Exchange Design and Efficiency*

Marzena Rostek† and Ji Hee Yoon‡

September 29, 2020

Abstract

Most assets clear independently rather than jointly. This paper presents a model based on the uniform-price double auction which accommodates arbitrary restrictions on market clearing, including independent clearing across assets (allowed when demand for each asset is contingent only on the price of that asset) and joint market clearing for all assets (required when demand for each asset is contingent on the prices of all assets). The introduction of additional trading protocols for traded assets or linking existent trading protocols—neutral when the market clears jointly—are generally not redundant innovations, even if all traders participate in all protocols. Multiple trading protocols that clear independently can be designed to be at least as efficient as joint market clearing for all assets. Independence in market clearing can enhance diversification and risk sharing. Except when the market is competitive, market characteristics should guide innovation in trading technology.

JEL Classification: D47, D53, G11, G12

Keywords: Imperfect competition, Trading technology, Market design, Innovation, Decentralized market, Liquidity, Price impact, Uniform-price auction, Efficiency

* We thank numerous colleagues and participants at the Conference on "What We Know About Information in Financial Markets" (UC Santa Barbara, LEAF), the Cowles Conference on Economic Theory, the Philadelphia Search and Matching Conference (Federal Reserve Bank of Philadelphia), Conference on Finance-Related Theory and Experiments (NYU), Conference on "The Changing Regulatory and Technological Landscape of Market Structure" (Vanderbilt, The Financial Markets Research Center), the Conference on Advances in Mechanism and Market Design (NYU Abu Dhabi), the 5th European Meeting on Networks (UCL), the Spring Workshop in Economic Theory (Queen Mary University of London), and seminar participants at Arizona State University, University of Calgary (Finance), Caltech, Carnegie Mellon University/Pittsburgh University, University of Chicago, Columbia University, ECARES, The Federal Reserve Board, HEC Paris (Finance), Higher School of Economics & ICEF (Moscow), IAQF &Thalesians (New York), INSEAD (Finance), Ohio State University, New Economic School (Moscow), Paris School of Economics, Princeton University, Stanford GSB, Stockholm School of Economics (Economics, Finance), UCL, UCSD, University of Cambridge, Washington University (St. Louis) for comments and suggestions. Rostek gratefully acknowledges the financial support from the NSF Grant SES-1851857. Yoon acknowledges the financial support of the University of Wisconsin-Madison and UCL.

† University of Wisconsin-Madison, Department of Economics; E-mail: `mrostek@ssc.wisc.edu`.

‡ University College London, Department of Economics; E-mail: `jihee.yoon@ucl.ac.uk`.

1 Introduction

Today's financial markets are comprised of coexistent trading protocols for the same or distinct assets. Venues for financial securities clear independently and, typically, the assets traded in each venue do as well: an order submitted for one asset cannot be made contingent on the prices of other assets. In some markets, such as those for spectrum, electricity, and electronic trading platforms for financial assets, traders can express their demands for one asset contingent on the prices of other assets.[1] When available, however, such contingent orders allow cross-asset conditioning among a limited number of assets. Feasibility might provide one rationale as to why cross-asset conditioning is relatively uncommon in practice—with contingent schedules, the market-clearing prices must be determined *jointly* for all assets, thus requiring coordination in market clearing among market makers or trading venues that are private entities. Advances in technology have increased interest in cross-asset conditioning.[2]

The objective of this paper is twofold. First, we investigate the implications of independent market clearing for equilibrium and welfare. Second, we examine the innovations in trading technology—defined by changes in market clearing—that cross-asset conditioning makes possible. As we will show, regulation that promotes joint clearing for some assets, if applied in disregard of market characteristics, can lower welfare in the Pareto sense. Moreover, multiple exchanges that clear independently can be designed to be at least as efficient as joint clearing for all assets *irrespective of* the characteristics of assets and traders. Thus, joint market clearing of all assets is inessential and can be suboptimal.

We dispense with the assumption that demand schedules are contingent—on which the standard competitive (e.g., general equilibrium) and imperfectly competitive models of equilibrium and asset pricing are based—in the canonical uniform-price double auction for $I < \infty$ strategic traders and $K < \infty$ assets (e.g., Wilson (1979), Klemperer and Meyer (1989), Kyle (1989), Vives (2011)). Our analysis is cast in the quadratic-Gaussian setting. Traders have private information about their endowments, which are independent across assets and possibly correlated across traders. The model encompasses the standard in theory but less so in practice *contingent schedules* $q_k^{i,c}(\cdot) : \mathbb{R}^K \to \mathbb{R}$, specifying the quantities demanded of each asset for any realization of the price vector (i.e., joint market clearing for all assets).

We first examine markets with *uncontingent schedules* $q_k^i(\cdot) : \mathbb{R} \to \mathbb{R}$, each specifying the quantities demanded for any price realization of a given asset (i.e., assets clear independently). To accommodate innovation in trading technology and more general market structures, we then generalize the model in two ways. Specifically, we permit arbitrary restrictions on cross-asset

[1] E.g., Active Trader Pro, Etrade, Street Smart, Tradehawk. Variants of cross-asset conditioning are available in futures and options markets (e.g., multi-leg orders). The Regulation National Market System and Unlisted Trading Privileges in US stock exchanges *de facto* induce contingent demand schedules; however, analogous rules do not apply in markets for other asset classes or stock markets abroad (see Budish, Lee, and Shim (2019)).

[2] Indeed, electronic trading platforms such as those listed in ft. 1 innovate on such orders.

demand conditioning "between" uncontingent and contingent and allow an asset to be traded in multiple venues. A *market structure* consists of *exchanges*, each defined by the subset of the K assets traded there; all traders participate in all exchanges (see also ft. 12). Demand schedules condition on the prices of the assets traded in an exchange and not on those in other exchanges; the market clears independently across exchanges. It is convenient to identify independent market clearing (i.e., a uniform-price trading protocol) with an exchange in the model; an exchange can thus represent either a trading protocol within a trading venue or the venue itself.

Limited demand conditioning requires a new technique for characterizing equilibrium. In contrast to contingent trading, we cannot rely on the method of characterizing *ex post* optimization. When a trader's demands are not contingent on the prices of all assets, they depend on the expected (rather than realized) trades of the assets in other exchanges. Due to cross-asset inference, the coefficients of a trader's own *best-response* demands must be characterized as a fixed point across assets.[3] Additionally, price impact is no longer a sufficient statistic for a trader's residual supply—the distribution matters as well.

The methodological contribution of the paper is the characterization of the Bayesian Nash equilibrium in markets with limited cross-asset conditioning. The equilibrium fixed point in demand schedules, we show, is equivalent to a fixed point in price impact matrices alone. That is, we endogenize all demand coefficients—including expected trades and the distributions of residual supplies—as functions of price impacts (Theorem 1). We prove the existence of a symmetric linear Bayesian Nash equilibrium in the uniform-price double auction for $K \geq 2$ assets (Theorem 2) and equilibrium uniqueness for $K = 2$ assets.

The paper's second contribution is its implications of independence in market clearing for equilibrium, welfare, and design, which underscore the role of imperfect competition. If joint clearing were feasible, would it maximize total welfare? If the market were competitive ($I \to \infty$), then joint market clearing would be weakly more efficient than any other market structure *regardless of* the characteristics of assets and traders—contingent schedules prevent information loss across exchanges. We show that in imperfectly competitive markets ($I < \infty$), independent market clearing across venues can lower the trading costs associated with per-unit price impact for a given asset and/or across assets. Thus, multi-venue trading changes, respectively, the traders' ability to share an asset's risk and diversify risk across assets. It can increase welfare despite the information loss due to limited demand conditioning.

Central to the effects of multi-venue trading that have no analogues with joint clearing is that it severs the proportionality between the equilibrium price impact and the fundamental assets

[3] In a multivariate optimization problem, contingent or uncontingent, a trader's first-order conditions define a fixed point for the trader's best-response demand schedules across assets. With demands contingent on the common price vector, the first-order conditions can be written as a single matrix equation and solved for the quantity vector pointwise with respect to the price vector. Thus, the best-response demand coefficients need not be characterized as a fixed point.

covariance that holds with contingent trading. First, the cross-venue price impact becomes zero,[4] which may or may not be conducive to efficiency (Example 1). Second, with multiple assets per venue, cross-asset price impacts are asymmetric when either asset covariance or market structure is heterogeneous across assets. Underlying the lack of proportionality are the cross-asset inference effects brought by independence in market clearing. We provide the comparative statics of price impact, which can be higher or lower than in the contingent market, with respect to the asset correlation and number of traders (Theorem 3, Proposition 5 in Appendix C.1).

We present three main results. First, once one departs from the assumption that demands are contingent, innovation in trading technology[5] which would be neutral for traders' equilibrium payoffs with joint clearing (or have no counterparts) is no longer redundant. Independent market clearing motivates two types of innovations that are nonredundant: innovations that reduce inference error for *all* assets (e.g., the linking of existing trading protocols by merging their assets or the inclusion of an asset in a trading protocol where it was not previously traded, such as asset listings) and innovations that alter inference *across* assets without letting demands of any assets be contingent on prices of additional assets (e.g., duplicating a traded asset in a new venue, Example 3). If the market were competitive, the latter type of innovation would be redundant irrespective of demand conditioning.

Proposition 3 shows that one can compare equilibrium and welfare across arbitrary market structures through a pair of sufficient statistics that correspond to a *single*-exchange counterfactual: *per-unit* price impact and cross-asset inference matrices (Proposition 3). Thus, this result also identifies nonredundant innovation. Intuitively, an innovation is not redundant if it alters the trading costs or inference *across assets*.

Second, markets with multiple trading protocols that clear independently can be at least as efficient as a single exchange that clears all assets jointly. We show that one can design a market with multiple protocols—none of which clears all assets—that can function like a single exchange. That is, equilibrium trades, prices, and traders' payoffs are the same with schedules simpler than the contingent ones. Thus, innovation in trading technology can bound welfare at the corresponding contingent level—with no knowledge of traders' preferences and endowments or asset distribution. Equilibrium is *ex post* even if demands in no exchange are contingent on the prices of all assets. Such designs involve sufficiently many trading protocols for different assets, which enables the conditioning variables to eliminate inference errors in traders' expectations.

[4]Nevertheless, equilibrium behavior and outcome (i.e., prices and trades) are not independent across trading venues—unless the asset payoffs are independent.

[5]Or in market structure, depending on whether market clearing is interpreted as applying to trading venues or the trading protocols that they provide—the model accommodates both. Neither alters the traders' endowments, assets' net supply (which we assume to be zero, for simplicity), or participation. Thus, invoking the notion of spanning: in contrast to the contingent model, innovation in market clearing for the traded assets which does not change their payoff span is no longer redundant with independent market clearing.

This equivalence result also characterizes the scope for innovation that can be introduced in a market and be nonredundant. For a market structure to implement equilibrium with joint clearing, one venue per pair of assets suffices—the maximal number of nonredundant protocols is $\frac{K(K-1)}{2}$. Moreover, even in market structures that are not payoff-equivalent to joint clearing, not all new trading protocols affect welfare (Theorem 4). Notably, a new trading protocol whose assets are not all jointly traded in any existing venue can be neutral when price impact matrix is symmetric.

Third, we ask which designs can strictly improve welfare relative to the welfare bound implemented by the designs equivalent to joint clearing. Hinting at the diversity of the trading protocols in practice, the market structure in which all assets clear jointly (i.e., contingent demands or a payoff-equivalent design) is not generally efficient; nor is the market structure in which every asset is traded in a separate venue (i.e., uncontingent demands) efficient irrespective of the market characteristics. In symmetric trading environments,[6] the extreme uncontingent/contingent market structures are optimal with, respectively, asset payoff substitutabilities/complementarities.

A key result (Corollary 3 of Theorem 3) relates the welfare effects of changes in market structure to corresponding changes in per-unit price impact. In two-asset markets, joint clearing minimizes the per-unit price impact for each asset—hence, the cost of risk sharing is the lowest. Thus, to increase welfare, multi-venue trading must lower the trading cost of diversification (i.e., cross-asset price impact). More generally, however, with multiple assets, innovation that increases demand conditioning can raise or lower the price impact costs of *both* diversification and risk sharing.

Our results recognize that the welfare-enhancing exchange design should respond to the number of traders, and *joint* substitutability of the asset payoffs and trading needs of market participants *across assets*.[7] Even if assets' payoffs are all either symmetric substitutes or complements, efficient design depends on whether the market is "one-sided"—i.e., traders want to buy or sell all assets (e.g., the primary market for Treasury securities)—or some assets are demanded while others are supplied (e.g., intra-dealer markets). In imperfectly competitive markets, given the assets and traders, a market structure with multiple venues is more efficient than joint clearing for some distributions of endowments (Proposition 4). In fact, any demerger (i.e., breaking up a single exchange for all assets into multiple venues) can increase welfare.

One might wonder—given that, when trading is dynamic, traders can often condition their demands in one trading venue on past outcomes from other venues—whether the independence in market clearing across trading venues has any effects. Conditioning on past outcomes allows information from past shocks to be (at least partially) incorporated into traders' demands—

[6] With respect to asset correlations, trading needs, and market structure.

[7] In a market with multiple venues, all innovations are neutral if and only if the payoffs of *all* assets are either perfectly correlated (i.e., cross-asset inference is perfect) or independent (i.e., cross-asset inference is absent), or (by design) equilibrium is *ex post*.

contingent or not. Our paper investigates how independence in market clearing affects the way current-round shocks impact behavior and outcomes.[8] Our results thus indicate a role for opaqueness in the form of independent market clearing (embodied in restrictions on cross-asset conditioning), which have implications distinct from transparency requirements (i.e., conditioning on past outcomes).

Other related literature. Our paper contributes to the literature on imperfectly competitive trading (Kyle (1989), Vayanos (1999), Vives (2011), Garleanu and Pedersen (2013), Rostek and Weretka (2015), Bergemann, Heumann, and Morris (2017), Sannikov and Skrzypacz (2016), Du and Zhu (2017a,b), Antill and Duffie (2019), Kyle, Obizhaeva, and Wang (2017), Kyle and Lee (2018), Duffie (2018), Zhang (2020), Zhu (2018a,b)). To our knowledge, we are the first to examine equilibrium and welfare with arbitrary restrictions on cross-asset conditioning and to characterize the (non)redundant exchange design. In fact, little is known about markets with multiple heterogeneous assets outside of settings with *ex post* equilibria. Contemporaneously, in a model with two assets and random supply, Wittwer (2019) shows that traders trade the same amounts with contingent and uncontingent demands if and only if traders' private signals are perfectly correlated and supplies are either zero or perfectly correlated across assets. In a one-asset model with strategic traders and noise traders, Chen and Duffie (2020) show that additional venues increase welfare relative to one venue and the welfare-maximizing number of exchanges is finite.

Apart from financial market applications, the techniques we introduce will be useful to researchers studying games in which agents interact through contracts over multiple goods, actions, or characteristics. One application is to package auctions with large traders, who have price impact. Our results suggest that package bids can be implemented via simpler-than-contingent schedules and limiting the allowable packages that traders can bid for can be efficient. The problem in which players submit uncontingent demand schedules in different trading venues is also related to those studied by the literature on "island" models (in competitive markets) and, more generally, the approach based on Nash-in-Nash.[9] A typical context

[8]With the gains from trade renewed by shocks (to endowment or information), if demands are contingent *and* traders are price-takers ($I \to \infty$), the outcome will be efficient in every round. In a dynamic model with imperfectly competitive traders ($I < \infty$), multiple rounds are needed to realize the gains from trade from a given round's shock. The effects of limited conditioning we identify will be present in all rounds. From the literature on dynamic trading (which is based on *contingent* demands or one-asset markets; e.g., Du and Zhu (2017b), Rostek and Yoon (2019)), two results can be extrapolated beyond contingent demands. First, even in the competitive market, the equilibrium outcome will differ with limited conditioning and contingent demands unless the frequency of the shocks renewing the gains from trade relative to the frequency of trading is low. Second, apart from its contemporaneous effects (this paper), limited conditioning will have temporal effects on price impact. With a finite number of traders ($I < \infty$) and trading rounds, the interaction between the dynamics of price impact and cross-exchange inference also contributes to the difference between the contingent and uncontingent outcomes. Whether the inefficiency of trade that stems from limited demand conditioning can be eliminated in some limit depends on the relative frequencies of the shocks renewing the gains from trade, market-clearing, and payoff realization (consumption).

[9]This solution concept, introduced by Horn and Wolinsky (1988), has become popular in the structural

where Nash-in-Nash has been applied is surplus division in bargaining or contracting with externalities—across contracts and agents—when negotiations are simultaneous. Likewise, in this paper, the demands that a player submits simultaneously for different assets are essentially contracts specifying the quantities demanded as a function of a subset of prices. There are two differences. In Nash-in-Nash, a player agrees to the price in one contract while holding fixed (i) the prices in his other contracts[10] and (ii) the prices to which other players agree. By its virtue of treating prices as contingent variables in traders' demands, the (noncooperative) game in demand functions allows accounting for the cross-contract and cross-player externalities in a Bayesian Nash equilibrium—without employing the Nash-in-Nash counterfactual (which holds prices fixed in other contracts) or restricting how beliefs can change off equilibrium (e.g., passive beliefs; Hart and Tirole (1990)).[11] Our model complements the Nash-in-Nash approach in applications where inefficiencies in surplus sharing arise due to limited inference and imperfect competition, and contracts involve multiple assets with cross-asset externalities. Accounting for imperfect competition along with private information sheds light on how the design of contracts over which agents bargain can enhance the equilibrium surplus when there are cross-contract externalities. With price-taking behavior, contingent contracts are efficient.

Our paper also contributes to the literature on decentralized trading. Markets with contingent schedules are *centralized* because a single market clearing applies to all assets. Accordingly, a market in which assets are traded in separate venues that clear independently is *decentralized*. The assumption that schedules are contingent is the only assumption of the centralized market model that we relax. In particular, assuming that all traders trade all assets with all other traders allows us to focus on those effects of decentralized trading that are due to incomplete conditioning as opposed to incomplete participation.[12] The literature recognizes several argu-

analysis of decentralized markets. As in this paper's model, the applications of Nash-in-Nash have typically considered negotiated contracts, given the set of agreements. See, e.g., Collard-Wexler, Gowrisankaran, and Lee (2019) and references there. We are grateful to an anonymous referee for suggesting we explore the link to the literature on "island" models.

[10]This is typically justified using the "delegated agent" interpretation: a player involved in multiple bilateral bargains relies on separate agents for each negotiation, and these agents cannot communicate with one another while bargaining.

[11]With price-elastic demands, all price realizations occur in equilibrium for some realizations of endowments.

[12]In the centralized market assumption, two assumptions are implicit. First, demand conditioning is complete (i.e., demands are contingent); then, a single aggregation applies to all assets. Second, trader participation in the market is complete in the sense that each trader trades all assets with all other traders. A growing literature on decentralized trading has explored the implications of incomplete participation modeled as fixed or random (hyper)graphs (e.g., Gale (1986a,b), Kranton and Minehart (2001), Duffie, Garleanu, and Pedersen (2005), Vayanos and Weill (2008), Afonso and Lagos (2015), Gofman (2018), Atkeson, Eisfeldt, and Weill (2015), Elliott (2015), Choi, Galeotti, and Goyal (2017), Condorelli, Galeotti, and Renou (2017), Hugonnier, Lester, and Weill (2020), Malamud and Rostek (2017), and Chang and Zhang (2018)). Babus and Kondor (2018), Babus and Parlatore (2017), and Malamud and Rostek (2017) study markets with limited participation and contingent contracts. Interestingly, with decentralized trading in the sense of limited demand conditioning (e.g., this paper) as well as limited participation, the equilibrium price covariance and price impact are not proportional to the asset covariance (see Malamud and Rostek (2017). Yet, the effects on price impact, as well as the underlying mechanisms, are distinct. Indeed, with limited participation and contingent schedules, equilibrium is *ex post*.

ments as to why decentralized trading might be more efficient: it may improve traders' learning about the asset value (Babus and Kondor (2018)) or asset price (Zhu (2014)); it may reduce inefficient screening (Glode and Opp (2016)) or inefficient information aggregation (Kawakami (2017)); it may redistribute risk towards less risk averse traders (Malamud and Rostek (2017)); and it may be more stable than the centralized market (Peivandi and Vohra (2020)). This paper contributes another argument: even if risk preferences are the same among all traders, decentralized trading may improve risk sharing and/or diversification by lowering the trading costs that are due to price impact.[13]

2 Model

Notation. We use the following notation: $(x_k)_k$ is a vector in which the k^{th} element is x_k, and $(y_{k\ell})_{k,\ell}$ is a matrix such that the $(k,\ell)^{\text{th}}$ element is $y_{k\ell}$; sets of the respective elements are denoted by $\{x_k\}_k$ and $\{y_{k\ell}\}_{k,\ell}$. Also, $diag(x_k)_k = diag(x_1, \cdots, x_K)$ is a diagonal matrix in $\mathbb{R}^{K\times K}$ where the k^{th} diagonal element is x_k. The $(k,\ell)^{\text{th}}$ element of matrix $\mathbf{M}$ is denoted by $m_{k\ell}$, and the k^{th} row is denoted by $\mathbf{M}_k$. To distinguish them from scalar variables, vectors and matrices are denoted in bold, and matrices are capitalized.

Market: traders, assets, and exchanges. Consider a market with $I \geq 3$ traders who trade K risky assets in N exchanges. An *exchange* is defined by the assets traded (*listed*) there; all traders participate in all exchanges. In Section 3, to ease exposition, we focus on markets with one asset per exchange, $N = K$; in Sections 4 and 5, we consider exchanges with multiple assets (Definition 4). We index traders by i, assets by k, and exchanges by n.

The payoffs of the K risky assets are jointly normally distributed $\mathbf{r} = (r_k)_k \sim \mathcal{N}(\boldsymbol{\delta}, \boldsymbol{\Sigma})$ with a vector of expected payoffs $\boldsymbol{\delta} = (\delta_k)_k \in \mathbb{R}^K$ and a positive semi-definite covariance matrix $\boldsymbol{\Sigma} \in \mathbb{R}^{K\times K}$. There is also a riskless asset with a zero interest rate (a numéraire).

Each trader i has a quadratic in the quantity of risky assets (mean-variance) utility:

$$u^i(\mathbf{q}^i) = \boldsymbol{\delta} \cdot (\mathbf{q}^i + \mathbf{q}_0^i) - \frac{\alpha^i}{2}(\mathbf{q}^i + \mathbf{q}_0^i) \cdot \boldsymbol{\Sigma}(\mathbf{q}^i + \mathbf{q}_0^i), \tag{1}$$

where $\mathbf{q}^i = (q_k^i)_k \in \mathbb{R}^K$ is trade, $\mathbf{q}_0^i = (q_{0,k}^i)_k \in \mathbb{R}^K$ represents the units of risky assets with which trader i is initially endowed, and $\alpha^i \in \mathbb{R}_+$ is trader i's risk aversion. Endowments $\{\mathbf{q}_0^i\}_i$ are traders' private information and are independent of asset payoffs $\mathbf{r}$. Gains from trade come from risk sharing and diversification: endowments are heterogeneous. All traders are strategic.

In keeping with the literature, to ensure that the per-capita aggregate endowment (equivalently, price) is random in the limit large market ($I \to \infty$), we allow for the common value component $\mathbf{q}_0^{cv} = (q_{0,k}^{cv})_k \in \mathbb{R}^K$ in traders' endowments. For each asset k, privately known

[13] Lester, Shourideh, Venkateswaran, and Zetlin-Jones (2018, 2019) examine the joint effects of an information friction and market power (induced by a search friction) in over-the-counter markets.

endowments $\{q^i_{0,k}\}_i$ are correlated among traders through $q^{cv}_{0,k} \sim \mathcal{N}(E[q^{cv}_{0,k}], \sigma_{cv})$: for each i,

$$q^i_{0,k} = q^{cv}_{0,k} + q^{i,pv}_{0,k}, \qquad q^{i,pv}_{0,k} \sim \mathcal{N}(E[q^{i,pv}_{0,k}], \sigma_{pv}),$$

where $q^{i,pv}_{0,k}$ are independent across i and k.[14] Trader i knows his endowment $\mathbf{q}^i_0$ but not its components $\mathbf{q}^{cv}_0$ or $\mathbf{q}^{i,pv}_0 = (q^{i,pv}_{0,k})_k \in \mathbb{R}^K$. The endowments $\{q^i_{0,k}\}_i$ and the common value $q^{cv}_{0,k}$ are independent across assets k.[15]

Double auction. Each exchange is organized as the uniform-price double auction in which traders submit strictly downward-sloping[16] (net) demand schedules. For $q^i_k > 0$, trader i is a buyer of asset k; for $q^i_k < 0$, he is a seller. We first consider two types of schedules: contingent and uncontingent. In Section 4, we analyze arbitrary cross-asset conditioning.

Definition 1 (Contingent and Uncontingent Schedules) *In a double auction with* contingent schedules, *each trader i submits K demand functions $\mathbf{q}^{i,c}(\cdot) \equiv (q^{i,c}_1(\mathbf{p}), \ldots, q^{i,c}_K(\mathbf{p}))$, each $q^{i,c}_k(\cdot) : \mathbb{R}^K \to \mathbb{R}$ specifying the quantity of asset k demanded for any price vector $\mathbf{p} = (p_1, \ldots, p_K)$.*

In a double auction with uncontingent schedules, *each trader i submits K demand functions $\mathbf{q}^i(\cdot) \equiv (q^i_1(p_1), \ldots, q^i_K(p_K))$, each $q^i_k(\cdot) : \mathbb{R} \to \mathbb{R}$ specifying the quantity of asset k demanded for any price p_k.*

How the market clears is determined by demand conditioning. With uncontingent schedules, the market clears independently across assets: the market-clearing price p_k sets the aggregate net demand in each exchange k to zero, $\sum_i q^i_k(p_k) = 0$. With contingent schedules, the K assets clear *jointly*: the equilibrium price vector is determined by $\sum_i \mathbf{q}^{i,c}(p_1, \cdots, p_K) = \mathbf{0} \in \mathbb{R}^K$. With either type of schedule, trader i trades $\{q^i_k\}_k$, pays $\sum_k p_k q^i_k$, and receives a payoff of $u^i(\mathbf{q}^i) - \mathbf{p} \cdot \mathbf{q}^i$.

Equilibrium. We study the Bayesian Nash equilibrium in linear demand schedules (hereafter, *equilibrium*).

Definition 2 (Equilibrium) *A profile of (net) demand schedules $\{\{q^i_k(\cdot)\}_k\}_i$ is a Bayesian Nash equilibrium if, for each i, $\{q^i_k(\cdot)\}_k$ maximizes the expected payoff:*

$$\max_{\{q^i_k(\cdot)\}_k} E[\boldsymbol{\delta} \cdot (\mathbf{q}^i + \mathbf{q}^i_0) - \frac{\alpha^i}{2}(\mathbf{q}^i + \mathbf{q}^i_0) \cdot \boldsymbol{\Sigma}(\mathbf{q}^i + \mathbf{q}^i_0) - \mathbf{p} \cdot \mathbf{q}^i | \mathbf{q}^i_0], \tag{2}$$

given the schedules of other traders $\{\{q^j_k(\cdot)\}_k\}_{j \neq i}$ and market clearing $\sum_j q^j_k(\cdot) = 0$ for all k.

[14] The common value component in $\{\mathbf{q}^i_0\}_i$ affects the magnitude of inference coefficients, but does not affect any results qualitatively.

[15] For simplicity, we assume the symmetry of variance across traders and the independence of endowments across assets; the results hold qualitatively without these assumptions. Our equilibrium characterization in Appendix A allows for correlated endowments across assets that are symmetrically correlated across traders.

[16] I.e., the Jacobian of demand schedules $\frac{\partial \mathbf{q}^i(\cdot)}{\partial \mathbf{p}} = \left(\frac{\partial q^i_k(\cdot)}{\partial p_\ell}\right)_{k,\ell} \in \mathbb{R}^{K \times K}$ is negative semi-definite. This rules out trivial equilibria with no trade.

As is well known, in markets with contingent schedules, equilibrium is invariant to the distribution of private endowments; i.e., the linear Bayesian Nash equilibrium with (possibly correlated) private endowments has an *ex post* property.[17] The contingent schedule allows a trader to choose his demand for each asset as a function of all prices to be realized, which map one-to-one to realizations of quantities traded of other assets. With uncontingent schedules, equilibrium is not generally *ex post*. Given the quasilinear-quadratic utility, traders face uncertainty both with respect to the price and payoff; their expected payoff (2) penalizes the latter, but not the former.

Competitive market. The competitive market will often serve as a benchmark when evaluating the effects of incomplete conditioning with imperfectly competitive traders.[18]

Definition 3 (Competitive Market, Competitive Equilibrium) *Consider a market with $I < \infty$ traders. The* competitive market *is the limit game as $I \to \infty$, holding fixed all other primitives. Letting $\{\mathbf{q}^{i,I}(\cdot)\}_i$ be the equilibrium in the market with $I < \infty$ traders, the* competitive equilibrium *$\{\mathbf{q}^i(\cdot)\}_i$ is the limit of equilibria $\{\mathbf{q}^{i,I}(\cdot)\}_i$ as $I \to \infty$:*

$$\mathbf{q}^i(\cdot) = \lim_{I\to\infty} \mathbf{q}^{i,I}(\cdot) \qquad \forall i.$$

3 Equilibrium: Contingent vs. Uncontingent Demands

In this section, we characterize equilibrium in markets with uncontingent demands (Proposition 2, Theorem 1, and Corollary 1). For the sake of comparison, we also review equilibrium with contingent demands.

Although the contingent and uncontingent models are quite different, equilibria in both models can be characterized through parallel conditions (Propositions 1 and 2). First, a key argument (Lemma 2 in Appendix B) shows that the well-known equivalence between individual trader optimization in demand functions (2) and *pointwise* optimization with respect to the realizations of $\mathbf{p} \in \mathbb{R}^K$ in the contingent model also holds in the uncontingent model with respect to the realizations of the relevant contingent variable, i.e., $p_k \in \mathbb{R}$. Both pointwise problems are motivated by the observation that when traders submit demand schedules contingent on price realizations (of any subset of assets), it is useful to adopt the perspective of an individual

[17] Equilibrium is *linear* if schedules have the functional form of $\mathbf{q}^i(\cdot) = \boldsymbol{a}^i + \mathbf{B}^i\mathbf{q}_0^i + \mathbf{C}^i\mathbf{p}$. Equilibrium is *ex post* if equilibrium schedules $\{q_k^i(\cdot;\mathbf{q}_0^i)\}_k$ are optimal for all i, given endowment realizations for *all* traders $\{\mathbf{q}_0^j\}_j$:

$$\{q_k^i(\cdot;\mathbf{q}_0^i)\}_k = \arg\max_{\{q_k^i(\cdot)\}_k} E[\boldsymbol{\delta}\cdot(\mathbf{q}^i+\mathbf{q}_0^i) - \frac{\alpha^i}{2}(\mathbf{q}^i+\mathbf{q}_0^i)\cdot\boldsymbol{\Sigma}(\mathbf{q}^i+\mathbf{q}_0^i) - \mathbf{p}\cdot\mathbf{q}^i|\{\mathbf{q}_0^j\}_j].$$

[18] The common value component $\mathbf{q}_0^{cv}$ in traders' endowments $\{\mathbf{q}_0^i\}_i$ ensures that the price (equivalently, the per-capita aggregate endowment) is random in the limit large market ($I \to \infty$). To make the price variance $Var[\mathbf{p}|\mathbf{q}_0^i] = \boldsymbol{\Sigma} Var[(\sum_j \frac{1}{\alpha^j})^{-1}\mathbf{q}_0^j|\mathbf{q}_0^i]\boldsymbol{\Sigma}'$ independent of the number of traders I, the risk aversion α^i in utility (1) can be scaled according to $\alpha^{i,I} \equiv \alpha^i \frac{I}{I-1}\sqrt{(\sigma_{cv}+\frac{1}{I}\sigma_{pv})^{-1}\sigma_{cv}}$. As $I\to\infty$, $\alpha^{i,I}\to\alpha^i>0$ for all i. More generally, one can jointly scale $(\alpha^i,\sigma_{cv},\sigma_{pv})$.

trader who optimizes against a profile of his residual supply functions, which is the sufficient statistic of a residual market $\{\{q_k^j(\cdot)\}_k\}_{j\neq i}$.[19] The residual supply $S_k^{-i,c}(\cdot):\mathbb{R}^K\to\mathbb{R}$ for asset k is a function of $\mathbf{p}$ if demands are contingent ($\{q_k^{i,c}(\cdot):\mathbb{R}^K\to\mathbb{R}\}_i$), and $S_k^{-i}(\cdot):\mathbb{R}\to\mathbb{R}$ is a function of p_k if demands are uncontingent ($\{q_k^i(\cdot):\mathbb{R}\to\mathbb{R}\}_i$). Second, in equilibrium, the residual supply functions are correct: $S_k^{-i}(\cdot)=-\sum_{j\neq i}q_k^j(\cdot)$ for all k by aggregation through market clearing of the other traders' submitted schedules.

3.1 Equilibrium with Contingent Demands

Consider the optimization problem (2) of trader i who submits demand schedules $\mathbf{q}^{i,c}(\cdot):\mathbb{R}^K\to\mathbb{R}^K$ contingent on price realizations for all assets $\mathbf{p}\in\mathbb{R}^K$.[20] It is well known that maximizing the expected payoff (2) is the same as maximizing the *ex post* payoff pointwise: for each asset k,

$$\max_{q_k^{i,c}\in\mathbb{R}}\{\boldsymbol{\delta}\cdot(\mathbf{q}^{i,c}+\mathbf{q}_0^i)-\frac{\alpha^i}{2}(\mathbf{q}^{i,c}+\mathbf{q}_0^i)\cdot\boldsymbol{\Sigma}(\mathbf{q}^{i,c}+\mathbf{q}_0^i)-\mathbf{p}\cdot\mathbf{q}^{i,c}\}\qquad\forall\mathbf{p}\in\mathbb{R}^K,\tag{3}$$

given the trader's demands for other assets $\{q_\ell^{i,c}(\cdot)\}_{\ell\neq k}$ and his residual supply function $\mathbf{S}^{-i,c}(\cdot):\mathbb{R}^K\to\mathbb{R}^K$ for all assets. In essence, the equivalence follows because the demand for each asset is measurable with respect to $\{\mathbf{p},\mathbf{q}_0^i\}$ (i.e., the contingent variable $\mathbf{p}$ and the privately known endowment vector $\mathbf{q}_0^i$) and, as we will show, price distribution has full support (see Remark 1).

Coupled with the requirement that the residual supply is correct, i.e., $\mathbf{S}^{-i,c}(\cdot)=-\sum_{j\neq i}\mathbf{q}^{j,c}(\cdot)$ for all i, pointwise optimization leads to an equilibrium characterization in terms of two simple conditions (Proposition 1).

Step 1 (Optimization, given price impact) The first-order condition with respect to the demand for each asset $q_k^{i,c}$ is: for each k,

$$\underbrace{\delta_k-\alpha^i(\sigma_{kk}(q_k^{i,c}+q_{0,k}^i)+\sum_{\ell\neq k}\sigma_{k\ell}(q_\ell^{i,c}+q_{0,\ell}^i))}_{\text{Marginal utility for asset }k}=\underbrace{p_k+\frac{dp_k}{dq_k^{i,c}}q_k^{i,c}+\sum_{\ell\neq k}\frac{dp_\ell}{dq_k^{i,c}}q_\ell^{i,c}}_{\text{Marginal payment for asset }k}\quad\forall\mathbf{p}\in\mathbb{R}^K.\tag{4}$$

In a linear equilibrium,[21] $\frac{dp_\ell}{dq_k^{i,c}}\equiv\lambda_{k\ell}^{i,c}$ is constant for each k,ℓ and i. Written in matrix form, the

[19] The idea of considering the pointwise optimization problem of a single trader, taking as given his residual market, goes back to Klemperer and Meyer (1989) and Kyle (1989). Rostek and Weretka (2015) introduced the equilibrium characterization in terms of the fixed point in price impacts (Proposition 1 below), showing equivalence between the equilibrium conditions in Definition 2 and Proposition 1. The characterization of equilibrium with contingent demands for heterogeneous risk aversions (Proposition 1) is from Malamud and Rostek (2017).

[20] A unilateral demand change of trader i is understood as a profile of arbitrary twice continuously differentiable functions $\{\Delta q_k^i(\cdot):\mathbb{R}^K\to\mathbb{R}\}_k$ so that $\mathbf{q}^i(\cdot)+\Delta\mathbf{q}^i(\cdot)$ is downward-sloping with respect to $\mathbf{p}\in\mathbb{R}^K$, i.e., the Jacobian $\frac{\partial(\mathbf{q}^i(\cdot)+\Delta\mathbf{q}^i(\cdot))}{\partial\mathbf{p}}\in\mathbb{R}^{K\times K}$ is negative semi-definite.

[21] More precisely, assuming that the best-response demands of traders $j\neq i$ are linear.

first-order conditions (4) become:

$$\boldsymbol{\delta} - \alpha^i \boldsymbol{\Sigma}(\mathbf{q}^{i,c} + \mathbf{q}_0^i) = \mathbf{p} + \boldsymbol{\Lambda}^{i,c}\mathbf{q}^{i,c} \qquad \forall \mathbf{p} \in \mathbb{R}^K, \tag{5}$$

where matrix

$$\boldsymbol{\Lambda}^{i,c} \equiv \frac{d\mathbf{p}}{d\mathbf{q}^{i,c}} = (\frac{dp_\ell}{dq_k^{i,c}})_{k,\ell} = \begin{bmatrix} \frac{dp_1}{dq_1^{i,c}} & \cdots & \frac{dp_K}{dq_1^{i,c}} \\ \vdots & \ddots & \vdots \\ \frac{dp_1}{dq_K^{i,c}} & \cdots & \frac{dp_K}{dq_K^{i,c}} \end{bmatrix} \in \mathbb{R}^{K\times K}$$

is the *price impact* of trader i. Its $(k,\ell)^{\text{th}}$ element $\lambda_{k\ell}^{i,c}$ represents the price change in asset ℓ following a demand change in asset k by trader i. The inverse of price impact is a common measure of *liquidity*: the lower the price impact, the smaller the price concession a trader must accept, the more liquid the market. From the first-order condition (5), the best-response demand of trader i is:

$$\mathbf{q}^{i,c}(\mathbf{p}) = (\alpha^i \boldsymbol{\Sigma} + \boldsymbol{\Lambda}^{i,c})^{-1}(\boldsymbol{\delta} - \mathbf{p} - \alpha^i \boldsymbol{\Sigma}\mathbf{q}_0^i) \qquad \forall \mathbf{p} \in \mathbb{R}^K, \tag{6}$$

given his price impact $\boldsymbol{\Lambda}^{i,c}$, which is a sufficient statistic for trader i's residual supply function (see Remark 2) and is endogenized in Step 2.

Step 2 (Correct price impact) In equilibrium, the price impact in the pointwise first-order condition (5) of trader i must be correct, i.e., must equal the transpose of the $K \times K$ Jacobian matrix of the trader's inverse residual supply function. Applying market clearing to the best-response demands (6) for traders $j \neq i$ yields the residual supply function $\mathbf{S}^{-i,c}(\cdot)$ of trader i:

$$\mathbf{S}^{-i,c}(\mathbf{p}) = -\sum_{j\neq i}(\alpha^j \boldsymbol{\Sigma} + \boldsymbol{\Lambda}^{j,c})^{-1}(\boldsymbol{\delta} - \alpha^j \boldsymbol{\Sigma}\mathbf{q}_0^j) + \sum_{j\neq i}(\alpha^j \boldsymbol{\Sigma} + \boldsymbol{\Lambda}^{j,c})^{-1}\mathbf{p} \qquad \forall \mathbf{p} \in \mathbb{R}^K. \tag{7}$$

The price impact of trader i is $\boldsymbol{\Lambda}^{i,c} \equiv (\frac{dp_\ell}{dq_k^{i,c}})_{k,\ell} = \Big(\Big(\frac{\partial \mathbf{S}^{-i,c}(\cdot)}{\partial \mathbf{p}}\Big)^{-1}\Big)'$.

Proposition 1 gives an *equivalent* characterization of the equilibrium in demand schedules by two conditions: (i) traders optimize, given their assumed price impacts, (ii) which are correct.

Proposition 1 (Equilibrium: Contingent Trading) A profile of (net) demand schedules $\{\mathbf{q}^{i,c}(\cdot)\}_i$ is a linear Bayesian Nash equilibrium if and only if, for each trader i,

(i) (*Optimization, given price impact*) Demand schedules $\mathbf{q}^{i,c}(\cdot) : \mathbb{R}^K \to \mathbb{R}^K$ are determined by pointwise equalization of marginal utility and marginal payment in Eq. (6), given his price impact $\boldsymbol{\Lambda}^{i,c}$, such that:

(ii) (*Correct price impact*) The price impact of trader i equals the transpose of the Jacobian of his inverse residual supply function:

$$\boldsymbol{\Lambda}^{i,c} = \Big(\Big(\sum\nolimits_{j\neq i}(\alpha^j \boldsymbol{\Sigma} + \boldsymbol{\Lambda}^{j,c})^{-1}\Big)^{-1}\Big)'. \tag{8}$$

The fixed point for price impact matrices defined by the system of I equations (8) can be solved in closed form when demands are contingent: for each i,

$$\mathbf{\Lambda}^{i,c} = \beta^{i,c}\alpha^i\mathbf{\Sigma}, \tag{9}$$

where $\beta^{i,c} = \frac{2-\alpha^i b+\sqrt{(\alpha^i b)^2+4}}{2\alpha^i b} \in \mathbb{R}_+$ and $b \in \mathbb{R}_+$ is the unique solution to $\sum_j(\alpha^j b + 2 + \sqrt{(\alpha^j b)^2+4})^{-1} = 1/2$. If risk aversions are symmetric (i.e., $\alpha^i = \alpha$ for all i), then price impact is $\mathbf{\Lambda}^{i,c} = \frac{\alpha}{I-2}\mathbf{\Sigma}$.

Analyzing price impact directly offers insights into the role of imperfectly competitive behavior. As $I \to \infty$, then $\mathbf{\Lambda}^{i,c} \to \mathbf{0}$ for all i,[22] and the competitive limit demand coincides with the inverse marginal utility, given the quasilinearity of the payoff function. When price impact is positive, $\mathbf{\Lambda}^{i,c} > \mathbf{0}$, trader i demands (or sells) less than if he had submitted his competitive schedule.

Remarks. We note four properties, which—with the exception of the second—do not hold when demands are not contingent.

1. By (9), the price impact of trader i derives from the utility concavity of the residual market $\{\alpha^j\mathbf{\Sigma}\}_{j\neq i}$[23] and, with contingent trading, the equilibrium price impact of every trader is *proportional* to the fundamental covariance matrix $\mathbf{\Sigma}$. This proportionality has important implications for how the contingent market functions and, as we will show, does not hold with limited demand conditioning (cf. Theorem 3, Proposition 5 in Appendix C.1).

2. All price realizations $\mathbf{p} \in \mathbb{R}^K$ occur in equilibrium for some realizations of endowments, given the traders' downward-sloping demands (i.e., the Jacobian $\frac{\partial \mathbf{q}^{i,c}(\cdot)}{\partial \mathbf{p}} = -(\alpha^i\mathbf{\Sigma}+\mathbf{\Lambda}^i)^{-1} < \mathbf{0}$).[24] Hence, the first-order conditions must hold for all prices, and the price impact of each trader is determined by the requirement that optimization, Bayesian inference, and market clearing hold in equilibrium *and* following a unilateral demand change. The market-clearing condition (Definition 2) is accounted for by condition (ii) for price impacts (Eq. (8)).

3. A trader's own price impact $\mathbf{\Lambda}^{i,c}$ is a sufficient statistic for the residual supply function in the best-response problem. This holds due to the one-to-one map between the contingent variable (i.e., price vector $\mathbf{p}$) and the residual supply's intercept (i.e., the vector $\mathbf{s}^{-i,c} \equiv -\sum_{j\neq i}(\alpha^j\mathbf{\Sigma} + \mathbf{\Lambda}^{j,c})^{-1}(\boldsymbol{\delta} - \alpha^j\mathbf{\Sigma}\mathbf{q}_0^j) \in \mathbb{R}^K$ in Eq. (7)) for all assets.

4. Equilibrium is *ex post*, given the one-to-one map.

[22] Price impact converges to zero as $I \to \infty$ so long as the risk aversion $\alpha^{i,I}$ increases slower than linearly, i.e., $\alpha^{i,I} = \alpha^i\gamma^I$ where $\gamma^I \sim o(I^{1-\varepsilon})$ for some $\varepsilon \in (0,1)$ (Lemma 3 in Appendix B). Aggregate endowment is random in the limit, provided that traders' endowments are correlated via $\mathbf{q}_0^{cv}$ ($\sigma_{cv} > 0$); see ft. 18.

[23] The riskier the assets traded and the more risk averse trader i's counterparties, the less elastic their marginal utilities, the less elastic the residual supply of trader i, and the larger the price concessions that i has to accept.

[24] If $\mathbf{M}$ is not invertible, $\mathbf{M}^{-1}$ is the Moore-Penrose pseudoinverse of $\mathbf{M}$.

3.2 Equilibrium with Uncontingent Demands

Consider the optimization problem (2) of trader i in a market with K exchanges, each for one asset, who submits demand schedules $\{q^i_k(\cdot) : \mathbb{R} \to \mathbb{R}\}_k$.[25] The trader's objective function is the same as with contingent trading; in particular, his information set (i.e., $\mathbf{q}^i_0$) is. However, the choice variable differs: the demand in the exchange for asset k is contingent on, and hence measurable with respect to, price p_k only. Consequently, maximizing expected payoff (2) is not the same as maximizing *ex post* payoff.

Proposition 2 establishes that, analogously to the contingent market (Proposition 1), a trader's pointwise optimization for each asset k, now with respect to $p_k \in \mathbb{R}$, is necessary and sufficient for optimization in demand functions (i.e., $\{q^i_k(\cdot)\}_k$): for each asset k,

$$\max_{q^i_k \in \mathbb{R}} E[\boldsymbol{\delta} \cdot (\mathbf{q}^i + \mathbf{q}^i_0) - \frac{\alpha^i}{2}(\mathbf{q}^i + \mathbf{q}^i_0) \cdot \boldsymbol{\Sigma}(\mathbf{q}^i + \mathbf{q}^i_0) - \mathbf{p} \cdot \mathbf{q}^i | p_k, \mathbf{q}^i_0] \qquad \forall p_k \in \mathbb{R}, \tag{10}$$

given his demands for other assets $\{q^i_\ell(\cdot)\}_{\ell \neq k}$ and a profile of residual supply functions $\{S^{-i}_\ell(\cdot) : \mathbb{R} \to \mathbb{R}\}_\ell$ for all assets. Proposition 2 also shows that asset by asset optimization is without loss of generality by the Fréchet differentiability of expected payoff (2) with respect to the profile of demands $\{q^i_k(\cdot)\}_k$ (Lemma 2 in Appendix B), and that the second-order condition holds, given downward-sloping demands.

Compared to contingent trading (Eq. (4)), the first-order condition differs in two ways:

$$\underbrace{\delta_k - \alpha^i(\sigma_{kk}(q^i_k + q^i_{0,k}) + \sum_{\ell \neq k} \sigma_{k\ell}(\underbrace{E[q^i_\ell | p_k, \mathbf{q}^i_0]}_{\substack{\text{Expected trade} \\ \text{of asset } \ell}} + q^i_{0,\ell}))}_{\text{Expected marginal utility for asset } k} = \underbrace{p_k + \underbrace{\lambda^i_k}_{\substack{\text{Zero cross-exchange} \\ \text{price impact}}} q^i_k}_{\text{Marginal payment for asset } k} \qquad \forall p_k \in \mathbb{R}, \tag{11}$$

where $\lambda^i_k \equiv \frac{dp_k}{dq^i_k} \in \mathbb{R}_+$ is the price impact of trader i in the exchange for asset k; in a linear equilibrium, λ^i_k is constant. First, a trader's demand for asset k depends on *expected rather than realized trades* of other assets $\ell \neq k$, $E[q^i_\ell | p_k, \mathbf{q}^i_0]$. Equilibrium is generally not *ex post*. Second, *the cross-exchange price impact is zero*: $\lambda^i_{k\ell} \equiv \frac{dp_\ell}{dq^i_k} = 0$ for all k and $\ell \neq k$, since the residual supply function $S^{-i}_k(\cdot; \{\mathbf{q}^j_0\}_{j \neq i}) : \mathbb{R} \to \mathbb{R}$ is contingent on p_k but not $\{p_\ell\}_{\ell \neq k}$. It follows that, in contrast to the contingent market, where the price impact matrices of all traders are proportional to the covariance matrix $\boldsymbol{\Sigma}$ (Eq. (9) and Remark 1), the price impacts of all traders are diagonal matrices: for each i,

$$\boldsymbol{\Lambda}^i \equiv \left(\frac{dp_\ell}{dq^i_k}\right)_{k,\ell} = diag(\lambda^i_k)_k \in \mathbb{R}^{K \times K}. \tag{12}$$

Although the cross-exchange price impact is eliminated, equilibrium behavior and outcome (i.e., prices and allocations) are not independent across exchanges—unless all assets' payoffs

[25] A unilateral demand change of trader i is understood as a profile of arbitrary twice continuously differentiable functions $\{\Delta q^i_k(\cdot) : \mathbb{R} \to \mathbb{R}\}_k$ so that $q^i_k(\cdot) + \Delta q^i_k(\cdot)$ is downward-sloping, i.e., $\frac{\partial(q^i_k(\cdot) + \Delta q^i_k(\cdot))}{\partial p_k} < 0$ for all k.

are independent (i.e., $\sigma_{k\ell} = 0$ for all $\ell \neq k$), in which case traders' utility Hessian is separable.

Proposition 2 takes the intercept of trader i's residual supply s_k^{-i} rather than price p_k as a contingent variable—s_k^{-i} is exogenous in the best-response problem of trader i. This allows us to separate the best response and equilibrium problems analogously to Proposition 1: (i) optimization by trader i, given i's residual supply (Step 1); (ii) which is correct (Step 2).

For each trader i, let $F((\mathbf{q}_0^j)_{j\neq i}|\mathbf{q}_0^i)$ be the joint distribution of other traders' endowments and let $F(\mathbf{s}^{-i}|\mathbf{q}_0^i)$ be the joint distribution of the intercepts $\mathbf{s}^{-i} \equiv (s_k^{-i})_k$ of the residual supplies of trader i—both conditional on trader i's privately known endowment. The former distribution is a primitive object; the latter is not, but it is taken as given in trader i's best-response problem. Given the linear demands $\{\mathbf{q}^j(\cdot)\}_{j\neq i}$, $F(\mathbf{s}^{-i}|\mathbf{q}_0^i)$ is jointly Normal.

Proposition 2 (Equilibrium: Uncontingent Trading) A profile of (net) demand schedules $\{\{q_k^i(\cdot)\}_k\}_i$ is a linear Bayesian Nash equilibrium if and only if, for each trader i,

(i) (*Optimization, given residual supply*) Demand schedules $q_k^i(\cdot) : \mathbb{R} \to \mathbb{R}$ are determined by equalization of expected marginal utility and marginal payment for each asset k pointwise to $p_k \in \mathbb{R}$:

$$\delta_k - \alpha^i \mathbf{\Sigma}_k E[\mathbf{q}^i + \mathbf{q}_0^i | s_k^{-i}, \mathbf{q}_0^i] \;=\; p_k + \lambda_k^i q_k^i \qquad \forall p_k \in \mathbb{R}, \tag{13}$$

given the trader's own demands for other assets $\{q_\ell^i(\cdot)\}_{\ell\neq k}$, the distribution $F(\mathbf{s}^{-i}|\mathbf{q}_0^i)$, and price impact $\mathbf{\Lambda}^i = diag(\lambda_k^i)_k$.[26]

(ii) (*Correct residual supply*) The residual supply function $S_k^{-i}(\cdot) : \mathbb{R} \to \mathbb{R}$ of trader i is determined by applying market clearing to the best responses of traders $j \neq i$ $\{q_k^j(\cdot)\}_{j\neq i}$ that satisfy condition (i): for each k,

$$S_k^{-i}(\cdot) = -\sum_{j\neq i} q_k^j(\cdot).$$

The price impact λ_k^i of trader i is characterized by the slope of $(S_k^{-i}(\cdot))^{-1}$. The distribution $F(\mathbf{s}^{-i}|\mathbf{q}_0^i)$ is characterized by the intercept of $S_k^{-i}(\cdot)$, given $F((\mathbf{q}_0^j)_{j\neq i}|\mathbf{q}_0^i)$.

Because of cross-asset inference $\{\{E[q_\ell^i|s_k^{-i}, \mathbf{q}_0^i]\}_{\ell\neq k}\}_k$, one cannot rely on the method of *ex post* optimization, and this makes the equilibrium characterization more challenging in two ways.

First, the price impact $\mathbf{\Lambda}^i$ is not by itself a sufficient statistic for the residual supply of trader i (cf. Remark 2)—the joint distribution of the conditioning variable $\mathbf{s}^{-i}$ (equivalently, $\mathbf{p}$) matters. Second, in a trader's best response (11) for asset k, expected trades $E[q_\ell^i|p_k, \mathbf{q}_0^i]$ depend on the distribution of his endogenous quantity traded of other assets, $\{q_\ell^i\}_{\ell\neq k}$. Therefore,

[26] Given the one-to-one map between s_k^{-i} and p_k, expected trades $E[q_\ell^i|p_k, \mathbf{q}_0^i] = E[q_\ell^i|s_k^{-i}, \mathbf{q}_0^i]$ for $\ell \neq k$ are characterized by the Projection Theorem, given $F(\mathbf{s}^{-i}|\mathbf{q}_0^i)$ and $\mathbf{\Lambda}^i$.

characterizing a trader's own best-response demands requires solving a fixed point for the trader's own demand coefficients *across assets*.

To elaborate on the latter point, in a trader's multivariate optimization problem (2)—contingent or uncontingent—the system of first-order conditions defines a fixed point problem among the trader's best-response schedules. With contingent demands, the system can be written as a single matrix equation, $\boldsymbol{\delta} - \alpha^i \boldsymbol{\Sigma}(\mathbf{q}^{i,c} + \mathbf{q}_0^i) = \mathbf{p} + \boldsymbol{\Lambda}^{i,c}\mathbf{q}^{i,c}$ for all $\mathbf{p} \in \mathbb{R}^K$ (Eq. (5)) for a quantity vector $\mathbf{q}^{i,c} = (q_k^{i,c})_k$ as a function of the common conditioning variable $\mathbf{p}$, and can be solved for $\mathbf{q}^{i,c}$ pointwise with respect to $\mathbf{p}$. (Eq. (6) gives the closed-form solution.) Thus, the coefficients of the best-response functions need not be characterized as a fixed point.

Theorem 1 in Section 3.2.2 endogenizes all demand coefficients (Step 1)—including expected trades $\{\{E[q_\ell^i|p_k, \mathbf{q}_0^i]\}_{\ell \neq k}\}_k$—and the distribution of the residual supply (Step 2) as functions of price impacts $\{\boldsymbol{\Lambda}^i\}_i$. It thus shows that *a fixed point in uncontingent demand schedules* $\{\{q_k^i(\cdot)\}_k\}_i$ *is equivalent to a fixed point in price impact matrices.*

3.2.1 Preview

Before introducing a technique to characterize equilibrium in uncontingent markets, Example 1 provides a preview of the results that follow.

Example 1 (Price Impact with Uncontingent Demands) Consider a market with two imperfectly correlated assets, $0 < |\rho_{12}| < 1$, $\rho_{12} = \frac{\sigma_{12}}{\sqrt{\sigma_{11}\sigma_{22}}}$:

$$\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}, \qquad \boldsymbol{\Lambda}^{i,c} = \begin{bmatrix} \lambda_1^{i,c} & \lambda_{12}^{i,c} \\ \lambda_{21}^{i,c} & \lambda_2^{i,c} \end{bmatrix}, \qquad \boldsymbol{\Lambda}^{i} = \begin{bmatrix} \lambda_1^{i} & 0 \\ 0 & \lambda_2^{i} \end{bmatrix}.$$

In the contingent market, by the proportionality of price impact $\boldsymbol{\Lambda}^{i,c}$ in the covariance $\boldsymbol{\Sigma}$ (Eq. (9) and Remark 1), the cross-asset price impact inherits the covariance's sign. When the assets are payoff substitutes, i.e., $\sigma_{12} > 0$, for the traders who take the same (buying or selling) position in both assets, the cross-asset price impacts $\lambda_{12}^{i,c} > 0$ and $\lambda_{21}^{i,c} > 0$ *increase* the marginal trading cost of each asset:

$$p_1 + \lambda_1^{i,c} q_1^{i,c} + \lambda_{12}^{i,c} q_2^{i,c}, \qquad p_2 + \lambda_2^{i,c} q_2^{i,c} + \lambda_{21}^{i,c} q_1^{i,c}, \tag{14}$$

thereby exacerbating the demand reduction relative to the competitive demand. When the assets are payoff complements, i.e., $\sigma_{12} < 0$, the negative cross-asset price impacts $\lambda_{12}^{i,c} < 0$ and $\lambda_{21}^{i,c} < 0$ *lower* the trading costs. These effects are absent with uncontingent demands—the cross-asset price impacts λ_{12}^i and λ_{21}^i are zero. Moreover, as we will show, the within-exchange price impacts λ_1^i and λ_2^i change (Theorem 3). □

As Example 1 indicates (and Corollary 4 will demonstrate), letting assets clear independently can increase welfare in some trading environments. Unlike the competitive market, the characteristics of traders and assets matter for which design is efficient. In particular, when trading is imperfectly competitive, neither the market structure in which all assets clear jointly nor that in which each asset is traded in a separate exchange is always efficient.

We will show that one can design a market with multiple venues that clear independently which, *for any characteristics of traders and assets*, is as efficient as a single exchange that clears all assets jointly (Section 4). Thus, suitable design can implement a bound on welfare with no knowledge of traders' preferences or endowments. In fact, multi-venue design can be strictly more efficient than joint clearing (Section 5). Underlying these results it is that innovation that would be neutral for traders' payoffs with joint clearing (if well defined at all) is no longer redundant—another consequence of the nonproportionality between the price impact matrix $\mathbf{\Lambda}^i$ and the covariance matrix $\mathbf{\Sigma}$.

3.2.2 Equilibrium as a Fixed Point in Price Impacts

This section presents our main characterization result, Theorem 1. To tackle the characterization of the fixed point problem for a trader's best-response schedules $\{q_k^i(\cdot)\}_k$, we first transform the system of first-order conditions (11) into a fixed point among the trader's demand coefficients, given the residual supplies, i.e., $\mathbf{\Lambda}^i$ and $F(\mathbf{s}^{-i}|\mathbf{q}_0^i)$ (Step 1). We then endogenize the distribution of the residual supply—and thus all demand coefficients, including expected trades $E[q_\ell^i|p_k,\mathbf{q}_0^i]$ for all $\ell \neq k$ and k—as a function of price impacts $\{\mathbf{\Lambda}^i\}_i$ and characterize equilibrium as the fixed point for $\{\mathbf{\Lambda}^i\}_i$ (Step 2).

Best-response problem (Step 1). In each exchange k, we parameterize a trader's conjectured best responses for other assets $\ell \neq k$ as linear functions of p_ℓ and $\mathbf{q}_0^i$:

$$q_\ell^i(p_\ell) \equiv a_\ell^i - \mathbf{b}_\ell^i\mathbf{q}_0^i - c_\ell^i p_\ell \qquad \forall p_\ell \in \mathbb{R}, \tag{15}$$

with the demand intercept $a_\ell^i \in \mathbb{R}$, the demand coefficients $\mathbf{b}_\ell^i \in \mathbb{R}^{1\times K}$ on $\mathbf{q}_0^i$, and the demand slope $c_\ell^i \in \mathbb{R}_+$ on p_ℓ.[27]

The parameterization of demands (15) allows us to characterize the change in the contingent variable (from p_k to s_k^{-i}) and endogenize expected trades in terms of variables that are exogenous in the trader's best-response problem for asset k. By market clearing, given the residual supply $S_\ell^{-i}(\cdot) = s_\ell^{-i} + (\lambda_\ell^i)^{-1}p_\ell$, we have $a_\ell^i - \mathbf{b}_\ell^i\mathbf{q}_0^i - c_\ell^i p_\ell = s_\ell^{-i} + (\lambda_\ell^i)^{-1}p_\ell$ for all $s_\ell^{-i} \in \mathbb{R}$, which gives p_ℓ as a linear function of s_ℓ^{-i} and

$$E[q_\ell^i|p_k,\mathbf{q}_0^i] = E[a_\ell^i - \mathbf{b}_\ell^i\mathbf{q}_0^i - \frac{c_\ell^i}{c_\ell^i + (\lambda_\ell^i)^{-1}}\Big(a_\ell^i - \mathbf{b}_\ell^i\mathbf{q}_0^i - s_\ell^{-i}\Big)|s_k^{-i},\mathbf{q}_0^i] \qquad \forall s_k^{-i} \in \mathbb{R}.$$

Theorem 1 shows that when traders' risk aversions are the same, the fixed point problem for the coefficients of best-response schedules $\{q_k^i(\cdot)\}_k$ in Eqs. (18)-(20) has a unique solution.

[27]Here, we used that the expected trades in exchange ℓ condition on price p_ℓ (the contingent variable in exchange ℓ) and endowment vector $\mathbf{q}_0^i$ (a trader's private information) by the one-to-one map between p_ℓ and s_ℓ^{-i} (to be established).

Equilibrium as a fixed point in price impacts (Step 2). With best-response coefficients $\{a_k^i, \mathbf{b}_k^i, c_k^i\}_k$ endogenized as functions of $\mathbf{\Lambda}^i$ and $F(\mathbf{s}^{-i}|\mathbf{q}_0^i)$, the equilibrium fixed point problem *across traders* becomes one among the traders' residual supplies $\{\mathbf{\Lambda}^i,\ F(\mathbf{s}^{-i}|\mathbf{q}_0^i)\}_i$. The fixed point problem for $\{\mathbf{\Lambda}^i,\ F(\mathbf{s}^{-i}|\mathbf{q}_0^i)\}_i$ across traders and assets is still complex and has a larger dimensionality.[28]

Nevertheless, the equilibrium distributions of the residual supply $\{F(\mathbf{s}^{-i}|\mathbf{q}_0^i)\}_i$ can also be characterized as functions of only price impacts $\{\mathbf{\Lambda}^i\}_i$, given the primitive distribution of endowments: Applying market clearing to the best-response schedules $\{q_k^j(\cdot)\}_{j\neq i}$ gives the residual supply functions of trader i (i.e., condition (ii) in Proposition 2): for each k,

$$S_k^{-i}(p_k) = \underbrace{-\sum_{j\neq i}(a_k^j - \mathbf{b}_k^j\mathbf{q}_0^j)}_{=s_k^{-i}} + \underbrace{\sum_{j\neq i} c_k^j}_{=(\lambda_k^i)^{-1}}\ p_k \qquad \forall p_k \in \mathbb{R}. \tag{16}$$

The fixed point problem for the joint distributions of the residual supply $\{F(\mathbf{s}^{-i}|\mathbf{q}_0^i)\}_i$ *across traders* becomes one for demand coefficients $\{\{a_k^i, \mathbf{b}_k^i\}_k\}_i$:

$$F(\mathbf{s}^{-i}|\mathbf{q}_0^i) = \mathcal{N}\Big(\Big(-\sum_{j\neq i}(a_k^j - \mathbf{b}_k^j E[\mathbf{q}_0^j|\mathbf{q}_0^i])\Big)_k, \Big(\sum_{j,h\neq i}\mathbf{b}_k^j Cov[\mathbf{q}_0^j, \mathbf{q}_0^h|\mathbf{q}_0^i](\mathbf{b}_\ell^h)'\Big)_{k,\ell}\Big), \tag{17}$$

which, by Step 1, are functions of price impacts $\{\{\lambda_k^i\}_k\}_i$. Finally, in each exchange, the equilibrium price impact $\lambda_k^i \equiv \frac{dp_k}{dq_k^i} \in \mathbb{R}_+$ must equal the slope of the inverse residual supply function: $\lambda_k^i = -\Big(\sum_{j\neq i}\frac{\partial q_k^j(\cdot)}{\partial p_k}\Big)^{-1} = (\sum_{j\neq i} c_k^j)^{-1}$ for all i and k.

Theorem 1 characterizes the equilibrium demand coefficients $\boldsymbol{a}^i \equiv (a_k^i)_k \in \mathbb{R}^K$, $\mathbf{B}^i \equiv (\mathbf{b}_k^i)_k \in \mathbb{R}^{K\times K}$, and $\mathbf{C}^i \equiv diag(c_k^i)_k \in \mathbb{R}^{K\times K}$ as functions of price impact—in matrix closed form—and characterizes equilibrium price impact in terms of primitives. In the main text, we present the characterization of the symmetric equilibrium[29] for simplicity of notation. In Appendix A, we state and prove the result for an asymmetric equilibrium. In what follows, we assume symmetric risk preferences.

Assumption (Symmetric Risk Preferences) Let $\alpha^i = \alpha$ for all i.

Notation. Let $[\cdot]_d : \mathbb{R}^{K\times K} \to \mathbb{R}^{K\times K}$ be an operator such that, for any matrix $\mathbf{M}$, $[\mathbf{M}]_d$ is a diagonal matrix with the $(k,\ell)^{\text{th}}$ element equal to zero for $k \neq \ell$ and the $(k,k)^{\text{th}}$ element equal

[28] In the contingent market, given the proportionality of the price impact matrix in the covariance, the equilibrium fixed point equations (i.e., price impact equations) become scalar equations; hence, the fixed point problem involves I scalar variables $\{\beta^{i,c}\}_i$ (Eq. (9)). When schedules are not contingent, since the price impact $\mathbf{\Lambda}^i$ is not by itself a sufficient statistic for the residual supply, the corresponding fixed point problem among the distributions of the residual supplies' intercepts involves $(K + \frac{K(K+1)}{2})I$ variables—i.e., K first moments $\{E[s_k^{-i}|\mathbf{q}_0^i]\}_k$ and $\frac{K(K+1)}{2}$ second moments $\{Cov[s_k^{-i}, s_\ell^{-i}|\mathbf{q}_0^i]\}_{k,\ell}$ for each i. Theorem 1 shows that equilibrium can be characterized by IK price impacts $\{\{\lambda_k^i\}_k\}_i$.

[29] Equilibrium is symmetric, if for all k, price impacts satisfy $\lambda_k^i \equiv \lambda_k$ for all i, demand coefficients satisfy $c_k^i \equiv c_k$ and $\mathbf{b}_k^i \equiv \mathbf{b}_k$ for all i, and a_k^i is a symmetric function of $\big\{\{E[\mathbf{q}_0^j]\}_{j\neq i}, E[\mathbf{q}_0^i]\big\}$ across traders. We will suppress the superscript i except where it is helpful.

to m_{kk} for any k.

Theorem 1 (Equilibrium: Fixed Point in Demand Schedules) In a symmetric equilibrium, the (net) demand schedules, defined by matrix coefficients $\{\boldsymbol{a}^i\}_i, \mathbf{B}$, and $\mathbf{C}$, and price impact $\boldsymbol{\Lambda}$ are characterized by the following conditions: for each trader i,

(i) (*Optimization, given price impact*) Given price impact matrix $\boldsymbol{\Lambda}$, the coefficients of (net) demands $\boldsymbol{a}^i, \mathbf{B}$, and $\mathbf{C}$ are characterized by:

$$\boldsymbol{a}^i = \mathbf{C}\underbrace{\left(\boldsymbol{\delta} - (\alpha\boldsymbol{\Sigma} - \mathbf{C}^{-1}\mathbf{B})E[\overline{\mathbf{q}}_0]\right)}_{=\mathbf{p}-\mathbf{C}^{-1}\mathbf{B}\overline{\mathbf{q}}_0} + \underbrace{\left((\alpha\boldsymbol{\Sigma} + \boldsymbol{\Lambda})^{-1}\alpha\boldsymbol{\Sigma} - \mathbf{B}\right)(E[\overline{\mathbf{q}}_0] - E[\mathbf{q}_0^i])}_{\text{Adjustment due to cross-asset inference}}, \qquad (18)$$

$$\mathbf{B} = \Big((1-\sigma_0^2)(\alpha\boldsymbol{\Sigma} + \boldsymbol{\Lambda}) + \underbrace{\sigma_0^2\mathbf{C}^{-1}}_{\substack{\text{Adjustment due to}\\ \text{cross-asset inference}}}\Big)^{-1}\alpha\boldsymbol{\Sigma}, \qquad (19)$$

$$\mathbf{C} = \Big[(\alpha\boldsymbol{\Sigma} + \boldsymbol{\Lambda})\underbrace{(\mathbf{B}\mathbf{B}')[\mathbf{B}\mathbf{B}']_d^{-1}}_{\substack{\text{Inference coefficient}\\ Var[\mathbf{s}^{-i}|\mathbf{q}_0^i][Var[\mathbf{s}^{-i}|\mathbf{q}_0^i]]_d^{-1}}}\Big]_d^{-1}, \qquad (20)$$

where $\overline{\mathbf{q}}_0 \equiv \frac{1}{I}\sum_j \mathbf{q}_0^j \in \mathbb{R}^K$ is the aggregate endowment and $\sigma_0 \equiv \frac{\sigma_{cv}+\frac{1}{I}\sigma_{pv}}{\sigma_{cv}+\sigma_{pv}} \in \mathbb{R}$.

(ii) (*Correct price impact*) Price impact $\boldsymbol{\Lambda}$ equals the transpose of the Jacobian of the trader's inverse residual supply function:

$$\boldsymbol{\Lambda} = \frac{1}{I-1}(\mathbf{C}^{-1})' = \frac{\alpha}{I-2}\Big[\boldsymbol{\Sigma}(\mathbf{B}\mathbf{B}')[\mathbf{B}\mathbf{B}']_d^{-1}\Big]_d. \qquad (21)$$

Note that the price slope $\mathbf{C}$ is a diagonal matrix in the uncontingent market. Appendix C.2 derives demand coefficients for $K = 2$.

Equilibrium outcome. Theorem 1 enables a direct comparison between the imperfectly competitive ($I < \infty$) and competitive ($I \to \infty$) outcomes. The competitive case is characterized by $\boldsymbol{\Lambda}^i \to \mathbf{0}$ for all i.

By Theorem 1, the implications of independence in market clearing that we will subsequently characterize can be understood through the structure of the endogenous price impact matrix. To begin, Corollary 1 shows how independence in market clearing affects equilibrium outcome. To ease the comparison, if the market clears jointly, then by Eqs. (6) and (9),

$$\mathbf{B}^c = (\alpha\boldsymbol{\Sigma} + \boldsymbol{\Lambda}^c)^{-1}\alpha\boldsymbol{\Sigma} = \frac{I-2}{I-1}\mathbf{Id}, \qquad \mathbf{C}^c = (\alpha\boldsymbol{\Sigma} + \boldsymbol{\Lambda}^c)^{-1}, \qquad \boldsymbol{\Lambda}^c = \frac{1}{I-2}\alpha\boldsymbol{\Sigma},$$

where $\mathbf{Id} \in \mathbb{R}^{K\times K}$ is the identity matrix. By contrast, in the uncontingent market, traders' demand coefficients depend on the *distribution* of traders' endowments, as do price impacts, which are also *not proportional* to the fundamental risk $\mathbf{\Sigma}$.[30]

Corollary 1 (Equilibrium Prices and Allocations) Given the equilibrium demand coefficients $\{\boldsymbol{a}^i\}_i, \mathbf{B}, \mathbf{C}$, and price impact $\mathbf{\Lambda}$ in Theorem 1, equilibrium prices and allocations are:

$$\mathbf{p} = \boldsymbol{\delta} - (\alpha\mathbf{\Sigma} - \mathbf{C}^{-1}\mathbf{B})E[\overline{\mathbf{q}}_0] - \mathbf{C}^{-1}\mathbf{B}\overline{\mathbf{q}}_0, \tag{22}$$

$$\mathbf{q}^i + \mathbf{q}_0^i = \Big((\alpha\mathbf{\Sigma} + \mathbf{\Lambda})^{-1}\alpha\mathbf{\Sigma} - \mathbf{B}\Big)(E[\overline{\mathbf{q}}_0] - E[\mathbf{q}_0^i]) + \mathbf{B}\overline{\mathbf{q}}_0 + (\mathbf{Id} - \mathbf{B})\mathbf{q}_0^i. \tag{23}$$

We highlight two implications of the lack of proportionality between $\mathbf{\Lambda}$ and $\mathbf{\Sigma}$. In contrast to the contingent market, where $\mathbf{p}^c = \boldsymbol{\delta} - \alpha\mathbf{\Sigma}\overline{\mathbf{q}}_0$, the second moment $Var[\mathbf{p}]$ of the distribution of equilibrium prices depends on the distribution of endowments—through the endogenous demand coefficient $\mathbf{C}^{-1}\mathbf{B}$—rather than only the exogenous asset covariance $\mathbf{\Sigma}$. In particular, the price covariance of any asset pair depends on the second moment of the joint distribution of *all* assets. (We explore the implications of this property in Section 3.2.3 and Example 3.) Additionally, the allocations' weights on the idiosyncratic and market risk (i.e., $\mathbf{Id} - \mathbf{B}$ and $\mathbf{B}$) depend on the asset covariance and the distribution of endowments. Thus, asset payoff substitutability itself factors in which assets' allocation is more efficient. The nonproportionality of $\mathbf{C}$ to $\mathbf{\Sigma}^{-1}$ continues to hold in the limit as $I \to \infty$.

Theorem 2 (Existence of Symmetric Equilibrium) There exists a symmetric linear Bayesian Nash equilibrium. When $K = 2$, equilibrium is unique.

In the contingent market, the proportionality of price impact to asset covariance reduces the fixed point problem for $\{\mathbf{\Lambda}^{i,c}\}_i$ to one for scalars $\{\beta^{i,c} \in \mathbb{R}\}_i$ (Eq. (9)). In the uncontingent market, the argument differs in two ways, due to cross-asset inference (i.e., inference coefficient $(\mathbf{BB}')[\mathbf{BB}']_d^{-1}$). First, price impact matrices are not proportional to the covariance, and ought to be found jointly for all assets and traders. Second, the mapping for price impact $\mathbf{\Lambda}^i$—defined by the fixed point Eqs. (19) and (21)—is not monotone in price impacts $\{\mathbf{\Lambda}^j\}_{j\neq i}$.

Given Theorem 1's result that a fixed point in demand schedules can equivalently (for $I < \infty$) be represented as a fixed point in price impact matrices, the existence of equilibrium follows from the Brouwer fixed point theorem (Theorem 2) with the bounds of price impact being matrices (rather than scalars).[31]

[30] The inference coefficient $(\mathbf{BB}')[\mathbf{BB}']_d^{-1} = Var[\mathbf{s}^{-i}|\mathbf{q}_0^i][Var[\mathbf{s}^{-i}|\mathbf{q}_0^i]]_d^{-1}$ in Eq. (20) is derived from the distribution of the residual supply intercepts $\mathbf{s}^{-i}$ in Eq. (17), given the distribution of endowments $F((\mathbf{q}_0^j)_{j\neq i}|\mathbf{q}_0^i)$.

[31] We do not provide a uniqueness result for $K > 2$. In the contingent model, the equilibrium uniqueness can be shown by applying the argument from a one-asset market, using the proportionality of price impact in the covariance matrix (Eq. (9)) (see Malamud and Rostek (2017)). Lambert, Ostrovsky, and Panov (2018) consider a game in which strategies are quantities (market orders) with one asset and one liquidity provider; the scalar price impact solves a quadratic equation that has a unique positive solution, which gives equilibrium uniqueness. We analyze games in demand and supply functions with multiple assets and price impacts characterized by a system of nonlinear (non-polynomial) matrix equations.

3.2.3 Comparative Statics of Price Impact

Thus far, we noted that, in contrast to when markets clear jointly, the *cross-exchange* price impacts are zero (by definition of uncontingent demands) and the *within-exchange* price impacts depend on cross-asset inference (Theorem 1). We now show that, due to cross-asset inference, the within-exchange price impacts $\{\lambda_k\}_k$ can be larger or smaller than their contingent counterparts. Theorem 3 provides a sufficient condition for $\{\lambda_k\}_k$ to be larger.

Price impact and cross-asset inference. Consider the counterfactual that defines trader i's price impact in exchange k: what is the effect of increasing the demand by trader i for asset k at a margin? Price p_k increases so that other traders are willing to sell the extra units and the market clears. This *direct effect* is present in the contingent market as well. When the market is uncontingent, equilibrium is not *ex post* and the change in price p_k has an *indirect inference effect.* Implicitly differentiating the first-order condition (11) of trader $j \neq i$ for asset k with respect to p_k characterizes the direct and inference effects on the marginal utility and the marginal payment:

$$-\alpha^j \sigma_{kk} \frac{\partial q_k^j(\cdot)}{\partial p_k} - \underbrace{\sum_{\ell \neq k} \alpha^j \sigma_{k\ell} \frac{\partial q_\ell^j(\cdot)}{\partial p_\ell} \frac{\partial E[p_\ell | p_k, \mathbf{q}_0^j]}{\partial p_k}}_{\substack{\text{Inference effect} \\ \text{on the marginal utility}}} = \underbrace{1}_{\substack{\text{Direct effect} \\ \text{on the marginal payment}}} + \lambda_k^j \frac{\partial q_k^j(\cdot)}{\partial p_k}. \tag{24}$$

Rewriting Eq. (24) decomposes the demand slope $\frac{\partial q_k^j(\cdot)}{\partial p_k}$, and hence trader i's price impact λ_k^i, into the direct and indirect inference effects: Using $\frac{\partial q_k^j}{\partial p_\ell} \equiv \frac{\alpha^j \sigma_{k\ell}}{\alpha^j \sigma_{kk} + \lambda_k^j}(-\frac{\partial q_\ell^j(\cdot)}{\partial p_\ell}) = \frac{\alpha^j \sigma_{k\ell}}{\alpha^j \sigma_{kk} + \lambda_k^j} c_\ell^j$,

$$\lambda_k^i = -\Big(\sum_{j \neq i} \underbrace{\frac{\partial q_k^j(\cdot)}{\partial p_k}}_{= -c_k^j}\Big)^{-1} = -\Big(\sum_{j \neq i} \Big(\underbrace{\underbrace{-\frac{1}{\alpha^j \sigma_{kk} + \lambda_k^j}}_{\equiv \frac{\partial q_k^j}{\partial p_k}}}_{\equiv \text{ Direct effect } (-)} + \underbrace{\sum_{\ell \neq k} \underbrace{\frac{\partial q_k^j}{\partial p_\ell}}_{sign(\sigma_{k\ell})} \underbrace{\frac{\partial E[p_\ell | p_k, \mathbf{q}_0^j]}{\partial p_k}}_{sign(Cov[p_k, p_\ell])}}_{\equiv \text{ Inference effect}} \Big)\Big)^{-1}. \tag{25}$$

To explain the inference effect in Eq. (25), in the counterfactual following the demand increase by i, consider traders' $j \neq i$ posterior conditioned on the higher price p_k. When asset payoffs are symmetric substitutes (i.e., $\sigma_{kk} = \sigma$ for all k and $\sigma_{k\ell} = \sigma\rho > 0$ for all k and $\ell \neq k$), then other traders, who assume that all others—including trader i—play equilibrium, would instead attribute the higher price p_k to a lower, on average, realization of endowments for all correlated assets, and expect higher prices and lower trades of those assets.[32] This further increases the price at which they are willing to sell units of the substitute asset k to trader i.[33]

[32]Price p_k affects the conditional expectation separately from endowments $\mathbf{q}_0^j$ in expected trades $E[q_\ell^j | p_k, \mathbf{q}_0^j]$ (equivalently, $E[p_\ell | p_k, \mathbf{q}_0^j]$) provided that asset payoffs are not independent: p_k contains information about endowments of other traders for *all* assets.

[33]The decomposition of equilibrium price impact—which, by definition, represents an off-equilibrium

Theorem 3 shows that when payoff correlations are symmetric, uncontingent trading increases the within-exchange price impact—the inference effect in Eq. (25) is *positive*. Let $\rho_{k\ell} \equiv Corr[r_k, r_\ell] = \frac{\sigma_{k\ell}}{\sqrt{\sigma_{kk}\sigma_{\ell\ell}}}$.

Theorem 3 (Price Impact: Comparative Statics) Suppose that asset covariances are symmetric: $\sigma_{\ell\ell} = \sigma$ for all ℓ and $\sigma_{\ell m} = \sigma\rho$ for all ℓ and $m \neq \ell$. The within-exchange price impact λ_k satisfies the following properties for each k:

(1) (*Magnitude*) With K assets, price impact λ_k maximally increases K-fold relative to $\lambda_k^c = \frac{\alpha}{I-2}\sigma_{kk}$:

$$\frac{\alpha}{I-2}\sigma_{kk} \leq \lambda_k \leq \frac{\alpha}{I-2}\sum_{\ell}\sigma_{\ell\ell}.$$

The upper bound of the K-fold increase is attained if and only if $|\rho| = 1$.

(2) (*Comparative statics*) Relative to the contingent market:

(i) $\frac{\partial(\lambda_k - \lambda_k^c)}{\partial I} < 0$, i.e., the inference effect is decreasing in the number of traders I;

(ii) $\frac{\partial(\lambda_k - \lambda_k^c)}{\partial |\rho|} > 0$, i.e., the inference effect is increasing in asset correlation $|\rho|$.

As a corollary, in two-asset markets, uncontingent trading always increases the within-exchange price impact.[34] Price impact λ_k increases less relative to $\lambda_k^c = \frac{\alpha}{I-2}\sigma_{kk}$ when the inference effect is weaker—i.e., with a larger number of traders I or smaller correlations $|\rho|$ (Fig. 1A). Price impact increases K-fold when the inference is perfect (i.e., $|\rho| = 1$). As $I \to \infty$, then $\mathbf{\Lambda}^i \to 0$ for all i (Lemma 3 in Appendix B).[35]

Endogenous price covariance. When asset correlations are *heterogeneous* ($K > 2$), uncontingent trading can *lower* the price impact λ_k for some assets relative to contingent trading (Fig. 1B). In the counterfactual below Eq. (25), when correlations are symmetric, the inferred price changes of all assets induced by a demand change for one asset have the same sign. With heterogeneous correlations, however, the inferred price changes may differ in sign, resulting in a *negative* inference effect.

---

counterfactual—captures how the cross-agent and cross-asset externalities are accounted for. This makes precise the difference with Nash-in-Nash (see Introduction).

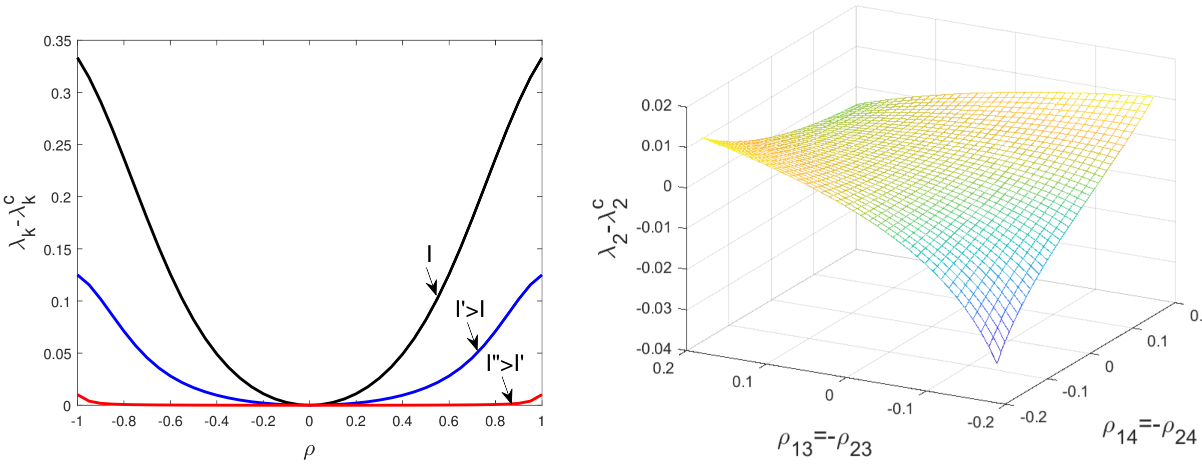
[34] When $K = 2$, Appendix C.2 characterizes $\frac{\partial E[p_\ell | p_k, \mathbf{q}_0^j]}{\partial p_k}$ as a closed-form function of price impact λ and simplifies Eq. (25) into Eq. (161):

$$\lambda_k = \underbrace{\frac{\alpha}{I-2}}_{=\lambda_k^c} + \underbrace{\underbrace{\frac{\alpha\rho}{I-2}}_{sign(\rho)} \underbrace{\frac{2xy}{x^2+y^2}}_{sign(\rho)}}_{\text{Inference } (+)},$$

where $x \equiv (1-\sigma_0)(1-\rho^2)\alpha + (1+(I-2)\sigma_0)\lambda$ and $y \equiv \rho(1+(I-2)\sigma_0)\lambda$.

[35] The conditions from ft. 22 apply to contingent, uncontingent, and general markets in the next section. When schedules are not contingent, the cross-asset inference is present in the limit (i.e., $\frac{\partial E[p_\ell|p_k,\mathbf{q}_0^j]}{\partial p_k} = \frac{Cov[p_\ell,p_k|\mathbf{q}_0^j]}{Var[p_k|\mathbf{q}_0^j]} \neq 0$ for $\ell \neq k$) even when the price impact becomes zero (i.e., $\lambda_k^i \to 0$).

Figure 1: Within-exchange price impact: inference effect



(A) (B)

Notes: Panel A: ($K = 2$) Price impact difference $\lambda_k - \lambda_k^c$ is determined by the inference effect (Eq. (25))—the direct effect is the same in contingent and uncontingent markets. The inference effect is larger (in absolute value) when assets are more strongly correlated (i.e., $|\rho|$ is larger) and the number of traders I is smaller. The black, blue and red curves assume, respectively, $I = 5$, $I' = 10$, and $I'' = 100$. Panel B: ($K > 2$) With heterogeneous correlations, price impact λ_k can be lower than λ_k^c. Assets 1 and 2 are heterogeneously correlated with other assets: $\rho_{13} = \rho_{14}$, $\rho_{23} = \rho_{24}$, $\rho_{12} = -0.1$, $\rho_{15} = \rho_{16} = -0.5$, and $\rho_{25} = \rho_{26} = 0.3$; $I = 10$. In both panels, $\sigma_{cv} = 0, \sigma_{pv} = 1, \alpha = 1$.

Underlying this result is the lack of proportionality between equilibrium price impact, and hence price covariance, and asset covariance $\mathbf{\Sigma}$ with limited demand conditioning, as seen in Eq. (22). Consequently, $\mathbf{\Lambda}$ and $Cov[p_k, p_\ell]$ depend on the covariance of *all* assets and, in fact, need not match the sign of asset correlation (i.e., $\sigma_{k\ell}$), e.g., prices of complementary assets ($\sigma_{k\ell} < 0$) can be positively correlated ($Cov[p_k, p_\ell] > 0$). The intuition can be seen in the price equation (22): $Cov[p_k, p_\ell]$ is determined by

$$\left(\mathbf{C}^{-1}\mathbf{B}\right)_{k\ell} = \left((\mathbf{C} + \kappa(\alpha\mathbf{\Sigma})^{-1})^{-1}\right)_{k\ell} = \alpha\sigma_{k\ell} - \alpha\mathbf{\Sigma}_k(\alpha\mathbf{\Sigma} + \kappa\mathbf{C}^{-1})^{-1} \cdot \alpha\mathbf{\Sigma}_\ell, \tag{26}$$

where $\kappa = \frac{1+(I-2)\sigma_0}{(I-1)(1-\sigma_0)} \in \mathbb{R}_+$. When demand coefficient $\mathbf{C}$ is not proportional to $(\alpha\mathbf{\Sigma})^{-1}$, one can have $sign(\frac{\partial E[p_\ell|p_k,\mathbf{q}_0^j]}{\partial p_k}) = sign(Cov[p_k, p_\ell]) \neq sign(\sigma_{k\ell})$ for some $\ell \neq k$, and as a result, $\lambda_k < \lambda_k^c$ by Eq. (25) (Fig. 1B). In the contingent market, the price covariance matrix is proportional to the asset covariance: substituting $\mathbf{C}^c = (\alpha\mathbf{\Sigma} + \mathbf{\Lambda}^c)^{-1} = \frac{I-2}{I-1}(\alpha\mathbf{\Sigma})^{-1}$ in Eq. (26), we have:

$$\left((\mathbf{C}^c + \kappa(\alpha\mathbf{\Sigma})^{-1})^{-1}\right)_{k\ell} = \frac{(I-1)\kappa}{(I-1)\kappa + (I-2)}\alpha\sigma_{k\ell} \qquad \forall k \ \ \forall \ell;$$

hence, $sign(Cov[p_k, p_\ell]) = sign(\sigma_{k\ell})$.

4 Changes in Market Structure

The endogenous—with limited conditioning—price covariance creates incentives for innovation in trading technology, defined by changes in market clearing. To fix ideas, we first discuss a particular example of such innovation.

Example 2 (Innovation in Trading Technology) Suppose a new exchange for one of the K traded assets is created to operate along with the existing exchanges *without altering traders' endowments of any asset*. In the contingent market, the corresponding innovation of duplicating a traded asset would be neutral for traders' equilibrium payoffs. This can be seen from the first-order condition (5) for contingent demands $\mathbf{q}^{i,c}(p_1, ..., p_K, p_{K+1}) : \mathbb{R}^{K+1} \to \mathbb{R}^{K+1}$,

$$(\alpha^i \boldsymbol{\Sigma}^+ + \boldsymbol{\Lambda}^{i,c})\mathbf{q}^{i,c} = \boldsymbol{\delta}^+ - \mathbf{p} - \alpha^i \boldsymbol{\Sigma}^+ \mathbf{q}_0^{i,+} \qquad \forall \mathbf{p} \in \mathbb{R}^{K+1}, \tag{27}$$

where the payoffs of $K+1$ assets are jointly Normally distributed according to $\mathcal{N}(\boldsymbol{\delta}^+, \boldsymbol{\Sigma}^+)$ with $\boldsymbol{\delta}^+ \in \mathbb{R}^{K+1}$ and $\boldsymbol{\Sigma}^+ \in \mathbb{R}^{(K+1)\times(K+1)}$, and endowments for the duplicated asset can be arbitrarily split, provided that $q_{0,k}^{i,+} + q_{0,K+1}^{i,+} = q_{0,k}^{i}$. Using the fact that price impact $\boldsymbol{\Lambda}^{i,c} = \beta^{i,c}\alpha^i \boldsymbol{\Sigma}^+ \in \mathbb{R}^{(K+1)\times(K+1)}$ is proportional to the covariance matrix for all i in the contingent market (Eq. (9)), and that the covariance matrix $\boldsymbol{\Sigma}^+$ is singular with the new asset, condition (27) has a continuum of solutions $\mathbf{q}^{i,c} \in \mathbb{R}^{K+1}$ pointwise with respect to the price vector $\mathbf{p} \in \mathbb{R}^{K+1}$, including zero trade of the new asset $q_{K+1}^{i,c}(\cdot) = 0$. Even if the asset in the new venue is traded, traders' equilibrium payoffs are the same as in the market with K assets. $\square$

With independent market clearing, innovation in trading technology that would be neutral for traders' payoffs with joint clearing is generally no longer redundant, i.e., traders' equilibrium payoffs change. In regard to innovation, we present two results: first, we characterize when innovation is not redundant (Proposition 3, Theorem 4); second, we show that markets with multiple exchanges that clear independently can be designed to function like a single exchange for all assets (Corollary 2). In Section 5, we examine how innovation affects welfare.

To accommodate various forms of innovation and a more general class of market structures, we extend the uncontingent model from Section 3.2: we allow arbitrary restrictions on cross-asset demand conditioning "between" uncontingent and contingent—this permits multiple assets per exchange—and we allow an asset to be traded in multiple venues. Given that all traders participate in all exchanges, we can identify an exchange with a subset of assets traded.

Definition 4 (Exchanges, Market Structure) *Consider a market with I traders and K assets. An* exchange *n is defined by the subset of assets traded $K(n) \subseteq K$. The* market structure *is described by a set of N exchanges; i.e., $N = \{K(n)\}_n$.*

Exchanges clear independently: in each exchange n, trader i submits a demand $q_{k,n}^i(\cdot) : \mathbb{R}^{K(n)} \to \mathbb{R}$ for each asset $k \in K(n)$ contingent on the prices of assets traded there, $\mathbf{p}_{K(n)} \equiv$

$(p_{\ell,n})_{\ell \in K(n)} \in \mathbb{R}^{K(n)}$. *The market-clearing price vector* $\mathbf{p}_{K(n)}$ *in exchange* n *is determined by* $\sum_j q^j_{k,n}(\mathbf{p}_{K(n)}) = 0$ *jointly for all assets* $k \in K(n)$ *traded in this exchange.*

Like in previous sections, the market clears independently across exchanges (but not necessarily across assets). The uncontingent market corresponds to K exchanges $N = \{\{k\}\}_k$, and the contingent market corresponds to a single exchange $N = \{K\}$.

We treat the same asset traded in different exchanges as distinct assets with perfectly correlated payoffs. For the fundamentals $\boldsymbol{\delta} \in \mathbb{R}^K, \boldsymbol{\Sigma} \in \mathbb{R}^{K \times K}$, and $\{\mathbf{q}^i_0 \in \mathbb{R}^K\}_i$, the superscript '+' indicates their counterparts in $\mathbb{R}^{\sum_n K(n)}$. Accordingly, the asset payoffs in N exchanges are jointly Normally distributed $\mathcal{N}(\boldsymbol{\delta}^+, \boldsymbol{\Sigma}^+)$, where $\boldsymbol{\delta}^+ \in \mathbb{R}^{\sum_n K(n)}$ and $\boldsymbol{\Sigma}^+ \in \mathbb{R}^{(\sum_n K(n)) \times (\sum_n K(n))}$. An asset's endowment can be split arbitrarily across exchanges.[36] This is because the trader's demand for each asset depends on his *total* endowment of all assets (Eq. (28)); hence, so do prices. Generalizing from the first-order condition (13) for one asset per exchange, the best-response demand schedule for asset $k \in K(n)$ in exchange n is determined by:

$$\delta^+_k - \alpha^i \boldsymbol{\Sigma}_k \mathbf{q}^i_0 - \alpha^i \boldsymbol{\Sigma}^+_k E[\mathbf{q}^i | \mathbf{p}_{K(n)}, \mathbf{q}^i_0] = p_{k,n} + (\boldsymbol{\Lambda}^i_{K(n)})_k \mathbf{q}^i_{K(n)} \qquad \forall \mathbf{p}_{K(n)} \in \mathbb{R}^{K(n)}, \tag{28}$$

given $\{q^i_{\ell,n}(\cdot)\}_{\ell \neq k, \ell \in K(n)}$, $\{\{q^i_{\ell,n'}(\cdot)\}_{\ell \in K(n')}\}_{n' \neq n}$, $F(\mathbf{s}^{-i}|\mathbf{q}^i_0)$, and $\{\boldsymbol{\Lambda}^i_{K(n)}\}_n$, where $\boldsymbol{\Lambda}^i_{K(n)} \in \mathbb{R}^{K(n) \times K(n)}$ is trader i's price impact in exchange n and $(\boldsymbol{\Lambda}^i_{K(n)})_k \in \mathbb{R}^{1 \times K(n)}$ is the k^{th} row of $\boldsymbol{\Lambda}^i_{K(n)}$.

To analyze equilibrium in markets with arbitrary demand conditioning across assets (Definition 4), we must extend Theorem 1. As with the simpler market structures characterized in Theorem 1, the fixed point in demand schedules is equivalent to a fixed point in traders' price impacts—now, block-diagonal matrices $\boldsymbol{\Lambda}^i \equiv diag(\boldsymbol{\Lambda}^i_{K(n)})_n \in \mathbb{R}^{(\sum_n K(n)) \times (\sum_n K(n))}$ for all i. Theorem 5 in Appendix A characterizes equilibrium; Proposition 5 in Appendix C.1 provides the comparative statics of equilibrium price impact. The proofs of Proposition 2 and Theorem 2 in Appendix B encompass general market structures.

4.1 Nonredundant Changes in Market Structure

Price impacts *per se* are not useful in comparing payoffs across arbitrary market structures, as they are defined for different exchanges and may have different dimensionality. Proposition 3 simplifies and illuminates the analysis of nonredundancy and welfare: it relates the payoffs across market structures with different conditioning variables, and hence different price impact $\boldsymbol{\Lambda}$, through a *single-exchange* counterfactual.

We define two statistics, $\widehat{\boldsymbol{\Lambda}}$ and $\widehat{\mathbf{B}}$, that match the moments of *total* equilibrium trade of each asset k across exchanges in a market structure $\{K(n)\}_n$, $\widehat{q}^i_k \equiv \sum_{\{n|k \in K(n)\}} q^i_{k,n}$. The *per-unit price impact* $\widehat{\boldsymbol{\Lambda}}$ corresponds to the unique positive semi-definite matrix, such that if the

[36] Given trader i's endowment $\mathbf{q}^i_0 = (q^i_{0,k})_k \in \mathbb{R}^K$, his endowment in $\mathbb{R}^{\sum_n K(n)}$ can be an arbitrary vector $\mathbf{q}^{i,+}_0 \equiv ((q^{i,+}_{0,k,n})_k)_n \in \mathbb{R}^{\sum_n K(n)}$ such that $q^i_{0,k} = \sum_{\{n|k \in K(n)\}} q^{i,+}_{0,k,n}$ for all k. The parts of the split endowment in different exchanges are perfectly correlated: $Corr[q^{i,+}_{0,k,n}, q^{i,+}_{0,k,n'}] = 1$ for any n, n' such that $k \in K(n) \cap K(n')$.

price impact in a market structure with a *single* exchange for K assets were $\widehat{\mathbf{\Lambda}}$, the expected trade of each asset $k \in K$ in the counterfactual exchange would equal the expected equilibrium total trade in the market structure $\{K(n)\}_n$. For all i and k,

$$E[\widehat{q}_k^i] \equiv \sum_{\{n|k\in K(n)\}} E[q_{k,n}^i] \; = (\alpha\mathbf{\Sigma} + \widehat{\mathbf{\Lambda}})_k^{-1}\alpha\mathbf{\Sigma}(E[\overline{\mathbf{q}}_0] - E[\mathbf{q}_0^i]). \tag{29}$$

In turn, the *cross-asset inference* $\widehat{\mathbf{B}}$ is the coefficient on the privately known endowment $\mathbf{q}_0^i$ in a trader's total demand that matches the variance of the equilibrium total trade (cf. Eq. (19)). For all i and k,

$$Var[\widehat{q}_k^i] \equiv Var\Big[\sum_{\{n|k\in K(n)\}} q_{k,n}^i\Big] = \widehat{\mathbf{B}}Var[\overline{\mathbf{q}}_0 - \mathbf{q}_0^i]\widehat{\mathbf{B}}' = \frac{I-1}{I}\sigma_{pv}\widehat{\mathbf{B}}\widehat{\mathbf{B}}'; \tag{30}$$

$(\widehat{\mathbf{B}}\widehat{\mathbf{B}}')_{k\ell}(\widehat{\mathbf{B}}\widehat{\mathbf{B}}')_{kk}^{-1}$ is the cross-asset inference coefficient in the expected total trade $E[\widehat{q}_\ell^i|\widehat{q}_k^i, \mathbf{q}_0^i]$.[37]

Proposition 3 shows that one can compare equilibrium payoffs across market structures through $(\widehat{\mathbf{\Lambda}}, \widehat{\mathbf{B}}) \in \mathbb{R}^{K\times K} \times \mathbb{R}^{K\times K}$ and, hence, identify nonredundant innovation with the change in $(\widehat{\mathbf{\Lambda}}, \widehat{\mathbf{B}})$. We introduce an indicator matrix $\mathbf{W}$ that represents a market structure $N = \{K(n)\}_n$.

Definition 5 (Indicator Matrix for Market Structure) *An indicator matrix* $\mathbf{W} \equiv (\mathbf{W}_n)_n \in \{0,1\}^{(\sum_n K(n))\times K}$ represents a market structure $N = \{K(n)\}_n$ *if for each exchange* n*, the* $(\ell, k)^{th}$ *element of* $\mathbf{W}_n \in \{0,1\}^{K(n)\times K}$ *equals one if the* ℓ^{th} *asset in exchange* n *is asset* k *and zero otherwise.*

We can now write $\boldsymbol{\delta}^+ = \mathbf{W}\boldsymbol{\delta}$ and $\mathbf{\Sigma}^+ = \mathbf{W}\mathbf{\Sigma}\mathbf{W}'$. Recall that $tr(\mathbf{M}) \equiv \sum_k m_{kk}$ is the trace of a matrix $\mathbf{M}$ (i.e., the sum of its diagonal elements).

Proposition 3 (Sufficient Statistic for Equilibrium Payoffs) Let $I < \infty$ and $K > 1$. Assume that $\mathbf{\Sigma}$ is not singular.[38] Fix a market structure $N = \{K(n)\}_n$.

(1) *(Expected payoffs)* The expected equilibrium payoff of trader i is characterized as a function of $\widehat{\mathbf{\Lambda}}$ and $\widehat{\mathbf{B}}$:

$$\begin{aligned} E[u^i(\mathbf{q}^i) - \mathbf{p}\cdot\mathbf{q}^i] \;\; &= \;\; \underbrace{E[\boldsymbol{\delta}\cdot\mathbf{q}_0^i - \frac{1}{2}\mathbf{q}_0^i\cdot\alpha\mathbf{\Sigma}\mathbf{q}_0^i]}_{\text{Payoff without trade}} + \underbrace{(E[\overline{\mathbf{q}}_0] - E[\mathbf{q}_0^i])\cdot\mathbf{\Upsilon}(\widehat{\mathbf{\Lambda}})(E[\overline{\mathbf{q}}_0] - E[\mathbf{q}_0^i])}_{\text{Equilibrium surplus from trade}} \\ &\quad + \underbrace{\frac{1}{2}\frac{I-1}{I}\sigma_{pv}tr\Big(\widehat{\mathbf{B}}'\alpha\mathbf{\Sigma} + \alpha\mathbf{\Sigma}\widehat{\mathbf{B}} - \widehat{\mathbf{B}}'\alpha\mathbf{\Sigma}\widehat{\mathbf{B}}\Big)}_{\text{Payoff term due to } Var[\overline{\mathbf{q}}_0|\mathbf{q}_0^i]>0}, \end{aligned} \tag{31}$$

[37] $\widehat{\mathbf{\Lambda}}$ and $\widehat{\mathbf{B}}$ are not defined as equilibrium objects in a single-exchange game.

[38] The proof allows for a singular covariance $\mathbf{\Sigma}$. Then, the uniqueness of $\widehat{\mathbf{\Lambda}}$ and $\widehat{\mathbf{B}}$ holds up to payoff equivalence.

where the per-unit price impact $\widehat{\mathbf{\Lambda}} \in \mathbb{R}^{K\times K}$ and cross-asset inference $\widehat{\mathbf{B}} \in \mathbb{R}^{K\times K}$ are defined by conditions (29) and (30), respectively, and characterized by:

$$\widehat{\mathbf{\Lambda}} \equiv \Big(\mathbf{W}'\underbrace{\mathbf{\Lambda}^{-1}}_{=(I-1)\mathbf{C}'}\mathbf{W}\Big)^{-1}, \quad \widehat{\mathbf{B}} \equiv \mathbf{W}'\Big((1-\sigma_0)(\alpha\mathbf{\Sigma}^{+} + \underbrace{\mathbf{\Lambda}}_{=\frac{1}{I-1}(\mathbf{C}^{-1})'}) + \sigma_0(I-1)\underbrace{\mathbf{\Lambda}'}_{=\frac{1}{I-1}\mathbf{C}^{-1}}\Big)^{-1}\mathbf{W}\alpha\mathbf{\Sigma}; \tag{32}$$

$\mathbf{\Upsilon}(\widehat{\mathbf{\Lambda}}) \equiv \frac{1}{2}\alpha\mathbf{\Sigma}(\alpha\mathbf{\Sigma}+\widehat{\mathbf{\Lambda}}')^{-1}(\alpha\mathbf{\Sigma}+\widehat{\mathbf{\Lambda}}+\widehat{\mathbf{\Lambda}}')(\alpha\mathbf{\Sigma}+\widehat{\mathbf{\Lambda}})^{-1}\alpha\mathbf{\Sigma} \in \mathbb{R}^{K\times K}$ represents the marginal payoff per unit of *ex ante* trading needs $E[\overline{\mathbf{q}}_0] - E[\mathbf{q}_0^i]$.

(2) *(Sufficient statistic and symmetry)* The sufficient statistic $(\widehat{\mathbf{\Lambda}}, \widehat{\mathbf{B}})$ for the equilibrium payoffs (31) reduces to $\widehat{\mathbf{\Lambda}}$, equivalently $\widehat{\mathbf{B}}$, if and only if the equilibrium price impact is a symmetric matrix, i.e., $\mathbf{\Lambda} = \mathbf{\Lambda}'$.

Crucially, while either price impact $\mathbf{\Lambda}$ itself or the demand coefficient $\mathbf{B}$ is a sufficient statistic for equilibrium payoffs in a market structure $N = \{K(n)\}_k$ (Theorems 1 and 5), their per-unit counterparts $\widehat{\mathbf{\Lambda}}$ and $\widehat{\mathbf{B}}$ are both required for the payoffs in two market structures N and N' to match, unless price impact is a symmetric matrix.

Theorem 4 shows that the asymmetry of price impact is the key to understanding which innovations are nonredundant. Mathematically, the relevance of the price impact asymmetry can be seen in Eq. (32): the inverse matrix in $\widehat{\mathbf{B}}$ is a harmonic mean of the demand coefficient $\mathbf{C} = \frac{1}{I-1}(\mathbf{\Lambda}^{-1})'$ and its transpose $\mathbf{C}'$ and, thus, it is not a linear function of $\mathbf{\Lambda}$ *unless* $\mathbf{\Lambda}$ *is symmetric*. The asymmetry is a new equilibrium property relative to both the contingent design ($\mathbf{\Lambda}^c$ is proportional to $\mathbf{\Sigma}$ and hence $\lambda_{k\ell} = \lambda_{\ell k}$, $\ell \neq k$, Eq. (9)) and the uncontingent design ($\lambda_{k\ell} = 0$, $\ell \neq k$). The asymmetry of the *cross-asset* price impacts is another consequence of the asymmetric, with limited conditioning, inference effect component of price impact (Section 3.2.3). Example 3 illustrates this link.

Theorem 4 (Nonredundancy of Changes in Market Structure: Conditions) Let $I < \infty$ and $K > 1$, and consider a market structure $N = \{K(n)\}_n$. Suppose a new exchange n' such that $K(n') \subset K(n)$ for some $n \in N$ is introduced. Exchange n' is redundant in an equilibrium if and only if one of the following conditions holds:

(i) (*Innovation mimics an exchange*) The set of assets traded in exchange n' is the same as in an existing exchange, i.e., $K(n') = K(n'')$ for some $n'' \in N$.

(ii) (*Symmetric price impact*) Price impact in an exchange n'' such that $K(n') \subset K(n'')$ is symmetric, i.e., $\mathbf{\Lambda}_{K(n'')} = \mathbf{\Lambda}'_{K(n'')}$.

(iii) (*Independent or perfectly correlated assets*) The payoffs of all assets in $K(n')$ are independent or perfectly correlated with those of the assets in $K(n) \setminus K(n')$, i.e., $|\rho_{k\ell}| \in \{0, 1\}$ for all $k \in K(n')$ and $\ell \in K(n) \setminus K(n')$.

By Proposition 3, an innovation is nonredundant if it changes the *relative* trading costs across assets $\widehat{\mathbf{\Lambda}}$ or cross-asset inference $\widehat{\mathbf{B}}$.[39] Intuitively, under any of Theorem 4's conditions, new exchanges do not create additional linearly independent conditioning variables in any asset's demand. Neither $\widehat{\mathbf{B}}$ nor $\widehat{\mathbf{\Lambda}}$ change. Example 3 illustrates Theorem 4.

Example 3 (Nonredundant Exchanges and Price Impact Asymmetry) Consider a market with two exchanges $N = \{\{1,2\},\{3\}\}$. For simplicity, assume that $\sigma_{11} = \sigma_{22} = \sigma_{33}$ and the assets are imperfectly correlated ($0 < |\rho_{k\ell}| < 1$ for all k and $\ell \neq k$). Per Theorem 4 (ii), the introduction of exchange $\{1\}$ is redundant if and only if the equilibrium price impact $\mathbf{\Lambda}_{\{1,2\}}$ is a symmetric matrix: i.e., the *cross-asset* price impacts coincide, $\lambda_{12} = \lambda_{21}$.

(i) When is price impact $\mathbf{\Lambda}_{\{1,2\}}$ symmetric? This is the case if and only if the covariances of assets 1 and 2 are symmetric:

$$\sigma_{13} = \sigma_{23} \;\Leftrightarrow Cov[p_2, p_3] = Cov[p_1, p_3]. \tag{33}$$

A closer look at the inference effect in price impact shows why. Using the relation between price impact and demand slope, $\mathbf{\Lambda}_{\{1,2\}} = \frac{1}{I-1}(\mathbf{C}^{-1}_{\{1,2\}})'$, i.e.,

$$\begin{bmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{bmatrix} = \frac{1}{I-1}\frac{1}{c_{11}c_{22} - c_{12}c_{21}} \begin{bmatrix} c_{22} & -c_{21} \\ -c_{12} & c_{11} \end{bmatrix}, \tag{34}$$

we decompose the off-diagonal demand coefficients $c_{12} \equiv \frac{\partial q_1^j(\cdot)}{\partial p_2}$ and $c_{21} \equiv \frac{\partial q_2^j(\cdot)}{\partial p_1}$ into direct and indirect effects (analogously to Eq. (25) in Section 3.2.3):

$$\underbrace{\frac{\partial q_1^j(\cdot)}{\partial p_2}}_{=-c_{12}} = \underbrace{\frac{\partial q_1^j}{\partial p_2}}_{\text{Direct effect } \sigma_{12}} + \underbrace{\underbrace{\frac{\partial q_1^j}{\partial p_3}}_{\sigma_{13}} \underbrace{\frac{\partial E[p_3|p_1, p_2, \mathbf{q}_0^j]}{\partial p_2}}_{Cov[p_2,p_3]}}_{\text{Inference effect}}, \tag{35}$$

$$\underbrace{\frac{\partial q_2^j(\cdot)}{\partial p_1}}_{=-c_{21}} = \underbrace{\frac{\partial q_2^j}{\partial p_1}}_{\text{Direct effect } \sigma_{21}} + \underbrace{\underbrace{\frac{\partial q_2^j}{\partial p_3}}_{\sigma_{23}} \underbrace{\frac{\partial E[p_3|p_1, p_2, \mathbf{q}_0^j]}{\partial p_1}}_{Cov[p_1,p_3]}}_{\text{Inference effect}}. \tag{36}$$

Since the direct effects in cross-asset price impact coincide, price impact is symmetric if the corresponding inference effects coincide. As we discussed in Section 3.2.3, because price impact is not proportional to covariance $\mathbf{\Sigma}$ with independent market clearing, the price impact between any pair of assets depends on the covariance of *all* assets. If assets 1 and 2, whose demands are contingent on the same prices, are symmetrically correlated with asset 3, the equilibrium price covariances with asset 3—and hence inference effects and *cross-asset* price impacts λ_{12} and λ_{21}—are the same.

[39] Both change if one does, generically in asset covariance $\mathbf{\Sigma}$ and market structure $N = \{K(n)\}_n$.

(ii) Why is exchange $\{1\}$ redundant when the cross-asset price impacts λ_{12} and λ_{21} are symmetric (Theorem 4 (ii))?

We first note that the inclusion of exchange $\{1\}$ preserves the symmetry of cross-asset price impacts of assets $1, 2$ vs. 3 in the new market structure. While the new venue $\{1\}$ changes cross-asset inference effects, it does not include prices of assets that are not traded in exchange $\{1, 2\}$.

Price impact symmetry plays a key role in the proof of Theorem 4 because the demand coefficient $\widehat{\mathbf{B}} = \mathbf{W}'\mathbf{B}$ is determined by a harmonic mean of $\mathbf{C}$ and $\mathbf{C}'$ (Eq. (32)). The equilibrium price is a linear function of the random variables—the aggregate endowments $\{\overline{q}_{0,k}\}_k$—with weights $\mathbf{C}^{-1}\mathbf{B}$ (Eq. (22)). When the price impact (equivalently $\mathbf{C}_{\{1,2\}}$) is symmetric, the demand matrix coefficients $\mathbf{B}$ and $\mathbf{C}$ are linked *proportionally*; thus, the relative weights across assets in $\mathbf{C}^{-1}\mathbf{B}$ are invariant to the changes in $\mathbf{B}$ and $\mathbf{C}$ induced by the new venue. In particular, the price of each asset is the same linear combination of $\{\overline{q}_{0,k}\}_k$ before and after the new venue is introduced. Consequently, the per-unit price impact $\widehat{\mathbf{\Lambda}}$ and cross-asset inference $\widehat{\mathbf{B}}$ do not change. Moreover, we show that, in any market structure, the prices of the same asset traded in different venues equalize if and only if the price impact submatrix for the corresponding exchanges is symmetric (Lemma 4 in Appendix B).

With asymmetric cross-asset price impacts, $\lambda_{12} \neq \lambda_{21}$, the demand coefficients $\mathbf{B}$ and $\mathbf{C}$ are not linked proportionally—neither before nor after the new venue is introduced. The prices of asset 1 in exchanges $\{1, 2\}$ and $\{1\}$ are different linear combinations of the random variables than its price before venue $\{1\}$ is created; the new exchange is not redundant.

(iii) On the other hand, the introduction of exchange $\{3\}$ is redundant irrespective of the symmetry of $\mathbf{\Lambda}$ (Theorem 4 (i)). Because the inference effects with respect to assets 1 and 2 are the same in both exchanges $\{3\}$, the price impact is the same in these venues—prices equalize, and traders split their demands for asset 3 equally between the two exchanges. □

We highlight additional insights of Proposition 3 and Theorem 4. These results underscore the role of imperfect competition for the nonredundancy of innovation. In imperfectly competitive markets, there are two types of nonredundant innovation:

- Exchanges n' whose assets are not a subset of another venue's assets ($K(n') \not\subset K(n)$ for all n), i.e., the total demand for some asset $k \in K$ is contingent on prices of new assets; e.g., exchange $\{2, 3\}$ in Example 3. The inference error weakly decreases for *all* assets.

- Exchanges n' whose assets are a proper subset of another venue's assets ($K(n') \subsetneq K(n)$ for some n), i.e., the *total demand* of each asset $k \in K$ is contingent on prices of the same assets; e.g., exchange $\{1\}$ in Example 3. However, with new contingent variables (i.e., additional prices of the same assets), *demands* for the same asset in different exchanges are contingent on distinct linear combinations of the random variables. This changes cross-asset inference, and hence $\widehat{\mathbf{\Lambda}}$ and $\widehat{\mathbf{B}}$. Inference error can increase for some assets and decrease for others.

The latter type of nonredundant innovation—which is present when price impact is asymmetric (Theorem 4 (ii))—has no counterparts in competitive markets. Indeed, when the price impact is symmetric (e.g., the zero matrix) only the former type of innovation can be nonredundant. Furthermore, even if information loss were zero (i.e., $(\sigma_{cv}, \sigma_{pv}) \to 0$, $\frac{\sigma_{pv}}{\sigma_{cv}} > 0$), the former type of innovation would be nonredundant when $I < \infty$ but not when $I \to \infty$, as it would change the equilibrium price impact.

More generally, apart from the introduction of new trading protocols (Example 3), independent market clearing motivates other forms of innovation, such as the linking of existing trading protocols (i.e., merging assets between venues), and the inclusion of an asset in a trading protocol where it was not previously traded (e.g., asset listings). When increasing the set of (imperfectly correlated) conditioning variables in traders' demands, these innovations lead to a market structure with more, fewer, and the same number of exchanges, respectively.[40]

At a primitive level, price impact is a symmetric matrix under a joint condition on the market structure and the asset covariance. Per Theorem 4 (ii), the symmetry of price impact is required only for the exchanges n'' whose assets are a *superset* of those in the new venue n'. For example, in the market $\{\{1,2\},\{3,4\}\}$, the price impact in exchange $\{3,4\}$ need not be symmetric for exchange $\{1\}$ to be redundant. The required symmetry condition ensures that the inference effects among the new assets $K(n')$ and between assets $K(n')$ and assets $K(n) \setminus K(n')$ and $K \setminus K(n)$ are symmetric.

4.2 Multiple Exchanges: Equivalence with Joint Market Clearing

In this section, we ask: what is the scope for innovation in trading technology that would not be redundant in the market? Corollary 2 characterizes the bound on the number of nonredundant exchanges in any market structure. The intuition for the general result can be gleaned from the following example.

Example 4 (Multi-Venue Market Can Be Equivalent to Market That Clears Jointly) Consider the market structure $\{\{1,2\},\{2,3\},\{3,1\}\}$. Even though the market is comprised of multiple exchanges, none of which contain all assets, traders' equilibrium payoffs are the same as in the market with a single exchange for all assets $\{\{1,2,3\}\}$.

To explain this result, we consider trader i's total demand for asset 1 behind the one-exchange counterfactual in Proposition 3, i.e., the sum of his demands for asset 1 in exchanges $\{1,2\}$ and $\{3,1\}$:

$$\widehat{q}^i_1(\mathbf{p}_{\{1,2\}}, \mathbf{p}_{\{3,1\}}) \equiv q^i_{1,\{1,2\}}(\mathbf{p}_{\{1,2\}}) + q^i_{1,\{3,1\}}(\mathbf{p}_{\{3,1\}}) \qquad \forall \mathbf{p}_{\{1,2\}} \in \mathbb{R}^2 \quad \forall \mathbf{p}_{\{3,1\}} \in \mathbb{R}^2.$$

[40]The argument from the proof of Theorem 4 applies to these other innovations and shows that an innovation is redundant if the price impact submatrix that corresponds to the affected exchanges is symmetric.

In either exchange, the expected trades are conditioned on the respective contingent variables in traders' demands—e.g., p_1 and p_2 in exchange $\{1,2\}$. In the total demand for asset 1, with additional expected trade terms contingent on a different subset of prices, the expected trades of assets 2 and 3 are linear combinations of all random variables, $\overline{q}_{0,1}, \overline{q}_{0,2}$, and $\overline{q}_{0,3}$.[41]

Crucially, since the total demand for *each* asset is conditioned on at least $K = 3$ prices that give linearly independent combinations of the K random variables $\{\overline{q}_{0,k}\}_k$, the inference errors cancel out; i.e., the sum of expected total trades $E[\widehat{q}^i_\ell | \mathbf{p}_{K(n)}, \mathbf{q}^i_0]$ in the total demand for asset k across exchanges $\{n | k \in K(n)\}$ is the same as $\widehat{q}^i_\ell$ for all k and $\ell \neq k$. Equilibrium is *ex post* even if in no exchange, traders' demands condition on prices of all assets (i.e., $K(n) \subsetneq K$ for all n) so that no expectation about trade is perfect, i.e., $E[q^i_{\ell,n'} | \mathbf{p}_{K(n)}, \mathbf{q}^i_0] \neq q^i_{\ell,n'}$ for all n and $n' \neq n$, and all i.

Furthermore, with perfect inference in total demands, price impact matrix is symmetric and the same as in the contingent market, $\widehat{\mathbf{\Lambda}} = \mathbf{\Lambda}^c = \frac{\alpha}{I-2}\mathbf{\Sigma}$ and $\widehat{\mathbf{B}} = \mathbf{B}^c = \frac{I-2}{I-1}\mathbf{Id}$ (Proposition 3). Equilibrium is as if traders could condition their demand for each asset on the price vector.[42]
□

Corollary 2 gives a condition on the market structure itself that characterizes the scope for nonredundant innovation.[43]

Corollary 2 (Redundancy of Changes in Market Structure: A Condition on Exchanges) Suppose that $0 < |\rho_{k\ell}| < 1$ for some k and $\ell \neq k$. When $I < \infty$, the following statements are equivalent:

(i) Introducing any additional exchange n' is redundant;

(ii) Equilibrium is *ex post*;

(iii) For every pair of assets k' and $\ell' \neq k'$ such that $0 < |\rho_{k'\ell'}| < 1$, there is an exchange n in which these assets are traded, i.e., $k', \ell' \in K(n)$.

The equivalence between conditions (ii) and (iii) answers the following question: Given the assets and traders, which market structures with multiple exchanges that clear independently function like a market that clears jointly, and when does equilibrium behavior differ? For all

[41] For example, from Eq. (28), the total expected trade for asset 2 in total demand $\widehat{q}^i_1(\cdot) = q^i_{1,\{1,2\}}(\cdot) + q^i_{1,\{3,1\}}(\cdot)$ is $((\alpha\mathbf{\Sigma}_{\{1,2\},\{1,2\}} + \mathbf{\Lambda}_{\{1,2\}})^{-1})_1 \alpha \mathbf{\Sigma}_{\{1,2\}} E[\widehat{q}^i_2 | \mathbf{p}_{\{1,2\}}, \mathbf{q}^i_0] + ((\alpha\mathbf{\Sigma}_{\{3,1\},\{3,1\}} + \mathbf{\Lambda}_{\{3,1\}})^{-1})_1 \alpha \mathbf{\Sigma}_{\{3,1\}} E[\widehat{q}^i_2 | \mathbf{p}_{\{3,1\}}, \mathbf{q}^i_0]$.

[42] Consider a market structure $\{\{1,2\},\{2,3\},\{3,1\},\{4\}\}$ and assume that the payoff of asset 4 is imperfectly correlated with those of other assets. This market structure is payoff-equivalent to $\{\{1,2,3\},\{4\}\}$ if and only if the inference effects of assets $1,2$ and 3 with respect to asset 4 are symmetric. Then, multiple venues that clear independently can implement joint clearing "locally" for assets $1,2$, and 3.

[43] Corollary 6 in Appendix B shows that as long as some assets in the market are *imperfectly* correlated, some innovations will not be redundant. With perfectly correlated assets, inference is perfect; with independent assets, inference is not payoff-relevant. Innovation then does not affect equilibrium price distribution.

market structures characterized in condition (iii) of Corollary 2, equilibrium is *ex post*.[44] This result shows that one can implement the contingent-market outcome via simpler schedules. Two assets per exchange suffice.

The equivalence between conditions (i) and (iii) puts a bound on the number of exchanges that can be introduced in a market or the ways in which trading protocols can be linked (by merging their assets) or asset listings and still be nonredundant. The maximal number of such nonredundant innovations is $\frac{K(K-1)}{2}$.

5 Welfare and Independence in Market Clearing

In this section, we consider the welfare impact of independence in market clearing. An important implication of Corollary 2 is that *when combined with suitable exchange design, markets with multiple exchanges that clear independently can be as efficient as a single exchange that clears all assets jointly, for any distributions of asset payoffs and endowments.* The main observation in this section is that markets with multiple exchanges can *strictly* improve *ex ante* welfare relative to joint market clearing.

We first ask: why might a market with multiple trading venues give rise to higher welfare than a single exchange for all assets? If the market were competitive ($I \to \infty$), then joint clearing would give higher welfare than any other market structure—it would eliminate information loss across exchanges. In imperfectly competitive markets, the benefit of lower trading costs associated with price impact can countervail the cost that stems from inference error.

We then inquire: In which trading environments is welfare higher with multi-venue trading? Unlike the competitive market, efficient design depends on market characteristics. Corollary 3, Proposition 4, and Examples 1 and 5 give and illustrate the conditions.

5.1 Welfare-improving Designs

The *ex ante* total welfare is given by the sum of the equilibrium payoffs (Eq. (31)) for all traders. By Proposition 3, the welfare effects of market structure can be understood in terms of the per-unit price impact matrix $\widehat{\mathbf{\Lambda}}$ and cross-asset inference $\widehat{\mathbf{B}}$. When the market structure changes, the corresponding welfare change can be decomposed into three effects related to: (1) the price impact for a given asset (i.e., diagonal elements of $\widehat{\mathbf{\Lambda}}$), (2) the cross-asset price impact (i.e., off-diagonal elements of $\widehat{\mathbf{\Lambda}}$), and (3) the inference error.[45] A corollary of Theorem 3 and

[44] In the market structures that satisfy condition (iii), while the cross-exchange price impacts are zero, the per-unit price impact matrix is the same as in the contingent market, $\widehat{\mathbf{\Lambda}} = \mathbf{\Lambda}^c$; in particular, it is proportional to the asset covariance $\mathbf{\Sigma}$. The cross-exchange inference effects mimic the contingent design's cross-asset price impact.

[45] The inference error in the last term of $\sum_i E[u^i(\mathbf{q}^i) - \mathbf{p}\cdot\mathbf{q}^i]$ in Eq. (31) can be characterized as:

$$(I-1)\sigma_{pv} tr\Big(\frac{1}{2}(\widehat{\mathbf{B}}-\mathbf{B}^c)'\alpha\mathbf{\Sigma}(\widehat{\mathbf{B}}-\mathbf{B}^c) + \frac{1}{I-1}\alpha\mathbf{\Sigma}(\widehat{\mathbf{B}}-\mathbf{B}^c)\Big), \tag{37}$$

Proposition 3 shows that it is important to distinguish between the trading cost components (1) and (2). Respectively, they represent the *trading cost of risk sharing* and the *trading costs of diversification.*

Corollary 3 (Price Impact and Market Structure) Consider two market structures N and N' such that if $\{k,\ell\} \subset K(n')$ for some $n' \in N'$, $\ell \neq k$, then $\{k,\ell\} \subset K(n)$ for some $n \in N$. Let $\widehat{\mathbf{\Lambda}}^N$ and $\widehat{\mathbf{\Lambda}}^{N'}$ be the corresponding per-unit price impact matrices. Assume that $\widehat{\mathbf{\Lambda}}^N \neq \widehat{\mathbf{\Lambda}}^{N'}$.[46]

(i) If $K = 2$, then $\widehat{\lambda}_k^N \leq \widehat{\lambda}_k^{N'}$ for each k.

(ii) If $K > 2$, $\widehat{\lambda}_k^N$ need not be lower than $\widehat{\lambda}_k^{N'}$for some k.

For two-asset markets, Corollary 3 establishes a trade-off between risk sharing and diversification. Namely, when $K = 2$, then limited conditioning increases the per-unit diagonal price impact for each asset (part (i)).[47] Although joint clearing minimizes the cost of risk sharing, multi-venue trading can strictly increase welfare by lowering the trading cost of diversification that stems from cross-asset price impact; Example 1 illustrates this. Corollary 4 in Section 5.2 provides the necessary and sufficient conditions for independent market clearing to increase welfare in two-asset markets.

More generally, in markets with $K > 2$, multi-venue trading can increase welfare by lowering the trading costs of risk sharing, diversification, or *both* (part (ii); Example 5). Likewise, so can the further limiting of demand conditioning in markets with multiple exchanges. The increase in welfare with multi-venue trading—relative to joint clearing or more generally—can be accomplished in the Pareto sense.

5.2 When Are Multi-Venue Markets More Efficient?

We further ask: in which trading environments is multi-venue trading more efficient than joint clearing? Proposition 4 translates Corollary 3's price-impact effects on risk sharing and diversification to market characteristics.

Proposition 4 (Welfare with Multiple Exchanges vs. Joint Market Clearing) Given $I < \infty$ traders and K assets such that $0 < |\rho_{k\ell}| < 1$ for some k and $\ell \neq k$, there exists a market structure with multiple exchanges for which the *ex ante* welfare is strictly larger than that in a single exchange for some distribution of endowments $\{\mathbf{q}_0^i\}_i$.

where $\mathbf{B}^c = \frac{I-2}{I-1}\mathbf{Id}$ is the coefficient of the contingent demand on $\mathbf{q}_0^i$. In the contingent market, using the fact that $\widehat{\mathbf{\Lambda}} = \mathbf{\Lambda}^c$, we have that $\Upsilon(\mathbf{\Lambda}^c) \equiv \frac{I(I-2)}{2(I-1)^2}\alpha\mathbf{\Sigma}$ in the *ex ante* welfare Eq. (31), and the inference error (37) is zero (equilibrium is *ex post*).

[46] That is, there exists $\{k',\ell'\}$ such that $\{k',\ell'\} \subset K(n)$ for some $n \in N$ but $\{k',\ell'\} \not\subset K(n')$ for all $n' \in N'$.

[47] Part (i) of Corollary 3 holds in more general markets with $K \geq 2$ assets, symmetric covariances (i.e., $\sigma_{kk} = \sigma$ for all k and $\sigma_{k\ell} = \sigma\rho$ for all k and $\ell \neq k$), and market structures defined by symmetric demergers (Definition 6). (See Proposition 5 in Appendix C.1.)

Proposition 4, Corollary 4, and Example 5 demonstrate that neither the contingent nor the uncontingent market structure is efficient irrespective of the market characteristics (see also Example 1 and Corollary 3). Here, we highlight three insights.

First, if the number of traders is sufficiently large, then joint clearing (or an equivalent design) is efficient. Given any K (imperfectly correlated) assets and $I < \infty$ traders, however, a market structure with multiple venues is more efficient than joint clearing for some distributions of endowments. The proof of Proposition 4 is constructive and provides a sufficient condition on the pertinent market structures: any demerger of a single exchange for all assets.

Definition 6 (Demerger) *A* demerger *of a single exchange for all assets is a market structure $N = \{K(n)\}_n$ whose exchanges partition the K assets traded: $K(n) \cap K(n') = \emptyset$ for all n and $n' \neq n$. A demerger is* symmetric *if its exchanges have the same number of assets.*

Thus, simply breaking up a single exchange for all assets into multiple venues can increase welfare. The intuition for the proof can be seen in Eq. (31): as we show, in any demerger of a single exchange, the price impact matrices in the surplus matrix difference $\mathbf{\Upsilon}(\mathbf{\Lambda}^c) - \mathbf{\Upsilon}(\widehat{\mathbf{\Lambda}})$ are not ranked in a positive semidefinite sense (Corollary 3 and Lemma 5 in Appendix B).

Second, which assets' demands should be linked and which exchanges should be introduced depends on the *joint distribution of asset payoffs and traders' endowments across assets.* More precisely, the condition involves the joint substitutability in asset payoffs (i.e., $\mathbf{\Sigma}$) and the trading needs of market participants (i.e., $\{|E[\overline{q}_{0,k}] - E[q^i_{0,k}]|\}_{i,k}$). In particular, even if all asset payoffs are substitutes or complements, efficient design depends on whether some traders buy and others sell assets (e.g., the primary market in Treasury auctions) or traders buy some assets and sell others (e.g., intra-dealer markets). See Example 5 (a) and (b).

The role of the joint condition can be seen in Eq. (31), where the equilibrium surplus is a quadratic matrix function of expected trading needs $\{E[\overline{\mathbf{q}}_0] - E[\mathbf{q}^i_0]\}_i$. A market with multiple trading venues yields higher welfare than a single exchange if $E[\overline{\mathbf{q}}_0] - E[\mathbf{q}^i_0]$ is proportional to an eigenvector of $\mathbf{\Upsilon}(\mathbf{\Lambda}^c) - \mathbf{\Upsilon}(\widehat{\mathbf{\Lambda}})$ that corresponds to a negative eigenvalue. Corollary 3 and Lemma 5 in Appendix B show that a negative eigenvalue exists for any market structure whose exchanges are demergers of a single venue for all assets. The surplus matrix difference $\mathbf{\Upsilon}(\mathbf{\Lambda}^c) - \mathbf{\Upsilon}(\widehat{\mathbf{\Lambda}})$ depends on the asset covariance.[48]

Finally, the *heterogeneity* in the asset payoffs' substitutability $\mathbf{\Sigma}$ as well as trading needs across assets $\{E[\overline{\mathbf{q}}_0] - E[\mathbf{q}^i_0]\}_i$ determines whether the net benefit from diversification and risk sharing dominates with multi-venue trading. The heterogeneity can favor a market structure "intermediate" between contingent or uncontingent—then, inducing asymmetries in trading costs can be beneficial. See Example 5 (c).

Corollary 4 illustrates these observations in two-asset markets, providing a necessary and

[48] As Example 1 illustrates, in markets with $K = 2$, the zero cross-asset price impact is beneficial when traders take the same (buying or selling) position for asset payoff substitutes or the opposite position for complements.

sufficient condition on $\{\mathbf{\Sigma}, \{E[\overline{\mathbf{q}}_0] - E[\mathbf{q}_0^i]\}_i\}$ for multi-venue trading to dominate joint clearing in welfare terms.

Corollary 4 (Welfare with Multiple Exchanges vs. Joint Market Clearing ($K = 2$)) Consider a market with two imperfectly correlated (i.e., $0 < |\rho| < 1$) assets whose variances are the same (i.e., $\sigma_{11} = \sigma_{22}$). Suppose there is no information loss: i.e., $(\sigma_{cv}, \sigma_{pv}) \to \mathbf{0}$ and $\sigma_0 < 1$. The *ex ante* welfare is strictly larger in the uncontingent market $\{\{1\}, \{2\}\}$ than in a single exchange for all assets $\{\{1, 2\}\}$ if and only if the following conditions hold:

(i) $sign(E[\overline{q}_{0,1}] - E[q_{0,1}^i])sign(E[\overline{q}_{0,2}] - E[q_{0,2}^i]) = sign(\rho)$, i.e., the market is one-sided when $\rho > 0$ or two-sided when $\rho < 0$; and

(ii) The trading needs are sufficiently symmetric across assets, i.e., there exist bounds $\underline{\xi}(\rho, I) < 1 < \overline{\xi}(\rho, I)$ on the relative trading needs such that

$$\underline{\xi}(\rho, I) \ < \ \left| \frac{E[\overline{q}_{0,k}] - E[q_{0,k}^i]}{E[\overline{q}_{0,\ell}] - E[q_{0,\ell}^i]} \right| \ < \ \overline{\xi}(\rho, I) \qquad \forall i.$$

By condition (i), the zero cross-asset price impact is beneficial; by condition (ii), the benefit of diversification (i) dominates the cost of risk sharing (cf. Example 1 and Corollary 3).[49]

Example 5 (Heterogeneity and Efficient Market Structure)
Consider a market with $K = 3$ assets, two of which (i.e., 2 and 3) are each symmetrically correlated with other assets and have symmetric *ex ante* trading needs. There are thirteen payoff-relevant market structures, including the contingent and uncontingent ones. Fig. 2 plots the welfare-maximizing design as a function of the heterogeneity in asset correlations $\rho_{12}/\rho_{23} = \rho_{13}/\rho_{23} \equiv \rho_H/\rho_L$ on the horizontal axis and the heterogeneity in trading needs $(E[\overline{q}_{0,1}] - E[q_{0,1}^i])/(E[\overline{q}_{0,2}] - E[q_{0,2}^i]) = (E[\overline{q}_{0,1}] - E[q_{0,1}^i])/(E[\overline{q}_{0,3}] - E[q_{0,3}^i]) \equiv q_H^i/q_L^i$ for all i on the vertical axis.

(a) If the asset correlations and trading needs are symmetric across assets (i.e., the point $(\frac{\rho_H}{\rho_L}, \frac{q_H^i}{q_L^i}) = (1, 1)$ in each panel), either the contingent or the uncontingent market structure is efficient.

(b) In one-sided markets (i.e., when traders either buy or sell all assets), efficient market structure depends on the asset payoff substitutability:

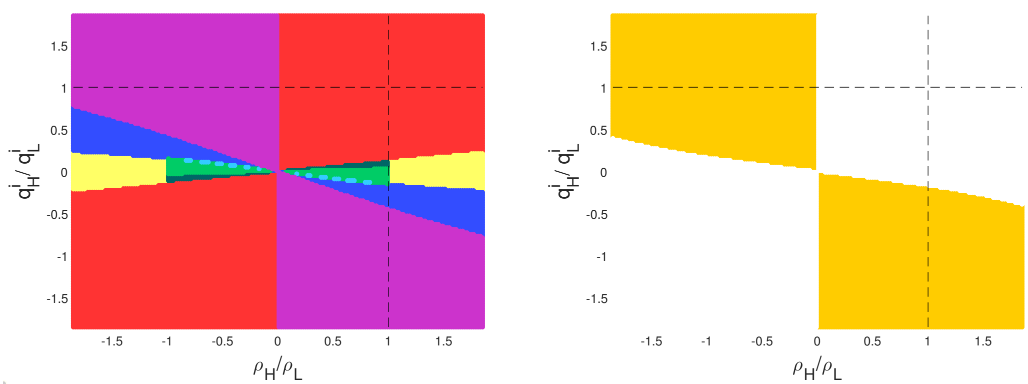
If asset payoffs are complements (i.e., $\rho_H < 0$ and $\rho_L < 0$; Fig. 2B) and traders buy both assets (i.e., $q_H^i > 0$ and $q_L^i > 0$), then the contingent market maximizes welfare, irrespective of the heterogeneity in $\{\rho_H, \rho_L\}$ and $\{q_H^i, q_L^i\}$.

[49] As $I \to \infty$, the bounds in condition (ii) $\underline{\xi}(\rho, I) \to 0$ and $\overline{\xi}(\rho, I) \to \infty$ for all ρ, and so the cost of risk sharing goes to zero regardless of the heterogeneity in trading needs across assets. However, $\mathbf{\Upsilon}(\mathbf{\Lambda}^c) - \mathbf{\Upsilon}(\widehat{\mathbf{\Lambda}}) \to \mathbf{0}$, i.e., the benefit from diversification relative to the contingent market also vanishes, and the *ex ante* welfare in $\{\{1\}, \{2\}\}$ and $\{\{1, 2\}\}$ can differ in the limit only due to the information loss (if $\sigma_{cv} > 0$ and $\sigma_{pv} > 0$).

If asset payoffs are substitutes (i.e., $\rho_H > 0$ and $\rho_L > 0$; Fig. 2A), then the heterogeneity in trading needs matters. If trading needs are symmetric, the uncontingent market is efficient. With sufficiently heterogeneous trading needs, a market structure other than the contingent or uncontingent ones maximizes welfare.

(c) Welfare-maximizing market structures in which some but not all assets clear jointly either link assets to reduce the trading cost of diversification for those assets (orange and blue areas) or link assets with the most heterogeneous trading needs to balance the tradeoff between risk sharing and diversification—even when linking the assets increases the trading costs due to diversification (yellow and purple areas).[50] □

FIGURE 2: HETEROGENEOUS ASSET CORRELATIONS AND TRADING NEEDS



(A) $\rho_L = 0.25$ (B) $\rho_L = -0.25$

Notes: Each color indicates which market structure provides the highest *ex ante* welfare. Red $= \{\{1\},\{2\},\{3\}\}$ (i.e., the uncontingent market); Orange $= \{\{1\},\{2,3\}\}$; Yellow $= \{\{1,2\},\{3\}\}$; Blue $= \{\{1,2\},\{1,3\}\}$; Light blue $= \{\{1,2\},\{1,3\},\{3\}\}$; Purple $= \{\{1,2\},\{1,3\},\{1\}\}$; Green $= \{\{1,2\},\{2\},\{3\}\}$; Olive $= \{\{1,2\},\{1\},\{3\}\}$; and White $= \{\{1,2,3\}\}$ (i.e., the contingent market). Information loss is sufficiently small not to dominate the welfare benefit from diversification ($\sigma_{cv} = 0, \sigma_{pv} = 0.01$). The number of traders is $I = 10$. The trading needs for assets 2 and 3 are $|E[\overline{q}_{0,L}] - E[q^i_{0,L}]| = 1$ for all i. Panel (A) assumes the asset payoff correlation $\rho_L = 0.2$ (i.e., substitutes), and panel (B) assumes $\rho_L = -0.2$ (i.e., complements).

Proposition 6 in Appendix C.1 generalizes Corollary 4 to $K \geq 2$ assets. It shows the link between asset payoff substitutes and complements and the optimality of an extreme market structure in symmetric trading environments.[51] The uncontingent market is the most efficient when asset payoffs are substitutes ($\rho > 0$ in condition (i)), whereas a single exchange for all assets is the most efficient when asset payoffs are complements ($\rho < 0$).

[50]Whether linking the trading protocols of some assets is efficient depends on the trading needs and the payoff substitutability *of all assets*—another result of the non-proportionality of price impact to asset covariance in markets other than the contingent one (Eqs. (21) and (26)).

[51]I.e., symmetric demergers (Definition 6), asset covariances (i.e., $\sigma_{kk} = \sigma$ for all k and $\sigma_{k\ell} = \sigma\rho$ for all $\ell \neq k$ and k), and trading needs across assets ($\frac{E[\overline{q}_{0,k}] - E[q^i_{0,k}]}{E[\overline{q}_{0,\ell}] - E[q^i_{0,\ell}]} = 1$ for all k, ℓ, and i in condition (ii) of Corollary 4).

6 Discussion

The effects identified in this paper contribute to the discussion concerning the impact of changes in trading technology (e.g., Pagano (1989), Budish, Cramton, and Shim (2015), Pagnotta and Philippon (2018), Budish, Lee, and Shim (2019), Cespa and Vives (2019)). By accommodating general cross-asset demand conditioning, our analysis takes a step towards exploring implications of innovation in trading technology defined by changes in market clearing that cross-asset conditioning makes possible. We conclude with directions for future research and further discussion of the model.

First, the non-neutrality of innovation in trading technology is a manifestation of a more general implication of independence in market clearing: equilibrium payoffs can be changed by innovations whose payoffs lie in the span of the existing assets. This paper's model and equilibrium characterization as a fixed point in price impacts can be adapted to the study of other innovations. In Rostek and Yoon (2018), we study one such class: derivatives, i.e., securities whose payoffs are defined as bundles (linear combinations) of the existing assets. We show that the equilibrium effects of the introduction of nonredundant derivatives differ from those produced by the innovation in trading technology studied in this paper. Thus, innovation in trading technology provides an instrument for impacting markets' performance separate from security innovation.

Second, it would be worthwhile to explore whether in a dynamic market, a joint design of the trading frequency and trading technology could further improve the lower bound on welfare relative to the contingent design. Dynamic trading provides *additional* reasons to innovate in trading technology. More generally, conditioning on simultaneously determined prices (i.e., this paper) and past prices within or outside an exchange will interact in nontrivial ways with the relative frequencies of shocks which renew the gains from trade, market clearing, and payoff realizations (i.e., consumption). The rich set of market design questions raised by these two types of demand conditioning merits a separate study to develop design principles for imperfectly competitive markets.

Third, the non-neutrality of innovation in trading technology could be leveraged to enhance revenue or other objectives in markets such as that for Treasury bills, in which securities are often traded simultaneously and independently.

Finally, there is room for further development of the asset-pricing implications of independence in market clearing as well as the equilibrium properties it induces, such as the transformation of risk made possible by the nonproportionality and asymmetry of price impact.

Independence of market clearing: incentives of traders vs. exchanges. While the paper does not characterize the endogenous formation of exchanges, the analysis suggests that it is interesting to distinguish between the incentives of traders and exchanges. If traders themselves could decide whether to submit contingent or uncontingent demands, individual optimization entails that a contingent demand would be a best response, taking as given the demands submitted by others. Submission of contingent demands by all traders would be the unique equilibrium, eliminating the welfare-improving (possibly in the Pareto sense) effects of limited conditioning. Thus, implementation of uncontingent trading involves a restriction of the cross-asset conditioning of simultaneously placed orders. This is the prevalent practice. Exchanges, however, endowed with an objective to maximize (e.g., revenue, volume, or liquidity) generally have at least weak incentives not to allow for full demand conditioning of assets traded. Too much innovation, through its externalities on the liquidity of the traded assets, can hinder the exchanges' objective.

Role of the uniform-price mechanism. One might wonder whether our conclusions rely on the uniform-price mechanism. The key to the welfare effects of new exchanges is the inefficiency of equilibrium allocation due to price impact—given incomplete demand conditioning—which new exchanges alter. With (two-sided) private information, one expects allocation to be inefficient and the effects of new exchanges to exist for other pricing mechanisms.

Heterogeneous participation.[52] Decentralizing a market by allowing some traders to participate in exchanges for only a subset of all assets or trade with only a subset of all traders (while submitting contingent schedules) can increase welfare by *reallocating risk across traders*, provided that traders' risk preferences differ; with symmetric risk preferences, the centralized market maximizes welfare (Malamud and Rostek (2017)). Our results suggest that restrictions on conditioning can provide an effective instrument to increase welfare *in the Pareto sense*—by improving risk sharing across traders and/or risk diversification across assets in ways not feasible with heterogeneous participation and contingent trading.

References

Afonso, G. and R. Lagos (2015): "Trade Dynamics in the Market for Federal Funds," Econometrica, 83: 263-313.

Antill, S. and D. Duffie (2019): "Augmenting Markets with Mechanisms," Working Paper.

Atkeson, A. G., A. L. Eisfeldt, and P.-O. Weill (2015): "Entry and Exit in OTC Derivatives Markets," Econometrica, 83: 2231-2292.

Babus, A. and P. Kondor (2018): "Trading and Information Diffusion in Over-the-Counter Markets," Econometrica, 86: 1727-1769.

Babus, A. and C. Parlatore (2017): "Strategic Fragmented Markets," Working Paper.

Bergemann, D., T. Heumann, and S. Morris (2017): "Information and Market Power," Working Paper.

Budish, E., P. Cramton, and J. Shim (2015): "The High-Frequency Trading Arms Race: Frequent Batch Auctions as a Market Design Response," Quarterly Journal of Economics, 130, 4: 1547-1621.

[52]Babus and Kondor (2017), Babus and Parlatore (2017), and Malamud and Rostek (2017) study markets with limited participation and contingent contracts.

Budish, E., R. Lee, and J. Shim (2019): "A Theory of Stock Exchange Competition and Innovation: Will the Market Fix the Market?" Working Paper.

Cespa, G. and X. Vives (2019): "Market Transparency and Fragility," Working Paper.

Chang, B. and S. Zhang (2018): "Endogenous Market Making and Network Formation," Working Paper.

Chen, D. and D. Duffie (2020): "Market Fragmentation," Working Paper.

Choi, M., A. Galeotti, and S. Goyal (2017): "Trading in Networks: Theory and Experiment," Journal of the European Economic Association, 15, 4: 784-817.

Collard-Wexler, A., G. Gowrisankaran, and R. S. Lee (2019): ""Nash-in-Nash" Bargaining: A Microfoundation for Applied Work," Journal of Political Economy, 127, 1: 163-195.

Condorelli , D., A. Galeotti, and L. Renou (2017): "Bilateral Trading in Networks," Review of Economic Studies, 84, 1: 82-105.

Du, S. and H. Zhu (2017a): "Bilateral Trading in Divisible Double Auctions," Journal of Economic Theory, 167: 285-311.

Du, S. and H. Zhu (2017b): "What is the Optimal Trading Frequency in Financial Markets?" Review of Economic Studies, 84, 4: 1606-1651.

Duffie, D. (2018): "Notes on LIBOR Conversion," Working Paper.

Duffie, D., N. Garleanu, and L. H. Pedersen (2005): "Over-the-Counter Markets," Econometrica, 73, 6: 1815-1847.

Elliott, M. (2015): "Inefficiencies in Networked Markets," American Economic Journal: Microeconomics, 7, 4: 43-82.

Gale, D. (1986a): "Bargaining and Competition Part I: Characterization," Econometrica, 54, 4: 785-806.

Gale, D. (1986b): "Bargaining and Competition Part II: Existence," Econometrica, 54, 4: 807-818.

Garleanu, N. and L. Pedersen (2013): "Dynamic Trading with Predictable Returns and Transaction Costs," Journal of Finance, 68, 6: 2309-2340.

Glode, V. and C. Opp (2016): "Asymmetric Information and Intermediation Chains," American Economic Review, 106, 9: 2699-2721.

Gofman, M. (2018): "A Network-Based Analysis of Over-the-Counter Markets," Working Paper.

Hart, O. and J. Tirole (1990): "Vertical Integration and Market Foreclosure." Brookings Papers on Economic Activity, Special Issue: 205-276.

Horn, H. and A. Wolinsky (1988): "Bilateral Monopolies and Incentives for Merger," The RAND Journal of Economics, 19, 3: 408-419.

Hugonnier, J., B. Lester, and P.-O. Weill (2020): "Frictional Intermediation in Over-the-Counter Markets," Review of Economic Studies, 87: 1432-1469.

Kawakami, K. (2017): "Welfare Consequences of Information Aggregation and Optimal Market Size," American Economic Journal: Microeconomics, 9, 4: 303-323.

Klemperer, P. and M. Meyer (1989): "Supply Function Equilibria in Oligopoly under Uncertainty," Econometrica, 57, 6: 1243-1277.

Kranton, R. E. and D. F. Minehart (2001): "A Theory of Buyer-Seller Networks," American Economic Review, 91, 3: 485-508.

Kyle, A. S. (1989): "Informed Speculation and Imperfect Competition," Review of Economic Studies, 56, 3: 517-556.

Kyle, A. S. and J. Lee (2018): “Information and Competition with Speculation and Hedging,” Working Paper.

Kyle, A. S., A. A. Obizhaeva, and Y. Wang (2017): “Smooth Trading with Overconfidence and Market Power,” Review of Economic Studies, 85, 1: 611-662.

Lambert, N., M. Ostrovsky, and M. Panov (2018): “Strategic Trading in Informationally Complex Environments,” Econometrica, 86, 4: 1119-1157.

Lester, B., A. Shourideh, V. Venkateswaran, and A. Zetlin-Jones (2018): “Market-making with Search and Information Frictions,” Working Paper.

Lester, B., A. Shourideh, V. Venkateswaran, and A. Zetlin-Jones (2019): “Screening and Adverse Selection in a Frictional Markets,” Journal of Political Economy, 127, 1: 338-377.

Malamud, S. and M. Rostek (2017): “Decentralized Exchange,” American Economic Review, 107, 11: 3320-3362.

Pagano, M. (1989): “Trading Volume and Asset Liquidity,” Quarterly Journal of Economics, 104, 2: 255-274.

Pagnotta, E. and T. Philippon (2018): “Competing on Speed,” Econometrica, 86, 3: 1067-1115.

Peivandi, A. and R. Vohra (2020): “On Fragmentation of Markets,” Working Paper.

Rostek, M. and M. Weretka (2015): “Dynamic Thin Markets,” Review of Financial Studies, 28, 10: 2946-2992.

Rostek, M. and J. H. Yoon (2018): “Innovation in Decentralized Markets,” Working Paper.

Rostek, M. and J. H. Yoon (2019): “Dynamic Imperfectly Competitive Markets with Private Information,” Working Paper.

Sannikov, Y. and A. Skrzypacz (2016): “Dynamic Trading: Price Inertia and Front-Running,” Working Paper.

Vayanos D. (1999): “Strategic Trading and Welfare in a Dynamic Market,” Review of Economic Studies, 66, 2: 219-54.

Vayanos, D. and P.-O. Weill (2008): “A Search-based Theory of the On-the-run Phenomenon,” Journal of Finance, 63, 3: 1351-1389.

Vives, X. (2011): “Strategic Supply Function Competition with Private Information,” Econometrica, 79, 6: 1919-66.

Wilson, R. (1979): “Auctions of Shares,” Quarterly Journal of Economics, 93, 4: 675-689.

Wittwer, M. (2019): “Connecting Disconnected Markets? An Irrelevance Result,” Working Paper.

Zhang, A. (2020): “Competition and Manipulation in Derivative Contract Markets,” Working Paper.

Zhu, H. (2014): “Do Dark Pools Harm Price Discovery?” Review of Financial Studies, 27, 3: 747-789.

Zhu, H. (2018a): “A Multi-maturity Clock Auction Design,” Market Design Proposal.

Zhu, H. (2018b): “Design of CCP Default Management Auctions,” Market Design Proposal.

Appendix

A Equilibrium Characterization

Theorem 5 characterizes equilibrium for general market structures (Definition 4). We allow endowments to be correlated across assets: $\mathbf{\Omega} = \Big(Cov[q^i_{0,k}, q^i_{0,\ell}]\Big)_{k,\ell} \in \mathbb{R}^{K\times K}$ is a positive definite matrix.[53] In a market structure $N = \{K(n)\}_n$, the distribution of asset returns is jointly Normal, $\mathcal{N}(\boldsymbol{\delta}^+, \mathbf{\Sigma}^+)$, where $\boldsymbol{\delta}^+ \in \mathbb{R}^{\sum_n K(n)}$ and $\mathbf{\Sigma}^+ \in \mathbb{R}^{(\sum_n K(n))\times(\sum_n K(n))}$.

Notation. We define an operator $[\cdot]_N : \mathbb{R}^{(\sum_n K(n))\times(\sum_n K(n))} \to \mathbb{R}^{(\sum_n K(n))\times(\sum_n K(n))}$ that maps a matrix $\mathbf{M}$ to a block-diagonal matrix $[\mathbf{M}]_N$ with $([\mathbf{M}]_N)_{K(n),K(n')} \equiv \mathbf{0}$ for $n \neq n'$ and $([\mathbf{M}]_N)_{K(n),K(n)} \equiv \mathbf{M}_{K(n),K(n)}$ for any n.

Theorem 5 (Equilibrium: Fixed Point in Demand Schedules; General Design) Consider a market with $N = \{K(n)\}_n$ exchanges. In equilibrium, the (net) demand schedules, defined by matrix coefficients $\{\boldsymbol{a}^i, \mathbf{B}^i, \mathbf{C}^i\}_i$, and price impacts $\{\mathbf{\Lambda}^i\}_i$ are characterized by the following conditions: for each trader i,

(i) (*Optimization, given price impact*) Given price impact matrices $\mathbf{\Lambda}^i \in \mathbb{R}^{(\sum_n K(n))\times(\sum_n K(n))}$, the coefficients of (net) demands $\boldsymbol{a}^i \in \mathbb{R}^{\sum_n K(n)}$, $\mathbf{B}^i \in \mathbb{R}^{(\sum_n K(n))\times K}$, and $\mathbf{C}^i \in \mathbb{R}^{(\sum_n K(n))\times(\sum_n K(n))}$ are characterized by:

$$\begin{aligned}\boldsymbol{a}^i = {} & \mathbf{C}^i\boldsymbol{\delta}^+ + \Big(\mathbf{B}^i - (\alpha^i\mathbf{\Sigma}^+ + \mathbf{\Lambda}^i)^{-1}\mathbf{W}\alpha^i\mathbf{\Sigma}\Big)E[\mathbf{q}^i_0] \\ & -(\mathbf{C}^i - (\alpha^i\mathbf{\Sigma}^+ + \mathbf{\Lambda}^i)^{-1})\underbrace{(\sum_j(\alpha^j\mathbf{\Sigma}^+ + \mathbf{\Lambda}^j)^{-1})^{-1}\sum_j(\alpha^j\mathbf{\Sigma}^+ + \mathbf{\Lambda}^j)^{-1}\mathbf{W}\alpha^j\mathbf{\Sigma}E[\mathbf{q}^j_0]}_{=\boldsymbol{\delta}^+ - E[\mathbf{p}]},\end{aligned} \tag{38}$$

$$\mathbf{B}^i = (\alpha^i\mathbf{\Sigma}^+ + \mathbf{\Lambda}^i)^{-1}\mathbf{W}\alpha^i\mathbf{\Sigma} - \underbrace{((\alpha^i\mathbf{\Sigma}^+ + \mathbf{\Lambda}^i)^{-1} - \mathbf{C}^i)(\sum_j\mathbf{C}^j)^{-1}\Big(\mathbf{B}^i + \frac{\sigma_{cv}}{\sigma_{cv}+\sigma_{pv}}(\sum_{j\neq i}\mathbf{B}^j)\Big)}_{\text{Adjustment due to cross-asset inference}}, \tag{39}$$

$$\Big[(\mathbf{Id} - (\alpha^i\mathbf{\Sigma}^+ + \mathbf{\Lambda}^i)\mathbf{C}^i)(\sum_j\mathbf{C}^j)^{-1}\underbrace{\Big(\sum_{j\neq i}\mathbf{B}^j\mathbf{\Omega}\Big(\mathbf{B}^j + \frac{\sigma_{cv}}{\sigma_{cv}+\sigma_{pv}}\sum_{h\neq i}\mathbf{B}^h\Big)'\Big)}_{\text{Inference coefficient } Var[\mathbf{s}^{-i}|\mathbf{q}^i_0]}\Big]_N = 0, \tag{40}$$

where $\mathbf{W} \in \{0,1\}^{(\sum_n K(n))\times K}$ is the indicator matrix in market N (Definition 5).

(ii) (*Correct price impact*) Price impact $\mathbf{\Lambda}^i$ equals the transpose of the Jacobian of trader i's inverse residual supply:

$$\mathbf{\Lambda}^i = \Big((\sum_{j\neq i}\mathbf{C}^j)^{-1}\Big)'. \tag{41}$$

Note. With one asset per exchange (i.e., $N = \{\{k\}\}_k$), the statement of Theorem 5 specializes to that of Theorem 1.

[53] The characterization results—Theorems 2 and 5, Proposition 2, Corollaries 1 and 5, and Lemmas 2 and 3—allow for general market structures. Theorem 5, Proposition 2, Corollary 1, Lemma 2 also allow for correlated endowments across assets, and heterogeneous risk preferences across traders.

Lemma 2 in Appendix B shows that asset by asset optimization[54] brings no loss of generality for optimization with respect to a profile of demands $\{\{q^i_{k,n}(\cdot)\}_{k\in K(n)}\}_n$.[55] Proposition 2 shows that pointwise optimization[56] is necessary and sufficient for optimization with respect to demand schedules $\mathbf{q}^i_{K(n)}(\cdot) \equiv (q^i_{k,n}(\cdot))_{k\in K(n)} : \mathbb{R}^{K(n)} \to \mathbb{R}^{K(n)}$ in each exchange n.

Proof of Theorem 5 (Equilibrium: Fixed Point in Demand Schedules; General Design).

Step 1 (Part (i): Optimization, given residual supply $\boldsymbol{\Lambda}^i$ and $F(\mathbf{s}^{-i}|\mathbf{q}^i_0)$) Because $\mathbf{p}_{K(n)}$ maps one-to-one to $\mathbf{s}^{-i}_{K(n)}$ (as we will show in Step 1.3), the price vector $\mathbf{p}_{K(n)}$ has full support. Proposition 2 and Lemma 2 establish that a trader's pointwise optimization for each asset k and each exchange n is necessary and sufficient for optimization in demand functions (i.e., $\{\{q^i_{k,n}(\cdot)\}_k\}_n$): for each k and n,

$$\max_{q^i_{k,n}\in\mathbb{R}} E[\boldsymbol{\delta}\cdot(\mathbf{q}^i+\mathbf{q}^i_0) - \frac{\alpha^i}{2}(\mathbf{q}^i+\mathbf{q}^i_0)\cdot\boldsymbol{\Sigma}(\mathbf{q}^i+\mathbf{q}^i_0) - \mathbf{p}\cdot\mathbf{q}^i|\mathbf{p}_{K(n)},\mathbf{q}^i_0] \qquad \forall p_{K(n)}\in\mathbb{R}^{K(n)}, \quad (42)$$

given his demands for other assets $\{q^i_{\ell,n}(\cdot)\}_{\ell\neq k\in K(n)}$ and $\{q^i_{\ell,n'}(\cdot)\}_{\ell\in K(n'),n'\neq n}$, and his residual supply functions for all assets: i.e., the distribution of the trader's residual supply intercepts $F(\mathbf{s}^{-i}|\mathbf{q}^i_0)$ and price impact $\boldsymbol{\Lambda}^i_{K(n)} \equiv \frac{d\mathbf{p}_{K(n)}}{d\mathbf{q}^i_{K(n)}} \in \mathbb{R}^{K(n)\times K(n)} > 0$ for all n.[57] The first-order condition of trader i in each exchange n is:

$$\boldsymbol{\delta}^+_{K(n)} - \alpha^i\boldsymbol{\Sigma}_{K(n)}\mathbf{q}^i_0 - \alpha^i\boldsymbol{\Sigma}^+_{K(n)}E[\mathbf{q}^i|\mathbf{p}_{K(n)},\mathbf{q}^i_0] = \mathbf{p}_{K(n)} + \boldsymbol{\Lambda}^i_{K(n)}\mathbf{q}^i_{K(n)} \quad \forall\mathbf{p}_{K(n)}\in\mathbb{R}^{K(n)}. \quad (43)$$

In the quadratic-Gaussian setting, $\boldsymbol{\Lambda}^i_{K(n)}$ is a constant matrix for all n, given the linearity of residual supply.

Because the first-order condition (43) in exchange n depends on expected trades $E[\mathbf{q}^i|\mathbf{p}_{K(n)},\mathbf{q}^i_0]$ of other assets, to characterize the best-response demands of trader i, $\{\mathbf{q}^i_{K(n)}(\cdot)\}_n$, we transform the fixed point among the trader's best-response demands into a fixed point among the trader's demand coefficients, given the residual supplies, i.e., $\boldsymbol{\Lambda}^i$ and $F(\mathbf{s}^{-i}|\mathbf{q}^i_0)$ (Step 1). We then endogenize the distribution of the residual supply—and thus all demand coefficients, including expected trades $E[\mathbf{q}^i_{K(n')}|\mathbf{p}_{K(n)},\mathbf{q}^i_0]$ for all $n'\neq n$ and n—as a function of price impacts $\{\boldsymbol{\Lambda}^i\}_i$ (Step 2).

Step 1.1 (Parameterization of trader i's demands in exchanges $n'\neq n$) Fix trader i's demands $\{\mathbf{q}^i_{K(n')}(\cdot)\}_{n'\neq n}$ in exchanges $n'\neq n$, and parameterize them as linear functions: for

[54] I.e., trader i's optimization with respect to demand $q^i_{k,n}(\cdot)$ for each asset $k\in K(n)$ in each exchange n, taking as given his demands for other assets $\{q^i_{\ell,n}(\cdot)\}_{\ell\in K(n),\ell\neq k}$ in exchange n and $\{\{q^i_{\ell,n'}(\cdot):\mathbb{R}^{K(n')}\to\mathbb{R}\}_{\ell\in K(n')}\}_{n'\neq n}$ in all other exchanges $n'\neq n$.

[55] A unilateral demand change of trader i is understood as a profile of arbitrary twice continuously differentiable functions $\{\Delta q^i_k(\cdot):\mathbb{R}^{K(n)}\to\mathbb{R}\}_k$ so that $\mathbf{q}^i_{K(n)}(\cdot)+\Delta\mathbf{q}^i_{K(n)}(\cdot)$ are downward-sloping with respect to the contingent variables, i.e., the Jacobian $\frac{\partial(\mathbf{q}^i_{K(n)}(\cdot)+\Delta\mathbf{q}^i_{K(n)}(\cdot))}{\partial\mathbf{p}_{K(n)}}\in\mathbb{R}^{K(n)\times K(n)}$ is negative semi-definite.

[56] I.e., optimization with respect to $\mathbf{q}^i_{K(n)}$ pointwise to each realization of $\mathbf{s}^{-i}_{K(n)}\in\mathbb{R}^{K(n)}$ in each exchange n

[57] Given the downward-sloping demands of traders $j\neq i$ (Step 2.2).

each $n' \neq n$,

$$\mathbf{q}^i_{K(n')}(\mathbf{p}_{K(n')}) = \boldsymbol{a}^i_{K(n')} - \mathbf{B}^i_{K(n')}\mathbf{q}^i_0 - \mathbf{C}^i_{K(n')}\mathbf{p}_{K(n')} \qquad \forall \mathbf{p}_{K(n')} \in \mathbb{R}^{K(n')}, \tag{44}$$

where $\boldsymbol{a}^i_{K(n')} \in \mathbb{R}^{K(n')}$, $\mathbf{B}^i_{K(n')} \in \mathbb{R}^{K(n')\times K}$, and $\mathbf{C}^i_{K(n')} \in \mathbb{R}^{K(n')\times K(n')}$.

Step 1.2 (Expected trades, given $F(\mathbf{s}^{-i}|\mathbf{q}^i_0)$) To endogenize the expected trades in a trader's demands in exchange n, we characterize the distributions of prices $\mathbf{p}_{K(n')}$ and the trader's quantity traded $\mathbf{q}^i_{K(n')}$ in other exchanges $n' \neq n$, using the parameterized demands (44).

The price vector $\mathbf{p}_{K(n')}$ is determined as a function of the residual supply intercept $\mathbf{s}^{-i}_{K(n')}$ by applying market clearing to trader i's demands (44) and residual supply $\mathbf{S}^{-i}_{K(n')}(\cdot) = \mathbf{s}^{-i}_{K(n')} + ((\boldsymbol{\Lambda}^i_{K(n')})^{-1})'\mathbf{p}_{K(n')} : \mathbb{R}^{K(n')} \to \mathbb{R}^{K(n')}$: for each $n' \neq n$,

$$\boldsymbol{a}^i_{K(n')} - \mathbf{B}^i_{K(n')}\mathbf{q}^i_0 - \mathbf{C}^i_{K(n')}\mathbf{p}_{K(n')} = \mathbf{s}^{-i}_{K(n')} + \left((\boldsymbol{\Lambda}^i_{K(n')})^{-1}\right)'\mathbf{p}_{K(n')} \qquad \forall \mathbf{s}^{-i}_{K(n')} \in \mathbb{R}^{K(n')}. \tag{45}$$

Given the downward-sloping demand in each exchange $n' \neq n$ (i.e., we assume $\mathbf{C}^i_{K(n')} > \mathbf{0}$), price vector $\mathbf{p}_{K(n')}$ maps one-to-one to the residual supply intercept vector $\mathbf{s}^{-i}_{K(n')}$, which we can thus treat as the contingent variable in trader i's demands in exchange n' (in place of $\mathbf{p}_{K(n')}$):

$$\mathbf{q}^{i*}_{K(n')}(\mathbf{s}^i_{K(n')}) = (\mathbf{C}^i_{K(n')}(\boldsymbol{\Lambda}^i_{K(n')})' + \mathbf{Id})^{-1}(\boldsymbol{a}^i_{K(n')} - \mathbf{B}^i_{K(n')}\mathbf{q}^i_0) + \mathbf{C}^i_{K(n')}(\boldsymbol{\Lambda}^i_{K(n')})'(\mathbf{C}^i_{K(n')}(\boldsymbol{\Lambda}^i_{K(n')})' + \mathbf{Id})^{-1}\mathbf{s}^{-i}_{K(n')}. \tag{46}$$

Eq. (46) characterizes the distribution of trades $\mathbf{q}^i_{K(n')}$ as a function of $F(\mathbf{s}^{-i}|\mathbf{q}^i_0)$, and $\boldsymbol{a}^i_{K(n')}, \mathbf{B}^i_{K(n')}$, and $\mathbf{C}^i_{K(n')}$. Denote this distribution by $F(\mathbf{q}^i_{K(n')}|\mathbf{q}^i_0)$. Moreover,

$$E[\mathbf{q}^i_{K(n')}(\mathbf{p}_{K(n')})|\mathbf{p}_{K(n)}, \mathbf{q}^i_0] = E[\mathbf{q}^{i*}_{K(n')}(\mathbf{s}^{-i}_{K(n')})|\mathbf{s}^{-i}_{K(n)}, \mathbf{q}^i_0]. \tag{47}$$

By Eq. (46), the expected trades vector $E[\mathbf{q}^{i*}_{K(n')}(\mathbf{s}^{-i}_{K(n')})|\mathbf{s}^{-i}_{K(n)}, \mathbf{q}^i_0]$ is a linear function of expected intercepts $E[\mathbf{s}^{-i}_{K(n')}|\mathbf{s}^{-i}_{K(n)}, \mathbf{q}^i_0]$. Applying the Projection Theorem to the distribution of intercepts $F(\mathbf{s}^{-i}|\mathbf{q}^i_0)$, the vector of expected intercepts is:

$$E[\mathbf{s}^{-i}_{K(n')}|\mathbf{s}^{-i}_{K(n)}, \mathbf{q}^i_0] = \mathbf{x}^{-i}_{n',n} + \mathbf{Y}^{-i}_{n',n}\mathbf{s}^{-i}_{K(n)} + \mathbf{Z}^{-i}_{n',n}\mathbf{q}^i_0, \tag{48}$$

where $\mathbf{x}^{-i}_{n',n} \in \mathbb{R}^{K(n')}$, $\mathbf{Y}^{-i}_{n',n} \in \mathbb{R}^{K(n')\times K(n)}$, and $\mathbf{Z}^{-i}_{n',n} \in \mathbb{R}^{K(n')\times K}$ are coefficients of the expected residual supply intercepts. We will endogenize these coefficients in Eqs. (56)-(58), having endogenized distribution $F(\mathbf{s}^{-i}|\mathbf{q}^i_0)$.

Substituting the expected intercepts (48) into Eq. (47) characterizes the expected trades $E[\mathbf{q}^i|\mathbf{s}^{-i}_{K(n)}, \mathbf{q}^i_0]$ as a function of the demand coefficients in other exchanges $\{\boldsymbol{a}^i_{K(n')}, \mathbf{B}^i_{K(n')}, \mathbf{C}^i_{K(n')}\}_{n'\neq n}$ and the inference coefficients $\{\mathbf{x}^{-i}_{n',n}, \mathbf{Y}^{-i}_{n',n}, \mathbf{Z}^{-i}_{n',n}\}_{n'\neq n}$.

Step 1.3 (Best response in exchange n is linear) Substituting the expected trades (47) into the first-order condition (43) gives the best-response demands $\mathbf{q}^i_{K(n)}(\cdot)$ in exchange n as a *linear* function of $\mathbf{p}_{K(n)}, \mathbf{s}^{-i}_{K(n)}$, and $\mathbf{q}^i_0$:

$$\begin{aligned}
&\left(\alpha^i \mathbf{\Sigma}^+_{K(n),K(n)} + \mathbf{\Lambda}^i_{K(n)}\right)\mathbf{q}^i_{K(n)} \\
&= \boldsymbol{\delta}^+_{K(n)} - \mathbf{p}_{K(n)} - \alpha^i \mathbf{\Sigma}_{K(n)}\mathbf{q}^i_0 - \sum_{n'\neq n} \alpha^i \mathbf{\Sigma}^+_{K(n),K(n')}(\mathbf{C}^i_{K(n')}(\mathbf{\Lambda}^i_{K(n')})' + \mathbf{Id})^{-1}(\boldsymbol{a}^i_{K(n')} - \mathbf{B}^i_{K(n')}\mathbf{q}^i_0) \\
&- \sum_{n'\neq n} \alpha^i \mathbf{\Sigma}^+_{K(n),K(n')}\mathbf{C}^i_{K(n')}(\mathbf{\Lambda}^i_{K(n')})'(\mathbf{C}^i_{K(n')}(\mathbf{\Lambda}^i_{K(n')})' + \mathbf{Id})^{-1}\Big(\mathbf{x}^{-i}_{n',n} + \mathbf{Y}^{-i}_{n',n}\mathbf{s}^{-i}_{K(n)} + \mathbf{Z}^{-i}_{n',n}\mathbf{q}^i_0\Big).
\end{aligned} \tag{49}$$

By the linearity of the downward-sloping best response $\mathbf{q}^i_{K(n)}(\cdot)$, equilibrium price $\mathbf{p}_{K(n)}$ is a strictly monotone linear function of $\mathbf{s}^{-i}_{K(n)}$ (Eq. (45) for exchange n), and hence maps one-to-one to $\mathbf{s}^{-i}_{K(n)}$. Thus, substituting $\mathbf{s}^{-i}_{K(n)}$ for $\mathbf{p}_{K(n)}$ in Eq. (49) gives the best response $\mathbf{q}^i_{K(n)}(\cdot)$ as a function of his private information $\mathbf{q}^i_0$ and contingent variable $\mathbf{p}_{K(n)}$. This allows us to parameterize $\mathbf{q}^i_{K(n)}(\cdot)$ as a linear function of $\mathbf{q}^i_0$ and $\mathbf{p}_{K(n)}$:

$$\mathbf{q}^i_{K(n)}(\mathbf{p}_{K(n)}) = \boldsymbol{a}^i_{K(n)} - \mathbf{B}^i_{K(n)}\mathbf{q}^i_0 - \mathbf{C}^i_{K(n)}\mathbf{p}_{K(n)} \qquad \forall \mathbf{p}_{K(n)} \in \mathbb{R}^{K(n)}. \tag{50}$$

Step 1.4 (Fixed point for each demand coefficient as a single matrix equation) Given the linearity of best response schedules $\{\mathbf{q}^i_{K(n)}(\cdot)\}_n$ in *all* exchanges (Step 1.3), we will write a profile of demand schedules $\mathbf{q}^i(\cdot) = (\mathbf{q}^i_{K(n)}(\cdot) : \mathbb{R}^{K(n)} \to \mathbb{R}^{K(n)})_n$ in matrix form (Eq. (51) below). The matrix form allows us to write the fixed point problem (49) for trader i's best-response demands in all exchanges as a system of matrix equations, given $\mathbf{\Lambda}^i$ and $F(\mathbf{s}^{-i}|\mathbf{q}^i_0)$.

Define demand coefficients for all N exchanges,

$$\boldsymbol{a}^i \equiv (\boldsymbol{a}^i_{K(n)})_n \in \mathbb{R}^{\sum_n K(n)},\ \mathbf{B}^i \equiv (\mathbf{B}^i_{K(n)})_n \in \mathbb{R}^{(\sum_n K(n))\times K},\ \mathbf{C}^i \equiv diag(\mathbf{C}^i_{K(n)})_n \in \mathbb{R}^{(\sum_n K(n))\times(\sum_n K(n))}.$$

With matrix coefficients $\{\boldsymbol{a}^i, \mathbf{B}^i, \mathbf{C}^i\}_i$, a profile of trader i's demand schedules $\mathbf{q}^i(\cdot)$ can be written as a function of the vector of the contingent variables in *all* exchanges $\mathbf{p} \in \mathbb{R}^{\sum_n K(n)}$ rather than the price vector $\mathbf{p}_{K(n)} \in \mathbb{R}^{K(n)}$ for exchange n:

$$\mathbf{q}^i(\mathbf{p}) = \boldsymbol{a}^i - \mathbf{B}^i\mathbf{q}^i_0 - \mathbf{C}^i\mathbf{p} \qquad \forall \mathbf{p} \in \mathbb{R}^{\sum_n K(n)}, \tag{51}$$

using that the matrix slope $\mathbf{C}^i = diag(\mathbf{C}^i_{K(n)})_n$ is a block-diagonal matrix; each block corresponds to an exchange in N. Similarly, we can write the inference coefficients (48) in trader i's expected intercepts $E[\mathbf{s}^{-i}_{K(n')}|\mathbf{s}^{-i}_{K(n)}, \mathbf{q}^i_0]$ in matrix form:

$$(\mathbf{x}^{-i}_{n',n})_{n'} \in \mathbb{R}^{\sum_n K(n)}, \quad \mathbf{Y}^{-i} \equiv (Y^{-i}_{n',n})_{n',n} \in \mathbb{R}^{(\sum_n K(n))\times(\sum_n K(n))}, \quad (\mathbf{Z}^{-i}_{n',n})_{n'} \in \mathbb{R}^{(\sum_n K(n))\times K},$$

where $\mathbf{x}^{-i}_{n,n} \equiv \mathbf{0}, \mathbf{Y}^{-i}_{n,n} \equiv \mathbf{Id}$, and $\mathbf{Z}^{-i}_{n,n} \equiv \mathbf{0}$ for all n and i.

Using the matrix demand coefficients $\{\boldsymbol{a}^i, \mathbf{B}^i, \mathbf{C}^i\}$ and the matrix inference coefficients $\{\{(\mathbf{x}^{-i}_{n',n})_{n'}, (\mathbf{Z}^{-i}_{n',n})_{n'}\}_n, \mathbf{Y}^{-i}\}$, the fixed point (49) for trader i's best-response demands across exchanges simplifies to three matrix equations, one for each demand coefficient:

$$(\alpha^i\mathbf{\Sigma}^+ + \mathbf{\Lambda}^i + (\mathbf{\Lambda}^i)')(\mathbf{C}^i(\mathbf{\Lambda}^i)' + \mathbf{Id})^{-1}\boldsymbol{a}^i = \boldsymbol{\delta}^+ - \left(\alpha^i\mathbf{\Sigma}^+_{K(n)}\mathbf{C}^i(\mathbf{\Lambda}^i)'(\mathbf{C}^i(\mathbf{\Lambda}^i)' + \mathbf{Id})^{-1}(\mathbf{x}^{-i}_{n',n})_{n'}\right)_n, \tag{52}$$

$$(\alpha^i\mathbf{\Sigma}^+ + \mathbf{\Lambda}^i + (\mathbf{\Lambda}^i)')(\mathbf{C}^i(\mathbf{\Lambda}^i)' + \mathbf{Id})^{-1}\mathbf{B}^i = \mathbf{W}\alpha^i\mathbf{\Sigma} + \left(\alpha^i\mathbf{\Sigma}^+_{K(n)}\mathbf{C}^i(\mathbf{\Lambda}^i)'(\mathbf{C}^i(\mathbf{\Lambda}^i)' + \mathbf{Id})^{-1}(\mathbf{Z}^{-i}_{n',n})_{n'}\right)_n, \tag{53}$$

$$\mathbf{\Lambda}^i\mathbf{C}^i + \left[\alpha^i\mathbf{\Sigma}^+\mathbf{C}^i(\mathbf{\Lambda}^i)'(\mathbf{C}^i(\mathbf{\Lambda}^i)' + \mathbf{Id})^{-1}\mathbf{Y}^{-i}\right]_N(\mathbf{\Lambda}^i)'(\mathbf{C}^i(\mathbf{\Lambda}^i)' + \mathbf{Id}) = \mathbf{Id}. \tag{54}$$

Step 2 (Correct residual supply $\{F(\mathbf{s}^{-i}|\mathbf{q}_0^i)\}_i$ and $\{\mathbf{\Lambda}^i\}_i$) Applying market clearing to the best-response demands (50) of traders $j \neq i$ gives the residual supply of trader i in exchange n: for each $\mathbf{p}_{K(n)} \in \mathbb{R}^{K(n)}$,

$$\mathbf{S}_{K(n)}^{-i}(\mathbf{p}_{K(n)}) \equiv -\sum_{j\neq i} \mathbf{q}_{K(n)}^j(\mathbf{p}_{K(n)}) = \underbrace{-\sum_{j\neq i}(\boldsymbol{a}_{K(n)}^j - \mathbf{B}_{K(n)}^j \mathbf{q}_0^j)}_{=\mathbf{s}_{K(n)}^{-i}} + \sum_{j\neq i} \mathbf{C}_{K(n)}^j \mathbf{p}_{K(n)}. \tag{55}$$

Step 2.1 (Correct distribution of residual supply intercepts $\{F(\mathbf{s}^{-i}|\mathbf{q}_0^i)\}_i$) We endogenize the distributions of the vector of the residual supply intercepts $\mathbf{s}^{-i}$ of each trader i as a function of price impacts $\{\mathbf{\Lambda}^j\}_j$. The vector of intercepts $\mathbf{s}_{K(n)}^{-i}$ in Eq. (55) in each exchange n is jointly Normally distributed:

$$F(\mathbf{s}^{-i}|\mathbf{q}_0^i) = \mathcal{N}\Big(-\sum_{j\neq i}(\boldsymbol{a}^j - \mathbf{B}^j E[\mathbf{q}_0^j|\mathbf{q}_0^i]), \sum_{j,h\neq i} \mathbf{B}^j Cov[\mathbf{q}_0^j, \mathbf{q}_0^h|\mathbf{q}_0^i](\mathbf{B}^h)'\Big),$$

given traders' $j \neq i$ demand coefficients $\{\boldsymbol{a}^j, \mathbf{B}^j\}_{j\neq i}$ and the primitive joint distribution of their endowments $F((\mathbf{q}_0^j)_{j\neq i}|\mathbf{q}_0^i)$. Applying the Projection Theorem to the joint distribution $F(\mathbf{s}^{-i}|\mathbf{q}_0^i)$ determines the inference coefficients $(\mathbf{x}_{n',n}^{-i})_{n'}$, $\mathbf{Y}^{-i}$, and $(\mathbf{Z}_{n',n}^{-i})_{n'}$ in expected intercepts $E[\mathbf{s}^{-i}|\mathbf{s}_{K(n)}^{-i}, \mathbf{q}_0^i]$ (Eq. (48)) as functions of demand coefficients $\{\boldsymbol{a}^j, \mathbf{B}^j\}_{j\neq i}$, given $\{\mathbf{\Lambda}^j\}_{j\neq i}$: for each n,

$$(\mathbf{x}_{n',n}^{-i})_{n'} = -\sum_{j\neq i}\Big(\boldsymbol{a}^j - (\mathbf{Y}_{n',n}^{-i})_{n'}\boldsymbol{a}_{K(n)}^j\Big) + \sum_{j\neq i}\Big(\mathbf{B}^j - (\mathbf{Y}_{n',n}^{-i})_{n'}\mathbf{B}_{K(n)}^j\Big)\Big(E[\mathbf{q}_0^j] - \frac{\sigma_{cv}}{\sigma_{cv}+\sigma_{pv}}E[\mathbf{q}_0^i]\Big), \tag{56}$$

$$(\mathbf{Z}_{n',n}^{-i})_{n'} = \frac{\sigma_{cv}}{\sigma_{cv}+\sigma_{pv}}\sum_{j\neq i}(\mathbf{B}^j - (\mathbf{Y}_{n',n}^{-i})_{n'}\mathbf{B}_{K(n)}^j), \tag{57}$$

$$\mathbf{Y}^{-i} = \Big(\sum_{j\neq i}\mathbf{B}^j\mathbf{\Omega}(\mathbf{B}^j + \frac{\sigma_{cv}}{\sigma_{cv}+\sigma_{pv}}\sum_{h\neq i}\mathbf{B}^h)'\Big)\Big[\sum_{j\neq i}\mathbf{B}^j\mathbf{\Omega}^+(\mathbf{B}^j + \frac{\sigma_{cv}}{\sigma_{cv}+\sigma_{pv}}\sum_{h\neq i}\mathbf{B}^h)'\Big]_N^{-1}. \tag{58}$$

Substituting these inference coefficients into Eqs. (52)-(54) gives the system of equations (38)-(40) for demand coefficients $\{\boldsymbol{a}^i, \mathbf{B}^i, \mathbf{C}^i\}_i$ as functions of price impact matrices $\{\mathbf{\Lambda}^i\}_i$.

Step 2.2 (Part (ii): Correct price impact $\{\mathbf{\Lambda}^i\}_i$) The transpose of the Jacobian of the trader's inverse residual supply $(\mathbf{S}^{-i}(\cdot))^{-1}$ characterizes equilibrium price impact $\mathbf{\Lambda}^i \equiv \text{diag}(\mathbf{\Lambda}_{K(n)}^i)_n$ by a single matrix equation for all exchanges: for each i,

$$\mathbf{\Lambda}^i = \Big((\sum_{j\neq i}\mathbf{C}^j)^{-1}\Big)'. \tag{59}$$

The system of equations (39)-(40) and (59) for all traders characterizes the fixed point problem for price impact $\{\mathbf{\Lambda}^i\}_i$. ■

"Exchange Design and Efficiency:"[1] Supplementary Material

B Other Proofs and Additional Results: General Design

Lemma 1 (Woodbury Matrix Identity) Suppose that $\mathbf{S} \in \mathbb{R}^{K\times K}$ and $\mathbf{T} \in \mathbb{R}^{L\times L}$ are square matrices, and $\mathbf{U} \in \mathbb{R}^{K\times L}$ and $\mathbf{V} \in \mathbb{R}^{L\times K}$ are real matrices. When $\mathbf{S}^{-1}$ and $\mathbf{T}^{-1}$ are (psedo)inverses of $\mathbf{S}$ and $\mathbf{T}$, respectively, the following matrix identity holds:

$$(\mathbf{S} + \mathbf{UTV})^{-1} = \mathbf{S}^{-1} - \mathbf{S}^{-1}\mathbf{U}(\mathbf{T}^{-1} + \mathbf{V}\mathbf{S}^{-1}\mathbf{U})^{-1}\mathbf{V}\mathbf{S}^{-1}.$$

We define demand $q^{i*}_{k,n}(\cdot) : \mathbb{R}^{K(n)} \to \mathbb{R}$ as a function of residual supply intercept $\mathbf{s}^{-i}_{K(n)}$ (rather than price $\mathbf{p}_{K(n)}$) for each $k \in K(n)$ and n.

Lemma 2 (Asset by Asset Optimization) Consider a market structure $N = \{K(n)\}_n$. Given the residual supply of trader i, i.e., price impact $\mathbf{\Lambda}^i$ and intercept distribution $F(\mathbf{s}^{-i}|\mathbf{q}^i_0)$, the following optimization problems are equivalent:

(1) a profile of demands $\{\{q^i_{k,n}(\cdot) : \mathbb{R}^{K(n)} \to \mathbb{R}\}_{k\in K(n)}\}_n$ maximizes the expected payoff (2);

(2) a profile of demands $\{\{q^{i*}_{k,n}(\cdot) : \mathbb{R}^{K(n)} \to \mathbb{R}\}_{k\in K(n)}\}_n$ maximizes the expected payoff (2);

(3) for each n and $k \in K(n)$, demand $q^{i*}_{k,n}(\cdot) : \mathbb{R}^{K(n)} \to \mathbb{R}$ maximizes the expected payoff (2), given trader i's demands for other assets $\{q^{i*}_{\ell,n}(\cdot)\}_{\ell\in K(n),\ell\neq k}$ in exchange n and other exchanges $\{\{q^{i*}_{\ell,n'}(\cdot)\}_{\ell\in K(n')}\}_{n'\neq n}$.

Proof of Lemma 2 (Asset by Asset Optimization). Consider a Banach space $\mathcal{X}$ of profiles of twice continuously differentiable downward-sloping demands $q^i_{k,n}(\cdot) : \mathbb{R}^{K(n)} \to \mathbb{R}$ for all $k \in K(n)$ and n. Similarly, we consider a Banach space $\mathcal{X}^*$ of profiles of twice continuously differentiable downward-sloping demands $q^{i*}_{k,n}(\cdot) : \mathbb{R}^{K(n)} \to \mathbb{R}$ for all $k \in K(n)$ and n. Specifically, the Jacobians of demands $\frac{\partial \mathbf{q}^i_{K(n)}(\cdot)}{\partial \mathbf{p}_{K(n)}} = \big(\frac{\partial q^i_{k,n}(\cdot)}{\partial p_{\ell,n}}\big)_{k,\ell} \in \mathbb{R}^{K(n)\times K(n)}$ and $\frac{\partial \mathbf{q}^{i*}_{K(n)}(\cdot)}{\partial \mathbf{s}^{-i}_{K(n)}} = \big(\frac{\partial q^i_{k,n}(\cdot)}{\partial s^{-i}_{\ell,n}}\big)_{k,\ell} \in \mathbb{R}^{K(n)\times K(n)}$ are negative definite for all n; they are negative semi-definite if some assets in exchange n are perfectly correlated.

(Part (1) $\Leftrightarrow$ (2)) We first show that $\mathbf{q}^i(\cdot) \equiv \{\{q^i_{k,n}(\cdot)\}_{k\in K(n)}\}_n \in \mathcal{X}$ maps one-to-one to $\mathbf{q}^{i*}(\cdot) \equiv \{\{q^{i*}_{k,n}(\cdot)\}_{k\in K(n)}\}_n \in \mathcal{X}^*$ that yields the same expected payoff (2), and endow the spaces $\mathcal{X}$ and $\mathcal{X}^*$ with a norm $\|\cdot\|_\infty$ that assign the same norm to $\mathbf{q}^i(\cdot)$ and $\mathbf{q}^{i*}(\cdot)$ when they are mapped. Then, the equivalence between problems (1) and (2) is immediate.

Central to the equivalence between problems (1) and (2)—equivalently, the existence of the one-to-one map between $\mathcal{X}$ and $\mathcal{X}^*$—is that $\mathbf{p}_{K(n)}$ maps one-to-one to $\mathbf{s}^{-i}_{K(n)}$ in each n. A function of $\mathbf{p}_{K(n)}$ ($q^i_{k,n}(\cdot)$) is measurable with respect to $\mathbf{s}^{-i}_{K(n)}$, and a function of $\mathbf{s}^{-i}_{K(n)}$ ($q^{i*}_{k,n}(\cdot)$) is measurable with respect to $\mathbf{p}_{K(n)}$.

[1] Marzena Rostek: University of Wisconsin-Madison, Department of Economics; mrostek@ssc.wisc.edu. Ji Hee Yoon: University College London, Department of Economics and Centre for Finance; jihee.yoon@ucl.ac.uk.

To construct a map between $\mathbf{q}^i(\cdot)$ and $\mathbf{q}^{i*}(\cdot)$, we first characterize the map between $\mathbf{p}_{K(n)}$ and $\mathbf{s}^{-i}_{K(n)}$, given residual supply and market clearing: The price vector $\mathbf{p}_{K(n)} \in \mathbb{R}^{K(n)}$ is determined applying market clearing to the demand of trader i and his residual supply in each exchange n:

$$\mathbf{q}^i_{K(n)}(\mathbf{p}_{K(n)}) = \mathbf{s}^{-i}_{K(n)} + \big((\mathbf{\Lambda}^i_{K(n)})'\big)^{-1}\mathbf{p}_{K(n)} \qquad \forall \mathbf{s}^{-i}_{K(n)} \in \mathbb{R}^{K(n)}. \tag{60}$$

By the continuity of the downward-sloping demand $\mathbf{q}^i_{K(n)}(\cdot)$, Eq. (60) uniquely determines price as continuous functions of intercepts' realizations $\mathbf{s}^{-i}_{K(n)} \in \mathbb{R}^{K(n)}$, which we denote by $\mathbf{p}^*_{K(n)}(\cdot) : \mathbb{R}^{K(n)} \to \mathbb{R}^{K(n)}$. Then, trader i's quantity demanded is uniquely determined by $\mathbf{q}^{i*}_{K(n)}(\cdot) = \mathbf{q}^i_{K(n)} \circ \mathbf{p}^*_{K(n)}(\cdot) : \mathbb{R}^{K(n)} \to \mathbb{R}^{K(n)}$ in each n. Conversely, given $\mathbf{q}^{i*}(\cdot)$, a profile of demands is uniquely determined by $\mathbf{q}^i_{K(n)}(\cdot) = \mathbf{q}^{i*}_{K(n)} \circ \big(\mathbf{p}^*_{K(n)}(\cdot)\big)^{-1}$ in each n when $(\mathbf{p}^*_{K(n)}(\cdot))^{-1}$ is the inverse of price function $\mathbf{p}^*_{K(n)}(\cdot)$. $\mathbf{q}^{i*}(\cdot)$ is downward-sloping if and only if $\mathbf{q}^i(\cdot)$ is downward-sloping, given the downward-sloping demands of traders $j \neq i$ (i.e., $\mathbf{\Lambda}^i_{K(n)}$ is positive semi-definite in Eq. (60)).

Moreover, the system of equations (2) and (60) that characterizes the expected payoff reduces to a single equation for $\mathbf{q}^{i*}(\cdot)$ (Eq. (62)). Market clearing (Eq. (60)) defines price as a function of trader i's quantity demanded $\mathbf{q}^{i*} \in \mathbb{R}^{\sum_n K(n)}$ and intercepts' realizations $\mathbf{s}^{-i} \in \mathbb{R}^{\sum_n K(n)}$:

$$\mathbf{p}^i(\mathbf{q}^{i*}, \mathbf{s}^{-i}) \equiv (\mathbf{\Lambda}^i)'(\mathbf{q}^{i*} - \mathbf{s}^{-i}) \qquad \forall \mathbf{q}^{i*} = (\mathbf{q}^{i*}_{K(n)})_n \in \mathbb{R}^{\sum_n K(n)} \quad \forall \mathbf{s}^{-i} = (\mathbf{s}^{-i}_{K(n)})_n \in \mathbb{R}^{\sum_n K(n)}. \tag{61}$$

Substituting $\mathbf{p}^*(\cdot) = (\mathbf{p}^i \circ \mathbf{q}^{i*})(\cdot)$ into the system of equations (2) and (61) characterizes the expected payoff as a function of $\mathbf{q}^{i*}(\cdot)$:

$$U(\mathbf{q}^{i*}(\cdot)) = E[\boldsymbol{\delta}^+ \cdot (\mathbf{q}^{i*} + \mathbf{q}^i_0) - \frac{\alpha^i}{2}(\mathbf{q}^{i*} + \mathbf{q}^i_0) \cdot \mathbf{\Sigma}^+(\mathbf{q}^{i*} + \mathbf{q}^i_0) - (\mathbf{q}^{i*} - \mathbf{s}^{-i}) \cdot \mathbf{\Lambda}^i \mathbf{q}^{i*} | \mathbf{q}^i_0] \quad \forall \mathbf{q}^{i*}(\cdot) \in \mathcal{X}^*. \tag{62}$$

The expected payoff $U(\mathbf{q}^i(\cdot))$ in the system of equations (2) and (61) satisfies $U(\mathbf{q}^i(\cdot)) = U(\mathbf{q}^{i*}(\cdot))$, given the map from $\mathbf{q}^i(\cdot) \in \mathcal{X}$ to $\mathbf{q}^{i*}(\cdot) \in \mathcal{X}^*$ defined by Eq. (60).

Endow the space $\mathcal{X}^*$ with a norm $\|\cdot\|_\infty$ defined by

$$||\mathbf{q}^{i*}(\cdot)\|_\infty \equiv \max_{k \in K(n), n} \|q^{i*}_{k,n}(\cdot)\| = \max_{k \in K(n), n} (E\big[|q^{i*}_{k,n}(\mathbf{s}^{-i}_{K(n)})|^2 \big| \mathbf{q}^i_0\big])^{1/2}, \tag{63}$$

given $\mathbf{\Lambda}^i$ and $F(\mathbf{s}^{-i}|\mathbf{q}^i_0)$. Because $\mathbf{q}^{i*}(\cdot) \in \mathcal{X}^*$ maps one-to-one to $\mathbf{q}^i(\cdot) \in \mathcal{X}$,[2] given $\mathbf{\Lambda}^i$ and $F(\mathbf{s}^{-i}|\mathbf{q}^i_0)$, the maximization of the expected payoff (2) with respect to a profile $\mathbf{q}^i(\cdot) = \{\{q^i_{k,n}(\cdot)\}_k\}_n \in \mathcal{X}$ subject to market clearing (60) is equivalent to the maximization of the expected payoff (62) with respect to a profile $\mathbf{q}^{i,*}(\cdot) = \{\{q^{i*}_{k,n}(\cdot)\}_k\}_n \in \mathcal{X}^*$.

(Part (2) $\Leftrightarrow$ (3)) We want to show that the maximization of expected payoff (62) with respect to a profile of demands $\{\{q^{i*}_{k,n}(\cdot)\}_k\}_n$ is equivalent to the maximization with respect to the demand $q^{i*}_{k,n}(\cdot)$, given the trader's demands for other assets, for all $k \in K(n)$ and n. By the Second Partial Derivative Test, to show the equivalence between problems (2) and (3) in the lemma, it suffices to show that the map $U(\cdot) : \mathcal{X}^* \to \mathbb{R}$ is twice (Fréchet) differentiable[3] and satisfies the second-order condition.

(Differentiability of expected payoff with respect to demand schedules) First, we will show

[2] We endow the space $\mathcal{X}$ (rather than $\mathcal{X}^*$) with a norm $\|\cdot\|_\infty$ defined by

$$||\mathbf{q}^i(\cdot)\|_\infty \equiv \max_{k \in K(n), n} \|q^i_{k,n}(\cdot)\| = \max_{k \in K(n), n} (E\big[|q^i_{k,n}(\mathbf{p}^*_{K(n)})|^2 \big| \mathbf{q}^i_0\big])^{1/2},$$

given $\mathbf{\Lambda}^i$ and $F(\mathbf{s}^{-i}|\mathbf{q}^i_0)$ and market clearing (60). By the definition of the norm in $\mathcal{X}^*$ in Eq. (63), $||\mathbf{q}^i(\cdot)\|_\infty = ||\mathbf{q}^{i*}(\cdot)\|_\infty$ when $\mathbf{q}^{i*}_{K(n)}(\cdot) = \mathbf{q}^i_{K(n)} \circ \mathbf{p}^*_{K(n)}(\cdot)$ in each n.

[3] Let V and W be normed vector spaces, and $U \subset V$ be an open subset of V. A function $f : U \to W$ is Fréchet differentiable at $x \in U$ if there exists a bounded linear operator $A : V \to W$ such that $\lim_{\|h\| \to 0} \frac{\|f(x+h) - f(x) - Ah\|_W}{\|h\|_V} = 0$. If such an operator A exists, it is unique. $Df(x) = A$ is the Fréchet derivative of f at x.

that $DU(\cdot) : \mathcal{X}^* \to \mathbb{R}^{\sum_n K(n)}$:

$$DU(\mathbf{q}^{i*}(\cdot)) = E\big[\boldsymbol{\delta}^+ - \alpha^i \boldsymbol{\Sigma}^+(\mathbf{q}^{i*} + \mathbf{q}_0^i) - \mathbf{p}^* - \boldsymbol{\Lambda}^i \mathbf{q}^{i*} \big| \mathbf{q}_0^i\big] \qquad \forall \mathbf{q}^{i*}(\cdot) \in \mathcal{X}^* \tag{64}$$

is the Fréchet derivative of $U(\cdot)$ with respect to $\mathbf{q}^{i*}(\cdot)$. Consider a demand change $\Delta\mathbf{q}^{i*}(\cdot) \equiv \{\{\Delta q^{i*}_{k,n}(\cdot)\}_k\}_n$ such that $\widetilde{\mathbf{q}}^{i*}(\cdot) \equiv \mathbf{q}^{i*}(\cdot) + \Delta\mathbf{q}^{i*}(\cdot)$ is in $\mathcal{X}^*$. Because $\widetilde{\mathbf{q}}^{i*}(\cdot)$ is downward-sloping, by the same argument as in (Part (1) $\Leftrightarrow$ (2)), we can define price $\widetilde{\mathbf{p}}^*(\cdot)$, that is a function of $\mathbf{s}^{-i}$, analogously to Eqs. (60)-(61). Substituting $\widetilde{\mathbf{p}}^*(\cdot)$ into Eq. (62) gives the expected payoff change (62) for a demand change $\Delta\mathbf{q}^i(\cdot)$:

$$U(\widetilde{\mathbf{q}}^{i*}(\cdot)) - U(\mathbf{q}^{i*}(\cdot)) = E\big[(\boldsymbol{\delta}^+ - \alpha^i\boldsymbol{\Sigma}^+(\mathbf{q}^{i*} + \mathbf{q}_0^i) - \mathbf{p}^* - \boldsymbol{\Lambda}^i\mathbf{q}^{i*})\cdot(\widetilde{\mathbf{q}}^{i*} - \mathbf{q}^{i*}) - (\widetilde{\mathbf{q}}^{i*} - \mathbf{q}^{i*})\cdot(\frac{\alpha^i}{2}\boldsymbol{\Sigma}^+ + \boldsymbol{\Lambda}^i)(\widetilde{\mathbf{q}}^{i*} - \mathbf{q}^{i*})\big|\mathbf{q}_0^i\big]. \tag{65}$$

By the convexity of the quadratic matrix function $(\widetilde{\mathbf{q}}^{i*} - \mathbf{q}^{i*}) \cdot (\frac{\alpha^i}{2}\boldsymbol{\Sigma}^+ + \boldsymbol{\Lambda}^i)(\widetilde{\mathbf{q}}^{i*} - \mathbf{q}^{i*})$, the Jensen's inequality implies an upper bound on the change in the expected payoff:

$$\begin{aligned} &\big|U(\widetilde{\mathbf{q}}^{i*}(\cdot)) - U(\mathbf{q}^{i*}(\cdot)) - E\big[(\boldsymbol{\delta}^+ - \alpha^i\boldsymbol{\Sigma}^+(\mathbf{q}^{i*} + \mathbf{q}_0^i) - \mathbf{p}^* - \boldsymbol{\Lambda}^i\mathbf{q}^{i*}) \cdot (\widetilde{\mathbf{q}}^{i*} - \mathbf{q}^{i*})\big|\mathbf{q}_0^i\big]\big| \\ &= \big|E\big[(\widetilde{\mathbf{q}}^{i*} - \mathbf{q}^{i*}) \cdot (\frac{\alpha^i}{2}\boldsymbol{\Sigma}^+ + \boldsymbol{\Lambda}^i)(\widetilde{\mathbf{q}}^{i*} - \mathbf{q}^{i*})\big|\mathbf{q}_0^i\big]\big| \leq \ (\mathbf{1} \cdot \big|\frac{\alpha^i}{2}\boldsymbol{\Sigma}^+ + \boldsymbol{\Lambda}^i\big|\mathbf{1})\big(\max_{k\in K(n),n}\big\{E\big[|\widetilde{q}^{i*}_{k,n} - q^{i*}_{k,n}|^2\big|\mathbf{q}_0^i\big]\big\}\big) \\ &\leq (\mathbf{1} \cdot \big|\frac{\alpha^i}{2}\boldsymbol{\Sigma}^+ + \boldsymbol{\Lambda}^i\big|\mathbf{1})\big(\big\|\widetilde{\mathbf{q}}^{i*}(\cdot) - \mathbf{q}^{i*}(\cdot)\big\|_\infty\big)^2. \end{aligned} \tag{66}$$

Finally, taking the limit of the payoff change (66) as $\big\|\widetilde{\mathbf{q}}^{i*}(\cdot) - \mathbf{q}^{i*}(\cdot)\big\|_\infty \to 0$, we have:

$$\begin{aligned} &\lim_{\|\widetilde{\mathbf{q}}^{i*}(\cdot) - \mathbf{q}^{i*}(\cdot)\|_\infty \to 0} \frac{\big|U(\widetilde{\mathbf{q}}^i(\cdot)) - U(\mathbf{q}^i(\cdot)) - E\big[(\boldsymbol{\delta}^+ - \alpha^i\boldsymbol{\Sigma}^+(\mathbf{q}^{i*} + \mathbf{q}_0^i) - \mathbf{p}^* - \boldsymbol{\Lambda}^i\mathbf{q}^{i*}) \cdot (\widetilde{\mathbf{q}}^{i*} - \mathbf{q}^{i*})\big|\mathbf{q}_0^i\big]\big|}{\big\|\widetilde{\mathbf{q}}^{i*}(\cdot) - \mathbf{q}^{i*}(\cdot)\big\|_\infty} \\ &\leq \lim_{\|\widetilde{\mathbf{q}}^{i*}(\cdot) - \mathbf{q}^{i*}(\cdot)\|_\infty \to 0} (\mathbf{1} \cdot \big|\frac{\alpha^i}{2}\boldsymbol{\Sigma}^+ + \boldsymbol{\Lambda}^i\big|\mathbf{1})\big\|\widetilde{\mathbf{q}}^{i*}(\cdot) - \mathbf{q}^{i*}(\cdot)\big\|_\infty = 0. \end{aligned}$$

Given that all elements of $|\frac{\alpha^i}{2}\boldsymbol{\Sigma}^+ + \boldsymbol{\Lambda}^i|$ are bounded, (64) is bounded (i.e., $|DU(\mathbf{q}^{i*}(\cdot))| < \infty$) for any $\mathbf{q}^{i*}(\cdot) \in \mathcal{X}^*$ such that $\|\mathbf{q}^{i*}(\cdot)\|_\infty < \infty$, and (64) is the Fréchet derivative of $U(\cdot)$.

(Second-order condition) We show that the second-order condition of the optimization problem (62) holds. The Hessian of $U(\cdot)$, $D^2U(\cdot) : \mathcal{X}^* \to \mathbb{R}^{(\sum_n K(n))\times(\sum_n K(n))}$, is:

$$D^2U(\mathbf{q}^{i*}(\cdot)) = -\alpha^i\boldsymbol{\Sigma}^+ - \boldsymbol{\Lambda}^i - (\boldsymbol{\Lambda}^i)' \qquad \forall \mathbf{q}^{i*}(\cdot) \in \mathcal{X}^*. \tag{67}$$

This is because, by the definition of the Fréchet derivative of $DU(\cdot)$, we have:

$$\begin{aligned} &\lim_{\|\widetilde{\mathbf{q}}^{i*}(\cdot) - \mathbf{q}^{i*}(\cdot)\|_\infty \to 0} \frac{\big\|DU(\widetilde{\mathbf{q}}^{i*}(\cdot)) - DU(\mathbf{q}^{i*}(\cdot)) - D^2U(\mathbf{q}^{i*}(\cdot))\Delta\mathbf{q}^i(\cdot)\big\|}{\|\widetilde{\mathbf{q}}^{i*}(\cdot) - \mathbf{q}^{i*}(\cdot)\|_\infty} \\ &= \lim_{\|\widetilde{\mathbf{q}}^{i*}(\cdot) - \mathbf{q}^{i*}(\cdot)\|_\infty \to 0} \frac{\big|E[-\alpha^i\boldsymbol{\Sigma}^+(\widetilde{\mathbf{q}}^{i*} - \mathbf{q}^{i*}) - \boldsymbol{\Lambda}^i(\widetilde{\mathbf{q}}^{i*} - \mathbf{q}^{i*}) - (\boldsymbol{\Lambda}^i)'(\widetilde{\mathbf{q}}^{i*} - \mathbf{q}^{i*}) + (\alpha^i\boldsymbol{\Sigma}^+ + \boldsymbol{\Lambda}^i + (\boldsymbol{\Lambda}^i)')(\widetilde{\mathbf{q}}^{i*} - \mathbf{q}^{i*})|\mathbf{q}_0^i]\big|}{\|\widetilde{\mathbf{q}}^{i*}(\cdot) - \mathbf{q}^{i*}(\cdot)\|_\infty} = 0. \end{aligned}$$

$D^2U(\cdot)$ is a constant (matrix) function on $\mathcal{X}^*$. Given the downward-sloping demands of traders $j \neq i$ (i.e., $\boldsymbol{\Lambda}^i$ is positive semi-definite), $D^2U(\cdot)$ is negative semi-definite. Hence, the second-order condition of the maximization problem (2) holds. The Second Partial Derivative Test then implies the equivalence between a trader's optimization with respect to a profile of demands $\{\{q^{i*}_{k,n}(\cdot)\}_{k\in K(n)}\}_n \in \mathcal{X}^*$ and asset by asset optimization with respect to $q^{i*}_{k,n}(\cdot)$, given his demands for assets $\ell \neq k$, for all k and n.

It is immediate that the second-order conditions in problems (1) and (3) hold, given that the second-order condition holds in problem (2). ■

Proof of Proposition 2 (Equilibrium: Uncontingent Trading). Let the market structure be

$N=\{K(n)\}_n$. Consider a trader who optimizes against a residual market $\{\{\mathbf{q}^j_{K(n)}(\cdot)\}_n\}_{j\neq i}$, for which the residual supply is the sufficient statistic. Assuming the linearity of other traders' demands, the trader's residual supply in each exchange n is parameterized as a linear function of the price vector $\mathbf{p}_{K(n)}$:

$$\mathbf{S}^{-i}_{K(n)}(\mathbf{p}_{K(n)})=\mathbf{s}^{-i}_{K(n)}+\big((\mathbf{\Lambda}^i_{K(n)})^{-1}\big)'\mathbf{p}_{K(n)}\qquad\forall\mathbf{p}_{K(n)}\in\mathbb{R}^{K(n)},$$

where $\mathbf{s}^{-i}_{K(n)}\equiv\mathbf{S}^{-i}_{K(n)}(0)\in\mathbb{R}^{K(n)}$ is the intercept of the trader's residual supply and $\mathbf{\Lambda}^i_{K(n)}=\big((\frac{\partial\mathbf{S}^{-i}_{K(n)}(\cdot)}{\partial\mathbf{p}_{K(n)}})^{-1}\big)'\in\mathbb{R}^{K(n)\times K(n)}$ is the transpose of the Jacobian of inverse residual supply.

(Part (i): "Only if") Suppose that a profile of (net) demands of trader i $\{\{q^i_{k,n}(\cdot)\}_{k\in K(n)}\}_n$ satisfies the first-order condition: for each $k\in K(n)$ and n,

$$\delta_k-\alpha^i\mathbf{\Sigma}_k\mathbf{q}^i_0-\alpha^i\mathbf{\Sigma}^+_kE[\mathbf{q}^i|\mathbf{s}^{-i}_{K(n)},\mathbf{q}^i_0]=p_{k,n}+(\mathbf{\Lambda}^i_{K(n)})_k\mathbf{q}^i_{K(n)}\qquad\forall\mathbf{s}^{-i}_{K(n)}\in\mathbb{R}^{K(n)}.\tag{68}$$

When written in matrix form, the first-order condition (68) in each exchange n becomes a single matrix equation:

$$\boldsymbol{\delta}^+_{K(n)}-\alpha^i\mathbf{\Sigma}_{K(n)}\mathbf{q}^i_0-\alpha^i\mathbf{\Sigma}^+_{K(n)}E[\mathbf{q}^i|\mathbf{s}^{-i}_{K(n)},\mathbf{q}^i_0]=\mathbf{p}_{K(n)}+\mathbf{\Lambda}^i_{K(n)}\mathbf{q}^i_{K(n)}\quad\forall\mathbf{s}^{-i}_{K(n)}\in\mathbb{R}^{K(n)},\tag{69}$$

where $\mathbf{\Lambda}^i_{K(n)}\equiv\frac{d\mathbf{p}_{K(n)}}{d\mathbf{q}^i_{K(n)}}\in\mathbb{R}^{K(n)\times K(n)}$ is his price impact in exchange n.

To demonstrate that the first-order conditions (69) computed pointwise with respect to each realization of $\mathbf{s}^{-i}_{K(n)}\in\mathbb{R}^{K(n)}$ are sufficient to the optimization of demand schedules $\mathbf{q}^i_{K(n)}(\cdot):\mathbb{R}^{K(n)}\to\mathbb{R}^{K(n)}$,[4] we show that a demand change $\Delta\mathbf{q}^i_{K(n)}(\cdot):\mathbb{R}^{K(n)}\to\mathbb{R}^{K(n)}$ does not increase the trader's payoff (2).[5] The payoff change following an arbitrary demand change $\Delta\mathbf{q}^i_{K(n)}(\cdot):\mathbb{R}^{K(n)}\to\mathbb{R}^{K(n)}$ that is a twice continuously differentiable function in $\mathbf{s}^{-i}_{K(n)}$ is (as characterized in the proof of Lemma 2, Eq. (65)):

$$E\big[(\boldsymbol{\delta}^+_{K(n)}-\alpha^i\mathbf{\Sigma}_{K(n)}\mathbf{q}^i_0-\alpha^i\mathbf{\Sigma}^+_{K(n)}\mathbf{q}^i-\mathbf{p}_{K(n)}-\mathbf{\Lambda}^i_{K(n)}\mathbf{q}^i_{K(n)})\cdot\Delta\mathbf{q}^i_{K(n)}\big|\mathbf{q}^i_0\big]-o(\|\Delta\mathbf{q}^i_{K(n)}\|^2_\infty).\tag{70}$$

Denoting the intercept distribution by $F(\mathbf{s}^{-i}_{K(n)}|\mathbf{q}^i_0)$, the payoff change (70) can be written as follows:

$$\int E\big[(\boldsymbol{\delta}^+_{K(n)}-\alpha^i\mathbf{\Sigma}_{K(n)}\mathbf{q}^i_0-\alpha^i\mathbf{\Sigma}^+_{K(n)}\mathbf{q}^i-\mathbf{p}_{K(n)}-\mathbf{\Lambda}^i_{K(n)}\mathbf{q}^i_{K(n)})\big|\mathbf{s}^{-i}_{K(n)},\mathbf{q}^i_0\big]\cdot\Delta\mathbf{q}^i_{K(n)}dF(\mathbf{s}^{-i}_{K(n)}|\mathbf{q}^i_0)-o(\|\Delta\mathbf{q}^i_{K(n)}\|^2_\infty).\tag{71}$$

If the integrand is zero for all intercept realizations $\mathbf{s}^{-i}_{K(n)}\in\mathbb{R}^{K(n)}$, i.e., if the pointwise first-order condition (69) holds, then the payoff change (71) is nonpositive for any demand change $\Delta\mathbf{q}^i_{K(n)}(\cdot)$. Given the one-to-one map between $\mathbf{p}_{K(n)}$ and $\mathbf{s}^{-i}_{K(n)}$ (i.e., $\mathbf{\Lambda}^i_{K(n)}>0$ in Eq. (60)), the first-order condition (69) is pointwise with respect to $\mathbf{p}_{K(n)}\in\mathbb{R}^{K(n)}$:

$$\boldsymbol{\delta}^+_{K(n)}-\alpha^i\mathbf{\Sigma}_{K(n)}\mathbf{q}^i_0-\alpha^i\mathbf{\Sigma}^+_{K(n)}E[\mathbf{q}^i|\mathbf{p}_{K(n)},\mathbf{q}^i_0]=\mathbf{p}_{K(n)}+\mathbf{\Lambda}^i_{K(n)}\mathbf{q}^i_{K(n)}\qquad\forall\mathbf{p}_{K(n)}\in\mathbb{R}^{K(n)}.\tag{72}$$

Given that the second-order condition $-\alpha^i\mathbf{\Sigma}^+-\mathbf{\Lambda}^i-(\mathbf{\Lambda}^i)'<\mathbf{0}$ holds (Lemma 2), pointwise optimization (69) is also sufficient for optimization with respect to $\mathbf{q}^i_{K(n)}(\cdot)$.

(Part (i): "If") We prove by contradiction that condition (i) is necessary for each trader's optimality

[4] As seen in the proof of Lemma 2, given downward-sloping and continuous $\mathbf{q}^i_{K(n)}(\cdot)$, Eq. (60) uniquely determines trader i's quantity demanded in each n as continuous functions of a realization of intercepts $\mathbf{s}^{-i}_{K(n)}\in\mathbb{R}^{K(n)}$, which we denote by $\mathbf{q}^{i*}_{K(n)}(\cdot):\mathbb{R}^{K(n)}\to\mathbb{R}^{K(n)}$. For simplicity, we omit the superscript '$*$' from the proof of Proposition 2.

[5] A unilateral demand change of trader i is understood as a profile of arbitrary twice continuously differentiable functions $\{\Delta q^i_k(\cdot):\mathbb{R}^{K(n)}\to\mathbb{R}\}_k$ so that $\mathbf{q}^i_{K(n)}(\cdot)+\Delta\mathbf{q}^i_{K(n)}(\cdot)$ are downward-sloping with respect to the contingent variables, i.e., the Jacobian $\frac{\partial(\mathbf{q}^i_{K(n)}(\cdot)+\Delta\mathbf{q}^i_{K(n)}(\cdot))}{\partial\mathbf{p}_{K(n)}}\in\mathbb{R}^{K(n)\times K(n)}$ is negative semi-definite.

of demand schedules in problem (2). Suppose that for some realization $\overline{\mathbf{s}}^{-i}_{K(n)} \in \mathbb{R}^{K(n)}$,

$$E\big[(\boldsymbol{\delta}^+_{K(n)} - \alpha^i \boldsymbol{\Sigma}_{K(n)} \mathbf{q}^i_0 - \alpha^i \boldsymbol{\Sigma}^+_{K(n)} \mathbf{q}^i - \mathbf{p}_{K(n)} - \boldsymbol{\Lambda}^i_{K(n)} \mathbf{q}^i_{K(n)})\big|\overline{\mathbf{s}}^{-i}_{K(n)}, \mathbf{q}^i_0\big] > 0. \tag{73}$$

The marginal payoff (i.e., the LHS of Eq. (73)) is continuous in $\mathbf{s}^{-i}_{K(n)}$ by the continuity of $\mathbf{q}^i_{K(n)}(\cdot)$ and $\mathbf{p}_{K(n)}(\cdot)$ with respect to $\mathbf{s}^{-i}_{K(n)}$. Hence, the marginal payoff is positive for all prices in a neighborhood of $\overline{\mathbf{s}}^{-i}_{K(n)}$: i.e., there exists $\varepsilon > 0$ such that

$$E\big[(\boldsymbol{\delta}^+_{K(n)} - \alpha^i \boldsymbol{\Sigma}_{K(n)} \mathbf{q}^i_0 - \alpha^i \boldsymbol{\Sigma}^+_{K(n)} \mathbf{q}^i - \mathbf{p}_{K(n)} - \boldsymbol{\Lambda}^i_{K(n)} \mathbf{q}^i_{K(n)})\big|\mathbf{s}^{-i}_{K(n)}, \mathbf{q}^i_0\big] > 0 \quad \forall \mathbf{s}^{-i}_{K(n)} \in R_\varepsilon(\overline{\mathbf{s}}^{-i}_{K(n)}),$$

where $R_\varepsilon(\overline{\mathbf{s}}^{-i}_{K(n)}) \equiv \{\mathbf{s}^{-i}_{K(n)} | \; \|\mathbf{s}^{-i}_{K(n)} - \overline{\mathbf{s}}^{-i}_{K(n)}\|_\infty < \varepsilon\}$ is an open set that contains $\overline{\mathbf{s}}^{-i}_{K(n)}$. Because the measure of $R_\varepsilon(\overline{\mathbf{s}}^{-i}_{K(n)})$ is nonzero, one can construct a demand change $\Delta\mathbf{q}^i_{K(n)}(\cdot)$ with a positive payoff change in Eq. (71): Pick a twice continuously differentiable, positive, bounded, and downward-sloping function $\boldsymbol{\eta}(\cdot) : \mathbb{R}^{K(n)} \to \mathbb{R}^{K(n)}_+$ such that the Jacobian $\frac{\partial \boldsymbol{\eta}(\cdot)}{\partial \mathbf{s}^{-i}_{K(n)}} \in \mathbb{R}^{K(n)\times K(n)}$ is negative semidefinite, $\boldsymbol{\eta}(\mathbf{s}^{-i}_{K(n)}) = 0$ for all $\mathbf{s}^{-i}_{K(n)} \notin R_\varepsilon(\overline{\mathbf{s}}^{-i}_{K(n)})$, and $\overline{\eta} \equiv \|\boldsymbol{\eta}(\cdot)\|_\infty < \infty$. For a demand change $\Delta\mathbf{q}^i_{K(n)}(\cdot) = \nu\boldsymbol{\eta}(\cdot)$ by trader i for some $\nu \in \mathbb{R}_+$, the payoff change (71) is:

$$\nu \int_{R_\varepsilon(\overline{\mathbf{s}}^{-i}_{K(n)})} E\big[(\boldsymbol{\delta}^+_{K(n)} - \alpha^i \boldsymbol{\Sigma}_{K(n)} \mathbf{q}^i_0 - \alpha^i \boldsymbol{\Sigma}^+_{K(n)} \mathbf{q}^i - \mathbf{p}_{K(n)} - \boldsymbol{\Lambda}^i_{K(n)} \mathbf{q}^i_{K(n)})\big|\mathbf{s}^{-i}_{K(n)}, \mathbf{q}^i_0\big] \cdot \boldsymbol{\eta}(\mathbf{s}^{-i}_{K(n)}) dF(\mathbf{s}^{-i}_{K(n)} | \mathbf{q}^i_0) - \nu^2 o(\overline{\eta}). \tag{74}$$

Eq. (74) is quadratic in ν with a negative quadratic coefficient and a *positive* linear coefficient. It follows that, for this $\nu > 0$, the payoff change (74) is strictly positive, and thus the demand increase $\Delta\mathbf{q}^i_{K(n)}(\cdot) = \nu\boldsymbol{\eta}(\cdot)$ is a strictly profitable deviation. This contradicts the optimality of $\mathbf{q}^i_{K(n)}(\cdot)$.

Similarly, we can show that if, for some realization $\overline{\mathbf{s}}^{-i}_{K(n)} \in \mathbb{R}^{K(n)}$,

$$E\big[(\boldsymbol{\delta}^+_{K(n)} - \alpha^i \boldsymbol{\Sigma}_{K(n)} \mathbf{q}^i_0 - \alpha^i \boldsymbol{\Sigma}^+_{K(n)} (\mathbf{q}^i + \mathbf{q}^i_0) - \mathbf{p}_{K(n)} - \boldsymbol{\Lambda}^i_{K(n)} \mathbf{q}^i_{K(n)})\big|\overline{\mathbf{s}}^{-i}_{K(n)}, \mathbf{q}^i_0\big] < 0, \tag{75}$$

then trader i can increase his payoff by reducing his demands around $\overline{\mathbf{s}}^{-i}_{K(n)}$ by $\Delta\mathbf{q}^i_{K(n)}(\cdot) = \nu\boldsymbol{\eta}(\cdot)$ for some $\nu < 0$ and $\boldsymbol{\eta}(\cdot) : \mathbb{R}^{K(n)} \to \mathbb{R}^{K(n)}_+$ with the same properties as above.

(Part (ii): "Only if") We show that condition (ii) is sufficient for equilibrium (Definition 2): given each trader's optimization problem that takes the residual supply as given (condition (i)), the requirement that the residual supply is correct (condition (ii)) is sufficient for each trader's optimization problem that takes other traders' demands as given (Definition 2).

In trader i's optimization problem, other traders' demands $\{\{\mathbf{q}^i_{K(n)}(\cdot)\}_n\}_{j\neq i}$ are payoff-relevant to his expected payoff (2) only via the price distribution $F(\mathbf{p}|\mathbf{q}^i_0)$ and price impact $\boldsymbol{\Lambda}^i \equiv \frac{d\mathbf{p}}{d\mathbf{q}^i}$. Because $F(\mathbf{p}|\mathbf{q}^i_0)$ and $\boldsymbol{\Lambda}^i$ are determined by applying market clearing to demand schedules $\mathbf{q}^i_{K(n)}(\cdot) + \sum_{j\neq i} \mathbf{q}^j_{K(n)}(\cdot) = 0$ in each exchange n, the sum of other traders' demands $\{\sum_{j\neq i} \mathbf{q}^j_{K(n)}(\cdot)\}_n$—equivalently, the residual supply $\{\mathbf{S}^{-i}_{K(n)}(\cdot) \equiv -\sum_{j\neq i} \mathbf{q}^j_{K(n)}(\cdot)\}_n$—is the sufficient statistic for $F(\mathbf{p}|\mathbf{q}^i_0)$ and $\boldsymbol{\Lambda}^i$, and thus the sufficient statistic for the profile of other traders' demands in trader i's optimization problem.

(Part (ii): "If") We show that condition (ii) is necessary for equilibrium: If the residual supply satisfies $\widetilde{\mathbf{S}}^{-i}_{K(n)}(\cdot) \neq -\sum_j \mathbf{q}^j_{K(n)}(\cdot)$ for some i and n for some realization of $\{\mathbf{q}^j_0\}_{j\neq i} \in \mathbb{R}^{(I-1)K}$, then trader i's demand $\widetilde{\mathbf{q}}^i_{K(n)}(\cdot)$ that is optimized when taking $\widetilde{\mathbf{S}}^{-i}_{K(n)}(\cdot)$ as given is not an equilibrium demand. The argument is by contradiction, and mimics the proof of Lemma 1 in Rostek and Weretka (2015).

Suppose that trader i submits demand functions $\{\widetilde{\mathbf{q}}^i_{K(n)}(\cdot) \equiv \mathbf{q}^i_{K(n)}(\cdot; \widetilde{\mathbf{S}}^{-i}(\cdot))\}_n$ for which $\widetilde{\mathbf{S}}^{-i}_{K(n')}(\cdot) \neq -\sum_j \mathbf{q}^j_{K(n')}(\cdot)$ for some n'. Then, either $\widetilde{\boldsymbol{\Lambda}}^i_{K(n')} \neq -\big(\big(\sum_{j\neq i} \frac{\partial \mathbf{q}^j_{K(n')}(\cdot)}{\partial \mathbf{p}_{K(n')}}\big)^{-1}\big)'$ or the residual supply intercept that defines $F(\widetilde{\mathbf{s}}^{-i}_{K(n')}|\mathbf{q}^i_0)$ is such that $\widetilde{\mathbf{s}}^{-i}_{K(n')} \neq -\sum_{j\neq i}\mathbf{q}^j_{K(n')}(0)$ for some realization of $\{\overline{\mathbf{q}}^j_0\}_{j\neq i} \in \mathbb{R}^{(I-1)K}$. Then, the first-order condition (69) of trader i in exchange n' that takes as given other traders' demands $\{\mathbf{q}^j(\cdot)\}_{j\neq i}$—rather than function $\widetilde{\mathbf{S}}^{-i}(\cdot)$—is violated at the realization $\{\overline{\mathbf{q}}^j_0\}_{j\neq i}$:

$$\boldsymbol{\delta}^+_{K(n')} - \alpha^i \boldsymbol{\Sigma}_{K(n')}\mathbf{q}^i_0 - \alpha^i \boldsymbol{\Sigma}^+_{K(n')} E[\widetilde{\mathbf{q}}^i|\mathbf{s}^{-i}_{K(n')},\mathbf{q}^i_0] - \mathbf{p}_{K(n')} - \boldsymbol{\Lambda}^i_{K(n')}\widetilde{\mathbf{q}}^i_{K(n')} \neq \mathbf{0}, \tag{76}$$

where $\boldsymbol{\Lambda}^i_{K(n')} \equiv -\big((\sum_{j\neq i}\frac{\partial \mathbf{q}^j_{K(n')}(\cdot))}{\partial \mathbf{p}_{K(n)}})^{-1}\big)'$ is the inverse of the transpose of the Jacobian of $-(\sum_{j\neq i}\mathbf{q}^j_{K(n')}(\cdot))$, and $\mathbf{s}^{-i}_{K(n')} \equiv -\sum_{j\neq i}\mathbf{q}^j_{K(n')}(0)$ is its intercept. Following the same argument as in the proof of (Part (i): Part If), one can construct a deviation $\Delta\mathbf{q}^i_{K(n')}(\cdot) = \nu\boldsymbol{\eta}(\cdot)$ for which the expected payoff change (Eq. (74)) is positive. It follows that $\{\mathbf{q}^i_{K(n)}(\cdot;\widetilde{\mathbf{S}}^{-i}(\cdot))\}_n$ is not a best response to the profile of other traders' demands $\{\mathbf{q}^j(\cdot)\}_{j\neq i}$, and hence is not an equilibrium. ■

Proof of Corollary 1 (Equilibrium Prices and Allocations). We characterize equilibrium prices and allocations as functions of the equilibrium demand coefficients $\{\mathbf{B}^i,\mathbf{C}^i\}_i$ and price impacts $\{\boldsymbol{\Lambda}^i\}_i$. Applying market clearing to demand schedules (51) yields the price vector:

$$\mathbf{p} = (\sum_i \mathbf{C}^i)^{-1}(\sum_i \boldsymbol{a}^i - \sum_i \mathbf{B}^i\mathbf{q}^i_0). \tag{77}$$

Summing the intercepts $\{\boldsymbol{a}^i\}_i$ in Eq. (38), we have:

$$\sum_i \boldsymbol{a}^i = (\sum_i \mathbf{C}^i)\big(\boldsymbol{\delta}^+ - (\sum_j(\alpha^j\boldsymbol{\Sigma}^+ + \boldsymbol{\Lambda}^j)^{-1})^{-1}\sum_j(\alpha^j\boldsymbol{\Sigma}^+ + \boldsymbol{\Lambda}^j)^{-1}\mathbf{W}\alpha^j\boldsymbol{\Sigma}E[\mathbf{q}^j_0]\big) + \sum_i \mathbf{B}^i E[\mathbf{q}^i_0]. \tag{78}$$

Substituting for $\sum_i \boldsymbol{a}^i$ from Eq. (78), the price equation (77) becomes:

$$\mathbf{p} = \boldsymbol{\delta}^+ - \underbrace{(\sum_j(\alpha^j\boldsymbol{\Sigma}^+ + \boldsymbol{\Lambda}^j)^{-1})^{-1}\sum_j(\alpha^j\boldsymbol{\Sigma}^+ + \boldsymbol{\Lambda}^j)^{-1}\mathbf{W}\alpha^j\boldsymbol{\Sigma}E[\mathbf{q}^j_0]}_{\equiv E[\mathbf{Q}^c]} - \underbrace{(\sum_j \mathbf{C}^j)^{-1}\sum_j \mathbf{B}^j(\mathbf{q}^j_0 - E[\mathbf{q}^j_0])}_{\equiv \mathbf{Q}-E[\mathbf{Q}]}. \tag{79}$$

$\mathbf{Q} \equiv (\sum_j \mathbf{C}^j)^{-1}\sum_j \mathbf{B}^j\mathbf{q}^j_0$ is the aggregate risk in the uncontingent market, while $\mathbf{Q}^c \equiv (\sum_j(\alpha^j\boldsymbol{\Sigma}^+ + \boldsymbol{\Lambda}^j)^{-1})^{-1}\sum_j(\alpha^j\boldsymbol{\Sigma}^+ + \boldsymbol{\Lambda}^j)^{-1}\mathbf{W}\alpha^j\boldsymbol{\Sigma}\mathbf{q}^j_0$ is the aggregate risk in the contingent markets, where we used $\mathbf{C}^{j,c} = (\alpha^j\boldsymbol{\Sigma}^+ + \boldsymbol{\Lambda}^j)^{-1}$ and $\mathbf{B}^{j,c} = (\alpha^j\boldsymbol{\Sigma}^+ + \boldsymbol{\Lambda}^j)^{-1}\mathbf{W}\alpha^j\boldsymbol{\Sigma}$ for all j.

To characterize the equilibrium quantity traded of trader i, substitute equilibrium price $\mathbf{p}$ and demand coefficient $\boldsymbol{a}^i$ into trader i's demand (51): for each i,

$$\mathbf{q}^i = (\alpha^i\boldsymbol{\Sigma}^+ + \boldsymbol{\Lambda}^i)^{-1}(E[\mathbf{Q}^c] - \mathbf{W}\alpha^i\boldsymbol{\Sigma}E[\mathbf{q}^i_0]) + \mathbf{C}^i(\mathbf{Q} - E[\mathbf{Q}]) - \mathbf{B}^i(\mathbf{q}^i_0 - E[\mathbf{q}^i_0]). \tag{80}$$

In the symmetric equilibrium (i.e., assuming $\alpha^i = \alpha$ for all i), equilibrium price (79) and quantity traded (80) become Eqs. (22) and (23), respectively. ■

Lemma 3 (Price Impact in Competitive Markets) Consider a market structure $N = \{K(n)\}_n$, let $\{\alpha^i\}_i$ be the profile of traders' risk aversions, and suppose $\{\boldsymbol{\Lambda}^{i,I}\}_i$ is a profile of the equilibrium price impacts for all $I < \infty$ and in the limit market as $I \to \infty$. The equilibrium price impact becomes zero as $I \to \infty$ if $\alpha^{i,I} = \alpha^i\gamma^I \in \mathbb{R}_+$ increases slower than linearly by a common factor $\gamma^I \sim o(I^{1-\varepsilon})$ for some $\varepsilon \in (0,1)$: for each i, $\boldsymbol{\Lambda}^i = \lim_{I\to\infty}\boldsymbol{\Lambda}^{i,I} = \mathbf{0}$.

Proof of Lemma 3 (Price Impact in Competitive Markets). By Definition 3, the equilibrium price impact in the competitive market is $\boldsymbol{\Lambda}^i = \lim_{I\to\infty}\boldsymbol{\Lambda}^{i,I}$. Theorem 5 characterizes the fixed point equation for price impact matrices $\{\boldsymbol{\Lambda}^{i,I}\}_i$ for $I < \infty$. We show that the price impact $\boldsymbol{\Lambda}^{i,I}$ is proportional to a factor $\gamma^I \in \mathbb{R}_+$ that is common to all traders $\alpha^{i,I} = \alpha^i \gamma^I$: i.e., $\{\mathbf{\Lambda}^{i,I}\}_i$ is a profile of equilibrium price impacts when traders' risk aversions are $\{\alpha^i \gamma^I\}_i$ if and only if $\{\widetilde{\mathbf{\Lambda}}^{i,I} \equiv \frac{1}{\gamma^I}\mathbf{\Lambda}^{i,I}\}_i$ is a profile of equilibrium price impacts when traders' risk aversions are $\{\alpha^i\}_i$ independently of the number of traders I. This can be shown by substituting $\alpha^{i,I} = \alpha^i \gamma^I$ for all i into Eqs. (39)-(41):

$$\mathbf{B}^{i,I} = (\alpha^i \mathbf{\Sigma}^+ + \frac{1}{\gamma^I}\mathbf{\Lambda}^{i,I})^{-1}\mathbf{W}\alpha^i\mathbf{\Sigma} - ((\alpha^i \mathbf{\Sigma}^+ + \frac{1}{\gamma^I}\mathbf{\Lambda}^{i,I})^{-1} - \gamma^I \mathbf{C}^{i,I})(\gamma^I \overline{\mathbf{C}}^I)^{-1}(\frac{\sigma_{pv}}{I(\sigma_{cv}+\sigma_{pv})}\mathbf{B}^{i,I} + \frac{\sigma_{cv}}{\sigma_{cv}+\sigma_{pv}}\overline{\mathbf{B}}^I),$$
$$[(\mathbf{Id} - (\alpha^i \mathbf{\Sigma}^+ + \frac{1}{\gamma^I}\mathbf{\Lambda}^{i,I})\gamma^I \mathbf{C}^{i,I})(\gamma^I \overline{\mathbf{C}}^I)^{-1}(\sum_{j\neq i}\mathbf{B}^{j,I}\mathbf{\Omega}(\mathbf{B}^{j,I} + \frac{\sigma_{cv}}{\sigma_{cv}+\sigma_{pv}}\sum_{h\neq i}\mathbf{B}^{h,I})')]_N = 0,$$
$$\frac{1}{\gamma^I}\mathbf{\Lambda}^{i,I} = (\sum_{j\neq i}\gamma^I \mathbf{C}^j)^{-1}. \tag{81}$$

Hence, with $\{\alpha^i\}_i$, equilibrium demand coefficients and price impacts are $\widetilde{\mathbf{C}}^{i,I} \equiv \gamma^I \mathbf{C}^{i,I}$, $\widetilde{\mathbf{B}}^{i,I} \equiv \mathbf{B}^{i,I}$, and $\widetilde{\mathbf{\Lambda}}^{i,I} \equiv \frac{1}{\gamma^I}\mathbf{\Lambda}^{i,I}$ for all i. The proportionality of the price impact matrix $\mathbf{\Lambda}^{i,I}$ to the common factor γ^I gives lower and upper bounds for the limit of the price impact matrix:

$$\mathbf{0} \leq \lim_{I\to\infty}\mathbf{\Lambda}^{i,I} = \lim_{I\to\infty}\gamma^I \lim_{I\to\infty}\widetilde{\mathbf{\Lambda}}^{i,I} = \lim_{I\to\infty}\gamma^I \lim_{I\to\infty}((\sum_{j\neq i}\widetilde{\mathbf{C}}^{i,I})^{-1})' \leq \lim_{I\to\infty}\frac{\gamma^I}{I-1}\max_i((\widetilde{\mathbf{C}}^{i,I})^{-1})'.$$

Given that equilibrium exists in the limit market ($I \to \infty$), the Jacobian of each trader's demand schedule is bounded: $\lim_{I\to\infty}(\widetilde{\mathbf{C}}^{i,I})^{-1}_{k\ell} < \infty$ for all k,ℓ, and i. We have $\lim_{I\to\infty}\mathbf{\Lambda}^{i,I} = \mathbf{0}$ when $\frac{\gamma^I}{I-1} \sim o(I^{-\varepsilon})$ decreases to zero as $I \to \infty$. We conclude that $\mathbf{\Lambda}^i = \mathbf{0}$ for all i. ■

In what follows, we assume that risk preferences are symmetric across traders, and endowments are independent across assets k unless specified otherwise. Then, $\mathbf{B}^i = \mathbf{B}, \mathbf{C}^i = \mathbf{C}$, and $\mathbf{\Lambda}^i = \mathbf{\Lambda}$ for all i, and $\mathbf{\Omega} = \mathbf{Id}$.

Assumption (Symmetric Risk Preferences) Let $\alpha^i = \alpha$ for all i.

In the symmetric equilibrium, Eqs. (38)-(41) in Theorem 5 simplify as summarized by Corollary 5.

Corollary 5 (Symmetric Equilibrium; General Design) Consider a market structure $N = \{K(n)\}_n$. In a symmetric equilibrium, (net) demand schedules, defined by matrix coefficients $\{\boldsymbol{a}^i\}_i, \mathbf{B}$, and $\mathbf{C}$, and price impact $\mathbf{\Lambda}$ are characterized by the following conditions: for each i,

(i) (*Optimization, given price impact*) Given price impact matrix $\mathbf{\Lambda}$, best-response coefficients $\boldsymbol{a}^i, \mathbf{B}$, and $\mathbf{C}$ are characterized by:

$$\boldsymbol{a}^i = \mathbf{C}(\boldsymbol{\delta}^+ - (\mathbf{W}\alpha\mathbf{\Sigma} - \mathbf{C}^{-1}\mathbf{B})E[\overline{\mathbf{q}}_0]) + ((\alpha\mathbf{\Sigma}^+ + \mathbf{\Lambda})^{-1}\mathbf{W}\alpha\mathbf{\Sigma} - \mathbf{B})(E[\overline{\mathbf{q}}_0] - E[\mathbf{q}_0^i]), \tag{82}$$
$$\mathbf{B} = ((1-\sigma_0)(\alpha\mathbf{\Sigma}^+ + \mathbf{\Lambda}) + \sigma_0\mathbf{C}^{-1})^{-1}\mathbf{W}\alpha\mathbf{\Sigma}, \tag{83}$$
$$\mathbf{C} = [(\alpha\mathbf{\Sigma}^+ + \mathbf{\Lambda})(\mathbf{B}\mathbf{B}')[\mathbf{B}\mathbf{B}']_N^{-1}]_N^{-1}, \tag{84}$$

where $\sigma_0 \equiv \frac{\sigma_{cv} + \frac{1}{I}\sigma_{pv}}{\sigma_{cv}+\sigma_{pv}}$.

(ii) (*Correct price impact*) Price impact $\mathbf{\Lambda}$ equals the transpose of the Jacobian of the trader's inverse residual supply function:

$$\mathbf{\Lambda} = \frac{1}{I-1}(\mathbf{C}^{-1})'. \tag{85}$$

Note that the price slope $\mathbf{C}^{-1} = diag(\mathbf{C}^{-1}_{K(n)})_n$ is a block-diagonal matrix.

Proof of Theorem 2 (Existence of Symmetric Equilibrium). Consider a market structure $N = \{K(n)\}_n$. Let $\mathcal{M}$ be the set of all $(\sum_n K(n))$-dimensional block-diagonal matrices, such that, for any $\mathbf{M} \in \mathcal{M}$, $\mathbf{M}_{K(n),K(n')} = 0$ for distinct exchanges $n \neq n'$. Given that price impact matrices are block-diagonal, we introduce a partial order on $\mathcal{M}$: $\mathbf{M}^1 \leq \mathbf{M}^2$ if $(\mathbf{M}^2 - \mathbf{M}^1)$ is positive semi-definite, or equivalently, if $\mathbf{M}^1_{K(n),K(n)} \leq \mathbf{M}^2_{K(n),K(n)}$ for all n.

(**Existence**) Substituting $\mathbf{B}$ from Eq. (83) into Eq. (84), the fixed point equation (85) for $\mathbf{\Lambda}$ becomes:

$$\underbrace{\big[(\alpha\mathbf{\Sigma}^+ + \mathbf{\Lambda} - (I-1)\mathbf{\Lambda}')\big(\alpha\mathbf{\Sigma}^+ + \mathbf{\Lambda} + \frac{(I-1)\sigma_0}{1-\sigma_0}\mathbf{\Lambda}'\big)^{-1}\mathbf{W}\alpha\mathbf{\Sigma}\alpha\mathbf{\Sigma}\mathbf{W}'\big(\alpha\mathbf{\Sigma}^+ + \mathbf{\Lambda}' + \frac{(I-1)\sigma_0}{1-\sigma_0}\mathbf{\Lambda}\big)^{-1}\big]_N}_{\equiv L(\mathbf{\Lambda})} = \mathbf{0}. \quad (86)$$

Define a map $L(\cdot) : \mathcal{M} \to \mathcal{M}$ using the LHS of Eq. (86). We want to show that there exists $\mathbf{\Lambda}$ such that $L(\mathbf{\Lambda}) = \mathbf{0}$. We first show that there exist two bounds $(\underline{\mathbf{\Lambda}}, \overline{\mathbf{\Lambda}})$, such that $L(\underline{\mathbf{\Lambda}}) \geq \mathbf{0}$ and $L(\overline{\mathbf{\Lambda}}) \leq \mathbf{0}$. Then, by the Brouwer fixed point theorem,[6] since the set of block-diagonal matrices defined by the bounds $(\underline{\mathbf{\Lambda}}, \overline{\mathbf{\Lambda}})$ is compact and the map $L(\mathbf{\Lambda})$ is continuous, there exists a solution $\mathbf{\Lambda}$ to the fixed point problem $L(\mathbf{\Lambda}) = \mathbf{0}$.

Let $\underline{\mathbf{\Lambda}} \equiv \mathbf{0}$ and $\overline{\mathbf{\Lambda}} \equiv \frac{\alpha}{I-2}N[\mathbf{\Sigma}^+]_N$. It is immediate that $\underline{\mathbf{\Lambda}}$ satisfies the desired condition: evaluating $L(\mathbf{\Lambda})$ at $\underline{\mathbf{\Lambda}} = \mathbf{0}$, we have $L(\underline{\mathbf{\Lambda}}) = [\alpha\mathbf{\Sigma}^+]_N \;\geq\; \mathbf{0}$ by the positive semi-definiteness of $\alpha\mathbf{\Sigma}^+$.

Evaluating $L(\mathbf{\Lambda})$ at $\overline{\mathbf{\Lambda}}$, we have:

$$L(\overline{\mathbf{\Lambda}}) = \alpha\big[(\mathbf{Id} + \frac{\kappa N}{I-2}[\mathbf{\Sigma}^+]_N(\mathbf{\Sigma}^+)^{-1})^{-1}(\mathbf{\Sigma}^+ - N[\mathbf{\Sigma}^+]_N)(\mathbf{Id} + \frac{\kappa N}{I-2}[\mathbf{\Sigma}^+]_N(\mathbf{\Sigma}^+)^{-1})^{-1}\big]_N, \quad (87)$$

where $\kappa \equiv \frac{1+(I-2)\sigma_0}{1-\sigma_0} \in \mathbb{R}_+$. Given that $\mathbf{\Sigma}^+$ is positive semi-definite, $(\mathbf{Id} + \frac{\kappa N}{I-2}[\mathbf{\Sigma}^+]_N(\mathbf{\Sigma}^+)^{-1})^{-1}$ is positive definite, where we used that the inverse of a positive definite matrix is positive definite. In turn, matrix $(\mathbf{\Sigma}^+ - N[\mathbf{\Sigma}^+]_N)$ in Eq. (87) is negative definite. It is negative semi-definite if and only if either some assets in an exchange are perfectly correlated or all $\sum_n K(n)$ assets are perfectly correlated. To prove this, observe that for any vector $\mathbf{v} \equiv (\mathbf{v}_{K(n)})_n \in \mathbb{R}^{\sum_n K(n)}$, we have:

$$Cov[\mathbf{v}_{K(n)} \cdot \mathbf{r}_{K(n)}, \mathbf{v}_{K(n')} \cdot \mathbf{r}_{K(n')}] \;\leq\; \frac{1}{2}Var[\mathbf{v}_{K(n)} \cdot \mathbf{r}_{K(n)}] + \frac{1}{2}Var[\mathbf{v}_{K(n')} \cdot \mathbf{r}_{K(n')}] \quad \forall n, n' \in N. \quad (88)$$

Using that $\mathbf{\Sigma}^+$ is the covariance matrix of the distribution of asset returns $\mathbf{r} = (\mathbf{r}_{K(n)})_n$, inequality (88) is equivalent to:

$$\mathbf{v}'_{K(n)}\mathbf{\Sigma}^+_{K(n),K(n')}\mathbf{v}_{K(n')} \;\leq\; \frac{1}{2}\mathbf{v}'_{K(n)}\mathbf{\Sigma}^+_{K(n),K(n)}\mathbf{v}_{K(n)} + \frac{1}{2}\mathbf{v}'_{K(n')}\mathbf{\Sigma}^+_{K(n'),K(n')}\mathbf{v}_{K(n')} \quad \forall n, n' \in N. \quad (89)$$

Summing each side of Eq. (89) over n and n' gives:

$$\mathbf{v}'\mathbf{\Sigma}^+\mathbf{v} = \sum_{n,n'}\mathbf{v}'_{K(n)}\mathbf{\Sigma}^+_{K(n),K(n')}\mathbf{v}_{K(n')} \;\leq\; N\sum_n \mathbf{v}'_{K(n)}\mathbf{\Sigma}^+_{K(n),K(n)}\mathbf{v}_{K(n)} = \mathbf{v}'(N[\mathbf{\Sigma}^+]_N)\mathbf{v},$$

and hence, $\mathbf{v}'(\mathbf{\Sigma}^+ - N[\mathbf{\Sigma}^+]_N)\mathbf{v} \leq 0$ for any vector $\mathbf{v}$.

By the positive semi-definiteness of $(\mathbf{Id} + \frac{\kappa N}{I-2}[\mathbf{\Sigma}^+]_N(\mathbf{\Sigma}^+)^{-1})^{-1}$ and the negative semi-definiteness of $(\mathbf{\Sigma}^+ - N[\mathbf{\Sigma}^+]_N)$, it follows that the RHS of Eq. (87) becomes:

$$(\mathbf{Id} + \frac{\kappa N}{I-2}[\mathbf{\Sigma}^+]_N(\mathbf{\Sigma}^+)^{-1})^{-1}(\mathbf{\Sigma}^+ - N[\mathbf{\Sigma}^+]_N)(\mathbf{Id} + \frac{\kappa N}{I-2}[\mathbf{\Sigma}^+]_N(\mathbf{\Sigma}^+)^{-1})^{-1} \;\leq\; \mathbf{0}. \quad (90)$$

[6]More precisely, the Brouwer fixed point theorem is applied to the equation $\mathbf{\Lambda} = L(\mathbf{\Lambda}) + \mathbf{\Lambda}$ on the partially ordered compact set $\{\mathbf{\Lambda} | \underline{\mathbf{\Lambda}} \leq \mathbf{\Lambda} \leq \overline{\mathbf{\Lambda}}\} \subset \mathbb{R}^{(\sum_n K(n)) \times (\sum_n K(n))}$.

Consequently, $L(\overline{\mathbf{\Lambda}}) \leq \mathbf{0}$; the equality holds if all $\sum_n K(n)$ assets are perfectly correlated.[7] This completes the argument.

(**Uniqueness for** $K=2$) We show that equilibrium in the uncontingent market for $K=2$ is unique. As Appendix C.2 shows, the equilibrium fixed point equation (86) for $\mathbf{\Lambda} = diag(\lambda, \lambda)$ simplifies to:

$$\lambda = \frac{\alpha}{I-2} + \frac{\alpha\rho}{I-2}\frac{2xy}{x^2+y^2}, \tag{91}$$

where $x \equiv (1-\sigma_0)(1-\rho^2)\alpha + (1+(I-2)\sigma_0)\lambda$ and $y \equiv \rho(1+(I-2)\sigma_0)\lambda$ characterize each row of $\mathbf{B}$ in Eq. (83): $\mathbf{b}_1 = (x,y)$ and $\mathbf{b}_2 = (y,x)$. Rearranging Eq. (91) gives a third-order polynomial of λ:

$$\begin{aligned} 0 \quad = \quad & -(I-2)(1+\rho^2)(1+(I-2)\sigma_0)^2\lambda^3 + \big(4-(1-\rho^2)(2I-1)+(I-2)(3+\rho^2)\sigma_0\big)(1+(I-2)\sigma_0)\alpha\lambda^2 \\ & +\big(4-(1-\rho^2)I+(I-2)(3+\rho^2)\sigma_0\big)(1-\sigma_0)(1-\rho^2)\alpha^2\lambda + \alpha^3(1-\sigma_0)^2(1-\rho^2)^2. \end{aligned}$$

By the Descartes' sign rule, there exists a unique positive solution λ. ■

Proof of Proposition 3 (Sufficient Statistic for Equilibrium Payoffs). Let $I < \infty$ and $K > 1$. Consider a market $N = \{K(n)\}_n$, represented by the indicator matrix $\mathbf{W} \in \{0,1\}^{(\sum_n K(n))\times K}$ (Definition 5). Let $\{q^i_{k,n}(\cdot)\}_{i,k,n}$ be a profile of equilibrium demands. We characterize the equilibrium payoff of each trader (Eq. (93)) as a function of per-unit price impact $\widehat{\mathbf{\Lambda}} \in \mathbb{R}^{K\times K}$ and the endowment coefficient of total demand $\widehat{\mathbf{B}} \in \mathbb{R}^{K\times K}$ (Eq. (32)).

(Part (1)) Substituting the equilibrium prices and trades from Eqs. (22) and (23) into $u^i(\cdot) - \mathbf{p}\cdot\mathbf{q}^i$ gives the *ex post* equilibrium payoff of trader i in N:

$$\begin{aligned} & u^i(\mathbf{q}^i) - \mathbf{p}\cdot\mathbf{q}^i = (\boldsymbol{\delta}\cdot\mathbf{q}^i_0 - \frac{\alpha}{2}\mathbf{q}^i_0\cdot\mathbf{\Sigma}\mathbf{q}^i_0) + (\boldsymbol{\delta}^+ - \mathbf{p} - \mathbf{W}\alpha\mathbf{\Sigma}\mathbf{q}^i_0)\cdot\mathbf{q}^i - \frac{\alpha}{2}\mathbf{q}^i\cdot\mathbf{\Sigma}^+\mathbf{q}^i \\ & = (\boldsymbol{\delta}\cdot\mathbf{q}^i_0 - \frac{\alpha}{2}\mathbf{q}^i_0\cdot\mathbf{\Sigma}\mathbf{q}^i_0) + \frac{1}{2}\big(2(\mathbf{W}\alpha\mathbf{\Sigma} - \mathbf{C}^{-1}\mathbf{B})(E[\overline{\mathbf{q}}_0] - \overline{\mathbf{q}}_0) - \big(\alpha\mathbf{\Sigma}^+(\alpha\mathbf{\Sigma}^+ + \mathbf{\Lambda})^{-1}\mathbf{W}\alpha\mathbf{\Sigma} - \alpha\mathbf{\Sigma}^+\mathbf{B}\big)\big(E[\overline{\mathbf{q}}_0] - E[\mathbf{q}^i_0]\big) \\ & \quad +(2\mathbf{W}\alpha\mathbf{\Sigma} - \alpha\mathbf{\Sigma}^+\mathbf{B})(\overline{\mathbf{q}}_0 - \mathbf{q}^i_0)\big)\cdot\big(((\alpha\mathbf{\Sigma}^+ + \mathbf{\Lambda})^{-1}\mathbf{W}\alpha\mathbf{\Sigma} - \mathbf{B})\big(E[\overline{\mathbf{q}}_0] - E[\mathbf{q}^i_0]\big) + \mathbf{B}(\overline{\mathbf{q}}_0 - \mathbf{q}^i_0)\big). \end{aligned} \tag{92}$$

Taking an expectation of the *ex post* payoff (92) with respect to $\{\mathbf{q}^j_0\}_j$, and using that trace satisfies $E[\mathbf{x}'\mathbf{M}\mathbf{x}] = E[tr(\mathbf{x}\mathbf{x}'\mathbf{M})] = tr(E[\mathbf{x}\mathbf{x}']\mathbf{M}) = tr(Var[\mathbf{x}]\mathbf{M}) + E[\mathbf{x}']\mathbf{M}E[\mathbf{x}]$ for any $\mathbf{x} \in \mathbb{R}^K$ and $\mathbf{M} \in \mathbb{R}^{K\times K}$, the *ex ante* equilibrium payoff of trader i is:

$$\begin{aligned} E[u^i - \mathbf{p}\cdot\mathbf{q}^i] \quad = \quad & \underbrace{E[\boldsymbol{\delta}\cdot\mathbf{q}^i_0 - \frac{1}{2}\mathbf{q}^i_0\cdot\alpha\mathbf{\Sigma}\mathbf{q}^i_0]}_{\text{Payoff without trade}} + \underbrace{(E[\overline{\mathbf{q}}_0] - E[\mathbf{q}^i_0])\cdot\mathbf{\Upsilon}^+(\mathbf{\Lambda})(E[\overline{\mathbf{q}}_0] - E[\mathbf{q}^i_0])}_{\text{Equilibrium surplus from trade}} \\ & + \underbrace{\frac{I-1}{I}\sigma_{pv} tr\big(\alpha\mathbf{\Sigma}\mathbf{W}'\mathbf{B} - \frac{1}{2}\mathbf{B}'\alpha\mathbf{\Sigma}^+\mathbf{B}\big)}_{\text{Payoff term due to } Var[\overline{\mathbf{q}}_0|\mathbf{q}^i_0]>0}, \end{aligned} \tag{93}$$

where $\mathbf{\Upsilon}^+(\mathbf{\Lambda}) \in \mathbb{R}^{(\sum_n K(n))\times(\sum_n K(n))}$ is a surplus matrix, which is a function of price impact:

$$\mathbf{\Upsilon}^+(\mathbf{\Lambda}) \equiv \alpha\mathbf{\Sigma}\mathbf{W}'(\alpha\mathbf{\Sigma}^+ + \mathbf{\Lambda})^{-1}\mathbf{W}\alpha\mathbf{\Sigma} - \frac{1}{2}\alpha\mathbf{\Sigma}\mathbf{W}'(\alpha\mathbf{\Sigma}^+ + \mathbf{\Lambda}')^{-1}\alpha\mathbf{\Sigma}^+(\alpha\mathbf{\Sigma}^+ + \mathbf{\Lambda})^{-1}\mathbf{W}\alpha\mathbf{\Sigma}. \tag{94}$$

First, applying the Woodbury Matrix Identity (Lemma 1) to matrix $(\alpha\mathbf{\Sigma}^+ + \mathbf{\Lambda})^{-1}$ in Eq. (94), the

[7]The equality in (90) implies that $\mathbf{\Lambda} = \frac{N}{I-2}[\mathbf{\Sigma}^+]_N$ is the solution to Eq. (86) if and only if all $\sum_n K(n)$ assets are perfectly correlated.

surplus matrix $\mathbf{\Upsilon}^{+}(\mathbf{\Lambda})$ can be written as:

$$\begin{aligned}\mathbf{\Upsilon}^{+}(\mathbf{\Lambda}) &= \alpha\mathbf{\Sigma}\mathbf{W}'\mathbf{\Lambda}^{-1}\mathbf{W}(\widehat{\mathbf{\Lambda}}^{-1}+(\alpha\mathbf{\Sigma})^{-1})^{-1}-\frac{1}{2}((\widehat{\mathbf{\Lambda}}^{-1})'+(\alpha\mathbf{\Sigma})^{-1})^{-1}\mathbf{W}'(\mathbf{\Lambda}^{-1})'\alpha\mathbf{\Sigma}^{+}\mathbf{\Lambda}^{-1}\mathbf{W}(\widehat{\mathbf{\Lambda}}^{-1}+(\alpha\mathbf{\Sigma})^{-1})^{-1}\\ &= \alpha\mathbf{\Sigma}(\alpha\mathbf{\Sigma}+\widehat{\mathbf{\Lambda}}')^{-1}(\frac{1}{2}\alpha\mathbf{\Sigma}+\widehat{\mathbf{\Lambda}}')(\alpha\mathbf{\Sigma}+\widehat{\mathbf{\Lambda}})^{-1}\alpha\mathbf{\Sigma}. \qquad (95)\end{aligned}$$

Therefore, the equilibrium surplus from trade in Eq. (93) is determined as a function of the per-unit price impact $\widehat{\mathbf{\Lambda}}$:

$$(E[\overline{\mathbf{q}}_0]-E[\mathbf{q}_0^i])\cdot\mathbf{\Upsilon}(\widehat{\mathbf{\Lambda}})(E[\overline{\mathbf{q}}_0]-E[\mathbf{q}_0^i]),$$

where

$$\mathbf{\Upsilon}(\widehat{\mathbf{\Lambda}})\equiv\frac{1}{2}\alpha\mathbf{\Sigma}(\alpha\mathbf{\Sigma}+\widehat{\mathbf{\Lambda}}')^{-1}(\alpha\mathbf{\Sigma}+\widehat{\mathbf{\Lambda}}+\widehat{\mathbf{\Lambda}}')(\alpha\mathbf{\Sigma}+\widehat{\mathbf{\Lambda}})^{-1}\alpha\mathbf{\Sigma}. \qquad (96)$$

Eq. (96) replaces the matrix $(\frac{1}{2}\alpha\mathbf{\Sigma}+\widehat{\mathbf{\Lambda}}')$ in Eq. (95) with its symmetric component $\frac{1}{2}(\alpha\mathbf{\Sigma}+\widehat{\mathbf{\Lambda}}+\widehat{\mathbf{\Lambda}}')$. Replacing $\mathbf{\Upsilon}^{+}(\mathbf{\Lambda})$ (Eq. (95)) by $\mathbf{\Upsilon}(\mathbf{\Lambda})$ (Eq. (93)) is innocuous when evaluating equilibrium surplus from trade in Eq. (93), which is a quadratic function of $E[\overline{\mathbf{q}}_0]-E[\mathbf{q}_0^i]$.

Second, we show that matrix $\alpha\mathbf{\Sigma}\mathbf{W}'\mathbf{B}-\frac{1}{2}\mathbf{B}'\alpha\mathbf{\Sigma}^{+}\mathbf{B}$ in Eq. (93) is determined as a function of $\widehat{\mathbf{B}}$. By the characterization of $\mathbf{B}$ in Eq. (83) of Corollary 5 in Appendix A, we have:

$$\alpha\mathbf{\Sigma}\mathbf{W}'\mathbf{B}-\frac{1}{2}\mathbf{B}'\alpha\mathbf{\Sigma}^{+}\mathbf{B}=\alpha\mathbf{\Sigma}-\frac{1}{2}\big(\mathbf{B}'\mathbf{W}-\mathbf{Id}\big)\alpha\mathbf{\Sigma}\big(\mathbf{W}'\mathbf{B}-\mathbf{Id}\big)=\alpha\mathbf{\Sigma}-\frac{1}{2}\big(\widehat{\mathbf{B}}-\mathbf{Id}\big)'\alpha\mathbf{\Sigma}\big(\widehat{\mathbf{B}}-\mathbf{Id}\big), \qquad (97)$$

where the second equality holds by the definition of $\widehat{\mathbf{B}}\equiv\mathbf{W}'\mathbf{B}$ (Eq. (32)). It follows that trader i's *ex ante* equilibrium payoff (93) is determined as a function of $(\widehat{\mathbf{\Lambda}},\widehat{\mathbf{B}})$:

$$\begin{aligned}E[u^i-\mathbf{p}\cdot\mathbf{q}^i] &= E[\boldsymbol{\delta}\cdot\mathbf{q}_0^i-\frac{1}{2}\mathbf{q}_0^i\cdot\alpha\mathbf{\Sigma}\mathbf{q}_0^i]+(E[\overline{\mathbf{q}}_0]-E[\mathbf{q}_0^i])\cdot\mathbf{\Upsilon}(\widehat{\mathbf{\Lambda}})(E[\overline{\mathbf{q}}_0]-E[\mathbf{q}_0^i])\\ &\quad+\frac{I-1}{I}\sigma_{pv}tr\big(\alpha\mathbf{\Sigma}\big)-\frac{I-1}{2I}\sigma_{pv}tr\big((\widehat{\mathbf{B}}-\mathbf{Id})'\alpha\mathbf{\Sigma}(\widehat{\mathbf{B}}-\mathbf{Id})\big). \qquad (98)\end{aligned}$$

We now show that equilibrium payoffs (98) of all traders coincide between market structures N and N' if and only if $\widehat{\mathbf{\Lambda}}$ and $\widehat{\mathbf{B}}$ coincide between N and N'.

(If and only if: $0\leq|\rho_{k\ell}|<1$ for all k and $\ell\neq k$) We first assume that no assets are perfectly correlated, i.e., $\mathbf{\Sigma}$ is invertible. Then, in Eq. (96), the per-unit price impact $\widehat{\mathbf{\Lambda}}$ is one-to-one with $\mathbf{\Upsilon}(\widehat{\mathbf{\Lambda}})$, while in Eq. (97), cross-asset inference $\widehat{\mathbf{B}}$ is one-to-one with $\alpha\mathbf{\Sigma}\mathbf{W}'\mathbf{B}-\frac{1}{2}\mathbf{B}'\alpha\mathbf{\Sigma}^{+}\mathbf{B}$. Consequently, the per-unit price impact $\widehat{\mathbf{\Lambda}}\in\mathbb{R}^{K\times K}$ is the sufficient statistic for the surplus matrix $\mathbf{\Upsilon}(\widehat{\mathbf{\Lambda}})$, while cross-asset inference $\widehat{\mathbf{B}}\in\mathbb{R}^{K\times K}$ is the sufficient statistic for the payoff term due to $Var[\overline{\mathbf{q}}_0|\mathbf{q}_0^i]$ in Eq. (93).

(If and only if: $|\rho_{k\ell}|=1$ for some k and $\ell\neq k$) Suppose that the payoffs of some assets are perfectly correlated, i.e., $\mathbf{\Sigma}$ is singular. When the asset payoffs of assets k and $\ell\neq k$ are perfectly correlated, equilibrium coincides with that in which asset k and ℓ are treated as the same asset, i.e., the asset payoffs are defined by $(r_m)_{m\neq\ell}\in\mathbb{R}^{K-1}$ that is jointly normally distributed according to $\mathcal{N}(\boldsymbol{\delta}^{-},\mathbf{\Sigma}^{-})$, where $\boldsymbol{\delta}^{-}\in\mathbb{R}^{K-1}$ and $\mathbf{\Sigma}^{-}\in\mathbb{R}^{(K-1)\times(K-1)}$. Given trader i's endowment $\mathbf{q}_0^i\in\mathbb{R}^K$, his endowment in $\mathbb{R}^{K-1}$ is $\mathbf{q}_0^{i,-}\equiv(q_{0,m}^{i,-})_m\in\mathbb{R}^{K-1}$ such that $q_{0,k}^{i,-}=q_{0,k}^i+sign(\rho_{k\ell})\frac{\sigma_{kk}}{\sigma_{\ell\ell}}q_{0,\ell}^i$ and $q_{0,m}^{i,-}=q_{0,m}^i$ for all $m\neq\ell,k$.

The same argument as for the case $0\leq|\rho_{k\ell}|<1$, $\ell\neq k$, applies with endowments defined in $\mathbf{R}^{(K-1)\times(K-1)}$ rather than $\mathbf{R}^{K\times K}$: Equilibrium payoff (92) is a function of $\mathbf{\Sigma}E[\mathbf{q}_0^i]=\mathbf{W}^{-}\mathbf{\Sigma}^{-}E[\mathbf{q}_0^{i,-}]$ and $\mathbf{\Sigma}\mathbf{q}_0^i=\mathbf{W}^{-}\mathbf{\Sigma}^{-}\mathbf{q}_0^{i,-}$, where the ℓ^{th} row $W_\ell^{-}=(w_{\ell m}^{-})_{m\neq\ell}$ of $\mathbf{W}^{-}\in\mathbb{R}^{K\times(K-1)}$ is such that $w_{\ell k}^{-}=sign(\rho_{k\ell})\frac{\sigma_{\ell\ell}}{\sigma_{kk}}$ and $w_{\ell m}^{-}=0$ for all $m\neq k$, and the $(K-1)\times(K-1)$ submatrix of $\mathbf{W}^{-}$ excluding the

ℓ^{th} row is the identity matrix. Then, the trade of asset k (and zero trade of asset ℓ) in the market with $K-1$ assets is the same as the total trade for assets k and ℓ, defined by $\widehat{q}_k^i = q_k^i + sign(\rho_{k\ell})\frac{\sigma_{kk}}{\sigma_{\ell\ell}} q_\ell^i$ in the market with K assets.

(Part (2)) We show that $\widehat{\mathbf{\Lambda}}$ maps one-to-one to $\widehat{\mathbf{B}}$ if and only if $\mathbf{\Lambda}$ is symmetric, i.e., $\mathbf{\Lambda} = \mathbf{\Lambda}'$. Then, $\widehat{\mathbf{\Lambda}}$ is a sufficient statistic for equilibrium payoffs (93).

By Eq. (32), the per-unit cross-asset inference $\widehat{\mathbf{B}}$ is characterized as follows:

$$\begin{aligned}\widehat{\mathbf{B}} &= \mathbf{W}'\big((1-\sigma_0)(\alpha\mathbf{\Sigma}^+ + \mathbf{\Lambda}) + \sigma_0(I-1)\mathbf{\Lambda}'\big)^{-1}\mathbf{W}\alpha\mathbf{\Sigma} \\ &= \big((1-\sigma_0)\alpha\mathbf{\Sigma} + \big(\mathbf{W}'((1-\sigma_0)\mathbf{\Lambda} + \sigma_0(I-1)\mathbf{\Lambda}')^{-1}\mathbf{W}\big)^{-1}\big)^{-1}\alpha\mathbf{\Sigma}. \end{aligned} \tag{99}$$

The second equality holds by applying the Woodbury Matrix Identity (Lemma 1) to $\big((1-\sigma_0)\alpha\mathbf{\Sigma}^+ + (1-\sigma_0)\mathbf{\Lambda} + \sigma_0(I-1)\mathbf{\Lambda}'\big)^{-1}$. Given the invertibility of $\mathbf{\Sigma}$, Eq. (99) shows that $\widehat{\mathbf{B}}$ maps one-to-one to $\mathbf{W}'((1-\sigma_0)\mathbf{\Lambda} + \sigma_0(I-1)\mathbf{\Lambda}')^{-1}\mathbf{W}$, which is a function of $\widehat{\mathbf{\Lambda}} = (\mathbf{W}'\mathbf{\Lambda}\mathbf{W})^{-1}$ if and only if $\mathbf{\Lambda} = \mathbf{\Lambda}'$:

$$\mathbf{W}'((1-\sigma_0)\mathbf{\Lambda} + \sigma_0(I-1)\mathbf{\Lambda}')^{-1}\mathbf{W} = \frac{1}{1+(I-2)\sigma_0}\widehat{\mathbf{\Lambda}}^{-1} \qquad \text{if and only if} \qquad \mathbf{\Lambda} = \mathbf{\Lambda}'.$$

Hence, the sufficient statistic $(\widehat{\mathbf{\Lambda}}, \widehat{\mathbf{B}})$ of equilibrium payoffs reduces to a single variable $\widehat{\mathbf{\Lambda}}$ or $\widehat{\mathbf{B}}$ if and only if $\mathbf{\Lambda} = \mathbf{\Lambda}'$. ■

Proof of Theorem 4 (Nonredundancy of Changes in Market Structure: Conditions). Suppose that $K > 1$ and $|\rho_{k\ell}| < 1$ for all k and $\ell \neq k$. By the same argument as in the proof of Proposition 3, it is without loss of generality to treat the perfectly correlated assets as the same asset. Given a market structure $N = \{K(n)\}_n$, let $\mathbf{\Lambda}^N$ be the equilibrium price impact. Suppose that an exchange n' is introduced such that $K(n') \subset K(n)$ for some $n \in N$ and define $N' \equiv N \cup \{n'\}$. Indicator matrices $\mathbf{W}^N$ and $\mathbf{W}^{N'}$ represent market structures N and N' (Definition 5), respectively.

(Part "If") We show that, when one of conditions (i)-(iii) holds, equilibrium payoffs in market N and N' coincide. By Proposition 3, it suffices to show that equilibrium price impact $\mathbf{\Lambda}^{N'}$ in market $N' \equiv N \cup \{n'\}$ satisfies $\widehat{\mathbf{\Lambda}}^{N'} = \widehat{\mathbf{\Lambda}}^N$ and $\widehat{\mathbf{B}}^{N'} = \widehat{\mathbf{B}}^N$. Given equilibrium price impact $\mathbf{\Lambda}^N$ in market N, we will first construct a block-diagonal matrix $\mathbf{\Lambda}^{N'} \in \mathbb{R}^{(\sum_n K(n)+K(n'))\times(\sum_n K(n)+K(n'))}$ that equalizes the per-unit price impact $\widehat{\mathbf{\Lambda}}^{N'} = \widehat{\mathbf{\Lambda}}^N$ and cross-asset inference $\widehat{\mathbf{B}}^{N'} = \widehat{\mathbf{B}}^N$. Then, we will show that such matrix $\mathbf{\Lambda}^{N'}$ is an equilibrium price impact in N'.

(Construction of matrix $\mathbf{\Lambda}^{N'}$ such that $\widehat{\mathbf{\Lambda}}^{N'} = \widehat{\mathbf{\Lambda}}^N$ and $\widehat{\mathbf{B}}^{N'} = \widehat{\mathbf{B}}^N$) Define a block-diagonal matrix $\mathbf{\Lambda}^{N'} = diag(\mathbf{\Lambda}^{N'}_{K(n'')})_{n''\in N'}$ such that

$$\big((\mathbf{\Lambda}^{N'}_{K(n)})^{-1}\big)_{\ell,m} = \big((\mathbf{\Lambda}^{N}_{K(n)})^{-1}\big)_{\ell,m} \qquad \forall \ell, m \in K(n) \text{ and } \{\ell, m\} \not\subset K(n'), \tag{100}$$

$$\big((\mathbf{\Lambda}^{N'}_{K(n')})^{-1}\big)_{\ell,m} = \xi\big((\mathbf{\Lambda}^{N}_{K(n)})^{-1}\big)_{\ell,m}, \ \big((\mathbf{\Lambda}^{N'}_{K(n)})^{-1}\big)_{\ell,m} = (1-\xi)\big((\mathbf{\Lambda}^{N}_{K(n)})^{-1}\big)_{\ell,m} \ \ \forall \ell, m \in K(n'), \tag{101}$$

$$\mathbf{\Lambda}^{N'}_{K(n'')} = \mathbf{\Lambda}^{N}_{K(n'')} \qquad \forall n'' \neq n, n', \tag{102}$$

for some $\xi \in (0,1)$ subject to $\mathbf{\Lambda}^{N'}_{K(n')} > 0$ and $\mathbf{\Lambda}^{N'}_{K(n)} > 0$. This implies that each trader's demand coefficient $\mathbf{C}^N_{K(n)} = \frac{1}{I-1}((\mathbf{\Lambda}^N_{K(n)})^{-1})'$ in exchange n of market N is a linear function of $\mathbf{C}^{N'}_{K(n)}$ and $\mathbf{C}^{N'}_{K(n')}$ in exchanges n and n' of market N'. Moreover, demands in other exchanges $n'' \neq n, n'$ coincide between markets N and N'.

First, $\mathbf{\Lambda}^{N'}$ defined in Eqs. (100)-(102) satisfies $\widehat{\mathbf{\Lambda}}^{N'} = \widehat{\mathbf{\Lambda}}^N$ and $\widehat{\mathbf{B}}^{N'} = \widehat{\mathbf{B}}^N$. By construction, $\widehat{\mathbf{\Lambda}}^{N'} = \big((\mathbf{W}^{N'})'(\mathbf{\Lambda}^{N'})^{-1}\mathbf{W}^{N'}\big)^{-1}$ is the same as the per-unit price impact $\widehat{\mathbf{\Lambda}}^N = \big((\mathbf{W}^N)'(\mathbf{\Lambda}^N)^{-1}\mathbf{W}^N\big)^{-1}$ in N

when the indicator matrix in N' is

$$\mathbf{W}^{N'} = \begin{bmatrix} \mathbf{W}^N \\ \mathbf{W}_{K(n')} \end{bmatrix}.$$

In addition, when one of conditions (i)-(iii) holds, the cross-asset inference coincides, i.e., $\widehat{\mathbf{B}}^{N'} = \widehat{\mathbf{B}}^N$. By Eq. (32), $\widehat{\mathbf{B}}^{N'} = \widehat{\mathbf{B}}^N$ if and only if $\mathbf{\Lambda}^{N'}$ and $\mathbf{\Lambda}^N$ satisfy

$$(\mathbf{W}^{N'})'((1-\sigma_0)\mathbf{\Lambda}^{N'} + \sigma_0(I-1)(\mathbf{\Lambda}^{N'})')^{-1}\mathbf{W}^{N'} = (\mathbf{W}^N)'((1-\sigma_0)\mathbf{\Lambda}^N + \sigma_0(I-1)(\mathbf{\Lambda}^N)')^{-1}\mathbf{W}^N. \quad (103)$$

Because the trader's demands in exchanges $n'' \neq n, n'$ coincide between markets N and N' (Eq. (102)), Eq. (103) simplifies to an equation for price impacts in exchanges n and n' alone:

$$\sum_{n'' \in \{n,n'\}} (\mathbf{W}^{N'}_{K(n'')})'(\mathbf{\Lambda}^{N'}_{K(n'')} + \kappa(\mathbf{\Lambda}^{N'}_{K(n'')})')^{-1}\mathbf{W}^{N'}_{K(n'')} = (\mathbf{W}^N_{K(n)})'(\mathbf{\Lambda}^N_{K(n)} + \kappa(\mathbf{\Lambda}^N_{K(n)})')^{-1}\mathbf{W}^N_{K(n)}, \quad (104)$$

where $\kappa \equiv \frac{\sigma_0(I-1)}{1-\sigma_0} \in \mathbb{R}_+$. When $K(n') = K(n'')$ (condition (i)), Eq. (104) holds because both $(\mathbf{\Lambda}^{N'}_{K(n)} + \kappa(\mathbf{\Lambda}^{N'}_{K(n)})')^{-1}$ and $(\mathbf{\Lambda}^{N'}_{K(n')} + \kappa(\mathbf{\Lambda}^{N'}_{K(n')})')^{-1}$ are proportional to $(\mathbf{\Lambda}^N_{K(n)} + \kappa(\mathbf{\Lambda}^N_{K(n)})')^{-1}$. When the payoff of assets $K(n')$ are independent of other assets in exchange n, $K(n) \setminus K(n')$ (condition (iii)), the demand coefficient $\mathbf{C}^N_{K(n)} = diag(\mathbf{C}^N_{K(n)\setminus K(n')}, \mathbf{C}^N_{K(n')})$ is a block diagonal matrix, and so do $(\mathbf{\Lambda}^{N'}_{K(n)} + \kappa(\mathbf{\Lambda}^{N'}_{K(n)})')^{-1}$ and $(\mathbf{\Lambda}^N_{K(n)} + \kappa(\mathbf{\Lambda}^N_{K(n)})')^{-1}$ in Eq. (104). Applying the same argument as in condition (i) to each block diagonal submatrices that correspond to $K(n')$ and $K(n) \setminus K(n')$ shows that Eq. (104) holds. Lastly, when $\mathbf{\Lambda}^N_{K(n)}$ is symmetric (condition (ii)), both $\mathbf{\Lambda}^{N'}_{K(n)}$ and $\mathbf{\Lambda}^{N'}_{K(n')}$ are symmetric by construction (Eqs. (100)-(101)), and Eq. (104) holds:

$$\sum_{n'' \in \{n,n'\}} \frac{1}{1+\kappa}(\mathbf{W}^{N'}_{K(n'')})'(\mathbf{\Lambda}^{N'}_{K(n'')})^{-1}\mathbf{W}^{N'}_{K(n'')} = \frac{1}{1+\kappa}(\mathbf{W}^N_{K(n)})'(\mathbf{\Lambda}^N_{K(n)})^{-1}\mathbf{W}^N_{K(n)}.$$

(Simplifying the equilibrium fixed point with $\widehat{\mathbf{\Lambda}}$ and $\widehat{\mathbf{B}}$) We now show that $\mathbf{\Lambda}^{N'}$ defined in Eqs. (100)-(102) is equilibrium price impact in $N' = N \cup \{n'\}$. We first simplify equilibrium fixed point (83)-(85) by decomposing the terms that coincide between market N and N' (Eq. (107) below). Applying the Woodbury Matrix Identity to $\mathbf{B}^{N'}$ gives:

$$\mathbf{B}^{N'} = \frac{1}{1-\sigma_0}\big((1-\sigma_0)\mathbf{\Lambda}^{N'} + \sigma_0(I-1)(\mathbf{\Lambda}^{N'})'\big)^{-1}\mathbf{W}^{N'}\big(\widehat{\mathbf{\Phi}} + ((1-\sigma_0)\alpha\mathbf{\Sigma})^{-1}\big)^{-1}, \quad (105)$$

where $\widehat{\mathbf{\Phi}} \equiv (\mathbf{W}^{N'})'((1-\sigma_0)\mathbf{\Lambda}^{N'} + \sigma_0(I-1)(\mathbf{\Lambda}^{N'})')^{-1}\mathbf{W}^{N'} \in \mathbb{R}^{K\times K}$ coincides between N and N' (Eq. (103)). Substituting $\mathbf{B}^{N'}$ into the LHS of equilibrium fixed point equation (Eq. (84)):

$$\big[(\alpha\mathbf{\Sigma}^+ + \mathbf{\Lambda}^{N'} - (I-1)(\mathbf{\Lambda}^{N'})')\big((1-\sigma_0)\mathbf{\Lambda}^{N'} + \sigma_0(I-1)(\mathbf{\Lambda}^{N'})'\big)^{-1}\mathbf{W}^{N'}\widehat{\mathbf{V}}(\mathbf{W}^{N'})'\big]_{N'}, \quad (106)$$

where $\widehat{\mathbf{V}} \equiv \big(\widehat{\mathbf{\Phi}} + ((1-\sigma_0)\alpha\mathbf{\Sigma})^{-1}\big)^{-1}\mathbf{\Omega}\big(\widehat{\mathbf{\Phi}}' + ((1-\sigma_0)\alpha\mathbf{\Sigma})^{-1}\big)^{-1} \in \mathbb{R}^{K\times K}$ represents the covariance of K linearly independent random variables that determines the residual supply intercepts (cf. Eq. (55)). The term $\widehat{\mathbf{V}}$ in Eq. (106) coincides in N and N'. Eq. (106) further simplifies using $\mathbf{\Lambda}^{N'} - (I-1)(\mathbf{\Lambda}^{N'})' = -\frac{1}{\sigma_0}\big((1-\sigma_0)\mathbf{\Lambda}^{N'} + \sigma_0(I-1)(\mathbf{\Lambda}^{N'})'\big) + \frac{1}{\sigma_0}\mathbf{\Lambda}^{N'}$:

$$\big[\mathbf{W}^{N'}(\alpha\mathbf{\Sigma}\widehat{\mathbf{\Phi}}\widehat{\mathbf{V}} - \frac{1}{\sigma_0}\widehat{\mathbf{V}})(\mathbf{W}^{N'})'\big]_{N'} + \frac{1}{\sigma_0}\big((1-\sigma_0)\mathbf{Id} + \sigma_0(I-1)(\mathbf{\Lambda}^{N'})'(\mathbf{\Lambda}^{N'})^{-1}\big)^{-1}\big[\mathbf{W}^{N'}\widehat{\mathbf{V}}(\mathbf{W}^{N'})'\big]_{N'}. \quad (107)$$

Eq. (107) is a function of a block diagonal matrix $\big((1-\sigma_0)\mathbf{Id} + \sigma_0(I-1)(\mathbf{\Lambda}^{N'})'(\mathbf{\Lambda}^{N'})^{-1}\big)^{-1}$ and terms that coincide in markets N and N'. $\mathbf{\Lambda}^{N'}$ is the equilibrium price impact in N' if and only if Eq. (107) equals $\mathbf{0}$.

($\mathbf{\Lambda}^{N'}$ in Eqs. (100)-(102) is equilibrium price impact in N') Each block submatrix $\mathbf{\Lambda}^{N'}_{K(n'')}$ of $\mathbf{\Lambda}^{N'}$ satisfies Eq. (107): For any exchange $n'' \neq n, n'$, $\mathbf{W}^{N'}_{K(n'')} = \mathbf{W}^{N}_{K(n'')}$ and $\mathbf{\Lambda}^{N'}_{K(n'')} = \mathbf{\Lambda}^{N}_{K(n'')}$, and hence, the block submatrix of Eq. (107) that corresponds to exchange n'' is the same as the corresponding submatrix in N. Therefore,

$$\mathbf{W}^{N'}_{K(n'')}\big(\alpha\mathbf{\Sigma}\widehat{\mathbf{\Phi}}\widehat{\mathbf{V}}-\frac{1}{\sigma_0}\widehat{\mathbf{V}}\big)(\mathbf{W}^{N'}_{K(n'')})'+\frac{1}{\sigma_0}\big((1-\sigma_0)\mathbf{Id}+\sigma_0(I-1)(\mathbf{\Lambda}^{N'}_{K(n'')})'(\mathbf{\Lambda}^{N'}_{K(n'')})^{-1}\big)^{-1}\mathbf{W}^{N'}_{K(n'')}\widehat{\mathbf{V}}(\mathbf{W}^{N'}_{K(n'')})' = \mathbf{0}.$$

In exchange n, when the price impact matrix is a symmetric matrix (condition (ii)), we have $\big((1-\sigma_0)\mathbf{Id}+\sigma_0(I-1)(\mathbf{\Lambda}^{N'}_{K(n)})'(\mathbf{\Lambda}^{N'}_{K(n)})^{-1}\big)^{-1} = \big((1-\sigma_0)\mathbf{Id}+\sigma_0(I-1)(\mathbf{\Lambda}^{N}_{K(n)})'(\mathbf{\Lambda}^{N}_{K(n)})^{-1}\big)^{-1} = \frac{1}{1+(I-2)\sigma_0}\mathbf{Id}$. This shows that the block submatrix in Eq. (107) corresponding to $K(n)$ equals to $\mathbf{0}$ in N', given that $\mathbf{\Lambda}^{N}_{K(n)}$ is equilibrium price impact in exchange n in market N. If condition (iii) holds, $\mathbf{\Lambda}^{N'}_{K(n)}$ is a block diagonal matrix, whose each block submatrix corresponds to assets $K(n')$ and $K(n) \setminus K(n')$. By construction, the block submatrix of $\mathbf{\Lambda}^{N'}_{K(n)}$ is proportional to the corresponding submatrix of $\mathbf{\Lambda}^{N}_{K(n)}$, and hence, the block submatrix in Eq. (107) for exchange n equals to $\mathbf{0}$. The same argument applies to condition (i). Finally, in the new exchange n', Eq. (107) is equivalent to the $K(n')\times K(n')$ submatrix of $\big(\alpha\mathbf{\Sigma}\widehat{\mathbf{\Phi}}\widehat{\mathbf{V}}-\frac{1}{\sigma_0}\widehat{\mathbf{V}}\big)_{K(n),K(n)}+\frac{1}{\sigma_0}\frac{1}{1+(I-2)\sigma_0}\widehat{\mathbf{V}}_{K(n),K(n)} = \mathbf{0}$. It follows that $\mathbf{\Lambda}^{N'}$ defined in Eqs. (100)-(102) is equilibrium price impact in N'.

(Part "Only if") We prove the contrapositive: Suppose that $K(n') \neq K(n'')$ for all $n'' \in N$ (condition (i)) and $0 < |\rho_{k\ell}| < 1$ for some assets $k \in K(n')$ and $\ell \in K(n)\setminus K(n')$ (condition (iii)). We show that if a block diagonal matrix $\mathbf{\Lambda}^{N'} \in \mathbb{R}^{(\sum_n K(n)+K(n'))\times(\sum_n K(n)+K(n'))}$ satisfies $\widehat{\mathbf{\Lambda}}^{N'} = \widehat{\mathbf{\Lambda}}^{N}$, then $\widehat{\mathbf{B}}^{N'} = \widehat{\mathbf{B}}^{N}$ generally does not hold unless $\mathbf{\Lambda}^{N}_{K(n)} = (\mathbf{\Lambda}^{N}_{K(n)})'$ for an exchange n such that $K(n') \subset K(n)$ (condition (ii)). Then, by Proposition 3, introducing new exchange n' in market N is nonredundant.

(Construction of $\mathbf{\Lambda}^{N'}$ that equalizes per-unit price impact $\widehat{\mathbf{\Lambda}}^{N'} = \widehat{\mathbf{\Lambda}}^{N}$) We first assume a block-diagonal matrix $\mathbf{\Lambda}^{N'} = diag(\mathbf{\Lambda}^{N'}_{K(n'')})_{n''\in N'}$ such that $\widehat{\mathbf{\Lambda}}^{N'} = \widehat{\mathbf{\Lambda}}^{N}$:

$$(\mathbf{W}^{N'})'(\mathbf{\Lambda}^{N'})^{-1}\mathbf{W}^{N'} = (\mathbf{W}^{N})'(\mathbf{\Lambda}^{N})^{-1}\mathbf{W}^{N}. \tag{108}$$

Given that $K(n') \subset K(n)$ for an existing exchange $n \in N$, the indicator matrix (Definition 5) $\mathbf{W}^{N'}$ in market $N' = N \cup \{n'\}$ can be represented as a function of $\mathbf{W}^{N}$:

$$\mathbf{W}^{N'} = \begin{bmatrix} \mathbf{W}^{N} \\ \mathbf{W}^{N'}_{K(n')} \end{bmatrix} = \begin{bmatrix} \mathbf{W}^{N} \\ \begin{bmatrix} \mathbf{0} & \mathbf{Id} \end{bmatrix}\mathbf{W}^{N} \end{bmatrix}. \tag{109}$$

This is because $\mathbf{W}^{N'}_{K(n')} \in \mathbb{R}^{K(n')\times K}$ for the new exchange n' is a submatrix of the matrix $\mathbf{W}^{N}$. Replacing $\mathbf{W}^{N'}$ by Eq. (109) simplifies Eq. (108) in terms of $\mathbf{W}^{N}$ itself rather than $\mathbf{W}^{N}$ and $\mathbf{W}^{N'}$:

$$(\mathbf{W}^{N})'\big((\mathbf{\Lambda}^{N})^{-1}-(\mathbf{\Lambda}^{N'}_{-n'})^{-1}\big)\mathbf{W}^{N} = (\mathbf{W}^{N})'\begin{bmatrix} \mathbf{0} & \mathbf{Id} \end{bmatrix}'(\mathbf{\Lambda}^{N'}_{K(n')})^{-1}\begin{bmatrix} \mathbf{0} & \mathbf{Id} \end{bmatrix}\mathbf{W}^{N}. \tag{110}$$

The subscript "$-n'$" denotes the existing exchanges $n'' \neq n'$ in market $N' = N \cup \{n'\}$: i.e., $\mathbf{\Lambda}^{N'}_{-n'} = diag(\mathbf{\Lambda}^{N'}_{K(n'')})_{n''\neq n'}$.

($\mathbf{\Lambda}^{N'}$ does not satisfy $\widehat{\mathbf{B}}^{N'} = \widehat{\mathbf{B}}^{N}$) We now show that $\mathbf{\Lambda}^{N'}$ that satisfies Eq. (110) does not satisfies $\widehat{\mathbf{B}}^{N'} = \widehat{\mathbf{B}}^{N}$ unless one of conditions (i)-(iii) holds. The same argument near Eq. (104) shows that $\widehat{\mathbf{B}}^{N'} = \widehat{\mathbf{B}}^{N}$ holds if and only if the following equation holds:

$$(\mathbf{W}^{N})'\Big(\big(\mathbf{\Lambda}^{N}+\kappa(\mathbf{\Lambda}^{N})'\big)^{-1}-\big(\mathbf{\Lambda}^{N'}_{-n'}+\kappa(\mathbf{\Lambda}^{N'}_{-n'})'\big)^{-1}\Big)\mathbf{W}^{N} = (\mathbf{W}^{N})'\begin{bmatrix} \mathbf{0} & \mathbf{Id} \end{bmatrix}'\big(\mathbf{\Lambda}^{N'}_{K(n')}+\kappa(\mathbf{\Lambda}^{N'}_{K(n')})'\big)^{-1}\begin{bmatrix} \mathbf{0} & \mathbf{Id} \end{bmatrix}\mathbf{W}^{N}. \tag{111}$$

Equalization of the per-unit price impact (Eq. (110)) gives a *linear* relation between the demand coefficients $\mathbf{C} = \frac{1}{I-1}(\boldsymbol{\Lambda}^{-1})'$ in N and N': $(\mathbf{W}^N)'(\mathbf{C}^{N'}_{-n'})'\mathbf{W}^N = \frac{1}{I-1}(\mathbf{W}^N)'(\boldsymbol{\Lambda}^{N'}_{-n'})^{-1}\mathbf{W}^N$ is a linear function of $(\mathbf{W}^N)'\mathbf{C}^N\mathbf{W}^N$ and $(\mathbf{W}^N)'\begin{bmatrix} \mathbf{0} & \mathbf{Id} \end{bmatrix}'\mathbf{C}^{N'}_{K(n')}\begin{bmatrix} \mathbf{0} & \mathbf{Id} \end{bmatrix}(\mathbf{W}^N)$ (Eq. (110)). However, given that $\mathbf{C} = \frac{1}{I-1}(\boldsymbol{\Lambda}^{-1})'$, Eq. (111) equalizes the *harmonic* means of $\mathbf{C}^{N'}_{-n'}$ and $(\mathbf{C}^{N'}_{-n'})'$ with the sum of the *harmonic* means of $\{\mathbf{C}^N, (\mathbf{C}^N)'\}$ and $\{\mathbf{C}^{N'}_{K(n')}, (\mathbf{C}^{N'}_{K(n')})'\}$.

Using the different relations—linear and harmonic mean—between the inverses of price impacts $(\boldsymbol{\Lambda}^N)^{-1}$ and $(\boldsymbol{\Lambda}^{N'})^{-1}$ in Eqs. (110)-(111), we show that if Eq. (110) holds then Eq. (111) generally does not hold. The RHS of Eq. (111) has zero elements for all $k, \ell \in K$, unless assets k and ℓ are both traded in the new exchange n', i.e., $\{k,\ell\} \subset K(n')$. If Eq. (111) holds, then the LHS of Eq. (111) must have zero elements for all k and ℓ such that $\{k,\ell\} \not\subset K(n')$. By applying the Woodbury Matrix Identity to $(\boldsymbol{\Lambda}^N + \kappa(\boldsymbol{\Lambda}^N)')^{-1}$ and $(\boldsymbol{\Lambda}^{N'}_{-n'} + \kappa(\boldsymbol{\Lambda}^{N'}_{-n'})')^{-1}$, the LHS of Eq. (111) can be represented as:

$$(\mathbf{W}^N)'(\mathbf{C}^N - \mathbf{C}^{N'}_{-n})\mathbf{W}^N - \kappa(\mathbf{W}^N)'\big(\mathbf{C}^N\big((\mathbf{C}^N)^{-1}(\mathbf{C}^N)' + \kappa\mathbf{Id}\big)^{-1} - \mathbf{C}^{N'}_{-n'}((\mathbf{C}^{N'}_{-n'})^{-1}(\mathbf{C}^{N'}_{-n'})' + \kappa\mathbf{Id})^{-1}\big)\mathbf{W}^N. \tag{112}$$

The first term in Eq. (112) has zero elements for all k, ℓ such that $\{k,\ell\} \not\subset K(n')$, while the second term in Eq. (112) has a zero $(k,\ell)^{\text{th}}$ element if and only if

$$\sum_{\{n'' \in N | \{k,\ell\} \subset K(n'')\}} \big(\mathbf{C}^N_{K(n'')}\big((\mathbf{C}^N_{K(n'')})^{-1}(\mathbf{C}^N_{K(n'')})' + \kappa\mathbf{Id}\big)^{-1} - \mathbf{C}^{N'}_{K(n'')}\big((\mathbf{C}^{N'}_{K(n'')})^{-1}(\mathbf{C}^{N'}_{K(n'')})' + \kappa\mathbf{Id}\big)^{-1}\big)_{k\ell} = 0. \tag{113}$$

By Eq. (110), however, the $\mathbf{C}^{N'}$ that matches the per-unit price impact satisfies

$$\sum_{\{n'' \in N | \{k,\ell\} \subset K(n'')\}} \big(c^N_{k\ell,n''} - c^{N'}_{k\ell,n''}\big) = 0.$$

The demand Jacobian $\mathbf{C}^{N'}_{K(n'')}$ in exchange n'' has a non-zero off-diagonal element $c^{N'}_{k\ell,n''}$ except when $[\mathbf{BB}']_{k,K(n'')}$ is proportional to $[\alpha\boldsymbol{\Sigma}\mathbf{BB}']_{k,K(n'')}$ (see Eq. (84) in Corollary 5), i.e., $c^{N'}_{k\ell,n''} \neq 0$ unless $\sigma_{k\ell} = 0$ (condition (iii)). When condition (iii) does not hold, for Eq. (113) to hold, for each $n'' \in N$, the demand coefficient $\mathbf{C}^N_{K(n'')}$ must either be the same in market structures N and N', i.e., $\mathbf{C}^N_{K(n'')} = \mathbf{C}^{N'}_{K(n'')}$, or must be symmetric, i.e., $\mathbf{C}^N_{K(n'')} = (\mathbf{C}^N_{K(n'')})'$ so that $\big((\mathbf{C}^N_{K(n'')})^{-1}(\mathbf{C}^N_{K(n'')})' + \kappa\mathbf{Id}\big)^{-1} = \frac{1}{1+\kappa}\mathbf{Id}$. The former condition cannot hold for all exchanges n'' such that $K(n') \subsetneq K(n'')$ unless condition (i) holds: When equilibrium exists, the demand coefficient in the new exchange n' is positive semi-definite $\mathbf{C}^{N'}_{K(n')} > \mathbf{0}$ (i.e., demands are downward-sloping); using Eq. (110), for each $\{k,\ell\} \subset K(n')$, there exists an exchange n'' such that $\{k,\ell\} \subset K(n') \cap K(n'')$ and $\mathbf{C}^N_{K(n'')} \neq \mathbf{C}^{N'}_{K(n'')}$. Hence, for such an exchange n'', $\mathbf{C}^N_{K(n'')}$ must be symmetric, i.e., condition (ii) must hold. ■

Lemma 4 (Price Equalization Across Exchanges) Given a market structure $N = \{K(n)\}_n$, the equilibrium prices of asset k are the same in the exchanges where k is traded,

$$p_{k,n} = p_{k,n'} \qquad \forall n,\ \forall n' \neq n \ \ s.t.\ \ k \in K(n) \cap K(n') \quad \forall k \quad \forall(\mathbf{q}^i_0)_i \in \mathbb{R}^{IK}, \tag{114}$$

if and only if price impact $\boldsymbol{\Lambda}$ is a symmetric matrix, i.e., $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}'$.

Proof of Lemma 4 (Price Equalization Across Exchanges). Using the indicator matrix $\mathbf{W}$ (Definition 5), we can write the equilibrium price equation (79) as follows:

$$\mathbf{p} = \boldsymbol{\delta}^+ - \mathbf{W}\alpha\boldsymbol{\Sigma}E[\overline{\mathbf{q}}_0] - \mathbf{C}^{-1}\mathbf{B}(\overline{\mathbf{q}}_0 - E[\overline{\mathbf{q}}_0]) = \mathbf{W}(\boldsymbol{\delta} - \alpha\boldsymbol{\Sigma}E[\overline{\mathbf{q}}_0]) - \mathbf{C}^{-1}\mathbf{B}(\overline{\mathbf{q}}_0 - E[\overline{\mathbf{q}}_0]). \tag{115}$$

Prices for each asset k are the same in all exchanges where k is traded if and only if there exists a price vector $\widehat{\mathbf{p}} \in \mathbb{R}^K$, such that $\mathbf{p} = \mathbf{W}\widehat{\mathbf{p}} = (\mathbf{W}_n\widehat{\mathbf{p}})_n$ for all realizations of endowments $(\mathbf{q}^i_0)_i \in \mathbb{R}^{IK}$. From

Eq. (115), the price equalization holds if and only if $\mathbf{C}^{-1}\mathbf{B} \in \mathbb{R}^{(\sum_n K(n))\times K}$ is characterized as $\mathbf{WM}$ for a matrix $\mathbf{M} \in \mathbb{R}^{K\times K}$.

We now show that $\mathbf{C}^{-1}\mathbf{B} = \mathbf{WM}$ if and only if $\mathbf{C}$ is a symmetric matrix (equivalently, $\mathbf{\Lambda} = \frac{1}{I-1}(\mathbf{C}^{-1})'$ is a symmetric matrix). Using demand coefficients $\mathbf{B}$ and $\mathbf{C}$ in Eqs. (83)-(84), the price weight matrix coefficient $\mathbf{C}^{-1}\mathbf{B} \in \mathbb{R}^{(\sum_n K(n))\times K}$ in Eq. (115) can be characterized as follows:

$$\begin{aligned}\mathbf{C}^{-1}\mathbf{B} &= \mathbf{C}^{-1}\big((1-\sigma_0)\mathbf{W}\alpha\mathbf{\Sigma}\mathbf{W}' + \frac{1-\sigma_0}{I-1}(\mathbf{C}^{-1})' + \sigma_0\mathbf{C}^{-1}\big)^{-1}\mathbf{W}\alpha\mathbf{\Sigma} \\ &= (\frac{1-\sigma_0}{I-1}(\mathbf{C}^{-1})'\mathbf{C} + \sigma_0\mathbf{Id})^{-1}\mathbf{W}\big(\mathbf{W}'(\frac{1}{I-1}(\mathbf{C}^{-1})' + \frac{\sigma_0}{1-\sigma_0}\mathbf{C}^{-1})^{-1}\mathbf{W} + (\alpha\mathbf{\Sigma})^{-1}\big)^{-1}, \qquad (116)\end{aligned}$$

where the second equality applies the Woodbury matrix identity (Lemma 1) to $\big((1-\sigma_0)\mathbf{W}\alpha\mathbf{\Sigma}\mathbf{W}' + \frac{1-\sigma_0}{I-1}(\mathbf{C}^{-1})' + \sigma_0\mathbf{C}^{-1}\big)^{-1}$. Eq. (116) shows that $\mathbf{C}^{-1}\mathbf{B} = \mathbf{WM}$ if and only if $(\frac{1-\sigma_0}{I-1}(\mathbf{C}^{-1})'\mathbf{C} + \sigma_0\mathbf{Id})^{-1}$ is a diagonal matrix whose diagonal elements corresponding to asset k are the same for all exchanges: i.e., $(\mathbf{C}^{-1})'\mathbf{C} = diag(m_{k,n})_{k,n}$, where $m_{k,n} = m_k$ for all n such that $k \in K(n)$. Given $\mathbf{C} > \mathbf{0}$, $m_{k,n} = 1$ for all k and n must hold, so $\mathbf{C} = \mathbf{C}'diag(m_{k,n})_{k,n} = \mathbf{C}'$. We conclude that $\mathbf{C}^{-1}\mathbf{B} = \mathbf{WM}$ if and only if $\mathbf{C}$ is a symmetric matrix, i.e., $\mathbf{C} = \mathbf{C}'$. ■

Proof of Corollary 2 (Redundancy of Changes in Market Structure: A Condition on Exchanges). Suppose that $0 < |\rho_{k\ell}| < 1$ for all k and $\ell \neq k$. This assumption is without loss of generality, as shown in the proof of Proposition 3.

(Part (ii) $\Leftrightarrow \widehat{\mathbf{\Lambda}} = \mathbf{\Lambda}^c$ and $\widehat{\mathbf{B}} = \mathbf{B}^c$) We show that equilibrium in the market with exchanges N is *ex post* if and only if traders' equilibrium payoffs are the same as in the contingent market: Equilibrium price and trades are characterized as a function of price impact $\mathbf{\Lambda}$:

$$\begin{aligned}\mathbf{p} &= \boldsymbol{\delta}^+ - (\mathbf{W}\alpha\mathbf{\Sigma} - \mathbf{C}^{-1}\mathbf{B})E[\overline{\mathbf{q}}_0] - \mathbf{C}^{-1}\mathbf{B}\overline{\mathbf{q}}_0, \qquad (117)\\ \mathbf{q}^i &= ((\alpha\mathbf{\Sigma}^+ + \mathbf{\Lambda})^{-1}\mathbf{W}\alpha\mathbf{\Sigma} - \mathbf{B})(E[\overline{\mathbf{q}}_0] - E[\mathbf{q}_0^i]) + \mathbf{B}\overline{\mathbf{q}}_0 - \mathbf{B}\mathbf{q}_0^i. \qquad (118)\end{aligned}$$

Equilibrium is *ex post* if and only if $\mathbf{B} = (\alpha\mathbf{\Sigma}^+ + \mathbf{\Lambda})^{-1}\mathbf{W}\alpha\mathbf{\Sigma}$ and $\mathbf{C}^{-1}\mathbf{B} = \mathbf{W}\alpha\mathbf{\Sigma}$ so that equilibrium price and total trades are independent of the distribution of endowments. Applying the Woodbury Matrix Identity (Lemma 1) to $(\alpha\mathbf{\Sigma}^+ + \mathbf{\Lambda})^{-1}$ and $\mathbf{B} = ((1-\sigma_0)(\alpha\mathbf{\Sigma}^+ + \mathbf{\Lambda}) + \sigma_0(I-1)\mathbf{\Lambda}')^{-1}\mathbf{W}\alpha\mathbf{\Sigma}$ (Eq. (83)), the matrix condition $\mathbf{B} = (\alpha\mathbf{\Sigma}^+ + \mathbf{\Lambda})^{-1}\mathbf{W}\alpha\mathbf{\Sigma}$ simplifies to:

$$(1-\sigma_0)\alpha\mathbf{\Sigma} + (\mathbf{W}'((1-\sigma_0)\mathbf{\Lambda} + \sigma_0(I-1)\mathbf{\Lambda}')^{-1}\mathbf{W})^{-1} = \alpha\mathbf{\Sigma} + \widehat{\mathbf{\Lambda}}. \qquad (119)$$

Eq. (119) holds if and only if $\widehat{\mathbf{\Lambda}} = \frac{\alpha}{I-2}\alpha\mathbf{\Sigma} = \mathbf{\Lambda}^c$ and $\mathbf{\Lambda} = \mathbf{\Lambda}'$. Then, by Eqs. (83)-(84), $\widehat{\mathbf{B}} = \frac{I-2}{I-1}\mathbf{Id} = \mathbf{B}^c$ and $\mathbf{C}^{-1}\mathbf{B} = \mathbf{W}\alpha\mathbf{\Sigma}$ also holds.

(Part (ii) $\Leftarrow$ (iii)) From (Part (ii) $\Leftrightarrow \widehat{\mathbf{\Lambda}} = \mathbf{\Lambda}^c$ and $\widehat{\mathbf{B}} = \mathbf{B}^c$), equilibrium in a market with exchanges N is *ex post* if and only if it is payoff equivalent to equilibrium in $N' = \{K\}$. Suppose that for every pair of assets k and $\ell \neq k$ such that $0 < |\rho_{k\ell}| < 1$, these assets are traded in a same exchange in N: i.e., $k, \ell \in K(n)$ for some $n \in N$.

We first show that, given market structure N, there exists a symmetric block-diagonal matrix $\mathbf{\Lambda} = diag(\mathbf{\Lambda}_{K(n)})_n \in \mathbb{R}^{(\sum_n K(n))\times(\sum_n K(n))}$ such that $\big(\mathbf{W}'\mathbf{\Lambda}^{-1}\mathbf{W}\big)^{-1} = \widehat{\mathbf{\Lambda}}^c = \frac{\alpha}{I-2}\mathbf{\Sigma}$, which we then show is the equilibrium price impact in market N. Given that $\mathbf{W}$ is the indicator matrix of market N, the $(k,\ell)^{\text{th}}$ element of $\mathbf{W}'\mathbf{\Lambda}^{-1}\mathbf{W} = (I-1)\mathbf{W}'\mathbf{C}'\mathbf{W}$ is the sum of demand coefficients $\sum_{\{n|k,\ell\in K(n)\}} c_{\ell k,n}$. Because condition (iii) implies that $\{n|k,\ell \in K(n)\} \neq \emptyset$, $\sum_{\{n|k,\ell\in K(n)\}} c_{\ell k,n} \neq 0$ for any k and $\ell \neq k$ except when $\rho_{k\ell} = 0$ (Proposition 5). Matching each element of $\mathbf{W}'\mathbf{C}'\mathbf{W}$ and $(\mathbf{C}^c)'$ gives the system of K^2 equations for $\sum_n (K(n))^2$ variables $\{\{c_{\ell k,n}\}_{k,\ell}\}_n$. Given $\sum_n (K(n))^2 \geq K^2$, there exist $\{c_{\ell k,n}\}_{\{n|k,\ell\in K(n)\}}$ such that $\sum_{\{n|k,\ell\in K(n)\}} c_{\ell k,n} = c^c_{\ell k}$ for all k and $\ell \neq k$. Moreover, when $\mathbf{\Lambda}$ satisfies $\left(\mathbf{W}'\mathbf{\Lambda}^{-1}\mathbf{W}\right)^{-1} = \widehat{\mathbf{\Lambda}}^c$, so does its symmetric counterpart $\frac{1}{2}(\mathbf{\Lambda}+\mathbf{\Lambda}')$, because $\widehat{\mathbf{\Lambda}}^c$ is symmetric. It follows that there exists a symmetric matrix $\mathbf{\Lambda} = \mathbf{\Lambda}'$ that satisfies $\left(\mathbf{W}'\mathbf{\Lambda}^{-1}\mathbf{W}\right)^{-1} = \widehat{\mathbf{\Lambda}}^c$.

We now show that a symmetric matrix $\mathbf{\Lambda}$ such that $\left(\mathbf{W}'\mathbf{\Lambda}^{-1}\mathbf{W}\right)^{-1} = \frac{\alpha}{I-2}\mathbf{\Sigma}$ is equilibrium price impact, by showing that it satisfies equilibrium fixed point Eqs. (84)-(85), i.e.,

$$\left[(\alpha\mathbf{\Sigma}^{+} + \mathbf{\Lambda} - (I-1)\mathbf{\Lambda}')\mathbf{B}\mathbf{\Omega}\mathbf{B}'\right]_N = \mathbf{0}. \tag{120}$$

Using $\mathbf{\Lambda} = \mathbf{\Lambda}'$, Eq. (105) for $\mathbf{B}$ simplifies to:

$$\mathbf{B} \;\; = \;\; \mathbf{\Lambda}^{-1}\mathbf{W}((1+(I-2)\sigma_0)(\alpha\mathbf{\Sigma})^{-1} + (1-\sigma_0)\widehat{\mathbf{\Lambda}}^{-1})^{-1}. \tag{121}$$

Substituting for $\mathbf{B}$ from Eq. (121) to Eq. (120), we have:

$$\left[\mathbf{W}(\alpha\mathbf{\Sigma}-(I-2)\widehat{\mathbf{\Lambda}})((1-\sigma_0)(\alpha\mathbf{\Sigma}+\widehat{\mathbf{\Lambda}})+\sigma_0(I-1)\widehat{\mathbf{\Lambda}})^{-1}\alpha\mathbf{\Sigma}\mathbf{\Omega}\alpha\mathbf{\Sigma}((1-\sigma_0)(\alpha\mathbf{\Sigma}+\widehat{\mathbf{\Lambda}})+\sigma_0(I-1)\widehat{\mathbf{\Lambda}})^{-1}\mathbf{W}'\right]_N = \mathbf{0}. \tag{122}$$

When condition (iii) holds, for any matrix $\mathbf{M} \in \mathbb{R}^{K\times K}$, $[\mathbf{W}\mathbf{M}\mathbf{W}']_N = \mathbf{0}$ if and only if $\mathbf{M} = \mathbf{0}$. This is because, $m_{k\ell} = 0$ for all $\ell \neq k$ and k if and only if $(\mathbf{W}\mathbf{M}\mathbf{W}')_{K(n)} = (m_{k\ell})_{k,\ell\in K(n)} = 0$ for all n. This establishes that matrix $\widehat{\mathbf{\Lambda}}$ satisfies Eq. (122) given market N if and only if $\widehat{\mathbf{\Lambda}}$ satisfies

$$(\alpha\mathbf{\Sigma}-(I-2)\widehat{\mathbf{\Lambda}})((1-\sigma_0)(\alpha\mathbf{\Sigma}+\widehat{\mathbf{\Lambda}})+\sigma_0(I-1)\widehat{\mathbf{\Lambda}})^{-1}\alpha\mathbf{\Sigma}\mathbf{\Omega}\alpha\mathbf{\Sigma}((1-\sigma_0)(\alpha\mathbf{\Sigma}+\widehat{\mathbf{\Lambda}})+\sigma_0(I-1)\widehat{\mathbf{\Lambda}})^{-1} = \mathbf{0}. \tag{123}$$

Given the positive definiteness of $\mathbf{\Sigma}$ and $\mathbf{\Omega}$, $\widehat{\mathbf{\Lambda}} = \frac{\alpha}{I-2}\mathbf{\Sigma} = \mathbf{\Lambda}^c$ is the unique matrix that satisfies Eq. (123). It follows that $\mathbf{\Lambda}$ such that $\widehat{\mathbf{\Lambda}} = \mathbf{\Lambda}^c$, which hence satisfies Eq. (123), is equilibrium price impact in N.

(Part (ii) $\Rightarrow$ (iii)) We prove by contradiction: Suppose that a pair of assets k and $\ell \neq k$ such that $0 < |\rho_{k\ell}| < 1$ is not traded in a same exchange in N: i.e., $\{k,\ell\} \not\subset K(n)$ for all $n \in N$. By Proposition 3, equilibrium payoffs in N coincide with *ex post* equilibrium payoffs only when $\left(\mathbf{W}'\mathbf{\Lambda}^{-1}\mathbf{W}\right)^{-1} = \frac{\alpha}{I-2}\mathbf{\Sigma}$, or equivalently, $\mathbf{W}'\mathbf{C}\mathbf{W} = \frac{I-2}{I-1}(\alpha\mathbf{\Sigma})^{-1}$. Following the argument in (Part (ii) $\Leftarrow$ (iii)), the $(k,\ell)^{\text{th}}$ element of $\mathbf{W}'\mathbf{C}\mathbf{W}$ is zero, because $\{n|k,\ell \in K(n)\} = \emptyset$. This contradicts the equality $\mathbf{W}'\mathbf{C}\mathbf{W} = \frac{I-2}{I-1}(\alpha\mathbf{\Sigma})^{-1}$, because $((\alpha\mathbf{\Sigma})^{-1})_{k\ell} \neq 0$: $\left(\mathbf{W}'\mathbf{\Lambda}^{-1}\mathbf{W}\right)^{-1} \neq \frac{\alpha}{I-2}\mathbf{\Sigma}$. Therefore, condition (iii) is necessary for *ex post* equilibrium.

(Part (i) $\Leftarrow$ (iii)) If condition (iii) holds in a market structure N, an additional exchange n' cannot change the set of conditioning variables in traders' total demands. More precisely, condition (iii) also holds in market structure $N' = N \cup \{n'\}$: for every pair of assets k and $\ell \neq k$ such that $0 < |\rho_{k\ell}| < 1$, there is an exchange n'' in which these assets are traded, i.e., $k,\ell \in K(n'')$ for some $n'' \in N'$. By (Part (ii) $\Leftrightarrow$ (iii)), the equilibrium payoffs in both N and N' coincide with those in the contingent market, and thus, are the same.

(Part (i) $\Rightarrow$ (ii)) We prove the contrapositive: if equilibrium is not *ex post*, then there exists a new exchange that is not redundant. Consider exchanges N. By the equivalence between (ii) and (iii), there exist imperfectly correlated assets k and $\ell \neq k$ that are not both traded in any exchange, i.e., there is no n such that $\{k,\ell\} \subset K(n)$. The $(k,\ell)^{\text{th}}$ element of the total demand's Jacobian $\widehat{\mathbf{C}} = \mathbf{W}\mathbf{C}\mathbf{W}'$ is zero, i.e., $\widehat{c}_{k\ell} = 0$. Suppose that a new exchange $n' \equiv \{k,\ell\}$ is introduced in the market structure N, i.e., $N' = N \cup \{n'\}$. We show that exchange n' is not redundant: The Jacobian $\mathbf{C}^{N'}_{K(n')} \in \mathbb{R}^{2\times 2}$ in exchange n' has a non-zero off-diagonal element $c^{N'}_{k\ell}$ except when $[\mathbf{B}\mathbf{B}']_{k,K(n')}$ is proportional to $[\alpha\mathbf{\Sigma}\mathbf{B}\mathbf{B}']_{k,K(n')}$

(Eq. (84) in Corollary 5), i.e., $c_{k\ell}^{N'} \neq 0$ unless $\sigma_{k\ell} = 0$. This shows that $\widehat{c}_{k\ell}^{N'} \neq \widehat{c}_{k\ell} = 0$, and hence, the Jacobians of the total demands in N' and N differ: $\widehat{\mathbf{C}}^{N'} = \mathbf{W}^{N'}\mathbf{C}^{N'}(\mathbf{W}^{N'})' \neq \widehat{\mathbf{C}}$. Equivalently, $\widehat{\mathbf{\Lambda}}^{N'} \neq \widehat{\mathbf{\Lambda}}$. By Proposition 3 (i), equilibrium payoffs differ in N and N'. ■

Corollary 6 (Nonredundancy of Changes in Market Structure: A Condition on Primitives) All market structures $\{K(n)\}_n$ give the same equilibrium payoff if and only if the payoffs of all assets are either perfectly correlated or independent.

Proof of Corollary 6 (Nonredundancy of Changes in Market Structure: A Condition on Primitives). The proof is immediate from Corollary 2. ■

Proof of Proposition 4 (Welfare with Multiple Exchanges vs. Joint Market Clearing). Suppose that there is no information loss: i.e., $\sigma_{cv} \to 0$, $\sigma_{pv} \to 0$, and $\sigma_0 \equiv \frac{\sigma_{cv}+\frac{1}{I}\sigma_{pv}}{\sigma_{cv}+\sigma_{pv}} < 1$. For a market structure N with multiple exchanges that is not payoff-equivalent to a single exchange, consider the difference in equilibrium surplus $U^c - U^N$:

$$U^c - U^N = \sum_i (E[\overline{\mathbf{q}}_0] - E[\mathbf{q}_0^i]) \cdot \big(\mathbf{\Upsilon}(\mathbf{\Lambda}^c) - \mathbf{\Upsilon}(\widehat{\mathbf{\Lambda}})\big)(E[\overline{\mathbf{q}}_0] - E[\mathbf{q}_0^i]), \tag{124}$$

which, by Proposition 3, is zero if the per-unit price impacts $\mathbf{\Lambda}^c$ and $\widehat{\mathbf{\Lambda}}$ are the same.

The equilibrium surplus difference (Eq. (124)) is a quadratic matrix function of $E[\overline{\mathbf{q}}_0] - E[\mathbf{q}_0^i]$ with a quadratic coefficient of $\mathbf{\Upsilon}(\mathbf{\Lambda}^c) - \mathbf{\Upsilon}(\widehat{\mathbf{\Lambda}})$. If the surplus matrix difference $\mathbf{\Upsilon}(\mathbf{\Lambda}^c) - \mathbf{\Upsilon}(\widehat{\mathbf{\Lambda}})$ has a negative eigenvalue $\mu < 0$, then there exist *ex ante* trading needs $\{E[\overline{\mathbf{q}}_0] - E[\mathbf{q}_0^i]\}_i \in \mathbb{R}^{IK}$ such that $U^c - U^N < 0$. Pick a distribution of endowments such that $E[\overline{\mathbf{q}}_0] - E[\mathbf{q}_0^i]$ is proportional to an eigenvector of matrix $\mathbf{\Upsilon}(\mathbf{\Lambda}^c) - \mathbf{\Upsilon}(\widehat{\mathbf{\Lambda}})$ (with a positive or a negative proportionality constant) associated with an eigenvalue μ: for all i,

$$\big(\mathbf{\Upsilon}(\mathbf{\Lambda}^c) - \mathbf{\Upsilon}(\widehat{\mathbf{\Lambda}})\big)(E[\overline{\mathbf{q}}_0] - E[\mathbf{q}_0^i]) = \mu(E[\overline{\mathbf{q}}_0] - E[\mathbf{q}_0^i]). \tag{125}$$

Substituting the vector of trading needs $\{E[\overline{\mathbf{q}}_0] - E[\mathbf{q}_0^i]\}_i$ that satisfies Eq. (125) into Eq. (124), we have:

$$\sum_i (E[\overline{\mathbf{q}}_0] - E[\mathbf{q}_0^i]) \cdot \big(\mathbf{\Upsilon}(\mathbf{\Lambda}^c) - \mathbf{\Upsilon}(\widehat{\mathbf{\Lambda}})\big)(E[\overline{\mathbf{q}}_0] - E[\mathbf{q}_0^i]) = \sum_i (E[\overline{\mathbf{q}}_0] - E[\mathbf{q}_0^i]) \cdot \mu(E[\overline{\mathbf{q}}_0] - E[\mathbf{q}_0^i]) < 0,$$

and hence, $U^N > U^c$.

Lemma 5 gives a sufficient condition for a negative eigenvalue to exist: any market structure whose exchanges are demergers (Definition 6) of a single venue for all assets. If the surplus matrix difference $\mathbf{\Upsilon}(\mathbf{\Lambda}^c) - \mathbf{\Upsilon}(\widehat{\mathbf{\Lambda}})$ does not have a negative eigenvalue, then $U^c - U^N \geq 0$ for any distribution of endowments. ■

Lemma 5 (Price Impacts in Multiple Exchanges vs. Joint Market Clearing) Let $K > 1$ and $I < \infty$. Consider a market structure $N = \{K(n)\}_n$ that consists of exchanges that partition the set of K assets: $K(n) \cap K(n') = \emptyset$ for all n and $n' \neq n$ and $\cup_n K(n) = K$. The equilibrium price impact $\mathbf{\Lambda}$ in N and the price impact in the contingent market $\mathbf{\Lambda}^c$ are not ranked in the positive semi-definite sense, i.e., neither $\mathbf{\Lambda} \geq \mathbf{\Lambda}^c$ nor $\mathbf{\Lambda} \leq \mathbf{\Lambda}^c$ holds, except when $\mathbf{\Lambda} = \mathbf{\Lambda}^c$.

Proof of Lemma 5 (Price Impacts in Multiple Exchanges vs. Joint Market Clearing). The equilibrium fixed point equation (84) for the equilibrium price impact $\mathbf{\Lambda} \in \mathbb{R}^{K\times K}$ can be written as follows:

$$\left[(\alpha\mathbf{\Sigma}^{+}+\mathbf{\Lambda}-(I-1)\mathbf{\Lambda}')\mathbf{B\Omega B}'\right]_N=0. \tag{126}$$

To demonstrate that $\mathbf{\Lambda}^c-\mathbf{\Lambda}$ is neither positive semi-definite nor negative semi-definite, we argue by contradiction: Suppose that $\mathbf{\Lambda}^c-\mathbf{\Lambda}=\frac{\alpha}{I-2}\mathbf{\Sigma}^{+}-\mathbf{\Lambda}$ is positive semi-definite. By the Trace Inequality for Matrix Product,[8] the trace of the matrix on the LHS of Eq. (126) is nonnegative:

$$tr\big((\alpha\mathbf{\Sigma}^{+}+\mathbf{\Lambda}-(I-1)\mathbf{\Lambda}')\mathbf{B\Omega B}'\big)\geq(I-2)\mu_K(\mathbf{B\Omega B}')\ tr\big(\frac{\alpha}{I-2}\mathbf{\Sigma}^{+}-\frac{1}{2}(\mathbf{\Lambda}+\mathbf{\Lambda}')\big)\ \geq\ 0, \tag{127}$$

where $\mu_K(\mathbf{M})\in\mathbb{R}$ is the lowest eigenvalue of matrix $\mathbf{M}$. Because matrix $\mathbf{B\Omega B}'$ is symmetric and positive definite, its lowest eigenvalue is positive, and hence, (127) holds with equality if and only if $\frac{\alpha}{I-2}\mathbf{\Sigma}^{+}=\frac{1}{2}(\mathbf{\Lambda}+\mathbf{\Lambda}')$, or equivalently $\mathbf{\Lambda}^c=\mathbf{\Lambda}$.

Except when $\mathbf{\Lambda}^c=\mathbf{\Lambda}$, however, Eq. (127) contradicts the equilibrium fixed point equation (126). Hence, by the definition of operator $[\cdot]_N$, the matrix trace must be zero:

$$tr\big((\alpha\mathbf{\Sigma}^{+}+\mathbf{\Lambda}-(I-1)\mathbf{\Lambda}')\mathbf{B\Omega B}'\big)=0.$$

An analogous argument shows that $\mathbf{\Lambda}^c-\mathbf{\Lambda}$ is not negative semi-definite except when $\mathbf{\Lambda}^c=\mathbf{\Lambda}$. ∎

C Symmetric Markets

C.1 Additional Results: Symmetric Markets

This Appendix presents results for markets that are symmetric in the following sense.

Definition 7 (Symmetric Market) *Assume $K=MN$ for some $M\geq 1$. A market structure $N=\{K(n)\}_n$ is* symmetric *if*

- *asset distribution is symmetric, i.e., $\sigma\equiv Var[r_k]$ for all k and $\rho\equiv Corr[r_k,r_\ell]$ for all k and $\ell\neq k$, and*
- *exchanges N partition the set of K assets into exchanges with the same number of assets, i.e., $K(n)\cap K(n')=\emptyset$ for all n and $n'\neq n$, and $K(n)=M$ for all n.*

For results in this part of the Appendix, we assume that traders' endowments are independent across assets: $\mathbf{\Omega}=\mathbf{Id}\in\mathbb{R}^{K\times K}$.

In a symmetric market, the asset covariance is $\mathbf{\Sigma}=\sigma((1-\rho)\mathbf{Id}+\rho\mathbf{1}\mathbf{1}')$ and the price impact matrix $\mathbf{\Lambda}=diag(\mathbf{\Lambda}_{K(n)})_n\in\mathbb{R}^{K\times K}$ is symmetric across exchanges and assets and can be written as follows: for all n,

$$\mathbf{\Lambda}_{K(n)}=(\lambda_k-\lambda_{k\ell})\mathbf{Id}+\lambda_{k\ell}\mathbf{1}\mathbf{1}'\in\mathbb{R}^{M\times M}, \tag{128}$$

where $\lambda_k\in\mathbb{R}_+$ is the diagonal price impact for asset k and $\lambda_{k\ell}\in\mathbb{R}$ is the off-diagonal price impact for assets $k,\ell\neq k$.

A counterpart of Theorem 3, Proposition 5 characterizes the within-exchange equilibrium price impact in symmetric markets.

[8] For a real matrix $\mathbf{S}\in\mathbb{R}^{K\times K}$ and a positive semi-definite matrix $\mathbf{T}\in\mathbb{R}^{K\times K}$ the following inequality holds:

$$\mu_K(\mathbf{S})tr(\mathbf{T})\leq tr(\mathbf{ST})=tr(\mathbf{TS})\leq\mu_1(\mathbf{S})tr(\mathbf{T}),$$

where $\mu_k(\mathbf{S})$ is the k^{th} largest eigenvalue of the Hermitian part $\frac{1}{2}(\mathbf{S}+\mathbf{S}')$.

Proposition 5 (Price Impact: Comparative Statics; Symmetric Markets, General Design) The within-exchange price impact $\mathbf{\Lambda}_{K(n)}$ satisfies the following properties for each n:

(1) (*Magnitude*) With K assets, the diagonal price impact λ_k maximally increases N-fold relative to $\lambda_k^c = \frac{\alpha}{I-2}\sigma$; this is the case if and only if $|\rho| = 1$.

$$\frac{\alpha}{I-2}\sigma \leq \lambda_k \leq \frac{\alpha}{I-2}N\sigma.$$

(2) (*Comparative statics*) Relative to the contingent market:

(i) $\frac{\partial(\lambda_k - \lambda_k^c)}{\partial I} \leq 0$ and $\frac{\partial(\lambda_{k\ell} - \lambda_{k\ell}^c)}{\partial I} \leq 0$ for all $k, \ell \in K(n)$;

(ii) $\frac{\partial(\lambda_k - \lambda_k^c)}{\partial|\rho|} \geq 0$ and $\frac{\partial(\lambda_{k\ell} - \lambda_{k\ell}^c)}{\partial|\rho|} \geq 0$ for all $k, \ell \in K(n)$. Either inequality holds with equality if and only if $\rho = 0$.

Note. With one asset per exchange (i.e., $N = \{\{k\}\}_k$ and $M = 1$), the proof of Proposition 5 specializes to that of Theorem 3.

Proof of Proposition 5 (Price Impact: Comparative Statics; Symmetric Markets, General Design).

(Scalar equations for price impact) By Corollary 5 in Appendix A, the price impact $\mathbf{\Lambda}$ is determined by Eqs. (83)-(85):

$$\left[(\alpha\mathbf{\Sigma} - (I-2)\mathbf{\Lambda})(\mathbf{Id} + \kappa(\alpha\mathbf{\Sigma})^{-1}\mathbf{\Lambda})^{-1}(\mathbf{Id} + \kappa\mathbf{\Lambda}(\alpha\mathbf{\Sigma})^{-1})^{-1}\right]_N = \mathbf{0}, \tag{129}$$

where $\kappa \equiv \frac{1+(I-2)\sigma_0}{1-\sigma_0} \in \mathbb{R}_+$.

We first rewrite the matrix fixed point equation (129) for $\mathbf{\Lambda}$ as a system of equations in $\mathbb{R}$ for λ_k and $\lambda_{k\ell}$ (Eqs. (135)-(136) below). Market symmetry simplifies Eq. (129). In particular, the symmetry of price impact $\mathbf{\Lambda}$ implies that vector $\mathbf{1} \in \mathbb{R}^K$ is an eigenvector of $\mathbf{\Lambda}$:

$$\mathbf{\Lambda 1} = \overline{\lambda}\mathbf{1}, \tag{130}$$

where $\overline{\lambda} \equiv \lambda_k + (M-1)\lambda_{k\ell}$ is the sum of elements in each row of $\mathbf{\Lambda}$. Using Eq. (130), the inverse matrix $(\mathbf{Id}+\kappa(\alpha\mathbf{\Sigma})^{-1}\mathbf{\Lambda})^{-1}$ in Eq. (129) can be decomposed as a linear combination of a block diagonal matrix $(\mathbf{Id} + \frac{\kappa}{\alpha\sigma(1-\rho)}\mathbf{\Lambda})^{-1}$ and matrix $\mathbf{11'} \in \mathbb{R}^{K\times K}$:

$$\begin{aligned}(\mathbf{Id} + \kappa(\alpha\mathbf{\Sigma})^{-1}\mathbf{\Lambda})^{-1} &= (\mathbf{Id} + \frac{\kappa}{\alpha\sigma(1-\rho)}\mathbf{\Lambda} - \frac{\kappa\rho\overline{\lambda}}{\alpha\sigma(1-\rho)(1+(K-1)\rho)}\mathbf{11'})^{-1} \\ &= (\mathbf{Id} + \frac{\kappa}{\alpha\sigma(1-\rho)}\mathbf{\Lambda})^{-1} + \frac{\kappa\overline{\lambda}\rho\overline{v}^2}{\alpha\sigma(1-\rho)(1+(K-1)\rho) - K\overline{v}\kappa\overline{\lambda}\rho}\mathbf{11'},\end{aligned} \tag{131}$$

where the second equality applies the Woodbury Matrix Identity (Lemma 1) to $(\mathbf{Id} + \frac{\kappa}{\alpha\sigma(1-\rho)}\mathbf{\Lambda} - \frac{\kappa\rho\overline{\lambda}}{\alpha\sigma(1-\rho)(1+(K-1)\rho)}\mathbf{11'})^{-1}$. Here, $\overline{v} \in \mathbb{R}_+$ is the eigenvalue of matrix $(\mathbf{Id} + \frac{\kappa}{\alpha\sigma(1-\rho)}\mathbf{\Lambda})^{-1}$ associated with the eigenvector $\mathbf{1}$:

$$\overline{v}\mathbf{1} = (\mathbf{Id} + \frac{\kappa}{\alpha\sigma(1-\rho)}\mathbf{\Lambda})^{-1}\mathbf{1}, \qquad \overline{v} = \frac{\alpha\sigma(1-\rho)}{\kappa(\lambda_k + (M-1)\lambda_{k\ell}) + \alpha\sigma(1-\rho)} = \frac{\alpha\sigma(1-\rho)}{\kappa\overline{\lambda} + \alpha\sigma(1-\rho)}. \tag{132}$$

Substituting $(\mathbf{Id}+\kappa(\alpha\mathbf{\Sigma})^{-1}\mathbf{\Lambda})^{-1}$ (Eq. (131)) into Eq. (129), the LHS of Eq. (129) can be decomposed as a linear combination of a block diagonal matrix and matrix $[\mathbf{11'}]_N \in \mathbb{R}^{K\times K}$:

$$\big[(\mathbf{Id}+\frac{\kappa}{\alpha\sigma(1-\rho)}\mathbf{\Lambda})^{-1}(\alpha\mathbf{\Sigma}-(I-2)\mathbf{\Lambda})(\mathbf{Id}+\frac{\kappa}{\alpha\sigma(1-\rho)}\mathbf{\Lambda})^{-1}\big]_N \tag{133}$$
$$+\big(\alpha\sigma(1+(K-1)\rho)-(I-2)\overline{\lambda}\big)K\big(\big(\frac{\kappa\overline{\lambda}\rho\overline{v}^2}{\alpha\sigma(1-\rho)(1+(K-1)\rho)-K\overline{v}\kappa\overline{\lambda}\rho}+\frac{\overline{v}}{K}\big)^2-\frac{\overline{v}^2}{K^2}\big)\big[\mathbf{11'}\big]_N=\mathbf{0}.$$

Because $(\mathbf{Id}+\frac{\kappa}{\alpha\sigma(1-\rho)}\mathbf{\Lambda})^{-1}$ is a block diagonal matrix, the matrix equation (133) for $\mathbf{\Lambda}=diag(\mathbf{\Lambda}_{K(n)})$ simplifies to a fixed point equation for $\mathbf{\Lambda}_{K(n)}$ in each exchange n:

$$(\mathbf{Id}+\frac{\kappa}{\alpha\sigma(1-\rho)}\mathbf{\Lambda}_{K(n)})^{-1}(\alpha\mathbf{\Sigma}_{K(n),K(n)}-(I-2)\mathbf{\Lambda}_{K(n)})(\mathbf{Id}+\frac{\kappa}{\alpha\sigma(1-\rho)}\mathbf{\Lambda}_{K(n)})^{-1} \tag{134}$$
$$+\big(\alpha\sigma(1+(K-1)\rho)-(I-2)\overline{\lambda}\big)K\big(\big(\frac{\kappa\overline{\lambda}\rho\overline{v}^2}{\alpha\sigma(1-\rho)(1+(K-1)\rho)-K\overline{v}\kappa\overline{\lambda}\rho}+\frac{\overline{v}}{K}\big)^2-\frac{\overline{v}^2}{K^2}\big)\mathbf{11'}=\mathbf{0}.$$

We remark that the second line of the LHS of Eq. (134) is proportional to matrix $\mathbf{11'}\in\mathbb{R}^{M\times M}$. Thus, for Eq. (134) to hold, its first line must be proportional to matrix $\mathbf{11'}$. Equivalently, because $(\mathbf{Id}+\frac{\kappa}{\alpha\sigma(1-\rho)}\mathbf{\Lambda}_{K(n)})^{-1}$ is invertible, multiplying the first line by $(\mathbf{Id}+\frac{\kappa}{\alpha\sigma(1-\rho)}\mathbf{\Lambda}_{K(n)})$ shows that matrix $(\alpha\mathbf{\Sigma}_{K(n),K(n)}-(I-2)\mathbf{\Lambda}_{K(n)})$ is proportional to $\mathbf{11'}$, and hence,

$$\alpha\sigma-(I-2)\lambda_k=\alpha\sigma\rho-(I-2)\lambda_{k\ell},\qquad \text{i.e.,}\quad \lambda_k-\lambda_{k\ell}=\frac{\alpha}{I-2}\sigma(1-\rho). \tag{135}$$

Furthermore, using Eq. (135) and $\overline{v}\kappa\overline{\lambda}=\alpha\sigma(1-\rho)(1-\overline{v})$ from Eq. (132), the matrix equation (134) simplifies to a fixed point equation for λ_k in $\mathbb{R}$:

$$\lambda_k-\lambda_k^c=\frac{\alpha\sigma\rho}{I-2}\ \big(\frac{K}{M+(K-M)\big(\frac{1+(K\overline{v}-1)\rho}{1+(K-1)\rho}\big)^2}-1\big). \tag{136}$$

(Part (1)) We are now ready to show the inequality $\lambda_k\geq\frac{\alpha}{I-2}\sigma=\lambda_k^c$. Because $\overline{\lambda}>0$ in Eq. (130), the following inequality holds for $\overline{v}$:

$$0<\ \overline{v}=\frac{\alpha\sigma(1-\rho)}{\kappa\overline{\lambda}+\alpha\sigma(1-\rho)}\ <1. \tag{137}$$

This implies that the term $\frac{1+(K\overline{v}-1)\rho}{1+(K-1)\rho}$ in the denominator of the RHS of (136) satisfies $sign\big(\frac{1+(K\overline{v}-1)\rho}{1+(K-1)\rho}-1\big)=-sign(\rho)$, and thus, $sign\big(\frac{K}{M+(K-M)\big(\frac{1+(K\overline{v}-1)\rho}{1+(K-1)\rho}\big)^2}-1\big)=sign(\rho)$. Hence, by Eq. (136), $\lambda_k\geq\frac{\alpha}{I-2}\sigma=\lambda_k^c$; $\lambda_k=\lambda_k^c$ if and only if $\rho=0$.

Furthermore, the proof of Theorem 2 demonstrated the existence of an upper bound $\overline{\mathbf{\Lambda}}=\frac{\alpha}{I-2}N\sigma\mathbf{Id}$ such that equilibrium price impact $\mathbf{\Lambda}$ satisfies $\mathbf{\Lambda}\leq\overline{\mathbf{\Lambda}}$. It follows that $\lambda_k\leq\frac{\alpha}{I-2}N\sigma$ for any k. The equality holds if and only if $|\rho_{k\ell}|=1$ for all k and $\ell\neq k$ as we showed in the proof of Theorem 2.

(Part (2i)) We prove the monotonicity of the inference effect with respect to the number of traders I. By Eq. (137), $\overline{v}\sim o(I^{-1+\varepsilon})$ for some $\varepsilon\in(0,1)$, given $\kappa=\frac{1+(I-2)\sigma_0}{1-\sigma_0}$. Then, Eq. (136) implies that $\frac{\partial(\lambda_k-\lambda_k^c)}{\partial I}<0$ because $\frac{\alpha\sigma\rho}{I-2}\sim o(I^{-1})$ and $\frac{K}{M+(K-M)\big(\frac{1+(K\overline{v}-1)\rho}{1+(K-1)\rho}\big)^2}\sim o(I^{1-\varepsilon})$.

From Eq. (135), the difference between off-diagonal and diagonal price impacts is the same:

$$\lambda_{k\ell}-\lambda_{k\ell}^c=\big(\lambda_k-\frac{\alpha}{I-2}\sigma(1-\rho)\big)-\frac{\alpha}{I-2}\sigma\rho=\lambda_k-\lambda_k^c. \tag{138}$$

Hence, $\frac{\partial(\lambda_{k\ell}-\lambda_{k\ell}^c)}{\partial I}<0$.

(Part (2ii)) For any $|\rho|>0$, $\frac{\partial}{\partial\rho}\frac{1+(K\overline{v}-1)\rho}{1+(K-1)\rho}<0$, because $K\overline{v}-1<K-1$ by Eq. (137) and

$1+(K-1)\rho>0$ by the positive definiteness of $\mathbf{\Sigma}$. This implies:

$$\frac{K}{M+(K-M)\left(\frac{1+(K\overline{v}-1)\rho}{1+(K-1)\rho}\right)^2}-1>0, \qquad \frac{\partial}{\partial\rho}\left(\frac{K}{M+(K-M)\left(\frac{1+(K\overline{v}-1)\rho}{1+(K-1)\rho}\right)^2}-1\right)>0.$$

Hence, in Eq. (136), $sign\left(\frac{\partial(\lambda_k-\lambda_k^c)}{\partial\rho}\right)=sign(\rho)$, i.e., $\frac{\partial(\lambda_k-\lambda_k^c)}{\partial|\rho|}>0$ when $|\rho|>0$. When $\rho=0$, $\frac{\partial(\lambda_k-\lambda_k^c)}{\partial|\rho|}=0$. From Eq. (138), $\frac{\partial(\lambda_{k\ell}-\lambda_{k\ell}^c)}{\partial|\rho|}>0$ when $|\rho|>0$ and $\frac{\partial(\lambda_{k\ell}-\lambda_{k\ell}^c)}{\partial|\rho|}=0$ when $\rho=0$ for all $k,\ell\in K(n)$. ■

Proof of Corollary 3 (Price Impact and Market Structure).
(Part (i)) Suppose that $K=2$ and consider market structures $N=\{K\}=\{\{1,2\}\}$ and $N'=\{\{k\}\}_k=\{\{1\},\{2\}\}$. We want to show that $\lambda_k^{N'}\geq\lambda_k^N$ for all k. For simplicity, we dispense with the superscript N' for the uncontingent market N' and use the superscript c for the contingent market N.

By the equilibrium fixed point equation (20) in the uncontingent market N', the demand slope $c_k=\frac{1}{I-1}\lambda_k^{-1}$ for asset k can be decomposed into the direct effect and the (indirect) inference effect:

$$c_k\equiv-\frac{\partial q_k^j(\cdot)}{\partial p_k}=-\Big(\underbrace{-\frac{I-2}{I-1}(\alpha\sigma_{kk})^{-1}}_{\text{Direct effect}}+\underbrace{\frac{I-2}{I-1}(\alpha\sigma_{kk})^{-1}\alpha\sigma_{k\ell}c_\ell(\mathbf{VV}')_{\ell k}((\mathbf{VV}')_{kk})^{-1}}_{\text{Indirect effect}}\Big), \qquad (139)$$

where $\mathbf{V}\equiv(1-\sigma_0)\mathbf{C}^{-1}\mathbf{B}=(\mathbf{C}+\kappa(\alpha\mathbf{\Sigma})^{-1})^{-1}$ and $\kappa\equiv\frac{1+(I-2)\sigma_0}{(I-1)(1-\sigma_0)}\in\mathbb{R}_+$. We will show that $sign(\sigma_{k\ell})=sign((\mathbf{VV}')_{\ell k})$.[9] Given the decomposition in Eq. (139), the inference effect in Eq. (139) is nonnegative, and hence $c_k\leq\frac{I-2}{I-1}(\alpha\sigma_{kk})^{-1}$, and $\lambda_k=\frac{c_k^{-1}}{I-1}\geq\lambda_k^c=\frac{\alpha}{I-2}\sigma_{kk}$ for all k.

We now characterize matrix $\mathbf{VV}'$. By the definition of $\mathbf{V}=(\mathbf{C}+\kappa(\alpha\mathbf{\Sigma})^{-1})^{-1}$, we have:

$$\mathbf{V}=\frac{\alpha\overline{\sigma}}{det(\mathbf{V}^{-1})}\begin{bmatrix}\alpha\overline{\sigma}c_2+\kappa\sigma_{11} & \kappa\sigma_{12}\\ \kappa\sigma_{12} & \alpha\overline{\sigma}c_1+\kappa\sigma_{22}\end{bmatrix}, \qquad (140)$$

where $\overline{\sigma}\equiv det(\mathbf{\Sigma})=\sigma_{11}\sigma_{22}-\sigma_{12}^2>0$ and $det(\mathbf{V}^{-1})=(\alpha\overline{\sigma}c_1+\kappa\sigma_{22})(\alpha\overline{\sigma}c_2+\kappa\sigma_{22})-\kappa^2\sigma_{12}^2>0$. Using Eq. (140), we compute $\mathbf{VV}'$, whose off-diagonal element is:

$$(\mathbf{VV}')_{12}=\frac{\alpha^2\overline{\sigma}^2}{det(\mathbf{V}^{-1})^2}\kappa\alpha\sigma_{12}(\alpha\overline{\sigma}(c_1+c_2)+\kappa(\sigma_{11}+\sigma_{22})). \qquad (141)$$

Because $\kappa>0$, Eq. (141) implies that $sign((\mathbf{VV}')_{\ell k})=sign(\sigma_{k\ell})$. Hence, $\lambda_k\geq\lambda_k^c$ for all k; the equality holds if and only if $\sigma_{12}=0$, because $(\mathbf{VV}')_{\ell k}=0$ if and only if $\sigma_{k\ell}=0$ in Eq. (141).

(Part (ii)) See Fig. 1B in Section 3.2.3 for an example of $\widehat{\lambda}^N>\widehat{\lambda}_k^{N'}$. ■

Proposition 6 (Efficient Market Structure in Symmetric Markets) Consider the class of symmetric markets (Definition 7). Assume that traders' *ex ante* trading needs are symmetric for all assets $E[\overline{q}_{0,k}]-E[q_{0,k}^i]=E[\overline{q}_{0,\ell}]-E[q_{0,\ell}^i]$ for all k and ℓ for each i, and there is no information loss: $(\sigma_{cv},\sigma_{pv})\to0$ and $\sigma_0<1$. When $\rho>0$, the uncontingent market maximizes total *ex ante* welfare; when $\rho<0$, the contingent market does.

Proof of Proposition 6 (Efficient Market Structure in Symmetric Markets). We first derive the *ex ante* equilibrium surplus (Eq. (93)) in a symmetric market. Given the symmetry of the *ex ante*

[9] We note that Eq. (139) is the counterpart of Eq. (25) for the demand coefficient $c_k=\frac{1}{I-1}\lambda_k^{-1}$ (rather than price impact λ_k). In Eq. (139), $(\mathbf{VV}')_{\ell k}=(1-\sigma_0)^2Cov[p_\ell,p_k|\mathbf{q}_0^i]$ determines the sign of price correlation for all k and $\ell\neq k$ (See Section 3.2.3).

trading needs across assets (i.e., $E[\overline{\mathbf{q}}_0]-E[\mathbf{q}_0^i]=e^i\mathbf{1}$ for some $e^i\in\mathbb{R}$), each trader's *ex ante* equilibrium surplus (Eq. (93)) is:

$$(E[\overline{\mathbf{q}}_0]-E[\mathbf{q}_0^i])\cdot\mathbf{\Upsilon}(\mathbf{\Lambda})(E[\overline{\mathbf{q}}_0]-E[\mathbf{q}_0^i])=K\alpha\sigma(1+(K-1)\rho)\big[1-\frac{\overline{\lambda}^2}{(\alpha\sigma(1+(K-1)\rho)+\overline{\lambda})^2}\big](e^i)^2, \quad (142)$$

where $\overline{\lambda}\equiv\lambda_k+(M-1)\lambda_{k\ell}\in\mathbb{R}_+$ is the eigenvalue of a symmetric matrix $\mathbf{\Lambda}$ that corresponds to eigenvector $\mathbf{1}$. The *ex ante* equilibrium surplus (142) is decreasing in $\overline{\lambda}$, because $\overline{\lambda}$ is nonnegative (i.e., $\mathbf{\Lambda}$ is positive semi-definite). Therefore, it suffices to show that $sign(\frac{\partial\overline{\lambda}}{\partial M})=sign(\rho)$.

In the proof of Proposition 5, Eq. (136) characterizes the price impact $\mathbf{\Lambda}=diag(\mathbf{\Lambda}_{K(n)})_n=(\lambda_k-\lambda_{k\ell})\mathbf{Id}+\lambda_{k\ell}[\mathbf{11'}]_N$ by a single scalar equation for λ_k and $\lambda_{k\ell}=\lambda_k-\frac{\alpha}{I-2}\sigma(1-\rho)$ (Eq. (135)): Replacing $\lambda_k=\frac{1}{M}\big(\overline{\lambda}+(M-1)\frac{\alpha}{I-2}\sigma(1-\rho)\big)$ in Eq. (136) gives a third-order equation for $\overline{\lambda}$ (a linear equation for M):

$$\begin{aligned}F(\overline{\lambda},M)\equiv{}&(K-M)(1-\rho)^2(\alpha\sigma(1-\rho)-(I-2)\overline{\lambda})(\alpha\sigma(1+(K-1)\rho)+\kappa\overline{\lambda})^2\\&+M(1+(K-1)\rho)^2(\alpha\sigma(1+(K-1)\rho)-(I-2)\overline{\lambda})(\alpha\sigma(1-\rho)+\kappa\overline{\lambda})^2=0,\end{aligned}\quad (143)$$

where $\kappa\equiv\frac{1+(I-2)\sigma_0}{1-\sigma_0}\in\mathbb{R}_+$ and $\overline{v}\equiv\frac{\alpha\sigma(1-\rho)}{\alpha\sigma(1-\rho)+\kappa\overline{\lambda}}\in\mathbb{R}_+$ (Eq. (132)). To characterize the partial $\frac{\partial\overline{\lambda}}{\partial M}$, we apply the Implicit Function Theorem to Eq. (143):

$$\frac{\partial\overline{\lambda}}{\partial M}=-\frac{\partial F/\partial M}{\partial F/\partial\overline{\lambda}}, \quad (144)$$

and each partial derivative of $F(\overline{\lambda},M)$ in Eq. (143) is:

$$\frac{\partial F}{\partial M} = -\frac{K}{M}(1-\rho)^2(\alpha\sigma(1-\rho)-(I-2)\overline{\lambda})(\alpha\sigma(1+(K-1)\rho)+\kappa\overline{\lambda})^2, \quad (145)$$

$$\begin{aligned}\frac{\partial F}{\partial\overline{\lambda}} ={}& -(I-2)\big(M(1+(K-1)\rho)^2-(K-M)(1+(K\overline{v}-1)\rho)^2\big)(\alpha\sigma(1-\rho)+\kappa\overline{\lambda})^2 \quad (146)\\&+2\kappa(K-M)(1-\rho)^2(\alpha\sigma(1-\rho)-(I-2)\overline{\lambda})(\alpha\sigma(1+(K-1)\rho)+\kappa\overline{\lambda})\\&+2\kappa M(1+(K-1)\rho)^2(\alpha\sigma(1+(K-1)\rho)-(I-2)\overline{\lambda})(\alpha\sigma(1-\rho)+\kappa\overline{\lambda}).\end{aligned}$$

To sign $\frac{\partial\overline{\lambda}}{\partial M}$ in Eq. (144), we now determine the signs of partials $\frac{\partial F}{\partial\overline{\lambda}}$ and $\frac{\partial F}{\partial M}$. First, in Eq. (143), the signs of the first and second lines of Eq. (143) must be opposite for $F(\overline{\lambda},M)=0$ to hold. Therefore, when $0<\rho$, the eigenvalue of equilibrium price impact $\overline{\lambda}$ satisfies $\alpha\sigma(1-\rho)<(I-2)\overline{\lambda}<\alpha\sigma(1+(K-1)\rho)$; when $\rho<0$, $\alpha\sigma(1-\rho)>(I-2)\overline{\lambda}>\alpha\sigma(1+(K-1)\rho)$ holds. It follows that $sign(\alpha\sigma(1-\rho)-(I-2)\overline{\lambda})=sign(-\rho)$, and so $sign(\frac{\partial F}{\partial M})=sign(\rho)$ in Eq. (145).

Second, multiplying the partial $\frac{\partial F}{\partial\overline{\lambda}}$ in Eq. (146) by $\frac{1}{(1+(K-1)\rho)2(\alpha\sigma(1-\rho)+\kappa\overline{\lambda})^2}>0$ shows that $\frac{\partial F}{\partial\overline{\lambda}}<0$ if and only if

$$\begin{aligned}&-(I-2)M-(I-2)(K-M)(\frac{1+(K\overline{v}-1)\rho}{1+(K-1)\rho})^2\\&+2\kappa\big((K-M)\frac{\alpha\sigma(1-\rho)-(I-2)\overline{\lambda}}{\alpha\sigma(1+(K-1)\rho)+\kappa\overline{\lambda}}+M\frac{\alpha\sigma(1+(K-1)\rho)-(I-2)\overline{\lambda}}{\alpha\sigma(1-\rho)+\kappa\overline{\lambda}}\big)(\frac{1+(K\overline{v}-1)\rho}{1+(K-1)\rho})^2 \;<\; 0.\end{aligned}\quad (147)$$

Proposition 5 (i) provides a lower bound for $\overline{\lambda}$: $\overline{\lambda}=\lambda_k+(M-1)\lambda_{k\ell}\geq\frac{\alpha(1+(M-1)\rho)}{I-2}$; the equality holds if and only if $M=K$. Replacing $\overline{\lambda}$ by its lower bound $\frac{\alpha(1+(M-1)\rho)}{I-2}$ in the numerators $\alpha\sigma(1-\rho)-(I-2)\overline{\lambda}$ and $\alpha\sigma(1+(K-1)\rho)-(I-2)\overline{\lambda}$ in Eq. (147) characterizes an upper bound of the LHS of Eq. (147):

$$-(I-2)M-(K-M)\frac{(I-2)(\alpha\sigma(1-\rho)+\kappa\overline{\lambda})(\alpha\sigma(1+(K-1)\rho)+\kappa\overline{\lambda})-2\kappa MK\alpha^2\sigma^2\rho^2}{(\alpha\sigma(1-\rho)+\kappa\overline{\lambda})(\alpha\sigma(1+(K-1)\rho)+\kappa\overline{\lambda})}(\frac{1+(K\overline{v}-1)\rho}{1+(K-1)\rho})^2<0.$$

It follows that $\frac{\partial F}{\partial \bar{\lambda}} < 0$, and hence, $sign\big(\frac{\partial \bar{\lambda}}{\partial M}\big) = sign(\rho)$ (Eq. (144)). ■

Proof of Corollary 4 (Welfare with Multiple Exchanges vs. Joint Market Clearing ($K = 2$)). Suppose the *ex ante* trading needs across assets $E[\overline{\mathbf{q}}_0] - E[\mathbf{q}_0^i]$ are proportional to $(\xi, 1)' \in \mathbb{R}^2$ for all i for some non-zero constant: i.e., $\xi \equiv \frac{E[\overline{q}_{0,1}] - E[q_{0,1}^i]}{E[\overline{q}_{0,2}] - E[q_{0,2}^i]}$. Given the symmetry of asset payoffs, the price impact in $\{\{1\},\{2\}\}$ is symmetric across assets: i.e., $\mathbf{\Lambda} = \lambda \mathbf{Id}$.

We characterize the difference between the *ex ante* equilibrium surplus (Eq. (93)) in the uncontingent market $\{\{1\},\{2\}\}$ and the contingent market $\{\{1,2\}\}$:

$$(E[\overline{\mathbf{q}}_0] - E[\mathbf{q}_0^i]) \cdot \big(\mathbf{\Upsilon}(\mathbf{\Lambda}) - \mathbf{\Upsilon}(\mathbf{\Lambda}^c)\big)(E[\overline{\mathbf{q}}_0] - E[\mathbf{q}_0^i]), \tag{148}$$

where $\mathbf{\Lambda}^c = \frac{I-2}{I-1}\mathbf{\Sigma}$. Substituting $\mathbf{\Lambda} = \lambda\mathbf{Id}$ and $\mathbf{\Sigma} = \alpha\sigma(1-\rho)\mathbf{Id} + \alpha\sigma\rho\mathbf{1}\mathbf{1}'$ into Eq. (96), we characterize the difference between surplus matrices $\mathbf{\Upsilon}(\mathbf{\Lambda}) - \mathbf{\Upsilon}(\mathbf{\Lambda}^c)$:

$$\mathbf{\Upsilon}(\mathbf{\Lambda}) - \mathbf{\Upsilon}(\mathbf{\Lambda}^c) = \frac{\alpha\sigma}{(I-1)^2}\begin{bmatrix} 1-x & \rho(1-y) \\ \rho(1-y) & 1-x \end{bmatrix}, \tag{149}$$

where $x \equiv \frac{(I-1)^2\lambda^2}{((\alpha\sigma+\lambda)^2-(\alpha\sigma\rho)^2)^2}((\alpha\sigma+\lambda)^2 - \alpha\sigma\rho^2(\alpha\sigma+2\lambda)) \geq 1$ and $y \equiv \frac{(I-1)^2\lambda^2}{((\alpha\sigma+\lambda)^2-(\alpha\sigma\rho)^2)^2}(\lambda^2 - (\alpha\sigma)^2(1-\rho^2)) \leq 1$. Substituting $\mathbf{\Upsilon}(\mathbf{\Lambda}) - \mathbf{\Upsilon}(\mathbf{\Lambda}^c)$ (Eq. (149)) and $E[\overline{q}_{0,1}] - E[q_{0,1}^i] = \xi(E[\overline{q}_{0,2}] - E[q_{0,2}^i])$ into Eq. (148) shows that the *ex ante* equilibrium payoff in the uncontingent market is higher than in the contingent market if and only if

$$x\xi^2 + 2\xi\rho y + x < \xi^2 + 2\xi\rho + 1. \tag{150}$$

The necessary and sufficient condition (150) has a solution $\xi \in (\underline{\xi}(\rho, I), \overline{\xi}(\rho, I))$ with $\overline{\xi}(\rho, I) > \underline{\xi}(\rho, I)$ for any asset correlation $\rho \neq \{0, \pm 1\}$ and any finite number of traders $I < \infty$:

$$\frac{(1-y)\rho - \sqrt{(1-y)^2\rho^2 - (x-1)^2}}{2(x-1)} \leq \xi \equiv \frac{E[\overline{q}_{0,1}] - E[q_{0,1}^i]}{E[\overline{q}_{0,2}] - E[q_{0,2}^i]} \leq \frac{(1-y)\rho + \sqrt{(1-y)^2\rho^2 - (x-1)^2}}{2(x-1)}. \tag{151}$$

Given that $|(1-y)\rho| > \sqrt{(1-y)^2\rho^2 - (x-1)^2}$, the bounds in the necessary and sufficient condition (150) are both positive when $\rho > 0$ and are both negative when $\rho < 0$. It follows that inequality (151) holds if and only if conditions (i) and (ii) hold with $\underline{\xi}(\rho, I) \equiv \frac{(1-y)\rho - \sqrt{(1-y)^2\rho^2-(x-1)^2}}{2(x-1)}$ and $\overline{\xi}(\rho, I) \equiv \frac{(1-y)\rho + \sqrt{(1-y)^2\rho^2-(x-1)^2}}{2(x-1)}$ when $\rho > 0$, and $\underline{\xi}(\rho, I) \equiv \left|\frac{(1-y)\rho + \sqrt{(1-y)^2\rho^2-(x-1)^2}}{2(x-1)}\right|$ and $\overline{\xi}(\rho, I) \equiv \left|\frac{(1-y)\rho - \sqrt{(1-y)^2\rho^2-(x-1)^2}}{2(x-1)}\right|$ when $\rho < 0$. ■

C.2 Symmetric Equilibrium Characterization in Markets with Two Assets: $K = 2$

Suppose that $\alpha^i = \alpha$ for all i, and $\mathbf{\Sigma} = (\sigma_{k\ell})_{k,\ell}$ is characterized by $\sigma_{11} = \sigma_{22} = 1$ and $\sigma_{12} = \sigma_{21} = \rho$. Then, the price impact is symmetric across traders and assets: $\lambda_k^i = \lambda$ for all k and i and $\lambda_{k\ell}^i = 0$ for all k, $\ell \neq k$, and i. Traders' demand coefficients in (15) are symmetric: $\mathbf{b}_k^i = \mathbf{b}_k$ and $c_k^i = c$ for all i and k. Observe that vector $\mathbf{b}_k$ is symmetric across k up to a permutation: i.e., if $\mathbf{b}_1 = (x, y)$, then $\mathbf{b}_2 = (y, x)$. We will continue to use the superscript i and subscript k when they are useful.

Equilibrium with uncontingent trading is characterized in two steps (Proposition 2). Step 1 characterizes the fixed point among trader i's demand coefficients for assets 1 and 2, taking as given his price impact λ and residual supply intercepts $F(\mathbf{s}^{-i}|\mathbf{q}_0^i)$. Step 2 endogenizes λ and $F(\mathbf{s}^{-i}|\mathbf{q}_0^i)$.

Step 1 (Optimization, given residual supply λ and $F(\mathbf{s}^{-i}|\mathbf{q}_0^i)$) Taking the derivative of the expected payoff (10) with respect to q_k^i gives the first-order conditions of trader i for each k:

$$E\big[\delta_1 - \alpha^i(\sigma_{11}(q_1^i + q_{0,1}^i) + \sigma_{12}(q_2^i + q_{0,2}^i))\big|p_1, \mathbf{q}_0^i\big] = p_1 + \lambda_1^i q_1^i \qquad \forall p_1 \in \mathbb{R}, \tag{152}$$

$$E\big[\delta_2 - \alpha^i(\sigma_{22}(q_2^i + q_{0,2}^i) + \sigma_{21}(q_1^i + q_{0,1}^i))\big|p_2, \mathbf{q}_0^i\big] = p_2 + \lambda_2^i q_2^i \qquad \forall p_2 \in \mathbb{R}. \tag{153}$$

Trader i's expected marginal utility for asset k depends on the demand coefficients of his schedule $q_\ell^i(\cdot)$ for asset $\ell \neq k$. The characterization of a trader's best-response demand $q_k^i(\cdot)$ requires solving a fixed point problem for trader i's own demand schedules $\{q_k^i(\cdot)\}_k$ across assets.

Step 1.1 (Parameterization of demands for asset $\ell \neq k$) To characterize the best-response demand of trader i for asset 1, assume that his demand for asset 2 is a linear function:

$$q_2^i(p_2) = a_2^i - \mathbf{b}_2^i \mathbf{q}_0^i - c_2^i p_2 \qquad \forall p_2 \in \mathbb{R}, \tag{154}$$

where $a_2^i \in \mathbb{R}, \mathbf{b}_2^i \in \mathbb{R}^{1\times 2}$, and $c_2^i \in \mathbb{R}_+$.

Step 1.2 (Price distribution and expected trades, given $F(\mathbf{s}^{-i}|\mathbf{q}_0^i)$) Market clearing for asset 2 characterizes the distribution of price p_2. Equalization of demand $q_2^i(\cdot)$ in Eq. (154) and residual supply $S_2^{-i}(\cdot) = s_2^{-i} + (\lambda_2^i)^{-1} p_2$ gives:

$$a_2^i - \mathbf{b}_2^i \mathbf{q}_0^i - c_2^i p_2 = s_2^{-i} + (\lambda_2^i)^{-1} p_2 \qquad \forall s_2^{-i} \in \mathbb{R}.$$

Price p_2 maps one-to-one to s_2^{-i}:

$$p_2 = \frac{1}{c_2^i + (\lambda_2^i)^{-1}}(a_2^i - \mathbf{b}_2^i \mathbf{q}_0^i - s_2^{-i}) \qquad \forall s_2^{-i} \in \mathbb{R}. \tag{155}$$

Eq. (155) characterizes price distribution $F(p_2|\mathbf{q}_0^i)$ as a function of the intercept distribution $F(s_2^{-i}|\mathbf{q}_0^i)$ and the coefficients $\{a_2^i, \mathbf{b}_2^i, c_2^i\}$ of trader i's own demand function $q_2^i(\cdot)$ for asset 2.

The one-to-one map between p_2 and s_2^{-i} (Eq. (155)) allows the expected trade $E[q_2^i|p_1, \mathbf{q}_0^i]$ in the first-order condition for asset 1 (Eq. (152)) to be characterized conditionally on s_1^{-i}:

$$E[q_2^i|p_1, \mathbf{q}_0^i] = E[q_2^i|s_1^{-i}, \mathbf{q}_0^i].$$

From the parameterization of $q_2^i(\cdot)$ in Eq. (154) and price distribution p_2 in Eq. (155),

$$E[q_2^i|s_1^{-i}, \mathbf{q}_0^i] = E\big[a_2^i - \mathbf{b}_2^i \mathbf{q}_0^i - c_2^i p_2\big|s_1^{-i}, \mathbf{q}_0^i\big] = a_2^i - \mathbf{b}_2^i \mathbf{q}_0^i - \frac{c_2^i}{c_2^i + (\lambda_2^i)^{-1}}\big(a_2^i - \mathbf{b}_2^i \mathbf{q}_0^i - E[s_2^{-i}|s_1^{-i}, \mathbf{q}_0^i]\big).$$

The conditional expectation $E[s_2^{-i}|s_1^{-i}, \mathbf{q}_0^i]$ is characterized by the intercept distribution $F(\mathbf{s}^{-i}|\mathbf{q}_0^i)$, which trader i takes as given.

Step 1.3 (Best response for asset k, given demands for $\ell \neq k$) Substituting the expected trade into the first-order condition (152) gives the following equation:

$$\delta_1 - \alpha^i\Big(\sigma_{11}(q_1^i + q_{0,1}^i) + \sigma_{12}\Big(a_2^i - \mathbf{b}_2^i \mathbf{q}_0^i - \frac{c_2^i}{c_2^i + (\lambda_2^i)^{-1}}\big(a_2^i - \mathbf{b}_2^i \mathbf{q}_0^i - E[s_2^{-i}|s_1^{-i}, \mathbf{q}_0^i]\big) + q_{0,2}^i\Big)\Big) = p_1 + \lambda_1^i q_1^i,$$

from which the best response $q_1^i(\cdot)$ is derived as a linear function of s_1^{-i} and p_1:

$$q_1^i(p_1, s_1^{-i}) = \frac{1}{\alpha^i \sigma_{11} + \lambda_1^i}\Big(\delta_1 - \alpha^i \mathbf{\Sigma}_1 \mathbf{q}_0^i - p_1 - \alpha^i \sigma_{12}\Big(a_2^i - \mathbf{b}_2^i \mathbf{q}_0^i - \frac{c_2^i}{c_2^i + (\lambda_2^i)^{-1}}\big(a_2^i - \mathbf{b}_2^i \mathbf{q}_0^i - E[s_2^{-i}|s_1^{-i}, \mathbf{q}_0^i]\big)\Big)\Big). \tag{156}$$

The demand schedule $q_1^i(\cdot)$ in Eq. (156) can be written as a function of both p_1 and s_1^{-i}. Using the one-to-one map between p_1 and s_1^{-i}:

$$q_1^i(p_1, s_1^{-i}) = s_1^{-i} + (\lambda_1^i)^{-1} p_1, \tag{157}$$

the best response $q_1^i(\cdot)$ in Eq. (156) is characterized as a function of only p_1 as an endogenous variable. Eqs. (156)-(157) characterize the demand coefficients in $q_1^i(p_1) = a_1^i - \mathbf{b}_1^i \mathbf{q}_0^i - c_1^i p_1$ as functions of $a_2^i, \mathbf{b}_2^i, c_2^i$, and $\{\lambda_k^i\}_k$. An analogous argument characterizes the demand coefficients $a_2^i, \mathbf{b}_2^i, c_2^i$ for asset 2 as functions of $a_1^i, \mathbf{b}_1^i, c_1^i$, and $\{\lambda_k^i\}_k$, which creates a fixed point for $\{a_k^i, \mathbf{b}_k^i, \mathbf{c}_k^i\}_k$.

Step 2 (Correct residual supply) Given other traders' demands (154) for all k and $j \neq i$, the correct residual supply of trader i is determined by $S_k^{-i}(\cdot) = -\sum_{j\neq i} q_k^j(\cdot)$.

Step 2.1 (Correct distribution of residual supply intercepts and expectations) The residual supply intercepts $s_k^{-i} = -\sum_{j\neq i}(a_k^j - \mathbf{b}_k^j \mathbf{q}_0^j)$ are jointly normally distributed. From the distribution of endowments $F((\mathbf{q}_0^j)_j | \mathbf{q}_0^i)$, the first and second moments of intercepts (s_1^{-i}, s_2^{-i}) are: for each k and ℓ,

$$\begin{aligned} E[s_k^{-i}|\mathbf{q}_0^i] &= -\sum_{j\neq i} a_k^j + \mathbf{b}_k \sum_{j\neq i}(E[\mathbf{q}_0^j] + \frac{\sigma_{cv}}{\sigma_{cv}+\sigma_{pv}}(\mathbf{q}_0^i - E[\mathbf{q}_0^i])), \\ Cov[s_k^{-i}, s_\ell^{-i}|\mathbf{q}_0^i] &= \mathbf{b}_k \sum_{j,h\neq i} Cov[\mathbf{q}_0^j, \mathbf{q}_0^h|\mathbf{q}_0^i]\mathbf{b}_\ell' = I(I-1)\sigma_{pv}\sigma_0 \mathbf{b}_k \cdot \mathbf{b}_\ell. \end{aligned}$$

Applying the Projection Theorem to this distribution of the residual supply intercepts $F(\mathbf{s}^{-i}|\mathbf{q}_0^i)$ gives the expected intercepts $E[s_\ell^{-i}|s_k^{-i}, \mathbf{q}_0^i]$:

$$E[s_\ell^{-i}|s_k^{-i}, \mathbf{q}_0^i] = E[s_\ell^{-i}|\mathbf{q}_0^i] + \frac{\mathbf{b}_k \cdot \mathbf{b}_\ell}{\mathbf{b}_k \cdot \mathbf{b}_k}(s_k^{-i} - E[s_k^{-i}|\mathbf{q}_0^i]).$$

Substituting $E[s_\ell^{-i}|s_k^{-i}, \mathbf{q}_0^i]$ into Eq. (156) characterizes trader i's demand coefficients $\{a_k^i, \mathbf{b}_k^i, c_k^i\}_k$ as functions of $\{a_k^j, \mathbf{b}_k^j, c_k^j\}_{k,j\neq i}$ and price impacts $\{\lambda_k^i\}_k$. This defines a fixed point for $\{a_k^i, \mathbf{b}_k^i, \mathbf{c}_k^i\}_{i,k}$ as a function of $\{\lambda_k^i\}_{i,k}$.

Step 2.2 (Fixed point for best response coefficients, given price impacts) By the symmetry across traders and assets, the fixed point for demand coefficients of trader i simplifies to:

$$a_k^i = c_k\delta_k - c_k(1-\sigma_0)\alpha\big((x(\alpha-(I-2)\lambda)+y\rho)E[\overline{q}_{0,k}] + (y(\alpha-(I-2)\lambda)+x\rho)E[\overline{q}_{0,l}]\big), \tag{158}$$

$$c_1 = c_2 = \big((\alpha+\lambda) + \alpha\rho\frac{\mathbf{b}_1\cdot\mathbf{b}_2}{\mathbf{b}_1\cdot\mathbf{b}_1}\big)^{-1}, \tag{159}$$

$$\mathbf{b}_1 = (x, y), \quad \mathbf{b}_2 = (y, x), \tag{160}$$

where $x \equiv (1-\sigma_0)(1-\rho^2)\alpha + (1+(I-2)\sigma_0)\lambda$ and $y \equiv \rho(1+(I-2)\sigma_0)\lambda$. The demand coefficients $a_k^i, \mathbf{b}_k$, and c_k are closed-form functions of λ (158)-(160).

Step 2.3 (Correct price impact) Price impact must equal the slope of the inverse residual supply, $\lambda_k = (\sum_{j\neq i}\frac{\partial q_k^j(\cdot)}{\partial p_k})^{-1} = \frac{1}{I-1}c_k^{-1}$ for all k. By Eqs. (159)-(160), the price impact $\lambda = \frac{1}{I-1}c_1^{-1} = \frac{1}{I-1}c_2^{-1}$ is characterized by:

$$\lambda = \frac{\alpha}{I-2} + \frac{\alpha\rho}{I-2}\frac{2xy}{x^2+y^2}. \tag{161}$$

Eq. (161) characterizes the equilibrium price impact, which in turn determines the demand coefficients in Eqs. (158)-(160).