A Model of Post-2008 Monetary Policy

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Abstract

Since the end of 2008, the Federal Reserve has been communicating its monetary policy in terms of two instruments under its direct control: the interest rate on bank reserves (IOR rate), and the size of its balance sheet. We introduce banks and bank reserves into the basic New Keynesian model to assess the main consequences of this policy change. We show that our model can account, in qualitative terms, for three key features of US inflation during the 2008-2015 zero-lower-bound (ZLB) episode: no significant deflation, little inflation volatility, and no significant inflation following quantitative-easing policies. Crucial to this result is our assumption that demand for bank reserves got close to satiation, but did not reach full satiation. We introduce liquid government bonds into the model to reconcile our non-satiation assumption with the fact that Treasury-bill rates dropped below the IOR rate during the ZLB episode. Looking ahead, we explore the implications of our model for the normalization of monetary policy and its future operational framework (floor vs. corridor system). In particular, we find that current and expected future IOR-rate hikes and balance-sheet contractions are always deflationary in our model, thus ruling out Neo-Fisherian effects.

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1 Introduction

Since the end of 2008, the Federal Reserve (the Fed, for short) has de-emphasized its intermediate target for the federal-funds rate, and communicated its monetary policy in terms of two instruments under its direct control: the interest rate on bank reserves (IOR rate), and the size of its balance sheet. In this paper, we analyze the implications of a simple monetary-policy model in which the central bank sets the IOR rate and the nominal stock of bank reserves (which is the only central-bank liability in the cashless model we use for most of our analysis). Looking backward, we show that the model can qualitatively account for key observations about US inflation and money-market rates during the 2008-2015 zero-lower-bound (ZLB) episode. Looking forward, we explore the model's implications for the pending normalization and future operational framework of monetary policy.

Commenting on US inflation during this ZLB episode, Cochrane (2018) argues that this "long period of quiet inflation at near-zero interest rates, with large quantitative easing, suggests that core monetary doctrines are wrong." Old Keynesian and New Keynesian (NK) models imply that hitting the ZLB for the policy rate should either generate significant deflationary pressures or make the inflation rate indeterminate; US data, however, reflect neither significant deflation nor signs of indeterminacy (such as volatility in the inflation rate), Cochrane observes. As to monetarist doctrine, Cochrane argues that the apparent lack of a significant inflationary response to the Fed's massive balance-sheet expansions casts doubt on models that emphasize the role of a stable money-demand equation.

Another challenge to a monetarist perspective on this ZLB episode is the fact that the federal-funds rate and T-bill returns dropped below the IOR rate during that period (and beyond). If this fact signals satiation of money demand, we lose the link (between the money supply and the price level) that is central to monetarist doctrine.

The model we propose in this paper introduces a monetarist element − bank reserves − into the basic NK model (Woodford, 2003, Galí, 2015). To generate a demand for reserves, we assume that holding reserves reduces, for banks, the costs associated with making loans. To generate a demand for bank loans, we assume that firms need to pay in advance their wage bill (or some fraction of it). In this model, the central bank sets both the IOR rate and the nominal stock of bank reserves. In the steady state, in particular, setting the IOR rate determines the demand for real reserves; and this demand pins down the price level, given the outstanding nominal stock of reserves.

The monetarist element in our model enables us to overcome the two challenges that Cochrane (2018) highlights for standard NK models − i.e., explaining little inflation volatility and no significant deflation at the ZLB. Unlike standard NK models, our model does not imply excessive inflation volatility during ZLB episodes. Nor does it imply severe deflationary pressures as we lengthen the duration of a liquidity trap caused by low natural real interest rates. Our model does not share these counterfactual implications because it delivers local-equilibrium determinacy under exogenous policy instruments.[1] In our model, the policy rate is the IOR rate; setting this rate exogenously does not constrain the supply of bank reserves; and, as we show, setting both the IOR rate and the supply of bank reserves exogenously always delivers local-equilibrium determinacy. We also show that this determinacy result under an exogenous IOR rate is essentially robust to making bank reserves endogenous through a "quantitative-easing rule," and to introducing household cash, alongside bank reserves, into the monetary base.

Having the central bank set the money supply also serves to rule out, in our model, the possibility of deflationary equilibrium paths analyzed by Benhabib, Schmitt-Grohé, and Uribe (2001a, 2001b). These paths are associated, in standard monetary-policy models, with an unintended deflationary steady state and a permanently binding ZLB constraint. By contrast, since the central bank sets the money supply in our model, we have a unique steady state, and inflation is equal to the money growth rate at this steady state. In a permanent ZLB episode, we can rule out deflationary equilibrium paths in (the flexible-price version of) our model as long as the central bank can commit not to shrink its balance sheet along such paths. We argue in the text that committing not to shrink the balance sheet at the ZLB does not involve the type of "blow-up-the-world threat" criticized by Cochrane (2011), because bank reserves serve as the medium of exchange and the unit of account in our model. These functions of money also make our arguments immune to Bassetto's (2002) criticism of the Fiscal Theory of the Price Level, when we invoke a non-Ricardian policy regime along off-equilibrium deflationary paths.

While the monetarist element of our model is helpful to account for Cochrane's (2018) first two observations about US inflation at the ZLB (i.e., little inflation volatility and no significant deflation), it may raise difficulties in accounting for his third observation (i.e., no significant inflation following quantitative-easing (QE) policies). The reason is that, to account for this last observation, we need a weak connection between the money supply and the price level in the short and medium run.

We show that our model can imply such a weak connection, and can thus account for the absence of significant inflation following QE policies, if two conditions are met: (1) the demand for reserves is "close to satiation" in a sense that we articulate, and (2) the monetary expansion is perceived as temporary. More specifically, we conduct non-linear numerical simulations of QE policies in our model, under a calibration to US data in November 2010 (i.e., at the start of the Fed's second round of QE). We find that large increases in the money supply (say, doubling the stock of reserves) can have very small inflationary effects (around twenty basis points per annum) if balance-sheet normalization is expected to occur in about five years and the marginal convenience yield of reserves is ten basis points per annum. It seems reasonable

[1] Carlstrom et al. (2015), Cochrane (2017), and Diba and Loisel (2020) highlight the connection between local-equilibrium indeterminacy and the puzzling implications of NK models about ZLB episodes.

to assume that the second and third QE rounds (QE2 and QE3) occurred close to satiation, were initially perceived as temporary, and were eventually absorbed by an increase in demand for reserves in anticipation of the Basel III liquidity-coverage requirements (implying that they would eventually not raise the price level even if they were permanent).[2] The first QE round (QE1), admittedly, did not occur close to satiation, but it seems reasonable to assume that it was immediately absorbed by the spike in demand for liquidity (in the midst of the crisis) that it was precisely intended to accommodate.

The ability of our model to account for these three salient features of US inflation hinges on the presumption that bank reserves kept a small but positive convenience yield (non-pecuniary return) during the ZLB episode. In reality, the convenience yield may arise from the banking sector's need for liquidity management or preference for safe assets; it may also reflect the usefulness of reserves for compliance with regulatory constraints (like liquidity-coverage requirements) and bank strategies for passing stress tests. Whatever the source of the convenience yield may be in reality, its presence is central to our analysis. Our arguments would fall apart if we assumed instead full satiation of demand for reserves. Since a number of prominent commentators (e.g., Cochrane, 2014, 2018; Reis, 2016) suggest that the Fed satiated the demand for reserves during the ZLB episode, this issue deserves discussion.

Empirical evidence presented by Ennis and Wolman (2015) and Reis (2016) does not support the satiation view during QE1. Reis's (2016) evidence, however, does not reject the satiation hypothesis during QE2 and QE3, when large increases in reserve balances had no apparent effect on expected inflation. This type of evidence is also the gist of Cochrane's (2018) criticism of monetarist doctrine, as we noted above. Our counter-argument, based on our numerical simulation of QE policies, is that this evidence may also be consistent with demand for reserves being close to satiation, rather than fully satiated. The distinction between these two possibilities (close-to-satiation versus fully satiated demand) does matter for monetary models; indeed, the implications of these models change discontinuously as we go from arbitrarily small convenience yields to a literal interpretation of full satiation. The distinction, however, may be difficult to make in practice. For example, in contrast to Reis's (2016) evidence about expected inflation, Krishnamurthy and Lustig (2019) find statistically significant effects of monetary policy, during and after QE2, on the convenience yield of US Treasury bills and the foreign-exchange value of the dollar.

As we noted above, besides Cochrane's observations about US inflation, observations about money-market rates pose a potential challenge to our model and its non-satiation assumption. The federal-funds rate and T-bill returns were below the IOR rate during the ZLB episode (and beyond, until October 2018). Isn't this fact prima-facie evidence that the convenience yield of

[2] Afonso et al. (2020) discuss the increase in demand for reserves reflecting Basel III regulations. Copeland et al. (2020) present evidence suggesting an apparent excess demand for reserves, despite a large Fed balance sheet, in more recent data.

reserves dropped to zero? We don't think so. Most of the trading activity in the federal-funds market over this period involved banks borrowing funds from entities that don't have direct access to the IOR rate (particularly, from Federal Home Loan Banks). Given the presence of such eager lenders, the federal-funds rate had to be below the IOR rate to incentivize the borrowers (banks with direct access to the IOR rate).

As to T-bill returns, the low rates could reflect strong demand by non-bank entities holding T-bills as collateral, or in response to regulatory constraints. To make this point formally, we present an extension of our benchmark model that allows T-bills to have a lower return than reserves without requiring that demand for reserves be fully satiated. In our extended model, workers get utility from holding government bonds, as a proxy for pension funds and money-market funds holding bonds and providing a service to households. Banks can use bonds instead of reserves for liquidity management, but they choose not to do so in equilibrium; so, the equilibrium of this extended model coincides with the equilibrium of our benchmark model, except for T-bill returns. Adding government bonds with liquidity services, thus, enables our model to account for T-bill returns below the IOR rate without altering any of the implications for inflation.

Looking ahead, we explore the implications of our model for the pending normalization and future operational framework of monetary policy. We find unambiguously negative effects of monetary-policy normalization on inflation: current and expected future IOR-rate hikes and balance-sheet contractions are always deflationary. Our model, thus, does not share the Neo-Fisherian implication of some equilibria in NK models – discussed in Schmitt-Grohé and Uribe (2017) and Bilbiie (2018), among others – that suggest policy-rate hikes may serve to raise inflation to target in economies that suffer from deflationary pressures.

We also find that the main two options currently under discussion for the Fed's future operational framework, namely a corridor system and a floor system, have substantially different implications for local-equilibrium determinacy. More specifically, under the corridor system, our model is isomorphic to the basic NK model; therefore, if the IOR rate reacts only to current inflation, it needs to react more than one-to-one in order to ensure determinacy (the so-called "Taylor principle"). Under a floor system, by contrast, we get determinacy for any non-negative response of the IOR rate to current inflation.

The floor system, however, may lead to global-equilibrium indeterminacy in (the flexible-price version of) our model. Indeed, we get dynamic equilibria with inflation below its steady-state value, under a floor system with no response of the IOR rate to inflation, if the central bank sets the IOR rate above the growth rate of nominal reserves – i.e., if the steady-state equilibrium has a positive real IOR rate. In particular, deflationary equilibria exist under constant nominal reserves if the (constant) nominal IOR rate is positive.[3] Interestingly, as we shrink the steady-

[3] Our global-indeterminacy result can be extended to other monetary models, like the MIU model analyzed in Obstfeld and Rogoff (1983), if we allow for interest payment on money. The transversality condition invoked in

state spread between the IOR and bond rates to zero, all these dynamic equilibria uniformly converge to the steady-state equilibrium with full satiation of demand for reserves. Thus, for an arbitrarily small spread, all these equilibria involve an inflation rate that is arbitrarily close to the growth rate of nominal reserves. We view this result as practically relevant because the "new normal" of US monetary policy may well involve "ample reserves" and small spreads between the IOR and interbank rates (Federal Open Market Committee, 2019).[4]

A few remarks may serve to put our contribution in the context of the literature. Cochrane (2018) proposes the Fiscal Theory of the Price Level to explain the three key features of US inflation during the 2008-2015 ZLB episode. In comparison, our model offers an alternative explanation that is purely monetary, in the sense that it presumes a Ricardian fiscal policy (except along off-equilibrium deflationary paths).

Our paper is also related to three papers that seek to explain how the price level is determined. Benigno (2020) shows how policies that set the IOR rate (following a rule subject to the ZLB constraint) as well as central bank remittances can ensure global equilibrium determinacy. Our emphasis on the role of money as unit of account follows his arguments (and the earlier work he cites). Canzoneri and Diba (2005) and Hagedorn (2018) assume that monetary policy exogenously sets the interest rate on a short-term government bond, and fiscal policy sets the nominal stock of these bonds. The resulting models are similar to ours in the sense that, in all these models, policy sets both the interest rate on, and the nominal stock of, an asset with some convenience yield.

A number of contributions – e.g., Gertler and Karadi (2011), Sims et al. (2020) – present models of monetary policy that highlight the asset side of the central bank's balance sheet, to focus on credit-market frictions. By contrast, we highlight the liability side, to focus on inflation and money-market rates. The link between reserves and banking costs in our model has some similarities with the one in Cúrdia and Woodford (2011). However, our modeling choices are different (to make our model analytically tractable), and we assume that demand for reserves is not satiated.[5]

Our model has also some similarities with the familiar Money-in-Utility (MIU) model. But it has more structure than the MIU model, and this structure serves two main purposes in our paper. First, it delivers sharper analytical results – in particular about monetary-policy normalization – under standard assumptions about the underlying utility and production func-

the literature does not rule out deflationary equilibria if the nominal interest rate on money is positive.

[4]Afonso et al. (2020) make a case, based on operational considerations, that the Fed should implement a floor system with "ample reserves." Their definition of ample reserves (as a range where the demand curve for reserves is fairly flat but still downward sloping) corresponds to our floor system with a small but positive convenience yield of reserves.

[5]In terms of modeling choices, two differences are worth emphasizing: (i) we link banking costs to time spent on banking activities, while Cúrdia and Woodford (2011) link them to goods consumed in banking activities; and (ii) the borrowers in our model are firms (borrowing the wage bill or some fraction of it), while they are impatient households in their model.

tions (like monotonicity and concavity). Second, modeling money explicitly as bank reserves serves to discipline the calibration of our QE experiment and to guide our extension with liquid government bonds.

Arce et al. (2019) and Piazzesi et al. (2019) develop NK models in which the central bank sets the IOR rate, in order to compare corridor and floor systems. Arce et al. (2019) incorporate matching frictions in the interbank market, and compare the risk of hitting the ZLB under the two systems. Piazzesi et al. (2019) relate the convenience yield of reserves, and other safe assets held by banks, to the liquidity services of the deposits that they issue (in a setting with collateralized deposit creation). They show that endogenous variation in convenience yields can deliver local-equilibrium determinacy even if the nominal stock of reserves is endogenous (e.g., in their corridor system) and if the policy rule for setting the IOR rate does not satisfy the Taylor Principle.

Finally, in Diba and Loisel (2020), we consider the log-linearized reduced forms of three monetary models, one of which is the model developed in the present paper. We show that the three models deliver local-equilibrium determinacy when both the interest rate on money and the supply of money are set exogenously; and that, as a result, they offer a resolution of four NK puzzles and paradoxes (like the "forward-guidance puzzle" of Del Negro, Giannoni, and Patterson, 2015). What is common to our two papers is the local-determinacy result under exogenous monetary-policy instruments. Beyond this common result, however, the two papers address largely distinct issues.

The rest of the paper is organized as follows. Section 2 presents our benchmark model. Sections 3, 4, and 5 show that this model can account for the three features of US inflation, discussed above, during the 2008-2015 ZLB episode. Section 6 introduces liquid government bonds into the model to also account for the negative spread between T-bill and IOR rates during this ZLB episode. Section 7 explores the implications of our model for the pending normalization and future operational framework of monetary policy. We then conclude and provide a technical appendix.

2 Benchmark Model

In this section, we present our benchmark model. This model will be extended in two different directions later in the paper: with household cash (Subsection 4.3) and with liquid government bonds (Section 6).

In our benchmark model, monopolistically competitive firms use labor to produce goods. They need to pay the wage bill (or some fraction of it) before they can produce and sell their output. They borrow the corresponding amount from banks. Banks incur costs making loans, and holding reserves mitigates these costs. The central bank sets both the interest rate on bank reserves and the quantity of bank reserves. We merge households and banks in our model, as there are no frictions between them. We make standard assumptions on utility and production functions (like monotonicity and concavity), without specifying any functional form.

2.1 Households (Reduced-Form Setup)

Each household consists of production workers and bankers. In this first subsection, we start from households' *reduced-form* utility function, whose arguments are consumption (c_t), hours worked by production workers (h_t), real loans (ℓ_t), and real reserves (m_t):

$$U_t = \mathbb{E}_t \left\{ \sum_{k=0}^{\infty} \beta^k \zeta_{t+k} \left[u\left(c_{t+k}\right) - v\left(h_{t+k}\right) - \Gamma\left(\ell_{t+k}, m_{t+k}\right) \right] \right\}, \tag{1}$$

where $\beta \in (0,1)$ and ζ_t denotes a stochastic exogenous discount-factor shock of mean one. The consumption-utility function u, defined over the set of positive real numbers $\mathbb{R}_{>0}$, is twice differentiable, strictly increasing ($u' > 0$), strictly concave ($u'' < 0$), and satisfies the usual Inada conditions $\lim_{c_t \to 0} u'(c_t) = +\infty$ and $\lim_{c_t \to +\infty} u'(c_t) = 0$. The labor-disutility function v, defined over the set of non-negative real numbers $\mathbb{R}_{\geq 0}$, is twice differentiable, strictly increasing ($v' > 0$), and weakly convex ($v'' \geq 0$).

The term $-\Gamma(\ell_t, m_t)$ in (1) comes from households acting as bankers. In Subsection 2.5, we will articulate how bankers produce loans using reserves and their own labor effort as inputs, and we will specify the *primitive* utility function of households. In the present subsection, we take a lighter approach to convey intuition: we simply work with the implied utility cost of making loans $\Gamma(\ell_t, m_t)$, reflecting bankers' disutility from work. This utility cost of banking is strictly increasing in loans ($\Gamma_\ell > 0$) because bankers have to work harder to make more loans. It is strictly decreasing in reserves ($\Gamma_m < 0$) because holding reserves reduces the labor effort needed to make a given amount of loans. It is also convex ($\Gamma_{\ell\ell} > 0$, $\Gamma_{mm} > 0$, $\Gamma_{\ell\ell}\Gamma_{mm} - (\Gamma_{\ell m})^2 \geq 0$), and such that $\Gamma_{\ell m} < 0$ (which says that a marginal increase in reserves decreases costs by more the larger are loans). Finally, it satisfies the limit properties $\lim_{m_t \to +\infty} \Gamma_m(\ell_t, m_t) = 0$ and $\lim_{m_t \to 0} \Gamma_\ell(\ell_t, m_t) = +\infty$ for any $\ell_t \in \mathbb{R}_{>0}$. The former property is a standard Inada condition, while the latter articulates a sense in which holding reserves is essential for banking.

In addition to making loans and holding reserve balances at the central bank, households trade bonds b_t (in zero net supply). Loans, reserves, and bonds are one-period non-contingent assets. We let I_t^ℓ, I_t^m, and I_t denote the corresponding gross nominal interest rates. We let P_t denote the price level, and $\Pi_t \equiv P_t / P_{t-1}$ the gross inflation rate. The household budget constraint, expressed in real terms, is then

$$c_t + b_t + \ell_t + m_t \leq \frac{I_{t-1}}{\Pi_t} b_{t-1} + \frac{I_{t-1}^\ell}{\Pi_t} \ell_{t-1} + \frac{I_{t-1}^m}{\Pi_t} m_{t-1} + w_t h_t + \tau_t, \tag{2}$$

where w_t represents the real wage and τ_t captures firm profits and lump-sum taxes or transfers.

Households choose b_t, c_t, h_t, ℓ_t, and m_t to maximize their reduced-form utility function (1) subject to their budget constraint (2), taking all prices (I_t, I_t^ℓ, I_t^m, P_t, and w_t) as given. Letting λ_t denote the Lagrange multiplier on the period-t budget constraint, the first-order conditions of the household optimization problem are

$$\lambda_t = \zeta_t u'(c_t), \tag{3}$$

$$\lambda_t = \beta I_t \mathbb{E}_t \left\{ \frac{\lambda_{t+1}}{\Pi_{t+1}} \right\}, \tag{4}$$

$$\lambda_t w_t = \zeta_t v'(h_t), \tag{5}$$

$$\zeta_t \Gamma_\ell (\ell_t, m_t) + \lambda_t = \beta I_t^\ell \mathbb{E}_t \left\{ \frac{\lambda_{t+1}}{\Pi_{t+1}} \right\},$$

$$\zeta_t \Gamma_m (\ell_t, m_t) + \lambda_t = \beta I_t^m \mathbb{E}_t \left\{ \frac{\lambda_{t+1}}{\Pi_{t+1}} \right\}.$$

Using (4), we can rewrite the last two conditions as

$$\frac{I_t^\ell}{I_t} = 1 + \frac{\zeta_t \Gamma_\ell (\ell_t, m_t)}{\lambda_t}, \tag{6}$$

$$\frac{I_t^m}{I_t} = 1 + \frac{\zeta_t \Gamma_m (\ell_t, m_t)}{\lambda_t}. \tag{7}$$

Condition (6) implies that loans pay more interest than bonds, because the marginal banking cost is positive ($\Gamma_\ell > 0$). Condition (7) implies that reserves pay less interest than bonds, because they serve to reduce banking costs ($\Gamma_m < 0$). The household optimization problem is also subject to a standard no-Ponzi-game constraint, and the transversality condition is

$$\lim_{k \to +\infty} \mathbb{E}_t \left\{ \beta^{t+k} \lambda_{t+k} a_{t+k} \right\} = 0, \tag{8}$$

where $a_t \equiv b_t + \ell_t + m_t$ denotes the household's total assets. The second-order conditions of the optimization problem are met because of the convexity of Γ.

2.2 Firms

There is a continuum of monopolistically competitive firms owned by households and indexed by $i \in [0, 1]$. Firm i uses $h_t(i)$ units of labor to produce

$$y_t(i) = f[h_t(i)] \tag{9}$$

units of output. The production function f, defined over $\mathbb{R}_{\geq 0}$, is twice differentiable, strictly increasing ($f' > 0$), and weakly concave ($f'' \leq 0$); it also satisfies $f(0) = 0$. To generate a demand for bank loans, we assume that firm i has to borrow a fraction $\phi \in (0, 1]$ of its nominal wage bill $W_t h_t(i)$ from banks, at the gross nominal interest rate I_t^ℓ, before it can produce and sell its output. Thus, the nominal value of firm i's loan $L_t(i)$ must satisfy

$$\phi W_t h_t(i) \leq L_t(i). \tag{10}$$

Following Calvo (1983), our sticky-price model assumes that at any given date, each firm (whatever its history) is not allowed to reset its price with probability $\theta \in [0,1)$. If allowed to reset its price at date t, firm i chooses its new price $P_t^*(i)$ to maximize the expected present value of the profits that this price will generate:

$$\mathbb{E}_t \left\{ \sum_{k=0}^{+\infty} (\beta\theta)^k \frac{\lambda_{t+k}}{\lambda_t \Pi_{t,t+k}} \left[P_t^*(i)\, y_{t+k}(i) - \frac{\beta\lambda_{t+k+1} I_{t+k}^{\ell} L_{t+k}(i)}{\lambda_{t+k}\Pi_{t+k+1}} - [W_{t+k} h_{t+k}(i) - L_{t+k}(i)] \right] \right\}.$$

The optimization is subject to the production function (9), the borrowing constraint (10), and the demand schedule

$$y_{t+k}(i) = \left[\frac{P_t^*(i)}{P_{t+k}} \right]^{-\varepsilon} y_{t+k}, \tag{11}$$

where $\Pi_{t,t+k} \equiv P_{t+k}/P_t$ for any $k \in \mathbb{N}$, $\varepsilon > 0$ denotes the elasticity of substitution between differentiated goods, and $y_t \equiv [\int_0^1 y_t(i)^{(\varepsilon-1)/(\varepsilon)} di]^{\varepsilon/(\varepsilon-1)}$.

Since the household first-order condition (6) implies $I_t^{\ell} > I_t$, the borrowing constraint of firms (10) is binding. Using the Euler equation (4) and the law of iterated expectations, we can write the first-order condition for the firm's optimization problem as

$$\mathbb{E}_t \left\{ \sum_{k=0}^{+\infty} (\beta\theta)^k \frac{\lambda_{t+k}}{\lambda_t \Pi_{t,t+k}} \left[P_t^*(i) - \left(\frac{\varepsilon}{\varepsilon - 1} \right) \left(\phi \frac{I_{t+k}^{\ell}}{I_{t+k}} + (1-\phi) \right) \frac{W_{t+k}}{f'[h_{t+k}(i)]} \right] y_{t+k}(i) \right\} = 0. \tag{12}$$

Under flexible prices ($\theta = 0$), and in a symmetric equilibrium (with $P_t^*(i) = P_t$ and $h_t(i) = h_t$), this first-order condition becomes

$$P_t = \frac{\varepsilon}{\varepsilon - 1} \left[\phi \frac{I_t^{\ell}}{I_t} + (1-\phi) \right] \frac{W_t}{f'(h_t)}. \tag{13}$$

2.3 Government

For simplicity, our benchmark model abstracts from government expenditures and government bonds. Introducing government expenditures into the model would not affect our results in any substantive way as long as these expenditures do not enter households' utility function or enter it in a separable way (as is standard in the literature). Government bonds would not matter at all if they serve only as a store of value, but may matter if they provide liquidity services; we will introduce liquid government bonds into our benchmark model in Section 6.

The central bank has two independent instruments: the (gross) nominal interest rate on reserves I_t^m, and the stock of nominal reserves M_t. In the absence of government bonds, the central bank injects reserves via lump-sum transfers (T_t). The nominal stock of reserves thus evolves according to

$$M_t = I_{t-1}^m M_{t-1} + T_t. \tag{14}$$

To capture a lower bound on I_t^m in a simple and stark way, we assume that vault cash (with no interest payments) is a perfect substitute for deposits at the central bank in terms of reducing banking costs. This introduces a zero lower bound (ZLB) for the net nominal IOR rate $I_t^m - 1$ in our model.[6] In an equilibrium with $I_t^m > 1$, banks will hold no cash. In an equilibrium with $I_t^m = 1$, the composition of reserve balances will be indeterminate, but also inconsequential; so, we will assume that banks hold no cash in equilibrium.

2.4 Market Clearing

The bond-market-clearing condition is

$$b_t = 0, \tag{15}$$

the reserve-market-clearing condition is

$$m_t = \frac{M_t}{P_t}, \tag{16}$$

and the goods-market-clearing condition is

$$c_t = y_t. \tag{17}$$

2.5 Households (Primitive Setup)

In this subsection, we briefly describe the primitive setup for households that leads to the reduced-form utility function (1). This brief description will be useful when we calibrate the model later in the paper (in Subsection 5.1). In this primitive setup, households get utility from consumption (c_t) and disutility from labor (h_t for production workers, h_t^b for bankers). Their intertemporal utility function is

$$U_t = \mathbb{E}_t \left\{ \sum_{k=0}^{\infty} \beta^k \zeta_{t+k} \left[u\left(c_{t+k}\right) - v\left(h_{t+k}\right) - v^b\left(h_{t+k}^b\right) \right] \right\}.$$

Like v, the labor-disutility function v^b is defined over $\mathbb{R}_{\geq 0}$, twice differentiable, strictly increasing ($v^{b\prime} > 0$), and weakly convex ($v^{b\prime\prime} \geq 0$). Bankers use their own labor h_t^b and (real) reserves at the central bank m_t to produce (real) loans ℓ_t according to the technology

$$\ell_t = f^b\left(h_t^b, m_t\right).$$

The production function f^b, defined over $\left(\mathbb{R}_{\geq 0}\right)^2$, is twice differentiable, strictly increasing ($f_h^b > 0$ and $f_m^b > 0$), homogeneous of degree $d \in (0, 1]$, and such that $f_{hh}^b < 0$, $f_{mm}^b < 0$, and $f_{hm}^b \geq 0$. These assumptions imply that f^b is concave, as we show in Appendix A.1. In addition, we assume that for any $h_t^b \in \mathbb{R}_{\geq 0}$, $\lim_{m_t \to +\infty} f_m^b(h_t^b, m_t) = 0$ and $\lim_{m_t \to 0} f_h^b(h_t^b, m_t) = 0$. The former assumption is a standard Inada condition, while the latter articulates a sense in which holding reserves is essential for banking.

[6] A more realistic model in which vault cash is substitutable to some extent for deposits at the central bank could imply a negative lower bound for the net nominal IOR rate. Whether the effective lower bound is zero or negative does not matter for most of our analysis below.

The function f^b is, of course, a convenient short cut to capture the role of bank reserves − which in reality may come, for example, from a maturity mismatch between banks' assets and liabilities. For the sake of generality, we do not impose any functional form for f^b. Examples of functional forms satisfying all the assumptions listed above include, in particular, constant-elasticity-of-substitution (CES) functions, and more generally (but not exclusively) CES functions raised to a power d such that $\max[(s-1)/s, 0] < d \leq 1$, where s denotes the elasticity of substitution. Moreover, we could relax some assumptions to some extent without affecting our results − for example, the assumption that labor and reserves are complements ($f^b_{hm} \geq 0$), or the assumption of decreasing or constant returns to scale ($d \leq 1$), which we make to simplify our analysis.[7]

Since $f^b_h > 0$, we can invert the production function of bankers f^b and get their labor hours as a function of loans and reserves: $h^b_t = g^b(\ell_t, m_t)$, where g^b is implicitly and uniquely defined by $\ell_t = f^b[g^b(\ell_t, m_t), m_t]$. Using this result to eliminate h^b_t in the primitive utility function, we get the reduced-form utility function (1), with the utility cost of banking

$$\Gamma\left(\ell_t, m_t\right) \equiv v^b\left[g^b\left(\ell_t, m_t\right)\right].$$

We establish some useful properties of the function g^b in Appendix A.2, and we show in Appendix A.3 that the function Γ has all the properties mentioned in Subsection 2.1.

3 Global Analysis Under Flexible Prices

In this section, we analyze the global perfect-foresight equilibria of our benchmark model under flexible prices and constant monetary-policy instruments. We show in particular that our model at the ZLB, like standard models with non-interest-bearing money (e.g. the MIU model of Obstfeld and Rogoff, 1983), has no deflationary equilibria as long as the nominal stock of reserves is not declining over time.

Deflationary equilibria, however, may exist away from the ZLB, as we also show in this section. Under a constant nominal stock of reserves, in particular, they do exist as soon as the IOR rate is above the ZLB. We will come back to this result in Section 7, when we discuss the implications of our model for the future operational framework of monetary policy.

3.1 Dynamic Equation Under Constant Instruments

We start by deriving the global dynamic equation of our benchmark model under flexible prices ($\theta = 0$), in the absence of discount-factor shocks ($\zeta_t = 1$), when the IOR rate I^m_t and the gross growth rate of nominal reserves $\mu_t \equiv M_t/M_{t-1}$ are permanently pegged ($I^m_t = I^m \geq 1$ and $\mu_t = \mu > 0$). To derive this dynamic equation, we first use (3), (5), (9), (10) holding with

[7] In an earlier, and longer, version of this paper (Diba and Loisel, 2017), we allow for increasing returns to scale ($d > 1$) when the function f^b is iso-elastic.

equality, (17), and $\zeta_t = 1$, to express loans ℓ_t as a function of employment h_t:

$$\ell_t = \mathcal{L}(h_t) \equiv \frac{\phi h_t v'(h_t)}{u'[f(h_t)]}. \tag{18}$$

The function $\mathcal{L}$, defined over $\mathbb{R}_{>0}$, is strictly increasing ($\mathcal{L}' > 0$), with $\lim_{h_t \to 0} \mathcal{L}(h_t) = 0$ and $\lim_{h_t \to +\infty} \mathcal{L}(h_t) = +\infty$. The reason is simply that loans are proportional to the wage bill (wage times employment), and the wage is increasing in employment.

Next, in Appendix A.4, we show that the equilibrium conditions (3), (5), (6), (9), (13), (17), and (18), together with $\zeta_t = 1$, implicitly and uniquely define a function $\mathcal{M}$ relating real reserves to employment:

$$m_t = \mathcal{M}(h_t). \tag{19}$$

This function is strictly increasing ($\mathcal{M}' > 0$). The reason is that under flexible prices, firms' profit maximization makes their real marginal cost equal to the inverse of their markup $(\varepsilon-1)/\varepsilon$; since real marginal cost depends positively on employment and negatively on real reserves (through borrowing costs), real reserves need to react positively to employment to keep real marginal cost equal to $(\varepsilon-1)/\varepsilon$. The function $\mathcal{M}$ is defined over $(0, \overline{h})$, where the upper bound $\overline{h} > 0$ is the limit value of employment when real reserves tend to infinity, and we have $\lim_{h_t \to \overline{h}} \mathcal{M}(h_t) = +\infty$.[8]

Finally, we use (18) and (19) to substitute for ℓ_t and m_t in households' first-order condition for reserves (7), and we use the resulting equation, together with (3), (9), (16), (17), $\zeta_t = 1$, $I_t^m = I^m$, and $\mu_t = \mu$, to rewrite households' first-order condition for bonds (4) as

$$1 + \frac{\Gamma_m[\mathcal{L}(h_t), \mathcal{M}(h_t)]}{u'[f(h_t)]} = \frac{\beta I^m}{\mu} \mathbb{E}_t \left\{ \frac{u'[f(h_{t+1})] \mathcal{M}(h_{t+1})}{u'[f(h_t)] \mathcal{M}(h_t)} \right\}. \tag{20}$$

In the next two subsections, we use this dynamic equation in employment, (20), to study the global perfect-foresight equilibria of our benchmark model under flexible prices and constant monetary-policy instruments.

3.2 Steady-State Equilibrium

We first show that, under a certain condition on I^m and μ, there exists a unique steady-state equilibrium, i.e. a unique equilibrium in which employment h_t is constant over time. When $h_{t+1} = h_t$, the dynamic equation (20) boils down to the static equation

$$\mathcal{F}(h_t) \equiv \frac{\Gamma_m[\mathcal{L}(h_t), \mathcal{M}(h_t)]}{u'[f(h_t)]} = -\left(1 - \frac{\beta I^m}{\mu}\right). \tag{21}$$

The function $\mathcal{F}$ is defined over $(0, \overline{h})$. We show in Appendix A.5 that it is strictly increasing ($\mathcal{F}' > 0$), with $\lim_{h_t \to 0} \mathcal{F}(h_t) = -\infty$ and $\lim_{h_t \to \overline{h}} \mathcal{F}(h_t) = 0$. So, for any policy satisfying $I^m <$

[8] This upper bound of employment $\overline{h}$ coincides with the frictionless employment level $h^\star$ in the case where the marginal banking cost Γ_ℓ converges to zero as real reserves tend to infinity. In general, however, we allow the marginal banking cost to converge to a positive value – in which case we have $\overline{h} < h^\star$, and our economy with the financial friction cannot attain the employment level of the frictionless economy.

μ/β, we have a unique equilibrium with constant employment. In this steady-state equilibrium, real reserves are constant, as follows from (19), and therefore inflation Π_t is constantly equal to μ. Households' first-order condition for bonds (4) then gives $I_t = I \equiv \mu/\beta$. Thus, the condition $I^m < \mu/\beta$ amounts to setting the IOR rate I^m below the steady-state interest rate on bonds I. When $I^m \geq I$, there is no equilibrium because banks would be tempted to issue infinite amounts of debt and deposit the proceeds at the central bank. When $I^m < I$, households' first-order condition for reserves (7) implies that the convenience yield of bank reserves is positive ($\Gamma_m < 0$), and this basically pins down the demand for real reserves. Since the nominal stock of reserves is exogenous, pinning down the demand for real reserves also pins down the price level.

The steady-state employment level $h \equiv \mathcal{F}^{-1}[-(1-\beta I^m/\mu)]$ is strictly increasing in the IOR rate I^m. This is because an increase in I^m reduces the opportunity cost of holding reserves I/I^m. The lower opportunity cost, in turn, increases real reserves, which decreases banking costs and hence borrowing costs, which in turn stimulates employment and output.

Following Benhabib, Schmitt-Grohé, and Uribe (2001a, 2001b), several contributions have analyzed deflationary liquidity traps in standard monetary models. These models assume that the central bank follows a Taylor-type rule for setting the interest rate on bonds (I_t). They typically have an unintended steady-state equilibrium in which the ZLB is binding ($I_t = 1$) and the price level falls over time (the gross inflation rate equals β). Our model, by contrast, has a unique steady-state equilibrium (provided that $1 \leq I^m < \mu/\beta$), and the gross inflation rate is equal to μ in this equilibrium. So, our model rules out steady-state deflation, even during a permanent ZLB episode ($I^m = 1$), provided that the central bank does not shrink its balance sheet over time ($\mu \geq 1$).

3.3 Dynamic Equilibria

We now assume that $I^m \in [1, \mu/\beta)$, and we characterize the perfect-foresight equilibria other than the unique steady-state equilibrium. We call them "dynamic equilibria," as they make employment h_t vary over time. If such equilibria exist, they must satisfy the dynamic equation (20), which we rewrite as

$$1 + \mathcal{F}(h_t) = \frac{\beta I^m}{\mu} \mathbb{E}_t \left\{ \frac{\mathcal{G}(h_{t+1})}{\mathcal{G}(h_t)} \right\},$$

or equivalently as

$$\mathcal{F}(h_t) - \mathcal{F}(h) = \frac{\beta I^m}{\mu} \mathbb{E}_t \left\{ \frac{\mathcal{G}(h_{t+1})}{\mathcal{G}(h_t)} - 1 \right\}, \tag{22}$$

where the function $\mathcal{G}$ is defined over $(0, \overline{h})$ by $\mathcal{G}(h_t) \equiv u'[f(h_t)]\mathcal{M}(h_t)$. We show in Appendix A.6 that this function is strictly increasing ($\mathcal{G}' > 0$), with $\lim_{h_t \to 0} \mathcal{G}(h_t) = 0$ and $\lim_{h_t \to \overline{h}} \mathcal{G}(h_t) = +\infty$.

Since the dynamic equation (22) is of order one, candidate perfect-foresight equilibria can be indexed by the initial value of employment, h_0. The steady-state equilibrium corresponds to

$h_0 = h$. Now consider a candidate equilibrium with $h_0 \in (0, h)$. Using (22), $\mathcal{F}' > 0$, and $\mathcal{G}' > 0$, we easily get by recurrence that employment h_t is strictly decreasing over time in this candidate equilibrium (as long as h_t is positive). Using (19) and $\mathcal{M}' > 0$, we then get that real reserves m_t are also strictly decreasing over time. Therefore, the inflation rate Π_t is higher than the growth rate of nominal reserves μ, or equivalently higher than the steady-state inflation rate $\Pi = \mu$, in such a candidate equilibrium. Alternatively, consider a candidate equilibrium with $h_0 \in (h, \overline{h})$. A similar reasoning shows that real reserves m_t are now strictly increasing over time. Therefore, the inflation rate is lower than its steady-state value in such a candidate equilibrium.

In terms of dynamic equilibria with "above-steady-state inflation" (i.e., inflation above its steady-state value), our model behaves similarly to other models with pure fiat money. Such models typically have dynamic equilibria in which money becomes eventually worthless, either at some finite date or asymptotically, unless it is made "essential" in some sense. In an endowment economy with separable utility $v(m_t)$ from holding real money balances, money is essential if the "super Inada condition" $\lim_{m_t \to 0} m_t v'(m_t) > 0$ is satisfied. Kingston (1982) and Obstfeld and Rogoff (1983) summarize earlier contributions suggesting that this condition is quite restrictive, and show that dynamic equilibria with above-steady-state inflation exist if this condition is not satisfied. Our model with pure fiat money is no exception to this rule. In Appendix A.7, we derive a condition that is the counterpart, in our production economy with banks, of the super Inada condition in endowment economies with separable utility. We show that if this condition is not satisfied, then there exists a countable infinity of dynamic equilibria with above-steady-state inflation, corresponding to a countable infinity of initial employment values $h_0 \in (0, h)$. All these equilibria are "hyperinflationary" in the sense that they make money worthless and prices infinite at a finite date; they also imply zero production from this date onwards.

We do not view these speculative hyperinflationary equilibria, which exist even when the money supply is constant or declining over time ($\mu \leq 1$), as particularly interesting for the purposes of our paper. The inflation question raised by the US experience since 2008 is not why we did not have speculative hyperinflation, but rather why we did not have significant inflation following the dramatic increase in money supply. For this reason, when we address this inflation question in Section 5, we will focus on equilibria driven by the fundamentals of money supply.

Now turn to the dynamic equilibria with "below-steady-state inflation" (i.e., inflation below its steady-state value). To study them, we consider a candidate equilibrium with $h_0 \in (h, \overline{h})$ and, therefore, with a strictly increasing sequence $(h_t)_{t \in \mathbb{N}}$. In this candidate equilibrium, we have

$$\frac{\mathcal{G}(h_{t+1})}{\mathcal{G}(h_t)} = 1 + \frac{\mu}{\beta I^m} \left[\mathcal{F}(h_t) - \mathcal{F}(h)\right] > 1 + \frac{\mu}{\beta I^m} \left[\mathcal{F}(h_0) - \mathcal{F}(h)\right] > 1$$

and therefore $\lim_{t \to +\infty} \mathcal{G}(h_t) = +\infty$. Since $\mathcal{G}' > 0$ and $\lim_{h_t \to \overline{h}} \mathcal{G}(h_t) = +\infty$, we get that employment h_t converges over time to its upper bound $\overline{h}$. In turn, using the dynamic equation (22) and $\lim_{h_t \to \overline{h}} \mathcal{F}(h_t) = 0$, we get that the ratio $\mathcal{G}(h_{t+1})/\mathcal{G}(h_t)$ converges over time to $\mu/\beta I^m$. Finally, using the definition of $\mathcal{G}$ and $\lim_{h_t \to \overline{h}} u'[f(h_{t+1})]/u'[f(h_t)] = 1$, we get that the gross growth rate of real reserves m_{t+1}/m_t also converges over time to $\mu/\beta I^m$. However, the household's transversality condition (8) states that real reserves cannot grow at the rate $1/\beta$ or faster in equilibrium, because λ_t converges to a constant as $h_t \to \overline{h}$ and the household's other assets are non-negative. So, our candidate equilibrium is an equilibrium only under policies setting $I^m > \mu$. Under such policies, thus, there exists a non-countable infinity of dynamic equilibria with below-steady-state inflation, indexed by $h_0 \in (h, \overline{h})$. Alternatively, under policies setting $I^m \leq \mu$, there exists no dynamic equilibrium with below-steady-state inflation.

This result has different implications for inflation at the ZLB ($I^m = 1$) and inflation away from the ZLB ($I^m > 1$), under a constant nominal stock of reserves ($\mu = 1$). We will come back to its implications for inflation away from the ZLB in Section 7, when we discuss the implications of our model for the future operational framework of monetary policy. In the next subsection, we focus on its implications for inflation at the ZLB, in order to explain how our benchmark model under flexible prices can account for the absence of significant deflation at the ZLB.[9]

3.4 Implications For Inflation at the ZLB

The results that we have just obtained straightforwardly imply that at the ZLB ($I^m = 1$), our model has no dynamic equilibria with below-steady-state inflation as long as the nominal stock of reserves is not declining over time ($\mu \geq 1$). If we assume away the speculative hyperinflationary equilibria discussed above, the only perfect-foresight equilibrium is the steady-state equilibrium. In this equilibrium, gross inflation is equal to μ and is, therefore, not lower than one.

Our argument for ruling out deflationary equilibria, in the context of a permanent ZLB episode, is essentially the same as earlier results with a constant stock of non-interest-bearing money (e.g., Obstfeld and Rogoff, 1983). A similar argument has also motivated proposals to switch from a rule for setting the interest rate on bonds to a policy fixing the money supply when the ZLB constraint becomes binding (e.g., Benhabib, Schmitt-Grohé, and Uribe, 2002). We do not think this argument entails what Cochrane (2011) calls a threat to "blow up the world" in our setup. If the IOR rate is, for any reason, dragged to the ZLB, the central bank can always refuse to shrink the nominal stock of reserves. At the ZLB, the central bank does not pay any interest on reserves, so there is no financial or resource cost associated with keeping the nominal stock of reserves constant or increasing it, even if below-steady-state inflation leads to growth of real reserves. This suggests that our proposed policy is feasible under all circumstances (on and off the equilibrium path).

Does our argument implicitly invoke the Fiscal Theory of the Price Level (FTPL) and assume a non-Ricardian regime? The answer depends on how one adds fiscal policy to our model. One

[9] The ZLB episode that we consider in the next subsection, in our non-linear benchmark model under flexible prices and constant policy instruments, corresponds to the permanent ZLB episode analyzed in the literature. We will consider a temporary ZLB episode in our log-linearized benchmark model under sticky prices in the next section.

way to do it, following Benigno (2020), is to add a separate fiscal authority and assume that the central bank does not guarantee redemption of the public debt. In this case, the fiscal authority would have to satisfy its present-value budget constraint, and we would have a Ricardian fiscal regime. Alternatively, we may envision an integrated public sector, with a central bank that fully backs the debt issued by the fiscal authority. In this case, the resulting policy regime could be Ricardian on the equilibrium path (absent deflation). On a deflationary path, the real value of the public debt would start to grow, and the debt would have to be redeemed or monetized until reserves are the only public-sector liability outstanding. At this point, the central bank could fix the nominal stock of reserves and avert deflation. This would imply a non-Ricardian policy regime along an off-equilibrium deflationary path (with real reserve balances growing).

In the latter case, despite implying a non-Ricardian policy regime along an off-equilibrium deflationary path, our argument is immune to Bassetto's (2002) criticism of the FTPL, because reserves constitute the only public sector liability. Bassetto's criticism pertains to fiscal commitments that require the government to issue debt. Since the public can refuse to lend to the government, he notes, such a commitment is not feasible under all circumstances. Bassetto's argument hinges on the fact that bonds have a maturity date and are not legal tender. By contrast, money has no maturity date and is legal tender. A policy that keeps the money stock constant is always feasible because the central bank need not "roll over" money. And a policy that issues more money during deflationary episodes will also be feasible anyway, because private agents will accept to hold the newly issued money (since it is legal tender).

4 Local Analysis Under Sticky Prices

We now allow prices to be sticky ($\theta \geq 0$), and we study the local equilibria of our benchmark model in the neighborhood of its unique steady state. More specifically, we show that the model delivers local-equilibrium determinacy under exogenous monetary-policy instruments, and we explain how this determinacy result can account for two features of US inflation during the recent ZLB episode: no significant deflation, and little inflation volatility.

At the end of the section, as a robustness check, we relax our assumption of an exogenous nominal stock of bank reserves, and consider instead a "quantitative-easing rule" setting the nominal stock of bank reserves as a function of output and the price level. We find that determinacy no longer obtains for all parameter values, but argue that it still obtains for all *reasonable* parameter values. We also show that our determinacy result is similarly robust to the introduction of household cash, alongside bank reserves, into the monetary base.

4.1 Local-Equilibrium Determinacy Under Exogenous Instruments

We consider a monetary policy setting its instruments I_t^m and μ_t exogenously. More specifically, I_t^m is set in the neighborhood of some value $I^m \in [1, 1/\beta)$, and μ_t in the neighborhood of the value $\mu = 1$.[10] In any steady state, real reserves are constant over time, by definition of a steady state; nominal reserves too are constant over time, because $\mu = 1$; therefore, prices are also constant over time, and there is no price dispersion. As a consequence, the set of steady states is the same as under flexible prices (with $\mu = 1$). From Subsection 3.2, then, we deduce that there exists a unique steady state.

In Appendix B.1, we log-linearize the equilibrium conditions of the model around its unique steady state and get the following IS equation, Phillips curve, and reserves-demand equation:

$$\widehat{y}_t = \mathbb{E}_t \{\widehat{y}_{t+1}\} - \frac{1}{\sigma} (i_t - \mathbb{E}_t \{\pi_{t+1}\} - r_t) \tag{23}$$

$$\pi_t = \beta \mathbb{E}_t \{\pi_{t+1}\} + \kappa (\widehat{y}_t - \delta_m \widehat{m}_t), \tag{24}$$

$$\widehat{m}_t = \chi_y \widehat{y}_t - \chi_i (i_t - i_t^m), \tag{25}$$

where variables with hats denote log-deviations from steady-state values, $\pi_t \equiv \log \Pi_t$, $i_t \equiv \widehat{I}_t$, $i_t^m \equiv \widehat{I}_t^m$, $r_t \equiv \widehat{\zeta}_t - \mathbb{E}_t\{\widehat{\zeta}_{t+1}\}$, and all the parameters (σ, β, κ, δ_m, χ_y, χ_i) are positive.

The IS equation (23) is exactly the same as in the basic NK model. It directly comes from the consumption Euler equation (3)-(4) and the goods-market-clearing condition (17). The Phillips curve (24) differs from its counterpart in the basic NK model in two ways. First, it involves real reserves $\widehat{m}_t$, because they reduce banking costs, which in turn lowers the borrowing costs of firms and hence their marginal cost of production. The parameter δ_m thus depends (positively) on $|\Gamma_{\ell m}|$. Second, the slope κ of the Phillips curve depends (positively) on $\Gamma_{\ell\ell}$, as an increase in output $\widehat{y}_t$ raises firms' marginal cost of production also through the resulting increase in loans and banking costs. Finally, the reserves-demand equation (25) states that the demand for reserves depends positively on loans and hence output, and negatively on the marginal opportunity cost of holding reserves, measured by the spread between the IOR rate and the interest rate on bonds. The parameter χ_y thus depends positively on $|\Gamma_{\ell m}|$, and the parameter χ_i negatively on Γ_{mm}.

Our model, given its structure, implies in particular that

$$\sigma < \chi_y < \frac{1}{\delta_m}, \tag{26}$$

as we show Appendix B.2. This double inequality will play a key role in our determinacy result below. The first inequality in (26) arises from the fact that bank loans serve to finance the wage bill (or some fraction of it). If output $\widehat{y}_t$ increases by 1%, the marginal utility of consumption decreases by σ%; so, the wage, the wage bill, and loans all increase by more than σ%; and, in

[10] Our results would be unchanged if we considered an arbitrary value for μ and assumed that non-optimized prices are indexed to steady-state inflation.

turn, so does the demand for reserves $\widehat{m}_t$ for a given spread $i_t - i_t^m$ (i.e., $\chi_y > \sigma$). The second inequality in (26) reflects how holding reserves mitigates the costs of banking. For a given spread $i_t - i_t^m$, a rise in output $\widehat{y}_t$ has two opposite effects on firms' marginal cost of production (i.e., on the term in factor of κ in the Phillips curve): a standard positive direct effect (with elasticity 1), and a negative indirect effect via the implied rise in reserves $\widehat{m}_t$ (with elasticity $\chi_y \delta_m$). The inequality states that the direct effect dominates the indirect one (i.e., $\chi_y \delta_m < 1$).

Under permanently exogenous monetary-policy instruments i_t^m and $\widehat{M}_t$, the IS equation (23), the Phillips curve (24), the reserves-demand equation (25), and the identities $\widehat{m}_t = \widehat{M}_t - \widehat{P}_t$ and $\pi_t = \widehat{P}_t - \widehat{P}_{t-1}$ lead to the following dynamic equation relating $\widehat{P}_t$ to $\mathbb{E}_t\{\widehat{P}_{t+2}\}$, $\mathbb{E}_t\{\widehat{P}_{t+1}\}$, $\widehat{P}_{t-1}$, and exogenous terms:

$$\mathbb{E}_t \left\{ L\mathcal{P}\left(L^{-1}\right) \widehat{P}_t \right\} = Z_t,$$

where

$$\begin{aligned} \mathcal{P}(X) \equiv\ & X^3 - \left[2 + \frac{1}{\beta} + \frac{\chi_y}{\sigma \chi_i} + \left(\frac{1}{\sigma} - \delta_m \right) \frac{\kappa}{\beta} \right] X^2 + \left[1 + \frac{2}{\beta} + \left(1 + \frac{1}{\beta} \right) \frac{\chi_y}{\sigma \chi_i} \right. \\ & \left. + \left(\frac{1}{\sigma} - \delta_m \right) \frac{\kappa}{\beta} + (1 - \delta_m \chi_y) \frac{\kappa}{\beta \sigma \chi_i} \right] X - \left(\frac{1}{\beta} + \frac{\chi_y}{\beta \sigma \chi_i} \right), \\ Z_t \equiv\ & \frac{-\kappa}{\beta \sigma} \left(i_t^m - r_t \right) + \left[\frac{1}{\sigma \chi_i} - \left(1 + \frac{\chi_y}{\sigma \chi_i} \right) \delta_m \right] \frac{\kappa}{\beta} \widehat{M}_t + \frac{\delta_m \kappa}{\beta} \mathbb{E}_t \left\{ \widehat{M}_{t+1} \right\}. \end{aligned}$$

Using the double inequality (26), we show in Appendix B.3 that the roots of the characteristic polynomial $\mathcal{P}(X)$ are three real numbers ρ, ω_1, and ω_2 such that $0 < \rho < 1 < \omega_1 < \omega_2$. With one eigenvalue inside the unit circle (ρ) for one predetermined variable ($\widehat{P}_{t-1}$), thus, our model satisfies Blanchard and Kahn's (1980) conditions and has a unique bounded solution under permanently exogenous monetary-policy instruments.

This determinacy result can be interpreted as follows. Under exogenous monetary-policy instruments i_t^m and $\widehat{M}_t$, the reserves-demand equation (25) makes the interest rate on bonds i_t a strictly increasing function of output and the price level:

$$i_t = \frac{\chi_y}{\chi_i} \widehat{y}_t + \frac{1}{\chi_i} \widehat{P}_t + \left(i_t^m - \frac{1}{\chi_i} \widehat{M}_t \right). \tag{27}$$

If output or the price level rises, demand for nominal money balances increases; and, given the exogenous policy instruments, the interest rate on bonds must increase to clear the money market. Thus, the reserves-demand equation (25) plays the same role as a "Wicksellian rule" for i_t. At the ZLB, the central bank has to peg the IOR rate i_t^m, which is its policy rate; but it is as if it could control the interest rate on bonds i_t and did set it according to the (shadow) Wicksellian rule (27). Wicksellian rules are well known to ensure determinacy in the basic NK model (as shown in Woodford, 2003, Chapter 4). In our model, however, not all Wicksellian rules would ensure determinacy. Our result, thus, is that the specific shadow Wicksellian rule (27) that arises under permanently exogenous monetary-policy instruments, given the restriction (26) that the model imposes on its coefficients, always delivers determinacy.

4.2 Implications For Inflation at the ZLB

We now explain how our determinacy result can account for two features of US inflation during the 2008-2015 ZLB episode: no significant deflation, and little inflation volatility. We start with the latter, and end with the former.

In standard NK models, local-equilibrium determinacy is typically obtained by assuming that the central bank will switch, after a temporary ZLB episode, to a policy-rate rule that sets a nominal anchor (e.g., a Taylor rule like $i_t = \phi\pi_t$ with $\phi > 1$). In this case, however, inflation depends on expected future shocks – occurring before the end of the ZLB episode – in a way that grows exponentially with the horizon of the shocks, regardless of the type of shocks considered (preference, supply, monetary, fiscal, etc.). As a result, inflation can be very volatile during the ZLB episode. For instance, if this episode is the consequence of a negative discount-factor shock (as is typically the case in the literature), then even small changes in the expected duration of the ZLB episode can have very large effects on inflation.

In Diba and Loisel (2020), we expose this puzzling implication of standard NK models and show that it is directly related to their property of generating indeterminacy under a permanently exogenous policy rate. By contrast, our model delivers determinacy under permanently exogenous monetary-policy instruments and, therefore, does not share this puzzling implication of standard NK models. To establish this result, we rewrite the dynamic equation as

$$\mathbb{E}_t\left\{\left(L^{-1}-\omega_1\right)\left(L^{-1}-\omega_2\right)(1-\rho L)\,\widehat{P}_t\right\} = Z_t$$

and use the method of partial fractions to solve this equation forward and get the unique bounded solution for $\widehat{P}_t - \rho\widehat{P}_{t-1}$:

$$\begin{aligned}\widehat{P}_t - \rho\widehat{P}_{t-1} &= \mathbb{E}_t\left\{\frac{Z_t}{\left(L^{-1}-\omega_1\right)\left(L^{-1}-\omega_2\right)}\right\} = \frac{\mathbb{E}_t}{\omega_2-\omega_1}\left\{\frac{\omega_1^{-1}Z_t}{1-(\omega_1 L)^{-1}} - \frac{\omega_2^{-1}Z_t}{1-(\omega_2 L)^{-1}}\right\} \\ &= \frac{\mathbb{E}_t}{\omega_2-\omega_1}\left\{\sum_{k=0}^{+\infty}\left(\omega_1^{-k-1}-\omega_2^{-k-1}\right)Z_{t+k}\right\}. \qquad (28)\end{aligned}$$

Using the price-level solution (28), the Phillips curve (24), and the identities $\widehat{m}_t = \widehat{M}_t - \widehat{P}_t$ and $\pi_t = \widehat{P}_t - \widehat{P}_{t-1}$, we then get

$$\pi_t = -(1-\rho)\,\widehat{P}_{t-1} + \frac{\mathbb{E}_t}{\omega_2-\omega_1}\left\{\sum_{k=0}^{+\infty}\left(\omega_1^{-k-1}-\omega_2^{-k-1}\right)Z_{t+k}\right\}, \qquad (29)$$

$$\widehat{y}_t = -\vartheta\widehat{P}_{t-1} + \delta_m\widehat{M}_t - \frac{\mathbb{E}_t}{(\omega_2-\omega_1)\,\kappa}\left\{\sum_{k=0}^{+\infty}\left(\xi_1\omega_1^{-k-1}-\xi_2\omega_2^{-k-1}\right)Z_{t+k}\right\}, \qquad (30)$$

where $\vartheta \equiv (1-\rho)(1-\beta\rho)/\kappa + \delta_m\rho$ and $\xi_j \equiv \beta(\omega_j+\rho-1)+\kappa\delta_m - 1$ for $j \in \{1,2\}$. The sums in (29)-(30) involve only ω_1^{-k} and ω_2^{-k} terms with $|\omega_1| > 1$ and $|\omega_2| > 1$. Therefore, the later shocks are expected to occur, the smaller their current effects in our model. More specifically, shocks occurring at date $t+k$ and announced at date t do not affect $\widehat{P}_{t-1}$; their effects on inflation and output at date t decay at an exponential rate with the horizon k, instead of growing exponentially as in standard NK models.

Consider, in particular, a temporary ZLB episode caused by a negative discount-factor shock between dates 0 and T ($r_t < 0$ for $0 \leq t \leq T$). We assume that cutting down the IOR rate to the ZLB only partially offsets this shock ($i_t^m - r_t = z^* > 0$ for $0 \leq t \leq T$). For simplicity, we also assume that the price level is at its steady-state value before the ZLB episode ($\widehat{P}_{-1} = 0$), reserves-supply policy is neutral during this episode ($\widehat{M}_t = 0$ for $0 \leq t \leq T$), and monetary policy is neutral afterwards ($i_t^m - r_t = \widehat{M}_t = 0$ for $t \geq T+1$). Under these assumptions, the exogenous driving term Z_t takes the value $-\kappa z^*/(\beta\sigma)$ between dates 0 and T, and the value 0 afterwards. Therefore, we can then rewrite (29) at date 0 as

$$\pi_0 = \frac{-\kappa z^*}{\beta\sigma\left(\omega_2 - \omega_1\right)} \sum_{k=0}^{T} \left(\omega_1^{-k-1} - \omega_2^{-k-1}\right). \tag{31}$$

Since $|\omega_1| > 1$ and $|\omega_2| > 1$, small changes in the duration T of the ZLB episode will have small effects on initial inflation π_0 in our model, instead of big effects as in standard NK models. As a result, inflation during a temporary ZLB episode will typically be much less volatile in our model than in standard NK models.

Equation (31) also shows that our model, unlike standard NK models, predicts no severe deflation during a temporary ZLB episode. Standard NK models, because of their puzzling implication (discussed above), predict that the deflation rate at the start of a temporary ZLB episode grows exponentially with the duration of this episode. By contrast, in our model, (31) implies that the initial deflation rate ($-\pi_0$) converges to the finite value $\kappa z^*/[\beta\sigma(\omega_1 - 1)(\omega_2 - 1)]$ as the duration T of the ZLB episode goes to infinity. For this reason, deflation during a temporary ZLB episode will typically be much lower in our model than in standard NK models. As such, our model offers an explanation $-$ missing from standard NK models $-$ for the absence of significant deflation during the recent ZLB episode in the US.

4.3 Robustness Analysis: Reserves-Supply Rule and Household Cash

So far, in this section, we have shown that our benchmark model delivers local-equilibrium determinacy under exogenous monetary-policy instruments, and we have used this determinacy result to explain the low volatility of inflation and the absence of significant deflation at the ZLB. We now briefly discuss (relegating the detailed analysis to Appendices C and D) how this determinacy result is essentially robust to the relaxation of two simplifying assumptions in turn: the exogeneity of nominal reserves, and the absence of household cash.

The first assumption that we relax is the exogeneity of nominal reserves. This assumption does not seem to us like a bad approximation of reality, given how the Fed has announced in advance a path for its balance sheet. Nonetheless, the alternative assumption of a reserves-supply rule may also seem relevant. In Appendix C, we assume that the central bank sets the stock of nominal reserves according to a rule that makes real reserve balances respond negatively to the price level for a given output level, and non-positively to the output level for a given price level. This specification nests, in particular, the previous case of exogenous nominal reserves. Log-linearizing the model around its unique steady state, we get the same IS equation (23), Phillips curve (24), and reserves-demand equation (25) as previously, plus now the log-linearized reserves-supply rule. So, we still get a "shadow Wicksellian rule" for the interest rate on bonds i_t: the reserves-demand equation (25) makes i_t depend positively on $\widehat{y}_t$ and negatively on $\widehat{m}_t$; in turn, $\widehat{m}_t$ now depends, through the reserves-supply rule, non-positively on $\widehat{y}_t$ and negatively on $\widehat{P}_t$.

This time, however, the implied shadow Wicksellian rule for i_t does not deliver determinacy for all parameter values. We derive a simple sufficient condition for determinacy, and we argue that this condition is likely to be met. More specifically, we set some steady-state variables to match some features of the US economy during the ZLB episode from December 2008 to December 2015; we calibrate the price-level coefficient of the reserves-supply rule from the QE policies conducted by the Fed during this ZLB episode; and we make conservative assumptions about the values of other steady-state variables. We find that our sufficient determinacy condition is met by a large margin even under our conservative assumptions. We conclude that setting exogenously the IOR rate and following the reserves-supply rule still delivers local-equilibrium determinacy, except for implausible calibrations.

Second, our benchmark model is specific in that households hold money only in the form of reserves, in their capacity as bankers. This makes our point stark because banks cannot collectively change the aggregate nominal quantity of reserves outstanding. In reality, bank reserves can fall if households demand more cash. In Appendix D, we show that our results do not unravel when we allow for such leakages out of reserve balances. More specifically, we introduce household cash into our benchmark model through a cash-in-advance constraint, and we study the consequences, for local-equilibrium determinacy, of setting exogenously the IOR rate and the monetary base (made of bank reserves and household cash). Log-linearizing the model around its unique steady state, we get the same IS equation (23) and reserves-demand equation (25) as previously, a different Phillips curve (because firms have to hold their cash from one period to the next), and a log-linearized money-market-clearing condition. So, we still get a "shadow Wicksellian rule" for the interest rate on bonds i_t: the reserves-demand equation (25) makes i_t depend positively on $\widehat{y}_t$ and negatively on $\widehat{m}_t$; in turn, $\widehat{m}_t$ now depends, through the money-market-clearing condition, negatively on $\widehat{y}_t$ and $\widehat{P}_t$.

As in the benchmark model with a reserves-supply rule, however, the implied shadow Wicksellian rule for i_t does not deliver determinacy for all parameter values. We derive a simple sufficient condition for determinacy, and we argue that this condition is likely to be met. More specifically, we set some steady-state variables to match some features of the US economy from December 2008 to December 2015, and we make standard or conservative assumptions about the values of other steady-state variables and parameters. We find that our sufficient determinacy condition is met by a large margin even under our conservative assumptions. We conclude that the introduction of household cash into the monetary base does not affect the ability of our model to deliver determinacy under an exogenous IOR rate and an exogenous monetary base, except for implausible calibrations.

5 Numerical Simulation of QE2

In this section, we conduct a non-linear numerical simulation of the second round of quantitative easing (QE2) in our benchmark model with sticky prices. Our main goal is to illustrate how our model, despite its monetarist features, can explain why no significant inflation was observed in the US following QE policies. More specifically, we show that large monetary expansions (say, doubling the nominal stock of reserves) can have very small inflationary effects in our model (around twenty basis points per annum) if: (1) the demand for reserves is close to satiation (in the sense that I^m is close to I, or equivalently Γ_m is close to 0), and (2) the monetary expansion is perceived as temporary (say, balance-sheet normalization is expected to occur in about five years). To make this point, we first calibrate our model to a steady-state equilibrium that matches some features of the US economy in November 2010, leading up to QE2; then, we consider the effects of large monetary expansions, from one up to four times QE2.

5.1 Calibration

For the calibration, we consider iso-elastic functional forms:

$$\begin{aligned} u(c_t) &\equiv (1-\sigma)^{-1}(c_t)^{1-\sigma}, \\ v(h_t) &\equiv V(1+\eta)^{-1}(h_t)^{1+\eta}, \\ v^b\left(h_t^b\right) &\equiv V_b(1+\eta)^{-1}\left(h_t^b\right)^{1+\eta}, \\ f(h_t) &\equiv A(h_t)^{\alpha}, \\ f^b\left(h_t^b, m_t\right) &\equiv A_b\left(h_t^b\right)^{1-\varsigma}(m_t)^{\varsigma}, \end{aligned}$$

where $\sigma > 0$, $V > 0$, $\eta \geq 0$, $V_b > 0$, $A > 0$, $0 < \alpha \leq 1$, $A_b > 0$, and $0 < \varsigma < 1$. These specifications imply

$$\begin{aligned} g^b(\ell_t, m_t) &= A_b^{\frac{-1}{1-\varsigma}}(\ell_t)^{\frac{1}{1-\varsigma}}(m_t)^{\frac{-\varsigma}{1-\varsigma}}, \\ \Gamma(\ell_t, m_t) &= V_b(1+\eta)^{-1}A_b^{\frac{-(1+\eta)}{1-\varsigma}}(\ell_t)^{\frac{1+\eta}{1-\varsigma}}(m_t)^{\frac{-\varsigma(1+\eta)}{1-\varsigma}}. \end{aligned}$$

We need to calibrate the parameters characterizing these functional forms (σ, V, η, V_b, A, α, A_b, ς), as well as the parameters β, ε, ϕ, θ, and I^m.

We have three degrees of freedom in our calibration, as we can freely pick units for output y_t and labor inputs (h_t and h_t^b). So, without any loss in generality, we can set arbitrarily any three of the following four parameters: A, A_b, V, and V_b. We choose to normalize A, A_b, and V to one.

We set standard values for the parameters σ, η, α, ε, and θ that appear in standard models. The utility function is logarithmic in consumption ($\sigma = 1$) and has a unitary Frisch elasticity of labor supply, for production workers as well as bankers ($\eta = 1$). The elasticity of output with respect to the labor input is $\alpha = 0.67$. For the price-setting nexus, we set the elasticity of substitution across differentiated goods to $\varepsilon = 6$ and the Calvo price-rigidity parameter to $\theta = 0.67$ (corresponding to "three-quarter price rigidity"). In addition, we assume that firms borrow the entire wage bill ($\phi = 1$), as in Christiano et al. (2005) and Ravenna and Walsh (2006). None of these values plays a major role in our simulation results.

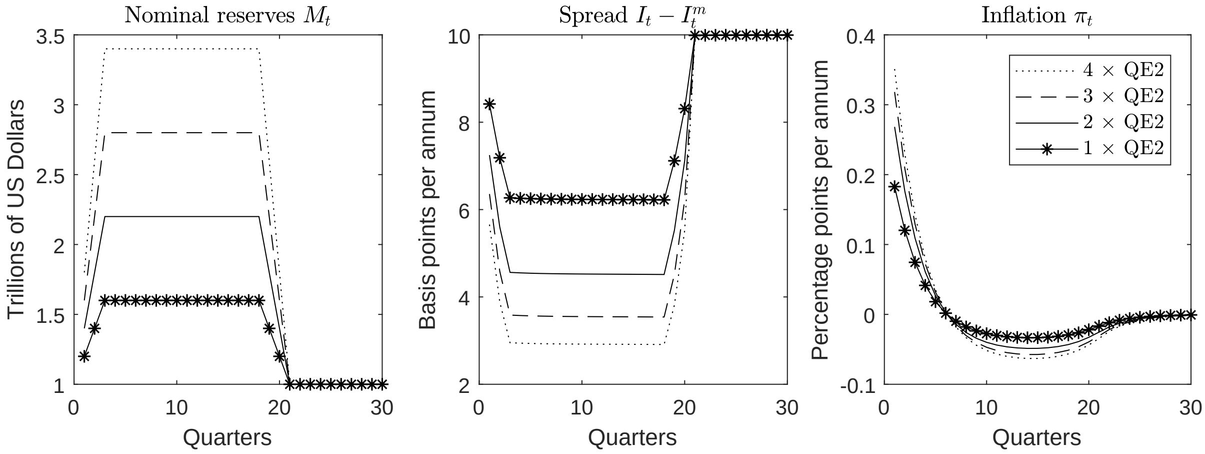
We set the net IOR rate $I^m - 1$ to 25 basis points per annum (the value prevailing in November 2010 in the US). To set a value for β, we need to take a stand on the value of the interest rate I. Nagel (2016) estimates a liquidity premium of 10 basis points per annum on T-bills in November 2010 in the US. As a benchmark, we assume the same figure applies to our $I - I^m$ spread, making the net interest rate $I - 1$ equal to 35 basis points per annum. This assumption does matter for our results, and we will discuss variations below. Since we have no inflation in the steady state, our target for I then pins down the discount factor to $\beta = 1/I = 0.999$ (on a quarterly basis).

We set the remaining two parameters, ς and V_b, so as to reach the following two steady-state targets: (i) the net interest rate on bank loans $I^\ell - 1$ is 3.25% per annum (the prime loan rate in November 2010 in the US); and (ii) the ratio of bank reserves to loans is $m/\ell = 1/9$ (the ratio of total reserves to bank credit of all commercial banks in November 2010 in the US). In Appendix A.8, we show how these targets pin down ς and V_b; we get $\varsigma = 0.0039$ and $V_b = 0.021$. Our simulation results, reported and discussed in the next subsection, are not sensitive to plausible variations in the values we pick for these targets.

5.2 Simulations

To assess the quantitative effects of large monetary expansions, we need to work with the non-linear version of our model. We use the "simul" command of Dynare for our non-linear simulation of a perfect-foresight equilibrium that asymptotically converges to the steady-state equilibrium. Figure 1 shows the effects of four alternative monetary expansions. One, like QE2, raises the balance-sheet size from an already large value (\$1 trillion) to a substantially larger one (\$1.6 trillion) in the course of 3 quarters (solid line with asterisks in Figure 1). The others raise the balance-sheet size by two, three, or four times as much, i.e. from \$1 to \$2.2, \$2.8, or \$3.4 trillion (solid, dashed, and dotted lines in Figure 1). All these monetary expansions are temporary: the balance-sheet size rises over 3 quarters, remains at its new value for 15 quarters, and goes back to its initial value over 3 quarters.

Figure 1 – Effect of a large temporary balance-sheet expansion



Note: The figure displays the effects of announcing at date 1 a large temporary balance-sheet expansion (left panel), starting from an already large balance-sheet size, on the spread (middle panel) and inflation (right panel) between dates 1 and 30.

As shown in Figure 1, the "single QE2" expansion makes the $I_t - I_t^m$ spread fall from 10 to 6.2 basis points per annum, and raises annualized inflation by only 18 basis points upon impact. And the "multiple QE2" expansions do not have much larger inflationary effects, given the decreasing returns of quantitative easing: following the "double, triple, and quadruple QE2" expansions, the spread falls to 4.5, 3.5, and 2.9 basis points, and inflation rises by only 27, 32, and 35 basis points respectively.

Our results are not sensitive to the values we assume for most of our parameters (although we could make the impact effects on inflation even smaller if we raised the price-rigidity parameter, say, to $\theta = 0.75$). Only two features really matter for the results.

First, our simulations start with a small spread $I_t - I_t^m$. This reflects a presumption that the already large level of reserve balances in the US prior to QE2 had "nearly satiated" the demand for real reserves (in the sense of bringing Γ_m close to 0). In this case, as the reserves-demand equation (7) makes clear, a large increase in nominal-reserves supply M_t can be absorbed by a small drop in the spread $I_t - I_t^m$, without changing the price level P_t by much. If we set the steady-state spread $I - I^m$ to 20 basis points (instead of 10), the inflationary impact of our "single QE2" expansion is 37 basis points (instead of 18); and if we did set the spread to 50 basis points, our inflation number would rise to 92 basis points. Conversely, of course, smaller spreads than our benchmark value of 10 basis points would strengthen our claim: cutting the spread to 5 basis points reduces the inflationary impact to 9 basis points.

The second assumption that matters for our low-inflation result is that the balance-sheet expansion is expected to be temporary. To see why this assumption matters, note that in the extreme case of a permanent increase in nominal reserves, our model would imply a proportional price increase in the long term. The reason is that the central bank does not change I^m in our QE experiment, and our representative-consumer setup pins down $I = 1/\beta$; so, the steady-state spread $I - I^m$ cannot shrink to raise the demand for real reserves if our monetary expansion is permanent, and the steady-state price level has to rise by the same amount so as to leave the steady-state real reserve balances unchanged. Our assumption that the unusual monetary expansion was not expected to last more than 5 years does not seem unreasonable to us, in light of commentary on how the crisis was not expected to last as long as it did. At any rate, the inflationary effects of temporary monetary expansions that are expected to last reasonably longer than 5 years are also modest in our model. For example, if our "single QE2" expansion is expected to last 10 years (instead of 5), then annualized inflation rises upon impact by only 40 basis points (instead of 18).

In reality, about ten years afterwards (at the time of this writing), the QE2 expansion has not been reversed, nor is it currently planned to be reversed. As we discussed in the Introduction, our view is that it lasted for longer than initially expected, and may have been eventually absorbed by an increase in demand for reserves in anticipation of the Basel III liquidity-coverage requirements (implying that it would eventually not raise the price level even if it were permanent).

6 Extension With Liquid Government Bonds

In the preceding three sections, we have shown that our benchmark model can broadly account for three key observations about US inflation during the 2008-2015 ZLB episode (no significant deflation, little inflation volatility, and no significant inflation following QE policies). In the present section, we first emphasize the key role played by one assumption in these results, namely the assumption that demand for reserves is close to satiation but not fully satiated. We then introduce liquid government bonds into the model in order to reconcile our non-satiation assumption with the observation that T-bill returns dropped below the IOR rate in the US during that period − while preserving the implications of the model for inflation at the ZLB.

6.1 Satiation vs. Non-Satiation of Demand for Reserves

Our results in Sections 3-5 rest on the assumption that demand for bank reserves got close to satiation, but did not reach full satiation, i.e. that bank reserves carried a small but positive convenience yield ($I_t^m < I_t$ and $\Gamma_m > 0$). If we allowed for a finite satiation point in the demand for reserves and if demand for reserves were fully satiated ($I_t^m = I_t$ and $\Gamma_m = 0$), then our results would fall apart as follows: (i) in the global flexible-price analysis of Section 3, the price level would be indeterminate; (ii) in the local sticky-price analysis of Section 4, our model would be isomorphic to the basic NK model ($\delta_m = 0$ and $i_t = i_t^m$), and would generate indeterminacy under a permanent interest-rate peg; and (iii) without price-level determinacy, the numerical simulation of QE2 in Section 5 would not be possible.

As we noted in the Introduction, our non-satiation assumption stands in contrast to views often expressed about the US economy in recent years. One argument making a case for satiation of demand for reserves is the fact that the second and third rounds of quantitative easing (QE2 and QE3) had no apparent inflationary consequences, as Reis (2016) and Cochrane (2018) point out. On this front, our counter-argument is simply that this fact may also be consistent with demand for reserves being close to satiation, rather than fully satiated, as our numerical simulation of QE2 in the previous section suggests.

In the present section, we address a second argument that goes against our non-satiation view. This argument is the fact that T-bill returns have been below the IOR rate during the 2008-2015 ZLB episode and for some time beyond. We do not think this fact contradicts our claim that reserves still had a positive marginal convenience yield during this period. The lower T-bill returns, we argue, could reflect strong demand by non-bank entities – using T-bills as, e.g., collateral or international reserve asset. We formalize our counter-argument by introducing government bonds providing liquidity services into our benchmark model. We show that our model with liquid bonds has an equilibrium in which the return on government bonds is below the IOR rate. Moreover, this equilibrium of our model with liquid bonds coincides with the equilibrium of our benchmark model (without liquid bonds), in the sense that all the endogenous variables that are common to both models, except the lump-sum transfer T_t, take the same equilibrium values. So, all the results that we have obtained in our benchmark model in Sections 3-5 also apply to our model with liquid bonds.

6.2 Liquidity of Government Bonds

Our benchmark model abstracts from government bonds and any role they may play in facilitating transactions. In reality, banks may hold government bonds (or other liquid assets), in addition to reserves, for liquidity management. Some regulatory constraints that give rise to a convenience yield for reserves – like the constraint on "high-quality liquid assets" imposed on US banks – can also be satisfied by holding government bonds. From this (regulatory) vantage point, bonds and reserves are perfect substitutes in satisfying liquidity needs. But government bonds are not as useful as reserves in satisfying the intra-day liquidity needs that arise from banking transactions, because bonds can either be sold for next-day settlement or used in repo transactions arranged to obtain liquidity, while reserves are readily available for any transaction – as Bush et al. (2019) elaborate.

Government bonds also provide a convenience yield to many non-bank entities (e.g. by serving as collateral or international reserve asset) and benefit from regulations (like restrictions on the asset portfolios of US money-market mutual funds). So, the observed returns on government bonds may reflect their convenience yield. If the returns are sufficiently attractive compared to the IOR rate, banks may hold government bonds to satisfy liquidity needs and regulatory constraints. If not, banks may hold mostly reserves for liquidity management.

Our model abstracts from non-bank financial institutions and foreign entities that may hold bonds. To formalize our main point, we will assume that workers get utility from government bonds (instead of modeling, say, a pension fund that holds bonds on workers' behalf). We will show that bankers may use government bonds for liquidity management if the IOR rate is sufficiently low compared to the equilibrium return from holding liquid bonds; but bankers will only use reserves for liquidity management when the IOR rate is sufficiently high. Although we don't explicitly model inside assets like federal-funds loans, we have in mind that our equilibrium with a relatively high IOR rate can also represent observed episodes in which banks don't lend federal funds, and the federal-funds rate is below the IOR rate. Our main point is that financial institutions that don't have direct access to the IOR rate may hold these assets in equilibrium, while banks hold reserves with a positive marginal convenience yield.

6.3 Equilibrium Conditions Related to Households

As in our benchmark model (presented in Section 2), the representative household consists of workers and bankers, and gets utility from consumption (c_t) and disutility from labor (h_t for workers, h_t^b for bankers). We now assume that workers also get utility from holding government bonds (b_t^w), and bankers may use government bonds (b_t^b) as well as reserves (m_t) to produce loans (ℓ_t). As before, we make a substitution for h_t^b in households' primitive utility function and get the following reduced-form utility function:

$$U_t = \mathbb{E}_t \left\{ \sum_{k=0}^{\infty} \beta^k \zeta_{t+k} \left[u\left(c_{t+k}\right) - v\left(h_{t+k}\right) - \Gamma\left(\ell_{t+k}, m_{t+k} + \eta b_{t+k}^b\right) + z\left(b_{t+k}^w\right) \right] \right\},$$

where $\eta \in (0, 1]$. The function z, defined over $\mathbb{R}_{>0}$, is twice differentiable, strictly increasing ($z' > 0$), and strictly concave ($z'' < 0$); it also satisfies the usual Inada conditions. Values of η below unity may capture the fact that in reality reserves are more useful than government bonds for liquidity management because they provide immediate intra-day liquidity to banks (as discussed above). We allow for $\eta < 1$ to show that T-bill returns can be below the IOR rate even when T-bills provide smaller liquidity services than reserves to banks.

In the interest of realism (to make sure some reserves are always held in equilibrium), we also assume that the central bank imposes reserve requirements on banks. Since our model consolidates bankers and workers into households (thus, abstracting from deposits), we specify the reserve requirement as

$$m_t \geq \psi \ell_t, \tag{32}$$

where $\psi > 0$. The household budget constraint, expressed in real terms, is

$$c_t + b_t + b_t^b + b_t^w + \ell_t + m_t \leq \frac{I_{t-1}}{\Pi_t} b_{t-1} + \frac{I_{t-1}^b}{\Pi_t}\left(b_{t-1}^b + b_{t-1}^w\right) + \frac{I_{t-1}^\ell}{\Pi_t}\ell_{t-1} + \frac{I_{t-1}^m}{\Pi_t} m_{t-1} + w_t h_t + \tau_t,$$

where I_t^b denotes the gross nominal interest rate on government bonds (and b_t represents a private bond in zero net supply, as we indicated in Section 2). We let λ_t and λ_t^r denote the Lagrange multipliers on the period-t budget constraint and reserve requirement (respectively). The optimality conditions are

$$\lambda_t = \zeta_t u'(c_t),$$

$$\lambda_t w_t = \zeta_t v'(h_t),$$

$$\lambda_t = \beta I_t \mathbb{E}_t \left\{ \frac{\lambda_{t+1}}{\Pi_{t+1}} \right\}, \tag{33}$$

$$\lambda_t = \zeta_t z'(b_t^w) + \beta I_t^b \mathbb{E}_t \left\{ \frac{\lambda_{t+1}}{\Pi_{t+1}} \right\}, \tag{34}$$

$$\zeta_t \Gamma_\ell \left(\ell_t, m_t + \eta b_t^b\right) + \lambda_t + \psi \lambda_t^r = \beta I_t^\ell \mathbb{E}_t \left\{ \frac{\lambda_{t+1}}{\Pi_{t+1}} \right\}, \tag{35}$$

$$\zeta_t \Gamma_m \left(\ell_t, m_t + \eta b_t^b\right) + \lambda_t = \lambda_t^r + \beta I_t^m \mathbb{E}_t \left\{ \frac{\lambda_{t+1}}{\Pi_{t+1}} \right\}, \tag{36}$$

and

$$(m_t - \psi \ell_t)\, \lambda_t^r = 0.$$

We must also have

$$\eta \zeta_t \Gamma_m \left(\ell_t, m_t + \eta b_t^b\right) + \lambda_t \geq \beta I_t^b \mathbb{E}_t \left\{ \frac{\lambda_{t+1}}{\Pi_{t+1}} \right\} \tag{37}$$

and $b_t^b \geq 0$, with complementary slackness.

6.4 Other Equilibrium Conditions

The remaining equilibrium conditions involve minor adjustments to our presentation in Subsections 2.2-2.4 (for firms, the government, and market clearing), as we describe below. The equilibrium conditions associated with firms don't change. For the government, we replace the consolidated budget constraint (14) by

$$M_t + B_t = I_{t-1}^m M_{t-1} + I_{t-1}^b B_{t-1} + T_t, \tag{38}$$

where B_t denotes the nominal stock of one-period public debt (held outside the central bank). A Ricardian fiscal policy adjusts the lump-sum transfer T_t to stabilize the real public debt around a steady-state target $b^* > 0$. The market-clearing conditions of Subsection 2.4 still apply $-$ in particular the condition (15), given that private bonds are in zero net supply. In addition, we now have the market-clearing condition for government bonds:

$$b_t^w + b_t^b = \frac{B_t}{P_t}. \tag{39}$$

6.5 Equilibrium of Interest

The household optimality conditions above admit a solution with $1 = I_t^m < I_t^b$ that may represent the period before interest payment on reserves in the US (i.e. before 2008). If η is large enough, such a solution may have binding reserve requirements and $b_t^b > 0$. In this case, banks use government bonds – and if we extended our model, they could use inside assets like commercial paper – to manage the liquidity of their portfolios.

We are, however, interested in a candidate equilibrium in which banks do not use government bonds ($b_t^b = 0$), the reserve requirement is not binding ($\lambda_t^r = 0$), and the IOR rate is above the government-bond yield ($I_t^m > I_t^b$). This candidate equilibrium may capture – admittedly, in a stark way – some features of US bank portfolios and T-bill returns in the aftermath of the financial crisis.

In Appendix E, we show that this candidate equilibrium is indeed, under some parameter restrictions, an equilibrium of our model with liquid bonds, and that it coincides with the equilibrium of our benchmark model (in the sense that all the endogenous variables that are common to both models, except the lump-sum transfer T_t, take the same equilibrium values). In essence, in this equilibrium, banks hold only reserves ($b_t^b = 0$) because they pay more interest than government bonds ($I_t^b < I_t^m$) and are at least as liquid as government bonds ($\eta \leq 1$). And because banks do not hold government bonds ($b_t^b = 0$) and face a non-binding reserve requirement ($\lambda_t^r = 0$), they behave in exactly the same way as in our benchmark model.

We need to impose two parameter restrictions to get this result. The first restriction is that the minimal reserves-to-loans ratio imposed by the central bank, ψ, should be lower than the steady-state value taken by the reserves-to-loans ratio in our benchmark model. This restriction is necessary for the reserve requirement (32) to be slack ($\lambda_t^r = 0$). The second restriction is that the steady-state marginal liquidity service of government bonds, $z'(b^*)$, should be large enough for the interest rate on government bonds to be lower than the IOR rate ($I_t^b < I_t^m$).

So, we conclude that the introduction of liquid government bonds into our benchmark model enables us to account for the negative spread between Treasury-bill and IOR rates observed during the 2008-2015 ZLB episode, without affecting in any way the ability of the model to account for the three key features of inflation during this ZLB episode (i.e., without affecting any of the results obtained in Sections 3-5).

7 Normalization and Operational Framework of Monetary Policy

In the preceding sections, we have looked backward and used our model to explain some key observations about inflation and money-market rates during the 2008-2015 ZLB episode. In the present section, we now look forward and explore the implications of our model for the pending normalization and the future operational framework of monetary policy.

7.1 Normalization of Monetary Policy

The issue of monetary-policy normalization, i.e. raising policy rates from the ZLB and shrinking the size of the central bank's balance sheet, has been at the center of attention in the past few years. During that time, the Fed started to normalize its monetary policy, raising the IOR rate from December 2015 to December 2018 and reducing the size of its balance sheet from October 2017 to August 2019, before reversing course.

Our model provides a simple framework to think about the effects of normalizing monetary policy. In particular, our model has unambiguous implications about the effects of normalizing monetary policy on inflation. More specifically, in our log-linearized model under exogenous monetary-policy instruments (studied in Section 4), current and expected future IOR-rate hikes and balance-sheet contractions always exert deflationary pressures. To establish this result, we use the definition of the exogenous driving term Z_t (as a function of $i_t^m - r_t$, $\widehat{M}_t$, and $\widehat{M}_{t+1}$) and the identity $\widehat{\mu}_t = \widehat{M}_t - \widehat{M}_{t-1}$ to rewrite (29) as

$$\begin{aligned}\pi_t \quad = \quad & -(1-\rho)\widehat{P}_{t-1} + \frac{(1-\delta_m\chi_y)\kappa}{\beta\sigma\chi_i(\omega_1-1)(\omega_2-1)}\widehat{M}_{t-1} \\ & + \frac{\kappa}{\beta(\omega_2-\omega_1)}\mathbb{E}_t\left\{-\frac{1}{\sigma}\sum_{k=0}^{+\infty}\left(\omega_1^{-k-1}-\omega_2^{-k-1}\right)\left(i_{t+k}^m - r_{t+k}\right)\right. \\ & \left. + \sum_{k=0}^{+\infty}\left[\left(\frac{1-\delta_m\chi_y}{\sigma\chi_i}\right)\left(\frac{\omega_1^{-k}}{\omega_1-1}-\frac{\omega_2^{-k}}{\omega_2-1}\right)+\delta_m\left(\omega_1^{-k}-\omega_2^{-k}\right)\right]\widehat{\mu}_{t+k}\right\}. \qquad (40)\end{aligned}$$

Since $\omega_2 > \omega_1 > 1$ and $\delta_m\chi_y < 1$ (as follows from the second inequality in (26)), the coefficient of i_{t+k}^m in (40) is negative, and the coefficient of $\widehat{\mu}_{t+k}$ is positive. Therefore, announcing at date t a positive i_{t+k}^m or a negative $\widehat{\mu}_{t+k}$, for any $k \geq 0$, lowers current inflation π_t.

So, in particular, our model does not share the Neo-Fisherian implication of some equilibria in NK models – discussed in Schmitt-Grohé and Uribe (2017) and Bilbiie (2018), among others – that suggest interest-rate hikes may serve to raise inflation to target in economies that suffer from deflationary pressures. Our result, of course, is about IOR-rate hikes, rather than hikes in the interest rate appearing in the IS equation of the NK models analyzed in the literature.

Note that we have obtained this result (that monetary-policy normalization always has deflationary effects) only because the "unstable eigenvalues" of the dynamic system, ω_1 and ω_2, are always positive real numbers. If ω_1 and ω_2 had instead been negative real numbers or conjugate complex numbers, then inflation would still have been characterized by (40), but the sign of the coefficients of i_{t+k}^m and $\widehat{\mu}_{t+k}$ in (40) would then have depended on the horizon k. Expected future IOR-rate hikes would have been deflationary for some hike horizons, and inflationary for others; and similarly for expected future balance-sheet contractions. In Diba and Loisel (2020), we show that other, less structured monetary models, in particular the familiar MIU model with separable or non-separable utility, allow for conjugate complex eigenvalues, unlike our model. Thus, the additional structure brought by our model is key to obtain our unambiguous result about the deflationary effects of monetary-policy normalization.

7.2 Future Operational Framework: Corridor vs. Floor System

In the rest of this section, we investigate the consequences of two alternative ways of conducting monetary policy away from the ZLB, motivated by recent discussions about the future operational framework of the Fed.

These discussions (e.g., Bernanke, 2015; Dudley, 2018; Powell, 2017) identify two main options that involve alternative plans for balance-sheet contraction. The first option is to shrink the balance sheet substantially, set an interbank-rate target depending on the state of the economy, and adjust endogenously the quantity of reserves to make the interbank rate hit this target. Some commentary refers to this first option as a "corridor system" (although a corridor system may also refer more specifically to the setting of a floor and a ceiling for the interbank rate). The policy-oriented discussions are not precise about the details of how the IOR rate may be set in this option. In particular, the IOR rate could be permanently set to zero (in net terms), as was the case in the US before the crisis; alternatively, it could be set at a fixed spread from the interbank-rate target, as was the case in the euro area before the crisis. The second option for the Fed's future operational framework is to keep the balance sheet large (or let it shrink slowly and predictably over time as central-bank assets mature), without actively managing the quantity of reserves, and set the interest rate on excess reserves (and, perhaps, the reverse-repo rate) depending on the state of the economy. Some commentary refers to this second option as a "floor system."

In the next two subsections, we show that these two options have substantially different implications for local-equilibrium determinacy in our model. First, we consider a corridor system in which the spread between the IOR rate and the interbank rate is kept fixed (by endogenously adjusting the stock of bank reserves) and policy sets a state-contingent target for the interbank rate. We show that under this corridor system, our model is isomorphic to the basic NK model if we treat the interest rate appearing in the IS equation as the interbank rate (for the sake of comparability). As a consequence, our model under this corridor system has the same implications for determinacy as the basic NK model. In particular, if the interbank-rate target reacts only to current inflation, it needs to react more than one-to-one in order to ensure determinacy (the so-called "Taylor principle"). Second, we consider a floor system in which policy sets the size of the balance sheet exogenously and sets the IOR rate depending on the state of the economy. We show that if the IOR rate reacts only to current inflation, then we get determinacy for any non-negative response, in contrast to what we get under the corridor system; and if the IOR rate also reacts to current output, determinacy conditions remain quite lax.

7.3 Corridor System: Fixed Spread and Interest-Rate Rule

The first option that we consider is, thus, a corridor system that maintains a fixed spread between the IOR rate and the interbank rate and sets an interbank-rate target depending on the state of the economy. We treat I_t as the interbank rate in our model, for the sake of comparability with the basic NK model, in which the interest rate that appears in the IS equation is also treated as the interbank rate.[11] It is easy to show, along the lines of Subsections 3.1 and 3.2, that our model has a unique zero-inflation steady state under such a corridor system.[12] We log-linearize the model around this steady state and, for simplicity, keep the same notations as previously. Under the corridor system, the reserves-demand equation (25) becomes

$$\widehat{m}_t = \chi_y \widehat{y}_t,$$

so that the Phillips curve (24) can be rewritten as

$$\pi_t = \beta \mathbb{E}_t \{\pi_{t+1}\} + \kappa (1 - \delta_m \chi_y) \widehat{y}_t. \tag{41}$$

The coefficient $\kappa(1-\delta_m\chi_y)$ of $\widehat{y}_t$ on the right-hand side of this Phillips curve is positive, as follows from the second inequality in (26). Therefore, the Phillips curve (41) of our model under the corridor system is isomorphic to the Phillips curve of the basic NK model. As a consequence, the reduced form of our model under the corridor system, made of the IS equation (23), the Phillips curve (41), and any given rule for the interbank-rate target i_t, is isomorphic to the reduced form of the basic NK model with the same rule for i_t.[13] We conclude that our model, under the corridor system, inherits all the implications of the basic NK model for local-equilibrium determinacy away from the ZLB. In particular, the Taylor principle applies: if the rule makes the interbank-rate target react non-negatively to current inflation ($i_t = \nu_\pi \pi_t$ with $\nu_\pi \geq 0$), then the reaction needs to be more than one-to-one ($\nu_\pi > 1$) to ensure determinacy.

7.4 Floor System: Exogenous Reserves and Taylor Rule for the IOR Rate

The second option that we consider is a floor system under which the central bank sets the growth rate of nominal reserves μ_t exogenously (around the value $\mu = 1$, as previously) and sets the IOR rate I_t^m according to the Taylor rule

$$I_t^m = \mathcal{R}(\Pi_t, y_t),$$

where the function $\mathcal{R}$, from $\mathbb{R}^2_{>0}$ to $[1, +\infty)$, is differentiable and non-decreasing in Π_t and y_t (i.e. $\mathcal{R}_\Pi \geq 0$ and $\mathcal{R}_y \geq 0$). Under this floor system, the set of steady states is still characterized

[11] A limitation common to the two models is, then, that they abstract from the convenience yield of interbank loans.

[12] This steady state does not depend on the particular rule considered for the interbank-rate target. In particular, the employment level at this steady state is equal to $\mathcal{F}^{-1}(-S)$, where $S \equiv (I - I^m)/I) > 0$ denotes the constant value of the spread. For the steady state to exist, though, the rule has to prescribe the value $1/\beta$ for the interbank-rate target at the steady state.

[13] Woodford (2003, Chapter 4) obtains a similar isomorphism result, though under some parameter restrictions, in the context of the non-separable MIU model.

by the flexible-price equation (20) with $\mu = 1$ and $h_{t+1} = h_t$, but now with I^m replaced by $\mathcal{R}[1, f(h_t)]$. The resulting equation is

$$\mathcal{F}(h_t) \equiv \frac{\Gamma_m [\mathcal{L}(h_t), \mathcal{M}(h_t)]}{u'[f(h_t)]} = \beta \mathcal{R}[1, f(h_t)] - 1.$$

Given the properties of function $\mathcal{F}$, a sufficient condition for existence of a steady state (which is also a necessary condition for existence and uniqueness of a steady state) is

$$\mathcal{R}\left[1, f\left(\overline{h}\right)\right] < \frac{1}{\beta}.$$

Log-linearizing the model around a steady state, we get the same IS equation (23), Phillips curve (24), and reserves-demand equation (25) as previously, plus now the Taylor rule

$$i_t^m = r_\pi \pi_t + r_y \widehat{y}_t, \tag{42}$$

where $r_\pi \equiv \mathcal{R}_\Pi / \mathcal{R} \geq 0$ and $r_y \equiv (\mathcal{R}_y y)/\mathcal{R} \geq 0$.

In Appendix B.4, we derive the necessary and sufficient condition for local-equilibrium determinacy under this floor system. We show in particular that a sufficient conditions is

$$r_y < \frac{1 - \delta_m \chi_y}{\delta_m \chi_i}, \tag{43}$$

where the right-hand side is positive (as follows from the second inequality in (26)). If the IOR rate reacts only to inflation (i.e. $r_y = 0$), then Condition (43) is necessarily met. In this case, determinacy obtains for any non-negative reaction to inflation (i.e. for any value of $r_\pi \geq 0$), and the Taylor principle does not apply. Alternatively, if the IOR rate reacts also to output (i.e. $r_y > 0$), then, to get a sense of how lax or stringent Condition (43) is, we consider the same calibration as in Subsection 5.1, which involves a large balance sheet (consistently with the discussion of a floor system for the Fed). Under this calibration, we get the value 15.7 for the right-hand side of (43). This threshold value seems comfortably high, given that the Taylor-rule coefficient on output is typically one order of magnitude lower in the literature. Thus, we view Condition (43) as likely to be met, and therefore determinacy as likely to prevail, under such a floor system in our model.

7.5 Floor System: Scarce vs. Ample Reserves

We have so far highlighted one implication of the floor system, which is that this system delivers local-equilibrium determinacy for any non-negative response of the IOR rate to inflation. This result obtains independently of whether the floor system involves ample or scarce reserves, i.e. whether the steady-state stock of real reserves is large or small; all that matters is that the stock of nominal reserves be set exogenously.

We now show that a global-equilibrium perspective provides a different view on the floor system than the local-equilibrium analysis conducted so far. More specifically, we show that the floor system may give rise to multiple dynamic perfect-foresight equilibria with below-steady-state inflation in (the flexible-price version of) our model; and we show that whether these equilibria exist, and how far below its steady-state value inflation is in these equilibria, depends on whether the floor system involves ample or scarce reserves.

For simplicity, and also to make clear that these multiple global equilibria do not arise from the ZLB constraint on the IOR rate, we focus on the case in which the IOR rate is set exogenously at a constant value $I^m > 1$ (i.e., the case with a zero response to inflation in the IOR-rate rule). We also assume that the stock of nominal reserves grows at a constant (gross) rate μ. We are, thus, in the case of constant monetary-policy instruments considered in Section 3. From Subsection 3.2, we know that there exists a unique steady-state equilibrium if and only if the constant IOR rate I^m is below μ/β (which is the steady-state value of the interest rate on bonds I). We also know that the steady-state stock of real reserves increases with I^m, or equivalently decreases with the steady-state spread I/I^m (which represents the steady-state opportunity cost of holding reserves). So, low values of I^m, close to the lower bound 1, correspond to floor systems with scarce reserves; alternatively, high values of I^m, close to the upper bound $I = \mu/\beta$, correspond to floor systems with ample reserves.

In Subsection 3.3, we showed that, under policies setting $I^m \leq \mu$ (i.e., under floor systems with relatively scarce reserves), there exists no dynamic equilibrium with below-steady-state inflation. Alternatively, under policies setting $I^m > \mu$ (i.e., under floor systems with relatively ample reserves), and in particular under a constant stock of nominal reserves ($\mu = 1$, while $I^m > 1$), there exists a non-countable infinity of dynamic equilibria with below-steady-state inflation, indexed by $h_0 \in (h, \overline{h})$. In these equilibria, the asymptotic gross growth rate of real reserves is $\mu/\beta I^m$, as we have seen. So, the asymptotic gross inflation rate is βI^m, which is lower than, and independent of, the steady-state gross inflation rate μ. The reason is that, in these equilibria, the economy converges over time to satiation ($\lim_{t \to +\infty} h_t = \overline{h}$); so, the real return on reserves, I^m/Π_t, converges over time to the real return on bonds, $1/\beta$; and, therefore, gross inflation Π_t converges over time to βI^m.

For given values of β and μ (and hence for a given value of the steady-state interest rate on bonds $I = \mu/\beta$), the higher the IOR rate I^m, the higher asymptotic inflation in these dynamic equilibria with below-steady-state inflation. As I^m gets closer to its upper bound $I = \mu/\beta$ (i.e. as the steady-state spread $I - I^m$ goes to zero), asymptotic inflation gets closer to its steady-state value μ. In fact, as $I^m \to I$, the whole inflation path, in any of these equilibria, uniformly converges to a limit path in which inflation is constantly equal to its steady-state value μ. The proof is the following. The static equation (21) and $\lim_{h_t \to \overline{h}} \mathcal{F}(h_t) = 0$ imply that steady-state employment h converges to its upper bound $\overline{h}$ as $I^m \to I$. Since $h_t \in (h, \overline{h})$ at all dates t in any of these equilibria, $\lim_{I^m \to I} h = \overline{h}$ implies in turn that the employment path, in any of these equilibria, uniformly converges (as $I^m \to I$) to a limit path in which employment is constantly equal to $\overline{h}$. Finally, given that $\lim_{I^m \to I} h = \overline{h}$ and that $\lim_{I^m \to I} h_t = \overline{h}$ at all dates t in any of these equilibria, the dynamic equation (22) implies in turn that the inflation path, in any of these equilibria, uniformly converges (as $I^m \to I$) to a limit path in which inflation is constantly equal to μ.

To sum up, under floor systems with sufficiently scarce reserves ($I^m \leq \mu$), there is no dynamic equilibrium with below-steady-state inflation. Alternatively, under floor systems with sufficiently ample reserves ($I^m > \mu$), there is an infinity of such equilibria. And, under floor systems with arbitrarily ample reserves (I^m arbitrarily close to $I = \mu/\beta$), inflation is arbitrarily close to steady-state inflation in any of these equilibria and at all dates. These results suggest that, in order to stabilize (gross) inflation around a given target μ, central banks should adopt a floor system that involves either scarce reserves, or on the contrary very ample reserves, rather than an intermediate amount of reserves.

8 Conclusion

Central banks conduct monetary policy by setting, ultimately, two instruments that they directly control: the interest rate on bank reserves and the size of their balance sheet. In this paper, we have taken this fact seriously and proposed a model in which the central bank sets these two instruments. Central-bank liabilities consist only of bank reserves in the benchmark version of the model, but include also household cash in an extended version. Our model is especially consistent, we argue, with the way in which the Fed has been communicating its monetary policy since the end of 2008.

We show that the model can account, in qualitative terms, for the three key observations made by Cochrane (2018) about US inflation during the 2008-2015 ZLB episode: no significant deflation, little inflation volatility, and no significant inflation following quantitative-easing policies. In addition, we show that with liquid government bonds, the model can also account for the negative spread between Treasury-bill and IOR rates observed during this ZLB episode. Looking ahead, we use our model to assess the consequences of the normalization and the future operational framework of monetary policy. On this front, we do not find any possibility of Neo-Fisherian effects during the monetary-policy normalization process, as current and expected future IOR-rate hikes and balance-sheet contractions always exert deflationary pressures in our model. And we highlight the important differences, in terms of determinacy conditions, between the corridor system and the floor system.

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Online Appendix to
"A Model of Post-2008 Monetary Policy"

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Appendix A: Non-Linear Benchmark Model

In this appendix, we establish the properties of the non-linear functions f^b, g^b, Γ, $\mathcal{M}$, $\mathcal{F}$, and $\mathcal{G}$ that appear in our benchmark model or play a role in its analysis, and we show the existence of Function $\mathcal{M}$. We also derive the necessary and sufficient condition for existence of global perfect-foresight equilibria with above-steady-state inflation in our benchmark model under flexible prices, and we characterize these equilibria when they exist. Finally, we show how our steady-state targets pin down the parameters in the calibration of our benchmark model under iso-elastic production and utility functions. To lighten up the notation, we sometimes omit function arguments when no ambiguity results, in this appendix as well as in the next ones.

A.1 Concavity of Function f^b

In this subsection and the next one, we also omit time subscripts to further lighten up the notation, thus writing h^b, ℓ, and m instead of h_t^b, ℓ_t, and m_t. Since f^b is homogeneous of degree d, we have $\forall x \in \mathbb{R}_{\geq 0}$, $f^b(xh^b, xm) = x^d f^b(h^b, m)$. Computing the first derivative of the left- and right-hand sides of this equation with respect to x at $x = 1$ leads to

$$df^b = h^b f_h^b + m f_m^b. \tag{A.1}$$

In turn, computing the first derivative of the left- and right-hand sides of the last equation with respect to h^b and m leads to

$$f_{hh}^b = -\frac{(1-d)\, f_h^b + m f_{hm}^b}{h^b} \quad \text{and} \quad f_{mm}^b = -\frac{(1-d)\, f_m^b + h^b f_{hm}^b}{m}.$$

Using these expression for f_{hh}^b and f_{hm}^b, as well as (A.1), we get

$$\begin{aligned} f_{hh}^b f_{mm}^b - \left(f_{hm}^b\right)^2 &= \frac{1-d}{h^b m}\left[(1-d)\, f_h^b f_m^b + f_{hm}^b \left(h^b f_h^b + m f_m^b\right)\right] \\ &= \frac{1-d}{h^b m}\left[(1-d)\, f_h^b f_m^b + d f^b f_{hm}^b\right] \\ &\geq 0, \end{aligned}$$

which implies (together with $f_{hh}^b \leq 0$ and $f_{mm}^b \leq 0$) that the function f^b is (weakly) concave.

A.2 Properties of Function g^b

Computing the first and second derivatives of the left- and right-hand sides of $\ell = f^b[g^b(\ell, m), m]$ with respect to ℓ and m gives

$$1 = f_h^b g_\ell^b, \quad 0 = f_h^b g_m^b + f_m^b, \quad 0 = f_{hh}^b \left(g_\ell^b\right)^2 + f_h^b g_{\ell\ell}^b,$$
$$0 = f_{hh}^b g_\ell^b g_m^b + f_{hm}^b g_\ell^b + f_h^b g_{\ell m}^b, \quad \text{and} \quad 0 = f_{hh}^b \left(g_m^b\right)^2 + 2 f_{hm}^b g_m^b + f_h^b g_{mm}^b + f_{mm}^b.$$

Using these equations and $f_h^b > 0$, $f_m^b > 0$, $f_{hh}^b < 0$, $f_{hm}^b \geq 0$, and $f_{mm}^b < 0$, we sequentially get

$$g_\ell^b = \frac{1}{f_h^b} > 0, \quad g_m^b = \frac{-f_m^b}{f_h^b} < 0, \quad g_{\ell\ell}^b = \frac{-f_{hh}^b}{\left(f_h^b\right)^3} > 0,$$
$$g_{\ell m}^b = \frac{f_m^b f_{hh}^b}{\left(f_h^b\right)^3} - \frac{f_{hm}^b}{\left(f_h^b\right)^2} < 0, \quad \text{and} \quad g_{mm}^b = \frac{-f_{hh}^b \left(f_m^b\right)^2}{\left(f_h^b\right)^3} + 2\frac{f_m^b f_{hm}^b}{\left(f_h^b\right)^2} - \frac{f_{mm}^b}{f_h^b} > 0.$$

Then, using these expressions for $g_{\ell\ell}^b$, g_{mm}^b, $g_{\ell m}^b$, and the concavity of f^b, we easily get

$$g_{\ell\ell}^b g_{mm}^b - \left(g_{\ell m}^b\right)^2 = \frac{f_{hh}^b f_{mm}^b - \left(f_{hm}^b\right)^2}{\left(f_h^b\right)^4} \geq 0,$$

which implies (together with $g_{\ell\ell}^b > 0$ and $g_{mm}^b > 0$) that the function g^b is (weakly) convex.

Moreover, since f^b is homogeneous of degree d, we have $\forall x \in \mathbb{R}_{\geq 0}$, $g^b(x^d \ell, xm) = x g^b(\ell, m)$. Computing the first derivative of the left- and right-hand sides of this equation with respect to x at $x = 1$ leads to

$$g^b = d\ell g_\ell^b + m g_m^b. \tag{A.2}$$

In turn, computing the first derivative of the left- and right-hand sides of the last equation with respect to ℓ and m leads to

$$g_{\ell\ell}^b = \frac{(1-d)\, g_\ell^b - m g_{\ell m}^b}{d\ell}, \tag{A.3}$$
$$g_{mm}^b = \frac{-d\ell g_{\ell m}^b}{m}. \tag{A.4}$$

Finally, as a direct consequence of $\lim_{m_t \to +\infty} f_m^b(h_t^b, m_t) = 0$ and $\lim_{m_t \to 0} f_h^b(h_t^b, m_t) = 0$, we get $\lim_{m \to +\infty} g_m^b(\ell, m) = 0$ and $\lim_{m \to 0} g_\ell^b(\ell, m) = +\infty$ for all $\ell \in \mathbb{R}_{\geq 0}$.

A.3 Properties of Function Γ

Computing the first and second derivatives of the left- and right-hand sides of $\Gamma(\ell, m) \equiv v^b[g^b(\ell, m)]$ with respect to ℓ and m gives

$$\Gamma_\ell = v^{b\prime} g_\ell^b > 0, \quad \Gamma_m = v^{b\prime} g_m^b < 0, \quad \Gamma_{\ell\ell} = v^{b\prime\prime} \left(g_\ell^b\right)^2 + v^{b\prime} g_{\ell\ell}^b > 0,$$
$$\Gamma_{\ell m} = v^{b\prime\prime} g_\ell^b g_m^b + v^{b\prime} g_{\ell m}^b < 0, \quad \text{and} \quad \Gamma_{mm} = v^{b\prime\prime} \left(g_m^b\right)^2 + v^{b\prime} g_{mm}^b > 0,$$

where the inequalities follow from $v^{b\prime} > 0$, $v^{b\prime\prime} \geq 0$, $g^b_\ell > 0$, $g^b_m < 0$, $g^b_{\ell\ell} > 0$, $g^b_{mm} > 0$, and $g^b_{\ell m} < 0$. In addition, using first (A.3)-(A.4) and then (A.2), we easily get

$$\begin{aligned}
\Gamma_{\ell\ell}\Gamma_{mm} - (\Gamma_{\ell m})^2 &= \left(v^{b\prime}\right)^2 \left[g^b_{\ell\ell} g^b_{mm} - \left(g^b_{\ell m}\right)^2\right] \\
&\quad + v^{b\prime} v^{b\prime\prime} \left[\left(g^b_\ell\right)^2 g^b_{mm} + \left(g^b_m\right)^2 g^b_{\ell\ell} - 2 g^b_\ell g^b_m g^b_{\ell m}\right] \\
&= \frac{-(1-d)\left(v^{b\prime}\right)^2 g^b_\ell g^b_{\ell m}}{m} \\
&\quad + \frac{v^{b\prime} v^{b\prime\prime}}{d\ell m} \left[-g^b_{\ell m}\left(d\ell g^b_\ell + m g^b_m\right)^2 + (1-d)\, m g^b_\ell \left(g^b_m\right)^2\right] \\
&= \frac{-(1-d)\left(v^{b\prime}\right)^2 g^b_\ell g^b_{\ell m}}{m} \\
&\quad + \frac{v^{b\prime} v^{b\prime\prime}}{d\ell m} \left[-\left(g^b\right)^2 g^b_{\ell m} + (1-d)\, m g^b_\ell \left(g^b_m\right)^2\right] \\
&\geq 0, \qquad\qquad (A.5)
\end{aligned}$$

which implies (together with $\Gamma_{\ell\ell} > 0$ and $\Gamma_{mm} > 0$) that the function Γ is (weakly) convex. Finally, as a direct consequence of $\lim_{m \to +\infty} g^b_m(\ell, m) = 0$ and $\lim_{m \to 0} g^b_\ell(\ell, m) = +\infty$, we get $\lim_{m_t \to +\infty} \Gamma_m(\ell_t, m_t) = 0$ and $\lim_{m_t \to 0} \Gamma_\ell(\ell_t, m_t) = +\infty$ for all $\ell \in \mathbb{R}_{\geq 0}$.

A.4 Existence and Properties of Function $\mathcal{M}$

Using (3), (5), (9), (13), (17), and (18), we can rewrite households' first-order condition for loans (6) as a relationship between reserves m_t and employment h_t:

$$\Gamma_\ell \left[\mathcal{L}(h_t), m_t\right] = \mathcal{A}(h_t) \equiv \frac{u'\left[f(h_t)\right]}{\phi} \left\{\left(\frac{\varepsilon - 1}{\varepsilon}\right) \frac{u'\left[f(h_t)\right] f'(h_t)}{v'(h_t)} - 1\right\}. \qquad (A.6)$$

Because the left-hand side of (A.6) is positive, we restrict the domain of the function $\mathcal{A}$ to $(0, h^\star)$, where $h^\star > 0$ is implicitly and uniquely defined by $u'[f(h^\star)]f'(h^\star)/v'(h^\star) = \varepsilon/(\varepsilon - 1)$. The value $h^\star$ is the value that h_t would take in the absence of financial frictions, i.e. if the marginal banking cost Γ_ℓ were zero. The function $\mathcal{A}$ is strictly decreasing ($\mathcal{A}' < 0$), with $\lim_{h_t \to 0} \mathcal{A}(h_t) = +\infty$ and $\lim_{h_t \to h^\star} \mathcal{A}(h_t) = 0$.

Since $\Gamma_{\ell\ell} > 0$, $\mathcal{L}' > 0$, $\Gamma_{\ell m} < 0$, and $\mathcal{A}' < 0$, Equation (A.6) implicitly and uniquely defines a function $\mathcal{M}$ such that

$$m_t = \mathcal{M}(h_t).$$

This function is strictly increasing ($\mathcal{M}' > 0$). Moreover, since $\lim_{m_t \to 0} \Gamma_\ell(\ell_t, m_t) = +\infty$, $\mathcal{M}$ is defined over $(0, \overline{h})$, where $\overline{h} \in (0, h^\star]$ is implicitly and uniquely defined by $\lim_{m_t \to +\infty} \Gamma_\ell[\mathcal{L}(\overline{h}), m_t] = \mathcal{A}(\overline{h})$. Finally, this last equation straightforwardly implies

$$\lim_{h_t \to \overline{h}} \mathcal{M}(h_t) = +\infty. \qquad (A.7)$$

A.5 Properties of Function $\mathcal{F}$

Using (A.6), we can rewrite $\mathcal{F}(h_t)$ as

$$\mathcal{F}(h_t) = \frac{1}{\phi}\mathcal{F}_1(h_t)\,\mathcal{F}_2(h_t),$$

where the functions $\mathcal{F}_1$ and $\mathcal{F}_2$ are defined over $(0,\overline{h})$ by

$$\begin{aligned}
\mathcal{F}_1(h_t) &\equiv \frac{\Gamma_m[\mathcal{L}(h_t),\mathcal{M}(h_t)]}{\Gamma_\ell[\mathcal{L}(h_t),\mathcal{M}(h_t)]} = \frac{g_m^b[\mathcal{L}(h_t),\mathcal{M}(h_t)]}{g_\ell^b[\mathcal{L}(h_t),\mathcal{M}(h_t)]},\\
\mathcal{F}_2(h_t) &\equiv \left(\frac{\varepsilon-1}{\varepsilon}\right)\frac{u'[f(h_t)]\,f'(h_t)}{v'(h_t)} - 1.
\end{aligned}$$

We have

$$\begin{aligned}
\left(g_\ell^b\right)^2\mathcal{F}_1' &= g_\ell^b\left(g_{\ell m}^b\mathcal{L}' + g_{mm}^b\mathcal{M}'\right) - g_m^b\left(g_{\ell\ell}^b\mathcal{L}' + g_{\ell m}^b\mathcal{M}'\right)\\
&= -g_{\ell m}^b\left(d\mathcal{L}g_\ell^b + \mathcal{M}g_m^b\right)\left(\frac{\mathcal{M}'}{\mathcal{M}} - \frac{\mathcal{L}'}{d\mathcal{L}}\right) - (1-d)\,g_\ell^b g_m^b\frac{\mathcal{L}'}{d\mathcal{L}}\\
&= -g^b g_{\ell m}^b\left(\frac{\mathcal{M}'}{\mathcal{M}} - \frac{\mathcal{L}'}{d\mathcal{L}}\right) - (1-d)\,g_\ell^b g_m^b\frac{\mathcal{L}'}{d\mathcal{L}},
\end{aligned}$$

where the second equality follows from (A.3)-(A.4), and the third equality from (A.2). Now, deriving the left- and right-hand sides of (A.6) with respect to h_t gives

$$\Gamma_{\ell m}\mathcal{M}' + \Gamma_{\ell\ell}\mathcal{L}' = \mathcal{A}' < 0.$$

Moreover, using (A.2) and (A.3), we get

$$\begin{aligned}
d\mathcal{L}\Gamma_{\ell\ell} + \mathcal{M}\Gamma_{\ell m} &= d\mathcal{L}\left[v^{b\prime\prime}\left(g_\ell^b\right)^2 + v^{b\prime}g_{\ell\ell}^b\right] + \mathcal{M}\left(v^{b\prime\prime}g_\ell^b g_m^b + v^{b\prime}g_{\ell m}^b\right)\\
&= v^{b\prime\prime}g_\ell^b\left(d\mathcal{L}g_\ell^b + \mathcal{M}g_m^b\right) + v^{b\prime}\left(d\mathcal{L}g_{\ell\ell}^b + \mathcal{M}g_{\ell m}^b\right)\\
&= v^{b\prime\prime}g^b g_\ell^b + (1-d)\,v^{b\prime}g_\ell^b\\
&\geq 0.
\end{aligned}$$

The last two inequalities together imply

$$\frac{\mathcal{M}'}{\mathcal{M}} > \frac{\mathcal{L}'}{d\mathcal{L}}, \tag{A.8}$$

from which we conclude that $\mathcal{F}_1' > 0$. Then, using $\mathcal{F}_1' > 0$, $\mathcal{F}_1 < 0$, $\mathcal{F}_2' < 0$, and $\mathcal{F}_2 > 0$, we get that the function $\mathcal{F}$ is strictly increasing ($\mathcal{F}' > 0$).

Moreover, $\mathcal{F}_1' > 0$ and $\mathcal{F}_1 < 0$ imply that $\lim_{h_t\to 0}\mathcal{F}_1(h_t) < 0$, while the Inada condition $\lim_{c_t\to 0}u'(c_t) = +\infty$ implies that $\lim_{h_t\to 0}\mathcal{F}_2(h_t) = +\infty$, so that

$$\lim_{h_t\to 0}\mathcal{F}(h_t) = -\infty.$$

Finally, both $\lim_{h_t\to\overline{h}}\mathcal{F}_1(h_t)$ and $\lim_{h_t\to\overline{h}}\mathcal{F}_2(h_t)$ are finite, since $\mathcal{F}_1$ is increasing and negative, and $\mathcal{F}_2$ decreasing and positive. If $\overline{h} < h^\star$, then (A.7) and the Inada condition $\lim_{m_t\to+\infty}\Gamma_m(\ell_t,m_t) = 0$ implies $\lim_{h_t\to\overline{h}}\mathcal{F}_1(h_t) = 0$. Alternatively, if $\overline{h} = h^\star$, then $\lim_{h_t\to\overline{h}}\mathcal{F}_2(h_t) = 0$. We conclude that, in both cases,

$$\lim_{h_t\to\overline{h}}\mathcal{F}(h_t) = 0.$$

A.6 Properties of Function $\mathcal{G}$

Using (18), we can rewrite $\mathcal{G}(h_t)$ as

$$\mathcal{G}\left(h_t\right) = \phi h_t v'\left(h_t\right) \frac{\mathcal{M}\left(h_t\right)}{\mathcal{L}\left(h_t\right)}. \tag{A.9}$$

Using (A.8), $0 < d \leq 1$, $\mathcal{L} > 0$, and $\mathcal{L}' > 0$, we get

$$\frac{\mathcal{M}'\left(h_t\right)}{\mathcal{M}\left(h_t\right)} > \frac{\mathcal{L}'\left(h_t\right)}{\mathcal{L}\left(h_t\right)},$$

from which we conclude that $\mathcal{M}(h_t)/\mathcal{L}(h_t)$ is strictly increasing in h_t. This last result, together with (A.9), $v'' \geq 0$, $v' > 0$, $\mathcal{L} > 0$, and $\mathcal{M} > 0$, implies that $\mathcal{G}$ is strictly increasing ($\mathcal{G}' > 0$). It also implies that $\lim_{h_t \to 0} \mathcal{M}(h_t)/\mathcal{L}(h_t)$ is finite and therefore, with (A.9), that

$$\lim_{h_t \to 0} \mathcal{G}\left(h_t\right) = 0.$$

Finally, using (A.7) and the fact that $\mathcal{L}(\overline{h})$ is finite, we get

$$\lim_{h_t \to \overline{h}} \mathcal{G}\left(h_t\right) = +\infty.$$

A.7 Dynamic Equilibria With Above-Steady-State Inflation

We consider a candidate equilibrium with $0 < h_0 < h$. We first show that we must have $h_T = 0$ for some finite date $T \geq 1$ in this candidate equilibrium (noting that zero labour can be an equilibrium outcome if the utility function takes a finite value for zero consumption). The proof is by contradiction. If we had $h_t > 0$ at all dates $t \in \mathbb{N}$, then, using (22), $\mathcal{F}' > 0$, and $\mathcal{G}' > 0$, we would get by recurrence that the sequence $(h_t)_{t\in\mathbb{N}}$ is strictly decreasing. This would imply that

$$\frac{\mathcal{G}\left(h_{t+1}\right)}{\mathcal{G}\left(h_t\right)} = 1 - \frac{\mu}{\beta I^m}\left[\mathcal{F}\left(h\right) - \mathcal{F}\left(h_t\right)\right] < 1 - \frac{\mu}{\beta I^m}\left[\mathcal{F}\left(h\right) - \mathcal{F}\left(h_0\right)\right] < 1,$$

and therefore that the sequence $\left[\mathcal{G}\left(h_t\right)\right]_{t\in\mathbb{N}}$ converges asymptotically to zero (without ever reaching zero). Since $\mathcal{G}' > 0$ and $\lim_{h_t \to 0} \mathcal{G}(h_t) = 0$, the sequence $(h_t)_{t\in\mathbb{N}}$ would then converge towards zero too (without ever reaching zero). However, the dynamic equation (22) and $\lim_{h_t \to 0} \mathcal{F}(h_t) = -\infty$ would then imply that the ratio $\mathcal{G}(h_{t+1})/\mathcal{G}(h_t)$ turns negative for h_t sufficiently close to 0, which is impossible.

Next, we jointly show that (i) if $h_T = 0$, then $h_t = 0$ at all dates $t \geq T$ in this candidate equilibrium, and (ii) $\lim_{h_t \to 0} \mathcal{B}(h_t) = 0$ is a necessary condition for this candidate equilibrium to be an equilibrium, where the function $\mathcal{B}$ is defined over $(0, \overline{h})$ by $\mathcal{B}(h_t) \equiv \mathcal{M}(h_t)\Gamma_m[\mathcal{L}(h_t), \mathcal{M}(h_t)]$. To that aim, we use the fact that $\mathcal{F}(h_t)\mathcal{G}(h_t) = \mathcal{B}(h_t)$ to rewrite the dynamic equation (22) as

$$\mathbb{E}_t\left\{\mathcal{G}\left(h_{t+1}\right)\right\} = \frac{\mu}{\beta I^m}\left[\mathcal{G}\left(h_t\right) + \mathcal{B}\left(h_t\right)\right]. \tag{A.10}$$

Since $h_T = 0$, we need to extend the domain of $\mathcal{G}$ and $\mathcal{B}$ to 0. By continuity, we have $\mathcal{G}(0) = \lim_{h_t \to 0} \mathcal{G}(h_t) = 0$ and, therefore, $\mathcal{G}(h_T) = 0$. The dynamic equation (A.10) at date $t = T$ thus becomes $\mathbb{E}_t\{\mathcal{G}(h_{T+1})\} = (\mu/\beta I^m)\mathcal{B}(h_T)$. This equation, together with $\mathcal{G}(h_{T+1}) \geq 0$ and $\mathcal{B}(h_T) \leq 0$, implies that $\mathcal{G}(h_{T+1}) = \mathcal{B}(h_T) = 0$. This result, in turn, has two implications. First, using $\mathcal{G}(h_{T+1}) = 0$, we get $h_{T+1} = 0$ and, recursively, $h_t = 0$ for all $t \geq T$. Second, using $\mathcal{B}(h_T) = 0$ and $h_T = 0$, we get $\mathcal{B}(0) = 0$ and, by continuity, $\lim_{h_t \to 0} \mathcal{B}(h_t) = 0$.

Finally, we characterize the path of employment h_t before reaching zero. Let us assume, without any loss in generality, that $T \geq 1$ is the *first* date at which employment is zero. Let us rewrite the dynamic equation (22) as

$$\mathbb{E}_t \{\mathcal{G}(h_{t+1})\} = \mathcal{C}(h_t), \tag{A.11}$$

where the function $\mathcal{C}$ is defined over $(0, \overline{h})$ by $\mathcal{C}(h_t) \equiv (\mu/\beta I^m)[1 + \mathcal{F}(h_t)]\mathcal{G}(h_t)$. We have $\mathcal{C}(h_t) < 0$ for $0 < h_t < H_0 \equiv \mathcal{F}^{-1}(-1)$, $\mathcal{C}(H_0) = 0$, and $\mathcal{C}(h_t) > 0$ for $H_0 < h_t < \overline{h}$. Using (A.11) and $h_T = 0$, we then get $h_{T-1} = H_0$. Let $\widetilde{\mathcal{C}} \equiv \mathcal{C}|_{[H_0,h]}$ denote the restriction of $\mathcal{C}$ to $[H_0, h]$. Since $\widetilde{\mathcal{C}}' > 0$, $\mathcal{G}' > 0$, $\widetilde{\mathcal{C}}(h_t) < \mathcal{G}(h_t)$ for $h_t \in [H_0, h)$, and $\widetilde{\mathcal{C}}(h) = \mathcal{G}(h)$, we can uniquely define the sequence $(H_n)_{n\in\mathbb{N}}$ by its initial value H_0 and the recurrence equation $H_{n+1} = \widetilde{\mathcal{C}}^{-1}[\mathcal{G}(H_n)]$; moreover, this sequence is strictly increasing and asymptotically converges to h (without ever reaching h). Using (A.11) and $h_{T-1} = H_0$, we then get sequentially $h_{T-2} = H_1$ (if $T \geq 2$), $h_{T-3} = H_2$ (if $T \geq 3$), etc., until $h_0 = H_{T-1}$.

To conclude, if $\lim_{h_t \to 0} \mathcal{B}(h_t) < 0$, i.e. equivalently if the condition

$$\lim_{h_t \to 0} \mathcal{M}(h_t) \Gamma_m [\mathcal{L}(h_t), \mathcal{M}(h_t)] < 0 \tag{A.12}$$

is met, then there does not exist any equilibrium with $0 < h_0 < h$. Alternatively, if $\lim_{h_t \to 0} \mathcal{B}(h_t) = 0$, i.e. equivalently if the condition (A.12) is not met, then there exists a countable infinity of equilibria with $0 < h_0 < h$, indexed by $n \in \mathbb{N}$ and such that $h_t = H_{n-t}$ for $0 \leq t \leq n$ and $h_t = 0$ for $t \geq n + 1$.

The condition (A.12) is the natural counterpart, in our production economy with banks, of the super Inada condition in endowment economies with separable utility that we discuss in Subsection 3.3. To provide some more insight on this condition, let us consider the specific case of iso-elastic functional forms for the production and utility functions. Using the same notations as in Subsection 5.1, we then get

$$\Gamma(\ell_t, m_t) = V_b (1+\eta)^{-1} A_b^{\frac{-(1+\eta)}{1-\varsigma}} (\ell_t)^{\frac{1+\eta}{1-\varsigma}} (m_t)^{\frac{-\varsigma(1+\eta)}{1-\varsigma}}, \qquad \mathcal{L}(h_t) = \phi V A^\sigma h_t^{1+\eta+\alpha\sigma},$$

$$\mathcal{M}(h_t) = \left(\frac{\phi V A^\sigma}{A_b}\right)^{\frac{1}{\varsigma}} \left[\frac{\frac{V_b}{(1-\varsigma)V}}{\left(\frac{\alpha A^{1-\sigma}}{V}\right)\left(\frac{\varepsilon-1}{\varepsilon}\right) h_t^{-[\eta+(1-\alpha)+\alpha\sigma]} - 1}\right]^{\frac{1-\varsigma}{(1+\eta)\varsigma}} h_t^{1+\frac{\eta+\alpha\sigma}{\varsigma}},$$

and therefore

$$\mathcal{M}(h_t) \Gamma_m [\mathcal{L}(h_t), \mathcal{M}(h_t)] = \varsigma V h_t^{1+\eta} - \left(\frac{\varepsilon - 1}{\varepsilon}\right) \varsigma \alpha A^{1-\sigma} h_t^{\alpha(1-\sigma)}.$$

As a consequence, in this specific case, the condition (A.12) boils down to $\sigma \geq 1$, where σ denotes the (constant) coefficient of relative risk aversion. The basic reason is that reserves are essential to get finite utility levels if and only if $\sigma \geq 1$. Indeed, our assumption $\lim_{m_t \to 0} f_h^b(h_t^b, m_t) = 0$ makes holding reserves essential for banking; in turn, our assumption $\phi > 0$ makes bank loans necessary for production; and finally, the condition $\sigma \geq 1$ makes non-zero production necessary to get finite utility levels. Relaxing one of these assumptions (e.g., taking out $\lim_{m_t \to 0} f_h^b(h_t^b, m_t) = 0$, or allowing for some production that does not require bank loans), we suspect, would make dynamic equilibria with above-steady-state inflation exist for all values of σ.

A.8 Calibration Under Iso-Elastic Production and Utility Functions

To see how our targets for I^ℓ and m/ℓ pin down the parameters ς and V_b, we first rewrite households' first-order conditions (6) and (7) as

$$\beta I^\ell = 1 + \frac{V_b}{(1-\varsigma)\lambda} A_b^{\frac{-(1+\eta)}{1-\varsigma}} \ell^{\frac{\eta+\varsigma}{1-\varsigma}} m^{\frac{-\varsigma(1+\eta)}{1-\varsigma}} \tag{A.13}$$

and

$$\beta I^m = 1 - \frac{\varsigma V_b}{(1-\varsigma)\lambda} A_b^{\frac{-(1+\eta)}{1-\varsigma}} \ell^{\frac{1+\eta}{1-\varsigma}} m^{\frac{-(1+\varsigma\eta)}{1-\varsigma}} \tag{A.14}$$

in the steady state. Equations (A.13) and (A.14) give parameter ς as a function of I^ℓ, m/ℓ, and already calibrated parameters:

$$\varsigma = \left(\frac{m}{\ell}\right)\left(\frac{1-\beta I^m}{\beta I^\ell - 1}\right).$$

Thus, the targets for I^ℓ and m/ℓ pin down ς; we get $\varsigma = 0.0039$.

Next, we rewrite firms' first-order condition under flexible prices (13) as

$$w = \alpha A \left(\frac{\varepsilon - 1}{\varepsilon}\right)\left[\phi \frac{I^\ell}{I} + (1-\phi)\right]^{-1} h^{-(1-\alpha)} \tag{A.15}$$

in the steady state, and we use (3), (9), and (17) to rewrite households' intra-temporal first-order condition (5) as

$$w = V A^\sigma h^{\eta + \alpha\sigma} \tag{A.16}$$

in the steady state. Equations (A.15) and (A.16) give the steady-state value of hours worked h as a function of I^ℓ and already calibrated parameters:

$$h = \left\{ \frac{\alpha A^{1-\sigma}}{V} \left(\frac{\varepsilon - 1}{\varepsilon}\right)\left[\phi \frac{I^\ell}{I} + (1-\phi)\right]^{-1} \right\}^{\frac{1}{\eta+\alpha\sigma+(1-\alpha)}}.$$

Thus, the target for I^ℓ pins down h. We plug the value obtained for h into either (A.15) or (A.16) to get the steady-state real wage w. Using the borrowing constraint (10) holding with equality, we then get the steady-state value of real loans $\ell = \phi w h$, from which we get in turn the steady-state value of real reserves $m = (m/\ell)\ell$. The value that we have obtained for h also gives us the steady-state value of consumption $c = y = Ah^{\alpha}$, from which we get in turn the steady-state value of the marginal utility of consumption $\lambda = c^{-\sigma}$. By plugging these values of ℓ, m, and λ, as well as the value that we have obtained for ς, into either (A.13) or (A.14), we recover the implied value of V_b; we get $V_b = 0.021$.

Appendix B: Log-Linearized Benchmark Model

In this appendix, we log-linearize our benchmark model around its unique steady state; we prove the double inequality (26) on its reduced-form parameters; we analyze the roots of its characteristic polynomial under exogenous monetary-policy instruments; and we derive the necessary and sufficient condition for local-equilibrium determinacy under a floor system.

B.1 Log-Linearization

The IS equation (23) is straightforwardly obtained by log-linearizing households' first-order conditions (3)-(4) and the goods-market-clearing condition (17). To derive the Phillips curve (24), we log-linearize firms' first-order condition (12), and use the definition of the real wage $w_t \equiv W_t/P_t$, to get

$$\widehat{P}_t^* = (1-\beta\theta)\,\mathbb{E}_t\left\{\sum_{k=0}^{+\infty}(\beta\theta)^k\left[\alpha_\phi\left(i_{t+k}^{\ell} - i_{t+k}\right) + \widehat{w}_{t+k} + \widehat{P}_{t+k} - \widehat{mp}_{t+k|t}\right]\right\}, \tag{B.1}$$

where $\alpha_\phi \equiv \phi I^{\ell}/[\phi I^{\ell} + (1-\phi)I] \in (0,1]$, variables with hats denote log deviations from steady-state values, $i_t^{\ell} \equiv \widehat{I}_t^{\ell}$, $i_t \equiv \widehat{I}_t$, and $mp_{t+k|t}$ denotes the marginal productivity in period $t+k$ for a firm whose price was last set in period t. Log-linearizing the production function (9) gives

$$\widehat{h}_t = \frac{f}{f'h}\widehat{y}_t, \tag{B.2}$$

so that we can rewrite $\widehat{mp}_{t+k|t}$ as

$$\begin{aligned}\widehat{mp}_{t+k|t} &= \frac{f''h}{f'}\widehat{h}_{t+k|t} = \widehat{mp}_{t+k} + \frac{f''h}{f'}\left(\widehat{h}_{t+k|t} - \widehat{h}_{t+k}\right)\\ &= \widehat{mp}_{t+k} + \frac{ff''}{(f')^2}\left(\widehat{y}_{t+k|t} - \widehat{y}_{t+k}\right) = \widehat{mp}_{t+k} - \frac{\varepsilon ff''}{(f')^2}\left(\widehat{P}_t^* - \widehat{P}_{t+k}\right),\end{aligned} \tag{B.3}$$

where mp_{t+k} denotes the average marginal productivity in period $t+k$. Using this result and

$$\pi_t \equiv \log(\Pi_t) = (1-\theta)\left(\widehat{P}_t^* - \widehat{P}_{t-1}\right),$$

and following the same steps as in, e.g., Galí (2015, Chapter 3), we can rewrite (B.1) as

$$\pi_t = \beta\mathbb{E}_t\{\pi_{t+1}\} + \frac{(1-\theta)(1-\beta\theta)}{\theta\left[1-\frac{\varepsilon ff''}{(f')^2}\right]}\left[\alpha_\phi\left(i_t^{\ell} - i_t\right) + \widehat{w}_t - \widehat{mp}_t\right]. \tag{B.4}$$

Now, log-linearizing the goods-market-clearing condition (17) gives

$$\widehat{c}_t = \widehat{y}_t. \tag{B.5}$$

Log-linearizing the first-order condition (6), and using (B.5), gives

$$i_t^\ell - i_t = \alpha_\ell \frac{\Gamma_{\ell\ell}\ell}{\Gamma_\ell}\widehat{\ell}_t + \alpha_\ell \frac{\Gamma_{\ell m}m}{\Gamma_\ell}\widehat{m}_t - \alpha_\ell \frac{u''y}{u'}\widehat{y}_t, \tag{B.6}$$

where $\alpha_\ell \equiv (I^\ell - I)/I^\ell \in (0,1)$. Log-linearizing the first-order condition (5), and using (B.2) and (B.5), gives

$$\widehat{w}_t = \left(-\frac{u''y}{u'} + \frac{v''h}{v'}\frac{f}{f'h}\right)\widehat{y}_t. \tag{B.7}$$

Log-linearizing the constraint (10) holding with equality, and using (B.2) and (B.7), gives

$$\widehat{\ell}_t = \left(-\frac{u''y}{u'} + \frac{v''h}{v'}\frac{f}{f'h} + \frac{f}{f'h}\right)\widehat{y}_t. \tag{B.8}$$

Moreover, we have

$$\widehat{mp}_t = \frac{ff''}{(f')^2}\left(\widehat{y}_t\right). \tag{B.9}$$

Using (B.6), (B.7), (B.8), and (B.9), we can then rewrite (B.4) as the Phillips curve (24):

$$\pi_t = \beta \mathbb{E}_t\left\{\pi_{t+1}\right\} + \kappa\left(\widehat{y}_t - \delta_m \widehat{m}_t\right),$$

where

$$\begin{aligned}
\kappa &\equiv \left[-\frac{u''y}{u'} + \frac{v''h}{v'}\frac{f}{f'h} - \frac{ff''}{(f')^2} - \alpha_\ell\alpha_\phi\frac{u''y}{u'} + \alpha_\ell\alpha_\phi\frac{\Gamma_{\ell\ell}\ell}{\Gamma_\ell}\left(-\frac{u''y}{u'} + \frac{v''h}{v'}\frac{f}{f'h} + \frac{f}{f'h}\right)\right]\Psi > 0, \\
\delta_m &\equiv -\alpha_\ell\alpha_\phi\left(\frac{\Gamma_{\ell m}m}{\Gamma_\ell}\right)\frac{\Psi}{\kappa} > 0,
\end{aligned}$$

where in turn

$$\Psi \equiv \frac{(1-\theta)(1-\beta\theta)}{\theta\left[1 - \frac{\varepsilon ff''}{(f')^2}\right]}.$$

To derive the reserves-demand equation (25), we log-linearize the first-order condition (7) and use (B.5) to get

$$i_t - i_t^m = \alpha_m \frac{\Gamma_{\ell m}\ell}{\Gamma_m}\widehat{\ell}_t + \alpha_m \frac{\Gamma_{mm}m}{\Gamma_m}\widehat{m}_t - \alpha_m \frac{u''y}{u'}\widehat{y}_t, \tag{B.10}$$

where $i_t^m \equiv \widehat{I}_t^m$ and $\alpha_m \equiv (I - I^m)/I^m > 0$. Using (B.8), we can rewrite (B.10) as the reserves-demand equation (25):

$$\widehat{m}_t = \chi_y \widehat{y}_t - \chi_i\left(i_t - i_t^m\right),$$

where

$$\begin{aligned}
\chi_y &\equiv \left[\frac{u''y}{u'} - \frac{\Gamma_{\ell m}\ell}{\Gamma_m}\left(-\frac{u''y}{u'} + \frac{v''h}{v'}\frac{f}{f'h} + \frac{f}{f'h}\right)\right]\left(\frac{\Gamma_{mm}m}{\Gamma_m}\right)^{-1} > 0, \\
\chi_i &\equiv \left(-\alpha_m\frac{\Gamma_{mm}m}{\Gamma_m}\right)^{-1} > 0.
\end{aligned}$$

B.2 Proof of the Double Inequality (26)

To show that $\chi_y < 1/\delta_m$, we define

$$\Omega \equiv \frac{\theta\left[1-\frac{\varepsilon f f''}{(f')^2}\right]\delta_m \kappa}{\alpha_m (1-\theta)(1-\beta\theta)\sigma\chi_i} > 0$$

and we write

$$\begin{aligned}
\Omega\left(\frac{1}{\delta_m}-\chi_y\right) &= \frac{u'}{u''y}\frac{\Gamma_{mm}m}{\Gamma_m}\left[\alpha_\ell\alpha_\phi\frac{\Gamma_{\ell\ell}\ell}{\Gamma_\ell}\left(-\frac{u''y}{u'}+\frac{v''h}{v'}\frac{f}{f'h}+\frac{f}{f'h}\right)\right.\\
&\quad \left.-(1+\alpha_\ell\alpha_\phi)\frac{u''y}{u'}+\frac{v''h}{v'}\frac{f}{f'h}-\frac{ff''}{(f')^2}\right]\\
&\quad +\alpha_\ell\alpha_\phi\frac{\Gamma_{\ell m}m}{\Gamma_\ell}\left[1-\frac{u'}{u''y}\frac{\Gamma_{\ell m}\ell}{\Gamma_m}\left(-\frac{u''y}{u'}+\frac{v''h}{v'}\frac{f}{f'h}+\frac{f}{f'h}\right)\right]\\
&= \frac{u'}{u''y}\frac{\Gamma_{mm}m}{\Gamma_m}\left[-\frac{u''y}{u'}+\frac{v''h}{v'}\frac{f}{f'h}-\frac{ff''}{(f')^2}\right]\\
&\quad +\frac{\alpha_\ell\alpha_\phi\ell m}{\Gamma_\ell\Gamma_m}\frac{u'}{u''y}\left(-\frac{u''y}{u'}+\frac{v''h}{v'}\frac{f}{f'h}+\frac{f}{f'h}\right)\left[\Gamma_{\ell\ell}\Gamma_{mm}-(\Gamma_{\ell m})^2\right]\\
&\quad +\alpha_\ell\alpha_\phi m\left(\frac{\Gamma_{\ell m}}{\Gamma_\ell}-\frac{\Gamma_{mm}}{\Gamma_m}\right).
\end{aligned}$$

The last expression is the sum of three terms (one per line). The first term is positive. So is the second one, given (A.5). And so is the third one, given that

$$\frac{\Gamma_{\ell m}}{\Gamma_\ell}-\frac{\Gamma_{mm}}{\Gamma_m}=\frac{\left(v^{b\prime}\right)^2\left(g_m^b g_{\ell m}^b-g_\ell^b g_{mm}^b\right)}{\Gamma_\ell\Gamma_m}=\frac{\left(v^{b\prime}\right)^2\left(f_h^b f_{mm}^b-f_m^b f_{hm}^b\right)}{\Gamma_\ell\Gamma_m\left(f_h^b\right)^3}>0. \tag{B.11}$$

Therefore, the whole expression is positive, which implies that $\chi_y < 1/\delta_m$.

To show that $\sigma \equiv -u''(c)y/u'(c) < \chi_y$, we write

$$\begin{aligned}
\frac{1}{\sigma\chi_i}(\chi_y-\sigma) &= \alpha_m\frac{\Gamma_{mm}m}{\Gamma_m}+\alpha_m-\alpha_m\frac{u'}{u''y}\frac{\Gamma_{\ell m}\ell}{\Gamma_m}\left(-\frac{u''y}{u'}+\frac{v''h}{v'}\frac{f}{f'h}+\frac{f}{f'h}\right)\\
&= \alpha_m-\alpha_m\frac{u'}{u''y}\frac{\Gamma_{\ell m}\ell}{\Gamma_m}\left(\frac{v''h}{v'}\frac{f}{f'h}+\frac{f}{f'h}\right)+\frac{\alpha_m}{\Gamma_m}\left(\ell\Gamma_{\ell m}+m\Gamma_{mm}\right).
\end{aligned}$$

The last expression is the sum of three terms. The first two terms are positive. And so is the third one, given that

$$\begin{aligned}
\ell\Gamma_{\ell m}+m\Gamma_{mm} &= (1-d)\,\ell\Gamma_{\ell m}+d\ell\Gamma_{\ell m}+m\Gamma_{mm}\\
&= (1-d)\,\ell\Gamma_{\ell m}+d\ell\left[v^{b\prime\prime}g_\ell^b g_m^b+v^{b\prime}g_{\ell m}^b\right]+m\left[v^{b\prime\prime}\left(g_m^b\right)^2+v^{b\prime}g_{mm}^b\right]\\
&= (1-d)\,\ell\Gamma_{\ell m}+v^{b\prime\prime}g_m^b\left(d\ell g_\ell^b+mg_m^b\right)+v^{b\prime}\left(d\ell g_{\ell m}^b+mg_{mm}^b\right)\\
&= (1-d)\,\ell\Gamma_{\ell m}+v^{b\prime\prime}g^b g_m^b\\
&\leq 0,
\end{aligned} \tag{B.12}$$

where the last equality follows from (A.2) and (A.4). Therefore, the whole expression is positive, which implies that $\sigma < \chi_y$.

B.3 Roots of Polynomial $\mathcal{P}(X)$

The polynomial $\mathcal{P}(X)$ can be rewritten as

$$\begin{aligned}\mathcal{P}(X) &= X^3 - \left(\frac{1+2\beta+\beta\Theta_1+\Theta_2}{\beta}\right)X^2 + \left[\frac{2+\beta+(1+\beta)\Theta_1+\Theta_2+\Theta_3}{\beta}\right]X - \left(\frac{1+\Theta_1}{\beta}\right) \\ &= (X-1-\Theta_1)\left[X^2 - \left(\frac{1+\beta+\Theta_2}{\beta}\right)X + \frac{1}{\beta}\right] - \left(\frac{\Theta_1\Theta_2-\Theta_3}{\beta}\right)X,\end{aligned}$$

where $\Theta_1 \equiv \chi_y/(\sigma\chi_i) > 0$, $\Theta_2 \equiv (1/\sigma - \delta_m)\kappa$, and $\Theta_3 \equiv (1-\delta_m\chi_y)\kappa/(\sigma\chi_i)$. The double inequality (26) implies $\Theta_2 > 0$, $\Theta_3 > 0$, and $\Theta_1\Theta_2 - \Theta_3 = (\chi_y - \sigma)\kappa/(\sigma^2\chi_i) > 0$. Therefore, we get $\mathcal{P}(0) = -(1+\Theta_1)/\beta < 0$, $\mathcal{P}(1) = \Theta_3/\beta > 0$, $\mathcal{P}(1+\Theta_1) = -(\Theta_1\Theta_2-\Theta_3)(1+\Theta_1)/\beta < 0$, and $\lim_{X\to+\infty}\mathcal{P}(X) = +\infty > 0$. As a consequence, the roots of $\mathcal{P}(X)$ are three real numbers ρ, ω_1, and ω_2 such that $0 < \rho < 1 < \omega_1 < 1+\Theta_1 < \omega_2$.

B.4 Determinacy Condition Under a Floor System

Using the IS equation (23), the Phillips curve (24), the reserves-demand equation (25), the Taylor rule (42), and the identities $\widehat{m}_t = \widehat{M}_t - \widehat{P}_t$ and $\pi_t = \widehat{P}_t - \widehat{P}_{t-1}$, we get a dynamic equation relating $\widehat{P}_t$ to $\mathbb{E}_t\{\widehat{P}_{t+2}\}$, $\mathbb{E}_t\{\widehat{P}_{t+1}\}$, $\widehat{P}_{t-1}$, and exogenous terms, whose characteristic polynomial is

$$\mathcal{P}_r(X) \equiv X^3 - a_2X^2 + a_1X - a_0$$

with

$$\begin{aligned}a_2 &\equiv 2 + \frac{1}{\beta} + \frac{(1-\sigma\delta_m)\kappa}{\beta\sigma} + \frac{\chi_y}{\sigma\chi_i} + \frac{r_y}{\sigma} > 0, \\ a_1 &\equiv 1 + \frac{2}{\beta} + \frac{(1-\sigma\delta_m)\kappa}{\beta\sigma} + \frac{(1+\beta)\chi_y}{\beta\sigma\chi_i} + \frac{(1-\delta_m\chi_y)\kappa}{\beta\sigma\chi_i} + \frac{\kappa r_\pi}{\beta\sigma} + \frac{(1+\beta-\delta_m\kappa)r_y}{\beta\sigma}, \\ a_0 &\equiv \frac{1}{\beta} + \frac{\chi_y}{\beta\sigma\chi_i} + \frac{\kappa r_\pi}{\beta\sigma} + \frac{r_y}{\beta\sigma} > 0,\end{aligned}$$

where the first inequality follows from the double inequality (26). Given that there is exactly one predetermined variable ($\widehat{P}_{t-1}$), the necessary and sufficient condition for local-equilibrium determinacy is that $\mathcal{P}_r(X)$ have exactly one root inside the unit circle. This root must be a real number (indeed, if it were a complex number, its conjugate would be another root inside the unit circle). We have $\mathcal{P}_r(0) = -a_0 < 0$ and $\mathcal{P}_r(1) = (1-\delta_m\chi_y-\delta_m\chi_i r_y)\kappa/(\beta\sigma\chi_i)$. In the following, we consider two alternative cases in turn, depending on the sign of $\mathcal{P}_r(1)$.

We first consider the case in which $\mathcal{P}_r(1) > 0$, that is to say equivalently the case in which

$$r_y < \zeta_1, \tag{B.13}$$

where

$$\zeta_1 \equiv \frac{1-\delta_m\chi_y}{\delta_m\chi_i} > 0,$$

where in turn the last inequality follows from the second inequality in (26). In this case, $\mathcal{P}_r(0)$ and $\mathcal{P}_r(1)$ are of opposite signs, so that $\mathcal{P}_r(X)$ has either one or three real roots inside $(0,1)$. Moreover, in this case, we have

$$a_1 = 1 + \frac{2}{\beta} + \frac{(1-\sigma\delta_m)\,\kappa}{\beta\sigma} + \frac{(1+\beta)\,\chi_y}{\beta\sigma\chi_i} + \frac{\kappa r_\pi}{\beta\sigma} + \frac{(1+\beta)\,r_y}{\beta\sigma} + \frac{(\zeta_1 - r_y)\,\delta_m\kappa}{\beta\sigma} > 0,$$

where the inequality comes from (26) and (B.13). In turn, $a_1 > 0$, together with $a_0 > 0$ and $a_2 > 0$, implies that $\mathcal{P}_r(X) < 0$ for all $X < 0$, and hence that $\mathcal{P}_r(X)$ has no negative real roots. So, $\mathcal{P}_r(X)$ has at least one real root inside $(0,1)$, which we denote by ρ, and its other two roots, which we denote by ω_1 and ω_2 with $|\omega_1| \leq |\omega_2|$, are either (i) both real and inside $(0,1)$, or (ii) both real and higher than 1, or (iii) both complex and conjugates of each other. Now, we have $\rho + \omega_1 + \omega_2 = a_2 > 3$, where the inequality follows from the double inequality (26) and from $\beta < 1$. Therefore, Case (i) is impossible, and in Case (iii) the common real part of ω_1 and ω_2 is higher than 1. So, in the remaining two possible cases, namely Cases (ii) and (iii), ω_1 and ω_2 lie outside the unit circle. As a consequence, we get local-equilibrium determinacy.

We now turn to the alternative case in which $\mathcal{P}_r(1) < 0$, that is to say equivalently the case in which Condition (B.13) is not met. In this case, $\mathcal{P}_r(0)$ and $\mathcal{P}_r(1)$ have the same sign, so that $\mathcal{P}_r(X)$ has either zero or two real roots inside $(0,1)$. Therefore, a necessary condition for local-equilibrium determinacy is then that $\mathcal{P}_r(-1)$ be of the opposite sign, i.e. $\mathcal{P}_r(-1) > 0$, so that $\mathcal{P}_r(X)$ can have either one or three real roots inside $(-1,0)$. This necessary condition for determinacy can be written as

$$[\delta_m\kappa - 2\,(1+\beta)]\,r_y > 4\,(1+\beta)\,\sigma + \frac{2\,(1+\beta)\,\chi_y}{\chi_i} + 2\,(1-\sigma\delta_m)\,\kappa + \frac{(1-\delta_m\chi_y)\,\kappa}{\chi_i} + 2\kappa r_\pi.$$

The right-hand side of this inequality is positive, given the double inequality (26). Therefore, the necessary condition for determinacy can be equivalently rewritten as

$$\delta_m\kappa > 2\,(1+\beta) \quad \text{and} \quad r_y > \zeta_2 + \zeta_3 r_\pi, \tag{B.14}$$

where

$$\begin{aligned} \zeta_2 &\equiv \frac{4\,(1+\beta)\,\sigma\chi_i + 2\,(1+\beta)\,\chi_y + 2\,(1-\sigma\delta_m)\,\kappa\chi_i + (1-\delta_m\chi_y)\,\kappa}{[\delta_m\kappa - 2\,(1+\beta)]\,\chi_i} > \zeta_1, \\ \zeta_3 &\equiv \frac{2\kappa}{\delta_m\kappa - 2\,(1+\beta)} > 0, \end{aligned}$$

where in turn the last two inequalities follow from the first inequality in (B.14). We now show that Condition (B.14) is not only necessary, but also sufficient for local-equilibrium determinacy in that case. To that aim, assume that this condition is met. Then, $\mathcal{P}_r(-1)$ and $\mathcal{P}_r(0)$ are of opposite signs, so that $\mathcal{P}_r(X)$ has either one or three real roots inside $(-1,0)$. Let ρ denote one root of $\mathcal{P}_r(X)$ inside $(-1,0)$. The other two roots of $\mathcal{P}_r(X)$, which we denote by ω_1 and ω_2 with $|\omega_1| \leq |\omega_2|$, can be either (i) both real and inside $(-1,0)$, or (ii) both real and inside $(0,1)$, or (iii) both real and outside $(-1,1)$, or (iv) both complex and conjugates of each other. Since

$\rho + \omega_1 + \omega_2 = a_2 > 3$, however, Cases (i) and (ii) are impossible, and in Case (iv) the common real part of ω_1 and ω_2 is higher than 1. Therefore, in the remaining two possible cases, namely Cases (iii) and (iv), ω_1 and ω_2 lie outside the unit circle. As a consequence, Condition (B.14) is, indeed, sufficient for local-equilibrium determinacy in that case.

From the results obtained in the two alternative cases considered, we get that there is local-equilibrium determinacy if and only if either Condition (B.13) is met, or Condition (B.13) is not met and Condition (B.14) is met. Now, Conditions (B.13) and (B.14) are mutually exclusive, given that $\zeta_2 > \zeta_1$. We conclude that there is local-equilibrium determinacy if and only if either Condition (B.13) or Condition (B.14) is met.

Appendix C: Benchmark Model With Reserves-Supply Rule

In this appendix, we consider a reserves-supply rule in our benchmark model, we derive a sufficient condition for local-equilibrium determinacy under this rule and an exogenous IOR rate, and we argue that this sufficient determinacy condition is likely to be met.

C.1 Reserves-Supply Rule

We assume that the central bank sets the stock of nominal reserves according to the rule

$$M_t = P_t \mathcal{Q}\left(P_t, y_t\right), \tag{C.1}$$

where the function $\mathcal{Q}$, from $\mathbb{R}^2_{>0}$ to $\mathbb{R}_{>0}$, is differentiable, decreasing in P_t ($\mathcal{Q}_P < 0$), and non-increasing in y_t ($\mathcal{Q}_y \leq 0$). This assumption ensures that real reserve balances respond negatively to the price level for a given output level, and non-positively to the output level for a given price level. This specification nests, in particular, the case of (constant) exogenous nominal reserves considered in the rest of the paper, which corresponds to $\mathcal{Q}\left(P_t, y_t\right) = M/P_t$ with $M > 0$.

This reserves-supply rule does not change any of the equilibrium conditions stated in Subsection 3.1, except the dynamic equation (20). This dynamic equation was obtained under the assumption that the (gross) growth rate of nominal reserves is exogenous and constant ($\mu_{t+1} \equiv M_{t+1}/M_t = \mu$); now, however, the rule (C.1) makes μ_{t+1} endogenous. Inverting the rule (C.1) leads to $P_t = \mathcal{S}(m_t, y_t)$, where the function $\mathcal{S}$, defined from $\mathbb{R}^2_{>0}$ to $\mathbb{R}_{>0}$, is differentiable, decreasing in m_t ($\mathcal{S}_m < 0$), and non-increasing in y_t ($\mathcal{S}_y \leq 0$). Therefore, the new flexible-price dynamic equation is

$$1 + \frac{\Gamma_m\left[\mathcal{L}\left(h_t\right), \mathcal{M}\left(h_t\right)\right]}{u'\left[f\left(h_t\right)\right]} = \beta I^m \mathbb{E}_t\left\{\frac{u'\left[f\left(h_{t+1}\right)\right] \mathcal{S}\left[\mathcal{M}\left(h_t\right), f\left(h_t\right)\right]}{u'\left[f\left(h_t\right)\right] \mathcal{S}\left[\mathcal{M}\left(h_{t+1}\right), f\left(h_{t+1}\right)\right]}\right\}.$$

In any steady state, this dynamic equation boils down to the static equation (21) with $\mu = 1$. Therefore, the necessary and sufficient condition on I^m for existence and uniqueness of a steady state is still $I^m < 1/\beta$, as previously. Log-linearizing the model around its unique steady state, we get the same IS equation (23), Phillips curve (24), and reserves-demand equation (25) as previously, plus now the rule

$$\widehat{m}_t = -q_P \widehat{P}_t - q_y \widehat{y}_t, \tag{C.2}$$

where $q_P \equiv -P\mathcal{Q}_P(P,y)/\mathcal{Q}(P,y) > 0$ and $q_y \equiv -y\mathcal{Q}_y(P,y)/\mathcal{Q}(P,y) \geq 0$ (variables without time subscript denote steady-state values).

C.2 Derivation of a Sufficient Determinacy Condition

Using the four log-linearized equations just mentioned and the identity $\pi_t = \widehat{P}_t - \widehat{P}_{t-1}$, we get the following dynamic system in $\widehat{m}_t$ and $\widehat{y}_t$:

$$\mathbb{E}_t \left\{ \left[\begin{array}{c} \widehat{m}_{t+1} \\ \widehat{m}_t \\ \widehat{y}_{t+1} \\ \widehat{y}_t \end{array} \right] \right\} = \mathbf{A} \left[\begin{array}{c} \widehat{m}_t \\ \widehat{m}_{t-1} \\ \widehat{y}_t \\ \widehat{y}_{t-1} \end{array} \right] + \mathbf{B} \left[\begin{array}{c} i_t^m \\ r_t \end{array} \right], \tag{C.3}$$

where

$$\begin{aligned}
\mathbf{A} \;\equiv\; & \left[\begin{array}{cccc} \frac{1+\beta-\delta_m\kappa}{\beta} & \frac{-1}{\beta} & \frac{\kappa}{\beta} & 0 \\ 1 & 0 & 0 & 0 \\ \frac{1-\delta_m\kappa}{\beta\sigma} - \frac{1}{\sigma\chi_i} & \frac{-1}{\beta\sigma} & 1 + \frac{\chi_y}{\sigma\chi_i} + \frac{\kappa}{\beta\sigma} & 0 \\ 0 & 0 & 1 & 0 \end{array} \right] + (q_P - 1) \left[\begin{array}{cccc} \frac{-\delta_m\kappa}{\beta} & 0 & \frac{\kappa}{\beta} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \\
& + \frac{q_P - 1}{q_P} \left[\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{-1}{\beta\sigma} & \frac{1}{\beta\sigma} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] + q_y \left[\begin{array}{cccc} \frac{\delta_m\kappa}{\beta\sigma} + \frac{1}{\sigma\chi_i} & 0 & \frac{1}{\beta} - \frac{\kappa}{\beta\sigma} - \frac{\chi_y}{\sigma\chi_i} & \frac{-1}{\beta} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \\
& + \frac{q_y}{q_P} \left[\begin{array}{cccc} \frac{-1}{\beta\sigma} & \frac{1}{\beta\sigma} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\beta\sigma} & \frac{-1}{\beta\sigma} \\ 0 & 0 & 0 & 0 \end{array} \right] + \frac{q_y^2}{q_P} \left[\begin{array}{cccc} 0 & 0 & \frac{-1}{\beta\sigma} & \frac{1}{\beta\sigma} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]
\end{aligned}$$

$$\text{and} \quad \mathbf{B} \equiv \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ \frac{1}{\sigma} & \frac{-1}{\sigma} \\ 0 & 0 \end{array} \right] + q_y \left[\begin{array}{cc} \frac{-1}{\sigma} & \frac{1}{\sigma} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{array} \right].$$

Since this system has two predetermined variables ($\widehat{m}_{t-1}$ and $\widehat{y}_{t-1}$) and two non-predetermined variables ($\widehat{m}_{t+1}$ and $\widehat{y}_{t+1}$), the necessary and sufficient condition for local-equilibrium determinacy is that the matrix $\mathbf{A}$ have two eigenvalues inside the unit circle and two eigenvalues outside. We write the characteristic polynomial of $\mathbf{A}$ as

$$\det(\mathbf{A} - X\mathbf{I}_4) = X\mathcal{Q}(X),$$

where $\mathbf{I}_4$ denotes the 4×4 identity matrix and

$$\begin{aligned}
\mathcal{Q}(X) \;\equiv\; & \mathcal{P}(X) - \left[\left(\frac{\delta_m\kappa}{\beta\sigma} + \frac{1}{\sigma\chi_i} \right) q_y - \frac{\delta_m\kappa}{\beta}(q_P - 1) \right] X^2 \\
& + \left[\left(\frac{\delta_m\kappa}{\beta\sigma} + \frac{1+\beta}{\beta\sigma\chi_i} \right) q_y + \left(\frac{-\delta_m\kappa}{\beta} + \frac{\kappa}{\beta\sigma\chi_i} - \frac{\delta_m\chi_y\kappa}{\beta\sigma\chi_i} \right)(q_P - 1) \right] X - \frac{q_y}{\beta\sigma\chi_i},
\end{aligned}$$

where in turn $\mathcal{P}(X)$ is defined in Subsection 4.1. Thus, the eigenvalues of $\mathbf{A}$ are 0 and the roots of $\mathcal{Q}(X)$. We have

$$\mathcal{Q}(0) = \frac{-1}{\beta} - \frac{\chi_y}{\beta\sigma\chi_i} - \frac{q_y}{\beta\sigma\chi_i} < -1 \quad \text{and} \quad \mathcal{Q}(1) = \frac{(1-\delta_m\chi_y)\,\kappa q_P}{\beta\sigma\chi_i} > 0,$$

where the second inequality follows from (26). Therefore, $\mathcal{Q}(X)$ has either one root or three roots in the real-number interval $[0,1]$. Now, the product of the three roots of $\mathcal{Q}(X)$ is equal to $-\mathcal{Q}(0) > 1$, so that $\mathcal{Q}(X)$ has at least one root outside the unit circle. As a consequence, $\mathcal{Q}(X)$ has exactly one root inside the real-number interval $[0,1]$.

The other roots of $\mathcal{Q}(X)$ are either two real numbers outside $[0,1]$, or two conjugate complex numbers. In the latter case, both are outside the unit circle, since $\mathcal{Q}(X)$ has at least one root outside it. Therefore, $\mathcal{Q}(X)$ has exactly two roots outside the unit circle if and only if it has no root inside the real-number interval $[-1,0)$. Since $\mathcal{Q}(0) < 0$, this condition is equivalent to $\mathcal{Q}(X) < 0$ for all $X \in [-1,0]$. Thus, the necessary and sufficient condition for determinacy is $\mathcal{Q}(X) < 0$ for all $X \in [-1,0]$.

A *sufficient* condition for determinacy is, therefore, that $\mathcal{Q}(X) < 0$ for all $X \in [-1,0]$ *and all* $\theta \in (0,1)$. To restate this sufficient condition, we rewrite $\mathcal{Q}(X)$ as

$$\mathcal{Q}(X) = \frac{\kappa}{\beta}\left[\mathcal{Q}_1(X) + \mathcal{Q}_2(X)\right],$$

where

$$\begin{aligned}
\mathcal{Q}_1(X) &\equiv \frac{-1}{\kappa}(1-X)(1-\beta X)\left(1 + \frac{\chi_y + q_y}{\sigma\chi_i} - X\right), \\
\mathcal{Q}_2(X) &\equiv \left(\frac{1+\delta_m q_y}{\sigma} - \delta_m q_P\right) X(1-X) + \left(\frac{1-\delta_m\chi_y}{\sigma\chi_i}\right) q_P X.
\end{aligned}$$

The only reduced-form parameter that depends on the degree of price stickiness θ is the slope of the Phillips curve κ, which is decreasing in θ. Therefore, whatever $X \in [-1,0]$, $\mathcal{Q}_1(X)$ is decreasing in θ, while $\mathcal{Q}_2(X)$ does not depend on θ. As a consequence, $\mathcal{Q}(X) < 0$ for all $X \in [-1,0]$ and all $\theta \in (0,1)$ if and only if $\mathcal{Q}(X) < 0$ for all $X \in [-1,0]$ as $\theta \to 0$. In turn, this condition is equivalent to $\mathcal{Q}_2(X) < 0$ for all $X \in [-1,0)$, since $\lim_{\theta\to 0}\kappa = +\infty$ implies $\lim_{\theta\to 0}\mathcal{Q}_1(X) = 0$. Now, we can rewrite $\mathcal{Q}_2(X)$ as

$$\mathcal{Q}_2(X) = X\left\{KX + \left[\left(\frac{1-\delta_m\chi_y}{\sigma\chi_i}\right) q_P - K\right]\right\},$$

where $K \equiv \delta_m q_P - (1+\delta_m q_y)/\sigma$. Therefore, $\mathcal{Q}_2(X) < 0$ for all $X \in [-1,0)$ if and only if

$$K < \left(\frac{1-\delta_m\chi_y}{2\sigma\chi_i}\right) q_P, \tag{C.4}$$

where the right-hand side is non-negative, as follows from the second inequality in (26).

Now consider the following condition:

$$q_P \leq \left\{\left(\frac{I^\ell - I}{I^\ell}\right)\left[\frac{-m\Gamma_{\ell m}(\ell,m)}{\Gamma_\ell(\ell,m)}\right]\right\}^{-1}. \tag{C.5}$$

This condition implies $\delta_m q_P \leq 1/\sigma$ because

$$\delta_m < \alpha_\ell \alpha_\phi \left(\frac{-\Gamma_{\ell m} m}{\Gamma_\ell}\right)\left(\frac{-u'' y}{u'}\right)^{-1} \leq \alpha_\ell \left(\frac{-\Gamma_{\ell m} m}{\Gamma_\ell}\right)\left(\frac{-u'' y}{u'}\right)^{-1} = \left(\frac{I^\ell - I}{I^\ell}\right)\left(\frac{-\Gamma_{\ell m} m}{\Gamma_\ell}\right)\frac{1}{\sigma}.$$

In turn, $\delta_m q_P \leq 1/\sigma$ implies $K < 0$, which in turn implies (C.4), which in turn implies that $\mathcal{Q}(X)$ has exactly two roots outside the unit circle, which finally implies determinacy. Therefore, Condition (C.5) is a sufficient condition for determinacy.

C.3 Assessment of the Sufficient Determinacy Condition

To assess whether Condition (C.5) is likely to be met or not, we proceed as follows. We consider, for simplicity, a Cobb-Douglas specification for the production function f^b:

$$f^b\left(h_t^b, m_t\right) \equiv A_b \left(h_t^b\right)^{1-\varsigma} (m_t)^\varsigma ,$$

where $A_b > 0$ and $0 < \varsigma < 1$. This specification implies that the steady-state elasticity of marginal banking costs to reserves, which appears in Condition (C.5), can be rewritten as

$$\frac{-m\Gamma_{\ell m}(\ell, m)}{\Gamma_\ell(\ell, m)} = \left(\frac{\varsigma}{1-\varsigma}\right)\left[1 + \frac{v^{b\prime\prime}\left(h^b\right) h^b}{v^{b\prime}\left(h^b\right)}\right].$$

Our Cobb-Douglas specification for f^b also implies that households' first-order conditions (6) and (7), at the steady state, can be combined to get

$$\varsigma = \left(\frac{m}{\ell}\right)\left(\frac{I - I^m}{I^\ell - I}\right).$$

Therefore, Condition (C.5) can be rewritten as

$$q_P \leq \frac{1 - \left(\frac{m}{\ell}\right)\left(\frac{I - I^m}{I^\ell - I}\right)}{\left(\frac{m}{\ell}\right)\left(\frac{I - I^m}{I^\ell}\right)\left[1 + \frac{v^{b\prime\prime}\left(h^b\right) h^b}{v^{b\prime}\left(h^b\right)}\right]}. \tag{C.6}$$

We set the steady-state variables I^m, I^ℓ, and m/ℓ to match some features of the US economy during the 2008-2015 ZLB episode. This episode lasted from December 16, 2008, to December 16, 2015; because we use monthly data, however, we consider the period from January 2009 to November 2015. We set the net IOR rate $I^m - 1$ to 0.25% per annum (the constant value of the interest rate on excess reserves over the period); the net interest rate on bank loans $I^\ell - 1$ to 3.25% per annum (the average value of the bank prime loan rate over the period); and the ratio of bank reserves to loans m/ℓ to 0.18 (the average ratio of total reserves of depository institutions to bank credit of all commercial banks over the period).

We calibrate the parameter q_P to match the increase in the stock of nominal reserves over the period. More specifically, we rewrite (C.2) as $\widehat{M}_t = (1 - q_P)\widehat{P}_t - q_y \widehat{y}_t$, and we assume conservatively that the Fed increased the stock of nominal reserves over the period only in response to low inflation, not in response to low output growth (i.e., $q_y = 0$). This assumption is conservative because it tends to overestimate q_P and, therefore, to make Condition (C.6) harder to satisfy. Under this assumption, we get

$$q_P = 1 - \frac{\widehat{M_t}}{\widehat{P_t}}. \tag{C.7}$$

Our model has constant reserves and prices at the steady state. In reality, however, reserves followed a positive trend before the crisis, and the Fed has a positive inflation target. The Fed increased the stock of nominal reserves *beyond its trend*, in response to inflation *below target*. A natural empirical counterpart of (C.7), over the January 2009-November 2015 period, is therefore

$$q_P = 1 - \frac{(\ln M_{2015:11} - \ln M_{2009:01}) - \Delta_M}{(\ln P_{2015:11} - \ln P_{2009:01}) - \Delta_P},$$

where M_t and P_t are measured by the total reserves of depository institutions and the consumer price index respectively, while Δ_M and Δ_P denote respectively the "neutral" trend growth rate in reserves and the targeted growth rate in prices over the January 2009-November 2015 period. We conservatively set Δ_M to zero, thus attributing all of the observed growth in reserves to the Fed's response to inflation below target. Like the assumption $q_y = 0$, the assumption $\Delta_M = 0$ is conservative because it tends to overestimate q_P and, therefore, to make Condition (C.6) harder to satisfy. And we set Δ_P to 14%, which corresponds to the Fed's 2% annual-inflation target over (almost) seven years. We then get $q_P = 50.0$.

Finally, we make conservative assumptions about the values of the steady-state variables $h^b v^{b\prime\prime}(h^b)/v^{b\prime}(h^b)$ and I. More specifically, we set $h^b v^{b\prime\prime}(h^b)/v^{b\prime}(h^b)$, the inverse of the steady-state Frisch elasticity of bankers' labor supply, to 5. The value 5 for the inverse of a Frisch elasticity of labor supply lies at the upper end of the range of microeconomic estimates, and is much higher than values commonly considered in macroeconomics. And we set the net interest rate $I - 1$ to 0.75% per annum, i.e. 50 basis points per annum above the net IOR rate $I^m - 1$. This value is much higher than the average value, over the January 2009-November 2015 period, of standard proxies for $I - 1$, like the 3-month T-bill rate or the 3-month AA (financial or non-financial) commercial paper rate. Our assumptions about $h^b v^{b\prime\prime}(h^b)/v^{b\prime}(h^b)$ and I are conservative because they tend to overestimate these two steady-state variables and, therefore, to make Condition (C.6) harder to satisfy.

The right-hand side of Condition (C.6) depends on the period length, through the ratio $(I - I^m)/I^\ell$. Since one-period bank loans in our model are working-capital loans, which are short-term loans in reality, we set the period length to one quarter. Thus, we express all the interest rates in the ratio $(I - I^m)/I^\ell$ as quarterly rates.

We then get the value 705.7 for the right-hand side of Condition (C.6). This value is one order of magnitude larger than the value 50.0 obtained for the left-hand side of Condition (C.6). We thus find that Condition (C.6) is met by a large margin even under our conservative assumptions.

We conclude that setting exogenously the IOR rate and following the reserves-supply rule still delivers local-equilibrium determinacy, except for implausible calibrations.

Appendix D: Extended Model With Household Cash

In this appendix, we extend our benchmark model with household cash (using a cash-in-advance constraint); we derive a sufficient condition for local-equilibrium determinacy under an exogenous IOR rate and an exogenous monetary base (made of bank reserves and household cash); and we argue that this sufficient determinacy condition is likely to be met.

D.1 Introducing Household Cash in the Benchmark Model

We assume that each period is made of a financial exchange followed by a goods exchange. Households acquire cash in the financial exchange and use it to buy goods in the goods exchange; firms receive this cash in the goods exchange and have to wait until the next period's financial exchange to spend it (repaying loans). Thus, households choose bonds b_t, consumption c_t, work hours h_t, loans ℓ_t, reserves m_t, and (now) cash m_t^c to maximize the same reduced-form utility function (1) as previously, subject to the budget constraint

$$m_t^c + b_t + \ell_t + m_t \leq \frac{m_{t-1}^c - c_{t-1}}{\Pi_t} + \frac{I_{t-1}}{\Pi_t} b_{t-1} + \frac{I_{t-1}^\ell}{\Pi_t} \ell_{t-1} + \frac{I_{t-1}^m}{\Pi_t} m_{t-1} + w_t h_t + \omega_t$$

and the cash-in-advance constraint

$$m_t^c \geq c_t, \tag{D.1}$$

taking all prices (I_t, I_t^ℓ, I_t^m, P_t, and w_t) as given. Letting λ_t and λ_t^c denote the Lagrange multipliers on these two constraints respectively, the first-order conditions of households' optimization problem are again (3), (4), (5), (6), (7), and now

$$\lambda_t^c + \frac{\beta \lambda_{t+1}}{\Pi_{t+1}} - \lambda_t = 0.$$

The objective of firm i is now to maximize

$$\mathbb{E}_t \left\{ \sum_{k=0}^{+\infty} (\beta\theta)^k \frac{\beta \lambda_{t+k+1}}{\lambda_t \Pi_{t,t+k+1}} \left[P_t^*(i)\, y_{t+k}(i) - I_{t+k}^\ell L_{t+k}(i) - \left[W_{t+k} h_{t+k}(i) - L_{t+k}(i) \right] \right] \right\},$$

since the firm has to wait until the next period to exchange its cash. The first-order condition for the firm's optimization problem is thus

$$\mathbb{E}_t \left\{ \sum_{k=0}^{+\infty} (\beta\theta)^k \frac{\beta \lambda_{t+k+1}}{\lambda_t \Pi_{t,t+k+1}} \left[P_t^*(i) - \left(\frac{\varepsilon}{\varepsilon - 1} \right) \left(\phi I_{t+k}^\ell + (1-\phi) \right) \frac{W_{t+k}}{f'\left[h_{t+k}(i)\right]} \right] y_{t+k}(i) \right\} = 0, \tag{D.2}$$

instead of (12). Under flexible prices (and in a symmetric equilibrium), this first-order condition becomes

$$P_t = \frac{\varepsilon}{\varepsilon - 1} \left[\phi I_t^\ell + (1-\phi) \right] \frac{W_t}{f'(h_t)}, \tag{D.3}$$

which replaces (13). None of the other equilibrium conditions stated in Section 2 is changed, except the reserve-market-clearing condition (16), which is replaced by the money-market-clearing condition

$$m_t + m_t^c = \frac{M_t}{P_t}, \tag{D.4}$$

since the monetary base M_t controlled by the central bank is now made not only of bank reserves, but also of household cash. As previously, the equilibrium conditions (3), (5), (9), (10) holding with equality, and (17) imply the relationship (18) between loans and employment.

To derive the necessary and sufficient condition for steady-state existence and uniqueness under a constant IOR rate ($I_t^m = I^m \geq 1$), a constant monetary base ($\mu_t = \mu = 1$), and no discount-factor shocks ($\zeta_t = 1$), we first note that the steady-state inflation rate Π is equal to one under a constant monetary base. In turn, $\Pi = 1$ and (4) together imply that the steady-state interest rate on bonds I is equal to $1/\beta$, as previously. Using (3), (5), (9), (17), (18), (D.3), and $I = 1/\beta$, we can rewrite households' first-order condition for loans (6) at the steady state as a relationship between steady-state real reserves m and steady-state employment h:

$$\Gamma_\ell \left[\mathcal{L}(h), m\right] = \widetilde{\mathcal{A}}(h) \equiv u'\left[f(h)\right] \left\{ \frac{\beta}{\phi} \left[\left(\frac{\varepsilon - 1}{\varepsilon} \right) \frac{u'\left[f(h)\right] f'(h)}{v'(h)} - (1 - \phi) \right] - 1 \right\}. \tag{D.5}$$

Because the left-hand side of (D.5) is positive, we restrict the domain of the function $\widetilde{\mathcal{A}}$ to $(0, \widetilde{h})$, where $\widetilde{h} > 0$ is implicitly and uniquely defined by $u'[f(\widetilde{h})]f'(\widetilde{h})/v'(\widetilde{h}) = [\phi/\beta + (1-\phi)]\varepsilon/(\varepsilon - 1)$. The function $\widetilde{\mathcal{A}}$ is strictly decreasing ($\widetilde{\mathcal{A}}' < 0$), with $\lim_{h_t \to 0} \widetilde{\mathcal{A}}(h_t) = +\infty$ and $\lim_{h_t \to \widetilde{h}} \widetilde{\mathcal{A}}(h_t) = 0$. Since $\Gamma_{\ell\ell} > 0$, $\mathcal{L}' > 0$, $\Gamma_{\ell m} < 0$, and $\widetilde{\mathcal{A}}' < 0$, Equation (D.5) implicitly and uniquely defines a function $\widetilde{\mathcal{M}}$ which is strictly increasing ($\widetilde{\mathcal{M}}' > 0$) and such that

$$m = \widetilde{\mathcal{M}}(h). \tag{D.6}$$

Using (3), (9), (17), (18), (D.6), and $I = 1/\beta$, we can then rewrite households' first-order condition for reserves (7) at the steady state as

$$\widetilde{\mathcal{F}}(h) \equiv \frac{\Gamma_m \left[\mathcal{L}(h), \widetilde{\mathcal{M}}(h)\right]}{u'\left[f(h)\right]} = \beta I^m - 1.$$

The same reasoning as in Appendix A.5, this time applied to $\widetilde{\mathcal{F}}$ rather than $\mathcal{F}$, shows that the function $\widetilde{\mathcal{F}}$ is strictly increasing from $-\infty$ to 0. Therefore, the necessary and sufficient condition for existence and uniqueness of a steady state is again $I^m < 1/\beta$.

Log-linearizing the model around its unique steady state, we get the same IS equation (23) and reserves-demand equation (25) as previously. Using the goods-market-clearing condition (17) and the binding cash-in-advance constraint (D.1) to rewrite the money-market-clearing condition (D.4), and then log-linearizing the resulting equation, leads to

$$\widehat{M}_t - \widehat{P}_t = (1 - \alpha_c)\,\widehat{m}_t + \alpha_c \widehat{y}_t, \tag{D.7}$$

where $\alpha_c \equiv f(h)/[f(h)+\mathcal{M}(h)] \in (0,1)$ denotes the steady-state share of household cash in the monetary base. Finally, we derive the Phillips curve by following the same steps as in Appendix B.1. More specifically, the log-linearized first-order condition under sticky prices is now

$$\widehat{P}_t^* = (1-\beta\theta)\,\mathbb{E}_t\left\{\sum_{k=0}^{+\infty}(\beta\theta)^k\left(\alpha_\phi i_{t+k}^\ell + \widehat{w}_{t+k} + \widehat{P}_{t+k} - \widehat{mp}_{t+k|t}\right)\right\},$$

which corresponds to (B.1) without the i_{t+k} term; and this log-linearized first-order condition leads to the Phillips curve

$$\pi_t = \beta\mathbb{E}_t\{\pi_{t+1}\} + \kappa\left(\widehat{y}_t - \delta_m\widehat{m}_t + \delta_i i_t\right) \tag{D.8}$$

with $\delta_i \equiv \alpha_\phi\Psi/\kappa > 0$, where the term $\delta_i i_t$ captures the opportunity cost for firms of holding their cash from one period to the next.

D.2 Derivation of a Sufficient Determinacy Condition

Using the IS equation (23), the reserves-demand equation (25), the money-market-clearing condition (D.7), the Phillips curve (D.8), and the identity $\pi_t = \widehat{P}_t - \widehat{P}_{t-1}$, we get the following dynamic system in $\widehat{m}_t$ and $\widehat{y}_t$:

$$\mathbb{E}_t\left\{\begin{bmatrix}\widehat{m}_{t+1}\\ \widehat{m}_t\\ \widehat{y}_{t+1}\\ \widehat{y}_t\end{bmatrix}\right\} = \widetilde{\mathbf{A}}\begin{bmatrix}\widehat{m}_t\\ \widehat{m}_{t-1}\\ \widehat{y}_t\\ \widehat{y}_{t-1}\end{bmatrix} + \mathbb{E}_t\left\{\widetilde{\mathbf{B}}\begin{bmatrix}i_t^m\\ \widehat{\mu}_{t+1}\\ \widehat{\mu}_t\\ r_t\end{bmatrix}\right\}, \tag{D.9}$$

where

$$\begin{aligned}\widetilde{\mathbf{A}} \equiv& \begin{bmatrix}\frac{1+\beta-\delta_m\kappa}{\beta} - \frac{\delta_i\kappa}{\beta\chi_i} & \frac{-1}{\beta} & \frac{\kappa}{\beta} + \frac{\delta_i\chi_y\kappa}{\beta\chi_i} & 0\\ 1 & 0 & 0 & 0\\ \frac{1-\delta_m\kappa}{\beta\sigma} - \frac{1}{\sigma\chi_i} - \frac{\delta_i\kappa}{\beta\chi_i\sigma} & \frac{-1}{\beta\sigma} & 1 + \frac{\chi_y}{\sigma\chi_i} + \frac{\kappa}{\beta\sigma} + \frac{\delta_i\chi_y\kappa}{\beta\chi_i\sigma} & 0\\ 0 & 0 & 1 & 0\end{bmatrix} + \alpha_c\begin{bmatrix}\frac{-1}{\beta\sigma} & \frac{1}{\beta\sigma} & \frac{1}{\beta\sigma} & \frac{-1}{\beta\sigma}\\ 0 & 0 & 0 & 0\\ \frac{-1}{\beta\sigma} & \frac{1}{\beta\sigma} & \frac{1}{\beta\sigma} & \frac{-1}{\beta\sigma}\\ 0 & 0 & 0 & 0\end{bmatrix}\\ &+\frac{\alpha_c}{1-\alpha_c}\begin{bmatrix}\frac{1}{\sigma\chi_i} + \frac{(1-\sigma)\delta_m\kappa}{\beta\sigma} + \frac{(1-\sigma)\delta_i\kappa}{\beta\chi_i\sigma} & 0 & \frac{-(1-\sigma)}{\beta\sigma} - \frac{\chi_y}{\sigma\chi_i} - \frac{(1-\sigma)\kappa}{\beta\sigma} - \frac{(1-\sigma)\delta_i\chi_y\kappa}{\beta\chi_i\sigma} & \frac{1-\sigma}{\beta\sigma}\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\end{aligned}$$

$$\text{and}\quad \widetilde{\mathbf{B}} \equiv \begin{bmatrix}\frac{\delta_i\kappa}{\beta} & 1 & \frac{-1}{\beta} & 0\\ 0 & 0 & 0 & 0\\ \frac{1}{\sigma} + \frac{\delta_i\kappa}{\beta\sigma} & 0 & \frac{-1}{\beta\sigma} & \frac{-1}{\sigma}\\ 0 & 0 & 0 & 0\end{bmatrix} + \frac{\alpha_c}{1-\alpha_c}\begin{bmatrix}\frac{-1}{\sigma} + \frac{(1-\sigma)\delta_i\kappa}{\beta\sigma} & 1 & \frac{1-\sigma}{\beta\sigma} & \frac{1}{\sigma}\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}.$$

Since this system has two predetermined variables ($\widehat{m}_{t-1}$ and $\widehat{y}_{t-1}$) and two non-predetermined variables ($\widehat{m}_{t+1}$ and $\widehat{y}_{t+1}$), the necessary and sufficient condition for local-equilibrium determinacy is that the matrix $\widetilde{\mathbf{A}}$ have two eigenvalues inside the unit circle and two eigenvalues outside. We note that $\widetilde{\mathbf{A}}$ can be obtained from $\mathbf{A}$ by replacing q_P, q_y, κ, and δ_m respectively by $1/(1-\alpha_c)$, $\alpha_c/(1-\alpha_c)$, $\widetilde{\kappa} \equiv (1+\delta_i\chi_y/\chi_i)\kappa$, and $\widetilde{\delta}_m \equiv (\delta_m\chi_i+\delta_i)/(\chi_i+\delta_i\chi_y)$ in $\mathbf{A}$. Therefore, we deduce from Appendix C.2 that $\widetilde{\mathbf{A}}$ has two eigenvalues inside the unit circle and two eigenvalues outside whatever $\theta \in (0,1)$ if and only if

$$\widetilde{K} \equiv \frac{-1}{\sigma} + \left[1 + \left(\frac{\alpha_c}{1-\alpha_c}\right)\left(\frac{\sigma-1}{\sigma}\right)\right]\widetilde{\delta}_m < \frac{1-\widetilde{\delta}_m\chi_y}{2\sigma\chi_i\left(1-\alpha_c\right)}, \tag{D.10}$$

where the right-hand side is positive since $1-\widetilde{\delta}_m\chi_y = (1-\delta_m\chi_y)\chi_i/(\chi_i+\delta_i\chi_y) > 0$ (as follows from the second inequality in (26)). We have

$$\begin{aligned}
\widetilde{K} &< \frac{-1}{\sigma} + \frac{\widetilde{\delta}_m}{1-\alpha_c} \\
&< \frac{-1}{\sigma} + \frac{1}{1-\alpha_c}\left(\delta_m + \frac{\delta_i}{\chi_i}\right) \\
&< \frac{1}{\sigma}\left[-1 + \frac{1}{1-\alpha_c}\left(-\alpha_\ell\frac{\Gamma_{\ell m}m}{\Gamma_\ell} - \alpha_m\frac{\Gamma_{mm}m}{\Gamma_m}\right)\right] \\
&\leq \overline{K} \equiv \frac{1}{\sigma}\left[-1 + \frac{1}{1-\alpha_c}\left(-\alpha_\ell\frac{\Gamma_{\ell m}m}{\Gamma_\ell} + \alpha_m\frac{\Gamma_{\ell m}\ell}{\Gamma_m}\right)\right],
\end{aligned}$$

where the last inequality follows from (B.12). In turn, using first (6) and (7), and then (10) with equality and (D.3), we get sequentially

$$\begin{aligned}
\overline{K} &= \frac{1}{\sigma}\left[-1 + \left(\frac{1-\beta I^m}{1-\alpha_c}\right)\left(\frac{I}{I^\ell}\frac{\Gamma_{\ell m}m}{\Gamma_m} + \frac{1}{\beta I^m}\frac{\Gamma_{\ell m}\ell}{\Gamma_m}\right)\right] \\
&= \frac{1}{\sigma}\left\{-1 + \left(\frac{1-\beta I^m}{1-\alpha_c}\right)\left[\left[\left(\frac{\varepsilon-1}{\varepsilon}\right)\left(\frac{f'h}{f}\right) - \left(\frac{1-\phi}{\phi}\right)\left(\frac{\ell}{y}\right)\right]^{-1}\left(\frac{m}{\beta y}\right) + \frac{1}{\beta I^m}\right]\frac{\Gamma_{\ell m}\ell}{\Gamma_m}\right\} \\
&\leq \frac{1}{\sigma}\left\{-1 + \left(\frac{1-\beta I^m}{1-\alpha_c}\right)\left[\frac{1}{\beta}\left(\frac{\varepsilon}{\varepsilon-1}\right)\left(\frac{f}{f'h}\right)\left(\frac{m}{y}\right) + \frac{1}{\beta I^m}\right]\frac{\Gamma_{\ell m}\ell}{\Gamma_m}\right\}.
\end{aligned}$$

Now consider the following condition, which states that the last expression is negative:

$$\left(\frac{1-\beta I^m}{1-\alpha_c}\right)\left\{\frac{1}{\beta}\left(\frac{\varepsilon}{\varepsilon-1}\right)\left[\frac{f(h)}{hf'(h)}\right]\left(\frac{m}{y}\right) + \frac{1}{\beta I^m}\right\}\left[\frac{\ell\Gamma_{\ell m}(\ell,m)}{\Gamma_m(\ell,m)}\right] < 1. \tag{D.11}$$

This condition implies $\overline{K} < 0$, which in turn implies $\widetilde{K} < 0$, which in turn implies (D.10), which in turn implies that $\widetilde{\mathbf{A}}$ has two eigenvalues inside the unit circle and two eigenvalues outside, which finally implies determinacy.

D.3 Assessment of the Sufficient Determinacy Condition

To assess whether Condition (D.11) is likely to be met or not, we proceed broadly along the same lines as in Appendix C.3. We consider, for simplicity, a Cobb-Douglas specification for the production function f^b:

$$f^b\left(h_t^b, m_t\right) \equiv A_b\left(h_t^b\right)^{1-\varsigma}(m_t)^\varsigma,$$

where $A_b > 0$ and $0 < \varsigma < 1$. This specification implies that the steady-state elasticity of Γ_m that appears in Condition (D.11) can be rewritten as

$$\frac{\ell\Gamma_{\ell m}(\ell,m)}{\Gamma_m(\ell,m)} = \left(\frac{1}{1-\varsigma}\right)\left[1 + \frac{\upsilon^{b\prime\prime}\left(h^b\right)h^b}{\upsilon^{b\prime}\left(h^b\right)}\right].$$

Our Cobb-Douglas specification for f^b also implies that households' first-order conditions (6) and (7), at the steady state, can be combined to get

$$\varsigma = \left(\frac{m}{\ell}\right)\left(\frac{I - I^m}{I^\ell - I}\right).$$

Therefore, Condition (D.11) can be rewritten as

$$\left(\frac{I - I^m}{1 - \alpha_c}\right)\left\{\left(\frac{\varepsilon}{\varepsilon - 1}\right)\left[\frac{f(h)}{hf'(h)}\right]\left(\frac{m}{y}\right) + \frac{1}{I^m}\right\}\left[\frac{1 + \frac{v^{b\prime\prime}(h^b)h^b}{v^{b\prime}(h^b)}}{1 - \left(\frac{m}{\ell}\right)\left(\frac{I - I^m}{I^\ell - I}\right)}\right] < 1. \tag{D.12}$$

We set the steady-state variables I^m, I^ℓ, m/ℓ, α_c, and m/y to match some features of the US economy during the 2008-2015 ZLB episode. More specifically, as in Appendix C.3, we set the net interest rates $I^m - 1$ and $I^\ell - 1$ to 0.25% and 3.25% per annum respectively, and the ratio m/ℓ to 0.18. We set the steady-state share of household cash in the monetary base, α_c, to 0.39, which is the average value of the ratio between the currency component of M1 and the monetary base from January 2009 to November 2015. And we set the ratio m/y to 0.40, which is the average value of the ratio between total reserves of depository institutions and quarterly GDP from 2009Q1 to 2015Q4.

We make standard assumptions about the steady-state elasticity of output to labor $hf'(h)/f(h)$ and the elasticity of substitution between goods ε. More specifically, we set the former to 0.66, and the latter to 6 (implying a 20% markup).

Finally, we make the same conservative assumptions as in Appendix C.3 about the values of the steady-state variables $h^b v^{b\prime\prime}(h^b)/v^{b\prime}(h^b)$ and I. More specifically, we set the elasticity $h^b v^{b\prime\prime}(h^b)/v^{b\prime}(h^b)$ to 5, and the net interest rate $I - 1$ to 0.75% per annum. These assumptions are conservative because they tend to overestimate these two steady-state variables and, therefore, to make Condition (D.12) harder to satisfy.

The left-hand side of Condition (D.12) depends on the period length through the ratio $(I - I^m)/I^m$ (and only through this ratio, since the ratio $(I - I^m)/y$ does not depend on the period length). We set the period length to one quarter, as in Appendix C.3. Thus, we express all the interest rates in the ratio $(I - I^m)/I^m$ as quarterly rates.

We then get the value 0.02 for the left-hand side of Condition (D.12). This value is one to two orders of magnitude smaller than 1. We thus find that Condition (D.12) is met by a large margin even under our conservative assumptions. We conclude that setting exogenously the IOR rate and the monetary base still delivers local-equilibrium determinacy, except for implausible calibrations, when the monetary base is made of both bank reserves and household cash.

Appendix E: Extended Model With Liquid Government Bonds

In this appendix, which is devoted to our extended model with liquid government bonds, we first characterize our equilibrium of interest (discussed in Subsection 6.5). Next, we show the existence of this equilibrium under two parameter restrictions. Finally, we show how one parameter restriction can be relaxed without affecting our results.

E.1 Characterization of our Equilibrium of Interest

Our equilibrium of interest in the model with liquid bonds has $b_t^b = \lambda_t^r = 0$ and $I_t^b < I_t^m$. We show that this equilibrium, if it exists, coincides with the equilibrium of our benchmark model (without liquid bonds), in the sense that all the endogenous variables that are common to both models, except the lump-sum transfer T_t, take the same values in both equilibria. To that aim, we first use (33) to rewrite households' optimality conditions (34), (35), (36), and (37) in the following simpler forms:

$$\frac{I_t^b}{I_t} = 1 - \frac{\zeta_t z'\left(b_t^w\right)}{\lambda_t}, \tag{E.1}$$

$$\frac{I_t^\ell}{I_t} = 1 + \frac{\zeta_t \Gamma_\ell\left(\ell_t, m_t + \eta b_t^b\right)}{\lambda_t} + \psi \frac{\lambda_t^r}{\lambda_t}, \tag{E.2}$$

$$\frac{I_t^m}{I_t} = 1 + \frac{\zeta_t \Gamma_m\left(\ell_t, m_t + \eta b_t^b\right)}{\lambda_t} - \frac{\lambda_t^r}{\lambda_t}, \tag{E.3}$$

and

$$\frac{I_t^b}{I_t} \leq (1-\eta) + \eta \frac{I_t^m}{I_t} + \eta \frac{\lambda_t^r}{\lambda_t}. \tag{E.4}$$

In our equilibrium of interest, because $b_t^b = \lambda_t^r = 0$, the equilibrium conditions (E.2) and (E.3) collapse to

$$\frac{I_t^\ell}{I_t} = 1 + \frac{\zeta_t \Gamma_\ell\left(\ell_t, m_t\right)}{\lambda_t} \quad \text{and} \quad \frac{I_t^m}{I_t} = 1 + \frac{\zeta_t \Gamma_m\left(\ell_t, m_t\right)}{\lambda_t}.$$

These conditions are identical to the equilibrium conditions (6) and (7) of our benchmark model. Therefore, our equilibrium of interest satisfies all the equilibrium conditions of our benchmark model (listed in Section 2), except the government budget constraint (14), which is replaced by (38). As a consequence, all the endogenous variables that are present in both models take the same values in our equilibrium of interest as in the equilibrium of our benchmark model, except the lump-sum transfer T_t appearing in the government budget constraint.

E.2 Existence of our Equilibrium of Interest

We prove the existence of our equilibrium of interest under two parameter restrictions. The first restriction is

$$\psi < \overline{\psi}, \tag{E.5}$$

where $\overline{\psi}$ denotes the steady-state value of the reserves-to-loans ratio m_t/ℓ_t in our benchmark model (without liquid bonds). As we will see, this restriction will ensure that the reserve requirement (32) is not binding in our model with liquid bonds. The second restriction is

$$\max\left\{1, \frac{1}{\beta} - \frac{z'(b^*)}{\beta\overline{\lambda}}\right\} < I^m < \frac{1}{\beta}, \tag{E.6}$$

where $\overline{\lambda}$ denotes the upper bound of the steady-state values taken by the marginal utility of consumption λ_t as I^m varies from 1 to $1/\beta$ in our benchmark model (this upper bound being reached for $I^m = 1$). As we will see, that restriction will ensure that the interest rate on government bonds I_t^b is lower than the IOR rate I_t^m in our model with liquid bonds. In fact, that restriction will turn out to be sufficient but not necessary for $I_t^b < I_t^m$; for simplicity, we relegate to Appendix E.3 the statement of the (more complex) parameter restriction that is necessary and sufficient for $I_t^b < I_t^m$.

We proceed in two steps: we show first the existence of our *steady-state* equilibrium of interest, and then the existence of our *dynamic* equilibrium of interest. In the first step, to show that our model with liquid bonds, under the parameter restrictions (E.5) and (E.6), has a steady-state equilibrium with $I^b < I^m$ and $b^b = \lambda^r = 0$, we start from a candidate steady-state equilibrium with $b^b = \lambda^r = 0$. In this candidate equilibrium, as follows from the analysis above, all the endogenous variables that also appear in our benchmark model, except the lump-sum transfer T_t, take the same steady-state values as in that model. Using these values and $b^b = 0$, we then get residually the steady-state values of the other endogenous variables: (i) b^w and B from the market-clearing condition (39) and the steady-state target $B/P = b^*$; (ii) I^b from the first-order condition (E.1); and (iii) T from the consolidated budget constraint of the government (38).

At this stage, all equality conditions for steady-state equilibrium are satisfied, and the steady-state value of all endogenous variables is pinned down. What remains to be shown is that: (i) the inequality conditions for steady-state equilibrium, i.e. the steady-state versions of (32) and (E.4), are satisfied as strict inequalities, implying that the candidate steady-state equilibrium is indeed a steady-state equilibrium; and (ii) this equilibrium has the property that $I^b < I^m$. We first establish this last inequality by using in turn the first-order condition (E.1) with $I = 1/\beta$ and $b^w = b^*$, the inequality $\lambda \leq \overline{\lambda}$, and the parameter restriction (E.6), to get

$$I^b = \frac{1}{\beta} - \frac{z'(b^*)}{\beta\lambda} \leq \frac{1}{\beta} - \frac{z'(b^*)}{\beta\overline{\lambda}} < I^m.$$

In turn, the property $I^b < I^m$, together with $I^m < I$ and $\lambda^r = 0$, implies that the steady-state version of (E.4) is satisfied as a strict inequality:

$$\frac{I^b}{I} < (1-\eta) + \eta\frac{I^m}{I} + \eta\frac{\lambda^r}{\lambda}.$$

Finally, the parameter restriction (E.5) straightforwardly implies that the steady-state version of (32) is satisfied as a strict inequality. We conclude that our model with liquid bonds does indeed have a steady-state equilibrium with $I^b < I^m$ and $b^b = \lambda^r = 0$ that coincides with the steady-state equilibrium of our benchmark model. In this equilibrium, banks hold only reserves ($b^b = 0$) because they pay more interest than government bonds ($I^b < I^m$) and are at least as liquid as government bonds ($\eta \leq 1$).

In the second step, we proceed similarly to show the existence of a dynamic equilibrium with $I_t^b < I_t^m$ and $b_t^b = \lambda_t^r = 0$. More specifically, we start from a candidate equilibrium with $b_t^b = \lambda_t^r = 0$. In this candidate equilibrium, as follows from the analysis above, all the endogenous variables that also appear in our benchmark model, except the lump-sum transfer T_t, take the same equilibrium values as in that model. Using these values and $b_t^b = 0$, we then get residually the equilibrium values of the other endogenous variables (expressed as log-deviations from their steady-state values, and denoted by letters with hats): (i) $\widehat{b}_t^w$ and $\widehat{B}_t$ from the log-linearized version of the market-clearing condition (39) and the fiscal-policy rule; (ii) $\widehat{I}_t^b$ from the log-linearized version of the first-order condition (E.1); and (iii) $\widehat{T}_t$ from the log-linearized version of the government's consolidated budget constraint (38).

At this stage, all equality conditions for equilibrium are satisfied, and the equilibrium value of all endogenous variables is pinned down. What remains to be shown is that: (i) the inequality conditions for equilibrium, i.e. (32) and (E.4), are satisfied as strict inequalities, implying that the candidate equilibrium is indeed an equilibrium; and (ii) this equilibrium has the property that $I_t^b < I_t^m$. Now, we have just shown that these three strict inequalities are satisfied at the steady state; therefore, by continuity, they are also satisfied in the neighborhood of this steady state, under the standard assumption of small enough shocks. As a consequence, our model with liquid bonds does indeed have a dynamic equilibrium with $I_t^b < I_t^m$ and $b_t^b = \lambda_t^r = 0$, and this equilibrium coincides with the dynamic equilibrium of our benchmark model.

E.3 Relaxation of a Parameter Restriction

To prove the existence of our equilibrium of interest in Appendix E.2, we have used the parameter restriction (E.6), which involves the reduced-form parameter $\overline{\lambda}$. This restriction, however, can be harmlessly relaxed to some extent, because our proof only rests on the weaker condition

$$\max\left\{1, \frac{1}{\beta} - \frac{z'(b^*)}{\beta\lambda}\right\} < I^m < \frac{1}{\beta}, \tag{E.7}$$

where the steady-state value λ depends on several parameters of the model – in particular β and I^m, but not b^*. To re-state (E.7) as a condition involving only parameters, we write λ as

$$\lambda = \Lambda(\beta I^m),$$

where the function Λ is defined by

$$\Lambda(x) \equiv u'\left\{f\left[\mathcal{F}^{-1}(x-1)\right]\right\}$$

for $x \in (-\infty, 1]$, where in turn the function $\mathcal{F}$ is defined in Subsection 3.2. Given the properties of $\mathcal{F}$, the function Λ is strictly decreasing ($\Lambda' < 0$), with $\lim_{x \to -\infty} \Lambda(x) = +\infty$. Therefore, there exists a unique $x^* \in (-\infty, 1)$ such that

$$1 - \frac{z'(b^*)}{\Lambda(x^*)} = x^*.$$

We can then re-state (E.7) as

$$\max\left\{1, \frac{x^*}{\beta}\right\} < I^m < \frac{1}{\beta},$$

where the reduced-form parameter x^* depends on several parameters of the model – in particular b^*, but not β nor I^m.

References

[1] Galí, J. (2015), *Monetary Policy, Inflation, and the Business Cycle: An Introduction to the New Keynesian Framework and its Applications*, Princeton: Princeton University Press, second edition.