affine if these have a convex shape.

The utility level $u^\star$ (defined in §2) admits a simple description when F^0 is affine: it is the unique $u \in \left[0, u^0\right]$ at which the frontiers are furthest apart,[24] as indicated in Figure 3. A *deadline mechanism* is one in which the agent's utility absent disclosure is at the efficient level u^0 before a deterministic deadline, and at the inefficiently low level $u^\star$ afterwards:

Definition 4. A mechanism (x, X) is a *deadline mechanism* iff

$$x_t = \begin{cases} u^0 & \text{for } t \leq T \\ u^\star & \text{for } t > T \end{cases} \qquad \text{for some } T \in [0, \infty].$$

Deadline mechanisms are simple: only two utility levels are used, with a single switch between them. And they form a small class of mechanisms, parametrised by a single number: the deadline T. (The utility levels u^0 and $u^\star$ are not free parameters, being pinned down by the technologies F^0, F^1.)

The agent's reward X upon disclosure in a deadline mechanism (equal to the present value of the remainder of the pre-disclosure flow x) is decreasing until the deadline, then constant at $u^\star$:

$$X_t = \begin{cases} \left(1 - e^{-r(T-t)}\right) u^0 + e^{-r(T-t)} u^\star & \text{for } t \leq T \\ u^\star & \text{for } t > T. \end{cases} \tag{$\diamondsuit$}$$

[24] $u^\star$ is a strict local maximum of the gap $F^1 - F^0$, which is concave when F^0 is affine.

5.1 Only deadline mechanisms are undominated

The affine case admits a sharp prediction: no matter what the shapes of the new frontier F^1 and breakthrough distribution G, the principal will choose a mechanism from the small and simple deadline class.

Theorem 2. If the old frontier F^0 is affine on $[0, u^0]$, then any undominated mechanism is a deadline mechanism.

The welfare implications are stark: ex-post Pareto efficiency in case of an early breakthrough, and surplus destruction otherwise.[25] In particular, absent a breakthrough, we have efficiency (at u^0) before the deadline, but surplus destruction (at $u^\star$) afterwards. Once the new technology arrives, it is deployed efficiently (on the downward-sloping part of F^1) if its arrival was early (while $X \geq u^1$),[26] and inefficiently otherwise.[27]

We prove Theorem 2 in appendix C. Below, we give an intuitive sketch.

Sketch proof. Fix a non-deadline mechanism (x, X) with $x \leq u^0$, and assume for simplicity that $x \geq u^\star$. We will show that (x, X) is dominated by the deadline mechanism $(x^\dagger, X^\dagger)$ whose deadline T satisfies

$$\underbrace{\left(1 - e^{-rT}\right) u^0 + e^{-rT} u^\star}_{= X_0^\dagger \quad \text{by } (\diamondsuit)} = X_0.$$

This mechanism is a *front-loading* of (x, X): the pre-disclosure flow has the same present value $X_0 = r \int_0^\infty e^{-rt} x_t \mathrm{d}t$, but is higher early and lower late, as depicted in Figure 5a. As time passes, the present value

$$X_t^\dagger = r \int_t^\infty e^{-r(s-t)} x_s^\dagger \mathrm{d}s$$

of the remainder of the front-loaded flow $x^\dagger$ rapidly diminishes, so that $X^\dagger$ is weakly below X in every period (see Figure 5b).

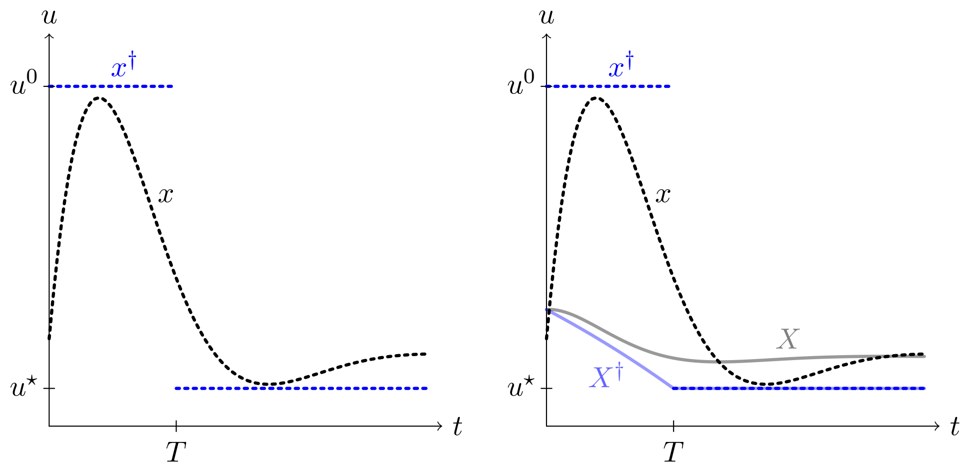
The principal's payoff may be written as

$$\Pi_G(x, X) = \mathbf{E}_G\Big(\underbrace{Y_0 - e^{-r\tau} Y_\tau}_{\text{pre-disclosure}} + \underbrace{e^{-r\tau} F^1(X_\tau)}_{\text{post-disclosure}}\Big),$$

[25] In the language of the screening literature, there is 'no distortion at the top', but 'lower' (i.e. later) types' allocations are distorted to reduce information rents.

[26] A detail: $X_t \geq u^1$ holds in early periods t only if the deadline is sufficiently late. We show in the next section that this must be the case in undominated mechanisms.

[27] Provided that $u^\star < u^1$, which holds e.g. if ($u^1 > 0$ and) F^1 has no kink at u^1.



(a) $x^\dagger$ is higher early and lower late. (b) $X^\dagger \leq X$, with equality at 0.

Figure 5: Sketch proof of Theorem 2: front-loading by a deadline mechanism.

where

$$Y_t := r \int_t^\infty e^{-r(s-t)} F^0(x_s) \mathrm{d}s$$

is her period-t continuation payoff if the agent never discloses. Qualitatively, front-loading has two effects. The first is a mechanical benefit: since the pre-disclosure flow is experienced only until the breakthrough, it is better that any given total present value X_0 be provided in a front-loaded fashion. (This is formalised below as an increase of $Y_0 - e^{-r\tau} Y_\tau$.) The second effect is ambiguous: lowering X alters the principal's post-disclosure payoff $F^1(X_\tau)$.

To assess these forces quantitatively, use the affineness of F^0 to write

$$Y_t = F^0\left(r \int_t^\infty e^{-r(s-t)} x_s \mathrm{d}s \right) = F^0(X_t),$$

so that

$$\Pi_G(x, X) = F^0(X_0) + \mathbf{E}_G\left(e^{-r\tau} \left[F^1 - F^0 \right](X_\tau) \right).$$

Front-loading lowers X toward $u^\star$, leaving X_0 unchanged. Since $F^1 - F^0$ is (strictly) decreasing on $[u^\star, u^0]$ by definition of $u^\star$, this improves the principal's payoff whatever the distribution G. The improvement is in fact strict for any full-support distribution. Thus $(x^\dagger, X^\dagger)$ dominates (x, X). ■

Theorem 2 provides a rationale for deadline mechanisms even when F^0 is not exactly affine. As we show in supplemental appendix L, the principal loses little by restricting attention to deadline mechanisms provided F^0 has only moderate curvature.

5.2 Undominated deadlines

Theorem 2 asserts that only deadline mechanisms are undominated when F^0 is affine, but does not adjudicate between deadlines. In fact, not every deadline mechanism is undominated. Consider a deadline T so early that $X_0 < u^1$. Since the disclosure reward X decreases over time in a deadline mechanism, we have $X_\tau < u^1$ whatever the time τ of the breakthrough.

The principal can do better by using the later deadline $\underline{T}$ that satisfies $X_0 = u^1$, or explicitly (using equation ($\Diamond$) on p. 16)

$$\left(1 - e^{-r\underline{T}}\right)u^0 + e^{-r\underline{T}}u^\star = u^1.$$

This raises the agent's disclosure reward X toward u^1, improving the principal's post-disclosure payoff $F^1(X_\tau)$ whatever the breakthrough time τ (strictly if $\tau < \underline{T}$). The principal also enjoys the high pre-disclosure flow $F^0(u^0) > F^0(u^\star)$ for longer, which is beneficial in case of a late breakthrough.

Undominatedness thus requires a deadline no earlier than $\underline{T}$. This condition is not only necessary, but also sufficient:

Proposition 1. If the old frontier F^0 is affine on $[0, u^0]$, then a mechanism is undominated exactly if it is a deadline mechanism with deadline $T \in [\underline{T}, \infty]$.

The proof is in appendix D.

5.3 Optimal deadlines

Proposition 1 narrows the search for an optimal mechanism to deadline mechanisms with a sufficiently late deadline. The optimal choice among these depends on the breakthrough distribution G.

A late deadline is beneficial if the breakthrough occurs late, as the efficient high utility u^0 is then provided for a long time. The cost is that in case of an early breakthrough, the agent must be given a utility of $X > u^1$ forever. A first-order condition balances this trade-off:

Proposition 2. Assume that the old frontier F^0 is affine on $[0, u^0]$, that F^1 is differentiable on $(0, u^0)$, and that $u^\star > 0$. A mechanism is optimal for G iff it is a deadline mechanism and satisfies $\mathbf{E}_G\big(F^{1\prime}(X_\tau)\big) = 0$.

In other words, the new technology should be operated optimally *on average*. This is a restriction on the deadline T because X is a function of it, as described by equation ($\diamondsuit$) on p. 16.

We prove Proposition 2 in appendix E by deriving a general first-order condition that is valid without any auxiliary assumptions, then showing that it can be written as $\mathbf{E}_G(F^{1\prime}(X_\tau)) = 0$ when F^0 is affine, F^1 is differentiable and $u^\star$ is interior.

In the same appendix, we derive comparative statics for optimal deadlines: they become later when the breakthrough distribution G becomes later in the sense of first-order stochastic dominance. This improves the agent's ex-ante payoff X_0, as can be seen from ($\diamondsuit$) on p. 16.

6 Optimal mechanisms in general

In this section, we show that optimal mechanisms in the general (non-affine) case exhibit a graduated deadline structure: absent disclosure, the agent's utility still declines from u^0 toward $u^\star$, but not necessarily abruptly. Given the breakthrough distribution, we describe the optimal path.

6.1 Qualitative features of optimal mechanisms

Recall from §2 that $u^\star$ denotes the greatest $u \in [0, u^0]$ at which the old and new frontiers F^0, F^1 have equal slopes, as depicted in Figure 1 (p. 6).

Theorem 3. Any mechanism (x, X) that is optimal for some distribution G with $G(0) = 0$ and unbounded support has x decreasing

$$\text{from} \quad \lim_{t\to 0} x_t = u^0 \quad \text{toward} \quad \lim_{t\to\infty} x_t = u^\star.$$[28]

That is, optimal mechanisms are just like deadline mechanisms, except that the transition from u^0 to $u^\star$ may be gradual. This graduality follows directly from relaxing affineness: when F^0 has a strictly concave shape, by definition, the principal prefers providing intermediate utility to providing only the extreme utilities $u^\star, u^0$. Theorem 3 is the combination of this mechanical effect with the front-loading insight expressed by Theorem 2. Formally, the proof in appendix G relies on a form of *local* front-loading.

Absent a breakthrough, efficiency deteriorates as we travel leftward along the upward-sloping part of the old frontier F^0. Once the new technology

[28] Recall that a mechanism has multiple *versions* (footnote 21, p. 10). Theorem 3 asserts that any optimal mechanism has a version with the stated properties. We focus on $\lim_{t\to 0} x_t$ rather than x_0 because '$x_0 = u^0$' is vacuous: *any* mechanism has a version satisfying it.

becomes available, it is operated efficiently (on the downward-sloping part of F^1) if its arrival was sufficiently early;[29] if not, then surplus is destroyed.[30]

The distributional hypotheses are mild: $G(0) = 0$ means that the new technology is unavailable initially, while unbounded support rules out an effectively finite horizon. The former's role is as a sufficient condition for $\lim_{t \to 0} x_t = u^0$,[31] while the latter is required by our proof strategy.

6.2 Optimal transition

Theorem 3 describes the distribution-free qualitative features of optimal mechanisms, but does not specify the precise manner in which the agent's utility ought to decline from u^0 toward $u^\star$. The optimal path, for a given breakthrough distribution, is characterised by an Euler equation:

Proposition 3. Assume that $u^\star > 0$ and that F^0, F^1 are differentiable on $(0, u^0)$. Then any mechanism (x, X) that is optimal for a distribution G with $G(0) = 0$ and unbounded support satisfies the initial condition $\mathbf{E}_G(F^{1\prime}(X_\tau)) = 0$ and the Euler equation

$$F^{0\prime}(x_t) \geq \mathbf{E}_G\Big(F^{1\prime}(X_\tau)\Big| \tau > t\Big) \quad \text{for each } t \in \mathbf{R}_+\text{, with equality if } x_t < u^0.$$[32]

The initial condition $\mathbf{E}_G(F^{1\prime}(X_\tau)) = 0$ demands that the new technology be used optimally on average, just like the first-order condition for an optimal deadline in the affine case (Proposition 2, p. 19). To understand the Euler equation, differentiate it and rearrange to obtain

$$\dot{x}_t = -\underbrace{\left(\frac{G'(t)}{1 - G(t)}\right)}_{\text{hazard rate}} \frac{F^{0\prime}(x_t) - F^{1\prime}(X_t)}{\underbrace{-F^{0\prime\prime}(x_t)}_{\text{curvature}}}.$$[33]

Thus the agent's pre-disclosure utility declines in proportion to the hazard rate, and in inverse proportion to the local curvature of the old frontier F^0. As the latter would suggest, x jumps over any affine segments ($F^{0\prime\prime} = 0$ and '$\dot{x} = \infty$'), and pauses at kinks ('$F^{0\prime\prime} = -\infty$' and $\dot{x} = 0$).

[29] We show in appendix H that $X_t > u^1$ holds in all sufficiently early periods t.

[30] Provided that $u^\star < u^1$, which holds e.g. if ($u^1 > 0$ and) F^1 has no kink at u^1.

[31] By Theorem 2 (p. 17), another sufficient condition is affineness of F^0 on $[0, u^0]$.

[32] Here $F^{j\prime}(0)$ ($F^{j\prime}(u^0)$) for $j \in \{0, 1\}$ denotes the right-hand (left-hand) derivative. Recall that a mechanism has multiple *versions* (footnote 21, p. 10). In full, the proposition asserts that some (any) version satisfies the Euler equation for (almost) every $t \in \mathbf{R}_+$.

[33] This expression is valid under the additional assumptions that G admits a continuous density and that F^0 possesses a continuous and strictly negative second derivative.

Without the interiority ($u^\star > 0$) and differentiability hypotheses, a superdifferential Euler equation characterises the optimal path. We prove in appendix H that this equation is necessary for optimality, whence Proposition 3 follows, and furthermore show that it is sufficient.

As for comparative statics, we show in supplemental appendix O that as the breakthrough distribution G becomes later in the sense of monotone likelihood ratio, the disclosure reward X increases in every period. (The pre-disclosure flow x need not increase pointwise.) It follows in particular that the agent's ex-ante payoff X_0 improves.

7 Application to unemployment insurance

The purpose of unemployment insurance ('UI') is to provide material support to involuntarily unemployed workers. Since job offers are typically unobservable, the state cannot easily distinguish the intended recipients of UI from workers who have access to an employment opportunity which they have chosen not to exercise. Unemployment insurance schemes must therefore be designed to incentivise workers to accept job offers. In this section, we shed light on this policy problem using our general theory of optimal screening for breakthroughs.

Many countries, including Germany, France and Sweden, use deadline benefit schemes: the short-term unemployed receive a generous benefit, while those remaining unemployed past a deadline see their benefit reduced to a much lower level. Using our results, we provide a potential rationale for such schemes by describing conditions under which they are close to optimal. We further argue that the particular deadlines used in Germany and France are broadly consistent with the recommendations from our analysis.

Related literature. The literature on optimal unemployment insurance has two main strands. The first studies the moral-hazard problem of incentivising job-search effort (Shavell & Weiss, 1979; Hopenhayn & Nicolini, 1997). We contribute to the second strand, which is concerned with the adverse-selection problem arising from privately observed job offers (Atkeson & Lucas, 1995).[34] To examine the implications of delay, we replace the usual assumption that offers expire immediately unless accepted with the opposite assumption that they remain valid indefinitely. Whereas the literature tends to focus on (often intricate) exactly optimal mechanisms, our results also

[34] See also Thomas and Worrall (1990), Atkeson and Lucas (1992), Hansen and İmrohoroğlu (1992) and Shimer and Werning (2008).

address the simple deadline schemes often used in practice.

7.1 Model

A worker (agent) is unemployed. At a random time $\tau \sim G$, she receives a job offer, which she can accept immediately, with a delay, or not at all. The worker's ability to delay acceptance is the distinguishing feature of our otherwise-standard model.

The worker's utility is $u = \phi(C) - \kappa(L)$, where $C \geq 0$ is her consumption and $L \geq 0$ her labour supply. We assume that ϕ and κ are strictly increasing, respectively strictly concave and strictly convex, continuous at zero with $\phi(0) = \kappa(0) = 0$, and that they possess derivatives satisfying

$$\lim_{C \to \infty} \phi'(C) = 0, \quad \lim_{C \to 0} \phi'(C) = \infty \quad \text{and} \quad \lim_{L \to 0} \kappa'(L) = 0.$$

We interpret $C = 0$ as the lowest socially acceptable standard of living. (This may differ across societies and eras.)

As is standard in the literature (e.g. Atkeson & Lucas, 1995), we assume that the state (principal) controls the worker's consumption C and labour supply L.[35] The state's objective is social welfare, which depends both on the worker's welfare u and on net tax revenue $wL - C$, where $w > 0$ is the wage, and $L = 0$ if the worker is unemployed. In particular, social welfare is $v = u + \lambda \times (wL - C)$, where $\lambda > 0$ is the shadow value of public funds.

The utility possibility frontiers for unemployed and employed workers are

$$\begin{aligned} & F^0(u) \coloneqq \max_{C \geq 0} \{u + \lambda(-C) : \phi(C) = u\} \\ \text{and} \quad & F^1(u) \coloneqq \max_{C,L \geq 0} \{u + \lambda(wL - C) : \phi(C) - \kappa(L) = u\}, \end{aligned}$$

respectively. These frontiers satisfy our model assumptions (§2):[36]

Lemma 1. In the application to unemployment insurance, the frontiers F^0, F^1 are strictly concave and continuous, with unique peaks u^0, u^1 that satisfy $u^1 < u^0$. The gap $F^1 - F^0$ is strictly decreasing, so that $u^\star = 0$.

The conflict of interest $u^1 < u^0$ arises because the social first-best requires employed workers to supply labour ($L > 0$), which they dislike.[37]

[35] The idea is that consumption is steered using taxes and benefits, while labour supply is controlled by setting work requirements enforced by sanctions (such as fines).

[36] We omit the elementary (but tedious) proof. It is given in Curello and Sinander (2021).

[37] Nor do they receive extra consumption: the social first-best gives the same consumption to all workers, since social welfare (and the worker's utility) are separable in C and L.

We shall use the term ‘unemployment insurance (UI) scheme’ for a mechanism. By Theorem 1 (p. 13), undominated schemes keep the worker only just willing promptly to accept an offer, so have the form (x, X).

A UI scheme (x, X) may be described in terms of benefits and taxes, as follows. The utility of a worker who remains unemployed at t is $x_t = \phi(b_t)$, where b_t is her unemployment benefit. If the worker accepts a job at t, then she earns utility $X_t = \phi(wL_t - \theta_t) - \kappa(L_t)$ in every subsequent period, where $\theta_t \coloneqq wL_t - C_t$ is her per-period tax bill.

7.2 Optimal unemployment insurance

Optimal UI schemes are described by Theorem 3 (p. 20): unemployment benefits $b_t = \phi^{-1}(x_t)$ decrease over time, from $b^0 \coloneqq \phi^{-1}(u^0)$ toward $0 = \phi^{-1}(u^\star)$. Thus workers enjoy socially optimal consumption at the beginning of an unemployment spell, but see their benefits reduced over time, with the long-term unemployed provided only with society’s lowest acceptable standard of living (‘consumption zero’).

Employed workers are rewarded with a higher continuation utility X_t the earlier they accept a job. This involves a mix of lower labour supply and more generous tax treatment of earnings (yielding higher consumption).

A *deadline UI scheme* is one in which a generous benefit of b^0 is paid to the short-term unemployed, while those remaining unemployed beyond a deadline receive a low benefit just sufficient to finance the minimum standard of living (‘consumption zero’). Such schemes are pervasive in practice, used in e.g. Germany, France and Sweden. In Germany, for instance, an unemployed worker can collect *Arbeitslosengeld I* (60% of her previous net salary) until a deadline, after which she is entitled only to the much lower *Arbeitslosengeld II* (€446 per month).[38] French workers similarly qualify for the fairly generous *allocation d’aide au retour à l’emploi* at the beginning of an unemployment spell, but only for the lower *allocation de solidarité spécifique* (about €510 per month) when unemployed for longer.

Our results speak to the desirability of such deadline schemes. Theorem 2 (p. 17) implies that a deadline scheme is approximately optimal if F^0 is close to affine, as we show in supplemental appendix L. This condition is satisfied if the worker’s consumption utility ϕ has limited curvature (which may be interpreted as low risk-aversion), or if the social value λ of tax revenue is moderate.[39] We are thus able to rationalise the use of a deadline scheme in any country in which either of these properties plausibly holds.

[38] All figures are given as of January 2021.

[39] We formalise and prove these claims in supplemental appendix L.

Conversely, where both assumptions are far from being satisfied, our analysis predicts substantial welfare gains from replacing these abrupt benefit reductions with more gradual tapering. Such tapering is rarer in practice, but occurs in Italy: as of the fourth month of unemployment, the amount of the *Nuova Assicurazione Sociale per l'Impiego* declines by 3% per month. (This continues until benefits reach a legal minimum, the *Reddito di Cittadinanza*.)

Given the pervasiveness of deadline schemes (whatever their merits), the choice of deadline is an important policy problem. Our analysis highlights labour-market prospects as a key consideration: a worker with worse chances (a later job-finding distribution G, in the sense of first-order stochastic dominance) should be set a later deadline.[40] Two implications are that older workers ought to face later deadlines and that extensions should be granted during recessions. These recommendations are broadly followed in Germany and France: workers older than about 50 face more lenient deadlines, and all workers' deadlines were prolonged by three months during the 2020 recession.

Appendices

A Background and notation

The Lebesgue integral is used throughout. In particular, for $s < t$ in $\mathbf{R}_+$ and a function $\phi : \mathbf{R}_+ \to [-\infty, \infty]$, $\int_s^t \phi$ denotes the Lebesgue integral $\int_{(s,t)} \phi \mathrm{d}\lambda$, where λ is the Lebesgue measure.

We rely on various facts about concave functions (see Rockafellar, 1970, esp. part V). For $j \in \{0,1\}$, recall that $F^j : [0,\infty) \to [-\infty,\infty)$ is concave and upper semi-continuous. Write $D^j \coloneqq \{u \in [0,\infty) : F^j(u) > -\infty\}$ for its effective domain (a convex set). We have $(0, u^0] \subseteq D^j$ by assumption.

The right- and left-hand derivatives of F^j are denoted by F^{j+} and F^{j-}, respectively. The former (latter) is well-defined on $D^j \cup \inf D^j$ (on $(D^j \cup \sup D^j) \setminus \{0\}$), but may take infinite values on the boundary. F^{j+} is right-continuous, and F^{j-} is left-continuous. If the derivative $F^{j\prime}$ exists at $u \in \operatorname{int} D^j$, then $F^{j\prime}(u) = F^{j+}(u) = F^{j-}(u)$, and $F^{j\prime}$ is continuous at u.

The directional derivatives F^{j+}, F^{j-} are decreasing, and satisfy

$$F^{j-}(u) \leq F^{j+}(u') \leq F^{j-}(u') \leq F^{j+}(u'') \quad \text{for any } u > u' > u'' \text{ in } \operatorname{int} D^j.$$

The first (last) inequality is strict iff F^j is not affine on $[u', u]$ (on $[u'', u']$), and the middle inequality is strict exactly if F^j has a kink at u'.

[40] In particular, the optimal deadline described by Proposition 2 (p. 19) is later when G is, as noted at the end of §5.3 and proved in appendix E (p. 32).

A *supergradient* of F^j at $u \in \operatorname{cl} D^j$ is an $\eta \in [-\infty, \infty]$ such that

$$F^j(u') \leq F^j(u) + \eta(u' - u) \quad \text{for every } u' \neq u \text{ in } [0, \infty).$$

(Note that ∞ and $-\infty$ can be supergradients.) F^j admits at least one supergradient at every $u \in \operatorname{cl} D^j$. For $u \in \operatorname{int} D^j$, $\eta \in [-\infty, \infty]$ is a supergradient of F^j at u exactly if $F^{j+}(u) \leq \eta \leq F^{j-}(u)$, while for $u = \inf D^j$ ($u = \sup D^j$) the former (latter) inequality by itself is necessary and sufficient.

B Proof of Theorem 1 (p. 13)

For any mechanism (x^0, X^1), let $h : \mathbf{R}_+ \to [-\infty, \infty]$ be given by $h(t) \coloneqq e^{-rt}(X_t^1 - X_t^0)$ for each $t \in \mathbf{R}_+$.[41] Theorem 1 asserts precisely that undominated IC mechanisms have h identically equal to zero.

Observation 1. A mechanism (x^0, X^1) is incentive-compatible exactly if h is (a) decreasing and (b) non-negative.

Proof. Part (a) (part (b)) of the definition of incentive-compatibility on p. 10 requires precisely that h be decreasing (non-negative). ■

Continuity lemma. Any undominated IC mechanism has h continuous.

Proof. We prove the contrapositive. Fix an IC mechanism (x^0, X^1).

Suppose that h is discontinuous at some $t \in (0, \infty)$. Since h is decreasing and X^0 is continuous, $\lim_{s \uparrow t} X_s^1$ and $\lim_{s \downarrow t} X_s^1$ exist and satisfy $\lim_{s \uparrow t} X_s^1 \geq X_t^1 \geq \lim_{s \downarrow t} X_s^1$, with one of the inequalities strict. We shall assume that

$$\lim_{s \uparrow t} X_s^1 = X_t^1 > \lim_{s \downarrow t} X_s^1,$$

omitting the similar arguments for the other two cases. If $\lim_{s \downarrow t} X_s^1 < u^1$, then we may increase X^1 toward u^1 on a small interval $(t, t+\varepsilon)$ while keeping h decreasing.[42] If instead $\lim_{s \downarrow t} X_s^1 \geq u^1$, then $\lim_{s \uparrow t} X_s^1 = X_t^1 > u^1$, so that we may decrease X^1 toward u^1 on a small interval $(t - \varepsilon, t]$ while keeping h decreasing.[43] In either case, IC is preserved, and the principal's payoff Π_G is (strictly) increased under any (full-support) distribution G.

---

[41] In case $X_t^1 = X_t^0 = \infty$, we let $h(t) \coloneqq 0$ by convention.

[42] Choose an $\varepsilon > 0$ small enough that $X^1 + \varepsilon < \min\{u^1, X_t^1\}$ on $(t, t+\varepsilon)$. Let $X_s^{1\dagger} \coloneqq X_s^1 - (s - t) + \varepsilon$ for $s \in (t, t+\varepsilon)$ and $X^{1\dagger} \coloneqq X^1$ off $(t, t+\varepsilon)$. Then $X^1 \leq X^{1\dagger} \leq u^1$, with the first inequality strict on $(t, t+\varepsilon)$. We have $h^\dagger \geq h \geq 0$, and $h^\dagger$ is clearly decreasing on $[0, t]$ and on (t, ∞). At t, we have $h^\dagger(t) - \lim_{s \downarrow t} h^\dagger(s) = e^{-rt}(X_t^1 - \lim_{s \downarrow t} X_s^1 - \varepsilon) \geq 0$.

[43] Choose an $\varepsilon \in (0, 1/r)$ small enough that $X^1 - \varepsilon > \lim_{s \downarrow t} X_s^1$ and $h > \varepsilon$ on $(t - \varepsilon, t]$. Let $X_s^{1\dagger} \coloneqq X_s^1 + t - s - \varepsilon$ for $s \in (t - \varepsilon, t]$ and $X^{1\dagger} \coloneqq X^1$ off $(t - \varepsilon, t]$. Then $u^1 \leq X^{1\dagger} \leq X^1$, with the second inequality strict on $(t - \varepsilon, t]$. Clearly $h^\dagger$ is non-negative, and is decreasing on $[0, t - \varepsilon]$ and on (t, ∞). It is decreasing on $[t - \varepsilon, t]$ since $h^\dagger(s) - h(s) = e^{-rs}(t - s - \varepsilon)$ is (by our choice of $\varepsilon < 1/r$). And at t, $h^\dagger(t) - \lim_{s \downarrow t} h^\dagger(s) = e^{-rt}(X_t^1 - \varepsilon - \lim_{s \downarrow t} X_s^1) \geq 0$.

Suppose instead that h is discontinuous at $t = 0$; then $X^1_0 > \lim_{s\downarrow 0} X^1_s$ by IC and the continuity of X^0. The case $\lim_{s\downarrow 0} X^1_s < u^1$ may be dealt with as above. If $\lim_{s\downarrow 0} X^1_s \geq u^1$, then lowering X^1_0 toward $\lim_{s\downarrow 0} X^1_s$ preserves IC and (strictly) increases Π_G for any distribution G (with $G(0) > 0$). ■

Proof of Theorem 1. Let (x^0, X^1) be an IC mechanism, so that h is non-negative and decreasing, and suppose that h is not identically zero. By the continuity lemma, we may assume that h (and thus X^1) is continuous.

We consider three cases. (The first two concern slack 'delay IC': Case 1 [Case 2] corresponds to the sketch proof's case (ii) [cases (i) and (iii)]. Case 3 is where 'delay IC' binds, but 'non-disclosure IC' is slack.) In each case, we shall construct an incentive-compatible mechanism $(x^{0\dagger}, X^{1\dagger})$ such that

$$\Pi_G\Big(x^{0\dagger}, X^{1\dagger}\Big) \geq (>)\, \Pi_G\Big(x^0, X^1\Big) \quad \text{for every (full-support) } G. \tag{D}$$

Define $A \coloneqq \{t \in \mathbf{R}_+ : h \text{ is differentiable at } t \text{ and } h'(t) < 0\}$.

Case 1: $\{t \in A : x^0_t < u^0\}$ is non-null. Since $h > 0$ on A,[44] there is an $\varepsilon > 0$ for which the set

$$A_\varepsilon \coloneqq \Big\{t \in A : x^0_t + \varepsilon < u^0,\ h(t) \geq \varepsilon \text{ and } h'(t) + r\varepsilon \leq 0\Big\}$$

is non-null.[45] Define $x^{0\dagger} \coloneqq x^0 + \varepsilon \mathbf{1}_{A_\varepsilon}$, and consider the mechanism $(x^{0\dagger}, X^1)$. Clearly $x^0 \leq x^{0\dagger} \leq u^0$, and $x^{0\dagger} \neq x^0$ on the non-null set A_ε, so that (D) holds by the strict monotonicity of F^0 on $[0, u^0]$. $h^\dagger$ is decreasing since for any $t < t'$ in $\mathbf{R}_+$,

$$\begin{aligned} h^\dagger(t') - h^\dagger(t) &= h(t') - h(t) + r\varepsilon \int_t^{t'} e^{-rs} \mathbf{1}_{A_\varepsilon}(s) \mathrm{d}s \\ &\leq \int_t^{t'} h' \mathbf{1}_{A_\varepsilon} + r\varepsilon \int_t^{t'} e^{-rs} \mathbf{1}_{A_\varepsilon}(s) \mathrm{d}s \leq 0, \end{aligned}$$

where the first inequality holds since h is decreasing,[46] and the second holds by definition of A_ε. As for non-negativity, we have $h^\dagger = h \geq 0$ on $(\sup A_\varepsilon, \infty)$, while $h^\dagger \geq 0$ on $[0, \sup A_\varepsilon)$ since $h^\dagger$ is decreasing and $h^\dagger \geq h - \varepsilon \geq 0$ on A_ε by definition of the latter. Thus $(x^{0\dagger}, X^1)$ is incentive-compatible.

Case 2: There are $t' < t''$ in $\mathbf{R}_+$ such that $h(t') > h(t'')$ and $X^1 \neq u^1$ on $[t', t'']$. Since X^1 is continuous, we have either $X^1 > u^1$ on $[t', t'']$ or $X^1 < u^1$

[44] Since $h \geq 0$, $h(t) = 0$ implies $\liminf_{t'\downarrow t} [h(t') - h(t)]/(t' - t) \geq 0$ and thus $t \notin A$.

[45] $A_0 = \bigcup_{n \in \mathbf{N}} A_{1/n}$ is non-null, so continuity of measures (with λ denoting the Lebesgue measure) yields $0 < \lambda(A_0) = \lim_{n\to\infty} \lambda(A_{1/n})$, whence $\lambda(A_{1/n}) > 0$ for some $n \in \mathbf{N}$.

[46] Recall the Lebesgue decomposition $h = h_a + h_s$ where h_a is decreasing and absolutely continuous and h_s is decreasing with $h'_s = 0$ a.e. (e.g. Stein & Shakarchi, 2005, p. 150).

on $[t', t'']$. We shall assume the former, omitting the similar argument for the latter case. Because $s \mapsto e^{rs}h(t'') + X^0_s$ is continuous and takes the value $X^1_{t''} > u^1$ at $s = t''$,

$$t^\star := \inf\left\{t \in [t', t''] : e^{rs}h(t'') + X^0_s \geq u^1 \text{ for all } s \in [t, t'']\right\}$$

is well-defined and strictly smaller than t''. Define

$$X^{1\dagger}_t := \begin{cases} e^{rt}h(t'') + X^0_t & \text{for } t \in [t^\star, t'') \\ X^1_t & \text{for } t \notin [t^\star, t''), \end{cases}$$

and consider the mechanism $(x^0, X^{1\dagger})$. This mechanism is IC since $h^\dagger = h + [h(t'') - h]\mathbf{1}_{[t^\star, t'')}$ is clearly decreasing and non-negative.

It remains to show that $(x^0, X^{1\dagger})$ satisfies (D). Since X^1 and $X^{1\dagger}$ differ only on $[t^\star, t'')$ and F^1 is strictly decreasing on $[u^1, \infty)$, it suffices to prove that

$$u^1 \leq X^{1\dagger}_t \leq (<)\; X^1_t \quad \text{for every (some) } t \in [t^\star, t'').$$[47]

The first inequality holds by definition of $t^\star$. For the second, observe that

$$X^{1\dagger}_t - X^1_t = e^{rt}\left[h^\dagger(t) - h(t)\right] = e^{rt}[h(t'') - h(t)] \leq 0 \quad \text{for } t \in [t^\star, t'')$$

since h is decreasing. We claim that the inequality is strict at $t = t^\star$. If $t^\star = t'$, then this is true because $h(t') > h(t'')$. And if not, then $t^\star \in (t', t'')$, in which case $X^{1\dagger}_{t^\star} = u^1 < X^1_{t^\star}$ by continuity of X^0 and $X^1 > u^1$.

Case 3: neither Case 1 nor Case 2. Since X^1 is continuous, every $t \in \mathbf{R}_+$ belongs either to a maximal open interval on which $X^1 \neq u^1$ or else to a maximal closed interval on which $X^1 = u^1$. h is increasing on any interval of the former kind since we are not in Case 2. We shall show that h is also increasing on each interval of the latter kind; then since h is continuous, it is increasing and thus constant.

So fix an interval I of the latter kind. Since h is decreasing, its derivative $h'(t) = re^{-rt}(x^0_t - u^1)$ exists a.e. on I. As we are not in Case 1, we have for a.e. $t \in I$ that either $h'(t) = 0$ or $x^0_t = u^0$, and in the latter case $h'(t) = re^{-rt}(u^0 - u^1) > 0$. Assuming wlog that $x^0 \leq u^0$,[48] the expression for h' implies that h is ru^0-Lipschitz on I. Thus h is increasing on I, as desired.

Since (by hypothesis) h is not identically zero, it is constant at some $k > 0$, so that $X^1_t = X^0_t + e^{rt}k$ for every $t \in \mathbf{R}_+$. Thus $X^{1\dagger} := \min\{X^1, X^0 + u^1\}$

[47] It is enough for the inequality to be strict at a single time $t \in [t^\star, t'')$, since it then holds strictly on a proper interval by the continuity of X^1 and $X^{1\dagger}$ on $[t^\star, t'')$.

[48] Otherwise the IC mechanism $(\min\{x^0, u^0\}, X^1)$ would satisfy (D).

is strictly smaller than X^1 after some time $T > 0$, so that $(x^0, X^{1\dagger})$ satisfies (D). And it is incentive-compatible.[49] ■

C Proof of Theorem 2 (p. 17)

Fix a non-deadline mechanism (x, X) with $x \leq u^0$ a.e.;[50] we will show that it is dominated by the deadline mechanism $(x^\dagger, X^\dagger)$ whose deadline T satisfies

$$\left(1 - e^{-rT}\right)u^0 + e^{-rT}u^\star \equiv X_0^\dagger = X_0 \vee u^\star,$$

where '$\vee$' denotes the pointwise maximum.

Claim. $X^\dagger \leq X \vee u^\star$.

Proof. For $t \geq T$, we have $X^\dagger = u^\star \leq X \vee u^\star$. For $t < T$, suppose first that $X_0^\dagger = X_0$; then since $x^\dagger = u^0 \geq x$ on $[0, t] \subseteq [0, T]$, we have

$$\begin{aligned} e^{-rt}X_t^\dagger &= X_0^\dagger - r\int_0^t e^{-rs}x_s^\dagger \mathrm{d}s \\ &\leq X_0 - r\int_0^t e^{-rs}x_s \mathrm{d}s = e^{-rt}X_t \leq e^{-rt}(X_t \vee u^\star). \end{aligned}$$

If instead $X_0^\dagger = u^\star$, then the fact that $x^\dagger \geq u^\star$ yields

$$\begin{aligned} e^{-rt}X_t^\dagger &= X_0^\dagger - r\int_0^t e^{-rs}x_s^\dagger \mathrm{d}s \\ &\leq u^\star - r\int_0^t e^{-rs}u^\star \mathrm{d}s = e^{-rt}u^\star \leq e^{-rt}(X_t \vee u^\star). \end{aligned}$$ □

The concave function $F^1 - F^0$ is uniquely maximised at $u^\star$, so is strictly increasing on $[0, u^\star]$ and strictly decreasing on $[u^\star, u^0]$. Since $u^\star \leq X^\dagger \leq X \vee u^\star$ by the claim, it follows that

$$\left[F^1 - F^0\right]\left(X^\dagger\right) \geq \left[F^1 - F^0\right](X \vee u^\star). \tag{1}$$

Since $X \vee u^\star \geq X$, and the two differ only when both are in $[0, u^\star]$, we have

$$\left[F^1 - F^0\right](X \vee u^\star) \geq \left[F^1 - F^0\right](X), \tag{2}$$

which chained together with the preceding inequality yields

$$\left[F^1 - F^0\right]\left(X^\dagger\right) \geq \left[F^1 - F^0\right](X). \tag{3}$$

[49] We have $h^\dagger(t) = e^{-rt}u^1 \in (0, h^\dagger(T))$ for $t > T$, and this expression is decreasing.
[50] IC mechanisms not of this form are dominated, by Lemma 0 and Theorem 1.

The facts that $X_0^\dagger = X_0 \vee u^\star \geq X_0$ and that F^0 is increasing on $[0, u^0]$ together imply

$$F^0\left(X_0^\dagger\right) \geq F^0(X_0). \tag{4}$$

Thus for any distribution G, using the expression for the principal's payoff derived in the sketch proof (p. 18), we have

$$\begin{aligned}
\Pi_G\left(x^\dagger, X^\dagger\right) &= F^0\left(X_0^\dagger\right) + \mathbf{E}_G\left(e^{-r\tau}\left[F^1 - F^0\right]\left(X_\tau^\dagger\right)\right) \\
&\geq F^0\left(X_0^\dagger\right) + \mathbf{E}_G\left(e^{-r\tau}\left[F^1 - F^0\right](X_\tau)\right) && \text{by (3)} \\
&\geq F^0(X_0) + \mathbf{E}_G\left(e^{-r\tau}\left[F^1 - F^0\right](X_\tau)\right) && \text{by (4)} \\
&= \Pi_G(x, X).
\end{aligned}$$

It remains show that $(x^\dagger, X^\dagger)$ delivers a *strict* improvement for some distribution G. We shall accomplish this by showing that the inequality (3) holds strictly on a non-null set of times, so that the first inequality in the above display is strict for any distribution G with full support. Since $X^\dagger \leq X \vee u^\star$ by the claim and $X, X^\dagger$ are continuous, there are two cases: either (a) $X^\dagger < X \vee u^\star$ on a non-null set of times, or (b) $X^\dagger = X \vee u^\star$.

Case (a): $X^\dagger < X \vee u^\star$ on a non-null set $\mathcal{T}$. In this case, the inequality (1) holds strictly on $\mathcal{T}$, and thus so does (3).

Case (b): $X^\dagger = X \vee u^\star$. Since the original mechanism (x, X) is not a deadline mechanism, there must be a non-null set of times on which $x \neq x^\dagger$, and thus $X \neq X^\dagger = X \vee u^\star$ on some non-null set $\mathcal{T}$, so that $X < X \vee u^\star$ on $\mathcal{T}$. Then (2) is strict on $\mathcal{T}$, and thus so is (3). ■

D Proof of Proposition 1 (p. 19)

Write (x^T, X^T) for the deadline mechanism with deadline T, and $\pi_G(T)$ for its payoff under a distribution G. By Theorem 2, any undominated mechanism is a deadline mechanism. We showed in the text (§5.2, p. 19) that those with deadline $T < \underline{T}$ are dominated, so it remains only to show that those with deadline $T \geq \underline{T}$ are not. By Theorem 2, it suffices to prove that (x^T, X^T) for $T \in [\underline{T}, \infty]$ is not dominated by another deadline mechanism.[51]

Part 1: finite deadlines. Fix a deadline $T \in [\underline{T}, \infty)$; we shall identify a distribution G under which the deadline T yields a strictly higher payoff

[51] Were (x^T, X^T) dominated, it would be dominated by an undominated mechanism (Proposition 5, supplemental appendix K), which by Theorem 2 must be a deadline mechanism.

than any other deadline. In particular, consider the point mass at $T - \underline{T}$. The mechanism $\left(x^T, X^T\right)$ has $x = u^0$ on $[0, T - \underline{T}] \subseteq [0, T]$ and

$$X^T_{T-\underline{T}} = \left(1 - e^{-r\underline{T}}\right)u^0 + e^{-r\underline{T}}u^\star = u^1$$

by ($\diamondsuit$) on p. 16 and the definition of $\underline{T}$. Thus $\left(x^T, X^T\right)$ provides flow payoff $F^0(u^0)$ before the breakthrough and $F^1(u^1)$ afterwards, which is the first-best. Any other deadline T' has $X^{T'}_{T-\underline{T}} \neq u^1$, so provides a strictly lower post-disclosure payoff and a no higher pre-disclosure payoff.

Part 2: the infinite deadline. Fix an arbitrary finite deadline $T \in [0, \infty)$; we must show that $\left(x^T, X^T\right)$ does not dominate (x^∞, X^∞). To that end, we shall identify a distribution G under which the former mechanism is strictly worse. In particular, let G^t denote the point mass at some $t \geq T$. Under this distribution, the payoff difference between the two mechanisms is

$$\begin{aligned}\pi_{G^t}(T) - \pi_{G^t}(\infty) = e^{-rt}\Big\{\Big[F^1(u^\star) - F^1\Big(u^0\Big)\Big] - \Big[F^0(u^\star) - F^0\Big(u^0\Big)\Big]\Big\} \\ + e^{-rT}\Big[F^0(u^\star) - F^0\Big(u^0\Big)\Big].\end{aligned}$$

The second term is strictly negative since F^0 is uniquely maximised at u^0 and $u^\star \leq u^1 < u^0$. By choosing $t \geq T$ large enough, we can make the first term as small as we wish, so that the payoff difference is strictly negative. ■

E Generalisation and proof of Proposition 2 (p. 19)

In this appendix, we obtain a general characterisation of optimal deadlines which entails Proposition 2 and which delivers comparative statics. Write $\left(x^T, X^T\right)$ for the deadline mechanism with deadline $T \in [0, \infty]$, and consider the first-order condition

$$\begin{aligned}&[1 - G(T)]\alpha + \int_{[0,T]} F^{1+}\Big(X^T_t\Big)G(\mathrm{d}t) \\ &\leq 0 \leq [1 - G(T-)]\alpha + \int_{[0,T)} F^{1-}\Big(X^T_t\Big)G(\mathrm{d}t),\end{aligned} \qquad (\partial)$$

where F^{1-} (F^{1+}) is the left-hand (right-hand) derivative of F^1,[52]

$$\alpha \coloneqq \frac{F^0(u^0) - F^0(u^\star)}{u^0 - u^\star},$$

and $G(T-) \coloneqq \lim_{t \uparrow T} G(t)$ for $T > 0$, $G(0-) \coloneqq G(0)$ and $G(\infty) \coloneqq 1$.

[52]These are well-defined since F^1 is concave.

Remark 2. If F^1 is differentiable on $(0, u^0)$, then (∂) reads

$$[G(T) - G(T-)]\Big[F^{1\prime}(u^\star) - \alpha\Big] \leq [1 - G(T)]\alpha + \int_{[0,T]} F^{1\prime}\Big(X_t^T\Big)G(\mathrm{d}t) \leq 0.^{53}$$

If in addition F^0 is affine on $[0, u^0]$ and $u^\star$ strictly exceeds zero, then

$$\alpha = F^{0\prime}(u^\star) = F^{1\prime}(u^\star) = F^{1\prime}\Big(X_t^T\Big) \quad \text{for any } t \geq T,$$

and thus (∂) may be written $\mathbf{E}_G\Big(F^{1\prime}\Big(X_\tau^T\Big)\Big) = 0$, as in Proposition 2.

Whether or not it is exactly optimal to use a deadline mechanism, (∂) is a necessary condition for optimal choice *among deadline mechanisms:*

Lemma 2. Among deadline mechanisms with finite deadline, the best for G satisfy (∂).

In the affine case, (∂) is both necessary and sufficient:

Proposition 2′. If the old frontier F^0 is affine on $[0, u^0]$, then a mechanism is optimal for G iff it is a deadline mechanism with deadline satisfying (∂).

In light of Remark 2, this result immediately implies Proposition 2. Finally, optimal deadlines are monotone in the distribution G:

Proposition 4 (comparative statics)**.** If the old frontier F^0 is affine on $[0, u^0]$ and G first-order stochastically dominates $G^\dagger$, then $T \geq T^\dagger$ for some deadlines T and $T^\dagger$ that are optimal for G and $G^\dagger$, respectively.

To prove the above results, we rely on two observations:

Observation 2. A deadline $T \in [0, \infty]$ satisfies (∂) for *some* distribution G exactly if it belongs to $[\underline{T}, \infty)$.[54]

Observation 3. Write $\pi_G(T)$ for the principal's payoff under G from deadline T. Letting '$\wedge$' denote the minimum, $\pi_G(T)$ is equal to

$$\int_{\mathbf{R}_+} \left[r \int_0^{t \wedge T} e^{-rs} F^0\Big(u^0\Big)\mathrm{d}s + r \int_{t \wedge T}^t e^{-rs} F^0(u^\star)\mathrm{d}s + e^{-rt} F^1\Big(X_t^T\Big) \right] G(\mathrm{d}t).$$

[53] In case $u^\star = 0$, we write $F^{1\prime}(0) \coloneqq F^{1+}(0)$ and assume that the latter is finite.

[54] Each deadline $T \in [\underline{T}, \infty)$ satisfies (∂) when G is the point mass at $T - \underline{T}$. Conversely, any $T < \underline{T}$ violates the first inequality in (∂) since then $X^T < u^1$ and thus $F^{1+}(X^T) > 0$, while $T = \infty$ violates the second inequality because $F^{1-}(X^\infty) \equiv F^{1-}(u^0) < 0$.

Its right- and left-hand derivatives are (for a constant $K > 0$)

$$\pi_G^+(T) = e^{-rT} K \left([1 - G(T)]\alpha + \int_{[0,T]} F^{1+}\left(X_t^T\right) G(\mathrm{d}t) \right) \qquad \text{for } T \in [0, \infty)$$

$$\pi_G^-(T) = e^{-rT} K \left([1 - G(T-)]\alpha + \int_{[0,T)} F^{1-}\left(X_t^T\right) G(\mathrm{d}t) \right) \quad \text{for } T \in (0, \infty).$$

Proof of Lemma 2. $\pi_G^+(T) \leq 0$ is necessary for $T \in [0, \infty)$ to be best, and this rules out $T = 0$ since $\pi_G^+(0) > 0$. Furthermore, $\pi_G^-(T) \geq 0$ is necessary for $T \in (0, \infty)$ to be best. So any best $T < \infty$ satisfies (∂). ■

Proof of Proposition 2′. All optimal mechanisms are deadline mechanisms by Theorem 2 (p. 17), and their deadlines satisfy (∂) if finite by Lemma 2. To rule out the infinite deadline, define $\phi_T \coloneqq F^{1-}(X^T)\mathbf{1}_{[0,T)}$ for each $T \in \mathbf{R}_+$, and note that $\phi_T \to F^{1-}(u^0)$ pointwise as $T \to \infty$ (since $X^T \uparrow u^0$ pointwise and F^{1-} is left-continuous) and that $(\phi_T)_{T \in \mathbf{R}_+}$ is uniformly bounded above by α.[55] Thus by Fatou's lemma,

$$\limsup_{T \to \infty} \int_{[0,T)} F^{1-}\left(X_t^T\right) G(\mathrm{d}t) = \limsup_{T \to \infty} \int_{\mathbf{R}_+} \phi_T \mathrm{d}G \leq F^{1-}\left(u^0\right) < 0,$$

so that $\pi_G^-(T) < 0$ for all sufficiently large $T \in \mathbf{R}_+$. Hence π_G is eventually strictly decreasing, so that any sufficiently late deadline $T < \infty$ is strictly better than ∞: namely, $\pi_G(T) > \lim_{T' \to \infty} \pi_G(T') = \pi_G(\infty)$.

For the converse, consider a deadline mechanism $\left(x^T, X^T\right)$ that satisfies (∂). Then $T \geq \underline{T}$ by Observation 2, so that $\left(x^T, X^T\right)$ is undominated by Proposition 1 (p. 19). It remains to show that $\left(x^T, X^T\right)$ maximises the principal's payoff under G, for which it suffices that T maximise π_G.[56]

Since T satisfies (∂), we need only show that this first-order condition is sufficient for maximisation of π_G, by establishing that π_G^+ and π_G^- are down-crossing.[57] It suffices to show that $T \mapsto e^{rT}\pi_G^+(T)$ and $T \mapsto e^{rT}\pi_G^-(T)$

[55] Since $X^T > u^\star$ on $[0, T)$, we need only show that $F^{1-} \leq \alpha$ on $(u^\star, u^0)$. If $u^\star = 0$, then $F^{1-} < \alpha$ on $(0, u^0)$ by definition of $u^\star$. And if $u^\star > 0$, then $F^{1-} \leq F^{1+}(u^\star) \leq \alpha$ on $(u^\star, u^0)$ since F^1 is concave and α is a supergradient of F^1 at $u^\star$ (by definition of $u^\star$).

[56] Then (x^T, X^T) is better under G than any other deadline mechanism. And it is better than any *non*-deadline mechanism (x, X) because any such is dominated by some undominated mechanism (by Proposition 5 in supplemental appendix K), which by Theorem 2 must be a deadline mechanism $(x^{T'}, X^{T'})$, so that $\Pi_G(x^T, X^T) \geq \Pi_G(x^{T'}, X^{T'}) \geq \Pi_G(x, X)$.

[57] I.e. that $\pi_G^+(T) \leq (<)\ 0$ implies $\pi_G^+(T') \leq (<)\ 0$ for any $T < T'$, and similarly for π_G^-.

are decreasing. For the former, take $T < T'$ and compute

$$\begin{aligned}\frac{e^{rT}\pi_G^+(T') - e^{rT}\pi_G^+(T)}{K} &= [-G(T') - G(T)]\alpha + \int_{(T,T']} F^{1+}\Big(X_t^{T'}\Big) G(\mathrm{d}t) \\ &\quad + \int_{[0,T]} \Big[F^{1+}\Big(X_t^{T'}\Big) - F^{1+}\Big(X_t^{T}\Big)\Big] G(\mathrm{d}t) \\ = \int_{(T,T']} \Big[F^{1+}\Big(X_t^{T'}\Big) - \alpha\Big] G(\mathrm{d}t) &+ \int_{[0,T]} \Big[F^{1+}\Big(X_t^{T'}\Big) - F^{1+}\Big(X_t^{T}\Big)\Big] G(\mathrm{d}t).\end{aligned}$$

The first term is non-positive since $F^{1+} \leq \alpha$ on $[u^\star, u^0] \ni X^{T'}$, and the second is non-positive since F^{1+} is decreasing and $X^{T'} \geq X^T$. Similarly,

$$\begin{aligned}&\frac{e^{rT}\pi_G^-(T') - e^{rT}\pi_G^-(T)}{K} \\ &= \int_{[T,T')} \Big[F^{1-}\Big(X_t^{T'}\Big) - \alpha\Big] G(\mathrm{d}t) + \int_{[0,T)} \Big[F^{1-}\Big(X_t^{T'}\Big) - F^{1-}\Big(X_t^{T}\Big)\Big] G(\mathrm{d}t),\end{aligned}$$

where the second term is non-positive since F^{1-} is decreasing. The first term is also non-positive because $F^{1-}(X_t^{T'}) \leq F^{1+}(u^\star) \leq \alpha$ for each $t \in [T, T')$, where the first inequality holds since F^1 is concave and $X_t^{T'} > u^\star$ for every $t < T'$, and the second holds by definition of $u^\star$. ■

Proof of Proposition 4. By Topkis's theorem,[58] it suffices to show that $\pi_G^+ \geq \pi_{G^\dagger}^+$ and $\pi_G^- \geq \pi_{G^\dagger}^-$ ('increasing differences'). We have for any $T \in \mathbf{R}_+$ that

$$\begin{aligned}\frac{e^{rT}\pi_G^+(T)}{K} &= \mathbf{E}_G\Big(\mathbf{1}_{[0,T]}(\tau) \times F^{1+}\Big(X_\tau^T\Big) + \mathbf{1}_{(T,\infty)}(\tau) \times \alpha\Big) \\ &\geq \mathbf{E}_{G^\dagger}\Big(\mathbf{1}_{[0,T]}(\tau) \times F^{1+}\Big(X_\tau^T\Big) + \mathbf{1}_{(T,\infty)}(\tau) \times \alpha\Big) = \frac{e^{rT}\pi_{G^\dagger}^+(T)}{K},\end{aligned}$$

where the equalities hold by Observation 3, and the inequality holds because G first-order stochastically dominates $G^\dagger$ and the map

$$t \mapsto \mathbf{1}_{[0,T]}(t) \times F^{1+}\Big(X_t^T\Big) + \mathbf{1}_{(T,\infty)}(t) \times \alpha$$

is increasing since F^{1+} and X^T are decreasing and we have $F^{1+} \leq \alpha$ on $[u^\star, u^0] \ni X^T$. A similar argument shows that $\pi_G^- \geq \pi_{G^\dagger}^-$: the map is

$$t \mapsto \mathbf{1}_{[0,T)}(t) \times F^{1-}\Big(X_t^T\Big) + \mathbf{1}_{[T,\infty)}(t) \times \alpha,$$

which is increasing since F^{1+} and X^T are decreasing and $F^{1-}(X_t^T) \leq F^{1+}(u^\star) \leq \alpha$ for $t < T$, where the first inequality holds since F^1 is concave and $X_t^T > u^\star$ for $t < T$, and the second follows from the definition of $u^\star$. ■

[58] See e.g. Theorem 2.8.1 in Topkis (1998, p. 76).

F A superdifferential Euler equation

In this appendix, we define a superdifferential Euler equation for the principal's problem, relate it to optimality (§F.1), and construct a solution (§F.2). These tools will be used in the next two appendices to prove Theorem 3 and Proposition 3 (pp. 20 and 21).

In this appendix and the two that follow it, we shall rely heavily on the convex-analysis concepts reviewed in appendix A (p. 25).

Definition 5. Given a distribution G, a mechanism (x, X) *satisfies the Euler equation (for G)* iff there is a measurable $\phi^0 : \mathbf{R}_+ \to [0, \infty]$ and a G-integrable $\phi^1 : \mathbf{R}_+ \to [-\infty, \infty]$ such that $\phi^0(t)$ is a supergradient of F^0 at x_t for almost all $t \in \mathbf{R}_+$ such that $G(t) < 1$, $\phi^1(t)$ is a supergradient of F^1 at X_t for G-almost all $t \in \mathbf{R}_+$, and

$$[1 - G(t)]\phi^0(t) + \int_{[0,t]} \phi^1 \mathrm{d}G = 0 \quad \text{for every } t \in \mathbf{R}_+. \tag{E}$$

For bounded ϕ^0, the backward-looking integral equation (E) is equivalent to a forward-looking integral equation plus an initial condition:

Observation 4. For a distribution G, a bounded and measurable $\phi^0 : \mathbf{R}_+ \to \mathbf{R}$ and a G-integrable $\phi^1 : \mathbf{R}_+ \to [-\infty, \infty]$, equation (E) holds iff $\mathbf{E}_G(\phi^1(\tau)) = 0$ and

$$\phi^0(t) = \mathbf{E}_G\Big(\phi^1(\tau)\Big|\tau > t\Big) \quad \text{for every } t \in \mathbf{R}_+ \text{ such that } G(t) < 1. \tag{5}$$

Proof. For any $t \in \mathbf{R}_+$, $\int_{(t,\infty)} \phi^1 \mathrm{d}G$ is finite since ϕ^1 is G-integrable, so we may add and subtract it to obtain

$$\begin{aligned} &[1 - G(t)]\phi^0(t) + \int_{[0,t]} \phi^1 \mathrm{d}G \\ &\quad = \begin{cases} [1 - G(t)]\left[\phi^0(t) - \mathbf{E}_G(\phi^1(\tau)|\tau > t)\right] + \mathbf{E}_G(\phi^1(\tau)) & \text{if } G(t) < 1 \\ \mathbf{E}_G(\phi^1(\tau)) & \text{if } G(t) = 1. \end{cases} \end{aligned}$$

Thus $\mathbf{E}_G(\phi^1(\tau)) = 0$ and (5) imply (E). Conversely, if (E) holds, then letting $t \to \infty$ and using the boundedness of ϕ^0 yields

$$0 = -\lim_{t\to\infty}[1 - G(t)]\phi^0(t) = \lim_{t\to\infty} \int_{\mathbf{R}_+} \phi^1 \mathbf{1}_{[0,t]} \mathrm{d}G = \int_{\mathbf{R}_+} \phi^1 \mathrm{d}G = \mathbf{E}_G\Big(\phi^1(\tau)\Big),$$

where the third equality holds by dominated convergence; thus (5) holds. ■

F.1 Optimality and the Euler equation

Let $\mathcal{X}$ be the set of all measurable maps $\mathbf{R}_+ \to [0, u^0]$. For a given breakthrough distribution G, define $\pi_G : \mathcal{X} \to \mathbf{R}$ by

$$\pi_G(x) \coloneqq \Pi_G(x, X) = \mathbf{E}_G\left(r \int_0^\tau e^{-rs} F^0(x_s)\mathrm{d}s + e^{-r\tau} F^1(X_\tau) \right).$$

This is the principal's payoff under G from the mechanism (x, X).

Euler lemma. Let G be any distribution, and suppose that a mechanism (x, X) with $x \in \mathcal{X}$ satisfies the Euler equation (with some ϕ^0, ϕ^1). Then $x \in \arg\max_{\mathcal{X}} \pi_G$. Moreover, any mechanism $(x^\dagger, X^\dagger)$ with $x^\dagger \in \arg\max_{\mathcal{X}} \pi_G$ satisfies the Euler equation with (the same) ϕ^0, ϕ^1.

The proof is in supplemental appendix M.

F.2 Constructing a solution of the Euler equation

Definition 6. F^0, F^1 are *simple* if they are strictly concave and possess bounded derivatives, $F^{1\prime}$ is Lipschitz continuous on $[u^\star, u^0]$, and $u^\star > 0$.

Observation 5. If F^0, F^1 are simple, then a mechanism (x, X) with $u^\star \leq x \leq u^0$ satisfies the Euler equation iff

$$[1 - G(t)] F^{0\prime}(x_t) + \int_{[0,t]} F^{1\prime}(X_s) G(\mathrm{d}s) = 0 \quad \text{for a.e. } t \in \mathbf{R}_+, \tag{6}$$

or equivalently (by Observation 4 in appendix F) $\mathbf{E}_G(F^{1\prime}(X_\tau)) = 0$ and

$$F^{0\prime}(x_t) = \mathbf{E}_G\left(F^{1\prime}(X_\tau) \middle| \tau > t \right) \quad \text{for a.e. } t \in \mathbf{R}_+ \text{ such that } G(t) < 1.$$

Proof. Fix (x, X) with $u^\star \leq x \leq u^0$. If (x, X) satisfies the Euler equation with ϕ^0, ϕ^1, then $\phi^1(s) = F^{1\prime}(X_s)$ for G-a.e. $s \in \mathbf{R}_+$, so that (E) reads

$$[1 - G(t)] \phi^0(t) + \int_{[0,t]} F^{1\prime}(X_s) G(\mathrm{d}s) = 0 \quad \text{for every } t \in \mathbf{R}_+,$$

and thus (6) holds since $\phi^0(t) = F^{0\prime}(x_t)$ for a.e. $t \in \mathbf{R}_+$ with $G(t) < 1$.

Suppose instead that (x, X) satisfies (6). Let $T \coloneqq \inf\{t \in \mathbf{R}_+ : G(t) = 1\}$ with the convention that $\inf \varnothing \coloneqq \infty$, and define $\phi^0, \phi^1 : \mathbf{R}_+ \to \mathbf{R}$ by

$$\phi^0(t) \coloneqq \begin{cases} -\frac{1}{1-G(t)} \int_{[0,t]} F^{1\prime}(X_s) G(\mathrm{d}s) & \text{for } t < T \\ F^{0\prime}(x_t) & \text{for } t \geq T \end{cases}$$

and $\phi^1(t) \coloneqq F^{1\prime}(X_t)$ for every $t \in \mathbf{R}_+$. Then $\phi^0(t) = F^{0\prime}(x_t)$ for a.e. $t \in \mathbf{R}_+$ by (6), and ϕ^0, ϕ^1 satisfy (E). ■

Let $\mathcal{X}'$ be the set of all decreasing maps $\mathbf{R}_+ \to [u^\star, u^0]$, endowed with the topology of pointwise convergence. Given a sequence of technologies $(F^0_n, F^1_n)_{n\in\mathbf{N}}$ satisfying our model assumptions, write u^0_n, $u^\star_n$, and $\mathcal{X}'_n$ for the analogues of u^0, $u^\star$ and $\mathcal{X}'$, respectively.

Observation 6. For technologies F^0, F^1 and $(F^0_n, F^1_n)_{n\in\mathbf{N}}$ such that $u^\star_n \to u^\star$ and $u^0_n \uparrow u^0$, any sequence $(x^n)_{n\in\mathbf{N}}$ with $x^n \in \mathcal{X}'_n$ for each $n \in \mathbf{N}$ admits a convergent subsequence with limit in $\mathcal{X}'$. (Thus $\mathcal{X}'$ is sequentially compact.)

Proof. The sequence $(x^n)_{n\in\mathbf{N}}$ lives in $[0, u^0]$ since $x^n \leq u^0_n \leq u^0$ for each $n \in \mathbf{N}$. Thus by the Helly selection theorem (e.g. Rudin, 1976, p. 167), $(x^n)_{n\in\mathbf{N}}$ admits a subsequence along which it converges pointwise to some decreasing $x : \mathbf{R}_+ \to [0, u^0]$. We have $x \geq u^\star$ since $x^n \geq u^\star_n$ for each $n \in \mathbf{N}$ and $u^\star_n \to u^\star$. By considering the constant sequence $(F^0_n, F^1_n) \equiv (F^0, F^1)$, we see that $\mathcal{X}'$ is sequentially compact. ■

The following three lemmata construct a solution of the Euler equation. Their (tedious) proofs are relegated to supplemental appendix N.

Lemma 3. If F^0, F^1 are simple and G has finite support, then there exists an $x \in \mathcal{X}'$ such that (x, X) satisfies the Euler equation.

Lemma 4. Let F^0, F^1 be simple, and let $(G_n)_{n\in\mathbf{N}}$ be a sequence of finite-support CDFs converging pointwise to a CDF G. Let $(x^n)_{n\in\mathbf{N}}$ be a sequence in $\mathcal{X}'$ such that (x^n, X^n) satisfies the Euler equation for (F^0, F^1, G_n) for each $n \in \mathbf{N}$, and suppose that $(x^n)_{n\in\mathbf{N}}$ converges pointwise to some $x \in \mathcal{X}'$. Then (x, X) satisfies the Euler equation for (F^0, F^1, G).

Lemma 5. Given F^0, F^1, there exists a sequence $(F^0_n, F^1_n)_{n\in\mathbf{N}}$ of simple technologies such that $u^0_n \uparrow u^0$ and $u^\star_n \to u^\star$ as $n \to \infty$ and, for any CDF G with unbounded support and any mechanism (x, X), if x is the pointwise limit of a sequence $(x^n)_{n\in\mathbf{N}}$ along which (x^n, X^n) satisfies the Euler equation for (F^0_n, F^1_n, G) and $x^n \in \mathcal{X}'_n$ for each $n \in \mathbf{N}$, then (x, X) satisfies the Euler equation for (F^0, F^1, G) with some increasing ϕ^0, ϕ^1.

The following will be used in the next appendix to prove Theorem 3.

Existence corollary. For any distribution G with unbounded support, there is a mechanism (x, X) with $x \in \mathcal{X}'$ which satisfies the Euler equation for G with some increasing $\phi^0, \phi^1 : \mathbf{R}_+ \to \mathbf{R}$.

Proof. Let $(F^0_n, F^1_n)_{n\in\mathbf{N}}$ be the simple technologies delivered by Lemma 5. Choose a sequence $(G_m)_{m\in\mathbf{N}}$ of finite-support distributions converging pointwise to G.

Fix an arbitrary $n \in \mathbf{N}$. For every $m \in \mathbf{N}$, Lemma 3 assures us of the existence of an $x^{nm} \in \mathcal{X}'_n$ such that (x^{nm}, X^{nm}) satisfies the Euler equation for (F^0_n, F^1_n, G_m). Since $\mathcal{X}'_n$ is sequentially compact by Observation 6, we may assume (passing to a subsequence if necessary) that x^{nm} converges pointwise as $m \to \infty$ to some $x^n \in \mathcal{X}'_n$. Since $u^0_n \to u^0$ and $u^\star_n \to u^\star$ as $n \to \infty$, Observation 6 permits us to assume (again passing to a subsequence if required) that x^n converges pointwise to some $x \in \mathcal{X}'$ as $n \to \infty$.

By Lemma 4, (x^n, X^n) satisfies the Euler equation for (F^0_n, F^1_n, G) for each $n \in \mathbf{N}$. Hence by Lemma 5, (x, X) satisfies the Euler equation for (F^0, F^1, G) with some increasing ϕ^0, ϕ^1. ■

G Proof of Theorem 3 (p. 20)

We shall make extensive use of the Euler equation (E) (appendix F, p. 35). Recall from §F.2 the definition of $\mathcal{X}'$.

Lemma 6. Suppose that G satisfies $G(0) = 0$ and has unbounded support. Let (x, X) with $x \in \mathcal{X}'$ satisfy the Euler equation with some ϕ^0, ϕ^1 such that ϕ^0 is increasing. Then $\lim_{t\to 0} x_t = u^0$ and $\lim_{t\to\infty} x_t = u^\star$.

Proof. Since x is decreasing with $u^\star \leq x \leq u^0$, the limits

$$\bar{u} := \lim_{t\to 0} x_t \quad \text{and} \quad \underline{u} := \lim_{t\to\infty} x_t$$

exist and satisfy $u^\star \leq \underline{u} \leq \bar{u} \leq u^0$. As G has unbounded support, ϕ^0 is a supergradient of F^0 at x_t for a.e. $t \in \mathbf{R}_+$.

To show that $\bar{u} \geq u^0$, note that for a.e. $t \in \mathbf{R}_+$, $\phi^0(t)$ is a supergradient at $x_t \leq \bar{u}$ of the concave function F^0, so that $\phi^0(t) \geq F^{0+}(x_t) \geq F^{0+}(\bar{u})$. Thus $\phi^0 \geq F^{0+}(\bar{u})$ on $(0, \infty)$ since ϕ^0 is increasing. Letting $t \to 0$ in (E) (p. 35) then yields

$$0 = \lim_{t\to 0} \phi^0(t) \geq F^{0+}(\bar{u}),$$

which implies that $\bar{u} \geq u^0$ since $F^{0+} > 0$ on $[0, u^0)$ by definition of u^0.

To show that $\underline{u} \leq u^\star$, assume without loss that $\underline{u} > 0$. Then ϕ^0 is bounded, since it is increasing and $\phi^0(t)$ is a supergradient of the concave function F^0 at x_t for a.e. $t \in \mathbf{R}_+$. Hence, we may use Observation 4 (p. 35) to obtain

$$F^{0+}(x_t) \leq \phi^0(t) = \mathbf{E}_G\left(\phi^1(\tau) \middle| \tau > t\right) \leq F^{1-}(\underline{u}) \quad \text{for a.e. } t \in \mathbf{R}_+,$$

where the first (second) inequality holds since $\phi^0(t)$ ($\phi^1(s)$) is a supergradient of the concave function F^0 at x_t for a.e. $t \in \mathbf{R}_+$ (of F^1 at $X_s \geq \underline{u}$ for

G-a.e. $s \in \mathbf{R}_+$). Then $F^{0+}(x_t) \leq F^{1-}(\underline{u})$ for *every* $t \in \mathbf{R}_+$ since F^{0+} and x are decreasing. Since F^{0+} is right-continuous, letting $t \to \infty$ yields $F^{0+}(\underline{u}) \leq F^{1-}(\underline{u})$, which implies that $\underline{u} \leq u^\star$ by definition of the latter. ■

Recall from appendix F the definitions of $\mathcal{X}$ and π_G, the Euler lemma, and the existence corollary.

Proof of Theorem 3. Let G be a distribution with $G(0) = 0$ and unbounded support. By the existence corollary, there is a mechanism $(x^\dagger, X^\dagger)$ with $x^\dagger \in \mathcal{X}'$ which satisfies the Euler equation for G with some increasing ϕ^0, ϕ^1. By the Euler lemma, $x^\dagger$ belongs to $\arg\max_{\mathcal{X}} \pi_G$.

Let (x, X) be optimal for G; we must show that it has the properties asserted by Theorem 3. By Lemma 0 (p. 12), it must be that $x \in \mathcal{X}$. Thus x belongs to $\arg\max_{\mathcal{X}} \pi_G$, so by the Euler lemma again, (x, X) satisfies the Euler equation with (the above increasing) ϕ^0, ϕ^1.

It suffices to show that some version[59] of x is decreasing and $\geq u^\star$, since it then belongs to $\mathcal{X}'$, so that the remaining properties $\lim_{t\to 0} x_t = u^0$ and $\lim_{t\to\infty} x_t = u^\star$ follow by Lemma 6.

Adopt the convention that $F^{0-}(0) \coloneqq \infty$.

Claim 0. $\phi^0 \leq F^{0-}(u^\star)$, strictly on a neighbourhood of $t = 0$.

Proof. The result is immediate if $u^\star = 0$, so suppose that $u^\star > 0$. Since $(x^\dagger, X^\dagger)$ satisfies the Euler equation with ϕ^0, ϕ^1 and G has unbounded support, $\phi^0(t)$ is a supergradient of F^0 at $x^\dagger_t$ for a.e. $t \in \mathbf{R}_+$. Thus since F^0 is concave and $x^\dagger \geq u^\star$ (because $x^\dagger \in \mathcal{X}'$), we have $\phi^0(t) \leq F^{0-}(x^\dagger_t) \leq F^{0-}(u^\star)$ for a.e. $t \in \mathbf{R}_+$. Hence $\phi^0 \leq F^{0-}(u^\star)$ since ϕ^0 is increasing.

Letting $t \to 0$ in (E) (appendix F, p. 35) yields $\lim_{t\to 0} \phi^0(t) = 0 < F^{0-}(u^\star)$, so that $\phi^0(0) \leq \phi^0(t) < F^{0-}(u^\star)$ for all sufficiently small $t > 0$. □

Write T for the (possibly infinite) time at which ϕ^0 hits $F^{0-}(u^\star)$:

$$T \coloneqq \inf\left\{t \in \mathbf{R}_+ : \phi^0(t) \geq F^{0-}(u^\star)\right\},$$

with the convention that $\inf \varnothing \coloneqq \infty$. T is strictly positive by claim 0.

The (increasing) function ϕ^0 is called *non-constant at* $t \in \mathbf{R}_+$ if $\phi^0(s) \neq \phi^0(t)$ for every $s \neq t$, and *constant at* t otherwise. Clearly if ϕ^0 is constant at t, then it is constant on a proper interval containing t.[60]

Claim 1. $x = x^\dagger$ a.e. on $\{t \in \mathbf{R}_+ : \phi^0 \text{ is non-constant at } t\}$.

[59]Recall from footnote 21 (p. 10) that $\widetilde{x}$ a *version* of x exactly if $\widetilde{x} = x$ a.e.
[60]But t need not be in the *interior* of such an interval.

A set of times is *prior to T* iff its intersection with (T, ∞) is empty. (The set of times in claim 1 is prior to T, by claim 0 and the definition of T.)

Claim 2. On any proper interval of $\mathbf{R}_+$ prior to T on which ϕ^0 is constant, some version of x is decreasing.

Claim 3. If $T < \infty$, then on $[T, \infty)$, some version of x is decreasing and bounded below by $u^\star$.

For each maximal proper interval of $\mathbf{R}_+$ prior to T on which ϕ^0 is constant at some $\alpha \in \mathbf{R}$, claim 2 delivers a version x^α of x that is decreasing on this interval. If $T < \infty$, then claim 3 provides a version $x^\star$ of x that is decreasing on $[T, \infty)$ and bounded below by $u^\star$. Define $\widetilde{x} : \mathbf{R}_+ \to \mathbf{R}$ by

$$\widetilde{x}_t \coloneqq \begin{cases} x_t^{\phi^0(t)} & \text{if } t < T \text{ and } \phi^0 \text{ is constant at } t \\ x_t^\dagger & \text{if } (t < T \text{ and}) \ \phi^0 \text{ is non-constant at } t \\ x_t^\star & \text{if } t \geq T. \end{cases}$$

We have $\widetilde{x} = x^\dagger = x$ a.e. on $\{t \in \mathbf{R}_+ : \phi^0 \text{ is non-constant at } t\}$ by claim 1. Thus $\widetilde{x}$ is a version of x.[61]

Let $\mathcal{T}$ be the set of times $t \in \mathbf{R}_+$ at which $\phi^0(t)$ is a supergradient of F^0 at $\widetilde{x}_t$. Its complement $\mathbf{R}_+ \setminus \mathcal{T}$ is null since $\widetilde{x}$ is a version of x, (x, X) satisfies the Euler equation with ϕ^0, ϕ^1, and G has unbounded support. It therefore suffices to show that $\widetilde{x}$ is decreasing and bounded below by $u^\star$ on $\mathcal{T}$.[62]

To see that $\widetilde{x}$ is decreasing on $\mathcal{T}$, fix any $s < t$ in $\mathcal{T}$; we must show that $\widetilde{x}_s \geq \widetilde{x}_t$. If $\phi^0(s) \neq \phi^0(t)$, then $\phi^0(s) < \phi^0(t)$ since ϕ^0 is increasing. Since s, t belong to $\mathcal{T}$ and F^0 is concave, it follows that

$$F^{0+}(\widetilde{x}_s) \leq \phi^0(s) < \phi^0(t) \leq F^{0-}(\widetilde{x}_t),$$

which implies that $\widetilde{x}_s \geq \widetilde{x}_t$ since F^0 is concave. If instead $\phi^0(s) = \phi^0(t)$, then we have either $s, t < T$ or $s, t \geq T$.[63] In the former (latter) case, $\widetilde{x}$ equals the decreasing function $x^{\phi^0(t)}$ (the decreasing function $x^\star$) on $[s, t]$.

It remains to show that $\widetilde{x} \geq u^\star$ on $\mathcal{T}$. If $T < \infty$, then this holds because $\widetilde{x}$ is decreasing and $\widetilde{x} = x^\star \geq u^\star$ on $[T, \infty)$. If instead $T = \infty$, then

$$F^{0+}(\widetilde{x}_t) \leq \phi^0(t) < F^{0-}(u^\star) \quad \text{for every } t \in \mathcal{T}$$

[61] Since ϕ^0 is increasing, it is constant on at most countably many intervals. So the definition of $\widetilde{x}$ has at most countably many cases, in each of which $\widetilde{x}$ equals a version of x.

[62] Then define $\bar{x}_t \coloneqq \sup_{[t,\infty)\cap\mathcal{T}} \widetilde{x}$ for each $t \in \mathbf{R}_+$. This $\bar{x}$ is a version of $\widetilde{x}$ (and thus of x), and is (everywhere) decreasing and bounded below by $u^\star$.

[63] s, t must be on the same side of T since ϕ^0 (being increasing) is constant on $[s, t]$, whereas $\phi^0(T - \varepsilon) < \phi^0(T)$ for any $\varepsilon \in [0, T)$ by (claim 0) and the definition of T.

by definition of $\mathcal{T}$ and the concavity of F^0 (weak inequality) and by definition of T (strict inequality). Since F^0 is concave, this implies that $\widetilde{x} \geq u^\star$ on $\mathcal{T}$.

The rest of the proof is devoted to establishing claims 1, 2 and 3. The argument for the first is straightforward, while those for the latter two are (local) 'front-loading' arguments similar to the proof of Theorem 2 (p. 17).

Proof of claim 1. Write $I := \{t \in \mathbf{R}_+ : \phi^0 \text{ is non-constant at } t\}$; we must show that $x = x^\dagger$ a.e. on I. By definition, ϕ^0 is strictly increasing on I.

Let A be the set of all $\alpha \in \mathbf{R}$ that are supergradients of F^0 at more than one $u \in [0, u^0]$. A is at most countable since F^0 is concave. Thus $I' := \{t \in I : \phi^0(t) \in A\}$ is null since ϕ^0 is strictly increasing on I.

Since (x, X) and $(x^\dagger, X^\dagger)$ satisfy the Euler equation with ϕ^0, ϕ^1 and G has unbounded support, $\phi^0(t)$ is a supergradient of F^0 at both x_t and $x_t^\dagger$ for a.e. $t \in \mathbf{R}_+$. The same therefore holds for a.e. $t \in I \setminus I'$, and $x_t = x_t^\dagger$ at each such t by definition of I' (and A). Thus $x = x^\dagger$ a.e. on I since I' is null. ☐

To prove claims 2 and 3, we shall utilise a forward-looking variant of Euler equation.[64] For any $t \in \mathbf{R}_+$, $\int_{(t,\infty)} \phi^1 \mathrm{d}G$ is finite since ϕ^1 is G-integrable, and $G(t) < 1$ since G has unbounded support. We may therefore add and subtract $\int_{(t,\infty)} \phi^1 \mathrm{d}G$ in (E) (p. 35) and divide by $1 - G(t)$ to obtain

$$\phi^0(t) = \mathbf{E}_G\Big(\phi^1(\tau)\Big|\tau > t\Big) - \frac{\mathbf{E}_G(\phi^1(\tau))}{1 - G(t)} \quad \text{for all } t \in \mathbf{R}_+. \tag{E$'$}$$

Moreover, (E) and the non-negativity of ϕ^0 imply that $\int_{[0,t]} \phi^1 \mathrm{d}G \leq 0$ for every $t \in \mathbf{R}_+$, so letting $t \to \infty$ and using dominated convergence yields[65]

$$\mathbf{E}_G\Big(\phi^1(\tau)\Big) \leq 0 \tag{∞}$$

We next prove claim 3. This requires a supporting claim:

Claim 4. If $T < \infty$, then $X_T \geq u^\star$.

Proof. The result is trivial if $u^\star = 0$, so suppose $u^\star > 0$. Fix any $\varepsilon \in (0, T)$. (Recall that $T > 0$, by claim 0.) ϕ^0 is not constant on $[T - \varepsilon, T + \varepsilon]$, and thus (E) (appendix F, p. 35) requires that $G(T - \varepsilon) < G(T + \varepsilon)$ since G has unbounded support. Then since (x, X) satisfies the Euler equation with ϕ^0, ϕ^1, it must be that $\phi^1(t)$ is a supergradient of F^1 at X_t for some $t \in (T - \varepsilon, T + \varepsilon]$.

[64] Similar to Observation 4 in appendix F (p. 35), but without boundedness of ϕ^0.

[65] In detail, $0 \geq \lim_{t\to\infty} \int_{\mathbf{R}_+} \phi^1 \mathbf{1}_{[0,t]} \mathrm{d}G = \int_{\mathbf{R}_+} \phi^1 \mathrm{d}G = \mathbf{E}_G\big(\phi^1(\tau)\big)$.

Fix any $u \in [0, u^\star)$. Since $u^\star$ is a strict local maximum of $F^1 - F^0$, $F^1 - F^0$ is not decreasing on $[u, u^\star]$, and thus there is a $u' \in [u, u^\star)$ at which $F^{1+}(u') > F^{0-}(u^\star)$.[66] Then

$$F^{1+}(u') > F^{0-}(u^\star) \geq \phi^0(t) = \mathbf{E}_G\Big(\phi^1(\tau)\Big|\tau > t\Big) - \frac{\mathbf{E}_G(\phi^1(\tau))}{1 - G(t)} \geq \phi^1(t)$$

where the second inequality holds by claim 0, the equality is (E′), and the last inequality holds by (∞) and the fact that ϕ^1 is increasing. Thus $X_t > u' \geq u$ since $\phi^1(t)$ is a supergradient at X_t of the concave function F^1.

Since $\varepsilon \in (0, T)$ and $u \in [0, u^\star)$ were arbitrary and X is continuous, it follows that $X_T \geq u^\star$. □

Write $\pi_t := \pi_{G^t}$ for $t > 0$, where G^t denotes the point mass at t.

Proof of claim 3. Let $u' \in \left[u^\star, u^0\right]$ be the largest $u \in \mathbf{R}_+$ at which F^0 admits $F^{0-}(u^\star)$ as a supergradient. We may have $u' = u^\star$; if not, then F^0 is affine on the interval $[u^\star, u']$, with slope $F^{0-}(u^\star)$.

We have $\phi^0 = F^{0-}(u^\star)$ on (T, ∞) by claim 0, the definition of T and the fact that ϕ^0 is increasing. Then since (x, X) satisfies the Euler equation with ϕ^0, ϕ^1 and G has unbounded support, we must have $x \leq u'$ a.e. on (T, ∞). It follows that $X \leq u'$ on $[T, \infty)$.

On the other hand, we have $X_T \geq u^\star$ by claim 4. If $u' = u^\star$, then we are done: $X = u^\star$ on $[T, \infty)$, and thus $x = u^\star$ a.e. on (T, ∞), which obviously has a version that is decreasing and bounded below by $u^\star$ on $[T, \infty)$.

It remains to consider the case in which $u' > u^\star$, meaning that F^0 has an affine segment with slope $F^{0-}(u^\star)$ extending from $u^\star$ to u'. We shall front-load the mechanism (x, X) over this affine segment, much as in the proof of Theorem 2 (p. 17). In particular, given a deadline $T' \in [T, \infty]$, consider

$$x^\star_t = \begin{cases} x_t & \text{for } t \in [0, T) \\ u' & \text{for } t \in [T, T') \\ u^\star & \text{for } t \in [T', \infty). \end{cases}$$

Since $u^\star \leq X_T \leq u'$, we may choose the deadline T' so that $X^\star_T = X_T$.

We will show that the front-loaded mechanism $(x^\star, X^\star)$ dominates (x, X) unless $X^\star = X$. This suffices because (x, X) is undominated (being optimal for G), so that we must have $X = X^\star$ and thus $x = x^\star$ a.e.; and $x^\star$ is decreasing and bounded below by $u^\star$ on $[T, \infty)$.

[66] If not, then $F^1 - F^0$ would be decreasing on $[u, u^\star]$ since $(F^1 - F^0)^+ = F^{1+} - F^{0+} \leq F^{1+} - F^{0-}(u^\star) \leq 0$ on $[u, u^\star)$, where the first inequality holds since F^0 is concave.

Clearly $\pi_t(x^\star) = \pi_t(x)$ for all $t \leq T$; we will show that for each $t > T$, we have $\pi_t(x^\star) \geq \pi_t(x)$, with equality only if $X^\star_t = X_t$. Define

$$\widehat{F}^0(u) \coloneqq F^0(u') - (u' - u)F^{0-}(u') \quad \text{for each } u \in \mathbf{R}_+.$$

We have $\widehat{F}^0 \geq F^0$ (with equality on $[u^\star, u']$) since $F^{0-}(u') = F^{0-}(u^\star)$ is a supergradient of F^0 at u' (at every $u \in [u^\star, u']$). Thus for any $t > T$, we have

$$\begin{aligned}
&\pi_t(x) - \pi_t(x^\star) \\
&\qquad = r\int_T^t e^{-rs}\Big[F^0(x_s) - F^0(x^\star_s)\Big]\mathrm{d}s + e^{-rt}\Big[F^1(X_t) - F^1(X^\star_t)\Big] \\
&\qquad \leq r\int_T^t e^{-rs}\Big[\widehat{F}^0(x_s) - \widehat{F}^0(x^\star_s)\Big]\mathrm{d}s + e^{-rt}\Big[F^1(X_t) - F^1(X^\star_t)\Big] \\
&\qquad = e^{-rt}\Big[\Big(F^1 - \widehat{F}^0\Big)(X_t) - \Big(F^1 - \widehat{F}^0\Big)(X^\star_t)\Big],
\end{aligned}$$

where the first equality holds since $x = x^\star$ on $[0, T]$, the inequality holds since $F^0 \leq \widehat{F}^0$ with equality on $[u^\star, u'] \ni x^\star$, and the final equality holds since $\widehat{F}^0$ is affine on $[0, u']$ and $X_T = X^\star_T$.

Since $u^\star$ is a strict local maximum of $F^1 - F^0$ and $F^1 - \widehat{F}^0 \leq F^1 - F^0$ with equality at $u^\star$, it must be that $u^\star$ is a strict local maximum of $F^1 - \widehat{F}^0$. Thus since $F^1 - \widehat{F}^0$ is concave, it is strictly increasing on $[0, u^\star]$ and strictly decreasing on $[u^\star, u']$. It thus suffices to show that $X^\star$ lies between X and $u^\star$. And this holds because $X \geq X^\star \geq u^\star$ on (T, T'),[67] while $X^\star = u^\star$ on $[T', \infty)$. □

It remains only to prove claim 2.

Proof of claim 2. Fix a maximal proper interval J of $\mathbf{R}_+$ prior to T on which ϕ^0 is constant, and let $\alpha \in \mathbf{R}$ be the value that ϕ^0 takes on J.

Since F^0 is concave, the set of $u \in \left[0, u^0\right]$ at which α is a supergradient of F^0 is an interval $[u', u'']$, where $u^\star \leq u' \leq u'' \leq u^0$.[68] Since (x, X) satisfies the

[67] The first inequality holds because for any $t \in (T, T')$,

$$X_t = e^{r(t-T)}X_T - r\int_T^t e^{-r(s-t)}x_s\mathrm{d}s \geq e^{r(t-T)}X_T - r\int_T^t e^{-r(s-t)}u'\mathrm{d}s = X^\star_t,$$

where the inequality holds since $x \leq u'$ a.e. on (T, ∞), and the last equality holds because $X_T = X^\star_T$ and $x^\star = u'$ on $[T, T')$.

[68] $u^\star \leq u'$ obtains since F^0 is concave and $F^{0+}(u') = \alpha = \phi^0 < F^{0-}(u^\star)$ on J, where the inequality holds since J is prior to T. As for $u'' \leq u^0$, letting $t \to 0$ in (E) (appendix F, p. 35) yields $\lim_{t\to 0}\phi^0(t) = 0$. Since ϕ^0 is increasing and J is a proper interval, it follows that $\alpha \geq 0$. Thus F^0 is increasing on $[u', u'']$, so that $u'' \leq u^0$ by definition of the latter.

Euler equation with ϕ^0, ϕ^1 and G has unbounded support, α is a supergradient of F^0 at x_t for a.e. $t \in J$. This implies that $u' \leq x \leq u''$ a.e. on J.

If $u' = u''$, then we are done: x is a.e. constant at $u'' = u' \geq u^\star$ on J, so obviously admits a version that is decreasing.

Suppose instead that $u' < u''$, meaning that F^0 has an affine segment with slope α extending from u' to u''. We shall front-load the mechanism (x, X) over this affine segment, imitating the proof of Theorem 2 (p. 17). In particular, given a deadline $T' \in \operatorname{cl} J$, define

$$x_t^\star \coloneqq \begin{cases} x_t & \text{for } t \notin J \\ u'' & \text{for } t \leq T' \text{ in } J \\ u' & \text{for } t > T' \text{ in } J. \end{cases}$$

Since $u' \leq x \leq u''$ a.e. on J, we may choose the deadline $T' \in \operatorname{cl} J$ so that $X^\star_{\inf J} = X_{\inf J}$.

We shall show that the front-loaded mechanism $(x^\star, X^\star)$ dominates (x, X) unless $X^\star = X$. This is sufficient because (x, X) is undominated (being optimal for G), so must then satisfy $x = x^\star$ a.e.; and $x^\star$ is decreasing on J.

We have $\pi_t(x^\star) = \pi_t(x)$ for every $t \notin J$ since F^0 is affine on $[u', u'']$ and $X^\star = X$ off J.[69] It remains to show that $\pi_t(x^\star) \geq \pi_t(x)$ for every $t \in J$, with equality only if $X_t^\star = X_t$. Define

$$\psi(u) \coloneqq F^1(u) - \alpha u \quad \text{for each } u \in \mathbf{R}_+.$$

Since F^0 is affine with slope α on $[u', u'']$ and $X^\star_{\inf J} = X_{\inf J}$, we have

$$\pi_t(x^\star) - \pi_t(x) = e^{-rt}\big[\psi(X_t^\star) - \psi(X_t)\big] \quad \text{for each } t \in J.$$

Since $X^\star_{\inf J} = X_{\inf J}$ and $u' \leq x \leq u''$ a.e. on J, we have $X^\star \leq X$ on J.[70] It therefore suffices to show that ψ is strictly decreasing on $[\inf_J X^\star, \infty)$.

Suppose that $X \geq u'$ on J. Then $X_{\sup J} \geq u'$ since X is continuous, so that $X^\star \geq u'$ on J as well. We need thus only show that ψ is strictly decreasing on $[u', \infty)$. It is strictly decreasing on $[u', u'']$ since there we have $\psi = \big(F^1 - F^0\big) + k$ for a constant $k \in \mathbf{R}$, and $F^1 - F^0$ is strictly decreasing on $[u^\star, u^0] \supseteq [u', u'']$ by definition of $u^\star$. Since ψ is concave, it must then be strictly decreasing on all of $[u', \infty)$.

It remains to consider the case in which $X_s < u'$ for some $s \in J$. Write

$$t' \coloneqq \inf J \quad \text{and} \quad t'' \coloneqq \sup J,$$

[69] Replicate the payoff-rewriting exercise in the sketch proof of Theorem 2 (p. 17).

[70] The idea is that front-loading lowers X pointwise; we saw this in the sketch proof of Theorem 2 (p. 17) and in footnote 67.

noting that $t' < t''$ since J is a proper interval. It must be that $t'' < \infty$, since otherwise we would have $x \geq u'$ a.e. on (t', ∞) and thus $X \geq u'$ on J. Since $X_{t''} \leq X$ on $[s, t'']$,[71] it suffices to show that ψ is strictly decreasing on $[X_{t''}, \infty)$. And for this, it is enough that $t \mapsto F^{1+}(X_t) - \alpha$ be strictly negative at, or arbitrarily close to, t''.[72]

Remark that since $\phi^0 = \alpha$ on (t', t''), letting $t \uparrow t''$ in (E′) on p. 41 yields

$$\mathbf{E}_G\Big(\phi^1(\tau)\Big|\tau \geq t''\Big) - \frac{\mathbf{E}_G(\phi^1(\tau))}{1 - \lim_{t\uparrow t''} G(t)} = \alpha. \tag{$\uparrow$}$$

Suppose first that G has an atom at t''. Then $\phi^1(t'')$ is a supergradient of F^1 at $X_{t''}$ since (x, X) satisfies the Euler equation with ϕ^0, ϕ^1. Since $F^{1+}(X_{t''}) \leq \phi^1(t'')$ (as F^1 is concave), it suffices to show that $\phi^1(t'') < \alpha$. So suppose toward a contradiction that $\phi^1(t'') \geq \alpha$. Then

$$\alpha \leq \phi^1(t'') \leq \mathbf{E}_G\Big(\phi^1(\tau)\Big|\tau \geq t''\Big) = \alpha + \frac{\mathbf{E}_G(\phi^1(\tau))}{1 - \lim_{t\uparrow t''} G(t)} \leq \alpha$$

since ϕ^1 is increasing (second inequality), by ($\uparrow$) (the equality) and by (∞) on p. 41 (final inequality). It follows that $\phi^1(t'') = \mathbf{E}_G(\phi^1(\tau)|\tau \geq t'') = \alpha$, so that $\phi^1 = \alpha$ G-a.e. on $[t'', \infty)$ since ϕ^1 is increasing. But then $\phi^0 = \alpha$ on (t', ∞) by (E′) on p. 41, which contradicts the fact that $t'' < \infty$.

Suppose instead that G has no atom at t''. Then t'' belongs to J since

$$\begin{aligned}\phi^0(t'') &= \mathbf{E}_G\Big(\phi^1(\tau)\Big|\tau > t''\Big) - \frac{\mathbf{E}_G(\phi^1(\tau))}{1 - G(t'')} \\ &= \mathbf{E}_G\Big(\phi^1(\tau)\Big|\tau \geq t''\Big) - \frac{\mathbf{E}_G(\phi^1(\tau))}{1 - \lim_{t\uparrow t''} G(t)} = \alpha\end{aligned}$$

by (E′) on p. 41 (first equality) and ($\uparrow$) (last equality). Fix any $\varepsilon > 0$. Since J is a maximal interval of constancy of ϕ^0 and t'' belongs to J, ϕ^0 is not constant on $[t'', t''+\varepsilon)$, and thus $[t'', t''+\varepsilon)$ is G-non-null by (E′). Since (x, X) satisfies the Euler equation with ϕ^0, ϕ^1, it follows that $\phi^1(t)$ is a supergradient of F^1 at X_t for some $t \in [t'', t''+\varepsilon)$.

Now, since G has no atom at t'' and ϕ^1 is increasing, we must have

$$\lim_{t\downarrow t''} \phi^1(t) < \mathbf{E}_G\Big(\phi^1(\tau)\Big|\tau \geq t''\Big),$$

[71] If $X_t = \min_{[s,t'']} X$ for $t \in [s, t'']$, then since $x \geq u' > X_s \geq X_t$ a.e. on $[s, t'']$, we have

$$X_t = r\int_t^{t''} e^{-r(z-t)} x_z \mathrm{d}z + e^{-r(t''-t)} X_{t''} \geq (1 - e^{-r(t''-t)}) X_t + e^{-r(t''-t)} X_{t''}.$$

[72] Since then $F^{1+} - \alpha < 0$ on $(X_{t''}, \infty)$, as F^{1+} is decreasing.

as otherwise ϕ^1 would be G-a.e. constant on (t'', ∞), which would contradict $t'' < \infty$ by the argument above. Thus for $\varepsilon > 0$ sufficiently small, we have

$$F^{1+}(X_t) \leq \phi^1(t) \leq \phi^1(t'' + \varepsilon)$$
$$< \mathbf{E}_G\left(\phi^1(\tau) \middle| \tau \geq t''\right) = \alpha + \frac{\mathbf{E}_G(\phi^1(\tau))}{1 - \lim_{t \uparrow t''} G(t)} \leq \alpha$$

by the concavity of F^1 (first inequality), the monotonicity of ϕ^1 (second inequality), ($\uparrow$) above (the equality) and (∞) on p. 41 (final inequality).

Since $\varepsilon > 0$ may be chosen arbitrarily small and t belongs to $[t'', t'' + \varepsilon)$, it follows that $F^{1+}(X_t) - \alpha < 0$ for arbitrarily small $t \geq t''$, as desired. □

With all three claims now established, the proof is complete. ■

H Generalisation and proof of Proposition 3 (p. 21)

Recall the (superdifferential) Euler equation defined in appendix F (p. 35).

Proposition 3′. Let G be a distribution with unbounded support. Any mechanism that is optimal for G satisfies the Euler equation for G. Any undominated mechanism that satisfies the Euler equation for G is optimal for G.

This result refines Proposition 3 in two ways: it provides that the Euler equation is necessary absent any auxiliary assumptions, and furthermore asserts sufficiency. To prove it, we shall rely on the Euler lemma and the existence corollary in appendix F (pp. 36 and 37).

Proof of Proposition 3′. Fix a distribution G. By Lemma 0 and Theorem 1 (pp. 12 and 13), any undominated mechanism has the form (x, X) with $x \in \mathcal{X}$. If (x, X) is undominated and satisfies the Euler equation for G, then it maximises the principal's payoff under G by (the first part of) the Euler lemma, so is optimal for G. Conversely, suppose that (x, X) is optimal for G. By the existence corollary, there is a(nother) mechanism that satisfies the Euler equation for G. So by (the second part of) the Euler lemma, (x, X) satisfies the Euler equation. ■

Proof of Proposition 3. Assume that $u^\star > 0$ and that F^0, F^1 are differentiable on $(0, u^0)$, and let (x, X) be optimal for a distribution G with $G(0) = 0$ and unbounded support. Then x is decreasing with $0 < u^\star \leq X \leq x \leq u^0$ and $\lim_{t \to \infty} x_t = u^\star \leq u^1 < u^0$ by Theorem 3 (p. 20), and (x, X) satisfies the Euler equation by Proposition 3′.

Thus $0 < X < u^0$, so that F^1 is differentiable at X_t for every $t \in \mathbf{R}_+$. Similarly, F^0 is differentiable at x_t for every $t \in \mathbf{R}_+$ at which $x_t < u^0$. Hence by Observation 4 in appendix F (p. 35), the Euler equation implies that $\mathbf{E}_G(F^{1\prime}(X_\tau)) = 0$ and

$$F^{0\prime}(x_t) = \mathbf{E}_G\Big(F^{1\prime}(X_\tau)\Big|\tau > t\Big) \quad \text{for a.e. } t \in \mathbf{R}_+ \text{ with } x_t < u^0,$$

and furthermore that

$$F^{0-}(x_t) \geq \mathbf{E}_G\Big(F^{1\prime}(X_\tau)\Big|\tau > t\Big) \quad \text{for a.e. } t \in \mathbf{R}_+ \text{ with } x_t = u^0,$$

since the left-hand derivative $F^{0-}(u^0)$ is the largest supergradient at u^0 of the concave function F^0. For any right-continuous version of x,[73] the above (in)equalities must hold for *every* $t \in \mathbf{R}_+$, since then both sides are right-continuous in t.[74] ■

Proposition 3′ implies the assertion made in footnote 29 on p. 21:

Corollary 1. If (x, X) is optimal for a distribution G with $G(0) = 0$ and unbounded support, then $X_0 > u^1$.

Proof. If $u^\star = u^1$, then $X_0 > u^\star = u^1$ by Theorem 3 (p. 20). Assume for the remainder that $u^\star < u^1$, and suppose toward a contradiction that $X_0 \leq u^1$. Then $X_t < u^1$ for all $t > 0$ since X is decreasing with $\lim_{t\to\infty} X_t = u^\star < u^1$ by Theorem 3 (p. 20), and thus $X < u^1$ G-a.e. since $G(0) = 0$. Since F^1 is strictly increasing on $[0, u^1]$, it follows that $F^{1+}(X) > 0$ G-a.e.

(x, X) satisfies the Euler equation with some ϕ^0, ϕ^1 by Proposition 3′, so $\phi^1(t)$ is a supergradient of F^1 at X_t for G-a.e. $t \in \mathbf{R}_+$, equation (E) (p. 35) holds, and ϕ^0 is non-negative. Thus for any $t \in \mathbf{R}_+$ with $G(t) > 0$, we have

$$0 < \int_{[0,t]} F^{1+}(X_s)G(\mathrm{d}s) \leq \int_{[0,t]} \phi^1 \mathrm{d}G = -[1 - G(t)]\phi^0(t) \leq 0,$$

which is absurd. ■

Supplemental appendices

I Extensions

In this appendix, we provide the details underlying the discussion in §2.2 of our model assumptions.

[73] E.g. $\widetilde{x}$ given by $\widetilde{x}_t = \sup_{s>t} x_s$ for each $t \in \mathbf{R}_+$.

[74] The right-hand side is right-continuous in t because G is right-continuous and $\phi^1(s) \coloneqq F^{1\prime}(X_s)$ is G-integrable, so that for $t_n \downarrow t$ we have $G(t_n) \to G(t)$ and (by dominated convergence) $\int_{\mathbf{R}_+} \phi^1 \mathbf{1}_{(t_n,\infty)} \mathrm{d}G \to \int_{\mathbf{R}_+} \phi^1 \mathbf{1}_{(t,\infty)} \mathrm{d}G$.

I.1 If $u^\star$ is not a strict local maximum

Our assumption that $u^\star$ is a strict local maximum of $F^1 - F^0$ requires merely that $u^\star$ be a strict local maximum on $[0, u^\star]$, as the same is true on $[u^\star, u^0]$ by definition of $u^\star$. This holds vacuously if $u^\star = 0$, while if $u^\star > 0$ it amounts essentially to ruling out a saddle point.[75]

In fact, nothing changes if we weaken our assumption that $u^\star$ is a strict local maximum of $F^1 - F^0$ to demand only that there be no proper interval $[u_\star, u^\star] \subseteq [0, u^0]$ on which F^0, F^1 are affine with equal slopes. Dropping this weaker assumption merely generates some uninteresting multiplicity. For concreteness, consider the case in which F^0 is affine, so that $F^1 - F^0$ is concave and thus attains its maximum over $[0, u^0]$ on an interval $[u_\star, u^\star]$.

Definition 7. A mechanism (x, X) is an *interval deadline mechanism* iff for some $T \in [0, \infty]$, we have $x_t = u^0$ for $t \leq T$ and $x_t \in [u_\star, u^\star]$ for $t > T$.

With small alterations, the proof of Theorem 2 in appendix C delivers

Theorem 2′. If the old frontier F^0 is affine on $[0, u^0]$, then any undominated mechanism is an interval deadline mechanism.

I.2 If some agent utility levels are infeasible

Our model does not require that every agent utility level $u \in [0, \infty)$ be feasible. Concretely, suppose that technology $j \in \{0, 1\}$ can only provide the agent with utility in an interval $I^j \subseteq [0, \infty)$.[76]

The frontier F^j is a concave and upper semi-continuous function $I^j \to \mathbf{R}$. (Recall that these assumptions are without loss.) It is innocuous to extend F^j continuously to $\operatorname{cl} I^j$.[77] (Note that F^j may take the value $-\infty$ off I^j.) Assume that F^0 has a unique peak $u^0 \in \operatorname{cl} I^0$. Assume without loss of generality that (i) $[0, u^0] \subseteq \operatorname{cl} I^0$,[78] (so that F^0 is finite on $(0, u^0]$,) and (ii) $I^0 \subseteq I^1$.[79]

[75] Precisely, $u^\star$ must be either a local maximum, a saddle point, or a point at which *both* F^0 *and* F^1 have a kink. We omit the details; see Curello and Sinander (2021).

[76] I^j is necessarily an interval because any convex combination of feasible utility levels can be attained by rapidly switching back and forth (or randomising).

[77] The principal can anyway attain utility arbitrarily close to $\lim_{u \downarrow \inf I^j} F^j(u)$ by choosing $u > \inf I^j$ small, and similarly for $\sup I^j$.

[78] Any mechanism (x^0, X^1) satisfies $x^0 \geq \inf I^0$ since utilities $< \inf I^0$ cannot be reached using the old technology. Thus IC mechanisms (x^0, X^1) have $X^1 \geq X^0 \geq \inf I^0$. So without loss, we may consider the translated model with agent utility $\widetilde{u} \coloneqq u - \inf I^0 \in [0, \infty)$.

[79] The new technology expands the set of available physical allocations, so any agent utility feasible before the breakthrough remains feasible afterwards.

We now impose the remaining model assumptions. First, $u^0 > 0$. Secondly, F^1 has a unique peak $u^1 \in \operatorname{cl} I^1$, which satisfies $u^1 < u^0$. Thirdly, $F^1 \geq F^0$ (without loss, recall). Finally, $u^\star$ is a strict local maximum of $F^1 - F^0$.

Extend F^j to all of $[0, \infty)$ by letting $F^j := -\infty$ off $\operatorname{cl} I^j$. Then F^0, F^1 satisfy our model assumptions. Since utility levels at which $F^j = -\infty$ are never chosen when using technology j, it is as if they were not feasible.

I.3 Participation constraint instead of non-negativity

Suppose that the agent's utility can take any value $u \in [-K, \infty)$, where $K > 0$ is (arbitrarily) large.[80] The agent can quit anytime, earning a continuation payoff worth zero (a normalisation). We focus on the interesting case in which the principal prefers for the agent never to quit, and therefore chooses among participation-inducing IC mechanisms.

The frontiers F^0, F^1 are now defined on $[-K, \infty)$. As in the text, $u^\star$ denotes the largest $u \in [0, u^0]$ at which F^0, F^1 have equal slopes, with $u^\star := 0$ if there is no such u. Note well that $u^\star$ is non-negative by definition.

Claim. All of our results remain valid (with $u^\star$ defined as above).

Proof. Consider the formally equivalent model in which the agent's utility is $\widetilde{u} := u + K \in [0, \infty)$, with frontiers $\widetilde{F}^j(\widetilde{u}) := F^j(\widetilde{u} - K)$ peaking at $\widetilde{u}^j := u^j + K$. Let $\widetilde{u}^\star$ be the largest $\widetilde{u} \in [0, \widetilde{u}^0]$ at which $\widetilde{F}^0, \widetilde{F}^1$ have equal slopes, with $\widetilde{u}^\star := 0$ if there is no such $\widetilde{u}$. It need *not* be that $\widetilde{u}^\star = u^\star + K$: rather, this holds iff $\widetilde{u}^\star \geq K$.[81] We next argue that this may be assumed without loss of generality.

The participation constraints read

$$\widetilde{X}_t^1 \geq K \quad \text{and} \quad \widetilde{X}_t^0 + \mathbf{E}_G\Big(e^{-r(\tau - t)}\Big(\widetilde{X}_\tau^1 - \widetilde{X}_\tau^0\Big)\Big|\tau > t\Big) \geq K \quad \text{for all } t \in \mathbf{R}_+.$$

Due to the first constraint, it is immaterial what values the new frontier $\widetilde{F}^1$ takes on $[0, K)$. So assume without loss that it equals the concave upper envelope of $\mathbf{1}_{[0,K)}\widetilde{F}^0 + \mathbf{1}_{[K,\infty)}\widetilde{F}^1$. Then $\widetilde{F}^1$ is weakly steeper than $\widetilde{F}^0$ on $[0, K)$,[82] so that $\widetilde{u}^\star \geq K$ and thus $\widetilde{u}^\star = u^\star + K$.

The principal's problem is as in the text, except that she must respect the participation constraints. We now show that these do not bind.

---

[80] The lower bound does not bind. We impose it merely to avoid integrability issues.

[81] If $\widetilde{u}^\star < K$, then $\widetilde{u}^\star < K \leq u^\star + K$. Conversely, if $\widetilde{u}^\star \geq K$, then $\widetilde{u}^\star$ is a fortiori the largest $\widetilde{u} \in [K, \widetilde{u}^0]$ at which $\widetilde{F}^0, \widetilde{F}^1$ have equal slopes, which is to say that $\widetilde{u}^\star - K$ is the largest $u \in [0, u^0]$ at which F^0, F^1 have equal slopes, which is the definition of $u^\star$.

[82] The greatest supergradient $\widetilde{F}^1$ weakly exceeds that of $\widetilde{F}^0$, and likewise for the smallest.

First, when F^0 is affine on $\left[0, u^0\right]$, any undominated mechanism $(\widetilde{x}^0, \widetilde{X}^1)$ in the relaxed problem that ignores the participation constraints (i.e. the problem in the text) satisfies $\widetilde{X}^1 = \widetilde{X}^0 \geq \widetilde{u}^\star \geq K$ by Theorems 1 and 2 (pp. 13 and 17). This implies the participation constraints. Thus undominated (optimal) mechanisms are characterised, in $\widetilde{u}$ units, by Theorem 2 and Proposition 1 (by Proposition 2).

Similarly, ignoring participation, any mechanism $(\widetilde{x}^0, \widetilde{X}^1)$ that is optimal for a distribution G with $G(0) = 0$ and unbounded support satisfies $\widetilde{X}^1 = \widetilde{X}^0 \geq \widetilde{u}^\star \geq K$ by Theorems 1 and 3 (pp. 13 and 20), so that the participation constraints hold. Thus Theorem 3 and Proposition 3 characterise optimal mechanisms, in $\widetilde{u}$ units.

These characterisations translate straightforwardly back to u units, except for one wrinkle: the long-run utility level appearing in Theorems 2 and 3 is $\widetilde{u}^\star - K$, not $u^\star$. We showed, however, that these two are equal. ■

I.4 Monetary transfers with limited liability

Our model can accommodate arbitrary monetary transfers to the agent. To see why, begin with a pair of frontiers F^0, F^1 describing utility possibilities absent transfers, and suppose that in addition to setting the agent's gross utility $u \in [0, \infty)$, the principal can pay her $w \geq 0$ (in an arbitrary history-dependent fashion). Net flow utilities are then $u + w$ for the agent and $F^j(u) - w$ for the principal, where $j \in \{0, 1\}$ is the technology used.

Allowing for transfers expands the utility possibility frontiers where they are steeply downward-sloping: F^j is replaced by the pointwise smallest function that exceeds F^j and has slope ≥ -1 everywhere,[83] as depicted in Figure 6. These expanded frontiers satisfy our model assumptions.

When transfers are permitted, it is optimal to use them only sparingly:

Observation 7. Suppose that the principal can pay the agent. Undominated mechanisms never pay before disclosure. If F^1 has slope ≥ -1 on $\left[0, u^0\right]$, then undominated mechanisms do not pay after disclosure, either.

Proof. The agent is paid exactly if she is to be provided with a utility at which the expanded frontier differs from the original, and undominated mechanisms never provide utility in excess of u^0 by Lemma 0 (p. 12). ■

This observation generalises an insight of Armstrong and Vickers (2010, §3.2): in a static example with an affine F^1, they showed that paying the agent is suboptimal whenever F^1 is sufficiently flat.

[83] 'Slope ≥ -1 everywhere' means 'admits a supergradient ≥ -1 at every $u \in [0, \infty)$'.

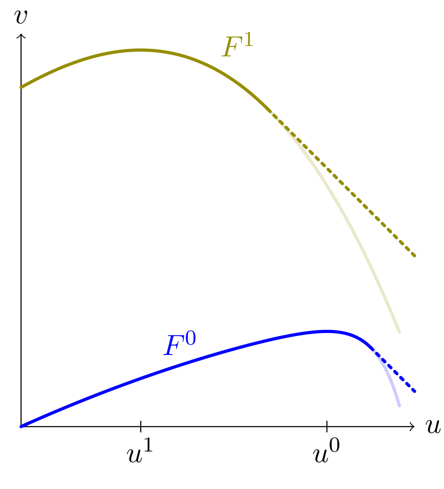


Figure 6: Utility possibility frontiers. Monetary transfers expand a frontier whenever its slope is < -1.

I.5 Uncertain technology

In our model, the new technology F^1 is known in advance—only its date of arrival is uncertain. In this appendix, we show that all of our results remain valid if the new technology is uncertain, provided the agent is not privately informed about its realisation.

Let $\mathcal{F}$ be a finite set of of concave and upper semi-continuous functions $[0, \infty) \to [-\infty, \infty)$ with unique peaks. The new frontier $\boldsymbol{F}$ is a random element of $\mathcal{F}$, drawn independently of the breakthrough time τ. Write $U^1(F)$ for the unique peak of $F \in \mathcal{F}$, and $u^1 \coloneqq \mathbf{E}(U^1(\boldsymbol{F}))$ for its expectation. We assume that there is a conflict of interest: $u^1 < u^0$.

The agent privately observes when the breakthrough occurs, but she does not learn the realised value of the new technology $\boldsymbol{F}$. This means that the agent cannot easily determine the payoff consequences for the principal of the new technology, which is natural in many (but not all) applications.

A mechanism specifies, for each period t, the agent's utility x_t^0 if she has not already disclosed, as well as the continuation utility $\widehat{X}_t(F)$ with which she is rewarded for disclosing at time t if the realised new technology is $F \in \mathcal{F}$. Since the agent does not know F prior to disclosure, only the expectation $X_t^1 \coloneqq \mathbf{E}(\widehat{X}_t(\boldsymbol{F}))$ matters for her incentives.

For a given value $X_t^1 = u$ of this expectation, the principal chooses $\widehat{X}_t : \mathcal{F} \to [0, \infty)$ to maximise $\mathbf{E}(\boldsymbol{F}(\widehat{X}_t(\boldsymbol{F})))$ subject to $\mathbf{E}(\widehat{X}_t(\boldsymbol{F})) = u$. We write $F^1(u)$ for the value of this problem.[84]

[84] A maximum exists (so that F^1 is well-defined) because the constraint set is compact in the pointwise topology (being a closed and bounded subset of the Euclidean space

To characterise the pre-disclosure flow x^0 and expected disclosure reward X^1 in undominated mechanisms, we may study the deterministic model in which the new technology is F^1. (The technology-contingent disclosure reward $\widehat{X}$ may be backed out from the above maximisation problem.) This deterministic model satisfies our model assumptions:

Lemma 7. F^1 is concave and upper semi-continuous, with unique peak at $u^1 = \mathbf{E}(U^1(\boldsymbol{F}))$.

Our results therefore remain valid, characterising the x^0 and X^1 of undominated mechanisms in the uncertain-technology model. We omit the (straightforward) proof of Lemma 7 (see Curello & Sinander, 2021).

J Revelation principle

A revelation principle for our environment must account for the verifiability of the agent's disclosures. A *direct mechanism* is one which solicits a cheap-talk report of the breakthrough's arrival, then instructs the agent when to deliver her hard evidence (her verifiable disclosure). The standard revelation principle (Myerson, 1982, Proposition 2) permits us to restrict attention to incentive-compatible direct mechanisms, meaning those in which the agent is willing to report promptly and to deliver her evidence at the appointed time.

It remains only to show that among such mechanisms, we may further restrict our attention to those involving prompt delivery of the evidence. Modulo differences in detail, this follows from Bull and Watson's (2007) revelation principle (their Theorem 2). The key requirement for their result, the 'normality' of evidence, is satisfied in our model: for each type of the agent (i.e. breakthrough time), there is a most-informative manner of verifiably disclosing: namely, disclosing promptly.

K Existence of undominated and optimal mechanisms

In this appendix, we prove that undominated and optimal mechanisms exist. We shall assume Lemma 0 and Theorem 1 (pp. 12 and 13), neither of whose proofs rely on any existence claim. In light of these, an undominated mechanism may be identified with a measurable map $x : \mathbf{R}_+ \to [0, u^0]$, with the post-disclosure reward X given by $X_t \coloneqq r \int_t^\infty e^{-r(s-t)} x_s \mathrm{d}s$.

Let $\mathcal{X}$ be the space of measurable maps $\mathbf{R}_+ \to [0, u^0]$, with the topology of pointwise convergence. $\mathcal{X}$ is compact, as it is a closed subset of the space

$[0, \infty)^{|\mathcal{F}|}$) and the maximand is upper semi-continuous since every element of $\mathcal{F}$ is.

of all functions $\mathbf{R}_+ \to [0, u^0]$,[85] which is compact by Tychonoff's theorem. Given a distribution G, write $\pi_G : \mathcal{X} \to \mathbf{R}$ for the principal's payoff:

$$\pi_G(x) \coloneqq \mathbf{E}_G\left(r \int_0^\tau e^{-rt} F^0(x_t) \mathrm{d}t + e^{-r\tau} F^1\left(r \int_t^\infty e^{-r(s-t)} x_s \mathrm{d}s \right)\right).$$

Observation 8. π_G is upper semi-continuous.

Proof. Fix a sequence $(x^n)_{n\in\mathbf{N}}$ in $\mathcal{X}$ converging to $x \in \mathcal{X}$. Since F^0 is bounded above, we may apply Fatou's lemma (twice) to obtain

$$\begin{aligned}\limsup_{n\to\infty} \mathbf{E}_G\left(r \int_0^\tau e^{-rt} F^0(x^n_t) \mathrm{d}t \right) &\leq \mathbf{E}_G\left(r \limsup_{n\to\infty} \int_0^\tau e^{-rt} F^0(x^n_t) \mathrm{d}t \right) \\ &\leq \mathbf{E}_G\left(r \int_0^\tau e^{-rt} \limsup_{n\to\infty} F^0(x^n_t) \mathrm{d}t \right) \leq \mathbf{E}_G\left(r \int_0^\tau e^{-rt} F^0(x_t) \mathrm{d}t \right),\end{aligned}$$

where the final inequality holds since F^0 is upper semi-continuous. Applying similar reasoning to F^1 yields $\limsup_{n\to\infty} \pi_G(x^n) \leq \pi_G(x)$. ■

Equip (the equivalence classes of) $\mathcal{X}$ with the partial order $\preceq$ defined by $x \preceq x^\dagger$ iff either x is dominated by $x^\dagger$ or $x = x^\dagger$ (a.e.).

Proposition 5. Any dominated mechanism is dominated by an undominated mechanism.

Proof. Let $x \in \mathcal{X}$ be dominated, and define $\mathcal{U} \coloneqq \{x' \in \mathcal{X} : x \prec x'\}$; we must show that this set admits a $\preceq$-maximal element. By Zorn's lemma, it suffices to show that every chain in $\mathcal{U}$ admits an upper bound in $\mathcal{U}$.

So fix a chain $\mathcal{C} \subseteq \mathcal{U}$, wlog one that does not contain an upper bound of itself. Let $(x_n)_{n\in\mathbf{N}} \subseteq \mathcal{C}$ be an $\preceq$-increasing sequence with no upper bound in $\mathcal{C}$. Since $\mathcal{X}$ is compact, we may assume (passing to a subsequence if necessary) that $(x_n)_{n\in\mathbf{N}}$ is convergent, denoting the limit by $x^\star \in \mathcal{X}$. Clearly $x^\star$ belongs to $\mathcal{U}$. It satisfies $x_n \preceq x^\star$ for every $n \in \mathbf{N}$ since π_G is upper semi-continuous for every G, and thus $x \preceq x^\star$ for every $x \in \mathcal{C}$ since $(x_n)_{n\in\mathbf{N}}$ has no upper bound in $\mathcal{C}$. So $x^\star$ is an upper bound of $\mathcal{C}$ in $\mathcal{U}$. ■

Corollary 2. An undominated mechanism exists.

Lemma 8. The set of undominated mechanisms is compact.

[85] It is closed because the pointwise limit of measurable functions is itself measurable (see e.g. Proposition 2.7 in Folland (1999)).

Proof. Since $\mathcal{X}$ is compact, it suffices to show that the subset of undominated mechanisms is closed. So take a sequence $(x_n)_{n\in\mathbf{N}} \subseteq \mathcal{X}$ of undominated mechanisms converging to $x^\star \in \mathcal{X}$; we will show that $x^\star$ is undominated. Fix an arbitrary $x \in \mathcal{X}$. Undominatedness along the sequence ensures that $x_n \not\prec x$ for every $n \in \mathbf{N}$, which since π_G is upper semi-continuous implies that $x^\star \not\prec x$. Since x was arbitrary, we have shown that $x^\star$ is undominated. ■

Corollary 3. For any distribution G, an optimal mechanism exists.

Proof. By Proposition 5, it suffices to show that π_G attains a maximum on the space of undominated mechanisms. This follows immediately from the upper semi-continuity of π_G and the non-emptiness and compactness of the space of undominated mechanisms (Corollary 2 and Lemma 8). ■

L Approximate variant of Theorem 2 (p. 17)

In this appendix, we extend Theorem 2 to show that *approximate* affineness of F^0 suffices for deadline mechanisms to be close to optimal, and derive the implications for our unemployment insurance application. Write

$$\underline{F}^0(u) \coloneqq F^0(u^\star) + \frac{u}{u^0}\left[F^0\left(u^0\right) - F^0(u^\star)\right] \quad \text{for } u \in \left[u^\star, u^0\right]$$

for the straight line connecting $(u^\star, F^0(u^\star))$ with $(u^0, F^0(u^0))$. Since $\underline{F}^0$ is the pointwise highest affine function everywhere below F^0 on $[u^\star, u^0]$, we call F^0 close to affine iff it is close to $\underline{F}^0$:

Definition 8. For $\varepsilon > 0$, the frontier F^0 is *ε-close to affine* on $[u^\star, u^0]$ iff $F^0 - \underline{F}^0 \leq \varepsilon$ on $[u^\star, u^0]$.

Corollary 4. If F^0 is ε-close to affine on $[u^\star, u^0]$, then for any distribution G, the best deadline mechanism is ε-optimal.

Proof. Fix a distribution G. The best deadline mechanism is exactly the mechanism that is optimal when F^0 is replaced by $\underline{F}^0$ on $[u^\star, u^0]$;[86] write Π^{d}_G for its value.[87] Any mechanism (x^0, X^1) is dominated by an undominated mechanism (x, X),[88] and thus

$$\begin{aligned}\Pi_G\left(x^0, X^1\right) \leq \Pi_G(x, X) = \mathbf{E}_G\left(r\int_0^\tau e^{-rt}\underline{F}^0(x_t)\mathrm{d}t + e^{-r\tau}F^1(X_\tau)\right) \\ + \mathbf{E}_G\left(r\int_0^\tau e^{-rt}\left[F^0 - \underline{F}^0\right](x_t)\mathrm{d}t\right) \leq \Pi^{\mathrm{d}}_G + \varepsilon,\end{aligned}$$

[86]This can be seen from Lemma 2 in appendix E (p. 32).
[87]By inspection, its value is the same whether or not F^0 is replaced by $\underline{F}^0$ on $[u^\star, u^0]$.
[88]By Proposition 5 in supplemental appendix K.

where the last inequality holds since $u^\star \leq x \leq u^0$ by Lemma 0 (p. 12). ■

L.1 Application to unemployment insurance (§7)

Since a linear ϕ would satisfy $\phi(C) - \phi'(C)C = 0$ for every C, we call ϕ *ε-close to linear* iff $\phi(C) - \phi'(C)C \leq \varepsilon$ for every $C \leq C^0$, where $C^0 \coloneqq (\phi')^{-1}(\lambda)$.

Corollary 5. In the application to unemployment insurance, let $\Pi^\star_G$ (Π^{d}_G) denote social welfare under the best (deadline) scheme.

(1) For any $\varepsilon > 0$, if ϕ is ε-close to linear, then $\Pi^{\mathrm{d}}_G \geq \Pi^\star_G - \varepsilon$.

(2) For any $\alpha \in (0,1)$, if $\lambda > 0$ is sufficiently small, then $\Pi^{\mathrm{d}}_G \,/\, \Pi^\star_G \geq \alpha$.

Proof. Calculation reveals that $u^0 = \phi(C^0)$ and $F^0(u^0) = \phi(C^0) - \phi'(C^0)C^0$. For (1), if ϕ is ε-close to linear, then F^0 is ε-close to affine since

$$F^0(u) - \underline{F}^0(u) \leq F^0\left(u^0\right) = \phi\left(C^0\right) - \phi'\left(C^0\right)C^0 \leq \varepsilon \quad \text{for any } u \in \left[0, u^0\right],$$

and thus $\Pi^{\mathrm{d}}_G \geq \Pi^\star_G - \varepsilon$ by Corollary 4.

For (2), let $\eta \coloneqq \phi'(C^0)C^0/\phi(C^0)$, and calculate

$$F^0(u) - \underline{F}^0(u) = \eta u - \lambda\phi^{-1}(u) \quad \text{for each } u \in \left[0, u^0\right]$$

This together with Corollary 4 yields the bound

$$\frac{\Pi^{\mathrm{d}}_G}{\Pi^\star_G} = 1 - \frac{\Pi^\star_G - \Pi^{\mathrm{d}}_G}{\Pi^\star_G} \geq 1 - \frac{\max_{u\in[0,u^0]}\left\{\eta u - \lambda\phi^{-1}(u)\right\}}{\Pi^\star_G}.$$

Note that $\Pi^\star_G \geq F^0(u^0) = (1-\eta)u^0$, where the inequality holds since social welfare $F^0(u^0)$ is attainable.[89] Thus

$$\frac{\Pi^{\mathrm{d}}_G}{\Pi^\star_G} \geq 1 - \frac{\max_{u\in[0,u^0]}\left\{\eta u - \lambda\phi^{-1}(u)\right\}}{(1-\eta)u^0}.$$

This implies (2) since as λ vanishes, $C^0, u^0 \to \infty$ and $\eta \to 0$, so that the right-hand side converges to 1. ■

[89] E.g. by the (non-IC) mechanism $\left(x^0, X^1\right) \equiv \left(u^0, 0\right)$ (in which no offer is accepted).

M Proof of the Euler lemma (appendix F.1)

Recall from appendix F the definitions of $\mathcal{X}$ and π_G, and note that the former is a convex set. For $j \in \{0,1\}$, define $F^{j\prime} : \mathbf{R}_+^2 \to \mathbf{R}_+$ by

$$F^{j\prime}(u,u') := \begin{cases} F^{j-}(u) & \text{if } u' < u \\ 0 & \text{if } u' = u \\ F^{j+}(u) & \text{if } u' > u, \end{cases}$$

where F^{j-} (F^{j+}) denotes the left-hand (right-hand) derivative. Write

$$\mathrm{D}\pi_G\Big(x, x^\dagger - x\Big) := \lim_{\alpha \downarrow 0} \frac{\pi_G\Big(x + \alpha\Big[x^\dagger - x\Big]\Big) - \pi_G(x)}{\alpha}$$

for the Gateaux derivative of π_G at x in direction $x^\dagger - x$.

Let $\mathcal{X}_G$ be the set of $x \in \mathcal{X}$ such that the maps $\psi^0_{x,u} : \mathbf{R}_+ \to [0,\infty]$ and $\psi^1_{X,u} : \mathbf{R}_+ \to [-\infty,\infty]$ defined by

$$\psi^0_{x,u}(t) := r\int_0^t e^{-rs} F^{0\prime}(x_s,u)\mathrm{d}s \quad \text{and} \quad \psi^1_{X,u}(t) := e^{-rt}F^{1\prime}(X_t,u)$$

are G-integrable for any $u \in \big(0,u^0\big)$. We require three lemmata.

Lemma 9. If $x \in \mathcal{X}$ and (x,X) satisfies the Euler equation for some ϕ^0, ϕ^1, then x belongs to $\mathcal{X}_G$, and the map $t \mapsto r\int_0^t e^{-rs}\phi^0(s)\mathrm{d}s$ is G-integrable.

Lemma 10. $\arg\max_{\mathcal{X}} \pi_G \subseteq \mathcal{X}_G$.

Gateaux lemma. For any $x, x^\dagger \in \mathcal{X}_G$, measurable $\phi^0 : \mathbf{R}_+ \to [0,\infty]$ such that $t \mapsto r\int_0^t e^{-rs}\phi^0(s)\mathrm{d}s$ is G-integrable, and G-integrable $\phi^1 : \mathbf{R}_+ \to [-\infty,\infty]$, the Gateaux derivative $\mathrm{D}\pi_G(x, x^\dagger - x)$ exists and is equal to

$$\begin{aligned} r\int_0^\infty e^{-rt}\left([1-G(t)]\phi^0(t) + \int_{[0,t]}\phi^1\mathrm{d}G\right)\Big(x^\dagger_t - x_t\Big)\mathrm{d}t \\ + \mathbf{E}_G\left(r\int_0^\tau e^{-rt}\Big[F^{0\prime}\Big(x_t, x^\dagger_t\Big) - \phi^0(t)\Big]\Big[x^\dagger_t - x_t\Big]\mathrm{d}t\right) \\ + \mathbf{E}_G\Big(e^{-r\tau}\Big[F^{1\prime}\Big(X_\tau, X^\dagger_\tau\Big) - \phi^1(\tau)\Big]\Big[X^\dagger_\tau - X_\tau\Big]\Big). \end{aligned}$$

Lemma 9 and the Gateaux lemma follow from standard arguments, which we omit (see Curello & Sinander, 2021). Lemma 10 is proved below.

Proof of the Euler lemma. Fix a distribution G. For the first part, suppose that $x \in \mathcal{X}$ and that (x, X) satisfies the Euler equation with ϕ^0, ϕ^1 (the former measurable, the latter G-integrable). Then by Lemma 9, x belongs to $\mathcal{X}_G$, and $t \mapsto r \int_0^t e^{-rs} \phi^0(s) \mathrm{d}s$ is G-integrable.

By Corollary 3 in supplemental appendix K (p. 54), there is a mechanism $(x^\star, X^\star)$ that is optimal for G. We must have $x^\star \in \mathcal{X}$ by Lemma 0 (p. 12), and thus $x^\star \in \arg\max_{\mathcal{X}} \pi_G$. So it suffices to show that $\pi_G(x^\star) \leq \pi_G(x)$.

By Lemma 10, $x^\star$ belongs to $\mathcal{X}_G$. Thus $x, x^\star$ and ϕ^0, ϕ^1 satisfy the hypotheses of the Gateaux lemma. Moreover, π_G is concave since F^0, F^1 are and the map $x \mapsto X$ is linear, so for any $\alpha \in (0,1)$, we have

$$\frac{\pi_G(x + \alpha[x^\star - x]) - \pi_G(x)}{\alpha} \geq \pi_G(x^\star) - \pi_G(x).$$

The left-hand side converges as $\alpha \downarrow 0$ by the Gateaux lemma, yielding

$$\mathrm{D}\pi_G(x, x^\star - x) \geq \pi_G(x^\star) - \pi_G(x).$$

It therefore suffices to show that $\mathrm{D}\pi_G(x, x^\star - x) \leq 0$. And indeed, the first term in the Gateaux lemma's expression for $\mathrm{D}\pi_G(x, x^\star - x)$ is zero by (E), while second (third) term is non-positive by definition of ϕ^0 (ϕ^1) and the concavity of F^0 (F^1).[90]

For the second part, fix an $x^\dagger \in \arg\max_{\mathcal{X}} \pi_G$. Since ϕ^0, ϕ^1 satisfy (E), what must be shown is merely that

- for a.e. $t \in \mathbf{R}_+$ with $G(t) < 1$, $\phi^0(t)$ is a supergradient of F^0 at $x_t^\dagger$, and
- for G-a.e. $t \in \mathbf{R}_+$, $\phi^1(t)$ is a supergradient of F^1 at $X_t^\dagger$.

$x^\dagger$ belongs to $\mathcal{X}_G$ by Lemma 10. So by the Gateaux lemma (with the roles of x and $x^\dagger$ reversed) and (E),

$$\begin{aligned}\mathrm{D}\pi_G\left(x^\dagger, x - x^\dagger\right) = \mathbf{E}_G\left(r \int_0^\tau e^{-rt}\left[F^{0\prime}\left(x_t^\dagger, x_t\right) - \phi^0(t)\right]\left[x_t - x_t^\dagger\right]\mathrm{d}t\right) \\ + \mathbf{E}_G\left(e^{-r\tau}\left[F^{1\prime}\left(X_\tau^\dagger, X_\tau\right) - \phi^1(\tau)\right]\left[X_\tau - X_\tau^\dagger\right]\right).\end{aligned}$$

We must have $\mathrm{D}\pi_G(x^\dagger, x - x^\dagger) \leq 0$ since π_G is maximised at $x^\dagger$. On the other hand, the two integrands

$$\begin{aligned} t &\mapsto \left[F^{0\prime}\left(x_t^\dagger, x_t\right) - \phi^0(t)\right]\left[x_t - x_t^\dagger\right] \\ \text{and} \quad t &\mapsto \left[F^{1\prime}\left(X_t^\dagger, X_t\right) - \phi^1(t)\right]\left[X_t - X_t^\dagger\right] \end{aligned}$$

[90] For the second term, $\phi^0(t)$ is a supergradient of the concave function F^0 at x_t for a.e. $t \in \mathbf{R}_+$ with $G(t) < 1$, and at each such t, $x_t^\star > x_t$ implies $F^{0\prime}(x_t, x_t^\star) = F^{0+}(x_t) \leq \phi^0(t)$ and $x_t^\star < x_t$ implies $F^{0\prime}(x_t, x_t^\star) = F^{0-}(x_t) \geq \phi^0(t)$. Analogously for the third term.

are non-negative at, respectively, a.e. $t \in \mathbf{R}_+$ with $G(t) < 1$ (the first integrand) and at G-a.e. every $t \in \mathbf{R}_+$ (the second).[91] Thus the first (second) integrand must be equal to zero at a.e. $t \in \mathbf{R}_+$ at which $G(t) < 1$ (at G-a.e. $t \in \mathbf{R}_+$). For a.e. $t \in \mathbf{R}_+$ with $G(t) < 1$ at which the first integrand is zero, $\phi^0(t)$ is a supergradient of F^0 at $x_t^\dagger$.[92] Similarly, $\phi^1(t)$ is a supergradient of F^1 at $X_t^\dagger$ for G-a.e. $t \in \mathbf{R}_+$ at which the second integrand is zero. ■

Proof of Lemma 10. Fix an $x \in \mathcal{X} \setminus \mathcal{X}_G$; we must show that it does not belong to $\arg\max_{\mathcal{X}} \pi_G$. By hypothesis, there is a $u \in (0, u^0)$ such that either $\psi^0_{x,u}$ or $\psi^1_{X,u}$ (defined on p. 56) fails to be G-integrable. Define $x^\dagger \equiv u$; it clearly belongs to $\mathcal{X}$. It suffices to show that $\mathrm{D}\pi_G(x, x^\dagger - x) = \infty$, since then

$$\pi_G\Big(x + \alpha\big[x^\dagger - x\big]\Big) > \pi_G(x) \quad \text{for } \alpha \in (0,1) \text{ small enough.}$$

Fix an $\varepsilon \in (0, u)$, and define

$$\mathcal{T} \coloneqq \{t \in \mathbf{R}_+ : x_t > u - \varepsilon\}.$$

Choose $\varepsilon' \in \big(0, u \wedge [u^0 - u]\big)$ so that $\varepsilon' < u^1 \wedge \big(u^0 - u^1\big)$ if $u^1 > 0$, and let

$$\mathcal{T}' \coloneqq \begin{cases} \{t \in \mathbf{R}_+ : X_t < u + \varepsilon'\} & \text{if } u^1 = 0 \\ \{t \in \mathbf{R}_+ : (u \wedge u^1) - \varepsilon' < X_t < (u \vee u^1) + \varepsilon'\} & \text{if } u^1 > 0. \end{cases}$$

(Here '$\wedge$' and '$\vee$' denote the minimum and maximum, respectively.)

Claim. $\mathrm{D}\pi_G(x, x^\dagger - x)$ exists in $[-\infty, \infty]$, and for some $C \in \mathbf{R}$,

$$\begin{aligned} \mathrm{D}\pi_G\Big(x, x^\dagger - x\Big) &= \varepsilon \mathbf{E}_G\bigg(r\int_0^\tau e^{-rt}\Big|F^{0\prime}(x_t, u)\Big|\mathbf{1}_{\mathbf{R}_+\setminus\mathcal{T}'}(t)\mathrm{d}t\bigg) \\ &\quad + \varepsilon'\mathbf{E}_G\Big(e^{-r\tau}\Big|F^{1\prime}(X_\tau, u)\Big|\mathbf{1}_{\mathbf{R}_+\setminus\mathcal{T}'}(\tau)\Big) + C. \end{aligned}$$

The claim follows from standard arguments, which we omit (see Curello & Sinander, 2021). Now, the maps

$$t \mapsto r\int_0^t e^{-rs}F^{0\prime}(x_s, u)\mathbf{1}_{\mathcal{T}}(s)\mathrm{d}s \quad \text{and} \quad t \mapsto e^{-rt}F^{1\prime}(X_t, u)\mathbf{1}_{\mathcal{T}'}(t)$$

[91] For the first integrand, $\phi^0(t)$ is a supergradient of the concave function F^0 at x_t for a.e. $t \in \mathbf{R}_+$ with $G(t) < 1$, and at every such t, $x_t < x_t^\dagger$ implies $F^{0\prime}(x_t^\dagger, x_t) = F^{0-}(x_t^\dagger) \leq F^{0+}(x_t) \leq \phi^0(t)$ and $x_t > x_t^\dagger$ implies $F^{0\prime}(x_t^\dagger, x_t) \geq \phi^0(t)$. Similarly for the second integrand.

[92] If the first integrand is zero at t, then either $F^{0\prime}(x_t^\dagger, x_t) = \phi^0(t)$ or $x_t = x_t^\dagger$. If the former, then $\phi^0(t)$ is a supergradient of F^0 at $x_t^\dagger$. And for almost every $t \in \mathbf{R}_+$ with $G(t) < 1$ at which the latter holds, $\phi^0(t)$ is a supergradient of F^0 at $x_t = x_t^\dagger$.

are G-integrable because F^0 is Lipschitz continuous on $[u-\varepsilon, u^0]$ and

$$F^1 \text{ is Lipschitz continuous on } \begin{cases} [0, u+\varepsilon'] & \text{if } u^1 = 0 \\ [(u \wedge u^1) - \varepsilon', (u \vee u^1) + \varepsilon'] & \text{if } u^1 > 0. \end{cases}$$

Since either $\psi^0_{x,u}$ or $\psi^1_{X,u}$ (defined on p. 56) fails to be G-integrable (as $x \notin \mathcal{X}_G$ by hypothesis), it must therefore be that one of the maps

$$t \mapsto r \int_0^t e^{-rs} F^{0\prime}(x_s, u) \mathbf{1}_{\mathbf{R}_+ \setminus \mathcal{T}}(s) \mathrm{d}s \quad \text{and} \quad t \mapsto e^{-rt} F^{1\prime}(X_t, u) \mathbf{1}_{\mathbf{R}_+ \setminus \mathcal{T}'}(t)$$

fails to be G-integrable. In either case, the claim implies that $\mathrm{D}\pi_G(x, x^\dagger - x) = \infty$, as desired. ■

N Proofs of the construction lemmata (appendix F.2)

In this appendix, we prove the lemmata in appendix F.2 used to construct a solution of the superdifferential Euler equation (appendix F).

N.1 Proof of Lemma 3 (p. 37)

Enumerate the support of G as $\operatorname{supp}(G) = \{t_k\}_{k=1}^K \subseteq \mathbf{R}_+$, where $K \in \mathbf{N}$ and

$$0 \leq t_1 < \cdots < t_K < \infty.$$

As $F^{0\prime}$ is continuous and strictly decreasing on $[u^\star, u^0]$, it admits a continuous and decreasing inverse $\operatorname{inv} F^{0\prime} : [F^{0\prime}(u^0), F^{0\prime}(u^\star)] \to [u^\star, u^0]$. Extend $\operatorname{inv} F^{0\prime}$ to $\mathbf{R}$ by making it constant on $(-\infty, F^{0\prime}(u^0)]$ and on $[F^{0\prime}(u^\star), \infty)$, so that continuity and monotonicity are preserved.

For $\lambda \in [u^\star, u^0]$, let $x^\lambda_{t_K} := X^\lambda_{t_K} := \lambda$ and, if $K > 1$, define a sequence $\{x^\lambda_{t_k}, X^\lambda_{t_k}\}_{k=1}^{K-1}$ in $[u^\star, u^0]$ recursively by

$$x^\lambda_{t_k} := \operatorname{inv} F^{0\prime}\Big(\mathbf{E}_G\Big(F^{1\prime}\Big(X^\lambda_\tau\Big)\Big|\tau > t_k\Big)\Big) \quad \text{and}$$
$$X^\lambda_{t_k} := \Big(1 - e^{r(t_k - t_{k+1})}\Big)x^\lambda_k + e^{r(t_k - t_{k+1})} X^\lambda_{t_{k+1}}.$$

Claim. The sequence $(x^\lambda_{t_k})_{k=1}^K$ is decreasing.

Proof. We prove that the sequence $(x^\lambda_{t_k})_{k=k'}^K$ is decreasing for every $k' \in \{1, \ldots, K-1\}$ by backward induction on k'. For the base case $k' = K-1$, we have

$$x^\lambda_{t_{K-1}} = \operatorname{inv} F^{0\prime}\Big(F^{1\prime}(\lambda)\Big) \geq \lambda = x^\lambda_{t_K},$$

where the inequality holds since $F^{0\prime} \geq F^{1\prime}$ on $[u^\star, u^0] \ni \lambda$.

For the induction step, suppose for $k' \in \{1, \ldots, K-2\}$ that $(x^\lambda_{t_k})^K_{k=k'+1}$ is decreasing; we must show that $x_{t_{k'}} \geq x_{t_{k'+1}}$. The induction hypothesis implies that $(X^\lambda_{t_k})^K_{k=k'+1}$ is also decreasing, which since $F^{1\prime}$ is a decreasing function implies that

$$F^{1\prime}\left(X^1_{t_{k'+1}}\right) \leq \mathbf{E}_G\left(F^{1\prime}\left(X^1_\tau\right) \middle| \tau > t_{k'+1}\right),$$

and thus

$$\begin{aligned}\mathbf{E}_G\left(F^{1\prime}\left(X^1_\tau\right) \middle| \tau > t_{k'}\right) &= \frac{G(t_{k'+1}) - G(t_{k'})}{1 - G(t_{k'})} F^{1\prime}\left(X^1_{t_{k'+1}}\right) \\ &\quad + \frac{1 - G(t_{k'+1})}{1 - G(t_{k'})} \mathbf{E}_G\left(F^{1\prime}\left(X^1_\tau\right) \middle| \tau > t_{k'+1}\right) \\ &\leq \mathbf{E}_G\left(F^{1\prime}\left(X^1_\tau\right) \middle| \tau > t_{k'+1}\right).\end{aligned}$$

Since $\operatorname{inv} F^{0\prime}$ is decreasing, it follows that $x_{t_{k'}} \geq x_{t_{k'+1}}$. □

Since $\operatorname{inv} F^{0\prime}$ and $F^{1\prime}$ are continuous, $\lambda \mapsto x^\lambda_{t_k}$ and $\lambda \mapsto X^\lambda_{t_k}$ are continuous on $[u^\star, u^0]$ for every $k \in \{1, \ldots, K\}$.[93] Thus the map $\psi : [u^\star, u^0] \to \mathbf{R}$ defined by

$$\psi(\lambda) \coloneqq \mathbf{E}_G\left(F^{1\prime}\left(X^\lambda_\tau\right)\right) \quad \text{for each } \lambda \in \left[u^\star, u^0\right]$$

is continuous. Since F^0 and F^1 are continuously differentiable and $u^\star > 0$, we have by definition of $u^\star$ that $F^{1\prime}(u^\star) = F^{0\prime}(u^\star)$ and $F^{1\prime}(u^0) \leq F^{0\prime}(u^0)$. Thus if $\lambda \in \{u^\star, u^0\}$, then $x^\lambda_{t_k} = \lambda$ for every $k \in \{1, \ldots, K\}$, so that $\psi(\lambda) = F^{1\prime}(\lambda)$.[94] It follows that

$$\psi(u^\star) = F^{0\prime}(u^\star) \geq 0 = F^{0\prime}\left(u^0\right) \geq \psi\left(u^0\right).$$

Hence the continuous function ψ has a root $\lambda_\star \in [u^\star, u^0]$ by the intermediate value theorem.

Let $x : \mathbf{R}_+ \to [u^\star, u^0]$ be given by

$$x_t \coloneqq \begin{cases} u^0 & \text{for } t \in [0, t_1) \\ x^{\lambda_\star}_{t_k} & \text{for } t \in [t_k, t_{k+1}) \text{ where } k \in \{1, \ldots, K-1\} \\ x^{\lambda_\star}_{t_K} & \text{for } t \in [t_K, \infty). \end{cases}$$

The mechanism (x, X) satisfies the Euler equation by Observation 5 in appendix F.2 (p. 36). ■

[93]Proceed by strong backward induction on $k \in \{1, \ldots, K\}$. Clearly continuity holds in the base case $k = K$. For the induction step, suppose for $k < K$ that $\lambda \mapsto X^\lambda_{t_{k'}}$ is continuous for all $k' > k$. Then $\lambda \mapsto x^\lambda_{t_k}$ is continuous, and thus so is $\lambda \mapsto X^\lambda_{t_k}$.

[94]This follows easily by backward induction on $k \in \{1, \ldots, K\}$.

N.2 Proof of Lemma 4 (p. 37)

Since F^0, F^1 are simple and x belongs to $\mathcal{X}'$, it suffices by Observation 5 in appendix F.2 (p. 36) to show that $\mathbf{E}_G(F^{1\prime}(X_\tau)) = 0$ and

$$F^{0\prime}(x_t) = \mathbf{E}_G\left(F^{1\prime}(X_\tau)\middle|\tau > t\right) \quad \text{for a.e. } t \in \mathbf{R}_+ \text{ such that } G(t) < 1. \tag{7}$$

By Observation 5 again, (x^n, X^n) and G_n satisfy $\mathbf{E}_{G_n}(F^{1\prime}(X^n_\tau)) = 0$ and (7) for each $n \in \mathbf{N}$.

To show that $\mathbf{E}_G(F^{1\prime}(X_\tau)) = 0$, note that by simplicity, $F^{1\prime}$ is L-Lipschitz on $[u^\star, u^0]$ for some $L > 0$. Thus for any $T \in \mathbf{R}_+$, we have

$$\begin{aligned}
&\left|\mathbf{E}_{G_n}\left(F^{1\prime}(X^n_\tau) - F^{1\prime}(X_\tau)\right)\right| \leq L\mathbf{E}_{G_n}(|X^n_\tau - X_\tau|) \\
&\quad\leq L\mathbf{E}_{G_n}\left(|X^n_\tau - X_\tau| \;\middle|\; \tau \leq T\right) + L[1 - G_n(T)]\left(u^0 - u^\star\right) \\
&\quad\leq L\mathbf{E}_{G_n}\left(e^{r\tau} r \int_\tau^\infty e^{-rt}|x^n_t - x_t|\mathrm{d}t \;\middle|\; \tau \leq T\right) + L[1 - G_n(T)]\left(u^0 - u^\star\right) \\
&\quad\leq Le^{rT} r \int_0^\infty e^{-rt}|x^n_t - x_t|\mathrm{d}t + L[1 - G_n(T)]\left(u^0 - u^\star\right).
\end{aligned}$$

Since $T \in \mathbf{R}_+$ was arbitrary and $G_n \to G$ and $x^n \to x$ pointwise, it follows that the left-hand side vanishes as $n \to \infty$.[95] Thus since (x^n, X^n) and G_n satisfy $\mathbf{E}_{G_n}(F^{1\prime}(X^n_\tau)) = 0$ for each $n \in \mathbf{N}$, we have

$$\begin{aligned}
&\left|\mathbf{E}_G\left(F^{1\prime}(X_\tau)\right)\right| = \left|\mathbf{E}_{G_n}\left(F^{1\prime}(X^n_\tau)\right) - \mathbf{E}_G\left(F^{1\prime}(X_\tau)\right)\right| \\
&\quad\leq \left|\mathbf{E}_{G_n}\left(F^{1\prime}(X^n_\tau) - F^{1\prime}(X_\tau)\right)\right| + \left|\mathbf{E}_{G_n}\left(F^{1\prime}(X_\tau)\right) - \mathbf{E}_G\left(F^{1\prime}(X_\tau)\right)\right| \to 0
\end{aligned}$$

as $n \to \infty$, where the second term vanishes because $G_n \to G$ pointwise (hence weakly) and X and $F^{1\prime}$ are bounded and continuous.

It remains to derive (7). Since (x^n, X^n) and G_n satisfy (7) for each $n \in \mathbf{N}$, we have for a.e. $t \in \mathbf{R}_+$ that

$$\begin{aligned}
&\left|F^{0\prime}(x_t) - \mathbf{E}_G\left(F^{1\prime}(X_\tau)\middle|\tau > t\right)\right| \\
&\quad\leq \left|F^{0\prime}(x_t) - F^{0\prime}(x^n_t)\right| + \left|\mathbf{E}_{G_n}\left(F^{1\prime}(X^n_\tau)\middle|\tau > t\right) - \mathbf{E}_G\left(F^{1\prime}(X_\tau)\middle|\tau > t\right)\right|.
\end{aligned}$$

[95] Fix any $\varepsilon > 0$; we seek an $N \in \mathbf{N}$ such that $|\mathbf{E}_{G_n}(F^{1\prime}(X^n_\tau) - F^{1\prime}(X_\tau))| < \varepsilon$ for all $n \geq N$. To that end, choose a $T \in \mathbf{R}_+$ large enough that $[1 - G(T)]L(u^0 - u^\star) < \varepsilon/3$. Since $G_n \to G$ and $x^n \to x$ pointwise, we may find an $N \in \mathbf{N}$ such that both $|G(T) - G_n(T)|L(u^0 - u^\star) < \varepsilon/3$ and $Le^{rT} r \int_0^\infty e^{-rt}|x^n_t - x_t|\mathrm{d}t < \varepsilon/3$ for all $n \geq N$.

The first term vanishes as $n \to \infty$ since $F^{0\prime}$ is continuous and $x^n \to x$ pointwise. The second term vanishes by a straightforward variation on the above argument, using the fact that since G_n converges pointwise to G, the same is true of the conditional CDFs given $\tau > t$.[96] ■

N.3 Proof of Lemma 5 (p. 37)

Choose a sequence $(F^0_n, F^1_n)_{n\in\mathbf{N}}$ of technologies satisfying the following:[97]

(a) F^0_n, F^1_n are simple for every $n \in \mathbf{N}$,

(b) $u^0_n \uparrow u^0$, $u^1_n \to u^1$ and $u^\star_n \to u^\star$ as $n \to \infty$,

(c) (i) for any $u \in (0, u^0]$ at which F^{1-} is finite, $(F^{0\prime}_n)_{n\in\mathbf{N}}$ and $(F^{1\prime}_n)_{n\in\mathbf{N}}$ are uniformly bounded below on $[0, u]$,

(ii) for any $u \in [0, u^0)$ at which F^{0+}, F^{1+} are finite, $(F^{0\prime}_n)_{n\in\mathbf{N}}$ and $(F^{1\prime}_n)_{n\in\mathbf{N}}$ are uniformly bounded above on $[u, u^0]$, and

(d) for both $j \in \{0, 1\}$ and any convergent sequence $(u_n)_{n\in\mathbf{N}}$ with $0 < u_n \leq u^0_n$ for each $n \in \mathbf{N}$, every subsequential limit of the sequence $(F^{j\prime}_n(u_n))_{n\in\mathbf{N}}$ is a supergradient of F^j at $\lim_{n\to\infty} u_n$.

Fix a mechanism (x, X) and a CDF G with unbounded support, and suppose that x is the pointwise limit of a sequence $(x^n)_{n\in\mathbf{N}}$ such that for each $n \in \mathbf{N}$, x^n belongs to $\mathcal{X}'_n$ and (x^n, X^n) satisfies the Euler equation for (F^0_n, F^1_n, G) and $x^n \in \mathcal{X}'_n$. Assume without loss that each x^n is decreasing and right-continuous.[98]

Since F^0_n, F^1_n are simple for each $n \in \mathbf{N}$ (by property (a)), we have by Observation 5 in appendix F.2 (p. 36) that

$$[1-G(t)]F^{0\prime}_n(x^n_t)+\int_{[0,t]} F^{1\prime}_n(X^n_s)G(\mathrm{d}s) = 0 \quad \text{for all } t \in \mathbf{R}_+ \text{ and } n \in \mathbf{N}. \quad (\mathcal{E})$$

(In particular, this holds for a.e. $t \in \mathbf{R}_+$ by Observation 5, and thus for every t since $F^{0\prime}$ is continuous and x^n is right-continuous.)

Claim. $X_0 < u^0$ unless $F^{1-}(u^0)$ is finite, and $X > 0$ unless $F^{0+}(0), F^{1+}(0)$ are finite.

[96] For all sufficiently large $n \in \mathbf{N}$, we have $G_n(t) < 1$ since $G_n(t) \to G(t) < 1$, so the conditional CDF $G_n/[1 - G_n(t)]$ is well-defined and converges pointwise to $G/[1 - G(t)]$.

[97] For an explicit example, see Curello and Sinander (2021).

[98] Each x^n admits a decreasing right-continuous version, e.g. $\widetilde{x}^n$ given by $\widetilde{x}^n_t := \sup_{s>t} x^n_t$.

Proof. For the first part, suppose toward a contradiction that $F^{1-}(u^0) = -\infty$ and $X_0 = u^0$. Then $x = X = u^0$ since $x \leq u^0$ (as $x \in \mathcal{X}'$), so that $x^n \to u^0$ pointwise and (thus) $X^n \to u^0$ pointwise. Fix a $t \in \mathbf{R}_+$ at which $G(t) > 0$. Property (d) implies that

$$\limsup_{n\to\infty} F_n^{0\prime}(x_t^n) \leq F^{0-}\left(u^0\right) \quad \text{and} \quad \lim_{n\to\infty} F_n^{1\prime}(X_s^n) = -\infty \quad \text{for any } s \in [0,t].$$

Since $u_n^1 \to u^1 < u^0$ by property (b), there is an $N \in \mathbf{N}$ such that $X_t^n \geq u_n^1$ for every $n \geq N$, and thus $X^n \geq u_n^1$ on $[0,t]$ since X^n is decreasing (as $x^n \in \mathcal{X}_n'$). It follows that $s \mapsto F_n^{1\prime}(X_s^n)$ is non-positive for every $n \geq N$, so that Fatou's lemma applies, yielding

$$\begin{aligned}
&\limsup_{n\to\infty}\left\{[1-G(t)]F_n^{0\prime}(x_t^n) + \int_{[0,t]} F_n^{1\prime}(X_s^n)G(\mathrm{d}s)\right\} \\
&\qquad\leq [1-G(t)]F^{0-}\left(u^0\right) + \limsup_{n\to\infty}\int_{[0,t]} F_n^{1\prime}(X_s^n)G(\mathrm{d}s) \\
&\qquad\leq [1-G(t)]F^{0-}\left(u^0\right) + \int_{[0,t]} \limsup_{n\to\infty} F_n^{1\prime}(X_s^n)G(\mathrm{d}s) = -\infty,
\end{aligned}$$

where the equality holds by $F^{0-}(u^0) < \infty$ and $G(t) > 0$. This is a contradiction with $(\mathcal{E})$.

For the second part, suppose toward a contradiction that $X_t = 0$ for some $t \in \mathbf{R}_+$ and that either $F^{0+}(0) = \infty$ or $F^{1+}(0) = \infty$. Choose $u'' \in \left[X_0, u^0\right]$ such that $X_0 < u'' < u^0$ if $X_0 < u^0$, and note that $F^{1-}(u'')$ is finite by the first part of the claim. Since $X^n \to X$ pointwise and $X^n \leq u_n^0 \leq u^0$ for each $n \in \mathbf{N}$ (by $x^n \in \mathcal{X}_n'$ and property (b)), there is an $N' \in \mathbf{N}$ such that $X_0^n \leq u''$ for all $n \geq N'$. Since X^n is decreasing, it follows that $X^n \leq u''$ for all $n \geq N'$. Thus by property (c), the sequence of maps $\left(s \mapsto F_n^{1\prime}(X_s^n)\right)_{n=N'}^{\infty}$ is uniformly bounded below, so satisfies the hypothesis of Fatou's lemma.

We have $X = x = 0$ on $[t,\infty)$ since x is decreasing. So by property (d),

$$\liminf_{n\to\infty} F_n^{0\prime}(x_s^n) \geq F^{0+}(0) \quad \text{and} \quad \liminf_{n\to\infty} F_n^{1\prime}(X_s^n) \geq F^{1+}(0) \quad \text{for any } s \geq t.$$

As G has unbounded support, there is a $t' > t$ with $G(t) < G(t') < 1$. Then

$$\begin{aligned}
&\liminf_{n\to\infty}\left\{[1-G(t')]F_n^{0\prime}(x_{t'}^n) + \int_{[0,t']} F_n^{1\prime}(X_s^n)G(\mathrm{d}s)\right\} \\
&\qquad\geq [1-G(t')]\liminf_{n\to\infty} F_n^{0\prime}(x_{t'}^n) + \int_{[0,t']} \liminf_{n\to\infty} F_n^{1\prime}(X_s^n)G(\mathrm{d}s) \\
&\qquad\geq [1-G(t')]F^{0+}(0) + G(t')F^{1+}(0) = \infty,
\end{aligned}$$

where the first inequality holds by Fatou's lemma. This contradicts $(\mathcal{E})$. $\square$

Define $\phi_n^0, \phi_n^1 : \mathbf{R}_+ \to \mathbf{R}$ by

$$\phi_n^0(t) \coloneqq F_n^{0\prime}(x_t^n) \quad \text{and} \quad \phi_n^1(t) \coloneqq F_n^{1\prime}(X_t^n) \quad \text{for each } t \in \mathbf{R}_+.$$

We shall show that for any $t \in \mathbf{R}_+$, $(\phi_n^0)_{n\in\mathbf{N}}$ and $(\phi_n^1)_{n\in\mathbf{N}}$ are uniformly bounded on $[0,t]$. So fix a $t \in \mathbf{R}_+$. Choose $u' \in [0, X_t]$ so that $0 < u' < X_t$ in case $X_t > 0$, and let $u'' \in \left[X_0, u^0\right]$ be such that $X_0 < u'' < u^0$ if $X_0 < u^0$. By the claim, $F^{0+}(u')$, $F^{1+}(u')$ and $F^{1-}(u'')$ are finite. Since $X^n \to X$ pointwise and $0 < u_n^\star \leq X^n \leq u_n^0 \leq u^0$ (by $x^n \in \mathcal{X}_n'$ and property (b)), there is an $N \in \mathbf{N}$ such that $X_0^n \leq u''$ and $X_t^n \geq u'$ for all $n \geq N$. Since x^n is decreasing for each $n \in \mathbf{N}$, it follows that

$$u' \leq x_s^n \leq u^0 \quad \text{and} \quad u' \leq X_s^n \leq u'' \quad \text{for all } s \in [0,t] \text{ and } n \geq N.$$

This together with property (c) and the fact that $\phi_n^0 \geq 0$ for each $n \in \mathbf{N}$ implies that $(\phi_n^0)_{n=N}^\infty$ and $(\phi_n^1)_{n=N}^\infty$ are uniformly bounded on $[0,t]$. Since ϕ_n^0, ϕ_n^1 are bounded for each $n \in \mathbf{N}$ by property (a), it follows that $(\phi_n^0)_{n\in\mathbf{N}}$ and $(\phi_n^1)_{n\in\mathbf{N}}$ are uniformly bounded on $[0,t]$, as desired.

For each $n \in \mathbf{N}$, ϕ_n^0, ϕ_n^1 are increasing since x^n is decreasing and F^0, F^1 are concave. Since $(\phi_n^0)_{n\in\mathbf{N}}$ and $(\phi_n^1)_{n\in\mathbf{N}}$ are also uniformly bounded on $[0,t]$ for any $t \in \mathbf{R}_+$, it follows that $(x^n)_{n\in\mathbf{N}}$ admits a subsequence along which ϕ_n^0, ϕ_n^1 converge pointwise to some increasing $\phi^0 : \mathbf{R}_+ \to [0,\infty]$ and $\phi^1 : \mathbf{R}_+ \to [-\infty, \infty]$, by the Helly selection theorem.[99]

Clearly ϕ^0 is measurable. Moreover, since $x^n \to x$ pointwise and (thus) $X^n \to X$ pointwise, the same is true along the subsequence. So by property (d), $\phi^0(t)$ ($\phi^1(t)$) is a supergradient of F^0 at x_t (of F^1 at X_t) for every $t \in \mathbf{R}_+$. Moreover, letting $n \to \infty$ in $(\mathcal{E})$ yields that ϕ^0, ϕ^1 satisfy (E) for each $t \in \mathbf{R}_+$ by bounded convergence.

It remains only to show that ϕ^1 is G-integrable. Note first that ϕ^1 is bounded below by $\phi^1(0) = \lim_{n\to\infty} \phi_n^1(0) \in \mathbf{R}$ since it is increasing. Hence

$$\phi^1(0) \leq \mathbf{E}_G\Big(\phi^1(\tau)\Big) \leq \liminf_{t\to\infty} \int_{[0,t]} \phi^1 \mathrm{d}G = -\limsup_{t\to\infty} [1 - G(t)]\phi^0(t) \leq 0$$

by Fatou's lemma (second inequality), (E) (the equality) and the non-negativity of ϕ^0 (final inequality). Hence ϕ^1 is G-integrable. ■

[99] E.g. Rudin (1976, p. 167). For any subsequence of $(x^n)_{n\in\mathbf{N}}$ and $t \in \mathbf{R}_+$, Helly yields a sub-subsequence along which $(\phi_n^0)_{n\in\mathbf{N}}$ converges on $[0,t]$; a diagonalisation argument yields a subsequence of $(x^n)_{n\in\mathbf{N}}$ along which $(\phi_n^0)_{n\in\mathbf{N}}$ converges pointwise on $\mathbf{R}_+$. The same reasoning yields a further subsequence along which $(\phi_n^1)_{n\in\mathbf{N}}$ converges on $\mathbf{R}_+$.

O Comparative statics

When the likely time of the breakthrough becomes later, the agent is optimally provided with a higher continuation utility X_t in every period t:

Comparative statics theorem. Suppose that F^0 is strictly concave. Let $G, G^\dagger$ be absolutely continuous distributions with equal, unbounded support. If G MLR-dominates $G^\dagger$,[100] then $X \geq X^\dagger$ for any mechanisms (x, X) and $(x^\dagger, X^\dagger)$ that are optimal for G and $G^\dagger$, respectively.

The restriction to absolutely continuous distributions $G, G^\dagger$ with equal support is merely for simplicity. The proof relies on the following two lemmata, which are proved below. Recall from appendix F the definitions of the (superdifferential) Euler equation, $\mathcal{X}$, $\mathcal{X}'$ and 'simple'.

Lemma 11. Suppose that F^0, F^1 are simple. Let $G, G^\dagger$ be finite-support distributions with $G(0) = G^\dagger(0) = 0$ and equal support. If G MLR-dominates $G^\dagger$,[101] then $X \geq X^\dagger$ for any mechanisms (x, X) and $(x^\dagger, X^\dagger)$ with $x, x^\dagger \in \mathcal{X}'$ that satisfy the Euler equation for G and $G^\dagger$, respectively.

Lemma 12. If F^0 is strictly concave and G has unbounded support, then a mechanism (x, X) which satisfies $x \in \mathcal{X}$ and the Euler equation is uniquely optimal for G.

We shall also use the construction lemmata (3, 4 and 5) in appendix F.2.

Proof of the comparative statics theorem. Let $(F^0_n, F^1_n)_{n\in\mathbf{N}}$ be the simple technologies delivered by Lemma 5. Choose sequences $(G_m)_{m\in\mathbf{N}}$ and $(G^\dagger_m)_{m\in\mathbf{N}}$ of finite-support CDFs converging pointwise to (respectively) G and $G^\dagger$ such that for each $m \in \mathbf{N}$, $G_m(0) = G^\dagger_m(0) = 0$, G_m and $G^\dagger_m$ have equal support, and the former MLR-dominates the latter.[102]

Fix an arbitrary $n \in \mathbf{N}$. For every $m \in \mathbf{N}$, Lemma 3 assures us of the existence of $x^{nm}, x^{\dagger,nm} \in \mathcal{X}'_n$ such that (x^{nm}, X^{nm}) and $(x^{\dagger,nm}, X^{\dagger,nm})$ satisfy the Euler equation for (F^0_n, F^1_n, G_m) and $(F^0_n, F^1_n, G^\dagger_m)$, respectively.

[100] I.e. the ratio $G'/G^{\dagger\prime}$ of their densities is increasing on the support.

[101] I.e. the ratio $g/g^\dagger$ of their probability mass functions is increasing on the support.

[102] For example: let $\{Q_n\}_{n=0}^\infty$ be an enumeration of $\operatorname{supp}(G) \cap \mathbf{Q}$ with $Q_0 = \min \operatorname{supp}(G)$ and $G(Q_1), G^\dagger(Q_1) > 0$, write $\{Q_k\}_{k=0}^m = \{q^m_k\}_{k=0}^m$ where $q^m_0 < \cdots < q^m_m$, and define

$$G^{(\dagger)}_m(t) = \frac{1}{G^{(\dagger)}(q^m_m)} \sum_{k=1}^m \mathbf{1}_{[0,q^m_k]}(t) \left[G^{(\dagger)}(q^m_k) - G^{(\dagger)}(q^m_{k-1})\right] \quad \text{for each } t \in \mathbf{R}_+.$$

Since $\mathcal{X}'_n$ is sequentially compact by Observation 6 in appendix F.2 (p. 37), we may assume (passing to a subsequence if necessary) that

$$x^{nm} \to x^n \quad \text{and} \quad x^{\dagger,nm} \to x^{\dagger,n} \quad \text{pointwise as } m \to \infty$$

for some $x^n, x^{\dagger,n} \in \mathcal{X}'_n$. Since $u^0_n \to u^0$ and $u^\star_n \to u^\star$, Observation 6 permits us to assume (again passing to a subsequence if required) that

$$x^n \to x \quad \text{and} \quad x^{\dagger,n} \to x^\dagger \quad \text{pointwise as } n \to \infty$$

for some $x, x^\dagger \in \mathcal{X}'$. We have $X^{nm} \geq X^{nm,\dagger}$ for any $n, m \in \mathbf{N}$ by Lemma 11, so that letting $m \to \infty$ and $n \to \infty$ yields $X \geq X^\dagger$.

(x^n, X^n) satisfies the Euler equation for (F^0_n, F^1_n, G) for each $n \in \mathbf{N}$ by Lemma 4, and thus (x, X) satisfies the Euler equation for (F^0, F^1, G) by Lemma 5. Hence (x, X) is uniquely optimal for G by Lemma 12. Similarly, $(x^\dagger, X^\dagger)$ is uniquely optimal for $G^\dagger$. ■

O.1 Proof of Lemma 11

Fix $x, x^\dagger \in \mathcal{X}'$ such that (x, X) and $(x^\dagger, X^\dagger)$ satisfy the Euler equation for G and $G^\dagger$, respectively; we must show that $X \geq X^\dagger$. Enumerate the (common) support of G and $G^\dagger$ as $\{t_k\}_{k=1}^K \subseteq \mathbf{R}_+$, where $K \in \mathbf{N}$ and

$$0 < t_1 < \cdots < t_K < \infty.$$

Since F^0, F^1 are simple and $x, x^\dagger \in \mathcal{X}'$, Observation 5 in appendix F.2 (p. 36) implies that for some $u_1 \geq \cdots \geq u_K$ in $[u^\star, u^0]$, we have

$$x_t = \begin{cases} u^0 & \text{for a.e. } t \in [0, t_1) \\ u_k & \text{for a.e. } t \in [t_k, t_{k+1}) \text{ where } k \in \{1, \dots, K-1\} \\ u_K & \text{for a.e. } t \in [t_K, \infty), \end{cases} \tag{S}$$

and that $x^\dagger$ satisfies also (S) with some $u^\dagger_1 \geq \cdots \geq u^\dagger_K$ in $[u^\star, u^0]$.

Claim. It suffices to show that $X_{t_k} \geq X^\dagger_{t_k}$ for every $k \in \{1, \dots, K\}$.

Proof. Suppose that $X_{t_k} \geq X^\dagger_{t_k}$ for every $k \in \{1, \dots, K\}$, and fix an arbitrary $t \in \mathbf{R}_+$; we shall show that $X_t \geq X^\dagger_t$. If $t \geq t_K$, then

$$X_t = X_{t_K} \geq X^\dagger_{t_K} = X^\dagger_t$$

by (S). Assume for the remainder that $t < t_K$.

Suppose first that for some $k \leq K$, we have $t \leq t_k$ and $x \geq x^\dagger$ a.e. on (t, t_k). (This holds if $t \leq t_1$, since $x = u^0 = x^\dagger$ a.e. on $[0, t_1)$ by (S).) Then

$$
\begin{aligned}
X_t - X_t^\dagger = r \int_t^{t_k} e^{-r(s-t)} \left(x_s - x_s^\dagger \right) \mathrm{d}s + e^{-r(t_k - t)} \left(X_{t_k} - X_{t_k}^\dagger \right) \\
\geq e^{-r(t_k - t)} \left(X_{t_k} - X_{t_k}^\dagger \right) \geq 0.
\end{aligned}
$$

Suppose instead that $t \in (t_k, t_{k+1})$ for some $k < K$ and that $x < x^\dagger$ on a non-null subset of (t_k, t_{k+1}). Then $x < x^\dagger$ a.e. on (t_k, t) by (S), so that

$$
\begin{aligned}
0 \leq X_{t_k} - X_{t_k}^\dagger = r \int_{t_k}^{t} e^{-r(s-t_k)} \left(x_s - x_s^\dagger \right) \mathrm{d}s + e^{-r(t - t_k)} \left(X_t - X_t^\dagger \right) \\
\leq e^{-r(t - t_k)} \left(X_t - X_t^\dagger \right). \quad \square
\end{aligned}
$$

To show that $X_{t_k} \geq X_{t_k}^\dagger$ for every $k \in \{1, \ldots, K\}$, suppose not; we shall derive a contradiction. Let k' denote the largest $k \in \{1, \ldots, K\}$ at which $X_{t_k} < X_{t_k}^\dagger$. We shall prove that for every $k \leq k'$, it holds that

$$
X_{t_k} < X_{t_k}^\dagger \tag{8}
$$

$$
\text{and} \quad \mathbf{E}_G \left(F^{1\prime}(X_\tau) \middle| \tau \geq t_k \right) > \mathbf{E}_{G^\dagger} \left(F^{1\prime}\left(X_\tau^\dagger \right) \middle| \tau \geq t_k \right). \tag{9}
$$

This suffices because it contradicts the fact that

$$
\begin{aligned}
\mathbf{E}_G \left(F^{1\prime}(X_\tau) \middle| \tau \geq t_1 \right) = \mathbf{E}_G \left(F^{1\prime}(X_\tau) \right) \\
= 0 = \mathbf{E}_{G^\dagger} \left(F^{1\prime}\left(X_\tau^\dagger \right) \right) = \mathbf{E}_{G^\dagger} \left(F^{1\prime}\left(X_\tau^\dagger \right) \middle| \tau \geq t_1 \right),
\end{aligned}
$$

which holds by Observation 5 in appendix F.2 (p. 36) since (x, X) and $(x^\dagger, X^\dagger)$ satisfy the Euler equation for G and $G^\dagger$. We proceed by (backward) induction on $k \in \{k', \ldots, 1\}$.

Base case: $k = k'$. Here (8) holds by hypothesis, so we need only derive (9). If $k' = K$, then we have by strict concavity of F^1 that

$$
\mathbf{E}_G \left(F^{1\prime}(X_\tau) \middle| \tau \geq t_k \right) = F^{1\prime}(X_{t_K}) > F^{1\prime}\left(X_{t_K}^\dagger \right) = \mathbf{E}_{G^\dagger} \left(F^{1\prime}\left(X_\tau^\dagger \right) \middle| \tau \geq t_k \right).
$$

Assume for the remainder that $k' < K$.

Since $X_{t_k} < X_{t_k}^\dagger$ and $X_{t_{k+1}} \geq X_{t_{k+1}}^\dagger$ by definition of k', (S) yields

$$
\begin{aligned}
\left(1 - e^{-r(t_{k+1} - t_k)} \right) u_k = X_{t_k} - e^{-r(t_{k+1} - t_k)} X_{t_{k+1}} \\
< X_{t_k}^\dagger - e^{-r(t_{k+1} - t_k)} X_{t_{k+1}}^\dagger = \left(1 - e^{-r(t_{k+1} - t_k)} \right) u_k^\dagger,
\end{aligned}
$$

so that $u_k < u_k^\dagger$. It follows by the strict concavity of F^0 that

$$\mathbf{E}_G\Big(F^{1\prime}(X_\tau)\Big|\tau > t_k\Big) = F^{0\prime}(u_k) > F^{0\prime}\Big(u_k^\dagger\Big) = \mathbf{E}_{G^\dagger}\Big(F^{1\prime}\Big(X_\tau^\dagger\Big)\Big|\tau > t_k\Big),$$

which is to say that (9) holds at $k+1$. Thus (9) holds at k:

$$\begin{aligned}
\mathbf{E}_G\Big(F^{1\prime}(X_\tau)\Big|\tau \geq t_k\Big) &= \mathbf{P}_G(\tau = t_k|\tau \geq t_k)F^{1\prime}(X_{t_k}) \\
&\quad + \mathbf{P}_G(\tau > t_k|\tau \geq t_k)\mathbf{E}_G\Big(F^{1\prime}(X_\tau)\Big|\tau > t_k\Big) \\
&\geq \mathbf{P}_{G^\dagger}(\tau = t_k|\tau \geq t_k)F^{1\prime}(X_{t_k}) \\
&\quad + \mathbf{P}_{G^\dagger}(\tau > t_k|\tau \geq t_k)\mathbf{E}_G\Big(F^{1\prime}(X_\tau)\Big|\tau > t_k\Big) \\
&> \mathbf{P}_{G^\dagger}(\tau = t_k|\tau \geq t_k)F^{1\prime}\Big(X_{t_k}^\dagger\Big) \\
&\quad + \mathbf{P}_{G^\dagger}(\tau > t_k|\tau \geq t_k)\mathbf{E}_{G^\dagger}\Big(F^{1\prime}\Big(X_\tau^\dagger\Big)\Big|\tau > t_k\Big) \\
&= \mathbf{E}_{G^\dagger}\Big(F^{1\prime}\Big(X_\tau^\dagger\Big)\Big|\tau \geq t_k\Big),
\end{aligned}$$

where the weak inequality holds since $G|_{\tau\geq t_k}$ MLR-dominates $G^\dagger|_{\tau\geq t_k}$ and

$$F^{1\prime}(X_{t_k}) \leq \mathbf{E}_G\Big(F^{1\prime}(X_\tau)\Big|\tau > t_k\Big) \quad \text{since } X \text{ and } F^{1\prime} \text{ are decreasing,}$$

and the strict inequality holds by (8) and strict concavity of F^1 (first term) and the fact that (9) holds at $k+1$ (second term).

Induction step: Assume that (8) and (9) hold at $k+1 \leq K$; we must show that they hold at k. Since (9) holds at $k+1$, we have

$$F^{0\prime}(u_k) = \mathbf{E}_G\Big(F^{1\prime}(X_\tau)\Big|\tau \geq t_{k+1}\Big) > \mathbf{E}_{G^\dagger}\Big(F^{1\prime}\Big(X_\tau^\dagger\Big)\Big|\tau \geq t_{k+1}\Big) = F^{0\prime}\Big(u_k^\dagger\Big),$$

so that $u_k < u_k^\dagger$ by strict concavity of F^0. Using (S) and the fact that (8) holds at $k+1$ yields

$$\begin{aligned}
X_{t_k} &= \Big(1 - e^{-r(t_{k+1}-t_k)}\Big)u_k + e^{-r(t_{k+1}-t_k)}X_{t_{k+1}} \\
&< \Big(1 - e^{-r(t_{k+1}-t_k)}\Big)u_k^\dagger + e^{-r(t_{k+1}-t_k)}X_{t_{k+1}}^\dagger = X_{t_k}^\dagger,
\end{aligned}$$

showing that (8) holds at k. Since (8) holds at k and (9) holds at $k+1$, the (exact) same argument as in the base case yields that (9) holds at k. ∎

O.2 Proof of Lemma 12

Recall the definitions of $\mathcal{X}$ and π_G from appendix F. Note that $\mathcal{X}$ is convex.

Observation 9. If F^0 is strictly concave and G has unbounded support, then $\arg\max_{\mathcal{X}} \pi_G$ has at most one element.

We omit the easy proof; see Curello and Sinander (2021).

Proof of Lemma 12. If (x, X) is an optimal mechanism, then we must have $x \in \mathcal{X}$ by Lemma 0 (p. 12), and thus x must belong to $\arg\max_{\mathcal{X}} \pi_G$.

By Corollary 3 in supplemental appendix K (p. 54), there is a mechanism $(x^\dagger, X^\dagger)$ that is optimal for G. Thus $\arg\max_{\mathcal{X}} \pi_G = \{x^\dagger\}$ by Observation 9.

Now, if a mechanism (x, X) satisfies $x \in \mathcal{X}$ and the Euler equation, then x belongs to $\arg\max_{\mathcal{X}} \pi_G$ by the Euler lemma in appendix F (p. 36), so (x, X) must be the uniquely optimal mechanism $(x^\dagger, X^\dagger)$. ■

References

Acharya, V. V., DeMarzo, P., & Kremer, I. (2011). Endogenous information flows and the clustering of announcements. *American Economic Review*, *101*(7), 2955–2979. https://doi.org/10.1257/aer.101.7.2955

Armstrong, M., & Vickers, J. (2010). A model of delegated project choice. *Econometrica*, *78*(1), 213–244. https://doi.org/10.3982/ECTA7965

Atkeson, A., & Lucas, R. E., Jr. (1992). On efficient distribution with private information. *Review of Economic Studies*, *59*(3), 427–453. https://doi.org/10.2307/2297858

Atkeson, A., & Lucas, R. E., Jr. (1995). Efficiency and equality in a simple model of efficient unemployment insurance. *Journal of Economic Theory*, *66*(1), 64–88. https://doi.org/10.1006/jeth.1995.1032

Baron, D. P., & Besanko, D. (1984). Regulation and information in a continuing relationship. *Information Economics and Policy*, *1*(3), 267–302. https://doi.org/10.1016/0167-6245(84)90006-4

Battaglini, M. (2005). Long-term contracting with Markovian consumers. *American Economic Review*, *95*(3), 637–658. https://doi.org/10.1257/0002828054201369

Ben-Porath, E., Dekel, E., & Lipman, B. L. (2019). Mechanisms with evidence: Commitment and robustness. *Econometrica*, *87*(2), 529–566. https://doi.org/10.3982/ECTA14991

Berkovitch, E., & Israel, R. (2004). Why the NPV criterion does not maximize NPV. *Review of Financial Studies*, *17*(1), 239–255. https://doi.org/10.1093/rfs/hhg023

Bird, D., & Frug, A. (2019). Dynamic non-monetary incentives. *American Economic Journal: Microeconomics*, *11*(4), 111–150. https://doi.org/10.1257/mic.20170025

Board, S. (2007). Selling options. *Journal of Economic Theory*, *136*(1), 324–340. https://doi.org/10.1016/j.jet.2006.08.005

Board, S., & Skrzypacz, A. (2016). Revenue management with forward-looking buyers. *Journal of Political Economy*, *124*(4), 168–198. https://doi.org/10.1086/686713

Bull, J., & Watson, J. (2007). Hard evidence and mechanism design. *Games and Economic Behavior*, *58*(1), 75–93. https://doi.org/10.1016/j.geb.2006.03.003

Campbell, A., Ederer, F., & Spinnewijn, J. (2014). Delay and deadlines: Freeriding and information revelation in partnerships. *American Economic Journal: Microeconomics*, *6*(2), 163–204. https://doi.org/10.1257/mic.6.2.163

Courty, P., & Li, H. (2000). Sequential screening. *Review of Economic Studies*, *67*(4), 697–717. https://doi.org/10.1111/1467-937X.00150

Curello, G. (2021). *Incentives for collective innovation* [working paper, 12 May 2021].

Curello, G., & Sinander, L. (2021). *Screening for breakthroughs: Omitted proofs* [working paper, Jul 2021, https://arXiv.org/abs/2104.02044v2].

Czigler, T., Reiter, S., Schulze, P., & Somers, K. (2020). Laying the foundation for zero-carbon cement. *McKinsey & Company Chemicals Insights*. https://www.mckinsey.com/industries/chemicals/our-insights/laying-the-foundation-for-zero-carbon-cement

de Clippel, G., Eliaz, K., Fershtman, D., & Rozen, K. (2021). On selecting the right agent. *Theoretical Economics*, *16*(2), 381–402. https://doi.org/10.3982/TE4027

Dilmé, F., & Li, F. (2019). Revenue management without commitment: Dynamic pricing and periodic flash sales. *Review of Economic Studies*, *86*(5), 1999–2034. https://doi.org/10.1093/restud/rdy073

Dye, R. A., & Sridhar, S. S. (1995). Industry-wide disclosure dynamics. *Journal of Accounting Research*, *33*(1), 157–174. https://doi.org/10.2307/2491297

Eső, P., & Szentes, B. (2007a). Optimal information disclosure in auctions and the handicap auction. *Review of Economic Studies*, *74*(3), 705–731. https://doi.org/10.1111/j.1467-937x.2007.00442.x

Eső, P., & Szentes, B. (2007b). The price of advice. *RAND Journal of Economics*, *38*(4), 863–880. https://doi.org/10.1111/j.0741-6261.2007.00116.x

Feng, F. Z., Taylor, C. R., Westerfield, M. M., & Zhang, F. (2021). *Setbacks, shutdowns, and overruns* [working paper, Jan 2021]. https://doi.org/10.2139/ssrn.3775340

Folland, G. B. (1999). *Real analysis: Modern techniques and their applications* (2nd). Wiley.

Frankel, A. (2016). Discounted quotas. *Journal of Economic Theory*, *166*, 396–444. https://doi.org/10.1016/j.jet.2016.08.001

Garrett, D. F. (2016). Intertemporal price discrimination: Dynamic arrivals and changing values. *American Economic Review*, *106*(11), 3275–3299. https://doi.org/10.1257/aer.20130564

Garrett, D. F. (2017). Dynamic mechanism design: Dynamic arrivals and changing values. *Games and Economic Behavior*, *104*, 595–612. https://doi.org/10.1016/j.geb.2017.04.005

Gershkov, A., Moldovanu, B., & Strack, P. (2018). Revenue-maximizing mechanisms with strategic customers and unknown, Markovian demand. *Management Science*, *64*(5), 1975–2471. https://doi.org/10.1287/mnsc.2017.2724

Glazer, J., & Rubinstein, A. (2004). On optimal rules of persuasion. *Econometrica*, *72*(6), 1715–1736. https://doi.org/10.1111/j.1468-0262.2004.00551.x

Glazer, J., & Rubinstein, A. (2006). A study in the pragmatics of persuasion: A game theoretical approach. *Theoretical Economics*, *1*(4), 395–410.

Green, B., & Taylor, C. R. (2016). Breakthroughs, deadlines, and self-reported progress: Contracting for multistage projects. *American Economic Review*, *106*(12), 3660–3699. https://doi.org/10.1257/aer.20151181

Grenadier, S. R., Malenko, A., & Malenko, N. (2016). Timing decisions in organizations: Communication and authority in a dynamic environment. *American Economic Review*, *106*(9), 2552–2581. https://doi.org/10.1257/aer.20150416

Grossman, S. J. (1981). The informational role of warranties and private disclosure about product quality. *Journal of Law & Economics*, *24*(3), 461–483. https://doi.org/10.1086/466995

Grossman, S. J., & Hart, O. D. (1980). Disclosure laws and takeover bids. *Journal of Finance*, *35*(2), 323–334. https://doi.org/10.2307/2327390

Guo, Y. (2016). Dynamic delegation of experimentation. *American Economic Review*, *106*(8), 1969–2008. https://doi.org/10.1257/aer.20141215

Guo, Y., & Hörner, J. (2020). *Dynamic allocation without money* [working paper, 26 Aug 2020].

Guo, Y., & Shmaya, E. (2021). *Project choice from a verifiable proposal* [working paper, 8 May 2021].

Guttman, I., Kremer, I., & Skrzypacz, A. (2014). Not only what but also when: A theory of dynamic voluntary disclosure. *American Economic Review*, *104*(8), 2400–2420. https://doi.org/10.1257/aer.104.8.2400

Hansen, G. D., & İmrohoroğlu, A. (1992). The role of unemployment insurance in an economy with liquidity constraints and moral hazard. *Journal of Political Economy*, *100*(1), 118–142. https://doi.org/10.1086/261809

Hart, S., Kremer, I., & Perry, M. (2017). Evidence games: Truth and commitment. *American Economic Review*, *107*(3), 690–713. https://doi.org/10.1257/aer.20150913

Hopenhayn, H. A., & Nicolini, J. P. (1997). Optimal unemployment insurance. *Journal of Political Economy*, *105*(2), 412–438. https://doi.org/10.1086/262078

Jackson, M. O., & Sonnenschein, H. F. (2007). Overcoming incentive constraints by linking decisions. *Econometrica*, *75*(1), 241–257. https://doi.org/10.1111/j.1468-0262.2007.00737.x

Li, J., Matouschek, N., & Powell, M. (2017). Power dynamics in organizations. *American Economic Journal: Microeconomics*, *9*(1), 217–241. https://doi.org/10.1257/mic.20150138

Lipnowski, E., & Ramos, J. (2020). Repeated delegation. *Journal of Economic Theory*, *188*. https://doi.org/10.1016/j.jet.2020.105040

Lublin, J. S. (2017). The danger of being too good at your job. *Wall Street Journal*. https://www.wsj.com/articles/the-danger-of-being-too-good-at-your-job-1491912000

Lyons, B. R. (2003). *Could politicians be more right than economists? a theory of merger standards* [working paper, Nov 2003].

Madsen, E. (2020). *Designing deadlines* [working paper, 13 Feb 2020].

Mierendorff, K. (2016). Optimal dynamic mechanism design with deadlines. *Journal of Economic Theory*, *161*, 190–222. https://doi.org/10.1016/j.jet.2015.10.007

Milgrom, P. (1981). Good news and bad news: Representation theorems and applications. *Bell Journal of Economics*, *12*(2), 380–391. https://doi.org/10.2307/3003562

Myerson, R. B. (1982). Optimal coordination mechanisms in generalized principal–agent problems. *Journal of Mathematical Economics*, *10*(1), 67–81. https://doi.org/10.1016/0304-4068(82)90006-4

Nocke, V., & Whinston, M. D. (2013). Merger policy with merger choice. *American Economic Review*, *103*(2), 1006–1033. https://doi.org/10.1257/aer.103.2.1006

Pai, M. M., & Vohra, R. (2013). Optimal dynamic auctions and simple index rules. *Mathematics of Operations Research*, *38*(4), 682–697. https://doi.org/10.1287/moor.2013.0595

Pavan, A., Segal, I., & Toikka, J. (2014). Dynamic mechanism design: A Myersonian approach. *Econometrica*, *82*(2), 601–653. https://doi.org/10.3982/ECTA10269

Roberts, K. (1983). *Long term contracts* [working paper, University of Warwick].

Rockafellar, R. T. (1970). *Convex analysis*. Princeton University Press.

Rudin, W. (1976). *Principles of mathematical analysis* (3rd). McGraw-Hill.

Shavell, S., & Weiss, L. (1979). The optimal payment of unemployment insurance benefits over time. *Journal of Political Economy*, *87*(6), 1347–1362.

Sher, I. (2011). Credibility and determinism in a game of persuasion. *Games and Economic Behavior*, *71*(2), 409–419. https://doi.org/10.1016/j.geb.2010.05.008

Shimer, R., & Werning, I. (2008). Liquidity and insurance for the unemployed. *American Economic Review*, *98*(5), 1922–1942. https://doi.org/10.1257/aer.98.5.1922

Stein, E. M., & Shakarchi, R. (2005). *Real analysis: Measure theory, integration, and Hilbert spaces*. Princeton University Press.

Thomas, J., & Worrall, T. (1990). Income fluctuation and asymmetric information: An example of a repeated principal–agent problem. *Journal of Economic Theory*, *51*(2), 367–390. https://doi.org/10.1016/0022-0531(90)90023-D

Topkis, D. M. (1998). *Supermodularity and complementarity*. Princeton University Press.