An algorithm to solve optimal stopping problems for one-dimensional diffusions

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October 8, 2019

Abstract

Considering a real-valued diffusion, a real-valued reward function and a positive discount rate, we provide an algorithm to solve the optimal stopping problem consisting in finding the optimal expected discounted reward and the optimal stopping time at which it is attained. Our approach is based on Dynkin's characterization of the value function. The combination of Riesz's representation of α-excessive functions and the inversion formula gives the density of the representing measure, being only necessary to determine its support. This last task is accomplished through an algorithm. The proposed method always arrives to the solution, thus no verification is needed, giving, in particular, the shape of the stopping region. Generalizations to diffusions with atoms in the speed measure and to non smooth payoffs are analyzed.

Mathematics Subject Classification (2010): 60G40, 60J60.

Keywords: Optimal stopping; diffusions; excessive representation.

1 Introduction

Given a diffusion $X = \{X_t : t \geq 0\}$ taking values in an interval $\mathcal{I} \subset \mathbb{R}$, a non-negative continuous reward function $g : \mathcal{I} \to \mathbb{R}$, and a discount factor $\alpha > 0$, consider the optimal stopping problem consisting in finding the *value function* $V(x)$ and the *optimal stopping rule* τ^*, such that

$$V(x) = \mathbb{E}_x \left(e^{-\alpha\tau^*} g(X_{\tau^*})\right) = \sup_{\tau \in \mathbf{T}} \mathbb{E}_x \left(e^{-\alpha\tau} g(X_\tau)\right). \tag{1}$$

Here $\mathbf{T}$ is the class of stopping times and we consider $g(X_\tau) = 0$ if $\tau = \infty$ (see section 2 for definitions). Optimal stopping for real-valued diffusions is a

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well established and rich area of research. It can be inscribed into the class of *markovian* stopping problems, existing many different approaches to solve them. One of the most popular ones, the *free boundary* approach (see Peskir and Shiryaev [2006] with the historical comments in pp. 50-52 and the corresponding references), when applicable, is very effective and consists in two steps: the solution of a free boundary differential equation to find a candidate solution; and the verification (usually through stochastic calculus) that the candidate is the true solution. The celebrated *smooth fit condition* becomes a key tool in this framework. A second approach we mention is Dynkin's characterization of the value function [Dynkin, 1963]. It was used for instance by Taylor [1968], and has a variation proposed in Dayanik and Karatzas [2003] (where concavity is used instead of excesiveness). Other several different approaches can be found in Chapter IV of Peskir and Shiryaev [2006].

Under the conditions assumed in this paper (see section 5), the optimal stopping rule exists and has the form

$$\tau^* = \inf\{t \geq 0\colon X_t \in \mathcal{S}\}, \tag{2}$$

where the *stopping region* is the closed set

$$\mathcal{S} = \{x \in \mathcal{I}\colon V(x) = g(x)\}. \tag{3}$$

The *continuation region* is $\mathcal{C} = \mathcal{I} \setminus \mathcal{S}$. A key role in our proposal is played by the *negative set*

$$\mathcal{N} = \{x \in \mathcal{I}\colon (\alpha - \mathcal{L})g(x) < 0\}.$$

where $\mathcal{L}$ is the infinitesimal generator of X.

Regarding applications, it must be noticed that in most of the problems where a solution can be found, the continuation region $\mathcal{C}$ is either a half-line, giving one-sided solutions, or a finite interval, giving rise to a two-threshold policy (or a two-sided solution). The first situation appears typically in perpetual American options (see for instance Mc Kean [1965] and Merton [1973]), also in the problem of disruption for a Wiener process (see section 4.4 in Shiryaev [2008]). The second one appears in the case of sequential testing of two simple hypotheses of the mean of a Wiener process (see section 4.2 in Shiryaev [2008]), in the quickest detection problem [Shiryaev, 2010], and also in the pricing of a perpetual straddle or strangle option [Gerber and Shiu, 1994]. The analysis of continuation intervals appears in Alvarez [2001] under restrictions on the shape of the continuation region. More recent references on one-sided and two-sided solutions are for instance Rüschendorf and Urusov [2008] and Lempa [2010]. Lamberton and Zervos [2013] obtain verification results in a framework of weak solutions of SDE with measurable coefficients and a state dependent discount.

As we mentioned above, the continuation region of solvable problems are usually half lines or intervals. More important, the shape of this set should be known in advance in order to solve the problem. Solved cases with different continuation regions are seldom treated in the literature, as in order to apply the smooth pasting condition, one has to guess first the structure of this set.

Furthermore, and perhaps more relevant to our discussion, examples are usually solved based on verification results (see the discussion in the Introduction in Lamberton and Zervos [2013] with the references therein).

The method that we propose in the present paper consists in the following steps:

(1) Apply Dynkin's characterization to obtain that the value function is the minimal excessive function that is a majorant of the reward.

(2) Represent this excessive function as an integral of the Green kernel of the process w.r.t. a representing measure.

(3) Identify the support of this representing measure as the stopping region $\mathcal{S}$, based on the fact that the value function is harmonic on the continuation region.

(4) Identify the density (w.r.t. the speed measure) of the representing measure through the inversion formula.

(5) Determine the support $\mathcal{S}$ of this measure through an algorithm, constructing $\mathcal{C} = \mathcal{I} \setminus \mathcal{S}$ as an enlargement of the set $\mathcal{N}$.

The proposed method departs from the scale function and speed measure (that determine the generator $\mathcal{L}$) and the increasing and decreasing solutions of the equation $\alpha u = \mathcal{L}u$ (that determine the Green kernel), and gives the complete solution of the problem without need of further verification. The main restriction of the method is that the negative set should be a finite union of intervals, i.e. $\mathcal{N} = \bigcup_{i=1}^{n} N_i$. It is important to note that $\mathcal{N}$ is directly computed from the data of the problem. The steps of the algorithm to construct the set $\mathcal{C}$ are the following:

1. (enlargment) for each N_i construct the largest possible interval $C_i \supset N_i$ contained in the continuation region (see Condition 2.3);

2. if C_i are pairwise disjoint intervals then $\mathcal{C} = \bigcup_i C_i$;

3. else (merge), denote by N_i each connected component of $\bigcup_i C_i$ and return to step 1. (Observe that the number of intervals strictly decreases.)

As a consequence, the algorithm's output are the connected components of the continuation region (whose number is smaller or equal than n) determining if the problem is one-sided, two-sided, or other (i.e. the continuation region is the union of several intervals). The value function is then written as an integral of the Green kernel w.r.t. the just obtained measure, and this integral gives the classical form of value function as a linear combination of the fundamental solutions in each continuation interval.

The representation of excessive functions in optimal stopping of diffusions was initiated by Salminen [1985], who represents the value function in terms of the Martin kernel. Afterwards, Mordecki and Salminen [2007] use the Green kernel for optimal stopping of Hunt and Lévy processes. The identification of the representing measure through the inversion formula was obtained in Crocce [2014], and appears in Crocce and Mordecki [2012] for one-sided problems, and in Christensen et al. [2019] for multidimensional diffusions. It can be traced back to formula (8.30) in Dynkin [1969], for the cases when the limit therein can be interchanged with the integral. More recently, also based in representation methods, disconnected stopping regions where obtained for optimal stopping problems for diffusions with discontinuous coefficients in Mordecki and Salminen [2019a,b].

The rest of the paper is as follows. In section 2 we introduce the necessary definitions and the main result, in section 3 we discuss possible generalizations in two separate directions: diffusions with atoms in the speed measure, and non-smooth (but still continuous) rewards. Section 4 presents the implementation of the algorithm and contains two examples, and section 5 contains the proof of the main result.

2 Main result

Consider a conservative and regular one-dimensional diffusion $X = \{X_t : t \geq 0\}$, in the sense of Itô and McKean Jr. [1974] (see also Borodin and Salminen [2002]). The state space of X is denoted by $\mathcal{I}$, an interval of the real line $\mathbb{R}$ with left endpoint $\ell = \inf \mathcal{I}$ and right endpoint $r = \sup \mathcal{I}$, where $-\infty \leq \ell < r \leq \infty$. The boundaries can be of any kind but killing. Denote by $\mathbb{P}_x$ the probability measure associated with X when starting from x, and by $\mathbb{E}_x$ the corresponding mathematical expectation. The set of stopping times $\mathbf{T}$ is considered with respect to the usual augmentation of the natural filtration generated by X (see I.14 in Borodin and Salminen [2002]).

Denote by $\mathcal{L}$ the *infinitesimal generator* of the diffusion X, and by $\mathcal{D}(\mathcal{L})$ its domain. For any stopping time τ and for any $f \in \mathcal{D}(\mathcal{L})$ the following discounted version of the Dynkin's formula holds:

$$f(x) = \mathbb{E}_x \left(\int_0^\tau e^{-\alpha t} (\alpha - \mathcal{L}) f(X_t) dt \right) + \mathbb{E}_x (e^{-\alpha\tau} f(X_\tau)). \tag{4}$$

The *resolvent* of the process X is the operator R_α defined by

$$R_\alpha u(x) = \int_0^\infty e^{-\alpha t}\, \mathbb{E}_x\, u(X_t) dt,$$

applied to a function $u \in \mathscr{C}_b(\mathcal{I}) = \{u : \mathcal{I} \to \mathbb{R}, u \text{ is continuous and bounded}\}$. The image of the operator R_α is independent of $\alpha > 0$ and coincides with the domain of the infinitesimal generator $\mathcal{D}(\mathcal{L})$. Moreover, for any $f \in \mathcal{D}(\mathcal{L})$, $R_\alpha(\alpha - \mathcal{L}) f = f$, and for any $u \in \mathscr{C}_b(\mathcal{I})$, $(\alpha - \mathcal{L}) R_\alpha u = u$. In other terms, R_α and $\alpha - \mathcal{L}$ are inverse operators (see Prop. VII.1.4 in Revuz and Yor [1999]). Denoting by s and m the scale function and the speed measure of the diffusion X respectively, we have that, for any $f \in \mathcal{D}(\mathcal{L})$, the lateral derivatives with respect to the scale function exist for every $x \in (\ell, r)$. Furthermore, they satisfy

$$\frac{\partial^+ f}{\partial s}(x) - \frac{\partial^- f}{\partial s}(x) = m(\{x\})\mathcal{L}f(x), \tag{5}$$

and the following identity holds for $z > y$:

$$\frac{\partial^+ f}{\partial s}(z) - \frac{\partial^+ f}{\partial s}(y) = \int_{(y,z]} \mathcal{L}f(x)m(dx).$$

This last formula allows to compute the infinitesimal generator of f at $x \in (\ell, r)$ by the Feller's differential operator [Feller, 1957]

$$\mathcal{L}f(x) = \frac{\partial}{\partial m}\frac{\partial^+}{\partial s}f(x). \tag{6}$$

Given a function $u\colon \mathcal{I} \to \mathbb{R}$, and $x \in (\ell, r)$ we give to $\mathcal{L}u(x)$ the meaning given in (6) if it makes sense. We also define $\mathcal{L}u(\ell) = \lim_{x \to \ell^+} \mathcal{L}u(x)$, if the limit exists. There exist two continuous functions $\varphi_\alpha\colon \mathcal{I} \mapsto \mathbb{R}^+$ decreasing, and $\psi_\alpha\colon \mathcal{I} \mapsto \mathbb{R}^+$ increasing, solutions of $\alpha u = \mathcal{L}u$, such that any other continuous function u is a solution of the differential equation if and only if $u = a\varphi_\alpha + b\psi_\alpha$, with a, b in $\mathbb{R}$. Denoting by $\hbar_z = \inf\{t\colon X_t = z\}$ the hitting time of level $z \in \mathcal{I}$, we have

$$\mathbb{E}_x(e^{-\alpha \hbar_z}) = \begin{cases} \frac{\psi_\alpha(x)}{\psi_\alpha(z)}, & x \leq z, \\ \frac{\varphi_\alpha(x)}{\varphi_\alpha(z)}, & x \geq z. \end{cases} \tag{7}$$

The functions φ_α and ψ_α, though not necessarily in $\mathcal{D}(\mathcal{L})$, also satisfy (5) for all $x \in (\ell, r)$, so that in case $m(\{x\}) = 0$, the derivative at x of both functions with respect to the scale exists. The α-*Green function* of X is defined by

$$G_\alpha(x, y) = \int_0^\infty e^{-\alpha t} p(t; x, y) dt,$$

where $p(t; x, y)$ is the transition density of the diffusion with respect to the speed measure $m(dx)$ (this density always exists, see Borodin and Salminen [2002]). The Green function may be expressed in terms of φ_α and ψ_α as follows:

$$G_\alpha(x, y) = \begin{cases} w_\alpha^{-1}\psi_\alpha(x)\varphi_\alpha(y), & x \leq y, \\ w_\alpha^{-1}\psi_\alpha(y)\varphi_\alpha(x), & x \geq y, \end{cases} \tag{8}$$

where w_α, the *Wronskian*, given by

$$w_\alpha = \frac{\partial \psi_\alpha^+}{\partial s}(x)\varphi_\alpha(x) - \psi_\alpha(x)\frac{\partial \varphi_\alpha^+}{\partial s}(x),$$

is positive and independent of x (Borodin and Salminen [2002]). For general reference on diffusions and Markov processes see Borodin and Salminen [2002], Itô and McKean Jr. [1974], Revuz and Yor [1999], Dynkin [1965], Karatzas and Shreve [1991].

A non-negative Borel function $u\colon \mathcal{I} \to \mathbb{R}$ is called α-*excessive* for the process X if $e^{-\alpha t}\,\mathbb{E}_x(u(X_t)) \leq u(x)$ for all $x \in \mathcal{I}$ and $t \geq 0$, and $\lim_{t\to 0}\mathbb{E}_x(u(X_t)) = u(x)$ for all $x \in \mathcal{I}$. A 0-excessive function is said to be *excessive*. Dynkin's characterization [Dynkin, 1963] states that, if the reward function is lower semi-continuous, V is the value function of the non-discounted optimal stopping problem with reward g if and only if V is the least excessive function such that $V(x) \geq g(x)$ for all $x \in \mathcal{I}$. Applying this result to a killed process [Crocce and Mordecki, 2012], we obtain that V, the value function of the problem (1), is characterized as the least α-excessive majorant of g .

A key feature of our proposal is the representation of excessive functions as integrals of the Green kernel. The Riesz's representation of an α-excessive function states that a function $u\colon \mathcal{I} \to \mathbb{R}$ is α-excessive if and only if there exist a non-negative Radon measure μ on $[\ell, r]$ such that

$$u(x) = \int_{(\ell,r)} G_\alpha(x,y)\mu(dy) + \mu(\{\ell\})\varphi_\alpha(x) + \mu(\{r\})\psi_\alpha(x). \tag{9}$$

Furthermore, the previous representation is unique. The measure μ is called the representing measure of u. Formula (9) is obtained from II.29 in Borodin and Salminen [2002].

We next formulate our main result in a smooth framework: the value function g satisfies the inversion formula (in particular $\mathcal{L}g(x)$ must be defined for all $x \in \mathcal{I}$):

$$g(x) = \int_{\mathcal{I}} G_\alpha(x,y)(\alpha - \mathcal{L})g(y)m(dy); \tag{10}$$

and the speed measure has no atoms. The proof of this result is deferred to section 5. A discussion of possible generalizations in presented in section 3. Denote

$$\sigma(dy) = (\alpha - \mathcal{L})g(y)m(dy). \tag{11}$$

Theorem 2.1. *Consider a diffusion X whose speed measure has no atoms. Assume that the reward function g satisfies the inversion formula (10) and that the negative set is a finite union of $n \geq 1$ disjoint intervals, i.e.*

$$\mathcal{N} = \cup_{i=1}^n N_i.$$

Then, the value function of the OSP is

$$V(x) = \int_{\mathcal{S}} G_\alpha(x,y)\sigma(dy), \tag{12}$$

where the continuation region $\mathcal{C} = \mathcal{I} \setminus \mathcal{S}$ is a finite union of $1 \leq m \leq n$ disjoint intervals C_i, i.e.

$$\mathcal{C} = \cup_{i=1}^m C_i,$$

s.t. $\mathcal{N} \subset \mathcal{C}$, and

(a) *if $\ell < \inf C_i$, then $\int_{C_i} \varphi_\alpha(y)\sigma(dy) = 0$,*

(b) *if* $\sup C_i < r$*, then* $\int_{C_i} \psi_\alpha(y)\sigma(dy) = 0$*,*

(c) *for* $x \in C_i$*,* $\int_{C_i} G_\alpha(x,y)\sigma(dy) \leq 0$*.*

Furthermore, the continuation region $\mathcal{C}$ *can be found by Algorithm 2.1, to be presented further on.*

Remark 2.2. If $x \in C_i = (\ell_i, r_i)$, according to (12), we have

$$
\begin{aligned}
V(x) &= \int_{\mathcal{I}\setminus\mathcal{C}} G_\alpha(x,y)\sigma(dy) \\
&= \int_{(\mathcal{I}\setminus\mathcal{C})\cap\{x<\ell_i\}} w_\alpha^{-1}\psi_\alpha(y)\varphi_\alpha(x)\sigma(dy) + \int_{(\mathcal{I}\setminus\mathcal{C})\cap\{x>r_i\}} w_\alpha^{-1}\psi_\alpha(x)\varphi_\alpha(y)\sigma(dy) \\
&= k_1^i\varphi_\alpha(x) + k_2^i\psi_\alpha(x).
\end{aligned}
$$

Applying the representation Lemma 5.2 we know that $V(\ell_i) = g(\ell_i)$ and $V(r_i) = g(r_i)$, obtaining

$$k_1^i = \frac{g(r_i)\psi_\alpha(\ell_i) - g(\ell_i)\psi_\alpha(r_i)}{\psi_\alpha(\ell_i)\varphi_\alpha(r_i) - \psi_\alpha(r_i)\varphi_\alpha(\ell_i)}, \tag{13}$$

and

$$k_2^i = \frac{g(\ell_i)\varphi_\alpha(r_i) - g(r_i)\varphi_\alpha(\ell_i)}{\psi_\alpha(\ell_i)\varphi_\alpha(r_i) - \psi_\alpha(r_i)\varphi_\alpha(\ell_i)}. \tag{14}$$

In the particular case in which $\ell_i = \ell$ we have $k_1^i = 0$ and $k_2^i = g(r_i)/\psi_\alpha(r_i)$, and if $r_i = r$ then $k_1 = g(\ell_i)/\varphi_\alpha(\ell_i)$ and $k_2 = 0$. We have then the classical alternative formula

$$V(x) = \begin{cases} g(x), & \text{for } x \notin \mathcal{C}, \\ k_1^i\varphi_\alpha(x) + k_2^i\psi_\alpha(x), & \text{for } x \in C_i \colon i = 1\dots m. \end{cases} \tag{15}$$

The coefficients in (13) and (14) appeared (in a slightly different form) in Alvarez [2001], and also in Lempa [2010], and Lamberton and Zervos [2013].

As we mentioned above, the continuation region is constructed as an enlargement of the negative set, and this is done by enlarging each of the intervals N_i of $\mathcal{N}$. Introduce $\mathcal{P} = \mathcal{I} \setminus \mathcal{N}$ the positive part of the support of σ, and denote by $\sigma^+(dx)$ the measure

$$\sigma^+(dx) := \sigma(dx)\mathbb{1}_{\mathcal{P}}(x),$$

where σ is given in (11), and, for an arbitrary interval $D \subset \mathcal{I}$ define the signed measure σ_D by

$$\sigma_D(dx) = \mathbb{1}_D(x)\sigma(dx) + \sigma^+(dx)\mathbb{1}_{\mathcal{I}\setminus D}. \tag{16}$$

Observe that σ_D is a positive measure outside D, equal to σ in D.

The following statement specifies what are the conditions that the enlarged interval C should satisfy.

Condition 2.3. We say that the pair of intervals (N, C): $N \subseteq C \subseteq \mathcal{I}$ satisfy Condition 2.3 if the following assertions hold:

(i) both, $\int_N \varphi_\alpha(x)\sigma(dx) \leq 0$ and $\int_N \psi_\alpha(x)\sigma(dx) \leq 0$;

(ii) if $\ell < \inf C$, then $\int_C \varphi_\alpha(x)\sigma_N(dx) = 0$;

(iii) if $\sup C < r$, then $\int_C \psi_\alpha(x)\sigma_N(dx) = 0$;

(iv) for every $x \in C$, $\int_C G_\alpha(x,y)\sigma_N(dy) \leq 0$.

The algorithm to construct the continuation region follows.

Algorithm 2.1. (Starting from a subset of the continuation region, in subsequent steps, increase the considered subset until finding the actual continuation region.)

BS. *(base step)* Consider disjoint intervals $N_1, \dots, N_n \subseteq \mathcal{I}$ such that

$$\mathcal{N} = \{x \in \mathcal{I} : (\alpha - \mathcal{L})g(x) < 0\} = \bigcup_{i=1}^{n} N_i.$$

Consider for each i, the interval C_i such that (N_i, C_i) satisfies Condition 2.3 (this can be done in virtue of Lemma 5.3). Define

$$\Theta = \{(N_i, C_i) : i = 1 \dots n\},$$

and go to the iterative step (IS) with Θ.

IS. *(iterative step)* At this step we assume given a set Θ of pair of intervals satisfying Condition 2.3. We assume the notation[1]

$$\Theta = \{(N_i = (a_i, b_i), C_i = (\bar{a}_i, \bar{b}_i)) \colon i = 1 \dots n\},$$

with $a_i < a_j$ if $i < j$ (the intervals N_i are ordered) and $b_i < a_{i+1}$ (the intervals N_i are disjoint)

– If for some j, $C_j = \mathcal{I}$, the algorithm is finished and the continuation region is $\mathcal{I}$.

– Else, if the intervals C_i are pairwise disjoint, the algorithm is finished and the continuation region is

$$\mathcal{C} = \bigcup_{i=1}^{n} C_i$$

– Else, if $\bar{a}_j = \ell$ for some $j > 1$, add to Θ the pair $(N = (\ell, b_j), C)$ satisfying Condition 2.3, and remove from Θ the pairs (N_i, C_i) for $i = 1 \dots j$. Observe that the existence of C is proved in Lemma 5.4. Return to the iterative step (IS).

[1] we remark that at different moments the algorithm execute this step, the notation refers to different objects, e.g. the set Θ is not always the same set.

– Else, if $\bar{b}_j = r$ for some $j < n$, add to Θ the pair $(N = (a_j, r), C)$ satisfying Condition 2.3, and remove from Θ the pairs (N_i, C_i) for $i = j \dots n$ (observe that the existence of C is proved in Lemma 5.5). Return to the iterative step (IS).

– Else, if for some j, $C_j \cap C_{j+1} \neq \emptyset$, remove from Θ the pairs j and $j+1$, and add to Θ the pair $(N = (a_j, b_{j+1}), C)$ satisfying Condition 2.3 (its existence is guaranteed, depending on the situation by Lemmas 5.6, 5.7, 5.8 or 5.9). Return to the iterative step (IS).

Finally note that, each time when we return to the iterative step the number of pairs of intervals in Θ decreases, the algorithm performs at maximum n steps.

3 Generalizations

3.1 Diffusions with atoms in the speed measure

The absence of atoms of the speed measure was required only for simplicity of exposition. A modification of the main result can be formulated also when the speed measure has a finite number of atoms. The main difference is that the functions

$$z \mapsto \int_{(z,b)} \varphi_\alpha(x)\sigma_N(dx), \quad z \mapsto \int_{(a,z)} \psi_\alpha(x)\sigma_N(dx),$$

in the proof of Lemma 5.3 can be discontinuous, having finite jumps at the atoms. Then, if one of the extremes of an interval happens to be an atom, in order to verify (ii) and (iii) in Condition 2.3, the representing measure should contain part of the mass of the atom, and the smooth fitting does not hold. This situation, with the presentation of corresponding examples, was examined in Crocce and Mordecki [2012].

3.2 More general reward functions

In many situations the reward function g is not regular enough to satisfy the inversion formula (10). Assume then that there exists a measure ν such that

$$g(x) = \int_{\mathcal{I}} G_\alpha(x, y)\nu(dy), \tag{17}$$

where g is non-negative and continuous, and $G_\alpha(x, y)$ is defined by (8). In these cases, considering the second derivative of the difference of two convex functions as a signed measure, it is possible to obtain a "generalized" inversion formula useful for our needs (see Dudley [2002] Problems 11 and 12 of Section 6.3).

Just to consider a simple example, assume that X is a standard Brownian motion, and consider the function $g\colon \mathbb{R} \to \mathbb{R}$ given by

$$g(x) := \begin{cases} 0, & x \le 0, \\ x, & 0 < x < 1, \\ 2 - x, & 1 \le x \le 2, \\ x - 2, & x > 2. \end{cases}$$

In this case, the differential operator is $\mathcal{L}f = \frac{1}{2}f''$ when f is in $\mathcal{D}(\mathcal{L})$. The inversion formula (10) would be

$$g(x) = \int_{\mathbb{R}} G_\alpha(x, y)(\alpha - \mathcal{L})g(y)m(dy)$$

where $m(dy) = 2dy$, so the candidate to be ν is $(\alpha - \mathcal{L})g(y)2dy$. The derivatives of g, in the general sense, would be

$$g'(x) = \begin{cases} 1, & x < 1 \\ -1, & 1 < x < 2 \\ 1 & x > 2 \end{cases}$$

and the second generalized derivative is the measure $-2\delta_1(dx) + 2\delta_2(dx)$ (where $\delta_a(dx)$ denotes the Dirac's delta measure at the point $x = a$). This lead us to consider

$$\nu(dy) = 2\alpha g(y)dy + 2\delta_1(dy) - 2\delta_2(dy). \tag{18}$$

The corresponding computations show that (17) holds with ν in (18).

Theorem 3.1. *Consider a one-dimensional diffusion X and a non-negative and continuous reward function $g\colon \mathcal{I} \to \mathbb{R}$ such that (17) holds, with ν a signed measure on $\mathcal{I}$. Suppose that $C_i\colon i = 1, \dots, m$ (m could be ∞) are pairwise disjoint subintervals of $\mathcal{I}$, such that*

(a) $\int_{C_i} \varphi_\alpha(y)\nu(dy) = 0$ *if there is some $x \in \mathcal{I}$ such that $x < y$ for all $y \in C_i$,*

(b) $\int_{C_i} \psi_\alpha(y)\nu(dy) = 0$ *if there is some $x \in \mathcal{I}$ such that $x > y$ for all $y \in C_i$.*

Define $\mathcal{S}$ by

$$\mathcal{S} = \mathcal{I} \setminus \cup_{i=1}^n C_i.$$

and $V\colon \mathcal{I} \to \mathbb{R}$ by

$$V(x) = \int_{\mathcal{S}} G_\alpha(x, y)\nu(dy).$$

If $\nu(dy) \ge 0$ in $\mathcal{S}$, and $V \ge g$ in $\mathcal{C} = \cup_{i=1}^m C_i$, then V is the value function associated with the OSP, and $\mathcal{S}$ is the stopping region.

Remark 3.2. With the same arguments given in Remark 2.2 we obtain the alternative representation for V, given in (15).

Proof. Based on Theorem 3.3.1 in Shiryaev [2008] we know that g satisfies Dynkin's characterization. The strategy for the proof is then to verify that V is the minimal α-excessive function that dominates the reward function g. By the definition of V, and taking into account that ν is a non-negative measure in $\mathcal{S}$, we conclude that V is an α-excessive function. Applying Lemma 5.2 with $W_\alpha := V$, we conclude that $V(x)$ and $g(x)$ are equal for $x \in \mathcal{S}$, which in addition to the hypothesis $V(x) \geq g(x)$ for all $x \in \mathcal{S}^c$ allow us to conclude that V is a majorant of the reward. So far, we know

$$\sup_\tau \mathbb{E}_x \left(e^{-\alpha\tau} g(X_\tau)\right) \leq V(x).$$

From Lemma 5.2 –in the first equality– we get

$$V(x) = \mathbb{E}_x \left(e^{-\alpha \hbar_{\mathcal{S}}} g(X_{\hbar_{\mathcal{S}}})\right) \leq \sup_\tau \mathbb{E}_x \left(e^{-\alpha\tau} g(X_\tau)\right),$$

that proves the other inequality holds as well. From the previous equation we also conclude that $\mathcal{S}$ is the stopping region. □

Comparing Theorem 2.1 and Theorem 3.1, it should be emphasized that the former gives a characterization of the solution and a method to find it, while the latter is just a verification theorem, which of course, also suggests a method to find the solution. However, Theorem 3.1 has less restrictive hypothesis and, although we do not include it here, an algorithm to find the continuation region may be developed, at least when the region in which the measure ν is negative, is a finite union of intervals; in fact, Algorithm 2.1 would be a particular case of this algorithm when considering $\nu(dy) = (\alpha - \mathcal{L})g(y)m(dy)$.

4 Examples

In order to apply Theorem 2.1 it is necessary to check the inversion formula. This requires essentially two conditions: enough smoothness and a proper behavior at infinity. A reasonable behavior at infinite is the following:

$$\lim_{z\uparrow r} \frac{g(z)}{\psi_\alpha(z)} = \lim_{z\downarrow \ell} \frac{g(z)}{\varphi_\alpha(z)} = 0, \tag{19}$$

(For other behaviors see Theorem 6.3 in Lamberton and Zervos [2013].) These conditions are useful to verify the inversion formula (10) for a smooth function, as stated in the following result.

Proposition 4.1. *Assume that $\mathcal{I} = (\ell, r)$, that for $g\colon \mathcal{I} \to \mathbb{R}$ the differential operator is defined for all $x \in \mathcal{I}$, and that*

$$\int_{\mathcal{I}} G_\alpha(x,y)|(\alpha - \mathcal{L})g(y)|m(dy) < \infty. \tag{20}$$

Take sequences $\ell_n \downarrow \ell$ and $r_n \uparrow r$ s.t. for each n there exists a function $g_n \in \mathcal{D}(\mathcal{L})$ such that $g_n(x) = g(x)$ for all $x \in (\ell_{n+1}, r_{n+1})$. Then, if (19) holds, the inversion formula (10) holds true.

Proof. Under the condition (20), an application of Fubini's Theorem gives

$$\int_{\mathcal{I}} G_\alpha(x,y)(\alpha-\mathcal{L})g(y)m(dy) = R_\alpha(\alpha-\mathcal{L})g(x).$$

Let τ_n be the hitting time of the set $\mathcal{I}\setminus(\ell_n,r_n)$, defined by

$$\tau_n := \inf\{t\geq 0\colon X_t\notin(\ell_n,r_n)\}.$$

Consider $x\in(r_n,\ell_n)$. We have $\tau_n=\inf\{\hbar_{r_n},\hbar_{\ell_n}\}$. By the continuity of the paths it can be concluded that $\tau_n\to\infty$, $(n\to\infty)$. Applying Dynkin's formula (4) to g_n and τ_n we obtain

$$g_n(x)=\mathbb{E}_x\left(\int_0^{\tau_n} e^{-\alpha t}(\alpha-\mathcal{L})g_n(X_t)dt\right)+\mathbb{E}_x\left(e^{-\alpha\tau_n}g_n(X_{\tau_n})\right),$$

taking into account that $g_n(x)=g(x)$ and $(\alpha-\mathcal{L})g(x)=(\alpha-\mathcal{L})g_n(x)$ for $\ell_{n+1}<x<r_{n+1}$, from the previous equality follows that

$$g(x)=\mathbb{E}_x\left(\int_0^{\tau_n} e^{-\alpha t}(\alpha-\mathcal{L})g(X_t)dt\right)+\mathbb{E}_x\left(e^{-\alpha\tau_n}g(X_{\tau_n})\right). \tag{21}$$

About the second term on the right-hand side of the previous equation we have

$$\begin{aligned}\mathbb{E}_x\left(e^{-\alpha\tau_n}g(X_{\tau_n})\right) &= \mathbb{E}_x\left(e^{-\alpha\hbar_{r_n}}g(X_{\hbar_{r_n}})\mathbb{1}_{\{\hbar_{r_n}<\hbar_{\ell_n}\}}\right)\\ &\quad+\mathbb{E}_x\left(e^{-\alpha\hbar_{\ell_n}}g(X_{\hbar_{\ell_n}})\mathbb{1}_{\{\hbar_{\ell_n}<\hbar_{r_n}\}}\right)\\ &\leq \mathbb{E}_x\left(e^{-\alpha\hbar_{r_n}}g(X_{\hbar_{r_n}})\right)+\mathbb{E}_x\left(e^{-\alpha\hbar_{\ell_n}}g(X_{\hbar_{\ell_n}})\right)\\ &= \psi_\alpha(x)\frac{g(r_n)}{\psi_\alpha(r_n)}+\varphi_\alpha(x)\frac{g(\ell_n)}{\varphi_\alpha(\ell_n)},\end{aligned}$$

by (7), which taking the limit as $n\to\infty$ vanishes, by (19). Finally, we can apply Fubini's theorem, and dominated convergence theorem to conclude that the limit as $n\to\infty$ of the first term on the right-hand side of (21) is

$$\int_{\mathcal{I}} G_\alpha(x,y)(\alpha-\mathcal{L})g(y)m(dy),$$

thus completing the proof. □

4.1 Implementation

To compute in practice the optimal stopping region, following the Algorithm 2.1, it can be necessary a computational implementation of some parts of the algorithm. In fact, to solve our examples we have implemented a script in R [R Core Team, 2012] that receives as input:

- the function $(\alpha-\mathcal{L})g$;
- the density of the speed measure m;

- the atoms of the speed measure m;
- the functions φ_α and ψ_α;
- two numbers a, b that are interpreted as the left and right endpoint of an interval N

and produce as output two numbers $a' \leq a$, $b' \geq b$ such that $(N, (a', b'))$ satisfy Condition 2.3. It is assumed that the interval N given as input satisfies the necessary conditions to ensure the existence of N'. To compute a' and b' we use a discretization of the given functions and compute the corresponding integrals numerically. We follow the iterative procedure presented in the proof of Lemma 5.3. Using this script the examples are easily solved following Algorithm 2.1.

4.2 Brownian motion with polynomial reward

Theorem 2.1 is specially suited for non-monotone reward functions. In the following two examples we consider the same process and reward function with to different discount values: $\alpha = 2$ and $\alpha = 1.5$. It is known that the stopping region increases with the discount (Prop. 1 in Mordecki and Salminen [2019a]). More interesting, the algorithm 2.1 finds no intersection in the first case (so it is not necessary to go back to the iterative step) but finds an intersection in the second case (and goes back to the iterative step). As a result in the first case the continuation region has three components, and in the second case two. Furthermore, it is clear that for α small enough, the problem is one sided.

Example 4.2 (α=2)**.** Consider a standard Brownian motion X. Consider the reward function g defined by

$$g(x) := -(x-2)(x-1)x(x+1)(x+2),$$

and the discount factor $\alpha = 2$. To solve the optimal stopping problem (1), by the application of Algorithm 2.1, we start by finding the set $(\alpha - \mathcal{L})g(x) < 0$. As the infinitesimal generator is given by $\mathcal{L}g(x) = g''(x)/2$, after computations, we find that

$$\mathcal{N} = \{x\colon (\alpha - \mathcal{L})g(x) < 0\} = \bigcup_{i=1}^{3} N_i,$$

with $N_1 \simeq (-2.95, -1.15)$, $N_2 \simeq (0, 1.15)$ and $N_3 \simeq (2.95, \infty)$. Computing C_i, as is specified in the (base step) of the algorithm in the proof of Theorem 2.1, we find $C_1 \simeq (-3.23, -0.50)$, $C_2 \simeq (-0.36, 1.43)$ and $C_3 \simeq (1.78, \infty)$. Observing that these intervals are disjoint we conclude that the continuation region is given by $C_1 \cup C_2 \cup C_3$. Now, by the application of equation (15), we find the value function, which is shown in Fig. 1. Note that the smooth fit principle holds in the five contact point.

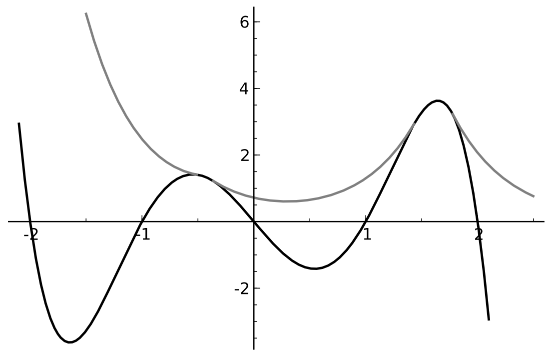

Figure 1: OSP for the standard BM and a 5th. degree polynomial: g (black), V (gray, when different from g). Parameter $\alpha = 2$.

Example 4.3 (α=1.5)**.** Consider the process and the reward as in the previous example but with a slightly smaller discount, $\alpha = 1.5$. We have again

$$\{x\colon (\alpha - \mathcal{L})g(x) < 0\} = \bigcup_{i=1}^{3} N_i,$$

but with $N_1 \simeq (-3.21, -1.17)$, $N_2 \simeq (0, 1.17)$ and $N_3 \simeq (3.21, \infty)$. Computing C_i we obtain $C_1 \simeq (-3.53, -0.31)$, $C_2 \simeq (-0.39, 1.46)$ and $C_3 \simeq (1.76, \infty)$. In this case $C_1 \cap C_2 \neq \emptyset$, therefore, according to the algorithm, we have to consider $N_1 \simeq (-3.21, 1.17)$, obtaining $C_1 \simeq (-3.53, 1.46)$. Now we have two disjoint intervals and the algorithm is completed. The continuation region, shown in Fig. 2, is

$$\mathcal{C} \simeq (-3.53, 1.46) \cup (1.76, \infty).$$

4.3 Example: A non-differentiable reward

Consider the OSP with reward $g\colon \mathbb{R} \to \mathbb{R}$ given by

$$g(x) = \begin{cases} x, & x < 1, \\ -x + 2, & 1 \leq x \leq 2, \\ x - 2 & x > 2. \end{cases}$$

This is the function already presented above. It satisfies (17) with ν given by (18). Consider the discount factor $\alpha = 1$. The measure ν is negative in $(-\infty, 0)$ and in $\{2\}$. Computing exactly in the first case, and by numerical approximation in the second (by following a variant of Algorithm 2.1), we manage to find two disjoint intervals $N_1 \simeq (-\infty, 1/\sqrt{2})$ and $N_2 \simeq (1.15, 2.85)$ that satisfy the

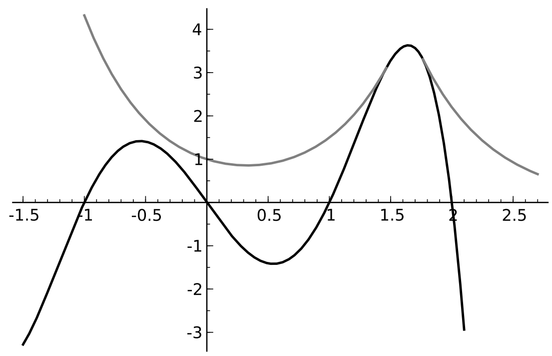

Figure 2: OSP for the standard BM and a 5th. degree polynomial: g (black), V (gray, when different from g). Parameter $\alpha = 1.5$.

conditions of Theorem 3.1. For V, we have the expression given in Remark 3.2, which considering $\psi_\alpha(x) = e^{\sqrt{2\alpha}x}$ and $\varphi_\alpha(x) = e^{-\sqrt{2\alpha}x}$ in the particular case $\alpha = 1$, renders[2]

$$V_1(x) = \begin{cases} k_2^1 e^{\sqrt{2}x}, & x < \frac{1}{\sqrt{2}}, \\ x, & \frac{1}{\sqrt{2}} \leq x \leq 1, \\ -x + 2, & 1 < x \leq 1.15, \\ k_1^2 e^{-\sqrt{2}x} + k_2^2 e^{\sqrt{2}x}, & 1.15 < x < 2.85, \\ x - 2, & x \geq 2.85; \end{cases}$$

with $k_2^1 = \frac{1}{e\sqrt{2}} \simeq 0.26$, $k_1^2 \simeq 3.96$ and $k_2^2 \simeq 0.013$. In Figure 3 we show the reward function g and the value function V_1.

5 Proof of the main result

To begin with the proof, we first observe that for a diffusions defined as in section 2, excessive functions are continuous (see 29 in Borodin and Salminen [2002]), and for the OSP in (1), Theorem 6 pp. 137 in Shiryaev [2008] is applicable, giving that the optimal stopping rule exists and has the form (2) with stopping set (3). In consequence, as both g and V are continuous functions, the set $\mathcal{S}$ is closed. We follow by presenting a few preliminary results.

Proposition 5.1 (Harmonicity)**.** (a) *Consider* $x \in [a, b] \subset \mathcal{I}$ *and*

$$\hbar_{ab} := \inf\{t\colon X_t = a\} \wedge \inf\{t\colon X_t = b\}.$$

Then, if $a > \ell$

$$\varphi_\alpha(x) = \mathbb{E}_x \left(e^{-\alpha \hbar_{ab}} \varphi_\alpha(X_{\hbar_{ab}})\right),$$

[2] we approximate the roots.

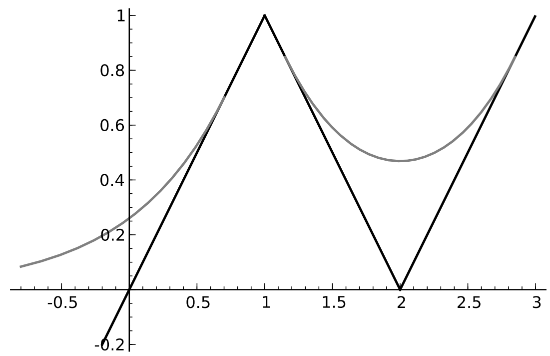

Figure 3: OSP for the standard BM and irregular reward: g (black), V_1 (gray, when different from g).

and, if $b < r$

$$\psi_\alpha(x) = \mathbb{E}_x \left(e^{-\alpha \hbar_{ab}} \psi_\alpha(X_{\hbar_{ab}}) \right).$$

(b) *Consider the function* $W_\alpha \colon \mathcal{I} \to \mathbb{R}$ *such that*

$$W_\alpha(x) = \int_S G_\alpha(x,y)\sigma(dy),$$

where σ *is a postive measure and the set* S *is*

$$S = \mathcal{I} \setminus \cup_{i=1}^n N_i,$$

where n *could be infinite, and* N_i *are disjoint intervals included in* $\mathcal{I}$*. Then, the function* W_α *satisfies*

$$W_\alpha(x) = \mathbb{E}_x \left(e^{-\alpha \hbar_S} W_\alpha(X_{\hbar_S}) \right).$$

Proof. (a) Let us proof the first statement, which is a direct consequence of the discounted Dynkin's formula for functions that belong to $\mathcal{D}(\mathcal{L})$. As $\varphi_\alpha \notin \mathcal{D}(\mathcal{L})$, we consider a function $h \in \mathscr{C}_b(\mathcal{I})$ such that $h(x) = 0$ for $x \geq a$ and $h(x) > 0$ for $x < a$. Then f defined by $f(x) := (R_\alpha h)(x)$ belongs to $\mathcal{D}(\mathcal{L})$ and there exist a constant $k > 0$ such that for $x \geq a$, $f(x) = k\varphi_\alpha(x)$ [see Itô and McKean Jr., 1974, section 4.6]. The discounted Dynkin's formula holds for f, so, for $x \geq a$,

$$f(x) - \mathbb{E}_x \left(e^{-\alpha \hbar_{ab}} f(X_{\hbar_{ab}}) \right) = \mathbb{E}_x \left(\int_0^{\hbar_{ab}} (\alpha - \mathcal{L}) f(X_t) dt \right).$$

From the continuity of the paths, for $t \in [0, \hbar_{ab}]$, $X_t \geq a$ and $(\alpha - \mathcal{L}) f(X_t) = h(X_t) = 0$, so the right-hand side of the previous equation vanishes. Finally taking into account the relation between f and φ_α the conclusion follows. The second statement is proved in an analogous way.

(b) If $x \in S$ the result is trivial, because $\hbar_S \equiv 0$. Let us consider the case $x \notin S$. In this case $x \in N_i$ for some i; we move on to prove that

$$G_\alpha(x,y) = \mathbb{E}_x \left(e^{-\alpha \hbar_S} G_\alpha(X_{\hbar_S}, y)\right)$$

for all y in S. To see this, let us denote by $a = \inf N_i$ and $b = \sup N_i$, and observe that $\hbar_S = \hbar_{ab}$. If $b < r$ and $y \geq b$ we have $G_\alpha(x,y) = w_\alpha^{-1}\psi_\alpha(x)\varphi_\alpha(y)$ and by (a) we get

$$\begin{aligned} G_\alpha(x,y) &= w_\alpha^{-1}\, \mathbb{E}_x \left(e^{-\alpha \hbar_{ab}} \psi_\alpha(X_{\hbar_{ab}})\right) \varphi_\alpha(y) \\ &= \mathbb{E}_x \left(e^{-\alpha \hbar_{ab}} G_\alpha(X_{\hbar_{ab}}, y)\right), \end{aligned}$$

where in the second equality we have used again (8) and the fact that $\hbar_{ab} \leq y$. In the case $y \leq a$ we have to do the analogous computation. Now we can write

$$\begin{aligned} W_\alpha(x) &= \int_S G_\alpha(x,y)\sigma(dy) = \int_S \mathbb{E}_x \left(e^{-\alpha \hbar_S} G_\alpha(X_{\hbar_S}, y)\right) \sigma(dy) \\ &= \mathbb{E}_x \left(e^{-\alpha \hbar_S} \int_S G_\alpha(X_{\hbar_S}, y)\sigma(dy)\right) = \mathbb{E}_x \left(e^{-\alpha \hbar_S} W_\alpha(X_{\hbar_S}, y)\right), \end{aligned}$$

and the result follows. ☐

Lemma 5.2 (Representation)**.** *Let X be a one-dimensional diffusion and consider the function $g\colon \mathcal{I} \to \mathbb{R}$ defined by*

$$g(x) := \int_{\mathcal{I}} G_\alpha(x,y)\sigma(dy),$$

where σ is a signed measure on $\mathcal{I}$. Consider the function $W_\alpha\colon \mathcal{I} \to \mathbb{R}$ defined by

$$W_\alpha(x) := \int_S G_\alpha(x,y)\sigma(dy), \tag{22}$$

where the set S is

$$S := \mathcal{I} \setminus \cup_{i=1}^m C_i,$$

where m could be infinite, the intervals $C_i \subset \mathcal{I}$ are pairwise disjoint, and

(a) $\int_{C_i} \varphi_\alpha(y)\sigma(dy) = 0$ *if there is some $x \in \mathcal{I}$ such that $x < y$ for all $y \in C_i$,*

(b) $\int_{C_i} \psi_\alpha(y)\sigma(dy) = 0$ *if there is some $x \in \mathcal{I}$ such that $x > y$ for all $y \in C_i$,*

Then $g(x) = W_\alpha(x)$ for all $x \in S$.

Proof. From the definitions of g and W_α we get

$$\begin{aligned} g(x) &= \int_{\mathcal{I}} G_\alpha(x,y)\sigma(dy) \\ &= W_\alpha(x) + \sum_{i=1}^m \int_{C_i} G_\alpha(x,y)\sigma(dy). \end{aligned}$$

To prove the result it is enough to verify that if $x \in S$, then

$$\int_{C_i} G_\alpha(x,y)\sigma(dy) = 0, \quad \text{for all } i.$$

Consider $x \in S$, then for any $i = 1 \dots n$, we have that $x \notin C_i$. Since C_i is an interval either $x < y$ for all y in C_i or $x > y$ for all y in C_i. Suppose the first case, from (8) we obtain

$$\int_{C_i} G_\alpha(x,y)\sigma(dy) = w_\alpha^{-1}\psi_\alpha(x)\int_{C_i} \varphi_\alpha(y)\sigma(dy) = 0,$$

where the second equality follows from hypothesis. The other case is analogous. □

5.1 Enlargment

Lemma 5.3 (Enlargment)**.** *Under the assumptions of Theorem 2.1, consider an interval $N \subseteq \mathcal{I}$, such that $\sigma(dx) < 0$ for $x \in N$. Then, there exists an interval C such that (N, C) satisfies Condition 2.3*

Proof. Consider N to be (a,b). Assertion (i) in Condition 2.3 is clearly fulfilled. Without loss of generality (denoting by φ_α the result of multiplying φ_α by the necessary positive constant) we may assume

$$\int_N \psi_\alpha(x)\sigma(dx) = \int_N \varphi_\alpha(x)\sigma(dx) < 0.$$

Under this assumption, $\varphi_\alpha(a) < \psi_\alpha(a)$ and $\varphi_\alpha(b) > \psi_\alpha(b)$. Consider

$$x_1 := \inf\left\{z \in [\ell, a] \colon \int_{(z,b)} \varphi_\alpha(x)\sigma_N(dx) < 0\right\}.$$

Since $\varphi_\alpha(x) > \psi_\alpha(x)$ for $x \leq a$ and $\sigma_N(dx)$ is non-negative in the same region we conclude that $\int_{(x_1,b)} \psi_\alpha(x)\sigma_N(dx) \leq 0$. Consider $y_1 > b$ defined by

$$y_1 := \sup\left\{z \in [b, r] \colon \int_{(x_1,z)} \psi_\alpha(x)\sigma_N(dx) < 0\right\}.$$

Now we consider $x_2 \geq x_1$ as

$$x_2 := \inf\left\{z \in [\ell, a] \colon \int_{(z,y_1)} \varphi_\alpha(x)\sigma_N(dx) < 0\right\}$$

and $y_2 \geq y_1$ as

$$y_2 := \sup\left\{z \in [b, r] \colon \int_{(x_2,z)} \psi_\alpha(x)\sigma_N(dx) < 0\right\}.$$

Following in the same way we obtain two non-decreasing sequences $\ell \leq \{x_n\} \leq a$ and $b \leq \{y_n\} \leq r$. By construction, the interval $C = (\lim x_n, \lim y_n)$ satisfies (ii) and (iii) in Condition 2.3. To prove (iv), first we find $k_1(x)$ and $k_2(x)$ such that

$$\begin{cases} k_1(x)\psi_\alpha(a) + k_2(x)\varphi_\alpha(a) = G_\alpha(x,a) \\ k_1(x)\psi_\alpha(b) + k_2(x)\varphi_\alpha(b) = G_\alpha(x,b). \end{cases}$$

Solving the system we obtain

$$k_1(x) = \frac{G_\alpha(x,b)\varphi_\alpha(a) - G_\alpha(x,a)\varphi_\alpha(b)}{\psi_\alpha(b)\varphi_\alpha(a) - \psi_\alpha(a)\varphi_\alpha(b)}$$

and

$$k_2(x) = \frac{G_\alpha(x,a)\psi_\alpha(b) - G_\alpha(x,b)\psi_\alpha(a)}{\psi_\alpha(b)\varphi_\alpha(a) - \psi_\alpha(a)\varphi_\alpha(b)}.$$

Let us see that $k_1(x), k_2(x) \geq 0$ for any $x \in C$: using the explicit formula for G_α it follows that

$$k_1(x) = \begin{cases} 0 & \text{for } x \leq a, \\ w_\alpha^{-1}\varphi_\alpha(b)\frac{\psi_\alpha(x)\varphi_\alpha(a) - \psi_\alpha(a)\varphi_\alpha(x)}{\psi_\alpha(b)\varphi_\alpha(a) - \psi_\alpha(a)\varphi_\alpha(b)} & \text{for } x \in (a,b), \\ w_\alpha^{-1}\psi_\alpha(x) & \text{for } x \geq b. \end{cases}$$

When $x \in (a,b)$ the numerator and denominator are non-negative because φ_α is decreasing and ψ_α increasing. The case of k_2 is analogous.

Considering $h(x,y) = k_1(x)\psi_\alpha(y) + k_2(x)\varphi_\alpha(y)$, it can be seen (discussing for the different positions of x and y with respect to a and b) that for all $x \in C$, $h(x,y) \leq G_\alpha(x,y)$ for $y \in (a,b)$ and $h(x,y) \geq G_\alpha(x,y)$ for $y \notin (a,b)$. From these inequalities we conclude that

$$\int_C G_\alpha(x,y)\sigma_N(dy) \leq \int_C h(x,y)\sigma_N(dy) \leq 0;$$

where the first inequality is consequence of $\sigma_N(dy) \geq 0$ in $\mathcal{I}\setminus N$ and $\sigma_N(dy) \leq 0$ in N; and the second one is obtained fixing x and observing that $h(x,y)$ is a linear combination of ψ_α and φ_α with non-negative coefficients. □

Lemma 5.4 (Left Enlargment). *Under the assumptions of Theorem 2.1, consider the interval $N = (a,b)$ and $C = (\ell, \bar{b})$ (with $\bar{b} < r$) such that (N,C) satisfy Condition 2.3. Then, there exists $b' \geq \bar{b}$ such that $(N' = (\ell,b), C' = (\ell,b'))$ satisfy Condition 2.3.*

Proof. First observe that, if $D \subset D'$, based on definition (16), we have that

$$\sigma_{D'}(dx) - \sigma_D(dx) \text{ is a negative measure.} \tag{23}$$

By hypothesis we know

$$\int_C \psi_\alpha(y)\sigma_N(dy) = 0.$$

It follows from (23) that

$$\int_C \psi_\alpha(y)\sigma_{N'}(dy) \le 0.$$

Consider $b' = \sup\{x \in [\bar{b}, r) \colon \int_{(\ell,x)} \psi_\alpha(y)\sigma_{N'}(dy) \le 0\}$. Let us check that $N' = (\ell, b), C' = (\ell, b')$ satisfies Condition 2.3. It is clear that

$$\int_{C'} \psi_\alpha(y)\sigma_{N'}(dy) \le 0,$$

with equality if $b' < r$. This proves (iii) in Condition 2.3. Observe that (ii) is automatic, as $\ell = \inf C'$. Now we prove (iv). Consider

$$\int_{C'} G_\alpha(x,y)\sigma_{N'}(dy) = \int_C G_\alpha(x,y)\sigma_N(dy) + \int_C G_\alpha(x,y)(\sigma_{N'} - \sigma_N)(dy) + \int_{C'\setminus C} G_\alpha(x,y)\sigma_{N'}(dy). \tag{24}$$

The first term on the right-hand side is non-positive by hypothesis. Let us analyze the sum of the remainder terms. Considering the previous decomposition with $\psi_\alpha(y)$ instead of $G_\alpha(x,y)$, and taking $\int_C \psi_\alpha(y)\sigma_N(dy) = 0$ into account, we obtain

$$\int_C \psi_\alpha(y)(\sigma_{N'} - \sigma_N)(dy) + \int_{C'\setminus C} \psi_\alpha(y)\sigma_{N'}(dy) \le 0. \tag{25}$$

Consider $k(x)$ such that $k(x)\psi_\alpha(\bar{b}) = G_\alpha(x,\bar{b})$; we have $k(x)\psi_\alpha(y) \le G_\alpha(x,y)$ if $y \le \bar{b}$ and $k(x)\psi_\alpha(y) \ge G_\alpha(x,y)$ if $y \ge \bar{b}$. Also note that $(\sigma_{N'} - \sigma_N)(dy)$ is non-positive in C and $\sigma_{N'}$ is non-negative in $C' \setminus C$. We get

$$\int_C G_\alpha(x,y)(\sigma_{N'} - \sigma_N)(dy) + \int_{C'\setminus C} G_\alpha(x,y)\sigma_{N'}(dy) \le k(x)\left(\int_C \psi_\alpha(y)(\sigma_{N'} - \sigma_N)(dy) + \int_{C'\setminus C} \psi_\alpha(y)\sigma_{N'}(dy)\right) \le 0. \tag{26}$$

This completes the proof of (iv). To prove (i), first observe that

$$\int_{N'} \psi_\alpha \sigma(dy) \le \int_{N'} \psi_\alpha \sigma_{N'}(dy) \le \int_{C'} \psi_\alpha \sigma_{N'}(dy) = 0. \tag{27}$$

To complete the proof, applying the same arguments in (27) to the decreasing solution φ_α, it is enough to see that

$$\int_{C'} \varphi_\alpha(y)\sigma_{N'}(dy) \le 0,$$

Now

$$\begin{aligned}
&\int_{C'} \varphi_\alpha(x)\sigma_{N'}(dy) \\
&\quad = \int_C \varphi_\alpha(x)\sigma_N(dy) + \int_C \varphi_\alpha(x)(\sigma_{N'} - \sigma_N)(dy) + \int_{C'\setminus C} \varphi_\alpha(x)\sigma_{N'}(dy) \\
&\qquad\qquad \le \int_C \varphi_\alpha(x)(\sigma_{N'} - \sigma_N)(dy) + \int_{C'\setminus C} \varphi_\alpha(x)\sigma_{N'}(dy) \\
&\qquad\qquad\quad \le k\left(\int_C \psi_\alpha(x)(\sigma_{N'} - \sigma_N)(dy) + \int_{C'\setminus C} \psi_\alpha(x)\sigma_{N'}(dy)\right) \le 0.
\end{aligned}$$

The last equality is (25). The first inequality is a consequence of the hypotesis. And in the second, $k > 0$ is such that $\varphi_\alpha(\bar b) = k\psi_\alpha(\bar b)$ and the same arguments as in (26) apply. This concludes the proof. □

Lemma 5.5 (Right enlargment)**.** *Under the assumptions of Theorem 2.1, consider the interval $N = (a, b)$ and $C = (\bar a, r)$ (with $\bar a > \ell$), such that (N, C) satisfies Condition 2.3. Then, there exists $a' \le \bar a$ such that $(N' = (a, r), C' = (a', r))$ satisfies Condition 2.3.*

Proof. Analogous to the proof of the previous lemma. □

5.2 Merge

Lemma 5.6 (Merge)**.** *Under the assumptions of Theorem 2.1, consider $N_1 = (a_1, b_1)$, $N_2 = (a_2, b_2)$ such that $b_1 < a_2$ and $(\alpha - \mathcal{L})g(x) \ge 0$ for x in (b_1, a_2). Let $C_1 = (\bar a_1, \bar b_1)$ and $C_2 = (\bar a_2, \bar b_2)$ be intervals such that $\bar a_1 > \ell$, $\bar b_1 < r$, $\bar a_2 > \ell$, $\bar b_2 < r$. Suppose that the two pairs of intervals (N_1, C_1), (N_2, C_2) satisfy Condition 2.3. Then, if $C_1 \cap C_2 \ne \emptyset$, considering $N = (a_1, b_2)$, there exists an interval C such that (N, C) satisfies Condition 2.3.*

Proof. By hypothesis

$$\int_{C_i} \varphi_\alpha(x)\sigma_{N_i}(dx) = \int_{C_i} \psi_\alpha(x)\sigma_{N_i}(dx) = 0, \quad i = 1, 2.$$

Then

$$\int_{C_1\cup C_2} \varphi_\alpha(x)\sigma(dx) = -\int_{C_1\cap C_2} \varphi_\alpha(x)\sigma^+(dx)$$

and

$$\int_{C_1\cup C_2} \psi_\alpha(x)\sigma(dx) = -\int_{C_1\cap C_2} \psi_\alpha(x)\sigma^+(dx).$$

We assume, without loss of generality, that

$$\int_{C_1\cap C_2} \varphi_\alpha(x)\sigma^+(dx) = \int_{C_1\cap C_2} \psi_\alpha(x)\sigma^+(dx) > 0$$

and therefore, denoting by (a', b') the interval $C_1 \cup C_2$, we get:

$$\int_{(a',b')} \varphi_\alpha(x)\sigma(dx) = \int_{(a',b')} \psi_\alpha(x)\sigma(dx) < 0;$$

$\psi_\alpha(a') \leq \varphi_\alpha(a')$; and $\psi_\alpha(b') \geq \varphi_\alpha(b')$. The same procedure in the proof of Lemma 5.3, allow us to construct an interval C such that (N, C) satisfy (i), (ii) and (iii) in Condition 2.3. Let us prove (iv): If $x < a_1$ we have $G_\alpha(x, y) = w_\alpha^{-1}\psi_\alpha(x)\varphi_\alpha(y)$ for $y \geq a_1$ and $G_\alpha(x, y) \leq w_\alpha^{-1}\psi_\alpha(x)\varphi_\alpha(y)$ for $y \leq a_1$; since $\sigma_N(dy)$ is non-negative in $y \leq a_1$ we find

$$\int_C G_\alpha(x, y)\sigma_N(dy) \leq w_\alpha^{-1}\psi_\alpha(x)\int_C \varphi_\alpha(y)\sigma_N(dy) \leq 0.$$

An analogous argument prove the assertion in the case $x > b_2$. Now consider $x \in N$, suppose $x < \min\{a_2, \bar{b}_1\}$ (in case $x > \max\{b_1, \bar{a}_2\}$ an analogous argument is valid), we get

$$\int_C G_\alpha(x, y)\sigma_N(dy) = \int_{C_1} G_\alpha(x, y)\sigma_{N_1}(dy) + \int_{C_1} G_\alpha(x, y)(\sigma_N - \sigma_{N_1})(dy) \\ + \int_{C\setminus C_1} G_\alpha(x, y)\sigma_N(dy),$$

where $\int_{C_1} G_\alpha(x, y)\sigma_{N_1}(dy) \leq 0$ by hypothesis. We move on to prove that the sum of the second and the third terms on the right-hand side of the previous equation are non-positive, thus completing the proof: Observe that

$$G_\alpha(x, y) \leq w_\alpha^{-1}\psi_\alpha(x)\varphi_\alpha(y)$$

and

$$G_\alpha(x, y) = w_\alpha^{-1}\psi_\alpha(x)\varphi_\alpha(y) \quad (y \geq \min\{a_2, \bar{b}_1\})$$

The measure $(\sigma_N - \sigma_{N_1})$ has support in N_2, where the previous equality holds. The measure $\sigma_N(dy)$ is positive for $y < a_1$ where we do not have the equality, then

$$\int_{C_1} G_\alpha(x, y)(\sigma_N - \sigma_{N_1})(dy) + \int_{C\setminus C_1} G_\alpha(x, y)\sigma_N(dy) \\ \leq w_\alpha^{-1}\psi_\alpha(x)\left(\int_{C_1} \varphi_\alpha(y)(\sigma_N - \sigma_{N_1})(dy) + \int_{C\setminus C_1} \varphi_\alpha(y)\sigma_N(dy)\right) \leq 0,$$

where the last inequality is a consequence of

$$\int_C \varphi_\alpha(y)\sigma_N(dy) = \int_{C_1} \varphi_\alpha(y)\sigma_{N_1}(dy) + \int_{C_1} \varphi_\alpha(y)(\sigma_N - \sigma_{N_1})(dy) \\ + \int_{C\setminus C_1} \varphi_\alpha(y)\sigma_N(dy) \leq 0,$$

and

$$\int_{C_1} \varphi_\alpha(y)\sigma_{N_1}(dy) = 0.$$

This completes the proof. □

Lemma 5.7 (Left merge). *Under the assumptions of Theorem 2.1, consider $N_1 = (\ell, b_1)$, $N_2 = (a_2, b_2)$ such that: $b_1 < a_2$; and $(\alpha - \mathcal{L})g(x) \geq 0$ for x in (b_1, a_2). Let $C_1 = (\ell, \bar{b}_1)$ and $C_2 = (\bar{a}_2, \bar{b}_2)$ be intervals such that: $\bar{b}_1 < r$; $\bar{a}_2 > \ell$; and $\bar{b}_2 < r$. Suppose that the two pairs of intervals (N_1, C_1), (N_2, C_2) satisfy Condition 2.3. If $C_1 \cap C_2 \neq \emptyset$ then, considering $N = (\ell, b_2)$, there exists $\bar{b}$ such that $(N, C = (\ell, \bar{b}))$ satisfies Condition 2.3.*

Proof. Define $\bar{b} = \sup\{x \in [\bar{b}_2, r) \colon \int_{(\ell,x)} \psi_\alpha(y)\sigma_N(dy) \leq 0\}$ (note that $\bar{b}_2$ belongs to the set). We have

$$\int_C \psi_\alpha(y)\sigma_N(dy) \leq 0, \tag{28}$$

with equality if $\bar{b} < r$, proving (ii) in Condition 2.3. To prove (iv) we split the integral as follows:

$$\begin{aligned} \int_C G_\alpha(x,y)\sigma_N(dy) = &\int_{C_1} G_\alpha(x,y)\sigma_{N_1}(dy) + \int_{C_2} G_\alpha(x,y)\sigma_{N_2}(dy) \\ &- \int_{C_1\cap C_2} G_\alpha(x,y)\sigma_N^+(dy) + \int_{C\setminus(C_1\cup C_2)} G_\alpha(x,y)\sigma_N(dy) \end{aligned} \tag{29}$$

where σ_N^+ is the positive part of σ_N. Considering the same decomposition as in (29) with $\psi_\alpha(y)$, instead of $G_\alpha(x,y)$, and also considering: equation (28); $\int_{C_1} \psi_\alpha(y)\sigma_{N_1}(dy) = 0$; and $\int_{C_2} \psi_\alpha(y)\sigma_{N_2}(dy) = 0$, we obtain

$$-\int_{C_1\cap C_2} \psi_\alpha(y)\sigma_N^+(dy) + \int_{C\setminus(C_1\cup C_2)} \psi_\alpha(y)\sigma_N(dy) \leq 0. \tag{30}$$

For every x consider $k(x) \geq 0$ such that $k(x)\psi_\alpha(\bar{b}_2) = G_\alpha(x, \bar{b}_2)$. We have $k(x)\psi_\alpha(\bar{b}_2) \leq G_\alpha(x, \bar{b}_2)$ for $y \leq \bar{b}_2$ and $k(x)\psi_\alpha(\bar{b}_2) \geq G_\alpha(x, \bar{b}_2)$ for $y \geq \bar{b}_2$ and therefore

$$\begin{aligned} &-\int_{C_1\cap C_2} G_\alpha(x,y)\sigma_N^+(dy) + \int_{C\setminus(C_1\cup C_2)} G_\alpha(x,y)\sigma_N(dy) \\ &= k(x)\left(-\int_{C_1\cap C_2} \psi_\alpha(y)\sigma_N^+(dy) + \int_{C\setminus(C_1\cup C_2)} \psi_\alpha(y)\sigma_N(dy)\right) \leq 0. \end{aligned}$$

The first two terms on the right-hand side of equation (29) are also non-positive, and we conclude that (iv) in Condition 2.3 holds. To prove (ii) we consider the decomposition in (29) with $\varphi_\alpha(y)$ instead of $G_\alpha(x,y)$ and $k \geq 0$ such that $k\psi_\alpha(\bar{b}_2) = \varphi_\alpha(\bar{b}_2)$; the same considerations done to prove (iv) conclude the result in this case. □

Lemma 5.8 (Right merge)**.** *Under the assumptions of Theorem 2.1, consider $N_1 = (a_1, b_1)$, $N_2 = (a_2, r)$ such that: $b_1 < a_2$; and $(\alpha - \mathcal{L})g(x) \geq 0$ for x in (b_1, a_2). Let $C_1 = (\bar{a}_1, \bar{b}_1)$ and $C_2 = (\bar{a}_2, r)$ intervals such that: $\bar{a}_1 > \ell$; $\bar{b}_1 < r$; and $\bar{a}_2 > \ell$. Suppose that the two pairs of intervals (N_1, C_1), (N_2, C_2) satisfy Condition 2.3. If $C_1 \cap C_2 \neq \emptyset$ then, considering $N = (a_1, r)$, there exists $\bar{a}$ such that $(N, C = (\bar{a}, r))$ satisfies Condition 2.3.*

Proof. Analogous to the previous lemma. □

Lemma 5.9 (Total merge)**.** *Under the assumptions of Theorem 2.1, consider $N_1 = (\ell, b_1)$, $N_2 = (a_2, r)$ such that: $b_1 < a_2$; and $(\alpha - \mathcal{L})g(x) \geq 0$ for x in (b_1, a_2). Let $C_1 = (\ell, \bar{b}_1)$ and $C_2 = (\bar{a}_2, r)$ intervals such that the two pairs of intervals (N_1, C_1), (N_2, C_2) satisfy Condition 2.3. If $C_1 \cap C_2 \neq \emptyset$ then for all $x \in \mathcal{I}$,*

$$\int_{\mathcal{I}} G_\alpha(x, y)\sigma(dy) \leq 0. \tag{31}$$

In consequence, the pair $(\mathcal{I}, \mathcal{I})$ satisfies Condition 2.3.

Proof. Consider the following decomposition of the integral

$$\int_{\mathcal{I}} G_\alpha(x, y)\sigma(dy) = \int_{C_1} G_\alpha(x, y)\sigma_{N_1}(dy) + \int_{C_2} G_\alpha(x, y)\sigma_{N_2}(dy) - \int_{C_1 \cap C_2} G_\alpha(x, y)\sigma^+(dy).$$

Observing that the three terms on the right-hand side are non-positive, the lemma is proved. □

Remark 5.10. Observe that this case can not happen under our hypothesis: as the inversion formula (10) holds, and we assume g non negative, condition (31) gives a contradiction (unless $g \equiv 0$).

5.3 Proof of Theorem 2.1

Proof. We first apply Algorithm 2.1 departing from $\mathcal{N} = \cup_{i=1}^n N_i$ to obtain a set of pairwise disjoint intervals $\{C_1, \ldots, C_m\}$. Observe that, under our hypothesis, the algorithm does not give as a result $C_1 = \mathcal{I}$ (see Remark 5.10). Denote $\mathcal{C} = \cup_{i=1}^m C_i$ and $\mathcal{S} = \mathcal{I} \setminus \mathcal{C}$. Condition $\mathcal{N} \subset \mathcal{C}$ follows by construction, as the algorithm enlarges the negative set. Furthermore, the intervals C_i that result from the algorithm satisfy conditions (a) and (b) as they satisfy (i) and (ii) in Condition 2.3, due to the fact that $\sigma_N = \sigma$ restricted to each final C_i resulting from the algorithm. Condition (c) is also satisfied, due to this same fact. It remains to prove that this is in fact the continuation region associated with the optimal stopping problem. We use the Dynkin's characterization as the minimal α-excessive majorant to prove that

$$V(x) := \int_{\mathcal{I} \setminus \mathcal{C}} G_\alpha(x, y)\sigma(dy)$$

is the value function. Since $\sigma(dy)$ is non-negative in $\mathcal{I} \setminus \mathcal{C}$ we have that V is α-excessive. For $x \in \mathcal{I}$, we have

$$g(x) = \int_{\mathcal{I}} G_\alpha(x,y)\sigma(dy) = V(x) + \sum_{i=1}^{m} \int_{C_i} G_\alpha(x,y)\sigma(dy). \qquad (32)$$

If $x \notin \mathcal{C}$, the sum in the r.h.s. of (32) vanishes by (a) and (b), as in the proof of Lemma 5.2. This gives $V(x) = g(x)$ for $x \in \mathcal{S}$. On the other hand, based on (c), this same sum is non-positive if $x \in \mathcal{C}$. This gives $V(x) \geq g(x)$ for $x \in \mathcal{C}$. We have then proved that V is a majorant of g. We have, up to now, $V(x) \geq \sup_\tau \mathbb{E}_x\left(e^{-\alpha\tau} g(X_\tau)\right)$. Finally observe that, denoting by $\mathcal{S}$ the set $\mathcal{I} \setminus \mathcal{C}$

$$V(x) = \mathbb{E}_x\left(e^{-\alpha \hbar_{\mathcal{S}}} V(X_{\hbar_{\mathcal{S}}})\right) = \mathbb{E}_x\left(e^{-\alpha \hbar_{\mathcal{S}}} g(X_{\hbar_{\mathcal{S}}})\right),$$

where the first equality is a consequence of Lemma 5.1. We conclude that V is the value function and that $\mathcal{S}$ is the stopping region, finishing the proof. □

Acknowledgement. The second author would like to thank the Institut Élie Cartan de Lorraine (IECL) and the Centre national de la recherche scientifique (CNRS) for hospitality and financial support, during the final period of the writing of this work.

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