Optimal Auctions for Dual Risk Averse Bidders: Myerson meets Yaari

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Abstract

We derive the revenue maximizing mechanism for a risk-neutral seller who faces Yaari's [1987] dual risk-averse bidders. The optimal mechanism offers "full-insurance" in the sense that each agent's utility is independent of other agents' reports. The seller excludes less types than under risk neutrality, and awards the object randomly to intermediate types. Subjecting intermediate types to a risky allocation while compensating them when losing allows the seller to collect larger payments from higher types. Relatively high types are anyway willing to pay more, and their allocation is efficient. Finally, a first-price auction maximizes revenue within the class of standard auctions.

1 Introduction

We introduce non-linear probability weighting to the analysis of optimal auctions. Specifically, we derive the revenue maximizing auction for a risk-neutral seller who faces (risk-averse) bidders with preferences respecting the axioms underlying Yaari's [1987] dual

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theory of choice under risk. The informational assumptions are standard, and follow the independent, private values paradigm.

Yaari's dual theory replaces the classical von Neumann-Morgenstern independence axiom behind the expected utility (EU) functional with another axiom concerning mixtures of comonotonic random variables. The mixture is along the payoff axis instead of the probability axis - hence the name dual - and the resulting utility functional uses a non-linear function to asses probabilities.[1] Equivalently, it weights each payoff by a weight that is decreasing in the size of the payoff, and thus Yaari's functional is a *rank-dependent* (or anticipated) utility functional a la Quiggin [1982]. Among other desirable properties, it disentangles attitudes towards risk from the marginal utility of money, that is constant. This property makes it rather appealing for auction settings where stakes are moderate: linearity of the bidders' utilities in monetary transfers can then coexist with any degree of risk aversion.

Our first main result is that, with dual risk-averse bidders, the search for an optimal procedure can be confined to the class of full-insurance mechanisms, where the utility of an agent is only a function of his type: it depends neither on the types of other agents, nor on the realization of other randomizations within the mechanism. In particular, this means that losing buyers must be compensated in order to make them indifferent to winning. The main technical insight behind this result is that comonotonic random variables have the largest sum in the convex stochastic order (and hence the highest variability) within the class of random variables with given marginals (see Meilijson and Nadas [1979]). We use this insight to show that, when agents are dual risk-averse, the incentive to deviate is minimal in a full-insurance mechanism among the class of all mechanisms which provide the agent with a given interim value.

With dual risk-averse bidders, the expected revenue becomes a concave function of the reduced-form allocation, and its maximization must be approached by variational methods. We first derive the optimal mechanism for a single bidder: instead of a classical take-it-or-leave offer for a risk-neutral buyer, we find that the seller awards the object randomly to intermediate types. Subjecting these types to a risky lottery while compensating them when they do not get the object allows the seller to collect larger payments from higher types, which is ultimately profitable. High types receive the object with probability one, as distorting their allocation is too costly. This is related to a monopolistic screening

[1] EU assumes that if the random variable X is preferred to X' then receiving X with probability $\frac{1}{2}$ and Y with probability $\frac{1}{2}$ is preferred to the equivalent mixture of X' and Y. Yaari preferences assume that if X is preferred to X' and Y is comonotonic with X and X' then getting $\frac{1}{2}$ the payoff of X and $\frac{1}{2}$ the payoff of Y is preferred to the equivalent weighted sum of payoffs of X' and Y.

problem a la Mussa and Rosen [1978][2]: even if the cost of producing any quality is zero, it is sometimes optimal for the monopolist to sell intermediate types a "damaged" good (that sometimes malfunctions) plus a full-insurance warranty. This class of contracts is consistent to observations from real-life insurance markets where even moderate risks are often fully insured: Cohen and Einav [2007] (house insurance) and Sydnor [2010] (car insurance), among others, empirically show that assuming EU yields implausibly large measures of risk parameters for a range of moderate risks. Most customers in their studies purchase low deductibles - de facto warranties - despite costs that are significantly above the expected value.

We next consider the n-agent allocation problem. Since the expected revenue is here concave in the probability with which an agent of a given type receives an object, it follows that, in symmetric n-bidder settings, the optimal mechanism is symmetric. The main complication of the n-bidder case relative to the single-bidder case is the feasibility constraint, binding across types, that restricts the reduced form allocation, i.e., the expected probability of obtaining an object for each bidder type (see Border [1991]).[3] Incorporating this constraint in the maximization exercise yields a variational *obstacle* problem that can be solved via the infinite-dimensional version of the Kuhn-Tucker theorem (see Luenberger [1997]).

The optimal allocation has features similar to that in the single-bidder case: in particular, even when the object is allocated, it is not always allocated efficiently, and payments are computed to yield full insurance. Moreover, the expected revenue increases when bidders become more risk-averse and when there are more bidders.

A well-known example of auctions where (some) losers are compensated are the so-called *premium* auctions: in such auctions the seller rewards one, or more, high losing bidders. Premium auctions have been around since the Middle Ages, and are still used today to sell houses, land, large equipment (e.g., boats, planes, machines) and inventories of insolvent businesses (see Goeree and Offerman [2004]). Although in practice not all losers who bid above a threshold are compensated (as would be required in our full insurance mechanisms), we note that the implied compensation in our mechanisms for low and intermediate types is relatively small because their chance of winning is also small. Thus, in our optimal mechanism only high-type losing bidders get substantial compensation, which is broadly consistent with the practice of premium auctions.

Since full-insurance mechanisms as described above may be sometimes difficult to im-

[2]This problem was analyzed by Matthews and Moore [1987] for a risk-averse agent with EU preferences.

[3]The current maximization problem is concave. See Gershkov et al. [2019] for the analysis of a convex revenue maximization problem (with the same obstacle) via the Fan-Lorentz integral inequality.

plement in practice, we also look at standard auctions where the seller cannot compensate losing bidders and where, conditional on allocating the object, the allocation is efficient. Within this standard class we show that a first-price auction with a reserve price is optimal (while other standard auction formats are not!).

In the first-price auction a bidder is uncertain whether he will win the object, a feature shared by all standard auctions. But, he does not have to pay in case of losing, as in an all-pay auction. Moreover, conditional on winning, the price is deterministic and equals his bid, contrasting the random price determined by other bidders, as in a second-price auction. These risk-reducing features yield the optimality of the first-price auction when facing dual risk-averse bidders.

1.1 Related Literature

There is ample empirical evidence both from stock markets (Kliger and Levy [2009]) and sport bets studies (see Snowberg and Wolfers [2010], and Andrikogiannopoulou and Papakonstantinou [2013]) that agents do use non-linear probability weighting.

Barseghyan et al [2016] find that stable Yaari and rank-dependent utility preferences cannot be rejected for the majority of households in a data set of car and home-insurance choices.[4] Several laboratory experiments also illustrated similar findings(see Bruhin, Fehr-Duda and Epper [2010] and, in particular, Goeree, Halt and Palfrey [2002] who find support for a quadratic probability weighting.)

Guriev [2001] offers a "microfoundation" for Yaari preferences. He shows that a risk neutral (expected utility) agent who faces a bid-ask spread in the credit market will behave as if he were a dual risk averse decision maker: bad outcomes where he will need to borrow are more heavily weighted than good outcomes where he will be able to save. The same happens for an agent that is linearly taxed on gains but not on losses.

The symmetric, independent and private values auction model with bidders whose preferences respect Yaari's axioms has been first analyzed by Volij [2002]. He studied standard auctions and established a "payoff equivalence" result for a class of mechanisms with deterministic allocation and transfers that includes the first-price, second-price and all-pay auctions. It is important to note that, with risk-averse buyers (stemming either from EU or from Yaari preferences), payoff equivalence for bidders does not imply revenue equivalence, and the latter does *not* hold. Moreover, when bidders use non-linear

[4] Looking at households that purchase property insurance Barseghyan et al [2011] reject the hypothesis that subjects have stable expected utility preferences for more than 3/4 of the households.This finding is confirmed for insurance coverage and 401(k) investment decisions in Einav et al[2012] .

probability weighting, the expected revenue of a mechanism implementing a given reduced form allocation rule is non-linear in the allocation probabilities. The combination of these two features requires an approach to revenue maximization that is different from the classical one under risk neutrality due to Myerson [1981] and Riley and Samuelon [1983]. There payoff equivalence implies revenue equivalence, and, moreover, revenue is linear in allocation probabilities. Even under standard expected utility preferences, the derivation of the revenue-maximizing auction mechanism is very complex unless bidders are risk-neutral and an explicit solution to the general problem is not known (see Maskin and Riley [1984]).

Almost all papers studying auctions where bidders have non-expected utility typically compare the performance of specific selling formats, e.g., the early contributions of Neilson [1984] and Karni and Safra [1989] and, more recently, Che and Gale [2006]. This last paper shows that, for a large class of risk-averse bidder preferences, including Yaari's, a first price auction yields a higher revenue than a second-price auction. These authors do not discuss optimal mechanisms.

The optimality of full-insurance mechanisms appears in the context of auctions with ambiguity-averse bidders (Bose et al. [2006])[5]. The maximization problem they solve is linear in probabilities, and thus the optimal auction in their framework is obtained by standard methods a la Myerson.

Similarly, most of the papers investigating auctions with risk-averse bidders (e.g. Matthews [1987], Baisa [2017]) do not aim to provide a characterization of the optimal mechanism. Revenue maximization with risk averse buyers under expected utility has been studied by Maskin and Riley [1984] and by Matthews [1983]. Matthews [1983] restricts attention to constant absolute risk aversion, and finds that the optimal mechanism resembles a modified first-price auction where the seller sells (partial) insurance to bidders with high valuation, but charges an entry fee to bidders with low valuation. Maskin and Riley [1984] allow for more general risk averse (EU) preferences and establish several properties of an optimal auction. In particular, they show that full insurance need not be optimal.

Maskin and Riley [1984] also show quite generally that, when buyers are risk averse the first-price auction yields more revenue than the second-price (or English) auction, but they do not discuss the optimality of the first-price auctions within the class of standard (or other) auctions.

[5]Insurance also plays a role in the robust mechanisms discussed by Bierbrauer and Netzer [2016] in settings where agents also care about intentions.

The literature on premium auctions identified several potential reasons for their use. Milgrom [2004] and Goeree and Offerman [2004] suggest that a premium auction format is used to encourage weak bidders to compete against strong bidders, and thus to increase the seller's revenue. Hu, Offerman and Zou [2011] studied a two-stage English premium auction model with symmetric, interdependent values, and showed that the use of premium is only profitable to the seller when bidders are risk-loving. Hu, Offerman and Zou [2017] showed that, if both the seller and the bidders are risk-averse, premium auctions allow risk sharing that may benefit all participants. All identified reasons where premium auctions may be beneficial (bidder asymmetry, risk loving-bidders, risk sharing between buyers and seller) are thus quite different from our insight that a risk-neutral seller can use premium auctions to provide insurance to risk-averse bidders.

The paper is organized as follows. In Section 2 we briefly review Yaari's dual theory. In Section 3 we introduce the auction model and feasible mechanisms. We show that the quest for an optimal mechanism can be restricted to the class of full-insurance mechanisms. In Section 4 we derive the revenue-maximizing mechanism for settings with a single bidder. The general $n-$bidder case is treated in Section 5, where we also derive several comparative statics results. In Section 6 we show that the first-price auction with a reserve price is optimal in the class of standard auctions (that do not offer insurance). Section 7 concludes.

2 Yaari's Dual Theory of Choice Under Risk

We briefly review here Yaari's theory. An appealing feature of the dual theory is that, unlike expected utility, attitudes towards risk are not entangled with attitudes towards wealth: the marginal utility of wealth is constant, but this is consistent with any attitude towards risk.

Let X be the set of all random variables defined on some given probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with values in the interval $[k, l]$, where $k, l \in (-\infty, \infty)$ and $k < l$. The values of the random variables are interpreted as payments, and each random variable is interpreted as a lottery. For any two random variables $x, y \in X$ and for any $\alpha \in [0, 1]$, let $\alpha x + (1-\alpha) y$ denote the random variable that equals the weighted sum $\alpha x(\omega) + (1-\alpha) y(\omega)$ in every state $\omega \in \Omega$.[6]

Two random variables, x and y, are said to be *comonotonic* if and only if for every

[6]This is NOT the probability mixture of x and y!

pair of states, ω and ω', it holds that

$$(x(\omega) - x(\omega')) \cdot (y(\omega) - y(\omega')) \geq 0.$$

For each $x \in X$, define the *decumulative* distribution function G_x by

$$G_x(s) = \mathbb{P}[x > s],\ k \leq s \leq 1.$$

Note that $\int_k^l G_x(s)ds + k = \mathbb{E}[x]$, and that the function G_x is non-increasing, right-continuous, and satisfies $G_x(1) = 0$.

Axioms Yaari imposed four axioms on a complete, weak order $\succsim$ over X:

1. *Neutrality*: If $G_x = G_y$, then $x \sim y$.
2. *Continuity* with respect to L_1 convergence.
3. *Monotonicity* : If $G_x \geq G_y$ then $x \succsim y$.
4. *Dual Independence*: If x, z, y are pairwise comonotonic, and if $x \succsim y$ then $\alpha x + (1-\alpha)z \succsim \alpha y + (1-\alpha)z$ for any $0 \leq \alpha \leq 1$.

Neutrality says that agents are indifferent between any two random variables that share the same outcome distribution. Continuity is a technical requirement, while monotonicity reflects classical (first-order) stochastic dominance. Dual independence is the *only* departure from the axiomatization of expected utility theory: instead of postulating independence for convex combinations formed along the probability axis, it is now postulated for convex combinations along the payment axis (hence the name *dual*). Yaari proved the following fundamental result:

Theorem 1 (Yaari, 1987) *A complete preference relation satisfies the above axioms if and only if there exists a continuous, non-decreasing function g, defined on the unit interval, such that, for any $x, y \in X$,*

$$x \succsim y \Leftrightarrow \int_k^l g(G_x(s))ds \geq \int_k^l g(G_y(s))ds.$$

Moreover, the function g is unique up to a positive affine transformation and can be selected such that, for all $0 \leq p \leq 1$,

$$[1; p] \sim [g(p); 1] \tag{1}$$

where $[s;p]$ denotes the lottery that yields s with probability p and the lower bound k with probability $1-p$.

Since it takes probabilities as an input, g is a "probability-evaluation" function rather than an utility function. But, given the above result, it can be said that the function g represents the agent's preferences on lotteries. Throughout, we denote by g the representation of the agents' preferences that satisfies (1), and by

$$\mathcal{U}(x) = k + \int_k^l g(G_x(s))ds$$

the utility from a random variable $x \in X$.

The theorem below, which is also due to Yaari, gathers several important implications repeatedly used below.

Theorem 2 (Yaari, 1987) *Consider a decision maker (DM) with Yaari preferences represented by g:*

1. *The DM is indifferent between the lottery x and getting the amount $\mathcal{U}(x)$ with certainty.*[7]

2. *For any comonotonic random variables $x, y \in X$ and any $a > 0, b \in \mathbb{R}$ it holds that*[8]

$$\mathcal{U}(x+y) = \mathcal{U}(x) + \mathcal{U}(y)$$
$$\mathcal{U}(ax+b) = a\,\mathcal{U}(x) + b$$

3. *A DM is risk averse if and only if g is convex.*[9]

4. *A DM with preferences represented by g_1 is more risk averse than a DM with preferences represented by g_2 if and only if there exists a convex function h, defined on the unit interval, such $g_1 = h \circ g_2$.*

[7]Thus, the DM choses among random variables as if he maximizes $\mathcal{U}$.

[8]Strictly speaking $\mathcal{U}(x+y)$ is not well defined since the random variable $x+y$ is not necessarily bounded to $[k,l]$. The generalization of Yaari's theory to unbounded random variables with finite means is in Guriev [2001]. The only additional axiom that needs to be imposed is that each random variable is bounded, from above and from below, by some constants (in the underlying order assumed on lotteries).

[9]Recall that a general definition of risk aversion is the aversion to mean-preserving spreads: a preference $\succsim$ is risk averse if $x \succ y$ whenever G_y is a mean-preserving spread of G_x. This is equivalent to $y \geq_{cx} x$, where cx denotes the convex stochastic order.

Note that, by integration by parts (assuming the necessary conditions), we also obtain that[10]

$$\mathcal{U}(x) = \mathbb{E}\left[xg'(G_x(x))\right]$$

Yaari's risk-averse dual utility functional can be thus seen as multiplying each payoff x by a weight $g'(G_x(x))$ that is decreasing in the quantile $1 - G_x(x)$ it has in the payoff distribution. This contrasts with standard expected utility where the DM maximizes

$$\mathbb{E}[u(x)]$$

and thus distorts payoffs according to a function u, based on their absolute value instead of associated quantile. Quiggin's [1982] general *rank-dependent* (or anticipated) utility goes one step further and allows for both a non-linear assessment of payoffs an a quantile dependent weighting.

Finally, a very useful probabilistic result needed for our treatment of incentive compatibility in auctions is a variability maximizing property of sums of comonotonic random variables:[11]

Theorem 3 (Meilijson and Nadas, 1979) *If the random vector $(y_1, ..., y_N)$ is comonotonic and has the same marginals as $(x_1, ..., x_N)$ then $\sum_{i=1}^N y_i \geq_{cx} \sum_{i=1}^N x_i$, where cx denotes the convex stochastic order.*

The following useful corollary follows immediately from Theorem 2-2 and Theorem 3:

Corollary 1 *If an agent with dual preferences is risk-averse, i.e. g is convex, then*

$$\mathcal{U}(x+y) \geq \mathcal{U}(x) + \mathcal{U}(y)\,.$$

for any two random variables $x, y \in X$.

3 The Auction Model

A risk-neutral seller has an indivisible object, and there are $n \geq 1$ potential buyers. The valuation (or type) of bidder i, $\theta_i \in [0,1]$, is drawn according to a distribution F_i with density $f_i > 0$, independently of other bidders' valuations. We assume that all distributions F_i are twice continuously differentiable.

[10] $\mathcal{U}(x) = k + \int_k^l g(G_x(s))ds = \int_k^l sg'(G_x(s))d(-G_x(s)) = \mathbb{E}\left[xg'(G_x(x))\right]$.

[11] See Kaas et al. [2002] for an intuitive proof.

Bidder i's dual preferences are represented by a convex, strictly increasing function $g_i : [0, 1] \to [0, 1]$. Hence, we follow the independent, private value (IPV) paradigm, but we allow for the bidders to be risk-averse in the sense of Yaari's dual theory.

For each bidder i, we use below the normalization $g_i(0) = 0$ (so that the utility from getting the object with probability zero and making a payment of zero is also zero), and $g_i(1) = 1$ (so that the utility of an agent with valuation θ_i who gets the object with certainty and pays t_i is given by $\theta_i - t_i$).

Note that if g is only non-decreasing, then g may be constant (and equal to zero) only at the bottom of the interval $[0, 1]$. The mechanism characterized below will continue to be optimal, but need not be unique.

The special case of risk-neutral bidders corresponds to $g_i(p) = p$ - this also coincides with standard risk neutrality under EU.

3.1 Mechanisms

In order to formally model both random allocations and random transfers that may depend on the realized allocation of the object, it will be useful to explicitly specify the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which these random variables are defined. We assume that all randomness in the mechanism is derived from a single random number $r \in [0, 1]$ that is drawn in addition to the draws of individual types.[12]

We denote by $\omega = (\boldsymbol{\theta}, r) \in \Omega = [0, 1]^{n+1}$ a realization of types and of the random number. We denote by $\mathbb{P}$ the probability measure on Ω defined by drawing θ_i independently according to F_i ,and r independently from the uniform distribution on $[0, 1]$. We denote by $\mathbb{E}$ the associated expectation operator.

We restrict attention to direct mechanisms where each agent i only reports her type θ_i. This is without loss of generality even for agents with non-expected utility preferences as long as the designer is either restricted to static mechanisms, or as long as each agent is sophisticated and can commit to a strategy in the mechanism.[13] We make this assumption to rule out dynamic mechanisms that exploit the agents' time-inconsistency.[14]

A *direct mechanism* $(\boldsymbol{q}, \boldsymbol{t})$ specifies for each agent i an *allocation rule* $q_i : [0, 1]^n \to [0, 1]$ and a *transfer* $t_i : [0, 1]^n \times [0, 1] \to [-m, m]$.[15] We require both q_i and t_i to be measurable

[12] This is without loss of generality by the general results of Halmos and von Neumann [1942].

[13] Under either assumption, each type of an agent can commit to follow the strategy of another type. This means that incentive compatibility of the original mechanism implies incentive compatibility of the direct mechanism implementing the same allocation and transfers. This is, for example, discussed in Bose & Daripa [2009].

[14] See Machina (1989) for an excellent discussion of this issue.

[15] We need to impose an upper bound on the transfer to ensure that a bidder's utility is bounded from

so that both allocation and transfer are well-defined random variables. To complete the description of the physical allocation, we define n non-overlapping sub-intervals of the unit interval, one for each agent i, by

$$W_i(\boldsymbol{\theta}) = \left[\sum_{j=1}^{i-1} q_j(\boldsymbol{\theta}), \sum_{j=1}^{i} q_j(\boldsymbol{\theta})\right) . \tag{2}$$

Agent i receives an object if and only if $r \in W_i(\boldsymbol{\theta})$, and pays the transfer $t_i(\boldsymbol{\theta}, r)$. Note that, conditional on the vector of types $\boldsymbol{\theta}$, the probability with which agent i receives the good, is

$$\mathbb{P}[r \in W_i(\boldsymbol{\theta}) \mid \boldsymbol{\theta}] = q_i(\boldsymbol{\theta}) .$$

Furthermore, for any realization of $(\boldsymbol{\theta}, r)$, at most one agent receives the object. Note also that, since it depends on the random number r, the transfer t_i of agent i may be random (even conditional on the agents' types θ and on the allocation of the good!)

Fix now a mechanism $(\boldsymbol{q}, \boldsymbol{t})$. The *ex-post payoff* $u_i : [0,1]^{n+2} \to [-m, 1+m]$ of agent i with type θ_i who reports that he has type θ_i' while all other agents report types $\boldsymbol{\theta_{-i}}$ is given by

$$u_i(\theta_i, \theta_i', \boldsymbol{\theta}_{-i}, r) = \mathbf{1}_{\{r \in W_i(\theta_i', \theta_{-i})\}} \theta_i - t_i(\theta_i', \boldsymbol{\theta}_{-i}, r) .$$

We slightly abuse notation by using $u_i(\boldsymbol{\theta}, r) = u_i(\theta_i, \theta_{-i}, r_i)$ instead of $u_i(\theta_i, \theta_i, \boldsymbol{\theta}_{-i}, r_i)$ for the case where agent i is truthful. Note that

$$u_i(\theta_i, \theta_i', \boldsymbol{\theta}_{-i}, r) = u_i(\theta_i', \boldsymbol{\theta}_{-i}, r) + \mathbf{1}_{\{r \in W_i(\theta_i', \theta_{-i})\}} (\theta_i - \theta_i') .$$

Assume that all agents other than i report truthfully, and that agent i has type θ_i, but reports type θ_i'. Then i's *dual utility* $V_i : [0,1]^2 \to [-m, 1+m]$ is given by

$$V_i(\theta_i, \theta_i') = -m + \int_{-m}^{1+m} g_i(\mathbb{P}[u_i(\theta_i, \theta_i', \boldsymbol{\theta}_{-i}, r) \geq s \mid \theta_i]) ds .$$

We again slightly abuse notation by using $V_i(\theta_i)$ instead of $V_i(\theta_i, \theta_i)$.

A mechanism $(\boldsymbol{q}, \boldsymbol{t})$ is *incentive compatible* if, for each agent i and for each pair of types θ_i and $\theta_i' \neq \theta_i$, it holds that:

$$V_i(\theta_i) = V_i(\theta_i, \theta_i) \geq V_i(\theta_i, \theta_i').$$

below so that her preferences are well defined. But, this upper bound can be arbitrarily large, and thus imposes no economically meaningful restriction.

Whenever we want to keep track of a mechanism that may vary, we shall also use the notation $V_i(\theta_i, \boldsymbol{q}, \boldsymbol{t})$ instead of $V_i(\theta_i)$.

3.2 Full-Insurance Mechanisms

Definition 1 *A full-insurance mechanism is one where the ex-post payoff of any bidder i with type θ_i who truthfully reports his own type is a constant. That is, $(\boldsymbol{q}, \boldsymbol{t})$ is a full insurance mechanism if and only if, for all i and all θ_i, $u_i(\theta_i, \boldsymbol{\theta}_{-i}, r)$ does not depend on $(\boldsymbol{\theta}_{-i}, r)$.*

Our first main result shows that, in order to search for the seller-optimal mechanism, we can restrict attention to full-insurance mechanisms.[16]

Proposition 1 *For any incentive compatible mechanism $(\boldsymbol{q}, \boldsymbol{t})$, there exists an incentive compatible, full-insurance mechanism that implements the same allocation and the same bidder utilities, and that is at least as profitable for the seller.*

Since the seller is risk neutral and the bidders are risk-averse, full-insurance maximizes the total social surplus. With dual risk-averse agents, full insurance also minimizes the cost of screening: in equilibrium, the optimal mechanism provides full-insurance for all types, but it does not provide full-insurance to those types who misreport. If a high type misreports to be of a lower, full-insured, type, the high type is still be exposed to risk. Since our agents distort the probability of obtaining the object according to convex functions, a small change in a relatively high probability has a more significant effect on utility. Thus agents have the least incentive to deviate when they can obtain a constant outcome by being truthful.

For any allocation rule $\boldsymbol{q}$, define now

$$Q_i(\theta_i) = \mathbb{E}[q_i(\theta_i, \boldsymbol{\theta}_{-i}) \mid \theta_i]$$

to be bidder's i induced interim probability of obtaining an object, given that he is of type θ_i. Observe that, by the law of iterated expectations, $Q_i(\theta_i)$ equals the interim probability $\mathbb{P}\left[r \in W_i(\theta_i, \theta_{-i}) \mid \theta_i\right]$ assigned by agent i to the event where he receives an object after observing his type θ_i. These expected probabilities are called *reduced form* allocations by Border [1991].

[16]The following result does not hold with standard (EU) risk-averse bidders

A reduced form allocation $\boldsymbol{Q} = (Q_1, Q_2, ...Q_n)$ is *feasible* if there exists an allocation function $\boldsymbol{q}$ that induces it. A feasible $\boldsymbol{Q}$ is *implementable* if there exists an incentive compatible mechanism $(\boldsymbol{q}, \boldsymbol{t})$ such that $\boldsymbol{q}$ induces $\boldsymbol{Q}$.

Lemma 1 (Implementable Reduced Form Allocations)

1. *A feasible, reduced form allocation* $\boldsymbol{Q}$ *can be implemented via a full-insurance mechanism if and only if each of its components is non-decreasing.*

2. *Let* $(\boldsymbol{q}, \boldsymbol{T})$ *be a full-insurance mechanism that implements* $\boldsymbol{Q}$*. Then, agent i's equilibrium (Yaari) utility is given by*

$$V_i(\theta_i) = V_i(0) + \int_0^{\theta} g_i(Q_i(t))dt.$$

   and

3. *The seller's expected revenue is given by*

$$R = \sum_{i=1}^{n} \int_0^1 \left[\theta_i Q_i(\theta_i) - g_i(Q_i(\theta_i)) \frac{1 - F_i(\theta_i)}{f_i(\theta_i)} \right] f_i(\theta_i)\, d\theta_i - \sum_{i=1}^{n} V_i(0)\,.$$

Note how the above revenue formula reduces to the standard one obtained by payoff and revenue equivalence for the case where bidders are risk-neutral, e.g. when g_i is the identity function for each i. Note also that, classical revenue equivalence does **not** generally hold in our setting! The result above, that expresses a particular revenue in terms of the reduced-form allocation function alone, holds here only for full-insurance mechanisms.

As our objective is to maximize the seller's revenue, it is optimal to always leave zero rent to the lowest type. Therefore, in the following analysis we only consider mechanisms where $V_i(0) = 0$ for all i.

4 The Single-Buyer Case

In this Section we consider the single buyer case where $n = 1$. We drop here the subscript i and the variables θ, Q, g, F refer here to the unique buyer. Here there is no additional (feasibility) constraint on Q à la Border. The seller's revenue maximization problem becomes then

$$\begin{aligned} &\max_Q \int_0^1 \left[\theta Q(\theta) - g(Q(\theta))\frac{1-F(\theta)}{f(\theta)}\right] f(\theta)d\theta \\ s.t. \qquad &Q \in [0,1] \text{ and } Q \text{ is non-decreasing} \end{aligned}$$

Assume that $\theta - g'(1)\frac{1-F(\theta)}{f(\theta)}$ is non-decreasing. Then, we can use point-wise maximization to solve the seller's problem. Since g is increasing and convex, g' is a.e. well-defined, non-negative, and non-decreasing.

Denote by Q^* the allocation rule that pointwise maximizes the principal's objective

$$Q^*(\theta) = \arg\max_{p\in[0,1]} \theta p - g(p)\frac{1-F(\theta)}{f(\theta)}\,.$$

As $\theta - g'(1)\frac{1-F(\theta)}{f(\theta)}$ is non-decreasing, $\theta p - g(p)\frac{1-F(\theta)}{f(\theta)}$ is super-modular in (θ, p)[17]: by the monotone selection theorem, we can always pick Q^* to be non-decreasing. Thus, Q^* is implementable and it constitutes the revenue maximizing allocation. Let $\theta_* = \inf\{\theta \mid g'(0) < \frac{\theta f(\theta)}{1-F(\theta)}\}$ and $\theta^* = \inf\{\theta \mid g'(1) < \frac{\theta f(\theta)}{1-F(\theta)}\}$.[18] The first-order condition yields

$$Q^*(\theta) = \begin{cases} 0 & \text{if } \theta < \theta_* \\ (g')^{-1}\left[\frac{\theta f(\theta)}{1-F(\theta)}\right] & \text{if } \theta \in [\theta_*, \theta^*] \\ 1 & \text{if } \theta > \theta^* \end{cases},$$

where $(g')^{-1}$ denotes the pseudo-inverse of g'.[19] The revenue-maximizing transfers conditional on receiving an object, T^w, and not receiving an object, T^l, are given by[20]

$$T^l(\theta) = \begin{cases} 0 & \text{if } \theta < \theta_* \\ -\int_{\theta_*}^{\theta} g\left((g')^{-1}\left[\frac{tf(t)}{1-F(t)}\right]\right)dt & \text{if } \theta \in [\theta_*, \theta^*] \\ -\int_{\theta_*}^{\theta^*} g\left((g')^{-1}\left[\frac{tf(t)}{1-F(t)}\right]\right)dt - g(1)(\theta - \theta^*) & \text{if } \theta > \theta^* \end{cases}.$$

[17] To see this, observe that the derivative with respect to p equals $\theta - g'(p)\frac{1-F(\theta)}{f(\theta)}$. As $g'(p) \leq g'(1)$ this function is increasing in θ whenever $\theta - g'(1)\frac{1-F(\theta)}{f(\theta)}$ is increasing.

[18] θ_* and θ^* are well-defined as g' is almost everywhere well-defined and non-decreasing.

[19] Formally, $(g')^{-1}(s) = \inf\{p \in [0,1] \mid g'(p) \leq s\}$.

[20] Note that we can define in an arbitrary way the losing payment $T^l(\theta)$ for $\theta > \theta^*$(since the buyer gets then the object with probability 1), and the winning payment $T^w(\theta)$ for $\theta < \theta_*$ (since the buyer gets then the object with probability zero).

and

$$T^{w}(\theta)=\theta-T^{l}(\theta)\,.$$

Example 1 *Assume that the bidder's type is uniformly distributed on the interval* $[0,1]$ *and let* $g(p)=p^{2}$*. Then* $\frac{\theta f(\theta)}{1-F(\theta)}=\frac{\theta}{1-\theta}$ *and* $g'(p)=2p$*. Therefore, the optimal allocation is given by:*

$$Q^{*}(\theta)=\begin{cases}\frac{1}{2}\frac{\theta}{1-\theta} & \textit{if } \theta\in[0,2/3]\\ 1 & \textit{if } \theta>2/3\end{cases}.$$

Due to risk-aversion, the buyer is willing to pay in order to get insured against a random allocation. The seller is risk-neutral, and hence she is willing to sell this insurance. This creates incentives for the seller to induce a random allocation that allows her to bundle the object with costly insurance, increasing revenue. If the valuation of the buyer is high enough, the buyer's willingness to pay in order to get the object for certain is high enough, and it is not profitable anymore to sell insurance. The payments in case of losing and winning are given by

$$T^{l}(\theta)=\begin{cases}-\frac{1}{4}\left(\frac{(\theta-2)\theta}{\theta-1}+2\log(1-\theta)\right) & \textit{if } \theta\in[0,\frac{2}{3}]\\ -\theta+\frac{2}{3}-\frac{1}{4}\left(\frac{8}{3}-2\log(3)\right) & \textit{if } \theta>\frac{2}{3}\end{cases}$$

and

$$T^{w}(\theta)=\begin{cases}\theta-\frac{1}{4}\left(\frac{(\theta-2)\theta}{\theta-1}+2\log(1-\theta)\right) & \textit{if } \theta\in[0,\frac{2}{3}]\\ +\frac{2}{3}-\frac{1}{4}\left(\frac{8}{3}-2\log(3)\right) & \textit{if } \theta>\frac{2}{3}\end{cases}$$

The above finding can be contrasted to the optimal allocation and transfers for a risk-neutral bidder $(g(p)=p)$ *, given by:*

$$Q_{r}(\theta)=\begin{cases}0 & \textit{if } \theta\in[0,\frac{1}{2}]\\ 1 & \textit{if } \theta>1/2\end{cases}.$$

and

$$T_{r}^{w}(\theta)=\frac{1}{2}\quad \textit{and}\quad T_{r}^{l}(\theta)=0\,.$$

In this case the revenue maximizing mechanism is deterministic, a take-it-or-leave-it offer at a price $\widehat{\theta}=0.5$ *(see Myerson [1981] or Riley & Zeckhauser [1983]). All types above* $\widehat{\theta}$ *get the object with certainty and pay a price of* $\widehat{\theta}$*, while all types below* $\widehat{\theta}$ *never get the object and do not pay. We illustrate the difference between the optimal mechanisms with and without risk aversion in Figure 1. Note that this scheme is incentive compatible even*

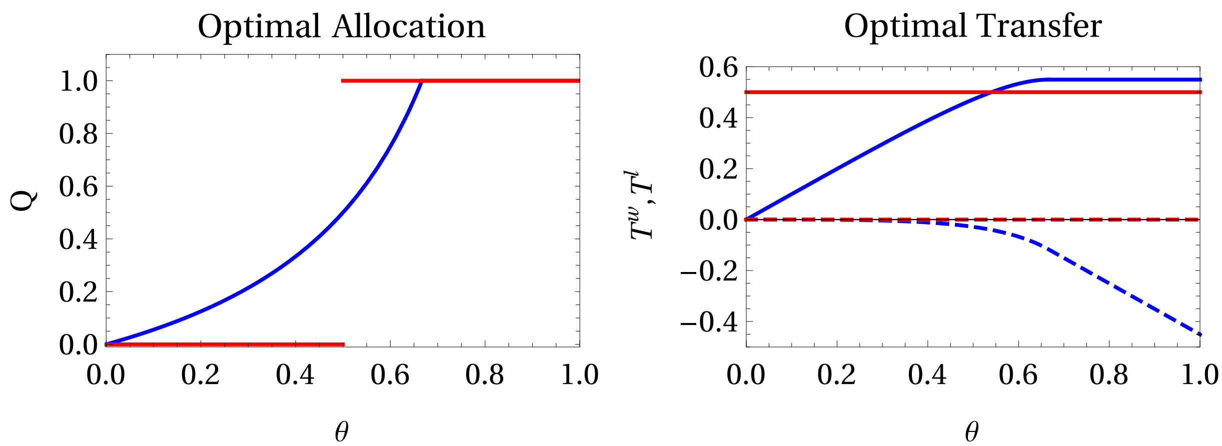

Figure 1: The optimal allocation and transfers in the risk averse case $g(p) = p^2$ (in blu) and the risk-neutral case $g(p) = p$ (in red) for θ uniformly distributed on $[0, 1]$. On the left is the probability of receiving an object as a function of the type. On the right is the transfer payed by the agent conditional on receiving an object (solid lines) and not receiving an object (dashed lines).

if the agent is dual risk-averse (as there is no uncertainty from the buyer's perspective). But, the seller increases her expected revenue by switching to the optimal mechanism we calculated above.

Remark: Our model can be also interpreted as a monopoly screening model where the designer can choose different qualities and terms of trade for different types. The standard Mussa-Rosen model with a population of (expected utility) risk averse buyers has been analyzed by Matthews and Moore [1987] who interpret quality as the probability of functioning. Matthews and Moore assume that the cost function is such that the monopolist never offers the highest quality (corresponding to functioning with probability one), and illustrate various properties of the optimal menu of offered qualities, prices, and warranties in case of malfunction. We showed above that, even if the cost of producing any quality is zero, in our model the monopolist sometimes provides a "damaged" good plus a full insurance warranty to intermediate types.

5 The General n-Bidder Case

We now return to the general case with $n > 1$ bidders. The main additional complication is the feasibility (or "Border") constraint on the vector of reduced form allocations $\boldsymbol{Q} = (Q_1, Q_2, ..., Q_n)$.

In this section we assume that the setting is *symmetric* in the sense that all bidders share the same distribution of values $F_1 = F_2 = \ldots = F$ and the same preference over lotteries $g_1 = g_2 = \ldots = g$. The seller's objective function

$$R = \sum_{i=1}^{n} \int_0^1 \left[\theta_i Q_i(\theta_i) - g(Q_i(\theta_i)) \frac{1 - F(\theta_i)}{f(\theta_i)} \right] f(\theta_i)\, d\theta_i$$

is concave in $(Q_i)_{i=1,2\ldots n}$ because, by assumption, g is convex. Thus, without loss of generality, we can restrict our attention to symmetric mechanisms, and we also drop below individual subscripts of interim allocation probabilities.[21]

In order to use a variational approach and the associated Euler-Lagrange conditions we shall only consider below allocation functions Q that are piece-wise continuous. Define the function $H : [0,1]^2 \to \mathbb{R}$ as follows:

$$H(\theta, p) = \left[\theta p - g(p) \frac{1 - F(\theta)}{f(\theta)} \right] f(\theta).$$

The seller's maximization problem over symmetric, implementable reduced-form allocation rules becomes:

$$\begin{aligned}
(R) \quad & \max_Q n \int_0^1 H(\theta, Q(\theta)) d\theta, \\
\text{s.t. } & \text{(a) } Q \in [0,1]\,; \\
& \text{(b) } Q \text{ non-decreasing} \\
& \text{(c) } \int_\theta^1 Q(t) f(t) dt \leq \int_\theta^1 F^{n-1}(t) f(t) dt \quad \text{for any } \theta \in [0,1].
\end{aligned}$$

The last constraint ensures that a monotonic, symmetric reduced form allocation rule can be indeed induced by an allocation rule q (see, for example, Maskin and Riley [1984]). If a regularity condition (defined precisely in the next Theorem) holds, then the optimal mechanism is a full-insurance mechanism whose reduced form allocation consists of two parts: for lower types, the seller uses the same allocation rule as in the single-buyer case. For higher types this becomes infeasible, and the seller allocates the object to the bidder

[21] To see this, consider, for example, the case of two bidders. Suppose there exists an optimal pair (Q_1^*, Q_2^*) such that $Q_1^* \neq Q_2^*$. By symmetry, (Q_2^*, Q_1^*) is also optimal. But then the symmetric allocation rule $(\frac{Q_2^*+Q_1^*}{2}, \frac{Q_2^*+Q_1^*}{2})$ is also feasible and it is at least as profitable for the seller (since R is concave in (Q_1, Q_2)). The generalization to more bidders is straightforward.

with the highest type. To formally state the result, we define:

$$\begin{aligned}\theta_* &= \inf\left\{\theta \mid g'(0) \leq \frac{\theta f(\theta)}{1-F(\theta)}\right\}\\ \theta_n^* &= \inf\left\{\theta \mid g'(F^{n-1}(\theta)) \leq \frac{\theta f(\theta)}{1-F(\theta)}\right\}.\end{aligned}$$

Theorem 4 (Optimal Allocation) *Assume that the function*

$$\theta \mapsto \theta - g'(F^{n-1}(\theta))\frac{1-F(\theta)}{f(\theta)}$$

is non-decreasing almost everywhere in $[0,1]$. *Then, an optimal mechanism is a full-insurance mechanism that implements the following reduced form allocation rule:*

$$Q^*(\theta) = \begin{cases} 0 & \text{if } \theta < \theta_* \\ (g')^{-1}\left(\frac{\theta f(\theta)}{1-F(\theta)}\right) & \text{if } \theta \in [\theta_*, \theta_n^*] \\ F^{n-1}(\theta) & \text{if } \theta > \theta_n^* \end{cases}.$$

The transfers in the optimal mechanism (T^w, T^l) *conditional on winning and not winning an object are given by*[22]

$$\begin{aligned} T^l(\theta) &= \begin{cases} 0 & \text{if } \theta < \theta_* \\ -\int_{\theta_*}^{\theta} g\left((g')^{-1}\left[\frac{tf(t)}{1-F(t)}\right]\right)dt & \text{if } \theta \in [\theta_*, \theta_n^*] \\ -\int_{\theta_*}^{\theta_n^*} g\left((g')^{-1}\left[\frac{tf(t)}{1-F(t)}\right]\right)dt - \int_{\theta_n^*}^{\theta} g\left(F^{n-1}(t)\right)dt & \text{if } \theta > \theta_n^* \end{cases}\\ \text{and } T^w(\theta) &= \theta - T^l(\theta).\end{aligned}$$

In the risk-neutral case where $g(p) = p$, the above regularity condition reduces to the standard requirement that the virtual value $\theta - \frac{1-F(\theta)}{f(\theta)}$ is strictly increasing. We know then from Myerson's analysis [1981] that the optimal mechanism allocates the object efficiently when the type of at least one agent exceeds an optimal cutoff (positive virtual value) and keeps the object in case all agents have types below the cutoff (negative virtual values). In particular, the allocation is deterministic.

In contrast, with dual risk averse buyers, the seller sells a "damaged" good to the intermediate types in order to reduce the information rent earned by the higher type: the

[22] As in the one-bidder case we specify $T^w(\theta)$ for $\theta < \theta_*$ and $T^l(\theta)$ for $\theta > \theta^*$ for the sake of completeness. This transfers play no role in those cases.

use of random allocation is a way to minimize the cost of screening. In addition, the seller also offers insurance to reduce the cost of exposing intermediate types to risk.

With a large number of bidders, the interval where the optimal allocation is random vanishes:

Corollary 2 *Assume that*

$$\theta \mapsto \theta - g'(F^{n-1}(\theta))\frac{1-F(\theta)}{f(\theta)}$$

is non-decreasing almost everywhere in $[0,1]$. *When* $n \to \infty$, *the interval where the optimal allocation is random,* $[\theta_*, \theta_n^*]$, *vanishes, and the limit optimal allocation rule assigns the object efficiently above a cutoff. Moreover, the limit interval of excluded types* $[0, \theta_*]$ *is a subset of the interval of excluded types under risk neutrality,* $[0, \theta_*^{rn}]$.

The first statment follows because θ_n^* is non-increasing in n, and because $\lim_{n\to\infty} \theta_n^* = \theta_*$. The second statement follows because θ_*^r solves the equation $1 = \frac{\theta f(\theta)}{1-F(\theta)}$ (that is independent of n) and because $g'(0) \leq 1$ for any increasing and convex g such that $g(0) = 0$ and $g(1) = 1$. In particular, no type is excluded if $g'(0) = 0$

While the interval of types obtaining a random allocation depends on n, the probability with which each intermediate type gets the object is independent of the number of the bidders. The needed randomization may be difficult to implement in practice since the seller needs commitment power. Imagine, for example, a realization where all bidders have intermediate types. Then, with positive probability, no bidder gets the object, and all bidders get positive transfers - but the seller actually prefers to sell to a single bidder. It is of course difficult to ex-post verify that a randomization was performed with the pre-committed probabilities. Yet, recall that the optimal mechanism above was specified in terms of an interim randomization. As we know from Theorem 3 in Gutmann et. al. [1991], or from Chen et. al. [2019], there exists a feasible and deterministic allocation rule q with given marginals Q^*. This means that one can always achieve revenue maximization via a mechanism that does not involve any randomization.

Our final Lemma in this Section shows that, under the increasing hazard rate condition, the regularity condition always holds as $n \to \infty$ if g'' is not too large.

Lemma 2 *Assume that* $g'' < e$. *Then, for any distribution* F *with an increasing hazard rate there exists* $\bar{n}$ *such that for all* $n \geq \bar{n}$, *the function*

$$\theta - g'(F^{n-1}(\theta))\frac{1-F(\theta)}{f(\theta)}$$

is non-decreasing almost everywhere in $[0,1]$. *Hence, the monotonicity requirement of Theorem 4 holds.*

5.1 Comparative Statics

We show here that the seller always (weakly) prefers to have more bidders.

Lemma 3 *The seller's expected revenue is non-decreasing in the number of bidders.*

In addition, we compare the expected (optimal) revenues from agents with different risk-attitudes, and show that the seller prefers to have bidders who are more risk averse.

Lemma 4 *Assume that* g_1 *represents preferences with a higher risk aversion than* g_2, *and fix an implementable allocation rule* q *with monotonic, reduced-form allocation rule* Q. *Let* $T_{g_i}(\theta)$, $i = 1, 2$, *be the full-insurance transfer implementing* Q *for the preferences represented by* g_i *with* $T^l_{g_i}(0) = 0$. *Then it holds that*

$$T^w_{g_1}(\theta) \geq T^w_{g_2}(\theta) \text{ and } T^l_{g_1}(\theta) \geq T^l_{g_2}(\theta) \text{ for any } \theta \in [0,1]$$

In particular, the optimal expected revenue when facing agents with risk preference g_1 *is higher than that obtained when facing agents with risk preference* g_2.

6 The Optimal Auction without Insurance

In many practical applications (e.g., public procurement), we observe the use of standard auctions (such as first-price, second-price, etc..) where the seller is obliged to award the object to the highest bidder (if she assigns the object at all), and where she is not allowed to compensate losing bidders. Thus, it is of interest to understand how such auctions perform when bidders are risk-averse in Yaari's dual sense. A first step in this direction has been made by Volij [2002] who proved a payoff equivalence result in a class of deterministic mechanisms. But, as already noted above, payoff equivalence is not the same as revenue equivalence. Recall that Che and Gale [2006] have shown that the first-price auction revenue outperforms the second-price auction in our symmetric model with dual risk averse bidders. In this section, we show that the first-price auction is actually revenue optimal in the class of standard auctions. We consider the general class of mechanisms where the seller makes no transfers to agents who do not receive an object.

Definition 2 *A mechanism* $(\boldsymbol{q}, \boldsymbol{t})$ *is a no-insurance mechanism if an agent* i *who does not receive the object receives no transfer from the principal*

$$r \notin W_i(\boldsymbol{\theta}) \Rightarrow t_i(\boldsymbol{\theta}, r) \geq 0\,.$$

Lemmas 6 and 7 in the Appendix show that, if the seller cannot make non-negative transfers to agents that do not receive an object, then the quest for the optimal auction can be confined to mechanisms where monetary transfers occur only between the seller and the agent who gets the object, and where this transfer is deterministic and independent of the types of the other agents. This result allows us to refine our search of the optimal mechanism to this particular class of mechanisms, for which the following lemma holds.

Lemma 5 *Suppose that the seller employs a mechanism of the form* $(\boldsymbol{q}, \boldsymbol{\tau})$ *where*

$$\tau_i(\boldsymbol{\theta}, r) = \begin{cases} \tilde{\tau}_i^w(\theta_i) & \textit{if } r \in W_i(\boldsymbol{\theta}) \\ 0 & \textit{else} \end{cases}\,.$$

and where the payment made by agent i *conditional on winning,* $\tilde{\tau}_i^w(\theta_i)$*, depends only on his own type. Then:*

1. *A feasible, reduced form allocation* $\boldsymbol{Q}$ *is implementable by such a mechanism if and only if each of its components is non-decreasing.*

2. *In any mechanism* $(\boldsymbol{q}, \boldsymbol{\tau})$ *as above that implements* $\boldsymbol{Q}$*, agent* i*'s equilibrium (Yaari) utility is given by*
$$V_i(\theta_i) = V_i(0) + \int_0^\theta g_i(Q_i(t))dt.$$

3. *In addition, if the lowest type obtain zero utility, then the seller's revenue in* $(\boldsymbol{q}, \boldsymbol{\tau})$ *is given by:*

$$R = n \int_0^1 \left[\theta_i Q_i(\theta_i) - \frac{Q_i(\theta_i)}{g_i(Q_i(\theta_i))} \int_0^{\theta_i} g_i(Q_i(t))dt\right] f_i(\theta_i)d\theta_i\,.$$

The above revenue formula, which holds only for this particular class of mechanisms, can be used to show that a first price auction is optimal among all no-insurance auctions.

Proposition 2 *Assume that bidders are symmetric, and that the common distribution of types* F *has an increasing hazard rate. Assume also that, conditional on awarding the* *object, the seller allocates it efficiently. That is, she implements* $\boldsymbol{Q}$ *such, that for every* i, $Q_i = Q$ *where*

$$Q(\theta) = \begin{cases} F^{n-1}(\theta) & \text{if } \theta \geq s \\ 0 & \text{else} \end{cases}$$

for some $s \in [0,1]$*. Then, a first-price auction with a reserve price* s *that solves*

$$-sF^{n-1}(s)f(s) + g(F^{n-1}(s)) \int_s^1 \frac{F^{n-1}(t)}{g(F^{n-1}(t))} f(t)dt = 0.$$

is revenue-maximizing within the class of no-insurance mechanism.

Conditional on winning, the price in a first-price auction is deterministic and equals the winner's bid. This contrast a random price determined by other bids (as in a second-price auction). Contrasting an all-pay auction, in the first-price auction a bidder does not have to pay in case of losing. Without insurance, a bidder gets a lower payoff when he loses than when he wins. Therefore, he assigns a higher weight to the probability of losing, relative to the probability of winning. Thus, not having to pay when he loses also decreases the degree of exposure to risk. These kind of risk-reducing features together yield the optimality of the first-price auction within the class of standard auctions that do not offer insurance.

Example 2 *Consider uniformly distributed types on the interval* $[0,1]$*, and let* $g_i(p) = g(p) = p^2$*. Then*[23]

$$\begin{aligned} R'(s) &= -s^n + s^{2n-2} \int_s^1 \frac{1}{t^{n-1}} dt = -s^n + s^{2n-2} \left[-\frac{t^{2-n}}{n-2} \right]_s^1 \\ &= -s^n - \frac{s^{2n-2}}{n-2} + \frac{s^n}{n-2} = -\frac{s^{2n-2}}{n-2} - s^n \frac{n-3}{n-2}. \end{aligned}$$

Hence, for $n \geq 3$*, the revenue is decreasing in a reserve price* s*, and the optimal reserve price is zero.*[24] *The first-price auction with no reserve price (which is here optimal) yields*

[23] The calculation below applies for $n \geq 3$. The analysis of the case $n = 2$ is similar and involves a logarithmic term.

[24] In the risk neutral case we obtain that

$$R'(s) = -sF^{n-1}(s)f(s) + F^{n-1}(s)(1 - F(s))$$

Then $s = 0$ is a local minimum. If the standard virtual value is monotonic, there is another solution to the F.O.C $s = \frac{1-F(s)}{f(s)}$ that maximizes revenue.

an expected revenue of

$$\begin{aligned}
&n\int_0^1\left[\theta Q(\theta)-\frac{Q(\theta)}{g(Q(\theta))}\int_0^\theta g(Q(t))dt\right]f(\theta)d\theta\\
=&n\int_0^1\left[\theta\theta^{n-1}-\frac{\theta^{n-1}}{(\theta^{n-1})^2}\int_0^\theta (t^{n-1})^2dt\right]d\theta\\
=&\frac{4n^2-4n}{2(n+1)(2n-1)}
\end{aligned}$$

The optimal full insurance mechanism always allocates the object efficiently (that is, in this example $\theta_n^*=\theta_*=0$*) and yields an expected revenue of*

$$\begin{aligned}
&n\int_0^1\left[\theta Q(\theta)-g(Q(\theta))\frac{1-F(\theta)}{f(\theta)}\right]f(\theta)\,d\theta\\
=&n\int_0^1\left[\theta\theta^{n-1}-\left(\theta^{n-1}\right)^2(1-\theta)\right]d\theta\\
=&\frac{4n^2-3n-1}{2(n+1)(2n-1)}
\end{aligned}$$

The difference between the above expected revenues clearly goes to 0 *as* $n\to\infty$*, but can be significant when* n *is small. For instance, when* $n=2$*, the optimal full-insurance mechanism yields an expected revenue of* $1/2$*,* 12.5% *more than that of the optimal no-insurance mechanism that yields* $4/9$*. Note that, in this example, the allocation rule is identical in both cases! Hence, the loss of revenue in a first-price auction is solely due to the inability of offering insurance to risk-averse bidders.*

7 Conclusion

We have explicitly derived the revenue maximizing mechanism in a framework where bidders use a non-linear probability weighting function and are risk-averse. In particular, bidders are not expected utility maximizers. The non-linearity in probabilities requires a relatively more complex maximization approach, focused on the limited supply constraint.

Our main results showed how the optimal mechanisms bundles the allocation of the physical good with the sale of insurance in order to increase revenue. Within the class of standard auctions that do not offer insurance, we showed that a first-price auction, augmented by a reserve price, outperforms any other mechanism.

We expect that our approach will be useful in other optimal mechanism design frameworks for agents who use a non-linear probability weighting function and other types of non-expected utility.

8 Appendix

Proof for Proposition 1: For an incentive compatible mechanism $(\boldsymbol{q}, \boldsymbol{t})$ associated with equilibrium utilities $V_1(\cdot, \boldsymbol{q}, \boldsymbol{t}), \ldots, V_n(\cdot, \boldsymbol{q}, \boldsymbol{t})$, we define another mechanism $(\boldsymbol{q}, \boldsymbol{T})$ such that the allocation rule remains the same, while the transfers in the new mechanism $(\boldsymbol{q}, \boldsymbol{T})$ only depend on whether agents receive the object or not. Moreover, once the agent knows her type, her utility in $(\boldsymbol{q}, \boldsymbol{T})$ is a constant:

$$T_i(\boldsymbol{\theta}, r) = \begin{cases} \theta_i - V_i(\theta_i, \boldsymbol{q}, \boldsymbol{t}) & \text{if } r \in W_i(\boldsymbol{\theta}) \\ -V_i(\theta_i, \boldsymbol{q}, \boldsymbol{t}) & \text{else} \end{cases}.$$

By construction, the mechanism $(\boldsymbol{q}, \boldsymbol{T})$ is a *full-insurance mechanism*. As the Yaari utility of a constant equals that constant[25], we obtain that, if $(\boldsymbol{q}, \boldsymbol{T})$ is incentive compatible, then it yields the same utility for each agent (given her type) as $(\boldsymbol{q}, \boldsymbol{t})$:

$$V_i(\theta_i, \boldsymbol{q}, \boldsymbol{T}) = V_i(\theta_i, \boldsymbol{q}, \boldsymbol{t}).$$

We show below that:

(a) $(\boldsymbol{q}, \boldsymbol{T})$ is incentive compatible and

(b) $(\boldsymbol{q}, \boldsymbol{T})$ is at least as profitable for the seller as $(\boldsymbol{q}, \boldsymbol{T})$.

To show **(a)**, we note that the agent's ex-post payoff from deviating in the original mechanism $(\boldsymbol{q}, \boldsymbol{t})$ and reporting θ_i' instead of θ_i equals

$$\begin{aligned} u_i(\theta_i, \theta_i', \boldsymbol{\theta}_{-i}, r; \boldsymbol{q}, \boldsymbol{t}) &= \mathbf{1}_{\{r \in W_i(\theta_i', \boldsymbol{\theta}_{-i})\}} \theta_i - t_i(\theta_i', \boldsymbol{\theta}_{-i}, r) \\ &= \mathbf{1}_{\{r \in W_i(\theta_i', \boldsymbol{\theta}_{-i})\}} \theta_i' - t_i(\theta_i', \boldsymbol{\theta}_{-i}, r) + \mathbf{1}_{\{r \in W_i(\theta_i', \theta_{-i})\}} (\theta_i - \theta_i') \\ &= u_i(\theta_i', \boldsymbol{\theta}_{-i}, r; \boldsymbol{q}, \boldsymbol{t}) + \mathbf{1}_{\{r \in W_i(\theta_i', \boldsymbol{\theta}_{-i})\}} (\theta_i - \theta_i'). \end{aligned}$$

The first part of the above sum is exactly the ex-post payoff agent i would obtain if her type would be truly θ_i'. Using Corollary 1, we obtain that the value $V_i(\theta_i, \theta_i')$ of agent i when she is of type θ_i, but reports to be of type θ_i' satisfies

$$V_i(\theta_i, \theta_i'; \boldsymbol{q}, \boldsymbol{t}) \geq V_i(\theta_i'; \boldsymbol{q}, \boldsymbol{t}) + \int_{-m}^{1+m} g_i(\mathbb{P}[\mathbf{1}_{\{r \in W_i(\theta_i', \theta_{-i})\}} (\theta_i - \theta_i') \geq s \mid \theta_i]) ds - m.$$

[25] This follows by Yaari's charaterization result together with the assumptions $g_i(0) = 0$ and $g_i(1) = 1$.

As the original mechanism $(\boldsymbol{q}, \boldsymbol{t})$ is incentive compatible we have that

$$V_i(\theta_i; \boldsymbol{q}, \boldsymbol{t}) \geq V_i(\theta_i'; \boldsymbol{q}, \boldsymbol{t}) + \int_{-m}^{1+m} g_i(\mathbb{P}[\mathbf{1}_{\{r \in W_i(\theta_i', \theta_{-i})\}}(\theta_i - \theta_i') \geq s \mid \theta_i])ds - m\,.$$

By construction, the agent's utility if everyone reports truthfully is the same in the mechanisms $(\boldsymbol{q}, \boldsymbol{t})$ and $(\boldsymbol{q}, \boldsymbol{T})$. This yields

$$\begin{aligned} V_i(\theta_i; \boldsymbol{q}, \boldsymbol{T}) &\geq V_i(\theta_i'; \boldsymbol{q}, \boldsymbol{T}) + \int_{-m}^{1+m} g_i(\mathbb{P}[\mathbf{1}_{\{r \in W_i(\theta_i', \theta_{-i})\}}(\theta_i - \theta_i') \geq s \mid \theta_i])ds - m \\ &= \int_{-m}^{1+m} g_i(\mathbb{P}[V_i(\theta_i'; \boldsymbol{q}, \boldsymbol{T}) + \mathbf{1}_{\{r \in W_i(\theta_i', \theta_{-i})\}}(\theta_i - \theta_i') \geq s \mid \theta_i])ds - m\,. \end{aligned}$$

where the last equality holds by Theorem 2-2. Since the right-hand-side is exactly the value the agent obtains from deviating in the mechanism $(\boldsymbol{q}, \boldsymbol{T})$, we have thus shown that $(\boldsymbol{q}, \boldsymbol{T})$ is incentive compatible.

(b) The ex-post payoff in mechanism $(\boldsymbol{q}, \boldsymbol{t})$ can be written as

$$u_i(\boldsymbol{\theta}, r_i; \boldsymbol{q}, \boldsymbol{t}) = u_i(\boldsymbol{\theta}, r; \boldsymbol{q}, \boldsymbol{T}) + T_i(\boldsymbol{\theta}, r) - t_i(\boldsymbol{\theta}, r)\,.$$

As the ex-post payoff in the mechanism $(\boldsymbol{q}, \boldsymbol{T})$ is independent of θ_{-i} and r, we obtain by Theorem 2-2 that

$$V_i(\theta_i; \boldsymbol{q}, \boldsymbol{t}) = V_i(\theta_i; \boldsymbol{q}, \boldsymbol{T}) + \mathcal{U}_i\big(\,T_i(\boldsymbol{\theta}, r) - t_i(\boldsymbol{\theta}, r)\,\big)$$

As $V_i(\theta_i; \boldsymbol{q}, \boldsymbol{t}) = V_i(\theta_i; \boldsymbol{q}, \boldsymbol{T})$ by construction, it must hold that

$$0 = \mathcal{U}_i\big(T_i(\boldsymbol{\theta}, r) - t_i(\boldsymbol{\theta}, r)\,\big)\,.$$

Note that the right-hand-side equals Yaari' utility from the random variable $T_i(\boldsymbol{\theta}, r) - t_i(\boldsymbol{\theta}, r)$. As the Yaari preference displays risk aversion (since g_i is convex), an upper bound on this value is given by the expectation of the random variable $T_i(\boldsymbol{\theta}, r) - t_i(\boldsymbol{\theta}, r)$:

$$0 = \mathcal{U}_i\big(\,T_i(\boldsymbol{\theta}, r) - t_i(\boldsymbol{\theta}, r)\,\big) \leq \mathbb{E}[T_i(\boldsymbol{\theta}, r) - t_i(\boldsymbol{\theta}, r) \mid \theta_i]\,.$$

We have thus established that the new mechanism $(\boldsymbol{q}, \boldsymbol{T})$ leads to a weakly higher expected revenue than the original mechanism $(\boldsymbol{q}, \boldsymbol{t})$.

Proof for Lemma 1 (1) (Sufficiency) Assume that $\boldsymbol{Q} = (Q_1, Q_2, ...Q_n)$ is feasible, and that each Q_i is non-decreasing. For any $\boldsymbol{q}$ that induces $\boldsymbol{Q}$, we construct a payment rule:

$$T_i(\boldsymbol{\theta}, r) = \begin{cases} -V_i(0) + \theta_i - \int_0^{\theta_i} g_i(Q_i(t))dt & \text{if } r \in W_i(\boldsymbol{\theta}) \\ -V_i(0) - \int_0^{\theta_i} g_i(Q_i(t))dt & \text{else} \end{cases}.$$

By construction, $(\boldsymbol{q}, \boldsymbol{T})$ is a full-insurance mechanism and

$$V_i(\theta_i) = V_i(0) + \int_0^{\theta_i} g_i(Q_i(t))dt. \tag{3}$$

As argued in the proof of Proposition 1, for any θ_i and θ_i' we have

$$\begin{aligned} V_i(\theta_i, \theta_i') &= V_i(\theta_i') + \int_{-m}^{1+m} g_i(\mathbb{P}[\mathbf{1}_{\{r \in W_i(\theta_i', \theta_{-i})\}}(\theta_i - \theta_i') \geq s \mid \theta_i])ds - m \\ &= V_i(\theta_i') + (\theta_i - \theta_i')g(Q_i(\theta_i')) \,. \end{aligned}$$

Assume that $\theta_i > \theta_i'$.[26]. Since Q_i and g_i are increasing, the expression for V_i given in (3) implies

$$V_i(\theta_i) - V_i(\theta_i') = \int_{\theta_i'}^{\theta_i} g_i(Q_i(t))dt \geq \int_{\theta_i'}^{\theta_i} g_i(Q_i(\theta_i'))dt = (\theta_i - \theta_i')g_i(Q_i(\theta_i')),$$

or, alternatively, that

$$V_i(\theta_i) \geq V_i(\theta_i') + (\theta_i - \theta_i')g_i(Q_i(\theta_i')) = V_i(\theta_i, \theta_i') \,.$$

Thus, truth-telling is optimal and $\boldsymbol{Q}$ is implementable.

(Necessity) Truthtelling requires that, for any θ_i and θ_i', it holds that

$$V_i(\theta_i, \theta_i') = V_i(\theta_i') + (\theta_i - \theta_i')g_i(Q_i(\theta_i')) \leq V_i(\theta_i)$$

and that

$$V_i(\theta_i', \theta_i) = V_i(\theta_i) + (\theta_i' - \theta_i)g_i(Q_i(\theta_i)) \leq V_i(\theta_i').$$

Assuming $\theta_i > \theta_i'$, we obtain

$$g_i(Q_i(\theta_i))(\theta_i - \theta_i') \geq V_i(\theta_i) - V_i(\theta_i') \geq g_i(Q_i(\theta_i'))(\theta_i - \theta_i').. \tag{4}$$

[26] The case with $\theta_i < \theta_i'$ is similar.

Since g is increasing with $g_i(0) = 0$, the above expression immediately implies that Q_i must be non-decreasing.

(2) Assuming $\theta_i > \theta'_i$, inequality (4) can be rewritten as follows: For any $\delta \in [0, 1-\theta']$

$$\delta g_i(Q_i(\theta'_i + \delta)) \geq V_i(\theta'_i + \delta) - V_i(\theta'_i) \geq \delta g(Q_i(\theta'_i))..$$

Since both Q_i and g_i are non-decreasing, $g_i \circ Q_i$ is also non-decreasing. Since g is also bounded, $g_i \circ Q_i$ is Riemann integrable. Letting $\delta \to 0$ yields

$$V_i(\theta_i) = V_i(0) + \int_0^{\theta_i} g(Q_i(t))dt..$$

(3) In a full-insurance mechanism implementing $\boldsymbol{Q}$ we have

$$T_i(\boldsymbol{\theta}, r) = \begin{cases} T_i^w(\theta_i) = \theta_i - \int_0^{\theta_i} g_i(Q_i(t))dt - V_i(0) & \text{if } r \in W_i(\boldsymbol{\theta}) \\ T_i^l(\theta_i) - \int_0^{\theta_i} g_i(Q_i(t))dt - V_i(0) & \text{else} \end{cases}.$$

The results follows by the following chain of equalities:

$$\begin{aligned} R + \sum_{i=1}^n V_i(0) &= \sum_{i=1}^n \int_0^1 \left[Q_i(\theta_i)T_i^w(\theta_i) + (1 - Q_i(\theta_i))T_i^l(\theta_i)\right] f_i(\theta_i)\, d\theta_i \\ &= \sum_{i=1}^n \int_0^1 \left[Q_i(\theta_i)\left(\theta_i - \int_0^{\theta_i} g_i(Q_i(t))dt\right)\right] d\theta_i \\ &\quad + \sum_{i=1}^n \int_0^1 \left[(1 - Q_i(\theta_i))\left(-\int_0^{\theta_i} g_i(Q_i(t))dt\right)\right] f_i(\theta_i)\, d\theta_i \\ &= \sum_{i=1}^n \int_0^1 \left[\theta_i Q_i(\theta_i) - \int_0^{\theta_i} g_i(Q_i(t))dt\right] f_i(\theta_i)\, d\theta_i \\ &= \sum_{i=1}^n \int_0^1 \left[\theta_i Q_i(\theta_i) - g_i(Q_i(\theta_i))\frac{1 - F_i(\theta_i)}{f_i(\theta_i)}\right] f(\theta_i)\, d\theta_i. \end{aligned}$$

Proof for Theorem 4: Note that $Q^*(\theta)$ satisfies constraints (a)-(c) of Problem (R). Therefore, in order to show that Q^* constitutes the solution to Problem (R), it suffices to show that it is the solution to the following, relaxed problem where we ignore the monotonicity constraint

$$
\begin{aligned}
&(R') \quad \max_{Q} n \int_0^1 H(\theta, Q(\theta))d\theta, \\
&\text{s.t. (a) } Q \in [0,1]\,; \\
&\qquad\text{(c) } \int_\theta^1 Q(t)f(t)dt \leq \int_\theta^1 F^{n-1}(t)f(t)dt \quad \text{for any } \theta \in [0,1].
\end{aligned}
$$

Since

$$\max_{Q} n \int_0^1 H(\theta, Q(\theta))d\theta \leq \max_{Q} n \int_0^{\theta_n^*} H(\theta, Q(\theta))d\theta + \max_{Q} n \int_{\theta_n^*}^1 H(\theta, Q(\theta))d\theta$$

it suffices to show that Q^* is the solution to the two problems below:

$$
\begin{aligned}
&(I) \quad \max_{Q} n \int_0^{\theta_n^*} H(\theta, Q(\theta))d\theta, \\
&\text{s.t. (a) } Q \in [0,1]
\end{aligned}
$$

and

$$
\begin{aligned}
&(II) \quad \max_{Q} n \int_{\theta_n^*}^1 H(\theta, Q(\theta))d\theta, \\
&\text{s.t. (c) } \int_\theta^1 Q(t)f(t)dt \leq \int_\theta^1 F^{n-1}(t)f(t)dt \quad \text{for any } \theta \in [0,1].
\end{aligned}
$$

Note, that we further relaxed our problem by ignoring the feasibility constraint in (I) and by ignoring the constraint that $Q \in [0,1]$ in (II).

Problem (I): By the same arguments used in the single buyer case, it can be verified that any feasible Q^I satisfying

$$Q^I(\theta) = \begin{cases} 0 & \text{if } \theta < \theta_* \\ (g')^{-1}\left[\frac{\theta f(\theta)}{1-F(\theta)}\right] & \text{if } \theta \in [\theta_*, \theta_n^*] \end{cases}.$$

solves Problem (I). So Q^* is also a solution to Problem (I), as desired.

Problem (II): The proof consists of two main steps.

Step 1: First, we show that, if Q solves Problem (II), then it must satisfy

$$\int_{\theta_n^*}^1 Q(t)f(t)dt = \int_{\theta_n^*}^1 F^{n-1}(t)f(t)dt\,.$$

By the definition of θ_n^* , and by the assumption that $\theta - g'(F^{n-1}(\theta))\frac{1-F(\theta)}{f(\theta)}$ is non-decreasing, we obtain that, for any $\theta > \theta_n^*$,

$$\frac{\partial H(\theta,p)}{\partial p}\mid_{p=F^{n-1}(\theta)} = \theta - g'(F^{n-1}(\theta))\frac{1-F(\theta)}{f(\theta)} > 0$$

Since H is concave in Q, we also obtain

$$\frac{\partial H(\theta,p)}{\partial p} > 0$$

for any $\theta > \theta_n^*$ and for any $Q(\theta) \leq F^{n-1}(\theta)$. This implies that, for any feasible Q that satisfies

$$\int_{\theta_n^*}^1 Q(t)f(t)dt < \int_{\theta_n^*}^1 F^{n-1}(t)f(t)dt,$$

we can construct another feasible $\tilde{Q}$ such that $\tilde{Q}(\theta) = 0$ for $\theta < \theta_n^*$, $Q(\theta) < \tilde{Q}(\theta) \leq F^{n-1}(\theta)$ on a set of positive measure in $[\theta_n^*, 1]$, and $\tilde{Q}(\theta) = Q(\theta)$ otherwise. As $\frac{\partial H(\theta,p)}{\partial p} > 0$ for any $\theta > \theta_n^*$ and for any $Q \leq F^{n-1}(\theta)$, it follows that, for any $\theta > \theta_n^*$, it holds that

$$H(\theta, Q(\theta)) \leq H(\theta, \tilde{Q}(\theta))$$

with strict inequality on a set of positive measure. This implies that

$$n\int_{\theta_n^*}^1 H(\theta, Q(\theta)d\theta < n\int_{\theta_n^*}^1 H(\theta, \tilde{Q}(\theta)d\theta$$

Thus, any such Q cannot be a solution to Problem (II).

Step 2: In view of the above step, the solution to Problem (II) can be found by solving the same problem augmented with an additional equality constraint:

$$\int_{\theta_n^*}^1 Q(t)f(t)dt = \int_{\theta_n^*}^1 F^{n-1}(t)f(t)dt$$

This means that the seller has to assign the object if all bidders' types are above θ_n^*

$\in [0,1)$. We show that $Q^*(\theta)$ is indeed a solution to:

$$
\begin{aligned}
(II') \quad & \max_{Q} n \int_{\theta_n^*}^{1} H(\theta, Q(\theta)) d\theta, \\
\text{s.t. (c)} \quad & \int_{\theta}^{1} Q(t) f(t) dt \leq \int_{\theta}^{1} F^{n-1}(t) f(t) dt \quad \text{for any } \theta \in [0,1] \\
\text{(d)} \quad & \int_{\theta_n^*}^{1} Q(t) f(t) dt = \int_{\theta_n^*}^{1} F^{n-1}(t) f(t) dt.
\end{aligned}
$$

Problem (II') can be now transformed into a standard calculus of variations problem. Define

$$\sigma(\theta) \equiv \int_{\theta}^{1} Q(t) f(t) dt$$

to obtain

$$\sigma'(\theta) = -f(\theta) Q(\theta), \text{and that } Q(\theta) = -\frac{\sigma'(\theta)}{f(\theta)}.$$

Let

$$
\begin{aligned}
h(\theta, \sigma'(\theta)) &= \left[-\frac{\theta \sigma'(\theta)}{f(\theta)} - g\Big(-\frac{\sigma'(\theta)}{f(\theta)} \Big) \frac{1 - F(\theta)}{f(\theta)} \right] f(\theta) \\
&= -\theta \sigma'(\theta) - g\Big(-\frac{\sigma'(\theta)}{f(\theta)} \Big) (1 - F(\theta)).
\end{aligned}
$$

Then, Problem (II') becomes

$$
\begin{aligned}
\max_{\sigma} J(\sigma) &= \int_{\theta_n^*}^{1} h(\theta, \sigma'(\theta)) d\theta \quad \text{s.t.} \\
(1)\ \sigma(\theta) &\leq \int_{\theta}^{1} F^{n-1}(t) dF(t) \text{ for any } \theta \\
(2)\ \sigma(\theta_n^*) &= \int_{\theta_n^*}^{1} F^{n-1}(t) f(t) dt \text{ and } \ \sigma(1) = 0
\end{aligned}
$$

To solve it, we apply the (infinite-dimensional) generalized Kuhn-Tucker Theorem (see for example Luenberger, 1969, p.249). The Lagrangian is

$$\mathcal{L}(\theta, \sigma, \sigma', \lambda) = -h(\theta, \sigma') + \lambda \left(\sigma - \int_{\theta}^{1} F^{n-1}(t) dF(t) \right).$$

Suppose that σ^* maximizes $\mathcal{L}$.[27] Then there exists a multiplier $\lambda : [0,1] \to \mathbb{R}_+$ such that

[27] Note that σ^* must be a regular point since there is only one inequality constraint.

(σ^*, λ) have to satisfy the following four necessary conditions:

1. The boundary conditions: $\sigma(\theta_n^*) = \int_{\theta_n^*}^1 F^{n-1}(t) f(t) d$ and $\sigma(1) = 0$.

2. The Euler–Lagrange condition (wherever it is well-defined):

$$\frac{\partial \mathcal{L}}{\partial \sigma}(\theta, \sigma(\theta), \sigma'(\theta), \lambda(\theta)) - \frac{d}{d\theta}\left[\frac{\partial \mathcal{L}}{\partial \sigma'}(\theta, \sigma(\theta), \sigma'(\theta), \lambda(\theta))\right] = 0\,. \tag{5}$$

3. $\lambda(\theta) \geq 0$.

4. The complementary slack condition $\lambda(\theta)\left(\sigma(\theta) - \int_\theta^1 F^{n-1}(t)dF(t)\right) = 0$.

We note that $\frac{\partial \mathcal{L}}{\partial \sigma} = \lambda$ and $\frac{\partial \mathcal{L}}{\partial \sigma'} = -\frac{\partial h}{\partial \sigma'} = \theta - g'(-\frac{\sigma'(\theta)}{f(\theta)})\frac{1-F(\theta)}{f(\theta)}$. The second condition thus simplifies to

$$\lambda(\theta) = \frac{d}{d\theta}\left[\theta - g'(Q(\theta))\frac{1 - F\,(\theta)}{f(\theta)}\right]. \tag{6}$$

If the function $\theta - g'(F^{n-1}(\theta))\frac{1-F(\theta)}{f(\theta)}$ is non-decreasing almost everywhere on $[0, 1]$, then it is easy to verify that any feasible Q^{II} satisfying $Q^{II}(\theta) = F^{n-1}(\theta)$ for all $\theta > \theta_n^*$ fulfills all four necessary conditions listed above. Moreover, this function is on the boundary of the convex feasible set of functions σ such that $\sigma(\theta) \leq \int_\theta^1 F^{n-1}(t)dF(t)$ for any $\theta > \theta_n^*$, i.e., the inequality constraint is binding for all $\theta \geq \theta_n^*$.

From the above reasoning, we obtain that any such Q^{II} is a local maximizer in Problem (II). To prove global optimality, we note that, by the convexity of g,

$$\frac{\partial^2 h}{\partial^2 \sigma'} = -\frac{1 - F(\theta)}{f^2(\theta)} g''\left(-\frac{\sigma'(\theta)}{f(\theta)}\right) \leq 0\,.$$

It follows that, for any σ_1, σ_2, and for any $\alpha \in [0, 1]$

$$\begin{aligned} J(\alpha\sigma_1 + (1-\alpha)\sigma_2) &= \int_{\theta_n^*}^1 h(\theta, \alpha\sigma_1' + (1-\alpha)\sigma_2'(\theta))d\theta \\ &\geq \alpha \int_{\theta_n^*}^1 h(\theta, \sigma_1'(\theta))d\theta + (1-\alpha)\int_{\theta_n^*}^1 h(\theta, \sigma_2'(\theta))d\theta \\ &= \alpha J(\sigma_1) + (1-\alpha)J(\sigma_2) \end{aligned}$$

Thus the functional J is also concave in σ. Then, by Proposition 1, Section 7.8, in Luenberger (1969), Q^* is also a global maximizer for Problem (II).

Having shown that Q^* is the solution to both the relaxed Problem (I) and the relaxed Problem (II), we conclude that it is also the solution to the original Problem (R).

Proof for Lemma 2: Taking the derivative of the function

$$\theta - g'(F^{n-1}(\theta))\frac{1-F(\theta)}{f(\theta)}$$

yields

$$1 - g'(F^{n-1}(\theta))\left[\frac{1-F(\theta)}{f(\theta)}\right]' - (n-1)F^{n-2}(\theta)g''(F^{n-1}(\theta))[1-F(\theta)]$$

As we have assumed a monotone hazard rate, the second terms is always non-negative. We now show that, for n large enough,

$$(n-1)F^{n-2}(\theta)g''(F^{n-1}(\theta))[1-F(\theta)] \leq 1$$

holds for any $\theta \in [0,1]$, which implies the result. Note that, over $p \in [0,1]$,

$$(n-1)p^{n-2}(1-p)$$

is maximized at $p = \frac{n-2}{n-1}$. Plugging in $p = \frac{n-2}{n-1}$ yields that

$$\max_{p\in[0,1]}(n-1)p^{n-2}(1-p) \leq \left(\frac{n-2}{n-1}\right)^{n-2}.$$

Hence

$$\begin{aligned} & \lim_{n\to\infty}\{(n-1)F^{n-2}(\theta)g''(F^{n-1}(\theta))[1-F(\theta)]\} \\ < \; & e \cdot \lim_{n\to\infty}\left(\frac{n-2}{n-1}\right)^{n-2} = 1 \end{aligned}$$

as desired.

Proof for Lemma 3: Let Q_n be the optimal reduced form allocation with n bidders, $i = 1, ..., n$. Assume now that there are $n+1$ bidders, $i = 1, ..., n, n+1$. Using the allocation Q_n for bidders $1, ..., i-1, i+1, ..., n+1$ while completely excluding bidder i is a feasible (i.e., incentive compatible and individually rational mechanism) in the $n+1$ bidder problem. Each of these proposed asymmetric allocations yields the same revenue as the optimal mechanism for the n bidder problem. Averaging these $n+1$ asymmetric allocations yields a symmetric, feasible allocation for the $n+1$ bidder problem. By the concavity of the revenue, the symmetric average allocation yields a higher revenue than each asymmetric one. In particular, the optimal allocation for the $n+1$ bidder problem (that is known to be symmetric) yields as least as much revenue as the optimal allocation for the n bidder problem.

Proof for Lemma 4: By Lemma 1 we have that

$$V_{g_i}(\theta_i) = V_{g_i}(0) + \int_0^\theta g_i(Q(t))dt,\ \ i = 1, 2.$$

Since the utility of type $\theta = 0$ is zero, the utility in any IC full-insurance mechanism is completely specified by the allocation Q. The implementing full-insurance transfer rule is given by

$$\begin{aligned} T^w_{g_i}(\theta) &= \theta_i - \int_0^\theta g_i(Q(t))dt, \\ T^l_{g_i}(\theta) &= -\int_0^\theta g_i(Q(t))dt,\ \ i = 1, 2. \end{aligned}$$

Note that $g_1 = h \circ g_2$ where h is convex. The convexity of h implies that, for any $Q \in [0, 1]$,

$$\begin{aligned} g_1(Q) &= h(g_2(Q)) = h[g_2(Q) \cdot 1 + (1 - g_2(Q)) \cdot 0] \\ &\leq g_2(Q) \cdot h(1) + (1 - g_2(Q)) \cdot h(0) \\ &= g_2(Q), \end{aligned}$$

where the last equality follows from $1 = g_1(1) = h(g_2(1)) = h(1)$ and from $0 = g_1(0) = h(g_2(0)) = h(0)$. Therefore, for any $\theta \in [0, 1]$

$$\int_0^\theta g_2(Q(t))dt \geq \int_0^\theta g_1(Q(t))dt$$

Hence, for any $\theta \in [0, 1]$, we obtain $T^w_{g_1}(\theta) \geq T^w_{g_2}(\theta)$ and $T^l_{g_1}(\theta) \geq T^l_{g_2}(\theta)$.

In particular, since the optimal allocation under preferences g_2 is a feasible allocation under preferences g_1, the expected optimal revenue when facing agents with risk preference g_1 is higher than that obtained when facing agents with risk preference g_2.

To prove Proposition 2, we first establish the following two Lemmas.

Lemma 6 *For any incentive compatible, no-insurance mechanism $(\boldsymbol{q}, \boldsymbol{t})$, there exist functions $T^w_i(\theta_i)$ and $T^l_i(\theta_i) \geq 0$ such that the mechanism $(\boldsymbol{q}, \boldsymbol{T})$ where*

$$T_i(\boldsymbol{\theta}, r) = \begin{cases} T^w_i(\theta_i) & \text{if } r \in W_i(\boldsymbol{\theta}) \\ T^l_i(\theta_i) & \text{else} \end{cases}$$

is also incentive compatible, and at least as profitable for the seller as $(\boldsymbol{q}, \boldsymbol{t})$.

Proof. Fix an incentive compatible mechanism $(\boldsymbol{q}, \boldsymbol{t})$ where payments are non-negative. Consider a mechanism of the form $(\boldsymbol{q}, \boldsymbol{t'})$ where

$$t'_i(\boldsymbol{\theta}, r) = \begin{cases} T_i^w(\theta_i) & \text{if } r \in W_i(\boldsymbol{\theta}) \\ t_i(\boldsymbol{\theta}, r) & \text{else} \end{cases}$$

such that $T_i^w(\theta_i)$ is deterministic and defined by

$$V_i(\theta_i; \boldsymbol{q}, \boldsymbol{t'}) = V_i(\theta_i; \boldsymbol{q}, \boldsymbol{t}).$$

Analogously, we construct another mechanism $(\boldsymbol{q}, \boldsymbol{T})$ where, for all i and all θ_i

$$T_i(\boldsymbol{\theta}, r) = \begin{cases} T_i^w(\theta_i) & \text{if } r \in W_i(\boldsymbol{\theta}) \\ T_i^l(\theta_i) & \text{else} \end{cases}$$

where $T_i^l(\theta_i)$ is deterministic and defined by

$$V_i(\theta_i; \boldsymbol{q}, \boldsymbol{t}) = V_i(\theta_i; \boldsymbol{q}, \boldsymbol{t'}) = V_i(\theta_i; \boldsymbol{q}, \boldsymbol{T}).$$

By the definition of a no-insurance mechanism, each $t_i(\cdot, r)$ is a non-negative, possibly random transfer whenever $r \notin W_i(\boldsymbol{\theta})$. Then, each $T_i^l(\theta_i)$ is non-negative by construction. By using very similar arguments to those used in the proof for Lemma 1, we can show that:

$$V_i(\theta_i, \theta'_i; \boldsymbol{q}, \boldsymbol{T}) = V_i(\theta'_i; \boldsymbol{q}, \boldsymbol{T}) + \int_{-m}^{1+m} g_i(\mathbb{P}[\mathbf{1}_{\{r \in W_i(\theta'_i, \boldsymbol{\theta}_{-i})\}}(\theta_i - \theta'_i) \geq s \mid \theta_i])ds - m\,.$$

and that

$$V_i(\theta_i, \theta'_i; \boldsymbol{q}, \boldsymbol{t}) \geq V_i(\theta'_i; \boldsymbol{q}, \boldsymbol{t}) + \int_{-m}^{m+1} g_i(\mathbb{P}[\mathbf{1}_{\{r \in W_i(\theta'_i, \boldsymbol{\theta}_{-i})\}}(\theta_i - \theta'_i) \geq s \mid \theta_i])ds - m..$$

Since $(\boldsymbol{q}, \boldsymbol{t})$ is incentive compatible by assumption, and because $V_i(\theta_i; \boldsymbol{q}, \boldsymbol{t}) = V_i(\theta_i; \boldsymbol{q}, \boldsymbol{t})$ by construction, it follows that, for any i, θ_i and θ'_i it holds that

$$\begin{aligned} V_i(\theta_i; \boldsymbol{q}, \boldsymbol{T}) =& V_i(\theta_i; \boldsymbol{q}, \boldsymbol{t}) \\ &\geq V_i(\theta_i, \theta'_i; \boldsymbol{q}, \boldsymbol{t}) \geq V_i(\theta_i, \theta'_i; \boldsymbol{q}, \boldsymbol{T}). \end{aligned}$$

That is, $(\boldsymbol{q}, \boldsymbol{T})$ is also incentive compatible. By using essentially the same arguments as in the proof for Lemma 1, we also obtain that $(\boldsymbol{q}, \boldsymbol{T})$ is at least as profitable as $(\boldsymbol{q}, \boldsymbol{t})$ for the seller. ■

We next show that, within the above constructed class of mechanisms, where losing bidders do not get positive transfers, it is also not optimal to ask losers to pay (for instance by charging an entry fee).

Lemma 7 *Consider an incentive compatible mechanism* $(\boldsymbol{q}, \boldsymbol{T})$ *where, for all* i *and for all* θ_i,

$$t_i(\boldsymbol{\theta}, r) = \begin{cases} T_i^w(\theta_i) & \text{if } r \in W_i(\boldsymbol{\theta}) \\ T_i^l(\theta_i) \geq 0 & \text{else} \end{cases}$$

Then, there exists an incentive compatible mechanism $(\boldsymbol{q}, \widetilde{\boldsymbol{T}})$ *where for all* i *and all* θ_i

$$\widetilde{T}_i(\boldsymbol{\theta}, r) = \begin{cases} \widetilde{T}_i^w(\theta_i) & \text{if } r \in W_i(\boldsymbol{\theta}) \\ 0 & \text{else} \end{cases}$$

that is at least as profitable for the seller as $(\boldsymbol{q}, \boldsymbol{t})$.

Proof. Fix any incentive compatible mechanism $(\boldsymbol{q}, \boldsymbol{T})$ as in the statement of the Lemma. Consider a mechanism $(\boldsymbol{q}, \widetilde{\boldsymbol{T}})$ with the same allocation as the mechanism $(\boldsymbol{q}, \boldsymbol{T})$, where

$$\widetilde{T}_i(\boldsymbol{\theta}, r) = \begin{cases} \widetilde{T}_i^w(\theta_i) & \text{if } r \in W_i(\boldsymbol{\theta}) \\ 0 & \text{else} \end{cases}$$

and where $\widetilde{T}_i^w(\theta_i)$ is implicitly defined by

$$V_i(\theta_i; \boldsymbol{q}, \widetilde{\boldsymbol{T}}) = V_i(\theta_i; \boldsymbol{q}, \boldsymbol{T}).$$

Such a mechanism clearly exists. By using very similar arguments as those used in the proof for Lemma 1, we can show that $(\boldsymbol{q}, \widetilde{\boldsymbol{T}})$ is incentive compatible.

The utility of bidder i with type θ_i in $(\boldsymbol{q}, \boldsymbol{T})$ is the sum of a fixed amount $-T_i^l(\theta_i)$ and of a lottery that pays $\theta_i - T_i^w(\theta_i) + T_i^l(\theta_i)$ with probability $Q_i(\theta_i)$ and pays 0 with probability $1 - Q_i(\theta_i)$. Therefore

$$V_i(\theta_i; \boldsymbol{q}, \boldsymbol{T}) = -T_i^l(\theta_i) + g_i(Q_i(\theta_i))\left(\theta_i - T_i^w(\theta_i) + T_i^l(\theta_i)\right)$$

Similarly

$$V_i(\theta_i;\boldsymbol{q},\widetilde{\boldsymbol{T}}) = g_i\left(Q_i\left(\theta_i\right)\right)\left(\theta_i - \tilde{T}_i^w(\theta_i)\right)$$

By construction, it holds that $V_i(\theta_i;\boldsymbol{q},\widetilde{\boldsymbol{T}}) = V_i(\theta_i;\boldsymbol{q},\boldsymbol{T})$ and hence that

$$\begin{aligned} -T_i^l(\theta_i)\left(1-g_i\left(Q_i\left(\theta_i\right)\right)\right)+g_i\left(Q_i\left(\theta_i\right)\right)\left(\theta_i-T_i^w(\theta_i)\right) &= g_i\left(Q_i\left(\theta_i\right)\right)\left(\theta_i-\tilde{T}_i^w(\theta_i)\right) \Longleftrightarrow \\ -T_i^l(\theta_i)\left(1-g_i\left(Q_i\left(\theta_i\right)\right)\right) &= g_i\left(Q_i\left(\theta_i\right)\right)\left(T_i^w(\theta_i)-\tilde{T}_i^w(\theta_i)\right) \end{aligned}$$

This yields $T_i^w(\theta_i) \leq \tilde{T}_i^w(\theta_i)$. Furthermore, $V_i(\theta_i;\boldsymbol{q},\widetilde{\boldsymbol{T}}) = V_i(\theta_i;\boldsymbol{q},\boldsymbol{T})$ implies that

$$T_i^l(\theta_i) = g_i\left(Q_i\left(\theta_i\right)\right)\left(\tilde{T}_i^w(\theta_i)-T_i^w(\theta_i)+T_i^l(\theta_i)\right) \leq Q_i\left(\theta_i\right)\left(\tilde{T}_i^w(\theta_i)-T_i^w(\theta_i)+T_i^l(\theta_i)\right) \tag{7}$$

where the last inequality follows because $\tilde{T}_i^w(\theta_i) - T_i^w(\theta_i) + T_i^l(\theta_i) \geq 0$, and because g_i is convex with $g_i\left(0\right)=0$ and $g_i\left(1\right)=1$. The last inequality implies that

$$T_i^l(\theta_i) + Q_i\left(\theta_i\right)\left(T_i^w(\theta_i) - T_i^l(\theta_i)\right) \leq Q_i\left(\theta_i\right)\tilde{T}_i^w(\theta_i)$$

and therefore the new mechanism is at least as profitable for the seller as $(\boldsymbol{q},\boldsymbol{T})$. ■

Proof for Lemma 5: The proofs for points 1 and 2 are essentially the same as those in Lemma 1, and we omit here the details.

By point 2, the agent's equilibrium utility is given by

$$V_i(\theta_i) = V_i(0) + \int_0^{\theta_i} g_i(Q_i(t))dt = \int_0^{\theta_i} g_i(Q_i(t))dt.$$

On the other hand, a direct computation of the agent's utility yields

$$\int_0^{\theta_i - \tilde{\tau}_i(\theta_i)} g_i(Q_i(\theta_i))du = (\theta_i - \tilde{\tau}_i^w(\theta_i))g_i(Q_i(\theta_i))$$

We therefore obtain that:

$$\begin{aligned} &\int_0^{\theta_i} g_i(Q_i(t))dt = (\theta_i - \tilde{\tau}_i^w(\theta_i))g_i(Q_i(\theta_i)) \\ &\Rightarrow \tilde{\tau}_i^w(\theta_i)g_i(Q_i(\theta_i)) = \theta_i g_i(Q_i(\theta_i)) - \int_0^{\theta_i} g_i(Q_i(t))dt \\ &\Rightarrow \tilde{\tau}_i^w(\theta_i)Q_i(\theta_i) = \theta_i Q_i(\theta_i) - \frac{Q_i(\theta_i)}{g_i(Q_i(\theta_i))}\int_0^{\theta_i} g_i(Q_i(t))dt\,. \end{aligned}$$

As the agent with type θ_i pays the transfer $\tilde{\tau}_i^w(\theta_i)$ only if he receives an object, which happens with probability $Q_i(\theta_i)$, the expected revenue from agent i equals $\int_0^1 Q_i(\theta_i)\tilde{\tau}_i^w(\theta_i)f(\theta_i)d\theta_i$ which completes the proof.

Proof for Proposition 2: By Lemmas 6 and 7, we can apply Lemma 5 and conclude that if the seller uses an optimal no insurance mechanism that implements Q as in the statement, then her expected revenue is given by

$$R(s) = n\int_0^1 \left[\theta F^{n-1}(\theta) - \frac{F^{n-1}(\theta)}{g(F^{n-1}(\theta))}\int_s^\theta g(F^{n-1}(t))dt\right] f(\theta)d\theta$$

Taking the derivative with respect to s yields

$$R'(s) = -F^{n-1}(s)f(s)\left[s - \frac{1}{f(s)}\frac{g(F^{n-1}(s))}{F^{n-1}(s)}\int_s^1 \frac{F^{n-1}(t)}{g(F^{n-1}(t))}f(t)dt\right]$$

Finally, we recall that Volij [2002] showed that the symmetric equilibrium bid function in a first-price auction with reserve s is given by

$$\beta(\theta) = \theta - \frac{\int_s^1 g(F^{n-1}(t))dt}{g(F^{n-1}(\theta))}$$

which is increasing. Thus, such an auction implements the efficient allocation whenever the object is sold. Moreover, the payment of bidder i only depends on his type (no further randomization). Thus, the first price auction achieves revenue $R(s)$ and is therefore optimal.

References

[2013] Andrikogiannopoulou, A., and F. Papakonstantinou (2013): Heterogeneity in Risk Preferences: Evidence from a Real-World Betting Market," SSRN working paper.

[2017] Baisa, B. (2017). Auction design without quasilinear preferences. *Theoretical Economics*, **12**(1), 53-78.

[2011] Barseghyan, L., Prince, J., & Teitelbaum, J. C. (2011). Are risk preferences stable across contexts? Evidence from insurance data. *American Economic Review*, **101**(2), 591-631.

[2016] Barseghyan, L., Molinari, F., & Teitelbaum, J. C. (2016). Inference under stability of risk preferences.*Quantitative Economics*, **7**(2), 367-409.

[2016] Bierbrauer, F., & Netzer, N. (2016). Mechanism design and intentions. *Journal of Economic Theory*, **163**, 557-603.

[1991] Border, K. (1991). Implementation of Reduced Form Auctions: A Geometric Approach, *Econometrica*, **59**(4), 1175-1187.

[2006] Bose, S., Ozdenoren, E., & Pape, A. (2006). Optimal Auctions with Ambiguity. *Theoretical Economics*, **1**(4), 411-438.

[2009] Bose, S., & Daripa, A. (2009). A Dynamic Mechanism and Surplus Extraction under Ambiguity. *Journal of Economic Theory*, **144**(5), 2084-2114.

[2010] Bruhin, A., Fehr-Duda, H., and T. Epper (2010) Risk and Rationality: Uncovering Heterogeneity in Probability Distortion, *Econometrica* **78**(4), 1375-1412.

[2006] Che, Y. K., & Gale, I. (2006). Revenue comparisons for auctions when bidders have arbitrary types, *Theoretical Economics*, **1**(1), 95-118.

[2019] Chen, Y. C., He, W., Li, J., & Sun, Y. (2019). Equivalence of Stochastic and Deterministic Mechanisms. *Econometrica*, **87**(4), 1367-1390.

[2007] Cohen, A., & Einav, L. (2007). Estimating risk preferences from deductible choice. *American Rconomic Review*, **97**(3), 745-788.

[2012] Einav, L., Finkelstein, A., Pascu, I., & Cullen, M. R. (2012). How general are risk preferences? Choices under uncertainty in different domains. *American Economic Review*, **102**(6), 2606-38.

[2019] Gershkov, A., Moldovanu, B., Strack, P. and Zhang, M. (2019). Auctions with Endogenous Valuations, discussion paper, University of Bonn.

[1991] Gutmann, S., Kemperman, J. H. B, Reeds, J. A. and Shepp, L. A. (1991) "Existence of Probability Measures with Given Marginals," The Annals of Probability **19**(4), 1781-1797.

[2001] Guriev, S. (2001). On Microfoundations of the Dual Theory of Choice. *The Geneva papers on risk and insurance theory*, **26**(2), 117-137.

[2002] Goeree, J. K., Holt, C. A., & Palfrey, T. R. (2002). Quantal Response Equilibrium and Overbidding in Private-Value Auctions. *Journal of Economic Theory*, **104**(1), 247-272.

[2004] Goeree, J. K., & Offerman, T. (2004). The Amsterdam Auction. *Econometrica* **72**(1), 281-294.

[1942] Halmos, P. R., & von Neumann, J. (1942). Operator Methods in Classical Mechanics, II. *Annals of Mathematics*, 332-350.

[2015] Hart, S. and Reny, P (2015): “Implementation of Reduced Form Mechanisms: a Simple Approach and a New Characterization,” *Economic Theory Bulletin* **3**(1), 1-8.

[2011] Hu, A., Offerman, T., & Zou, L. (2011). Premium Auctions and Risk Preferences. *Journal of Economic Theory* **146**(6), 2420-2439.

[2017] Hu, A., Offerman, T., & Zou, L. (2017). How Risk Sharing May Enhance Efficiency of English Auctions. *The Economic Journal*, **128**(610), 1235-1256.

[2002] Kaas, R., Dhaene, J., Vyncke, D., Goovaerts, M. J., & Denuit, M. (2002). A Simple Geometric Proof that Comonotonic Risks Have the Convex-Largest Sum. *ASTIN Bulletin*: The Journal of the IAA, **32**(1), 71-80.

[1979] Kahneman, D. and Tversky, A. (1979): “Prospect Theory: An Analysis of Decision Under Risk,” *Econometrica* **47**(2), 263–291.

[1989] Karni, E. and Safra, Z. (1986): “Dynamic Consistency, Revelations in Auctions and the Structure of Preferences,” *Review of Economic Studies* ***56***,421-434.

[2009] Kliger, D. and O. Levy (2009), ”Theories of Choice Under Risk: Insights from Financial Markets,” *Journal of Economic Behavior and Organization* **71**, 330-346.

[1997] Luenberger, D. G. (1997). *Optimization by vector space methods*. John Wiley & Sons.

[1989] Machina, M. J. (1989). ”Dynamic Consistency and Non-Expected Utility Models of Choice under Uncertainty”. *Journal of Economic Literature* **27**(4), 1622-1668.

[1984] Maskin, E., & Riley, J. (1984). Optimal Auctions with Risk Averse Buyers. *Econometrica*, 1473-1518.

[1987] Matthews, S., & Moore, J. (1987). Monopoly Provision of Quality and Warranties: An Exploration in the Theory of Multidimensional Screening. *Econometrica*, 441-467.

[1987] Matthews, S. (1987). Comparing Auctions for Risk Averse Buyers: A Buyer's Point of View. *Econometrica* 633-646.

[1983] Matthews, S. A. (1983). Selling to Risk Averse Buyers with Unobservable Tastes. *Journal of Economic Theory*, **30**(2), 370-400.

[1984] Matthews, S. A. (1984), "On the Implementability of Reduced Form Auctions," *Econometrica* **52**, 1519–1522.

[2004] Milgrom, P. (2004). *Putting Auction Theory to Work*, Cambridge: Cambridge University Press.

[1978] Mussa, M., & Rosen, S. (1978). Monopoly and product quality. *Journal of Economic Theory* **18**(2), 301-317.

[1981] Myerson, R.B. (1981) "Optimal Auction Design," *Mathematics of Operations Research*, **6**(1), 58–73.

[1984] Neilson, W. (1994) "Second Price Auctions without Expected Utility," *Journal of Economic Theory* **62**, 136-151

[1979] Meilijson, I., & Nádas, A. (1979). Convex Majorization with an Application to the Length of Critical Paths. *Journal of Applied Probability* **16**(3), 671-677.

[1982] Quiggin, J. (1982). A Theory of Anticipated Utility. J*ournal of Economic Behavior & Organization* **3**(4), 323-343.

[1983] Riley, J., & Zeckhauser, R. (1983). Optimal Selling Strategies: When to Haggle, When to Hold firm. *The Quarterly Journal of Economics*, **98**(2), 267-289.

[1981] Riley, J. and Samuelson, W. (1981), "Optimal Auctions," *American Economic Review*, **71**(3), 381-392.

[1992] Sagan, H. (1992): *Introduction to the Calculus of Variations*, New York: McGraw-Hill.

[1987] Seierstad, A. and Sydsæter, K. (1987): *Optimal Control Theory with Economic Applications*, North Holland.

[2010] Snowberg, E. and J. Wolfers (2010), "Explaining the Favorite-Long Shot Bias: Is it Risk-Love or Misperceptions?" *Journal of Political Economy*, **118**(4), 723-746.

[2010] Sydnor, J. (2010). (Over) insuring Modest Risks. *American Economic Journal: Applied Economics*, **2**(4), 177-199.

[2002] Volij, O. (2002). Payoff Equivalence in Sealed Bid Auctions and the Dual Theory of Choice under Risk. *Economics Letters*, **76**(2), 231-237.

[1987] Yaari, M. E. (1987). The Dual Theory of Choice under Risk. *Econometrica*: 95-115.