Equilibrium CEO Contract with Belief Heterogeneity*

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Abstract

We consider a firm owned by shareholders with heterogeneous beliefs who delegate to a manager the choice of a production plan. Shareholders can trade claims that are contingent on all possible realizations of the firm's output. Shareholders cannot observe the chosen production plan and design a contract for the manager that specifies her compensation as a function of the firm's output and possibly some restrictions to trade in the financial market. The contract is designed so that at equilibrium the manager chooses the plan preferred by shareholders and reveals it truthfully. Our first result is that such contract should restrict the manager from trading in the asset market. Second, we show that the marginal utility of manager's compensation should be proportional to that of the representative shareholder at the equilibrium plan. An implication is that, relative to a linear compensation, the manger should be induced to attach larger weights to extreme output realizations.

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1 Introduction

Corporations are often owned by a large number of investors, who may hold different beliefs say on the likelihood of success of a new product, on the payoffs associated to an investment opportunity, or on the future prospects of the market. It is well known that, under some conditions, shareholders with different beliefs can reach an agreement on the preferred plan of action by trading contingent claims in a complete asset market. Even when this is the case, however, shareholders often have to delegate their decisions to a manager and so they need to design a contract that would make the manager act in their interest. A typical complication is that the manager may have private information; in particular, it may be very costly or impossible for shareholders to observe the actions taken by the manager.

In this paper, we consider a firm owned by shareholders with different beliefs and run by a manager. There is only one production and consumption date T. The manager chooses a production plan, whose output is uncertain. Shareholders cannot observe the plan chosen by the manager, they only observe the realized production. Once the manager announces her plan, shareholders and possibly the manager can trade assets in a financial market that consists in all the assets that are contingent on the realized production. A contract for the manager specifies her compensation as a function of the firm output together with possibly some restrictions to trade in the financial market. We ask whether shareholders can design such a contract so that the manager truthfully reveals the plan she has chosen, and the chosen plan coincides with the plan preferred by shareholders (that we will call the consensus plan).

Bianchi, Dana and Jouini (2019) formalized this question by introducing the concept of manager-shareholders equilibrium, defined as follows. Given a compensation scheme and a price, the manager considers the indirect utility associated to each possible production plan, that is the maximal utility she can obtain by choosing a consumption plan subject to her budget constraint. She then chooses a production plan that maximizes her indirect utility and the associated optimal consumption plan and announces the production plan to shareholders. Each shareholder chooses a consumption plan so as to maximize her expected utility subject to her budget constraint. In addition to the usual market clearing conditions, at equilibrium, the manager should have no incentive to misreport the chosen plan and shareholders should unanimously prefer the manager's plan to any other feasible plan. In this paper, we extend this definition by allowing possible restrictions to trade on the manager. In our framework, a manager-shareholders equilibrium (in short m-s equilibrium) is a list of a compensation scheme, a space of allowed transactions, a production plan, consumption plans for shareholders and the manager and a price that fulfill the properties above.

Our analysis builds on, and it is motivated by, Bianchi et al. (2019). They show that, absent trading restrictions on the manager, the m-s equilibrium exists if and only if the manager has the same characteristics as the representative shareholder at the production equilibrium of the initial economy without manager. Furthermore, when the m-s equilibrium exists, the manager is given a linear compensation and does not trade. However, in practice, it may be hard or impossible to find a manager with the same characteristics as the representative agent. This begs the question of whether commonly observed restrictions to insider trading for the manager can help reaching an equilibrium when the manager has very general beliefs. If this is the case, the next question is what are the properties of the associated compensation scheme and in particular how the optimal compensation should deviate from linearity.

The main contribution of this paper is to provide a framework where indeed an equilibrium can be reached if shareholders can restrict the manager's trades in the asset market. In addition, we characterize the associated compensation scheme, and highlight some of its properties that are qualitatively valid irrespective of the exact form of belief heterogeneity.

We start by describing some properties of our equilibrium, showing how it relates to the standard concept of production equilibrium where the initial production set is replaced by a net (of manager's compensation) production set and how a representative agent associated to such a production equilibrium can be defined. We then move to our main task of defining a contract for the manager that can lead to the equilibrium. As implied by Bianchi et al. (2019), trading restrictions are necessary when the manager is not the representative agent. In fact, in order to characterize the equilibrium compensation, we focus on the case in which the manager is not allowed to trade in the financial market. As we show, the compensation schemes defined in this way are the only candidate equilibrium schemes, even when the manager faces milder trading restrictions. Our key result is that the manager implements the consensus plan (and truthfully reveals it) only if her compensation is designed so that the marginal utility of her compensation is proportional to that of the representative shareholder at the consensus plan. This provides a differential equation that the compensation function should satisfy. However, as the net production set is defined with the compensation function, both the production equilibrium (if it exists) and the representative agent's characteristics depend on the compensation function. This leads to a kind of fixed-point problem which, furthermore, is not well defined. We address this problem by taking a reverse approach: we take the net production set as given and restrict our attention to the case in which there exists a production equilibrium for that set. Solving the differential equation, that is now independent of the compensation, we obtain a compensation function and construct the gross production set. We show that the derived compensation function is an equilibrium compensation when the production set corresponds to the derived gross production set.

As an application, we show that, for any manager whose beliefs are between the "most optimistic" and the "most pessimistic" (in a sense made precise below) of the shareholders, the compensation should increase the weight that the manager attaches to very low or very large levels of production.

We conclude by providing an example of a net production set for which we can explicitly compute the production equilibrium and, given the belief of the manager, provide a formula to define the equilibrium compensation. Numerical illustrations are then provided.

We think our results have important implications for the study of agency problems and in particular of optimal executive compensation. While most of the literature has studied agency problem from the perspective of a single "representative" principal, we highlight the importance of modeling explicitly shareholder heterogeneity and the equilibrium process leading to the definition of a representative shareholder. In doing so, we obtain novel and different insights on the shape of the optimal contract for the manager.

We show it is necessary to impose trading restrictions to the manager in a setting in which the action she takes cannot be observed by shareholders. Without those restrictions, it would be impossible to find a compensation scheme which induces the manager to choose the consensus plan and to truthfully reveal her choice. This provides a rationale for the commonly observed restrictions both to insider trading and to non-exclusive contracts.

We also qualify the view that agency conflicts are minimized when the manager owns a substantial part of the firm's shares, which has motivated the rise in stock compensation. Our analysis instead emphasizes that the compensation rate should vary with the level of production, and provide conditions under which it should induce the manager to overweight the occurrence of extreme realizations. This result stands in contrast to the argument in favor of compensations -such as call options- which encourage risk taking (see e.g. Kadan and Swinkels (2008)).

1.1 Related Literature

Our paper builds on the literature on aggregation of preferences and beliefs in asset markets.[1] Our focus on agency problems between a manager and shareholders is however novel in this literature. Similarly, managerial compensation has typically been studied under the perspective of a representative shareholder (see e.g. Murphy (1999) and Murphy (2012) for reviews). We provide new insights by embedding the choice of the compensation in a stock market equilibrium with heterogeneous shareholders.[2]

In line with the literature on optimal contracting, we emphasize that, in a setting with asymmetric information, it may be beneficial to prevent the manager from trading in the stock market (Fischer (1992)).[3] Our novelty

[1] Recent contributions include Detemple and Murthy (1994); Gollier and Zeckhauser (2005); Jouini and Napp (2007); Jouini, Marin and Napp (2010); Cvitanić, Jouini, Malamud and Napp (2012); Xiong and Yan (2010); Bhamra and Uppal (2014).

[2] Alternative equilibrium models have instead focused on the labor market equilibrium (e.g. Gabaix and Landier (2008)) or on financial market equilibrium with a representative agent (e.g. Diamond and Verrecchia (1982)).

[3] This literature has also pointed out at beneficial aspects of insider trading, such as improving the informational efficiency of market prices (e.g. Leland (1992)). We abstract from this issue as in our settings there are no investors apart from shareholders. We

is to show how trading restrictions, and the form of optimal compensation, are determined by the interaction between moral hazard and shareholder heterogeneity.

The way information asymmetry is introduced makes our paper in line with the probability approach to general equilibrium developed by Magill and Quinzii (2009). Indeed, we assume that shareholders do not observe states of nature but only the production outcomes, so from their point of view, production plans only differ by the outcomes probability distribution. Accordingly, we consider contracts that are contingent on the possible realizations firm's production as opposed to being contingent on exogenous states of nature.[4]

Finally, we relate to the literature on firms' objectives when shareholders are heterogeneous. Magill and Quinzii (2002) review fundamental problems posed by market incompleteness, as well as classic contributions addressing these problems. Bisin, Gottardi and Ruta (2014) study competitive equilibria in a production economy with incomplete markets and agency frictions and derive fundamental welfare properties.[5] We instead focus on the design of the compensation scheme and keep shareholders' objective as simple as possible by assuming complete markets, or - more precisely - full spanning. As explained in Magill and Quinzii (2009), this assumption is typically much weaker than market completeness, and it means that it is possible to find a portfolio of assets that pays one unit if a given outcome for the firm is realized, and nothing otherwise.

2 Model

We consider a firm owned by a group of shareholders with heterogenous beliefs and run by a manager. There is only one production and consumption date T. The information structure is modeled by a probability space

refer to Bhattacharya (2014) for a recent review of these issues.

[4] As underlined by Magill and Quinzii (2009) "That this assumption is realistic seems to be confirmed by the striking fact that the contracts which are used to finance investment and share production risks— bonds, equity and derivative securities—are either non contingent or based on realized profits and prices, rather than on exogenous events with fixed probabilities."

[5] Other recent contributions include Demichelis and Ritzberger (2011), Magill, Quinzii and Rochet (2015), Crès and Tvede (2014).

$(\Omega, \mathcal{F}, P)$. The firm produces a consumption good, which we use as numeraire, according to a production plan y. This plan is a random variable and $y(\omega)$ defines the production of the firm at date T in state ω.

We denote by X the space of $\mathcal{F}$ measurable production and consumption random variables x. We denote by X' the space of state-price densities p where, for a given state of the world $\omega \in \Omega$, $p(\omega)$ corresponds to the price of one unit of consumption at date T in state ω. For a given price p, the value of the consumption plan x is $p \cdot x = E[px]$, where E is the expectation operator under the probability P.

For these expectations to be well defined, we have to further impose that production and consumption plans in X are such that $E|x|^p < \infty$ and that prices in X' are such that $E|p|^q < \infty$, where p and q are such that $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$.[6] We denote by X_+ and X'_+ the set of nonnegative pairs, respectively, in X and X'. For $Y \subset X$, we denote by Y_+ the set $Y \cap X_+$.

In terms of notations, while x and y will be used to denote random consumption and production pairs taking their values in $\mathbb{R}^2$, z will be used to denote vectors in $\mathbb{R}^2$ and, henceforth, generic values taken by x or y. As usual, $x \leq x'$ $(x << x')$ means $x(\omega) \leq x'(\omega)$ $(x(\omega) < x'(\omega))$ almost everywhere for all t, and $x < x'$ means $x \leq x'$ and $x \neq x'$. Finally, we denote by μ the Lebesgue measure on $\mathbb{R}$.

2.1 Production

We let $Y \subset X$ denote the set of production plans. Denote with $N_Y(y)$ the normal cone of Y at y,

$$N_Y(y) = \{p \in X' : p \cdot (y' - y) \leq 0, \forall\, y' \in Y\},$$

which corresponds to the set of linear forms that reach their maximum on Y at y. We will say that $y \in Y$ is positively exposed if there exists $p >> 0$ such that $p \in N_Y(y)$. Note that a positively exposed production plan y is efficient in the sense that it not dominated by other feasible production

[6] The space X equipped with the norm $\|x\| = \left(\sum_{t=0}^{T} E|x_t|^p\right)^{1/p}$ is then a Banach space whose dual (the space of continuous linear forms on X) is X'.

plans : $\nexists y' \in Y$, $y' > y$. We denote by $\text{Eff}^+(Y)$ the set of positively exposed production plans.

We say that Y is smooth if, for all $\bar{y} \in \text{Eff}^+(Y)$ and all t, there exists $p >> 0$ such that $N_Y(\bar{y}) = \{\lambda p : \lambda \geq 0\}$. This condition states that at positively exposed plans, the tangent cone (i.e. the polar of the normal cone) is a half space and it ensures that Y has no outward kink.

We make the following assumptions:

Assumption (P)

P1 $Y = K - X_+$ *where* $K \subset X_+$,

P2 Y *is closed and smooth,*

P3 *If* $y \in \text{Eff}^+(Y)$, *the random variable* y *has a density* h_y *with* $h_y > 0$, $\mu-$*a.e. on* $(0, \infty)$.

Assumption P1 implies the classical free disposal assumption, $Y - X_+ \subset Y$. Assumption P2 is standard in the general equilibrium literature in finite dimension.[7] Assumption P3 states that for every positively exposed production plan, all positive values are possible.[8] Indeed, this implies that by observing a given realization $(y(\omega))$ of a given production plan in $\text{Eff}^+(Y)$, shareholders are not able to exclude any plan $y \in Y$ from the set of possibly chosen plans. This assumption underlies the information asymmetry between the manager and the shareholders in our model. Bianchi et al. (2019) provide an illustration of Assumptions (P2) and (P3).

2.2 Shareholders

The firm is owned by a group of N shareholders, $i = 1, ..., N$. We denote with ν^i agent i's initial endowment of shares, and we assume $\nu^i > 0$ for all i. Shareholders have no other endowments, and they are heterogeneous in their subjective probabilities Q^i. All subjective probabilities are assumed

[7] Note that this assumption is automatically satisfied when the production set is of the form $Y = \{y \in X : E[F(y)] \leq 0\}$ where F is a given function with a bounded derivative. In such a setting, we have $N_Y(y) = \{\lambda F'(y) : \lambda \geq 0\}$ for y such that $E[F(y)] = 0$ and $N_Y(y) = \{0\}$ for y such that $E[F(y)] < 0$.

[8] In this assumption $(0, \infty)$ might be replaced by some (A, B) for $0 < A < B$. In this case, all the considered functions that are defined on $(0, \infty)$ are replaced by functions that are only defined on (A, B).

to be equivalent to P and we denote by M^i the density of Q^i with respect to P, $M^i = \frac{dQ^i}{dP}$.

A key ingredient in our analysis is that shareholders do not observe the plan y chosen by the manager nor the state of the world ω. In state ω, their information is given by the realization $(y(\omega))$. As already mentioned, by Assumption P3, the observation of a given trajectory does not allow them to infer the chosen plan nor the state of the world. It follows that shareholders can only trade assets whose payoffs are contingent on $y(\omega)$. More formally, let $\mathcal{C}$ be the set of contingent contracts $C : X_+ \to X_+$ whose payoffs for a given y are of the form $c(\omega) = C(y(\omega))$, for some measurable functions $C : \mathbb{R}_+^* \to \mathbb{R}_+$. Given y, shareholders only trade consumption plans in $\mathcal{C}(y) = \{C(y), C \in \mathcal{C}\}$.

All shareholders have the same consumption space X_+ and they are assumed to be expected utility maximizers. The expected utility of agent i for a contingent consumption plan c is defined as

$$U^i(c) = E\left[M^i u(c)\right], \tag{1}$$

in which u is a CRRA instantaneous utility function (the same for all shareholders). That is

$$u(x) = \frac{1}{\gamma}x^\gamma, \tag{2}$$

for some $\gamma < 1$. We further assume the following:

Assumption (C)

1. *For all i, M^i and $M^i\varsigma^{\gamma-1}$ belong to X'* for all $\varsigma \in K$,
2. *For all i, $M^i\varsigma^\gamma$ belong to $L^1(\Omega, \mathcal{F}, P)$.*

Assumption (C) assures that shareholders' marginal utility is well defined in all directions and that their utility is well defined on K.

2.3 Manager

The firm is run by a manager with characteristics similar to those of the shareholders. She is an expected utility maximizer with instantaneous utility u, as defined in (2), she has a subjective probability Q^m equivalent to P

with density M^m. Her expected utility of a contingent plan c is therefore defined by

$$U_m(c) = E\left[M_t^m u(c_t)\right].$$

The manager is given a contract (Φ, W) described by a compensation scheme $\Phi : X_+ \to X_+$ and of a set W of transaction plans she is allowed to make in the contingent claim market, which we describe below. As shareholders can only observe the realized production, the compensation at date t can only depend on the realization $y(\omega)$. Hence $\Phi(y)$ must be of the form

$$\Phi(y)(\omega) = \phi(y(\omega)), \tag{3}$$

for some $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ assumed to be continuous almost everywhere. From now on, we will use the same notation ϕ for $\Phi : X_+ \to X_+$ and for $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ with the convention $\phi(y)(\omega) = \phi(y(\omega))$.

As the compensation cannot exceed the available quantity of consumption good, we necessarily have $\phi(z) \leq z$ for all $z \in \mathbb{R}_+$ and, in particular, $\phi(0) = 0$.

We next describe W, the space of transactions allowed to the manager in the contingent claim market. We assume that W is a closed subspace of X. When $W = \{0\}$, then the manager has no access to the market while when $W = X$, the manager has access to all contingent contracts and she can trade just as shareholders. Given a production plan y, the manager's set of feasible adapted consumptions plans $\mathcal{C}^m(y)$ is

$$\mathcal{C}^m(y) = (\phi(y) + W) \cap \mathcal{C}(y).$$

In particular, when $W = \{0\}$ then $\mathcal{C}^m(y) = \{\phi(y)\}$. When $W = X$, then $\mathcal{C}^m(y) = \mathcal{C}(y)$.

Assumption (F)

1. *For all* $y \in K$ and all $w \in W$, $M_m \phi(y)^{\gamma-1} w$ *belongs to* $L^1(\Omega, \mathcal{F}, P)$,

2. *For all* $y \in K$, $M_m \phi(y)^{\gamma}$ *belong to* $L^1(\Omega, \mathcal{F}, P)$.

Assumption (F) insures that the manager's marginal utility is well defined in all feasible directions and that her utility is well defined when $y \in K$.

Shareholders delegate to the manager the choice of the production plan. To explain how the manager makes her choices, let us introduce the concept of indirect utility of production plans for a given price. Given a production plan y and a price $q \in \mathcal{C}(y)$, let $V_m(y, q)$ be the maximal utility of the consumption plans that the manager can obtain by trading her compensation under her market constraint $c \in \mathcal{C}^m(y)$ and her budget constraint $q \cdot c \leq q \cdot \phi(y)$,

$$V_m(y, q) = \max\{U_m(c),\ c \in \mathcal{C}^m(y),\ q \cdot c \leq q \cdot \phi(y)\}. \tag{4}$$

Similarly, let $V^i(y, q)$ be the maximal utility of the consumption plans that shareholder i can obtain by trading her share of production under her market and budget constraints:

$$V^i(y, q) = \max\{U^i(c),\ c \in \mathcal{C}(y),\ q \cdot c \leq \nu^i(q \cdot (y - \phi(y))\}, \tag{5}$$

where ν^i denotes her initial share and $y - \phi(y)$ is the production left to shareholders after having paid the manager. Equations (4) and (5) show how the manager and the shareholders, respectively, assess the utility associated to the various alternative production plans. They compare their indirect utility under y to the one they would have obtained under any alternative $y' \in Y$, by taking prices q as given.[9]

2.4 Equilibrium

Let us now define our concept of equilibrium between shareholders and the manager. We have in mind a setting with a large number of non-strategic agents. We take a general equilibrium approach in which resource allocation is decentralized through prices and which we adapt so as to account for the information asymmetry between the manager and the shareholders.

Shareholders appoint a manager with a contract (ϕ, W) and delegate to her the choice of the production plan. As mentioned, we do not model explicitly why shareholders need to delegate this choice to a manager. A

[9]Price taking is important to be able to define a consensus plan (see e.g. Grossman and Stiglitz (1980) for a discussion on price taking behaviors and unanimity). Price taking could also be derived by considering a setting with a large number of identical firms. The analysis would not be affected.

standard argument is that they lack the time or the skills needed to implement the plan, which may require continuous adjustments over time.

Given her compensation and a price q, the manager chooses a production plan y that maximizes her indirect utility $V_m(\cdot, q)$ over Y and an optimal consumption plan $C_m(y)$. The manager announces the chosen plan to shareholders, who maximize the utility of their consumption plans under their market and budget constraints. Due to Assumption P3, when y is in $\text{Eff}^+(Y)$, shareholders cannot verify the truth of the announcement. At equilibrium, the manager should have no incentive to misreport the chosen plan, markets should clear and finally, shareholders should unanimously prefer the manager's plan to any other plan feasible y.

Definition 1 *A manager-shareholders equilibrium (in short m-s equilibrium) is defined by a contract* (ϕ, W), $\phi \neq 0$, *a production plan* $\hat{y} \in Y$, *a list of contingent contracts* $(\hat{C}^i)_i$, *a contingent contract* $\hat{C}_m$, *and a price* $\hat{q} \in \mathcal{C}(\hat{y})$ *such that:*

1. $\hat{c}^i = \hat{C}^i(\hat{y})$ *maximizes* $U^i(c)$ *s.t.* $c \in \mathcal{C}(\hat{y})$, $\hat{q} \cdot c \leq \nu^i(\hat{q} \cdot (\hat{y} - \phi(\hat{y})))$,
2. $\hat{c}_m = \hat{C}_m(\hat{y})$ *maximizes* $U_m(c)$ *s.t.* $c \in \mathcal{C}^m(\hat{y})$, $\hat{q} \cdot c \leq \hat{q} \cdot \phi(\hat{y})$,
3. $\sum_i \hat{c}^i + \hat{c}_m = \hat{y}$,
4. $V_m(\hat{y}, \hat{q}) > V_m(y, \hat{q})$ *for all* $y \in Y$,
5. $U_m(\hat{C}_m(\hat{y})) = \max_{y \in Y} U_m(\hat{C}_m(y))$,
6. $V^i(\hat{y}, \hat{q}) = \max_Y V^i(y, \hat{q})$.

Our definition of m-s equilibrium is adapted from Bianchi et al. (2019) by considering that the manager in our setting receives a contract (ϕ, W) that may specify some trading constraints W, while this was not the case in Bianchi et al. (2019).

3 Equilibrium Properties

We first show that the manager's equilibrium consumption equals her equilibrium compensation. In other words, in equilibrium the manager does not trade in the financial market. We deduce from the no-trade result that, for a given compensation scheme ϕ, the manager chooses a production plan that strictly maximizes the utility of her compensation on the production set Y. The shareholders solve a production equilibrium problem given the net production set $Y^\phi = \{y - \phi(y) : y \in Y\}$.

In order to state our results, let us recall the concept of production equilibrium associated to the production set $\mathcal{Y}$. Note that shareholders are not constrained to trade in $\mathcal{C}(y^*)$.

Definition 2 *A production equilibrium associated to the production set $\mathcal{Y}$ is given by a production plan y^*, $y^* \in \mathcal{Y}$, a set of individual consumption plans $(c^{*i})_i \in X^N$ and a price $q^* \in X'$ such that*

1. $c^{*i} = \text{argmax}\, U^i(c),\ q^* \cdot c \leq \nu^i(q^* \cdot y^*)$ *for all* i,
2. $y^* = \arg\max_{\mathcal{Y}} q^* \cdot y$,
3. $\sum c^{*i} = y^*$.

We now list some properties of a manager-shareholders equilibrium given a contract (ϕ, W).

Theorem 1 *Assume (P) and (C). Let $((\phi, W), (\hat{c}^i)_i, (\hat{c}_m), \hat{q}, \hat{y})$ be a m-s equilibrium then*

1. *The manager does not trade, i.e.* $\hat{c}_m = \phi(\hat{y})$.
2. *The production plan* $\hat{y}$ *strictly maximizes* $U_m(\phi(y))$ *over* Y, $U_m(\phi(\hat{y})) > U_m(\phi(y))$ *for all* $y \in Y$,
3. *The triple* $((\hat{c}^i)_i, \hat{q}, \hat{y} - \phi(\hat{y}))$ *is a production equilibrium associated to the production set* $Y^\phi = \{y - \phi(y) : y \in Y\}$ *that fulfills* $\hat{q} \in \mathcal{C}(\hat{y})$.
4. $((\phi, W'), (\hat{c}^i)_i, (\hat{c}_m), \hat{q}, \hat{y})$ *is a m-s equilibrium for any* $W' \subseteq W$. *In particular, it is a* $(\phi, \{0\})$ *m-s equilibrium.*

Remark 2 *In point 3., the condition $\hat{q} \in \mathcal{C}(\hat{y})$ is redundant since it is already in the definition of a m-s equilibrium. However, it will be useful in order to derive a (partial) converse result.*

From Item 4, $W = \{0\}$ plays a special role since any m-s equilibrium $((\phi, W), (\hat{c}^i)_i, (\hat{c}_m), \hat{q}, \hat{y})$ is still a m-s equilibrium when the manager is not allowed to trade, i.e. when W is replaced by $\{0\}$. We will take advantage of this property in the next section, where we characterize the equilibrium compensation ϕ by setting $W = \{0\}$. The next theorem provides a characterization of the m-s equilibria associated to a contract $(\phi, \{0\})$.

Theorem 3 *Assume (C) and (P), then the list $((\phi, \{0\}), (\hat{c}^i)_i, (\hat{c}_m), \hat{q}, \hat{y})$ is a m-s equilibrium if and only if Assertions 1 to 3 of Theorem 1 are fulfilled.*

4 Equilibrium contract

4.1 Representative shareholder

As shown in Theorem 1 if $((\phi, W), (\hat{c}^i)_i, (\hat{c}_m), \hat{q}, \hat{y})$ is a m-s equilibrium then $((\hat{c}^i)_i, \hat{q}, \hat{y} - \phi(\hat{y}))$ is a production equilibrium associated to the net production set $Y^\phi = \{y - \phi(y) : y \in Y\}$. Hence, there exists a representative shareholder associated to this production equilibrium. As standard, this is a fictitious agent who - if endowed with the entire equilibrium production - would have no incentive to trade at equilibrium prices.

In this section, we derive a key result of the paper: a necessary condition for the existence of m-s equilibria is that the manager's compensation leads her to mimic the behavior of the representative shareholder at equilibrium or equivalently to have the same marginal utility as the representative shareholder at equilibrium.

We start by characterizing the representative shareholder in our setting. Let

$$\Lambda = \left\{ \lambda \in \mathbb{R}^n_+ : \sum_i (\lambda^i)^{\frac{1}{1-\gamma}} = 1 \right\}, \; N(\lambda) = \frac{\left(\sum_i (\lambda^i M^i)^{\frac{1}{1-\gamma}}\right)^{1-\gamma}}{E\left[\left(\sum_i (\lambda^i M^i)^{\frac{1}{1-\gamma}}\right)^{1-\gamma}\right]}.$$

As seen in Theorem 1, if $((\phi, W)(\hat{c}^i)_i, (\hat{c}_m), \hat{q}, \hat{y})$ is a m-s equilibrium, then $((\hat{c}^i)_i, \hat{q}, \hat{y} - \phi(\hat{y}))$ is a production equilibrium associated to the production set $Y^\phi = \{y - \phi(y) : y \in Y\}$ and there exists a unique vector of equilibrium utility weights $\left(\hat{\lambda}^i\right)$ in Λ and $\nu > 0$ such that

$$\hat{\lambda}^i M^i (\hat{c}^i)^{\gamma-1} = \nu \hat{q} \text{ or } \hat{c}^i = (\nu\hat{q})^{\frac{1}{\gamma-1}} \left(\hat{\lambda}^i M^i\right)^{\frac{1}{1-\gamma}} \text{ for all } i.$$

Summing over i, we obtain:

$$N(\hat{\lambda})\,(\hat{y} - \phi(\hat{y}))^{\gamma-1} = \nu'\hat{q} \text{ for } \nu' = \nu E\left[\left(\sum_i \left(\hat{\lambda}^i M^i\right)^{\frac{1}{1-\gamma}}\right)^{1-\gamma}\right]^{-1} > 0.$$

Therefore, the representative agent has instantaneous utility u and a density $\tilde{M}$ determined by

$$\tilde{M} = N(\hat{\lambda}). \tag{6}$$

The following corollary establishes a link between the compensation function ϕ, the characteristics of the manager and those of the representative agent.

Corollary 4 *Let $((\phi, W)(\hat{c}^i)_i, (\hat{c}_m), \hat{q}, \hat{y})$ be a m-s equilibrium. We then have*

$$\phi'(\hat{y}) M^m u'(\phi(\hat{y})) = \mu(1 - \phi'(\hat{y}))\tilde{M} u'(\hat{y} - \phi(\hat{y})) \tag{7}$$

for some $\mu > 0$.

4.2 Equilibrium contract

We now consider the possibility of designing a contract (ϕ, W) such that there exists $((\hat{c}^i)_i, (\hat{c}_m), \hat{q}, \hat{y})$ for which $((\phi, W)(\hat{c}^i)_i, (\hat{c}_m), \hat{q}, \hat{y})$ is a m-s equilibrium. When this is the case, we say that (ϕ, W) is an equilibrium contract and that ϕ is an equilibrium compensation.

Without trading restrictions on the manager ($W = X$), as seen in Bianchi et al. (2019), equilibrium requires that the compensation is linear. At the same time, from (7), under linear compensation, the manager shares the same belief as the representative shareholder. When this is not the case, there exists no equilibria and therefore the manager should bear some trading restrictions.

Corollary 5 *Let $((\phi, W)(\hat{c}^i)_i, (\hat{c}_m), \hat{q}, \hat{y})$ be a m-s equilibrium. If $W = X$, ϕ is linear. If ϕ is linear then $M^m = \tilde{M}$. If $M^m \neq \tilde{M}$ then $W \subsetneq X$.*

We then consider the other extreme case $W = \{0\}$. We show that, under suitable conditions, there is a unique equilibrium compensation scheme ϕ, up to a positive constant. From Theorem 1, item 4, such compensation schemes are the only candidate equilibrium schemes when the manager faces milder trading restrictions, $W \neq \{0\}$.

Let $((\phi, W)(\hat{c}^i)_i, (\hat{c}_m), \hat{q}, \hat{y})$ be a m-s equilibrium and let $\tilde{y} = \hat{y} - \phi(\hat{y}) = (\mathrm{Id} - \phi)(\hat{y})$ be the equilibrium net production plan. Let $M^{m,\tilde{y}} = E\left[M^m \middle| \tilde{y}\right]$ and $N^{\tilde{y}}(\tilde{\lambda}) = E\left[N(\tilde{\lambda}) \middle| \tilde{y}\right]$. Then there exists a measurable function $h : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\frac{N^{\tilde{y}}(\tilde{\lambda})}{M^{m,\tilde{y}}} = h(\tilde{y}).$$

As $Id - \phi$ is strictly increasing, it has an inverse and $C(\tilde{y}) = C(\hat{y})$. Taking the conditional expectations with respect to $\tilde{y}$ in (7), we obtain

$$\phi'(\hat{y})u'(\phi(\hat{y})) = \mu(1 - \phi'(\hat{y}))h(\tilde{y})u'(\hat{y} - \phi(\hat{y})), \tag{8}$$

or

$$\phi'(z)\phi(z)^{\gamma-1} = \mu(1 - \phi'(z))(z - \phi(z))^{\gamma-1}h(z - \phi(z)), \text{ for all } z \tag{9}$$

which gives us a differential equation that, up to a constant, ϕ should satisfy. Although this equation seems to be a standard differential equation, it raises a fixed point problem.[10] Indeed the solutions ϕ of the differential equation above depends on h. The function h depends on the net production equilibrium which in turn depends on ϕ since ϕ determines the net production set. When h and ϕ are independent, Equation (9) may be solved to determine ϕ. This is the case in the homogeneous case where $h = 1$. It is also the case when the net production set (and the net production equilibrium) are taken as primitive as h is then exogenously given. The argument is detailed in the next section.

[10] Along the same line, to show that $((\phi, W), (\hat{c}^i)_i, (\hat{c}_m), \hat{q}, \hat{y})$ is a m-s equilibrium, given ϕ, we must prove existence of a production equilibrium $((\hat{c}^i)_i, \hat{q}, \tilde{y})$ associated to Y^ϕ. We then have to find a ϕ such that the maximum of $U_m(\phi(y))$ on Y exists and is reached at $\hat{y}$ defined by $\hat{y} - \phi(\hat{y}) = \tilde{y}$. We are thus led to find a zero of a correspondence $\phi \rightrightarrows (\mathrm{Id} - \phi)^{-1}(\tilde{y}) - \arg\max_Y U_m(\phi(y))$. However, this correspondence may not be well defined as for a given ϕ the existence of a production equilibrium associated to Y^ϕ is not guaranted nor that of a solution to the problem $\max_Y U_m(\phi(y))$.

5 From Gross to Net Production

In order to construct a m-s equilibrium associated to a production set Y, summarizing previous results, we may take $W = \{0\}$.

In this section, we construct m-s equilibria assuming that the net production set is given as primitive. To this end, we consider a set $\mathcal{Y}$, the net production set for which there exists a production equilibrium $((\hat{c}^i)_i, \hat{q}, \tilde{y})$. We show that we can find a compensation ϕ and a gross production set Y such that $((\phi, \{0\}), (\hat{c}^i)_i, \phi(\hat{y}), \hat{q}, \hat{y})$ with $\hat{y} - \phi(\hat{y}) = \tilde{y}$ is a m-s equilibrium for Y.

The method we use is to transform Equation (9) into an equation that can be explicitly solved, first, by making a change of variable $z \to z - \phi(z)$ and next, by making a change of function: we consider the compensation as a function of net production, in other words, the function ψ such that $\phi(\hat{y}) = \psi(\hat{y} - \phi(\hat{y}))$. These transformations will enable us to carry on the analysis in terms of $\hat{y} - \phi(\hat{y})$, the net equilibrium production left to the shareholders after the manager has been paid, instead of the gross equilibrium production $\hat{y}$.

Let $((\hat{c}^i)_i, \hat{q}, \tilde{y})$ be a production equilibrium associated to $\mathcal{Y}$ and $\tilde{\lambda}$ be the associated Lagrange multipliers. We assume that the random variable $\tilde{y}$ has a positive density on $(0, a)$ for $a \in R \cup \{\infty\}$. Let $M^{m,\tilde{y}} = E\left[M^m \middle| \tilde{y}\right]$ and $N^{\tilde{y}}(\tilde{\lambda}) = E\left[N(\tilde{\lambda}) \middle| \tilde{y}\right]$. There exists a measurable function $h : (0, a) \to R_+$ such that

$$\frac{N^{\tilde{y}}(\tilde{\lambda})}{M^{m,\tilde{y}}} = h(\tilde{y}).$$

We assume that h satisfies the following condition[11]:

Assumption (H) For $\gamma > 0$, the integral of $h(u)u^{\gamma-1}$ is convergent at 0. For $\gamma < 0$, the integral of $h(u)u^{\gamma-1}$ is divergent at 0 and convergent at a when $a = \infty$.

For a given $\mu > 0$, we can then define ψ_μ by

$$\begin{cases} \psi_\mu(z)^\gamma = \gamma\mu \int_0^z h(u)u^{\gamma-1}du & \text{when } \gamma > 0, \\ \psi_\mu(z)^\gamma = C - \gamma\mu \int_z^a h(u)u^{\gamma-1}du & \text{when } \gamma < 0. \end{cases} \tag{10}$$

[11] The Assumptions at 0 are, for instance, satisfied if there exists some $\varepsilon > 0$ such that $\lim_{z\to 0} h(z)z^{\gamma-\varepsilon} = 0$ when $\gamma > 0$, and $h(z)z^\gamma$ is bounded away from 0 at 0 when $\gamma < 0$.

Thanks to Assumption (H) ψ_μ is well defined, continuous and increasing. In fact, ψ_μ is our candidate compensation scheme as a function of the net production. More precisely, if z_N is a given level of net production and $\psi_\mu(z_N)$ the associated compensation, the gross production is given by $z_G = z_N + \psi_\mu(z_N)$. Conversely, if z_G is a given level of gross production, $z_N = (\mathrm{Id} + \psi_\mu)^{-1}(z_G)$ is the corresponding net production and $\psi_\mu(z_N)$ (resp. $\psi_\mu((\mathrm{Id} + \psi_\mu)^{-1}(z_G))$) define the compensation as a function of the net (resp. gross) production. It is then natural to define ϕ_μ by

$$\phi_\mu(z) = \psi_\mu((\mathrm{Id} + \psi_\mu)^{-1}(z)), \tag{11}$$

which is well defined and satisfies $\phi_\mu(0) = 0$. As usual $\phi_\mu : X_+ \to X_+$ and $\psi_\mu : X_+ \to X_+$ are defined by $\phi_\mu(y)(\omega) = \phi_\mu(y(\omega))$ and $\psi_\mu(y)(\omega) = \psi_\mu(y(\omega))$ a.e.

Finally, we define Y_μ by $Y_\mu = \mathcal{Y} + \psi_\mu(\mathcal{Y})$. This set is the natural gross production set when the net production set is given by $\mathcal{Y}$ and when the compensation in terms of net production is given by ψ_μ. We have the following:

Theorem 6 *Let $\mathcal{Y}$ be smooth and let $((\hat{c}^i)_i, \hat{q}, \tilde{y})$ be a production equilibrium associated to $\mathcal{Y}$ such that $\hat{q} \in \mathcal{C}(\tilde{y})$ and $M^m \in \mathcal{C}(\tilde{y})$. If $\mathcal{Y}$ is considered as the net production set and if ψ_μ defined by (10) describes the compensation as a function of the net production, then the gross production set is given by $Y_\mu = \mathcal{Y} + \psi_\mu(\mathcal{Y})$ and the compensation as a function of the gross production is given by ϕ_μ as defined by (11). Assume (C) and (H) and $u \circ \psi_\mu$ strictly concave, then $((\phi_\mu, \{0\}), (\hat{c}^i)_i, \phi_\mu(\hat{y}), \hat{q}, \hat{y})$ with $\hat{y} = \tilde{y} + \psi_\mu(\tilde{y})$ is a m-s equilibrium associated to the production set $Y_\mu = \mathcal{Y} + \psi_\mu(\mathcal{Y})$.*

Remark 7 *Note that when the set $\mathcal{Y}$ is defined by $\mathcal{Y} = \{y : E[G(y)] \leq 0\}$ for some differentiable function G, the condition $\hat{q} \in \mathcal{C}(\tilde{y})$ is automatically fulfilled at a production equilibrium. Indeed, for such a set we have $N_{\mathcal{Y}}(\tilde{y}) = \{\lambda G'(\tilde{y})\}$ and, at equilibrium, we have $\hat{q} \in N_{\mathcal{Y}}(\tilde{y})$ which gives $\hat{q} = \lambda G'(\tilde{y}) \in \mathcal{C}(\tilde{y})$ for some $\lambda > 0$.*

To summarize, to construct a m-s equilibrium, it suffices to start with a set $\mathcal{Y}$ with a production equilibrium $((\hat{c}^i)_i, \hat{q}, \tilde{y})$ such that $\hat{q} \in C(\tilde{y})$ and

$M^m \in C(\tilde{y})$ and to solve for ψ_μ using (10). Then if $u \circ \psi_\mu$ is strictly concave, $((\phi_\mu, \{0\}), (\hat{c}^i)_i, \phi_\mu(\hat{y}), \hat{q}, \hat{y})$ with ϕ_μ defined by (11) and $\hat{y} = \tilde{y} + \psi_\mu(\tilde{y})$ is a m-s equilibrium associated to the production set $Y_\mu = \mathcal{Y} + \psi_\mu(\mathcal{Y})$.

In order to provide an existence theorem for m-s equilibria, let us now introduce a definition and an assumption. We will say that $\mathcal{Y}$ is strictly convex from above if for $(y_1, y_2) \in \mathcal{Y}^2$ and $t \in (0,1)$, there exists $y \in \mathcal{Y}$ such that $ty_1 + (1-t)y_2 < y$.[12]

Assumption (P')

- P'1 $Y = K - X_+$ where $K \subset X_+$ and such that $0 < \varsigma \leq K \leq \Xi$,
- P'2 Y is closed, strictly convex from above and smooth,
- P'3 If $y \in Eff^+(Y)$, the random variable y has a positive density on $(0, a_y)$ for some $a_y \in R \cup \{\infty\}$.
- P'4 For all i, $M^i \varsigma^{\gamma-1}$ belongs to X' and $M^i \varsigma^\gamma$ and $M^i \Xi^\gamma$ belong to $L^1(\Omega, \mathcal{F}, P)$.

Assumption (P') provides conditions on $\mathcal{Y}$ under which a production equilibrium exists. The following corollary can then be easily derived.

Corollary 8 *Assume (P'), then there exists a production equilibrium $((\hat{c}^i)_i, \hat{q}, \tilde{y})$ associated to $\mathcal{Y}$. Let ψ_μ be defined by (10) and ϕ_μ by (11). Assume (H), $\hat{q} \in \mathcal{C}(\tilde{y})$, $M^m \in \mathcal{C}(\tilde{y})$ and $u \circ \psi_\mu$ strictly concave. Then $((\phi_\mu, \{0\}), (\hat{c}^i)_i, \phi_\mu(\hat{y}), \hat{q}, \hat{y})$ with $\hat{y} = \tilde{y} + \psi_\mu(\tilde{y})$ is a m-s equilibrium associated to the production set $Y_\mu = \mathcal{Y} + \psi_\mu(\mathcal{Y})$.*

6 Properties of the equilibrium compensation

In this section, we consider the same setting as in the previous section and we wish to highlight some properties that the compensation schemes

[12] Let us recall that Y is strictly convex when for all $(y_1, y_2) \in Y^2$ and $t \in (0,1)$, there exists $ty_1 + (1-t)y_2 \in \text{int}(y)$. It is immediate that strict convexity implies strict convexity from above. However strict convexity is a much stronger condition and requires, in particular, a nonempty interior.

defined by Equations (10) and (11) should have. We first focus on the case of homogeneous beliefs and then consider a case in which beliefs are heterogenous.

6.1 Common Beliefs

Suppose that shareholders and the manager have identical beliefs, $M^i = M^m$ for all i. In that case $h(z) = 1$ and from (10) and (11), we obtain the following result.

Corollary 9 *If $M^i = M^m$ for all i, and if $((\phi, W)(\hat{c}^i)_i, (\hat{c}_m), \hat{q}, \hat{y})$ is a manager-shareholders equilibrium then*

1. *if $\gamma > 0, \psi$ and ϕ are linear,*
2. *if $\gamma < 0, \psi^\gamma = C + \mu z^\gamma$ and $\phi(z)^\gamma = C + \mu\left(z - \phi(z)\right)^\gamma$ for $C, \mu > 0$. In particular, for $C = 0$, we have ψ and ϕ linear.*

6.2 Heterogeneous Beliefs

We now highlight the effect of heterogenous beliefs on the shape of the compensation. We introduce the following assumption that implies some boundary properties on h. We say that agent i is more optimistic than agent j (or that agent j is more pessimistic than agent i) with respect to the net production y if[13]

$$\lim_{y\to\infty} \frac{M^i}{M^j} = \infty \text{ and } \lim_{y\to 0} \frac{M^i}{M^j} = 0. \tag{12}$$

We assume that there exists some agent i more optimistic than the manager and some agent j more pessimistic than the manager with respect to $\tilde{y} = \hat{y} - \phi(\hat{y})$. In particular, there exists a shareholder i and a shareholder j such that

$$\lim_{\tilde{y}\to\infty} \frac{M^i}{M^m} = \infty \text{ and } \lim_{\tilde{y}\to 0} \frac{M^m}{M^j} = 0. \tag{13}$$

[13] We assume that the support of the net production plan is $(0, \infty)$. All the results can be easily adapted to the $(0, a)$ case.

It is then easy to check that,

$$\lim_{\tilde{y}\to\infty} \frac{N(\tilde{\lambda})}{M^m} = \lim_{\tilde{y}\to 0} \frac{N(\tilde{\lambda})}{M^m} = \infty. \tag{14}$$

The reason is that

$$\frac{N(\tilde{\lambda})}{M^m} > \frac{\tilde{\lambda}^i M^i}{M^m} \text{ for all } i,$$

and that holds in particular for any agent who is more optimistic than the manager. That gives the result in (14) for $\tilde{y} \to \infty$. Similarly, we have

$$\frac{M^m}{N(\tilde{\lambda})} < \frac{M^m}{\tilde{\lambda}^j M^j},$$

where j is the most pessimistic agent. That gives the result in (14) for $\tilde{y} \to 0$. Therefore, under Assumption (13), from equation (7) the manager acts as if she had the representative belief $N(\tilde{\lambda})$ only if the ratio of her instantaneous marginal utility at $\phi(\hat{y})$ and that of the representative agent at $\tilde{y}$ is very large when equilibrium production $\tilde{y}$ is either very large or very low. From equation (8), we must have $h(\tilde{y}) \to \infty$ as $\tilde{y} \to \infty$ or $\tilde{y} \to 0$ or equivalently

$$\lim_{z\to 0} h(z) = \lim_{z\to\infty} h(z) = \infty. \tag{15}$$

Therefore, the compensation should increase the weight that the manager attaches to extreme realizations of the production. This can be expressed by considering the compensation rate, defined by the function $\chi : (0, \infty) \to \mathbb{R}_+$ such that

$$\chi(z) = \frac{\psi(z)}{z}, \; z \in (0, \infty).$$

We can show the following result.

Corollary 10 *Assume that $M^i \neq M^j$ for some i and j. Assume (H) and (13) hold. If there exists a m-s equilibrium, then the compensation rate in terms of net production should verify*

$$\begin{aligned} \lim\nolimits_{z\to 0} \chi(z) = \lim\nolimits_{z\to\infty} \chi(z) = 0, \; \textit{for } \gamma < 0, \\ \lim\nolimits_{z\to 0} \chi(z) = \lim\nolimits_{z\to\infty} \chi(z) = \infty, \; \textit{for } \gamma > 0. \end{aligned} \tag{16}$$

Equivalently, we prove that $\chi(z)^\gamma \to \infty$ as $z \to 0$ or $z \to \infty$, which gives the properties of $\chi(z)$ depending on the sign of γ as stated in (16). Let us see the consequences of (16) on ϕ, that is the compensation defined in terms of gross production, and let us define

$$\varkappa(z) = \frac{\phi(z)}{z},\ z \in (0, \infty).$$

Corollary 11 *Assume $M^i \neq M^j$ for some i and j. Suppose Assumption (H) and (13) hold. If there exists a m-s equilibrium, then the compensation rate should verify*

$$\begin{aligned} &\lim_{z\to 0} \varkappa(z) = \lim_{z\to\infty} \varkappa(z) = 0,\ \textit{for } \gamma < 0, \\ &\lim_{z\to 0} \varkappa(z) = \lim_{z\to\infty} \varkappa(z) = 1,\ \textit{for } \gamma > 0. \end{aligned}$$

7 Example

Let the net production set Y^ϕ be given by

$$\mathcal{Y}_{a,b,\theta_0} = \left\{ \exp\left(m(\theta)t + \theta\sqrt{t}\widetilde{x} \right) : \theta \geq 0 \right\},$$

where $\widetilde{x} \sim \mathcal{N}(0,1),\ m(\theta) = a - b\left(\theta - \theta_0\right)^2$ and a, b and θ_0 are given positive constants. The value of θ is set by the manager and a given choice of θ generates the production plan $y_\theta = \exp\left(m(\theta)t + \theta\sqrt{t}\widetilde{x}\right)$.

We assume there are two shareholders with CRRA utility as in (2) with $\gamma = -1$. Shareholders only consume at time t and they have heterogeneous beliefs indexed by β. An agent of type β believes that $\widetilde{x} \sim \mathcal{N}(\beta, 1)$ and we assume $\beta \in \{\delta, -\delta\}$ with $\delta > 0$. We have

$$M^\beta = \frac{1}{\sqrt{2\pi}} \exp -\frac{(\widetilde{x} - \beta)^2}{2},\ \beta \in \{\delta, -\delta\}.$$

All densities are expressed with respect to Lebesgue measure. We denote with ν_β the proportion of stocks of shareholder of type β. In order to insure the existence of a production equilibrium, we assume that

$$\frac{1}{\delta^2}(1 - 2b) + \frac{1}{1 + \exp -\frac{\delta^2}{2}} < 0. \tag{17}$$

Proposition 12 *Under (17), there exists a unique net production equilibrium* $(\hat{c}^{\delta}, \hat{c}^{-\delta}, \hat{q}, \tilde{y})$ *such that* $\hat{q} \in C(\tilde{y})$ *with* $\tilde{y} = \exp\left(m(\tilde{\theta})t + \tilde{\theta}\sqrt{t}\tilde{x}\right)$ *and*

$$\tilde{\theta} = \frac{2b}{2b+1}\theta_0 + \frac{1}{\sqrt{t}}\frac{\delta}{2b+1}\left(2\nu_{\delta} - 1\right), \tag{18}$$

$$\begin{aligned}
\hat{q} &= \left(\lambda_{\tilde{\theta}} \exp -\frac{(\tilde{x}-\delta)^2}{4} + \exp -\frac{(\tilde{x}+\delta)^2}{4}\right)^2 (\tilde{y})^{-2}, \\
\hat{c}^{\delta} &= \frac{\lambda_{\tilde{\theta}} \exp -\frac{(\tilde{x}-\delta)^2}{4}}{\lambda_{\tilde{\theta}} \exp -\frac{(\tilde{x}-\delta)^2}{4} + \exp -\frac{(\tilde{x}+\delta)^2}{4}}\tilde{y}, \\
\hat{c}^{-\delta} &= \frac{\exp -\frac{(\tilde{x}+\delta)^2}{4}}{\lambda_{\tilde{\theta}} \exp -\frac{(\tilde{x}-\delta)^2}{4} + \exp -\frac{(\tilde{x}+\delta)^2}{4}}\tilde{y}
\end{aligned}$$

where $\lambda_{\tilde{\theta}}$ *the square root of agent's* δ *equilibrium utility weight when that of agent* $-\delta$ *is normalized to one is the nonnegative solution of*

$$\lambda^2 \exp(-\tilde{\theta}\delta\sqrt{t})\nu_{-\delta} + \lambda \exp(-\frac{\delta^2}{2})(2\nu_{-\delta} - 1) + \exp(\tilde{\theta}\delta\sqrt{t})(\nu_{-\delta} - 1) = 0. \tag{19}$$

We notice from (18) that the equilibrium production plan induces a larger θ and so a larger exposure to the random variable $\tilde{x}$ when shareholder of type δ has at least half of the shares ($2\nu_{\delta} > 1$) and is very optimistic (δ is large). If shareholders have the same endowment ($2\nu_{\delta} = 1$), $\hat{\theta}$ is always lower than θ_0 since any $\theta > \theta_0$ would increase the exposure to risk and at the same time decrease $m(\theta)$.

Let the representative agent belief be defined by

$$\tilde{M} = \frac{\left(\lambda_{\tilde{\theta}} \exp -\frac{(\tilde{x}-\delta)^2}{4} + \exp -\frac{(\tilde{x}+\delta)^2}{4}\right)^2}{E\left[\left(\lambda_{\tilde{\theta}} \exp -\frac{(\tilde{x}-\delta)^2}{4} + \exp -\frac{(\tilde{x}+\delta)^2}{4}\right)^2\right]}.$$

Note that as $C(\tilde{y}) = C(\tilde{x})$, $\hat{q} \in C(\tilde{y})$ and $\tilde{M} \in C(\tilde{y})$ and this example fulfills the hypotheses of the previous section.

From Corollary 5, if $((\phi, W), (\hat{c}^{\delta}, \hat{c}^{-\delta}), \hat{c}_m, \hat{q}, \hat{y})$ with $\hat{c}_m = \phi(\hat{y})$ and $\tilde{y} = \hat{y} - \phi(\hat{y}))$ is a m-s equilibrium without trading restrictions on the manager, then ϕ must be linear. Furthermore if ϕ is linear, then the manager implements the equilibrium plan $\hat{y}$ only if she has the representative agent belief ($M^m = \tilde{M}$). This means that unless the manager belief is given by $\tilde{M}$, we have $W \neq X$ and the existence of an m-s equilibrium requires the manager to be constrained.

Remark 13 *Suppose that the manager receives a linear compensation. Let $E^m[\widetilde{x}]$ and $Var^m[\widetilde{x}]$ denote respectively the mean and the variance of $\widetilde{x}$ as perceived by the manager. We then have*

$$E^m[\widetilde{x}] = \frac{(\lambda_{\tilde{\theta}}^2 - 1)\delta}{\lambda_{\tilde{\theta}}^2 + 1 + 2\lambda_{\tilde{\theta}} \exp -\frac{\delta^2}{2}}.$$

$$\mathrm{VAR}^m[\widetilde{x}] = \frac{\left(\lambda_{\tilde{\theta}}^2 + 1\right)(1+\delta^2) + 2\lambda_{\tilde{\theta}} \exp\left(-\frac{\delta^2}{2}\right)}{\lambda_{\tilde{\theta}}^2 + 1 + 2\lambda_{\tilde{\theta}} \exp -\frac{\delta^2}{2}} - \left(\frac{(\lambda_{\tilde{\theta}}^2 - 1)\delta}{\lambda_{\tilde{\theta}}^2 + 1 + 2\lambda_{\tilde{\theta}} \exp -\frac{\delta^2}{2}}\right)^2.$$

We remark (see the appendix) that $E^m[\widetilde{x}]$ is a weighted average between the mean as perceived by the optimistic shareholder (δ) and that perceived by the pessimistic shareholder ($-\delta$), with weight depending on ν_δ. When $\nu_\delta = 0$, $\lambda_{\tilde{\theta}} = 0$ and $E^m[\widetilde{x}] = -\delta$ while if $\nu_\delta = 1$, $\lambda_{\tilde{\theta}} = \infty$ and $E^m[\widetilde{x}] = \delta$. Moreover, $E^m[\widetilde{x}] > 0$ if and only if $\lambda_{\tilde{\theta}} > 1$, which is true if ν_δ is sufficiently large. On the other hand, while shareholders agree that $\mathrm{VAR}[\widetilde{x}] = 1$, we have (see the appendix) $\mathrm{VAR}^m[\widetilde{x}] > 1$. Hence the manager should overestimate the variance of $\widetilde{x}$. In other words, a manager with a linear compensation evaluating the risk as shareholders would underestimate the level of risk relative to what would lead her to choose $\hat{y}$. Hence, without constraints ($W = X$), the m-s equilibrium exists only if the manager overestimate the risk with respect to the shareholders.

Assuming next that the manager cannot trade, let us search for a compensation ψ in terms of net production such that $((\phi, \{0\}), (\hat{c}^\delta, \hat{c}^{-\delta}), \hat{c}_m, \hat{q}, \hat{y})$ with ϕ defined by (11), $\hat{y} = \tilde{y} + \psi(\tilde{y})$ and $\hat{c}_m = \phi(\hat{y})$ is a m-s equilibrium. Let us assume that the manager has a belief which coincides with the objective one ($\widetilde{x} \sim \mathcal{N}(0,1)$) and that shareholders have the same endowment,

$$\nu_\delta = \nu_{-\delta} = \frac{1}{2}. \tag{20}$$

Condition (20) implies

$$\lambda_{\tilde{\theta}} = \exp\left(\delta\tilde{\theta}\sqrt{t}\right).$$

It is also convenient to define the variable

$$k = \frac{\delta}{2\tilde{\theta}\sqrt{t}}.$$

In order to ensure that the resulting compensation is nonnegative for all $y \in (0, \infty)$, we need to impose $2k \leq 1$. The compensation is given by Equation (10) with

$$h(y) = \exp\left(4k\tilde{\theta}^2 t\right) \exp(-2k\mu(\tilde{\theta})t) y^{2k} + \exp(2km(\tilde{\theta})t) y^{-2k} + \exp\left(2k\tilde{\theta}^2 t\right).$$

From (10), we obtain

$$\psi_{(\mu,C)}(y) = \frac{\mu y}{C\mu y + 1 + \frac{y^{-2k}}{2k+1}\exp(2kt(m(\tilde{\theta}) - \tilde{\theta}^2)) - \frac{y^{2k}}{2k-1}\exp(2kt(\tilde{\theta}^2 - m(\tilde{\theta})))}, \tag{21}$$

where μ and C are arbitrary positive scalars. Note that $u \circ \psi_{(\mu,C)} = \frac{1}{\psi_{(\mu,C)}}$ is a positive combination of a constant and of three negative power functions. Hence $u \circ \psi_{(\mu,C)}$ is concave. We can then state the following:

Proposition 14 *Suppose that* $\nu_\delta = 1/2$, $2k \leq 1$ *and that the manager correctly believes that* $\tilde{x} \sim \mathcal{N}(0, 1)$. *There exists a two parameters family of compensation functions of net production which implement a m-s equilibrium associated to the net production set* $\mathcal{Y}$ *given by Equation (21).*

The corresponding compensation in terms of gross production follows from (11):

$$\phi_{(\mu,C)}(z) = \psi_{(\mu,C)}((\mathrm{Id} + \psi_{(\mu,C)})^{-1}(z))$$

It is easy to verify that $\lim_{z\to 0} h(z) = \lim_{z\to\infty} h(z) = \infty$. Hence, the compensation in (21) has the properties highlighted in Corollary 10: the compensation rate $\chi(y)$ goes to zero as $y \to 0$ or $y \to \infty$. Further properties are illustrated in a simple numerical example.

Suppose $C = 0, a = \frac{1}{2}(\theta_0 - 1)$ and $b = \frac{1}{2(\theta_0 - 1)}$. That gives $\tilde{\theta} = 1$ and $m(\tilde{\theta}) = 0$. Suppose also $\mu = 1$, $t = 1$, $\theta_0 = 3/2$ and $\delta = 1/2$ (shareholders' beliefs deviate from the objective belief by $\frac{1}{2}$ standard deviation). The resulting compensation rate $\chi(y)$ is displayed as solid line in Figure 1 and the total compensation $\psi(y)$ is displayed as solid line in Figure 2. We observe that the compensation rate $\chi(y)$ is inverted U-shaped while the compensation $\psi(y)$ is convex as $y \to 0$ and concave as $y \to \infty$.

In order to see the effect of shareholder heterogeneity, notice first that if shareholders agreed on the true distribution of $\widetilde{x}$ (that is, $\delta = -\delta = 0$) we would have $k = 0$ and expression (21) would give $\psi(y) = \tau y$ and (the linear compensation). In Figures 1 and 2, we also plot an optimal compensation rate $\chi(y)$ and compensation $\psi(y)$ for $\delta = 1/4$ (dashed lines) and $\delta = 3/4$ (dotted lines). As intuitive, the larger shareholders heterogeneity is (the larger δ), the larger are the required deviations from linearity.

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8 Appendix

Proof of Proposition 1. STEP 1 Monotonicity

We show first that $z \to \phi(z)$ and $z \to z - \phi(z)$ are both nondecreasing a.e. in the sense $\mu \otimes \mu\left(\{(z,z') \in \mathbb{R}^2 : (z-z')(\varphi(z) - \varphi(z')) < 0\}\right) = 0$.

We have $\hat{c}_m \in \mathcal{C}(\hat{y})$ and there exists a nonnegative measurable function C such that $\hat{c}_m = C(\hat{y})$. By Lebesgue Theorem, the derivative of $z \to \int_0^z C(s)ds$ exists and is equal to $C(z)$ almost everywhere. Let $F(z) = E\left[M_m | \hat{y} = z\right]$ and let us assume that $\mu(G) = 0$ where $G = \{z : F(z) = 0\}$. We have $0 = \int_{z \in G} F(z) h_y(z) dz = E\left[M_m 1_{\hat{y} \in G}\right]$ and $M_m > 0$ a.e. which gives $M_m = 0$ a.e. on G and since $M_m > 0$ a.e., we have $\mu(G) = 0$.

Let D the set of points z such that $\left(\int_0^z C(s)ds\right)' = C(z)$, ϕ is continuous at z, $F(z) > 0$ and $h_y(z) > 0$. We have $\mu\left(\mathbb{R}_+ \backslash D\right) = 0$.

Let us assume that ϕ is not nondecreasing and let

$$A = \left\{(z,z') \in \left(\mathbb{R}_+^*\right)^2 : (z-z')(\phi(z) - \phi_T(z')) < 0\right\}.$$

We have, $\mu^2(A) > 0$. Without loss of generality, we may replace A by $A \cap C^2$.

Let $(a,b) \in A$ with $a < b$ and then

$$\phi(a) > \phi(b)\ .$$

For $\eta > 0$, there exists $\varepsilon > 0$ such that $|\phi(z) - \phi(a)| < \eta\phi(a)$ for $z \in I = [a-\varepsilon, a+\varepsilon)$ and $|\phi(z) - \phi(b)| < \eta\phi(b)$ for $z \in J = [b-\varepsilon, b+\varepsilon)$. If η and ε are chosen such that $\frac{1-\eta}{1+\eta}\frac{\phi(a)}{\phi(b)} > 1$ and $\varepsilon < \frac{b-a}{2}$, we have $I < J$ and $\phi(I) > \phi(J)$. It is immediate that $\phi(a) < \frac{1}{1-\eta}\phi(z)$ for $z \in I$ and $\phi(b) < \frac{1}{1-\eta}\phi(z)$ for $z \in J$.

Let $f : \mathbb{R} \to \mathbb{R} \setminus J$ be defined by $f(z) = \frac{1}{2}(a-\varepsilon) + \frac{1}{2}z$ on I, $f(z) = a + \frac{1}{2}(z - b + \varepsilon)$ on J and $f(z) = z$ elsewhere. We have $f(z) \leq z$ for all z and f admits an inverse denoted by g.

If $z \in I$ then $f(z) \in I$, and $|\phi(z) - \phi(f(z))| < 2\eta\phi(a) < \frac{2\eta}{1-\eta}\phi(z)$. We further impose $\frac{2\eta}{1-\eta} < 1$ and we have $\phi(f(z)) > (1 - \frac{2\eta}{1-\eta})\phi(z)$.

If $z \in J$ then $f(z) \in I$ and

$$\phi(z) - \phi(f(z)) < (1+\eta)\,\phi(b) - (1-\eta)\,\phi(a) < 0. \tag{22}$$

We also have $\phi(f(z)) < (1+\eta)\,\phi(a) < k\phi(z)$ with $k = \frac{1+\eta}{1-\eta}\frac{\phi(a)}{\phi(b)} > 1$.

Elsewhere, we have $\phi(z) = \phi(f(z))$.

Let us define the random variable $\tilde{y}$ by $\tilde{y} = f(\hat{y})$. We have $\tilde{y} \leq \hat{y}$ then $\tilde{y} \in Y$. By definition of g, we have $\hat{y} = g(\tilde{y})$. They generate then the same information structure and we have $\mathcal{C}(\tilde{y}) = \mathcal{C}(\hat{y})$.

By definition of the indirect utility function, $V_m(\hat{y}, \hat{q}) = U_m(\hat{c}_m)$. Let $\hat{w}$ be defined by $\hat{w} = \hat{c}_m - \phi(\hat{y})$, we have $\hat{w} \in \mathcal{C}(\hat{y}) \cap W$ and $\hat{q} \cdot \hat{w} \leq 0$. If we define $\tilde{c}_m$ by $\tilde{c}_m = \phi(\tilde{y}) + (1 - \varepsilon\frac{2\eta}{1-\eta})\hat{w}$, $\tilde{c}_m$ is a function of $\tilde{y}$. We further impose $\varepsilon < 1$.

On $\{\hat{y} \in I\}$, $\tilde{c}_m = \phi(f(\hat{y})) + (1-\varepsilon\frac{2\eta}{1-\eta})\hat{w} > (1-\frac{2\eta}{1-\eta})\left(\phi(\hat{y}) + \hat{w}\right) \geq (1-\frac{2\eta}{1-\eta})\hat{c}_m \geq 0$. From there, we derive $u(\hat{c}_m) - u(\tilde{c}_m) \leq \left(1 - (1-\frac{2\eta}{1-\eta})^\gamma\right) u(\hat{c}_m)$. By the market clearing condition, we have $\hat{c}_m \leq \hat{y}$ and $u(\hat{c}_m) - u(\tilde{c}_m) \leq \left(1 - (1-\frac{2\eta}{1-\eta})^\gamma\right) u(\hat{y})$.

On $\{\hat{y} \in J\}$, $\tilde{c}_m = \phi(f(\hat{y})) + (1-\varepsilon\frac{2\eta}{1-\eta})\hat{w} > \phi(\hat{y}) + (1-\frac{2\eta}{1-\eta})\hat{w} > (1-\frac{2\eta}{1-\eta})\left(\phi(\hat{y}) + \hat{w}\right) \geq 0$. Furthermore, since u is increasing and concave, we have

$$\begin{aligned}
&u(\hat{c}_m) - u(\tilde{c}_m) \\
&\leq u\left(\phi(\hat{y}) + \hat{w}\right) - u\left(\phi(f(\hat{y})) + (1-\varepsilon\frac{2\eta}{1-\eta})\hat{w}\right) \\
&\leq \left(\phi(\hat{y}) - \phi(f(\hat{y})) + \varepsilon\frac{2\eta}{1-\eta}\hat{w}\right) u'(c).
\end{aligned}$$

where $c = \hat{c}_m$ if $\phi(\hat{y}) - \phi(f(\hat{y})) + \varepsilon\frac{2\eta}{1-\eta}\hat{w} \geq 0$ and $c = \phi(f(\hat{y})) + (1-\varepsilon\frac{2\eta}{1-\eta})\hat{w} \leq k\left(\phi(\hat{y}) + \hat{w}\right)$ elsewhere.

From there and using (22), we have

$$\begin{aligned}
&u(\hat{c}_m) - u(\tilde{c}_m) \\
&\leq \sup_{h=1,k}\left((1+\eta)\,\phi(b) - (1-\eta)\,\phi(a) + \varepsilon\frac{2\eta}{1-\eta}\hat{w}\right) u'\left(h\hat{c}_m\right)
\end{aligned}$$

and since $(1+\eta)\,\phi(b)-(1-\eta)\,\phi(a)<0$, we have

$$
\begin{aligned}
& u(\hat{c}_m)-u(\tilde{c}_m) \\
& \leq k^{\gamma-1}\left((1+\eta)\,\phi(b)-(1-\eta)\,\phi(a)\right)u'\left(\hat{c}_m\right)+\varepsilon\frac{2\eta}{1-\eta}\left(k^{\gamma-1}\hat{w}1_{\hat{w}\geq 0}+\hat{w}1_{\hat{w}\geq 0}\right)u'\left(\hat{c}_m\right) \\
& \leq k^{\gamma-1}\left((1+\eta)\,\phi(b)-(1-\eta)\,\phi(a)\right)u'\left(\hat{y}\right)+\varepsilon\frac{2\eta}{1-\eta}\left(k^{\gamma-1}\hat{w}1_{\hat{w}\geq 0}+\hat{w}1_{\hat{w}\geq 0}\right)u'\left(\hat{c}_m\right)
\end{aligned}
$$

Finally, on $\{\hat{y}\notin I\cup J\}$, $\tilde{c}_m=\phi(\hat{y})+(1-\varepsilon\frac{2\eta}{1-\eta})\hat{w}\geq(1-\varepsilon\frac{2\eta}{1-\eta})\left(\phi(\hat{y})+\hat{w}\right)=(1-\varepsilon\frac{2\eta}{1-\eta})\hat{c}_m\geq 0$. Furthermore, since u is increasing $u(\hat{c}_m)-u(\tilde{c}_m)\leq u(\hat{c}_m)-u\left((1-\varepsilon\frac{2\eta}{1-\eta})\hat{c}_m\right)\leq u(\hat{c}_m)\left(1-(1-\varepsilon\frac{2\eta}{1-\eta})^\gamma\right)\leq u(\hat{y})\left(1-(1-\varepsilon\frac{2\eta}{1-\eta})^\gamma\right)$.

We have then $\tilde{c}_m\geq 0$ a.e. and then $\tilde{c}_m\in\mathcal{C}^m(\tilde{y})$. Furthermore, we have $\hat{q}\cdot\tilde{c}_m=\hat{q}\cdot\phi(\tilde{y})+\hat{q}\cdot\hat{w}\leq\hat{q}\cdot\phi(\tilde{y})$ which gives $V_m(\tilde{y},\hat{q})\geq U_m(\tilde{c}_m)$. From Condition 4 in the equilibrium definition, we have

$$
U_m(\hat{c}_m)=V_m(\hat{y},\hat{q})>V_m(\tilde{y},\hat{q})\geq U_m(\tilde{c}_m). \tag{23}
$$

However, from the above inequalities on $u(\hat{c}_m\left(\omega\right))-u(\tilde{c}_m\left(\omega\right))$, we have

$$
\begin{aligned}
& \frac{1}{2\varepsilon}\left[U_m(\hat{c}_m)-U_m(\tilde{c}_m)\right] \\
& \quad\leq\frac{1}{2\varepsilon}(1-(1-\frac{2\eta}{1-\eta})^\gamma)E\left[M_m u(\hat{y})1_{\hat{y}\in I}\right] \\
& \quad+\frac{1}{2\varepsilon}k^{\gamma-1}\left((1+\eta)\,\phi(b)-(1-\eta)\,\phi(a)\right)E\left[M_m u'\left(\hat{y}\right)1_{\hat{y}\in J}\right] \\
& \quad+\frac{\eta}{1-\eta}E\left[M_m\left(k^{\gamma-1}\hat{w}1_{\hat{w}\geq 0}+\hat{w}1_{\hat{w}\geq 0}\right)u'(\hat{c}_m)1_{\hat{y}\in J}\right] \\
& \quad+\frac{1}{2\varepsilon}(1-(1-\varepsilon\frac{2\eta}{1-\eta})^\gamma)E\left[M_m u(\hat{c}_m)1_{\hat{y}\notin I\cup J}\right].
\end{aligned} \tag{24}
$$

Let us show that $M_m u'\left(\hat{c}_m\right)\hat{w}\in L^1(\Omega,\mathcal{F},P)$ and for this purpose let us consider separately its positive part and its negative part. Since u is increasing, the sign of $M_m u'\left(\hat{c}_m\right)\hat{w}$ only depends on the sign of $\hat{w}$. Let us define c_ε by $c_\varepsilon=\phi(\hat{y})+(1-\varepsilon)\,\hat{w}$, we have $c_\varepsilon\in\mathcal{C}(\hat{y})\cap W$ and $\hat{q}\cdot c_\varepsilon\leq\hat{q}\cdot\phi(\hat{y})$ which gives $U(c_\varepsilon)\leq U(\hat{c}_m)$. For $\hat{w}\geq 0$, we have $0\leq\frac{u(\hat{c}_m)-u(c_\varepsilon)}{\varepsilon}\leq u'\left(\phi(\hat{y})\right)\hat{w}$ and since $\frac{u(\hat{c}_m)-u(c_\varepsilon)}{\varepsilon}\rightarrow u'\left(\hat{c}_m\right)\hat{w}$, Assumption (F) and the dominated convergence Theorem give $\lim_{\varepsilon\rightarrow 0}E\left[M_m\frac{u(\hat{c}_m)-u(c_\varepsilon)}{\varepsilon}1_{\hat{w}\geq 0}\right]=E\left[M_m u'\left(\phi(\hat{y})\right)\hat{w}1_{\hat{w}\geq 0}\right]<\infty$. Let us assume that $-E\left[M_m u'\left(\hat{c}_m\right)\hat{w}1_{\hat{w}\leq 0}\right]=\infty$. Since u is increasing, we have $\left(u(c_\varepsilon)-u(\hat{c}_m)\right)1_{\hat{w}\leq 0}\geq 0$ and by Fatou's Lemma, we have

$\liminf_{\varepsilon\to 0} E\left[M_m \frac{u(c_\varepsilon)-u(\hat{c}_m)}{\varepsilon} 1_{\hat{w}\leq 0}\right] \geq -E\left[u'(\hat{c}_m)\hat{w}1_{\hat{w}\leq 0}\right] = \infty$. From there $\lim_{\varepsilon\to 0} E\left[M_m \frac{u(c_\varepsilon)-u(\hat{c}_m)}{\varepsilon}\right] = \infty$ which gives $U(c_\varepsilon) > U(\hat{c}_m)$ for ε sufficiently small. We have then a contradiction and then $-E\left[M_m u'(\phi(\hat{y}))\hat{w}1_{\hat{w}\leq 0}\right] < \infty$ which gives $L^1(\Omega, \mathcal{F}, P)$. As a consequence, we have

$\lim_{\varepsilon\to 0} E\left[M_m\left(k^{\gamma-1}\hat{w}1_{\hat{w}\geq 0} + \hat{w}1_{\hat{w}\geq 0}\right)u'(\hat{c}_m)1_{\hat{y}\in J}\right] = 0$.

Taking the limit when ε goes to 0, in (24), we obtain

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\left[U_m(\hat{c}_m) - U_m(\tilde{c}_m)\right] \\
&= (1-(1-\frac{2\eta}{1-\eta})^\gamma)F(a)u(a) \\
&+ k^{\gamma-1}\left((1+\eta)\phi(b) - (1-\eta)\phi(a)\right)F(a)u'(a) \\
&+ \gamma\frac{\eta}{1-\eta}E\left[M_m u(\hat{c}_m)\right].
\end{aligned}$$

Taking the limit when η goes to 0, we obtain $k^{\gamma-1}\left(\phi(b)-\phi(a)\right)F(a)u'(a) < 0$.

Hence, for η and ε sufficiently small, $U_m(\phi(\hat{y})+\hat{w}) < U_m(\phi(\tilde{y})+\hat{w})$ which contradicts 23. Therefore, the compensation $z\to\phi(z)$ is increasing.

Similarly, replacing the manager by one of the shareholders, we obtain that the net production $z\to z-\phi(z)$ is increasing.

Let us show that $M_m u'(\hat{c}_m)\hat{w}\in L^1$ and for this purpose let us consider separately its positive part and its negative part. Since u is increasing, the sign of $M_m u'(\hat{c}_m)\hat{w}$ only depends on the sign of $\hat{w}$. Let us define c_ε by $c_\varepsilon = \phi(\hat{y})+(1-\varepsilon)\hat{w}$, we have $(1-\varepsilon)\hat{w}\in\mathcal{C}(\hat{y})\cap W$ and $\hat{q}\cdot c_\varepsilon \leq \hat{q}\cdot\phi(\hat{y})$ which gives $U(c_\varepsilon)\leq U(\hat{c}_m)$. For $\hat{w}\geq 0$, we have $0\leq \frac{u(\hat{c}_m)-u(c_\varepsilon)}{\varepsilon} \leq u'(\phi(\hat{y}))\hat{w}$ and since $\frac{u(\hat{c}_m)-u(c_\varepsilon)}{\varepsilon}\to u'(\hat{c}_m)\hat{w}$, Assumption (F) and the dominated convergence Theorem give $\lim_{\varepsilon\to 0} E\left[M_m\frac{u(\hat{c}_m)-u(c_\varepsilon)}{\varepsilon}1_{\hat{w}\geq 0}\right] = E\left[M_m u'(\phi(\hat{y}))\hat{w}1_{\hat{w}\geq 0}\right] < \infty$. Let us assume that $-E\left[M_m u'(\hat{c}_m)\hat{w}1_{\hat{w}\leq 0}\right] = \infty$. Since u is increasing, we have $\left(u(c_\varepsilon)-u(\hat{c}_m)\right)1_{\hat{w}\leq 0}\geq 0$ and by Fatou's Lemma, we have $\liminf_{\varepsilon\to 0} E\left[M_m \frac{u(c_\varepsilon)-u(\hat{c}_m)}{\varepsilon} 1_{\hat{w}\leq 0}\right] \geq -E\left[u'(\hat{c}_m)\hat{w}1_{\hat{w}\leq 0}\right] = \infty$. From there $\lim_{\varepsilon\to 0} E\left[M_m \frac{u(c_\varepsilon)-u(\hat{c}_m)}{\varepsilon}\right] = \infty$ which gives $U(c_\varepsilon) > U(\hat{c}_m)$ for ε sufficiently small. We have then a contradiction and then $-E\left[M_m u'(\phi(\hat{y}))\hat{w}1_{\hat{w}\leq 0}\right] < \infty$ which gives $u'(\hat{c}_m)\hat{w}\in L^1$.

Step 2. Differentiability

Since $z\to\phi(z)$ and $z\to z-\phi(z)$ are nondecreasing then $z\to\phi(z)$ is

1-Lipshitz and then differentiable a.e.

STEP 3. An approximation Lemma.

When some consumption plan c does not satisfy the market constraints $c \in \mathcal{C}(y)$ for some $y \in Y$, the following Lemma establishes that the market constraints are satisfied for some slight perturbation of y.

Lemma 15 *For $(x, x') \in X_+^2$ and $\varepsilon > 0$, there exists $x'' \in X_+$ such that $0 \le x' - x'' \le \varepsilon x'$ and $\mathcal{C}(x) \subset \mathcal{C}(x'')$. In particular, if $x' \in Y$ then $x'' \in Y$.*

■

Proof of the Lemma. Let $s_n = (1+\varepsilon)^{\frac{n}{2}}$ for $n \in \mathbb{Z}$. The family $S = (s_n)_{n\in\mathbb{Z}}$ is an ordered family with $\lim_\infty s_n = \infty$, $\lim_{-\infty} s_n = 0$. Let $z \ge 0$ and $z' > 0$. There exits $n \in \mathbb{Z}$ such that $s_n \le z' < s_{n+1}$ and we define h by $h(z, z') = s_n - (s_n - s_{n-1}) \frac{z}{1+z}$. We have $0 < (1-\varepsilon) z' < (1-\varepsilon) s_{n+1} \le s_{n-1} < h(z, z') \le s_n \le z' < s_{n+1}$. Let us suppose now that we know $h(z, z')$ without knowing z nor z'. The inequalities $s_{n-1} < h(z, z') \le s_n$ uniquely define a pair (s_n, s_{n-1}). z is uniquely determined by the equation $h(z, z') = s_n - (s_n - s_{n-1}) \frac{z}{1+z}$. Let $(x, x') \in X_+^2$, and let $x'' = h(x, x')$. We have $(1-\varepsilon) x' < x'' \le x'$ or $0 \le x' - x'' \le \varepsilon x'$. Furthermore, knowing x'' permits to determine x and we have $\mathcal{C}(x) \subset \mathcal{C}(x'')$. Finally, if $x' \in Y$ then $x'' \le x'$ leads to $x'' \in Y$.

STEP 4. Shareholder's income maximization

We first show that at equilibrium, the price $\hat{q}$ should be strictly positive. We then show that $\hat{y}$ should maximize on Y_+ any shareholder's income at price $\hat{q}$,i.e. $\hat{q} \cdot (\hat{y} - \phi(\hat{y})) = \max_Y \hat{q} \cdot (y - \phi(y))$ and $\hat{y} \in \text{Eff}^+(Y)$. Let us first show that $\hat{q} \gg 0$. Indeed if for some t, $\hat{q}_t \le 0$ on a set A of positive measure, the shareholders' and manager's demand could be arbitrarily large on A violating assertion 3 of the definition of equilibrium.

Let (ϕ, W) be given and $((\hat{c}^i)_i, \hat{c}_m, \hat{q}, \hat{y})$ be a manager-shareholder equilibrium. From Assertions 2 and 4 in Definition 1, the manager solves the following problem:

$$\max_{y\in Y} \max_{c} U_m(c) \text{ s.t. } c \in \mathcal{C}^m(y) \text{ and } \hat{q} \cdot c \le \hat{q} \cdot \phi(y). \tag{25}$$

As $\hat{c}_m$ solves Assertion 2 and U_m is increasing, $\hat{c}_m = \phi(\hat{y}) + \hat{w}$ for some $\hat{w} \in W \cap \mathcal{C}(\hat{y})$ and $\hat{q} \cdot \hat{c}_m = \hat{q} \cdot \phi(\hat{y})$. Let c be a consumption plan of the form,

$c = \phi(y) + \hat{w}$ for some $y \in K$ such that $U_m(c)$ is well defined with $U_m(c) > U_m(\hat{c}_m)$. Since ϕ is nondecreasing, we have $\min\left[u\left(\phi(y)+\hat{w}\right), u\left(\phi(y)\right)\right] \leq u\left(\phi(y) + (1-\varepsilon)\,\hat{w}\right) \leq \max\left[u\left(\phi(y)+\hat{w}\right), u\left(\phi(y)\right)\right]$ and $U_m(\phi(y)+(1-\varepsilon)\,\hat{w})$ is well defined. We have $u((1-\varepsilon)\left(\phi(y)+\hat{w}\right)) = (1-\varepsilon)^{\gamma}\, u(\phi(y)+\hat{w})$ and $U_m((1-\varepsilon)\left(\phi(y)+\hat{w}\right))$ is also well-defined and $U_m((1-\varepsilon)\left(\phi(y)+\hat{w}\right)) > U_m(\hat{c}_m)$ for ε sufficiently small. Let $x = \hat{y}$ and $x' = \phi(y)$, by Lemma 15 there exists $x'' \in X_+$ such that $0 \leq \phi(y) - x'' \leq \varepsilon\phi(y)$ and $\mathcal{C}(\hat{y}) \subset \mathcal{C}(x'')$. Since ϕ is continuous nondecreasing with $\phi(0) = 0$ and since we have $x'' \leq \phi(y)$, we may define y'' pointwise by $y''(\omega) = \inf\{z : \phi(z) = x''(\omega)\}$. We have $\phi(y'') = x''$, $\mathcal{C}(\hat{y}) \subset \mathcal{C}(x'') \subset \mathcal{C}(y'')$ and $y'' \leq y$ which gives $y'' \in Y$ If we take $c'' = x'' + (1-\varepsilon)\,\hat{w} = \phi(y'') + (1-\varepsilon)\,\hat{w}$, we have

$$(1-\varepsilon)\left(\phi(y)+\hat{w}\right) \leq c'' \leq \phi(y) + (1-\varepsilon)\,\hat{w}$$

hence $U_m(c'')$ is well defined with $U_m(c'') > U_m(\hat{c}_m)$, $\hat{w} \in \mathcal{C}(y'')$ and $\hat{q} \cdot (\phi(y'') + (1-\varepsilon)\,\hat{w}) = \hat{q} \cdot \phi(y'')$ which contradicts (25). Therefore, we have

$$U_m(\phi(\hat{y}) + \hat{w}) \geq \max_K U_m(\phi(y) + \hat{w}).$$

Assume that there exists $y' \in K$ such that $\hat{q}\cdot(y'-\phi(y')) > \hat{q}\cdot(\hat{y}-\phi(\hat{y}))$. As ϕ is $1-$Lipschitz and by Lemma 15, there exists y'' close to y' such that $\hat{q}\cdot(y''-\phi(y'')) > \hat{q}\cdot(\hat{y}-\phi(\hat{y}))$, $\mathcal{C}(\hat{y}) \subset \mathcal{C}(y'')$ and $y'' \in Y$. From the definition of the indirect utility, we have $V^i(y'', \hat{q}) > V^i(\hat{y}, \hat{q})$ which violates assertion 4 of the definition of equilibrium. Therefore, $\hat{q}\cdot(\hat{y}-\phi(\hat{y})) \geq \max_K \hat{q}\cdot(y-\phi(y))$.

Let us assume that there exists $y' \in Y$ such that $y' > \hat{y}$. Without loss of generality, we may assume that $y' \in K$. We have $U_m(\phi(\hat{y}) + \hat{w}) \geq U_m(\phi(y') + \hat{w})$ and since U_m is increasing we have $\phi(\hat{y}) = \phi(y')$ on $A = \{y' > \hat{y}\}$. This gives $\hat{q} \cdot (y' - \phi(y'))1_A > \hat{q} \cdot (\hat{y} - \phi(\hat{y}))1_A$ and $\hat{q} \cdot (y' - \phi(y'))1_{\Omega\setminus A} = \hat{q}\cdot(\hat{y}-\phi(\hat{y}))1_{\Omega\setminus A}$ which contradicts the fact that $\hat{q}\cdot(\hat{y}-\phi(\hat{y})) \geq \max_K \hat{q} \cdot (y - \phi(y))$. Hence $\hat{y} \in K$ and from there

$$\hat{q} \cdot (\hat{y} - \phi(\hat{y})) = \max_K \hat{q} \cdot (y - \phi(y)) = \max_Y \hat{q} \cdot (y - \phi(y)), \tag{26}$$

$$U_m(\phi(\hat{y}) + \hat{w}) = \max_K U_m(\phi(y) + \hat{w}) = \max_Y U_m(\phi(y) + \hat{w}). \tag{27}$$

Let $\partial\phi(z) = \overline{co}\left\{\lim_{h\to 0} \frac{\phi(z+h)-\phi(z)}{h}\right\}$. As ϕ is nondecreasing and $1-$Lipschitz, we have $\partial\phi(z) = \left[\phi'_-(z), \phi'_+(z)\right] \subset [0,1]$ for all z with $\phi'_-(z) = \phi'_+(z) = \phi'(z)$ when $\phi'(z)$ exists (which is the case almost everywhere on $\mathbb{R}_+$). Let $y \in Y$, $v = y - \hat{y}$ and $0 < \varepsilon < 1$. We have $\hat{y} + \varepsilon v \in Y$ and $\hat{q}\cdot(\hat{y} - \phi(\hat{y})) - \hat{q}\cdot(\hat{y} + \varepsilon v - \phi(\hat{y} + \varepsilon v)) = -\hat{q}\cdot\phi(\hat{y})) - \hat{q}\cdot(\varepsilon v - \phi(\hat{y} + \varepsilon v)) \geq 0$ or

$$-\hat{q}\cdot v + \hat{q}\cdot\frac{\phi(\hat{y} + \varepsilon v) - \phi(\hat{y})}{\varepsilon} \geq 0.$$

We have $0 \leq \hat{q}\left(1 - \frac{\phi(\hat{y}+\varepsilon v)-\phi(\hat{y})}{\varepsilon v}\right) v 1_{v\geq 0}$ hence, by Fatou's Lemma, $E\left[\hat{q}v1_{v\geq 0}\right] - E\left[\hat{q}\phi'_+(\hat{y})v1_{v\geq 0}\right] \leq E\left[\hat{q}v1_{v\geq 0}\right] - \limsup_{\varepsilon\to 0} E\left[\hat{q}\frac{\phi(\hat{y}+\varepsilon v)-\phi(\hat{y})}{\varepsilon}1_{v\geq 0}\right]$ or $E\left[\hat{q}\phi'_+(\hat{y})v1_{v\geq 0}\right] \geq \limsup_{\varepsilon\to 0} E\left[\hat{q}\frac{\phi(\hat{y}+\varepsilon v)-\phi(\hat{y})}{\varepsilon}1_{v\geq 0}\right]$. Similarly, from $0 \leq -\hat{q}\frac{\phi(\hat{y}+\varepsilon v)-\phi(\hat{y})}{\varepsilon v}v1_{v\leq 0}$ we derive $-E\left[\hat{q}\phi'_-(\hat{y})v1_{v\leq 0}\right] \leq -\liminf_{\varepsilon\to 0} E\left[\hat{q}\frac{\phi(\hat{y}+\varepsilon v)-\phi(\hat{y})}{\varepsilon}1_{v\leq 0}\right]$ or $E\left[\hat{q}\phi'_-(\hat{y})v1_{v\leq 0}\right] \geq \liminf_{\varepsilon\to 0} E\left[\hat{q}\frac{\phi(\hat{y}+\varepsilon v)-\phi(\hat{y})}{\varepsilon}1_{v\leq 0}\right]$. Therefore,

$$\begin{aligned}
&E\left[\hat{q}\phi'_+(\hat{y})v1_{v\geq 0}\right] + E\left[\hat{q}\phi'_-(\hat{y})v1_{v\leq 0}\right] \\
&\geq \limsup_{\varepsilon\to 0} E\left[\hat{q}\frac{\phi(\hat{y}+\varepsilon v)-\phi(\hat{y})}{\varepsilon}1_{v\geq 0}\right] + \liminf_{\varepsilon\to 0} E\left[\hat{q}\frac{\phi(\hat{y}+\varepsilon v)-\phi(\hat{y})}{\varepsilon}1_{v\leq 0}\right] \\
&\geq \liminf_{\varepsilon\to 0} E\left[\hat{q}\frac{\phi(\hat{y}+\varepsilon v)-\phi(\hat{y})}{\varepsilon}\right] \qquad (28) \\
&\geq \hat{q}v.
\end{aligned}$$

To prove (28), it suffices to consider a sequence (ε_n) such that $E\left[\hat{q}\frac{\phi(\hat{y}+\varepsilon_n v)-\phi(\hat{y})}{\varepsilon}1_{v\leq 0}\right]$ converges to $\liminf_{\varepsilon\to 0} E\left[\hat{q}\frac{\phi(\hat{y}+\varepsilon v)-\phi(\hat{y})}{\varepsilon}1_{v\leq 0}\right]$. The sequence $E\left[\hat{q}\frac{\phi(\hat{y}+\varepsilon_n v)-\phi(\hat{y})}{\varepsilon}1_{v\geq 0}\right]$ is bounded above by $E\left[\hat{q}v1_{v\geq 0}\right]$ and (ε_n) admits a subsequence that converges to some $\ell \leq \limsup_{\varepsilon\to 0} E\left[\hat{q}\frac{\phi(\hat{y}+\varepsilon v)-\phi(\hat{y})}{\varepsilon}1_{v\geq 0}\right]$. Along this subsequence, $E\left[\hat{q}\frac{\phi(\hat{y}+\varepsilon_n v)-\phi(\hat{y})}{\varepsilon}\right]$ converges to $\ell + \liminf_{\varepsilon\to 0} E\left[\hat{q}\frac{\phi(\hat{y}+\varepsilon v)-\phi(\hat{y})}{\varepsilon}1_{v\leq 0}\right]$ which gives $\ell + \liminf_{\varepsilon\to 0} E\left[\hat{q}\frac{\phi(\hat{y}+\varepsilon v)-\phi(\hat{y})}{\varepsilon}1_{v\leq 0}\right] \geq \liminf_{\varepsilon\to 0} E\left[\hat{q}\frac{\phi(\hat{y}+\varepsilon v)-\phi(\hat{y})}{\varepsilon}\right]$ then (28).

To summarize, we have $\hat{q}(1 - \phi'_+(\hat{y})1_{v\geq 0} - \phi'_-(\hat{y})1_{v\leq 0})\cdot v \leq 0$ for all v. Therefore, we have $p_1 = \hat{q}(1 - \phi'_+(\hat{y})1_{v\geq 0} - \phi'_-(\hat{y})1_{v\leq 0}) \in N_Y(\hat{y})$.

We also have $U_m(\phi(\hat{y}) + \hat{w}) - U_m(\phi(y + \varepsilon v) + \hat{w}) \geq 0$ and $\frac{U_m(\phi(\hat{y})+\hat{w})-U_m(\phi(\hat{y}+\varepsilon v)+\hat{w})}{\varepsilon v}v \geq 0$ and, by Fatou's Lemma, $E\left[\left(\phi'_+(\hat{y})1_{v\geq 0} + \phi'_-(\hat{y})1_{v\leq 0}\right)M_m u'(\phi(\hat{y}) + \hat{w})v\right] \leq 0$ for all v which leads to $p_2 = \left(\phi'_+(\hat{y})1_{v\geq 0} + \phi'_-(\hat{y})1_{v\leq 0}\right)M_m u'(\phi(\hat{y}) + \hat{w}) \in N_Y(\hat{y})$. Note that $p_2 = 0$ if and only if $\left(\phi'_+(\hat{y})1_{v\geq 0} + \phi'_-(\hat{y})1_{v\leq 0}\right) = 0$

which implies $p_2 \neq 0$. We have then $0 \ll \hat{p} = p_1 + p_2 \in N_Y(\hat{y})$ and $\hat{y} \in \text{Eff}^+(Y)$. Therefore, $\hat{y}$ has a density $h_{\hat{y}}$ with $h_{\hat{y}} > 0$, $\mu-$a.e. on $\mathbb{R}_+^*$ and $\phi'_+(\hat{y}) = \phi'_-(\hat{y}) = \phi'(\hat{y})$ a.e. Since $\hat{y} \in \text{Eff}^+(Y)$, $N_Y(\hat{y})$ is generated by a single vector, $(1 - \phi'(\hat{y}))$ and $\phi'(\hat{y})U'_m(\phi(\hat{y}) + \hat{w})$ are proportional.

STEP 5. Proof of the Theorem

From Assertion 5 of Definition 1

$$\max_{y \in Y} E[M^m u(\hat{C}_m(y)] = U_m(\hat{C}_m(\hat{y}))].$$

Let us show that this implies that $\hat{C}_m(z)$ is nondecreasing in z. Let us define $A = \left\{(z, z') \in \left(\mathbb{R}_+^*\right)^2 : (z - z')(\hat{C}_m(z) - \hat{C}_m(z')) < 0\right\}$ and let us assume that $\mu \otimes \mu\,(A) > 0$. By Fubini, there exists z^* such that $\mu\,(B) > 0$ with $B = \left\{z > z^* : \hat{C}_m(z^*) > \hat{C}_m(z))\right\}$. Let us consider $\hat{y}'$ such that $\hat{y}' = z^*$ on $\{\hat{y} \in B\}$ and $\hat{y}' = \hat{y}$ elsewhere. We have $\hat{y}' \leq \hat{y}$ and then $\hat{y}' \in Y$ and $\hat{C}_m(\hat{y}') > \hat{C}_m(\hat{y})$ which contradicts the fact that $\hat{y}$ maximizes $U_m(\hat{C}_m(\hat{y}))$ on Y. Hence, $\hat{C}_m(z)$ is nondecreasing and $\hat{C}_m(z)$ admits a derivative with respect to z almost everywhere. Let us denote by $\partial\hat{C}_m(z) = \overline{co}\left\{\lim_{h\to 0} \frac{\phi(z+h)-\phi(z)}{h}\right\}$. As $\hat{C}_m$ is nondecreasing, we have $\partial\phi(z) = \left[\hat{C}'_{m,-}(z), \hat{C}'_{m,+}(z)\right] \subset [0, \infty]$ for all z with $\hat{C}'_{m,-}(z) = \hat{C}'_{m,+}(z) = \hat{C}'_m(z)$ almost everywhere on $\mathbb{R}_+$. Let $y \in Y$, $v = y - \hat{y}$ and $0 < \varepsilon < 1$. We have $\hat{y} + \varepsilon v \in Y$ and $U_m(\hat{C}_m(\hat{y})) \geq U_m(\hat{C}_m(\hat{y}) + \varepsilon v)$ and $\frac{U_m(\hat{C}_m(\hat{y})) - U_m(\hat{C}_m(\hat{y}) + \varepsilon v)}{\varepsilon v} v \geq 0$ which, by Fatou's Lemma gives $E\left[\left(\hat{C}'_{m,+}(\hat{y})1_{v\geq 0} + \hat{C}'_{m,-}(\hat{y})1_{v\leq 0}\right) M_m u'(\hat{C}_m(\hat{y})(\hat{y}))v\right] \leq 0$ for all v and then $\hat{C}'_{m,+}(\hat{y})1_{v\geq 0} + \hat{C}'_{m,-}(\hat{y})1_{v\leq 0} \in N_Y(\hat{y})$. As $\hat{y} \in \text{Eff}^+(Y)$, it has a density $h_{\hat{y}}$ with $h_{\hat{y}} > 0$, $\mu-$a.e. on $\mathbb{R}_+^*$and $\hat{C}'_{m,-}(\hat{y}) = \hat{C}'_{m,+}(\hat{y}) = \hat{C}'_m(\hat{y})$ almost everywhere on Ω which gives $\hat{C}'_m(\hat{y}) \in N_Y(\hat{y})$ or

$$\hat{C}'_m(\hat{y}) M^m u'(\hat{c}_m) = \nu \hat{p}, \tag{29}$$

for some $\nu > 0$. From Equations (**??**) and (29), we thus obtain that :

$$\left(\hat{C}'_m - \nu\phi'\right)(\hat{y}) M^m u'(\hat{c}_m) = 0 \tag{30}$$

As $\hat{y} \in \text{Eff}^+(Y)$, it takes all possible values in $(0, \infty)$ and we have

$$\hat{C}'_m(z) = \nu\phi'(z),\ a.e.$$

Integrating with respect to z, we obtain

$$\hat{C}_m(z) = \nu\phi(z) + k \tag{31}$$

for some constant k. From Assertion 3 of Definition 1, we have $\hat{C}_m(z) \leq z$ hence $\hat{C}_m(0) = 0$ which implies $k = 0$ and

$$\hat{C}_m(z) = \nu\phi(z), \ \forall z \in \mathbb{R}_+. \tag{32}$$

As $\hat{q} \cdot \hat{C}_m = \hat{q} \cdot \phi(\hat{y})$, $\nu = 1$ and $\hat{C}_m = \phi$.

To prove Assertion 2, from item 2 of Definition 1 and from the last result, we have

$$V_m(\hat{y}, \hat{q}) = U_m(\hat{C}_m(\hat{y})) = U_m(\phi(\hat{y})).$$

As $\phi(y)$ is a feasible consumption plan for the manager when she is given y, $V_m(y, \hat{q}) \geq U_m(\phi(y))$, therefore

$$V_m(\hat{y}, \hat{q}) = U_m(\phi(\hat{y})) > V_m(y, \hat{q}) \geq U_m(\phi(y)), \forall y \in Y - \{\hat{y}\},$$

proving Assertion 2.

From the first item in the definition of the m-s equilibrium, $\hat{c}^i$ maximizes $U^i(c)$ s.t. $c \in \mathcal{C}(\hat{y})$, $\hat{q} \cdot c \leq \nu^i(\hat{q} \cdot (\hat{y} - \phi(\hat{y})))$. Let $c \notin \mathcal{C}(\hat{y})$ such that $\hat{q} \cdot c \leq \nu^i(\hat{q} \cdot (\hat{y} - \phi(\hat{y})))$. Taking the expectation conditional to $\hat{y}$, we have $\hat{q} \cdot E\left[c|\, \hat{y}\right] \leq \nu^i(\hat{q} \cdot (\hat{y} - \phi(\hat{y})))$. Since $E\left[c|\, \hat{y}\right] \in \mathcal{C}(\hat{y})$, $U^i(E\left[c|\, \hat{y}\right]) \leq U^i(\hat{c}^i)$ and, by concavity of U^i, $U^i(c) \leq U^i(\hat{c}^i)$. We therefore have that $\hat{c}^i$ maximizes $U^i(c)$ s.t. $\hat{q} \cdot c \leq \nu^i(\hat{q} \cdot (\hat{y} - \phi(\hat{y})))$, for all i. We conclude the proof of assertion 3 thanks to Proposition **??** and the market clearing condition.

In order to prove assertion 4, it suffices to remark that if the manager does no trade when allowed to make transactions in W, she will not trade when only allowed to make transactions in $W' \subset W$ and since all the other conditions do not involve W, we have the result. ■

Proof of Theorem 3. Let $\hat{q} \in \mathcal{C}(\hat{y})$ and $((\hat{c}^i)_i, \hat{q}, \hat{y} - \phi(\hat{y}))$ be a production equilibrium. Then $\hat{c}^i$ maximizes $U^i(c)$ s.t. $\hat{q} \cdot c \leq \nu^i(\hat{q} \cdot (\hat{y} - \phi(\hat{y})))$. As $\hat{q} \cdot \hat{c}^i = \nu^i(\hat{q} \cdot (\hat{y} - \phi(\hat{y})))$, taking the conditional expectation with respect to $\hat{y}$, we obtain that $\hat{q} \cdot E\left[\hat{c}^i|\, \hat{y}\right] = \nu^i(\hat{q} \cdot (\hat{y} - \phi(\hat{y})))$. Hence $E\left[\hat{c}^i|\, \hat{y}\right]$ is budget feasible and from Jensen's conditional inequality, as U^i is strictly concave,

$U^i(\hat{c}^i) \leq U^i(E[\hat{c}^i|\,\hat{y}])$ with a strict inequality unless $\hat{c}^i = E[\hat{c}^i|\,\hat{y}]$. As the strict inequality would contradict the definition of $\hat{c}^i$, we have $\hat{c}^i = E[\hat{c}^i|\,\hat{y}]$ and therefore $\hat{c}^i \in \mathcal{C}(\hat{y})$ for every i. It also follows from this remark and Proposition **??** that $V^i(\hat{y}, \hat{q}) \geq V^i(y, \hat{q})$ for any y and i. The remainder is immediate. ■

Proof of Corollary 4. From Theorem **??**, we have $\hat{q} \cdot (\hat{y} - \phi(\hat{y})) = \max_Y \hat{q} \cdot (y - \phi(y))$. From the first order condition, we have $(1 - \phi'(\hat{y}))\hat{q} \in N_Y(\hat{y})$. We also have $U_m(\phi(\hat{y})) = \max_Y U_m(\phi(y))$ which gives $\phi'(\hat{y})M_m u(\phi(\hat{y})) \in N_Y(\hat{y})$. Finally, by definition of the representative agent, we have $N(\lambda^*)u'(\hat{y} - \phi(\hat{y})) = \nu\hat{q}$. Since all elements in $N_Y(\hat{y})$ are positively proportional, we have the result. ■

Proof of Corollary 5. The 2 first assertions are from Bianchi et al. (2019). The third one is immediate. ■

Proof of the strict monotonicity of ϕ and $\mathrm{Id} - \phi$. Since ϕ is nondecreasing, let I an interval on which it is constant. We have $\phi' = 0$ on I and then $\mu(1 - \phi'(z))(z - \phi(z))^{\gamma-1}g(z) = 0$ on I. Since all the beliefs are equivalent and since $\hat{y}$ is such that $h_{\hat{y}} > 0$, we have $g > 0$ a.e. on I. Furthermore, $\phi'(z) = 0 \neq 1$ on I and ϕ cannot be equal to z a.e. on I. We have then $\mu = 0$ and $\phi' = 0$ everywhere and ϕ is constant. Since $\phi(0) = 0$ we have $\phi = 0$ which is excluded by condition 4 in the definition of a m-s equilibrium. We have then that ϕ is increasing. The result on $z \to z - \phi(z)$ is derived similarly. ■

Proof of Theorem 6. From $\phi_\mu(z) = \psi_\mu((\mathrm{Id} + \psi_\mu)^{-1}(z))$ we derive $\phi_\mu((\mathrm{Id} + \psi_\mu)(z)) = \psi_\mu(z)$ and $(\mathrm{Id} + \psi_\mu)(z) - \phi_\mu((\mathrm{Id} + \psi_\mu)(z)) = z$ or $(\mathrm{Id} - \phi_\mu) \circ (\mathrm{Id} + \psi_\mu) = \mathrm{Id}$. By construction, we have $\psi'_\mu(z)\psi_\mu(z)^{\gamma-1} = \mu z^{\gamma-1}h(z)$ which can be rewritten as

$$\psi'_\mu(z - \phi(z))\psi_\mu(z - \phi(z))^{\gamma-1} = \mu(z - \phi_\mu(z))^{\gamma-1}g(z) \tag{33}$$

with g such that $g(z) = h(z - \phi_\mu(z))$.

We have $\hat{y} = \tilde{y} + \psi_\mu(\tilde{y})$ or equivalently $\tilde{y} = \hat{y} - \phi_\mu(\hat{y})$ hence $\hat{y}$ and $\tilde{y}$ generate the same filtration. We have then $h(\tilde{y}) = g(\hat{y}) = \frac{N^{\tilde{y}}(\tilde{\lambda})}{M^{m,\tilde{y}}} = \frac{N^{\hat{y}}(\tilde{\lambda})}{M^{m,\hat{y}}}$ and ϕ_μ satisfies Equations (**??**).

By definition of the representative agent, $\tilde{y}$ maximizes $E\left[N(\tilde{\lambda})u(y)\right]$ on $\mathcal{Y}$ or, equivalently, $\hat{y}$ maximizes $E\left[N(\tilde{\lambda})u(y - \phi(y))\right]$ on Y_μ. Hence, we have $(1-\phi'(\hat{y}))N(\tilde{\lambda})u'(\hat{y}-\phi(\hat{y})) \in N_{Y_\mu}(\hat{y}) = (1-\phi'(\hat{y}))N_{\mathcal{Y}}(\hat{y}-\phi(\hat{y}))$. By definition of a production equilibrium, we have $\hat{q} \in N_{\mathcal{Y}}(\hat{y}-\phi(\hat{y}))$ and since $\mathcal{Y}$ is smooth, we have $N(\tilde{\lambda})u'(\hat{y}-\phi(\hat{y})) = \alpha\hat{q}$ for $\alpha > 0$. Since $\hat{q} \in \mathcal{C}(\tilde{y})$ we have $N(\tilde{\lambda}) \in \mathcal{C}(\tilde{y})$ and $N^{\hat{y}}(\tilde{\lambda})u'(\hat{y}-\phi(\hat{y})) = \alpha\hat{q}$ or $g(\hat{y})u'(\hat{y}-\phi(\hat{y})) = \alpha\frac{\tilde{q}}{M^{m,\tilde{y}}}$ and since $M^m \in \mathcal{C}(\tilde{y})$ and by (33), $M^m\psi_\mu'(\hat{y}-\phi(\hat{y}))\psi_\mu(\hat{y}-\phi(\hat{y}))^{\gamma-1} = \frac{\alpha}{\mu}\tilde{q}$ or $M^m\psi'(\tilde{y})u'(\psi(\tilde{y})) \in N_{\mathcal{Y}}(\tilde{y})$. When $u \circ \psi$ is strictly concave, this last condition gives that $\tilde{y}$ strictly maximizes $u(\psi(y))$ on $\mathcal{Y}$. Hence $\hat{y}$ strictly maximizes $u(\phi(y))$ on Y. From Theorem 3, $((\phi,\{0\}),(\hat{c}^i)_i,(\hat{c}_m),\hat{q},\hat{y})$ is a m-s equilibrium. ■

Proof of Proposition 8. The existence is adapted from Bianchi et al. (2019). The rest is immediate. ■

Proof of Corollary 9. For $\gamma > 0$, $\psi(z)^\gamma = \gamma\mu\int_0^z u^{\gamma-1}du = \mu z^\gamma$ and $\psi(z) = \mu^{\frac{1}{\gamma}}z$ and $\phi(z) = \psi(z-\phi(z)) = \mu^{\frac{1}{\gamma}}(z-\phi(z))$ which gives $\phi(z) = \frac{\mu^{\frac{1}{\gamma}}}{1+\mu^{\frac{1}{\gamma}}}z$. For $\gamma < 0$, we have $\psi(z)^\gamma = C - \gamma\mu\int_z^\infty u^{\gamma-1}du = C + \mu z^\gamma$ and $\psi(z) = (C+\mu z^\gamma)^{\frac{1}{\gamma}}$ and $\phi(z) = (C+\mu(z-\phi(z))^\gamma)^{\frac{1}{\gamma}}$ which gives $\phi(z)^\gamma = C+\mu(z-\phi(z))^\gamma$. ■

Proof of 10. For $\gamma > 0$,, $\psi(z)^\gamma = \gamma\mu\int_0^z h(u)u^{\gamma-1}du$ and $\frac{\psi(z)^\gamma}{z} \sim_0 \gamma\mu h(z)z^{\gamma-1}$. Hence $\left(\frac{\psi(z)}{z}\right)^\gamma = z^{1-\gamma}\frac{\psi(z)^\gamma}{z} \sim_0 \gamma\mu h(z)$ and $\frac{\psi(z)}{z} \to_0 \infty$. Let u be given, for u sufficiently large, we have $h(u) > M$ and $\frac{\psi(z)^\gamma}{z} = \gamma\mu\frac{1}{z}\int_0^z h(u)u^{\gamma-1}du > \mu M z^{\gamma-1}$ for z sufficiently large. Hence, for z sufficiently large, we have $\left(\frac{\psi(z)}{z}\right)^\gamma > \gamma\mu M$ and $\frac{\psi(z)}{z} \to_\infty \infty$. The case $\gamma < 0$ is treated similarly. ■

Proof of Corollary 11. We have $\phi(z) = \psi((\mathrm{Id}+\psi)^{-1}(z))$ and $\varkappa(z) = \frac{\psi((\mathrm{Id}+\psi)^{-1}(z))}{(\mathrm{Id}+\psi)^{-1}(z)}\frac{(\mathrm{Id}+\psi)^{-1}(z)}{z}$. If $x = (\mathrm{Id}+\psi)^{-1}(z)$, we have $\psi(x)+x = z$ then $x \leq z$ and, in particular, $\frac{(\mathrm{Id}+\psi)^{-1}(z)}{z} \leq 1$ and $\lim_{z\to 0}(\mathrm{Id}+\psi)^{-1}(z) = 0$. Since ψ is nondecreasing and defined on $\mathbb{R}_+$, it is also easy to check that $\lim_{z\to\infty}(\mathrm{Id}+\psi)^{-1}(z) = \infty$ If $\gamma < 0$, $\lim_{z\to 0,\infty}\frac{\psi(z)}{z} = 0$ and then $\lim_{z\to 0,\infty}\frac{\psi((\mathrm{Id}+\psi)^{-1}(z))}{(\mathrm{Id}+\psi)^{-1}(z)} = 0$ which gives $\lim_{z\to 0,\infty}\varkappa(z) = 0$. If $\gamma > 0$, we have $\frac{x\chi(x)}{z} + \frac{x}{z} = 1$ and the second term is negligible with respect to the first one for x (or equivalently for z) near to 0 or to ∞. Since we have $\frac{x\chi(x)}{z} = \frac{\psi(x)}{z} = \varkappa(z)$, it comes $\lim_{z\to 0,\infty}\varkappa(z) = 1$. ■

Proof of Proposition 12. Let $y_\theta = \exp\left(m(\theta)t + \theta\sqrt{t}\widetilde{x}\right)$ where θ is given. Let us show that the y_θ-exchange equilibrium $\left(c^{\theta,\delta}, c^{\theta,-\delta}, q_\theta\right)$ is given by

$$q_\theta = \left(\frac{y_\theta}{\lambda_\theta \exp -\frac{(\widetilde{x}-\delta)^2}{4} + \exp -\frac{(\widetilde{x}+\delta)^2}{4}} \right)^{-2}$$
$$c^{\theta,\delta} = \frac{\lambda_\theta \exp -\frac{(\widetilde{x}-\beta)^2}{4}}{\lambda_\theta \exp -\frac{(\widetilde{x}-\delta)^2}{4} + \exp -\frac{(\widetilde{x}+\delta)^2}{4}} y_\theta$$
$$c^{\theta,-\delta} = \frac{\exp -\frac{(\widetilde{x}+\delta)^2}{4}}{\lambda_\theta \exp -\frac{(\widetilde{x}-\delta)^2}{4} + \exp -\frac{(\widetilde{x}+\delta)^2}{4}} y_\theta,$$

where λ_θ the square root of agent δ equilibrium utility weight (when that of the other agent is normalized to one) is the nonnegative solution of

$$\lambda^2 \exp(-\theta\delta\sqrt{t})\nu_{-\delta} + \lambda \exp(-\frac{\delta^2}{2})(2\nu_{-\delta} - 1) + \exp(\theta\delta\sqrt{t})(\nu_{-\delta} - 1) = 0. \quad (34)$$

We have

$$q_\theta = \lambda_\theta^2 M_\delta u'\left(c^{\theta,\delta}\right) = M_{-\delta} u'(c^{\theta,-\delta}) \text{ and } y_\theta = c^{\theta,\delta} + c^{\theta,-\delta}$$

and $\left(c^{\theta,\delta}, c^{\theta,-\delta}\right)$ satisfies the first-order conditions for utility maximization as well as the market clearing condition. We need to check that the budget constraint is also satisfied, i.e. $E\left[q_\theta c^{\theta,-\delta}\right] = \nu_{-\delta} E\left[q_\theta y_\theta\right]$. After simple calculations, this constraint appears to be equivalent to (34). It is also straightforward to show that (34) admits only one positive solution. Let us finally find $\hat{\theta}$ such that $(\hat{c}^\delta, \hat{c}^{-\delta}, \hat{q}, \tilde{y})$ is a production equilibrium. Since $\left(c^{\tilde{\theta},\delta}, c^{\tilde{\theta},-\delta}, q_{\tilde{\theta}}\right)$ is already a $y_{\tilde{\theta}}$-exchange equilibrium, we need only take care of the profit maximization constraint. For this purpose, let us define $g(\theta, \sigma) = E\left[q_\theta y_\sigma\right]$ and let $\tilde{\theta}$ be such that $g_\sigma(\tilde{\theta}, \tilde{\theta}) = 0$ which can be rewritten as

$$\tilde{\theta} = \frac{2b}{2b+1} s_0 + \frac{1}{\sqrt{t}} \frac{\delta}{2b+1} \left(1 - 2\nu_{-\delta}\right).$$

For such a $\tilde{\theta}$, $\tilde{y} = y_{\tilde{\theta}}$ satisfies the first order condition for profit maximization. Under condition (17), we can show that $g_\sigma(\tilde{\theta}, \sigma)$ is positive for $\sigma < \tilde{\theta}$ and negative for $\sigma > \tilde{\theta}$ which means that $g(\tilde{\theta}, \sigma)$ reaches its maximum for $\sigma = \tilde{\theta}$ and $\left(c^{\tilde{\theta},\delta}, c^{\tilde{\theta},-\delta}, q_{\tilde{\theta}}\right)$ is a production equilibrium,

■

Proof of Remark 13. The analytic expressions of $E^m\left[\widetilde{x}\right]$ and $\text{VAR}^m\left[\widetilde{x}\right]$

are obtained through straightforward computations. From there, we easily derive,

$$-\delta \leq \frac{\lambda_{\tilde{\theta}} - 1}{\lambda_{\tilde{\theta}} + 1}\delta \leq E^m\left[\widetilde{x}\right] \leq \frac{\lambda_{\tilde{\theta}}^2 - 1}{\lambda_{\tilde{\theta}}^2 + 1}\delta \leq \delta,$$

hence if $\lambda_{\tilde{\theta}} < 1$ and $\lambda_{\tilde{\theta}} \neq 0$ (or $\nu_\delta \neq 0$), we have

$$\text{VAR}^m\left[\widetilde{x}\right] > 1 + \delta^2 \frac{(\lambda_{\tilde{\theta}}^2 + 1)}{(\lambda_{\tilde{\theta}} + 1)^2} - \delta^2 \frac{(\lambda_{\tilde{\theta}} - 1)^2}{(\lambda_{\theta^*} + 1)^2} > 1.$$

While for $\lambda_{\tilde{\theta}} \geq 1$ and if $\lambda_{\tilde{\theta}} \neq \infty$ (or $\nu_\delta \neq 1$), we have

$$\text{VAR}^m\left[\widetilde{x}\right] > 1 + \delta^2 \frac{(\lambda_{\tilde{\theta}}^2 + 1)}{(\lambda_{\tilde{\theta}} + 1)^2} - \delta^2 \left(\frac{\lambda_{\tilde{\theta}}^2 - 1}{\lambda_{\tilde{\theta}}^2 + 1}\right)^2 > 1.$$

■