

DYNKIN GAMES WITH INCOMPLETE AND ASYMMETRIC INFORMATION

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ABSTRACT. We study Nash equilibria for a two-player zero-sum optimal stopping game with incomplete and asymmetric information. In our set-up, the drift of the underlying diffusion process is unknown to one player (incomplete information feature), but known to the other one (asymmetric information feature). We formulate the problem and reduce it to a fully Markovian setup where the uninformed player optimises over stopping times and the informed one uses randomised stopping times in order to hide their informational advantage. Then we provide a general verification result which allows us to find Nash equilibria by solving suitable quasi-variational inequalities with some non-standard constraints. Finally, we study an example with linear payoffs, in which an explicit solution of the corresponding quasi-variational inequalities can be obtained.

1. INTRODUCTION

The primary focus in this paper is to devise methods to establish the existence of Nash equilibria for two-player Dynkin games with incomplete and asymmetric information. The process underlying the game is a one-dimensional linear diffusion X . Both players observe the path of X and Player 2 (the *informed* player) knows exactly the drift and diffusion coefficient of the process. Player 1 (the *uninformed* player) has incomplete information in the sense that she cannot observe directly the drift coefficient of X but has a prior distribution for it and can improve upon her initial estimate by sequential observation of the process. For simplicity and clarity of exposition we consider the case in which the drift can be either of two continuous functions of the state process, denoted $\mu_0(\cdot)$ and $\mu_1(\cdot)$.

Crucially, the one-sided lack of information introduces an *asymmetry* in the game because, contrarily to the informed player, the uninformed one

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cannot compute the true expected payoff of the game (for each given stopping rule).

In line with the literature on *differential* games with asymmetric information, it turns out that the informed player must use randomised stopping strategies in order to maximise the benefits of their informational advantage. The effect of randomisation is indeed to ‘hide’ the true drift from the uninformed player in an attempt to mislead her into acting in a way that benefits the informed player. On the contrary, the uninformed player cannot improve her performance by using randomisation (see Remark 2.6) and therefore will simply rely on stopping times for the filtration generated by X .

The key contributions of the paper are: (i) we give an explicit Markovian formulation of the problem and show its equivalence with a three-player nonzero-sum game of singular control and optimal stopping (Section 3); the resulting nonzero-sum game features a peculiar structure of the interaction among the players (Remark 3.6); moreover, the game is non-standard in the sense that the singular controls (played by Player 2) are not observable by Player 1; (ii) building on the previous item we formulate a verification theorem that allows us to construct Nash equilibria for the original game with incomplete and asymmetric information (Section 5); the verification result (Theorem 5.1) is formulated in terms of a quasi-variational inequality with a set of non-standard constraints; it appears that such constraints are a special feature of the asymmetric information setting; (iii) using the quasi-variational inequality approach we solve explicitly (up to numerical root-finding) a version of our game with linear payoffs (Section 6); the example illustrates how ‘reflected adjusted likelihood ratios’ are used to hide information in an optimal way.

To the best of our knowledge all the three items above are new in the literature. In particular, we would like to emphasise that the majority of papers on zero-sum games with asymmetric information focus on the existence of a value for the game, whereas the construction of Nash equilibria is mostly overlooked (we will elaborate more on this point in the literature review below). In this sense, we depart from the existing literature and present a feasible method for the characterisation of Nash equilibria. Moreover, the value of the game is normally obtained when both players are allowed to use randomised strategies; here instead we only allow pure strategies for the uninformed player. Finally, the approach developed in Section 3 is not specific to our setting and it can be used more broadly to link zero-sum Dynkin games with asymmetric information to n -player nonzero-sum games of singular control and stopping.

1.1. Motivations and literature review. Dynkin games were originally introduced in [15] as a *game variant* of optimal stopping problems. Their popularity in the last two decades is largely due to their applications to finance. Indeed many financial contracts are equipped with exit strategies

that allow one or several parties to abandon their obligations early but at an additional cost. These ‘exit options’ embedded in the contracts are known in the mathematical finance literature as *game options*.

In 2000 Kifer [25] showed that the arbitrage-free price of a game option can be found by solving a related Dynkin game. In the full information case, general conditions under which a Nash equilibrium for the game exists were derived in [28] (in a martingale setting) and in [17] (in a Markovian set-up).

Acknowledging the importance of information in applications of such games, more recent literature has considered games with asymmetric information structures. For example, asymmetric information about the time horizon of the game was considered in [27], who concluded that, in the setting of that paper, the more you know, the longer you wait. Moreover, Grün [23] studied the effect of asymmetric information about the payoff structure of the game; motivated by earlier studies (see [6] and [7]) of differential games with asymmetric information as well as by an explicit example, Grün allowed the informed player to use randomised stopping strategies to manipulate the beliefs of the uninformed player, and she characterised the value of the game as the unique viscosity solution of a related variational inequality. Note that the concept of ‘value’ here (and in the existing literature on zero-sum games with asymmetric information) coincides with the expected payoff *in equilibrium* of the uninformed player in our paper (see Remark 3.4 for further details). A more general situation was considered in [22], in which each player has access to stopping times with respect to different filtrations. In such a scenario each player must learn about the state of the world from the actions (or inaction) of the other player. Again, a variational characterisation of the value of the game is obtained in a similar form to [23]. The article [23] does not consider the question of existence for Nash equilibria but constructs optimal randomised stopping times for the informed player. Nash equilibria, instead, are addressed in [22] for non-diffusive dynamics.

It is important to notice that the setting in [23] is different from ours because in that paper the observable dynamics are fully known. Then, in contrast to our setting, in [23] there is no learning from the observation of the process. It is also worth noticing that the variational problem in [23] (and the one in [22]) looks very different from ours: Grün obtains a single variational inequality (as opposed to our coupled variational problem in Theorem 5.1) which involves three nested obstacle problems of the type ‘max-max-min’. Existence of smooth solutions to such variational problems remains an open question and the explicit motivating example of [23] does not include a random dynamic. It does not seem trivial to show a clear connection between our variational problem and that in [23]. However, our method allows to solve an example with diffusive dynamics by proving that the associated quasi-variational inequality has a unique classical solution (see Section 6).

Another formulation of asymmetric information within a Dynkin game was treated in [19], who provided conditions for the existence of a Nash

equilibrium in stopping times for a setting where learning for the uninformed player is not considered. In [19] both players use stopping times, although those of the informed player are taken with respect to a larger filtration which includes extra information on the structure of the game. Randomisation is not needed in their setting because the informed player does not need to hide the information. It is indeed stated in Section 3.1, p. 288, of [19] that the uninformed player ‘does not care about’ or ‘is not allowed to use’ the additional information. This stands in sharp contrast with our setting (as well as that of [23, 22], among others).

Finally, in [16] a Dynkin game in which both players had differing beliefs about the drift of the underlying process was studied. However, in that article, information is fully symmetric and complete, with both players agreeing to disagree.

In comparison to [16] and [19], where the set-up involves no learning, and [23] and [22], where the players learn only from the actions of the opponent, our players are faced with a more complex, two-source, learning situation. In particular, the uninformed player learns about the drift of the underlying process by continuous observations of the process itself *and* from the actions of the informed player.

Since learning is a key ingredient in our problem formulation, we naturally draw on the literature on stochastic filtering. Early contributions in the area include treatments of statistical problems in sequential analysis, see for example [3], [8] and [35]. A general treatment of stochastic filtering can be found in [29], and some important early work on the application of such techniques to investment problems with incomplete information can be found in [30] and [31]. More recent contributions along the financial lines include [2], [4], [10], [14], [33] and [39] (and the references therein). For incomplete information in the context of optimal stopping, an early reference is [13] which treats the effect of incomplete information on American-style option valuation; see also [21] and [38]. An optimal liquidation problem with unknown drift was studied in [18], and with an unknown jump intensity in [32]. A Dynkin game with symmetric and incomplete information was studied in [12], in which the existence of a Nash equilibrium was established. Finally, a related paper from the economics literature is [9], which considers the problem of a privately informed seller attempting to trade in a market of less informed buyers, and where information about the asset’s type (‘good’ or ‘bad’) is gradually revealed to them. In this setting, the market places offers based on this information and on the observation of the offers rejected by the seller so far. In this sense the learning process occurs from two sources as in our paper. However, the key difference with the current paper is that the buyers (i.e., the market) are non-strategic. Hence the mathematical formulation of the problem in [9] may be interpreted as a one-dimensional singular control problem with discretionary stopping for the seller, where the reaction of the market to new information is fully prescribed by a function of the underlying process.

1.2. Comments on the example with linear payoff and outline of the paper. As mentioned above we obtain an explicit solution of the game (in Section 6) in a particular case with linear payoffs. This includes explicit strategies for both players as well as computable equilibrium payoffs.

In this example the underlying process is a geometric Brownian motion with drift μ . Then the uninformed player (Player 1) has a two-point prior distribution for μ (i.e., $\mu \in \{\mu_0, \mu_1\}$, with $\mu_0 < 0 < \mu_1$) whose support contains the true drift. Player 1 knows standard filtering theory and updates her belief about μ by computing, at each time $t > 0$, the likelihood ratio Φ_t . This represents the ratio between the (conditional) probability that the true drift is μ_1 and the probability that it is μ_0 , given the observation of $(X_s)_{0 \leq s \leq t}$ (see Section 3). However, Player 1 also observes the (lack of) actions of Player 2 (informed) and she should modify her estimate of Φ_t accordingly. This happens particularly if the informed player does not stop when the process X is in some specific regions of the state space (loosely speaking, if Player 2 does not stop at points where she would be supposed to stop in a game with full information with $\mu = \mu_1$, then Player 1 may be more inclined to believe that $\mu = \mu_0$). Following this rationale, in equilibrium, Player 1 will produce an *adjusted likelihood ratio* Φ^* (see (33)), which depends on both the observation of the process and the lack of actions from the opponent.

As it turns out in our analysis, in equilibrium the informed player stops according to a generalised intensity specified in such a way that the adjusted likelihood ratio process is *reflecting* at an upper boundary, and the uninformed player stops when this process falls below a certain threshold. To prove the existence of such a Nash equilibrium we solve explicitly the associated quasi-variational inequalities and then invoke our verification result (Theorem 5.1). Our verification result indicates that reflection of the adjusted likelihood ratio plays a vital role also in the general case; however, reflection is then along curved boundaries that need to be determined as part of the solution of the variational inequalities. It is hoped that our analysis in the specific example with linear payoffs can be used to inform future work on more general optimal stopping games and on the solvability of the quasi-variational inequality that we derive in Section 5.

We conclude with an outline of the material in the paper. In Section 2 we formulate the general Dynkin game and introduce the class of randomised stopping times used by the informed player. The learning dynamics are derived and the game is reformulated as an equivalent game of stopping and singular control in Section 3. In Section 4 we explain how the informed player manipulates the beliefs of the uninformed one. A verification result based on quasi-variational inequalities is provided in Section 5. Finally, Section 6 investigates the example with linear payoffs in detail, and Section 7 illustrates the Nash equilibrium in this example numerically with a base-case set of parameters providing intuition for the optimal strategies used in equilibrium.

2. SETTING

Assume that on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ we have two random variables θ and \mathcal{U} together with a standard Wiener process W mutually independent of each other, and such that $\mathbb{P}(\theta = 1) = \pi$ and $\mathbb{P}(\theta = 0) = 1 - \pi$ where $\pi \in (0, 1)$ and \mathcal{U} is uniformly distributed on $[0, 1]$. We consider an optimal stopping game written on an underlying process X with dynamics

$$(1) \quad dX_t = ((1 - \theta)\mu_0(X_t) + \theta\mu_1(X_t)) dt + \sigma(X_t) dW_t$$

on a (possibly unbounded) interval \mathcal{I} . Here $\mu_0(\cdot)$, $\mu_1(\cdot)$ and $\sigma(\cdot) > 0$ are given continuous functions such that the state space of X is \mathcal{I} on both events $\{\theta = 0\}$ and $\{\theta = 1\}$. Then (1) admits a weak solution which is also unique in law, and to avoid further technicalities we assume that the boundary points of \mathcal{I} are unattainable.

The game is specified by Player 1 choosing a (random) time τ and Player 2 choosing a (random) time γ , and at $\tau \wedge \gamma$, Player 1 receives the amount

$$R(\tau, \gamma) := f(X_\tau)1_{\{\tau < \gamma\}} + g(X_\gamma)1_{\{\tau \geq \gamma\}}$$

from Player 2. Here the *payoff functions* f and g are two given functions satisfying $g \geq f \geq 0$. The objective of Player 1 (2) is to choose τ (γ) from a set of admissible stopping strategies to maximize (minimize) the expected value of $R(\tau, \gamma)$. The notion of admissible stopping strategies will be specified below. To avoid further technical complications, we will assume continuity of the payoff functions.

Assumption 2.1. *The payoff functions f and g are continuous on \mathcal{I} .*

Both players observe the process X , however we assume that Player 1 is partially informed whereas Player 2 is fully informed: initially, the only available information for Player 1 is the distribution of θ given above, while Player 2 knows the true value of θ already at the start of the game (the opposite case can be treated similarly). This asymmetry is modeled by letting the information available to Player 1 be given by the augmentation with \mathbb{P} -null sets of the filtration

$$\mathcal{F}_t^X := \sigma(X_s, 0 \leq s \leq t),$$

whereas the information available to Player 2 is given by the augmentation of the filtration

$$\mathcal{F}_t^{X, \theta} := \sigma(\theta, X_s, 0 \leq s \leq t).$$

We stress that the set-up of the game is known to both players. In particular, the initial probability π and the functions μ_0 , μ_1 and σ are known to both players, Player 1 is aware that Player 2 knows the true value of the drift, and Player 2 knows the distribution of θ from the perspective of Player 1.

When considering games with asymmetric information, hiding of information plays a crucial role. This is modeled mathematically by allowing stopping strategies to be randomised stopping times. In fact, below we allow

the informed player (Player 2) to use randomised stopping times, whereas the uninformed one (Player 1) uses strategies that are stopping times.

The following notations will be needed in the rest of the paper. We let $\mathcal{F}^X = (\mathcal{F}^X)_{t \geq 0}$ and $\mathcal{F}^{X,\theta} = (\mathcal{F}_t^{X,\theta})_{t \geq 0}$ and denote

$$\mathcal{T} := \{\tau: \tau \text{ is a } \mathbb{P}\text{-a.s. finite } \mathcal{F}^X\text{-stopping time}\}$$

$$\overline{\mathcal{T}} := \{\tau: \tau \text{ is an } \mathcal{F}^X\text{-stopping time}\}$$

$$\mathcal{A} := \{\Gamma: (\Gamma_t)_{t \geq 0} \text{ is } \mathcal{F}^X\text{-adapted and a.s. right-continuous,} \\ \text{non-decreasing, with } \Gamma_{0-} = 0 \text{ and } \Gamma_\infty \leq 1\}$$

$$\mathcal{A}^\theta := \{\Gamma: (\Gamma_t)_{t \geq 0-} \text{ is } \mathcal{F}^{X,\theta}\text{-adapted and a.s. right-continuous,} \\ \text{non-decreasing, with } \Gamma_{0-} = 0 \text{ and } \Gamma_\infty \leq 1\}.$$

In the definitions above we use $\Gamma_{0-} = 0$ to indicate that $\Gamma_0 > 0$ can only be achieved by a jump of the process at time zero.

Clearly, $\mathcal{A} \subseteq \mathcal{A}^\theta$; also note that $\Gamma \in \mathcal{A}^\theta$ if and only if $\Gamma = \Gamma^0 \mathbb{1}_{\{\theta=0\}} + \Gamma^1 \mathbb{1}_{\{\theta=1\}}$ for some $\Gamma^0, \Gamma^1 \in \mathcal{A}$.

To define randomised stopping times (see, e.g., [37]), recall that \mathcal{U} is a random variable which is independent of W and θ and Uniformly(0,1)-distributed.

Definition 2.2 (Randomised stopping times).

(A) An \mathcal{F}^X -randomised stopping time is a random variable γ given by

$$(2) \quad \gamma = \inf\{t \geq 0 : \Gamma_t > \mathcal{U}\}, \quad \text{for some } \Gamma \in \mathcal{A}.$$

We denote the set of \mathcal{F}^X -randomised stopping times by \mathcal{T}_R .

(B) An $\mathcal{F}^{X,\theta}$ -randomised stopping time is a random variable γ_θ given by

$$(3) \quad \gamma_\theta = \inf\{t \geq 0 : \Gamma_t > \mathcal{U}\}, \quad \text{for some } \Gamma \in \mathcal{A}^\theta.$$

We denote the set of $\mathcal{F}^{X,\theta}$ -randomised stopping times by \mathcal{T}_R^θ .

We then have

$$\mathcal{T} \subseteq \overline{\mathcal{T}} \subseteq \mathcal{T}_R \subseteq \mathcal{T}_R^\theta.$$

Indeed, the first inclusion is clear by definition and the third inclusion is immediate from $\mathcal{A} \subseteq \mathcal{A}^\theta$; moreover, if $\tau \in \overline{\mathcal{T}}$, then the construction (2) with

$$\Gamma_t = \begin{cases} 0 & t < \tau \\ 1 & t \geq \tau \end{cases}$$

gives a randomised stopping time that coincides with τ , which proves the middle inclusion.

Furthermore, any $\gamma_\theta \in \mathcal{T}_R^\theta$ can be decomposed as

$$\gamma_\theta = \gamma_0 \mathbb{1}_{\{\theta=0\}} + \gamma_1 \mathbb{1}_{\{\theta=1\}}$$

for some $(\gamma_0, \gamma_1) \in \mathcal{T}_R \times \mathcal{T}_R$. We say that $\gamma \in \mathcal{T}_R$ is *generated* by $\Gamma \in \mathcal{A}$ if γ is defined as in (2). Similarly, $\gamma_\theta \in \mathcal{T}_R^\theta$ is generated by $\Gamma \in \mathcal{A}^\theta$ if γ_θ is

defined as in (3). For future reference, given a $\gamma \in \mathcal{T}_R$ generated by $\Gamma \in \mathcal{A}$, we also introduce \mathcal{F}^X -stopping times (i.e., members of $\overline{\mathcal{T}}$)

$$(4) \quad \gamma(z) := \inf\{t \geq 0 : \Gamma_t > z\}, \quad \text{for all } z \in [0, 1].$$

Definition 2.3. *A randomised stopping pair is a pair $(\tau, \gamma_\theta) \in \mathcal{T} \times \mathcal{T}_R^\theta$. A couple $(\tau, \Gamma) \in \mathcal{T} \times \mathcal{A}^\theta$ or a triple $(\tau, \Gamma^0, \Gamma^1) \in \mathcal{T} \times \mathcal{A} \times \mathcal{A}$ are equivalent characterisations of a randomised stopping pair.*

With a slight abuse of notation, we sometimes write $\gamma_\theta = \Gamma = (\Gamma^0, \Gamma^1)$, where (Γ^0, Γ^1) is the decomposition of Γ that generates γ_θ . Given a randomised stopping pair $(\tau, \gamma_\theta) \in \mathcal{T} \times \mathcal{T}_R^\theta$, the expected payoff of the game from the point of view of the uninformed player is

$$(5) \quad \mathcal{J}(\tau, \gamma_\theta) = \mathcal{J}(\tau, \Gamma^0, \Gamma^1) := \mathbb{E}[R(\tau, \gamma_\theta)].$$

(See Remark 3.4 for further details around this interpretation of \mathcal{J} .) The lower value \underline{v} and the upper value \overline{v} of the game (for Player 1) are defined by

$$(6) \quad \underline{v} := \sup_{\tau \in \mathcal{T}} \inf_{\gamma_\theta \in \mathcal{T}_R^\theta} \mathcal{J}(\tau, \gamma_\theta) \leq \inf_{\gamma_\theta \in \mathcal{T}_R^\theta} \sup_{\tau \in \mathcal{T}} \mathcal{J}(\tau, \gamma_\theta) =: \overline{v},$$

and we say that a value v exists if $\underline{v} = \overline{v}$.

Definition 2.4. *A randomised stopping pair $(\tau^*, \gamma_\theta^*) \in \mathcal{T} \times \mathcal{T}_R^\theta$ is a Nash equilibrium if*

$$\mathbb{E}[R(\tau, \gamma_\theta^*)] \leq \mathbb{E}[R(\tau^*, \gamma_\theta^*)] \leq \mathbb{E}[R(\tau^*, \gamma_\theta)]$$

for all other pairs $(\tau, \gamma_\theta) \in \mathcal{T} \times \mathcal{T}_R^\theta$.

It is immediate to check that the existence of a Nash equilibrium implies the existence of a value. Although the definition above only makes use of the expected payoff of the uninformed player, it will be shown in Proposition 3.5 that this is the right concept of Nash equilibrium in our context.

Remark 2.5. *Note that we restrict our attention to stopping times in \mathcal{T} , i.e. stopping times that are finite \mathbb{P} -a.s. This has the advantage that the notation and calculations become easier. Moreover, a Nash equilibrium $(\tau^*, \gamma_\theta^*) \in \mathcal{T} \times \mathcal{T}_R^\theta$ (as in Definition 2.4) would also be a Nash equilibrium for the corresponding game with strategies in $\overline{\mathcal{T}} \times \mathcal{T}_R^\theta$ and with expected payoff*

$$\mathcal{J}'(\tau, \gamma_\theta) := \mathbb{E}[R(\tau, \gamma_\theta) \mathbf{1}_{\{\tau \wedge \gamma_\theta < \infty\}}].$$

Remark 2.6. *As mentioned above, there is no benefit for Player 1 in choosing a randomised stopping time if the game has a value (compare, e.g., [26]). Indeed, first note that*

$$\sup_{\tau \in \overline{\mathcal{T}}} \mathcal{J}'(\tau, \gamma_\theta) = \sup_{\tau \in \mathcal{T}} \mathcal{J}(\tau, \gamma_\theta)$$

for any $\gamma_\theta \in \mathcal{T}_R^\theta$ by Fatou's lemma. Consequently, for any $\gamma_\theta \in \mathcal{T}_R^\theta$ and $\gamma \in \mathcal{T}_R$, recalling (4), we have

$$\begin{aligned} \mathcal{J}'(\gamma, \gamma_\theta) &= \int_0^1 \mathcal{J}'(\gamma(z), \gamma_\theta) dz \leq \sup_{z \in [0,1]} \mathcal{J}'(\gamma(z), \gamma_\theta) \\ &\leq \sup_{\tau \in \overline{\mathcal{T}}} \mathcal{J}'(\tau, \gamma_\theta) = \sup_{\tau \in \overline{\mathcal{T}}} \mathcal{J}(\tau, \gamma_\theta). \end{aligned}$$

The inequality above implies

$$\underline{v} \leq \sup_{\gamma \in \mathcal{T}_R} \inf_{\gamma_\theta \in \mathcal{T}_R^\theta} \mathcal{J}'(\gamma, \gamma_\theta) \leq \inf_{\gamma_\theta \in \mathcal{T}_R^\theta} \sup_{\gamma \in \mathcal{T}_R} \mathcal{J}'(\gamma, \gamma_\theta) \leq \overline{v},$$

which validates our claim, provided that $\underline{v} = \overline{v}$.

Remark 2.7. For bounded payoff functions f and g , the set-up and results of the present article straightforwardly extend to the opposite case when instead Player 1 knows the drift and Player 2 only has partial information. However, additional care is needed for unbounded payoffs; in particular, one needs to be careful with the specification of the payoff at time infinity, as well as in specifying appropriate transversality conditions as in Theorem 5.1 below.

3. AN EQUIVALENT GAME OF STOPPING AND SINGULAR CONTROL

Here we formulate the game in a Markovian setting and show that it is equivalent to a 3-player nonzero-sum game of singular control and stopping. We begin by rewriting the expected cost functional in a more explicit form, which takes into account Player 1's learning of the true drift through observations of the process X .

For $t \geq 0$ denote by

$$(7) \quad \Pi_t := \mathbb{P}(\theta = 1 | \mathcal{F}_t^X)$$

the conditional expectation of the larger drift given observations of the underlying process X . By standard filtering theory (see [29, Chapter 9]) we have

$$dX_t = (\mu_0(X_t)(1 - \Pi_t) + \mu_1(X_t)\Pi_t) dt + \sigma(X_t) dB_t, \quad X_0 = x$$

and

$$(8) \quad d\Pi_t = \omega(X_t)\Pi_t(1 - \Pi_t) dB_t, \quad \Pi_0 = \pi.$$

Here the *innovation process*

$$B_t := \int_0^t \frac{1}{\sigma(X_s)} dX_s - \int_0^t \frac{\mu_0(X_s) + (\mu_1(X_s) - \mu_0(X_s))\Pi_s}{\sigma(X_s)} ds$$

is a $(\mathbb{P}, \mathcal{F}^X)$ -Brownian motion and $\omega(\cdot) := (\mu_1(\cdot) - \mu_0(\cdot))/\sigma(\cdot)$ is referred to as the *signal-to-noise ratio*.

Now the process $(X_t, \Pi_t)_{t \geq 0}$ is Markovian and adapted to \mathcal{F}^X . In what follows, for $(x, \pi) \in \mathcal{I} \times (0, 1)$, we will denote

$$\mathbb{P}_{x,\pi}(\cdot) := \mathbb{P}(\cdot | X_0 = x, \Pi_0 = \pi) \quad \text{and} \quad \mathbb{E}_{x,\pi}[\cdot] := \mathbb{E}[\cdot | X_0 = x, \Pi_0 = \pi].$$

Also, in (5) we use $\mathcal{J}_{x,\pi}(\tau, \gamma_\theta)$ to emphasise the dependence of the expected game payoff on the initial data.

In preparation for the reduction of our game to one of control and stopping, we introduce integrals of the form

$$\int_0^\tau Y_{t-} d\Gamma_t := Y_0\Gamma_0 + \int_{(0,\tau]} Y_{t-} d\Gamma_t,$$

for $\Gamma \in \mathcal{A}$ and Y a right-continuous, non-negative process adapted to \mathcal{F}^X . Integrals of this type are to be interpreted in the Stieltjes sense, and it is important to remark that, in this context, both the (possible) initial and terminal jumps of the process Γ are accounted for. Moreover, recalling (4) and using [34, Prop. 4.9, Ch. 0], we have

$$(9) \quad \int_0^1 g(X_{\gamma(z)}) \mathbf{1}_{\{\gamma(z) \leq \tau\}} dz = \int_0^\tau g(X_t) d\Gamma_t$$

for $\tau \in \mathcal{T}$.

Proposition 3.1. *For $(x, \pi) \in \mathcal{I} \times (0, 1)$ and any $(\tau, \gamma_\theta) \in \mathcal{T} \times \mathcal{T}_R^\theta$ we have*

$$(10) \quad \mathcal{J}_{x,\pi}(\tau, \gamma_\theta) = \mathbb{E}_{x,\pi} \left[(1 - \Pi_\tau)(1 - \Gamma_\tau^0) f(X_\tau) + (1 - \Pi_\tau) \int_0^\tau g(X_t) d\Gamma_t^0 \right] \\ + \mathbb{E}_{x,\pi} \left[\Pi_\tau(1 - \Gamma_\tau^1) f(X_\tau) + \Pi_\tau \int_0^\tau g(X_t) d\Gamma_t^1 \right],$$

where $(\Gamma^0, \Gamma^1) \in \mathcal{A} \times \mathcal{A}$ is the couple that generates γ_θ .

Proof. By definition of the game's payoff and by the definition of \mathcal{T}_R^θ we have

$$(11) \quad \mathcal{J}_{x,\pi}(\tau, \gamma_\theta) = \mathbb{E}_{x,\pi} [f(X_\tau) \mathbf{1}_{\{\tau < \gamma_\theta\}} + g(X_{\gamma_\theta}) \mathbf{1}_{\{\gamma_\theta \leq \tau\}}] \\ = \mathbb{E}_{x,\pi} [f(X_\tau) \mathbf{1}_{\{\tau < \gamma_0\} \cap \{\theta=0\}} + g(X_{\gamma_0}) \mathbf{1}_{\{\gamma_0 \leq \tau\} \cap \{\theta=0\}}] \\ + \mathbb{E}_{x,\pi} [f(X_\tau) \mathbf{1}_{\{\tau < \gamma_1\} \cap \{\theta=1\}} + g(X_{\gamma_1}) \mathbf{1}_{\{\gamma_1 \leq \tau\} \cap \{\theta=1\}}].$$

With the aim of using the tower property in the expression above, we claim that

$$(12) \quad \mathbb{E}_{x,\pi} [f(X_\tau) \mathbf{1}_{\{\tau < \gamma_0\} \cap \{\theta=0\}} | \mathcal{F}_\tau^X] = (1 - \Pi_\tau) f(X_\tau) (1 - \Gamma_\tau^0)$$

$$(13) \quad \mathbb{E}_{x,\pi} [f(X_\tau) \mathbf{1}_{\{\tau < \gamma_1\} \cap \{\theta=1\}} | \mathcal{F}_\tau^X] = \Pi_\tau f(X_\tau) (1 - \Gamma_\tau^1)$$

$$(14) \quad \mathbb{E}_{x,\pi} [g(X_{\gamma_0}) \mathbf{1}_{\{\gamma_0 \leq \tau\} \cap \{\theta=0\}} | \mathcal{F}_\tau^X] = (1 - \Pi_\tau) \int_0^\tau g(X_t) d\Gamma_t^0$$

$$(15) \quad \mathbb{E}_{x,\pi} [g(X_{\gamma_1}) \mathbf{1}_{\{\gamma_1 \leq \tau\} \cap \{\theta=1\}} | \mathcal{F}_\tau^X] = \Pi_\tau \int_0^\tau g(X_t) d\Gamma_t^1.$$

Taking conditional expectation inside (11) and using the above expressions we obtain (10).

It therefore only remains to prove the formulae above. Let us start by noticing that

$$(16) \quad \{\Gamma_\tau^0 < \mathcal{U}\} \subseteq \{\tau < \gamma_0\} \subseteq \{\Gamma_\tau^0 \leq \mathcal{U}\}.$$

Since X_τ is \mathcal{F}_τ^X -measurable, using simple properties of conditional expectation and (7) we have

$$\mathbb{E}_{x,\pi} [f(X_\tau) \mathbf{1}_{\{\tau < \gamma_0\} \cap \{\theta=0\}} | \mathcal{F}_\tau^X] = f(X_\tau) \mathbb{P}_{x,\pi} (\tau < \gamma_0 | \mathcal{F}_\tau^X, \theta = 0) (1 - \Pi_\tau).$$

Then, by definition of γ_0 , using that \mathcal{U} is independent of θ , Γ_τ^0 is \mathcal{F}_τ^X -measurable and (16), we also obtain

$$\mathbb{P}_{x,\pi} (\tau < \gamma_0 | \mathcal{F}_\tau^X, \theta = 0) = \mathbb{P}_{x,\pi} (\Gamma_\tau^0 \leq \mathcal{U} | \mathcal{F}_\tau^X, \theta = 0) = (1 - \Gamma_\tau^0).$$

Combining the last two expressions leads to (12). Clearly (13) follows by the same argument.

For (14) we follow a similar approach and we also recall $\gamma(u)$ as in (4) and (9). Then we have

$$(17) \quad \begin{aligned} & \mathbb{E}_{x,\pi} [g(X_{\gamma_0}) \mathbf{1}_{\{\gamma_0 \leq \tau\} \cap \{\theta=0\}} | \mathcal{F}_\tau^X] \\ &= \mathbb{E}_{x,\pi} [g(X_{\gamma_0}) \mathbf{1}_{\{\gamma_0 \leq \tau\}} | \mathcal{F}_\tau^X, \theta = 0] (1 - \Pi_\tau) \\ &= \mathbb{E}_{x,\pi} \left[\int_0^1 g(X_{\gamma_0(z)}) \mathbf{1}_{\{\gamma_0(z) \leq \tau\}} dz | \mathcal{F}_\tau^X, \theta = 0 \right] (1 - \Pi_\tau) \\ &= (1 - \Pi_\tau) \int_0^1 g(X_{\gamma_0(z)}) \mathbf{1}_{\{\gamma_0(z) \leq \tau\}} dz \\ &= (1 - \Pi_\tau) \int_0^\tau g(X_t) d\Gamma_t^0, \end{aligned}$$

where in the penultimate equality we used that $g(X_{\gamma_0(z)}) \mathbf{1}_{\{\gamma_0(z) \leq \tau\}}$ is \mathcal{F}_τ^X -measurable for all $z \geq 0$, and the last equality is due to (9).

The proof of (15) is analogous. \square

It will be convenient in what follows to also use the *likelihood ratio* process $\Phi_t := \Pi_t / (1 - \Pi_t)$, whose dynamics under \mathbb{P} are derived from (8) and Itô's formula as

$$(18) \quad \frac{d\Phi_t}{\Phi_t} = \omega(X_t) (dB_t + \Pi_t \omega(X_t) dt), \quad \Phi_0 = \varphi,$$

where $\varphi = \pi / (1 - \pi)$. The dynamics of the two-dimensional diffusion (X, Φ) are somewhat involved under \mathbb{P} , and we prefer instead to use the measures \mathbb{P}^0 and \mathbb{P}^1 specified by

$$\mathbb{P}^i(A) := \mathbb{P}(A | \theta = i)$$

for $A \in \mathcal{F}_\infty^X$. It is well-known (see [29, Chapter 9]) that

$$(19) \quad \begin{aligned} \frac{d\mathbb{P}^0}{d\mathbb{P}} \Big|_{\mathcal{F}_t^X} &= \frac{1 - \Pi_t}{1 - \pi} = \frac{1 + \varphi}{1 + \Phi_t} \\ &= \exp \left(-\frac{1}{2} \int_0^t \omega^2(X_s) \Pi_s^2 ds - \int_0^t \omega(X_s) \Pi_s dB_s \right), \end{aligned}$$

$$(20) \quad \begin{aligned} \frac{d\mathbb{P}^1}{d\mathbb{P}} \Big|_{\mathcal{F}_t^X} &= \frac{\Pi_t}{\pi} = \exp \left(-\frac{1}{2} \int_0^t \omega^2(X_s) (1 - \Pi_s)^2 ds \right. \\ &\quad \left. + \int_0^t \omega(X_s) (1 - \Pi_s) dB_s \right), \end{aligned}$$

and that X and Φ satisfy

$$(21) \quad \begin{cases} dX_t = \mu_i(X_t) dt + \sigma(X_t) dW_t^i \\ d\Phi_t = \omega(X_t) \Phi_t dW_t^0 \\ \quad = \omega^2(X_t) \Phi_t dt + \omega(X_t) \Phi_t dW_t^1, \end{cases}$$

where

$$W_t^i := - \int_0^t \omega(X_s) (i - \Pi_s) ds + B_t$$

is a \mathbb{P}^i -Brownian motion. Note that the system (21) is semi-decoupled in the sense that the dynamics of X do not depend on Φ .

We now rewrite our problem under the measure \mathbb{P}^0 . In what follows we set $\mathbb{E}^i[\cdot]$ for the expectation under the measure \mathbb{P}^i , with $i = 0, 1$.

Corollary 3.2 (The expected payoff for the uninformed player). *For $(x, \pi) \in \mathcal{I} \times (0, 1)$ and any $(\tau, \gamma_\theta) \in \mathcal{T} \times \mathcal{T}_R^\theta$ we have*

$$(22) \quad \begin{aligned} \mathcal{J}_{x, \pi}(\tau, \gamma_\theta) &= \frac{1}{1 + \varphi} \left(\mathbb{E}_{x, \pi}^0 \left[(1 - \Gamma_\tau^0) f(X_\tau) + \int_0^\tau g(X_t) d\Gamma_t^0 \right] \right. \\ &\quad \left. + \mathbb{E}_{x, \pi}^0 \left[(1 - \Gamma_\tau^1) \Phi_\tau f(X_\tau) + \int_0^\tau \Phi_t g(X_t) d\Gamma_t^1 \right] \right), \end{aligned}$$

where $\varphi = \pi / (1 - \pi)$.

Proof. We start by looking at the first term on the right-hand side of (10). For any $\tau \in \mathcal{T}$, the \mathbb{P} -martingale property of Π_t and (19) give

$$\begin{aligned} &\mathbb{E} \left[(1 - \Pi_\tau) (1 - \Gamma_\tau^0) f(X_\tau) 1_{\{\tau \leq t\}} \right] \\ &= \mathbb{E} \left[\mathbb{E} \left[1 - \Pi_t \mid \mathcal{F}_{t \wedge \tau}^X \right] (1 - \Gamma_\tau^0) f(X_\tau) 1_{\{\tau \leq t\}} \right] \\ &= \mathbb{E} \left[(1 - \Pi_t) (1 - \Gamma_\tau^0) f(X_\tau) 1_{\{\tau \leq t\}} \right] = (1 - \pi) \mathbb{E}^0 \left[(1 - \Gamma_\tau^0) f(X_\tau) 1_{\{\tau \leq t\}} \right]. \end{aligned}$$

Since $\frac{d\mathbb{P}^0}{d\mathbb{P}} \Big|_{\mathcal{F}_t^X}$ is bounded (see (19)), $\mathbb{P}(\tau < \infty) = 1$ implies that $\mathbb{P}^0(\tau < \infty) = 1$. Consequently, letting $t \rightarrow \infty$ yields

$$\mathbb{E} \left[(1 - \Pi_\tau) (1 - \Gamma_\tau^0) f(X_\tau) \right] = (1 - \pi) \mathbb{E}^0 \left[(1 - \Gamma_\tau^0) f(X_\tau) \right]$$

by monotone convergence. By the same argument we also obtain

$$\mathbb{E} \left[(1 - \Pi_\tau) \int_0^\tau g(X_t) d\Gamma_t^0 \right] = (1 - \pi) \mathbb{E}^0 \left[\int_0^\tau g(X_t) d\Gamma_t^0 \right]$$

For the remaining terms in (10) we notice that

$$\begin{aligned} \mathbb{E} [\Pi_\tau (1 - \Gamma_\tau^1) f(X_\tau)] &= \mathbb{E} [(1 - \Pi_\tau) (1 - \Gamma_\tau^1) \Phi_\tau f(X_\tau)] \\ &= (1 - \pi) \mathbb{E}^0 [(1 - \Gamma_\tau^1) \Phi_\tau f(X_\tau)] \end{aligned}$$

and

$$\mathbb{E} \left[\Pi_\tau \int_0^\tau g(X_t) d\Gamma_t^1 \right] = (1 - \pi) \mathbb{E}^0 \left[\int_0^\tau \Phi_t g(X_t) d\Gamma_t^1 \right].$$

Combining the above expressions we obtain (22) upon noticing that $1 - \pi = (1 + \varphi)^{-1}$. \square

The next corollary follows in a similar way using (19) and (20) in the first and second term on the right-hand side of (10), respectively.

Corollary 3.3 (The expected cost for the informed player). *For $(x, \pi) \in \mathcal{I} \times (0, 1)$ and any $(\tau, \gamma_\theta) \in \mathcal{T} \times \mathcal{T}_R^\theta$ we have*

$$(23) \quad \mathcal{J}_{x,\pi}(\tau, \gamma_\theta) = (1 - \pi) \mathcal{J}_{x,\pi}^0(\tau, \Gamma^0) + \pi \mathcal{J}_{x,\pi}^1(\tau, \Gamma^1),$$

where

$$(24) \quad \mathcal{J}_{x,\pi}^0(\tau, \Gamma^0) := \mathbb{E}_{x,\pi}^0 \left[(1 - \Gamma_\tau^0) f(X_\tau) + \int_0^\tau g(X_t) d\Gamma_t^0 \right]$$

and

$$(25) \quad \mathcal{J}_{x,\pi}^1(\tau, \Gamma^1) := \mathbb{E}_{x,\pi}^1 \left[(1 - \Gamma_\tau^1) f(X_\tau) + \int_0^\tau g(X_t) d\Gamma_t^1 \right].$$

Remark 3.4. *The expression in (23) offers a nice interpretation of the functional $\mathcal{J}_{x,\pi}$. Imagine that before the game starts (i.e., at time $t = 0-$), neither of the players knows θ . However, they both know that as soon as the game starts (i.e., at time $t = 0$) Player 2 will learn the true value of θ . Then, we can think of $\mathcal{J}_{x,\pi}$ as the expected payoff for both players at time $t = 0-$ (given the randomised stopping pair (τ, γ_θ)). As one would expect in this context, the payoff at time $t = 0-$ is the average according to the prior distribution of θ of the payoffs in the two possible scenarios.*

As soon as the game starts at time $t = 0$, the payoff of the informed player ‘collapses’ into either $\mathcal{J}_{x,\pi}^0$ or $\mathcal{J}_{x,\pi}^1$ because she learns the true value of θ . On the contrary, the expected payoff of Player 1 remains $\mathcal{J}_{x,\pi}$, which also clarifies our terminology for eqs. (5)–(6).

It is worth noting that many papers in the literature on asymmetric games (see, e.g., [6, 7, 23, 22]) only use the payoff $\mathcal{J}_{x,\pi}$ for their analysis. The ‘value’ of the game in those papers corresponds in our setting to the expected equilibrium payoff for the uninformed player.

We now come to the final formulation of the game's expected payoff, which is also the one that we find most convenient for our solution method. For $(\tau, \gamma_\theta) \in \mathcal{T} \times \mathcal{T}_R^\theta$ and $\varphi = \pi/(1 - \pi)$, let us denote

$$(26) \quad \widehat{\mathcal{J}}_{x,\varphi}(\tau, \gamma_\theta) := (1 + \varphi)\mathcal{J}_{x,\pi}(\tau, \gamma_\theta),$$

and notice that (23) now gives

$$(27) \quad \widehat{\mathcal{J}}_{x,\varphi}(\tau, \gamma_\theta) = \mathcal{J}_{x,\varphi}^0(\tau, \Gamma^0) + \varphi\mathcal{J}_{x,\varphi}^1(\tau, \Gamma^1).$$

Let a stopping time τ be given. It follows from Corollary 3.3 that a strategy $\Gamma = (\Gamma^0, \Gamma^1)$ of the informed player minimizes $\widehat{\mathcal{J}}_{x,\varphi}(\tau, \gamma_\theta)$ if and only if Γ^0 and Γ^1 minimize $\mathcal{J}_{x,\varphi}^0(\tau, \Gamma^0)$ and $\mathcal{J}_{x,\varphi}^1(\tau, \Gamma^1)$, respectively. Consequently, we have the following characterisation of a Nash equilibrium.

Proposition 3.5. *Let $(x, \varphi) \in \mathcal{I} \times \mathbb{R}_+$ be given. A randomised stopping pair $(\tau^*, \gamma_\theta^*) \in \mathcal{T} \times \mathcal{T}_R^\theta$ is a Nash equilibrium if and only if, letting $(\Gamma^{*,0}, \Gamma^{*,1}) \in \mathcal{A} \times \mathcal{A}$ be the couple that generates γ_θ^* , we have*

$$(28) \quad \mathcal{J}_{x,\varphi}^0(\tau^*, \Gamma^{*,0}) \leq \mathcal{J}_{x,\varphi}^0(\tau^*, \Gamma^0),$$

$$(29) \quad \mathcal{J}_{x,\varphi}^1(\tau^*, \Gamma^{*,1}) \leq \mathcal{J}_{x,\varphi}^1(\tau^*, \Gamma^1)$$

and

$$(30) \quad \widehat{\mathcal{J}}_{x,\varphi}(\tau, \gamma_\theta^*) \leq \widehat{\mathcal{J}}_{x,\varphi}(\tau^*, \gamma_\theta^*)$$

for all randomised stopping pairs $(\tau, \Gamma) \in \mathcal{T} \times \mathcal{A}^\theta$.

We refer to $\widehat{\mathcal{J}}_{x,\varphi}(\tau^*, \gamma_\theta^*)$, $\mathcal{J}_{x,\varphi}^0(\tau^*, \Gamma^{*,0})$ and $\mathcal{J}_{x,\varphi}^1(\tau^*, \Gamma^{*,1})$ as the corresponding equilibrium payoffs.

Remark 3.6. *We observe that Proposition 3.5 gives an interpretation of the game as a 3-player nonzero-sum game between a stopper and two controllers. Notice that the stopper plays simultaneously against both controllers, whereas each controller only plays against the stopper. This is in parallel with classical results on games with incomplete information, see [24].*

We also remark that, in contrast with more usual formulations of controller-stopper games, the processes Γ^0 and Γ^1 are not to be considered observables for the stopper and are not part of the state process. Nevertheless, the uninformed player may of course calculate an equilibrium strategy $(\Gamma^{,0}, \Gamma^{*,1})$ (should one exist) of the other player.*

It is intuitively clear that the informed player (Player 2) should never stop in a case where the drift is favourable for her. We end this section with a result along these lines that is useful for constructing a Nash equilibrium in certain cases; see Section 6 below for an application in a particular example.

Proposition 3.7. *Fix $(x, \varphi) \in \mathbb{R}_+ \times \mathbb{R}_+$ and assume that $(\tau^*, \Gamma^{*,0}, \Gamma^{*,1}) \in \mathcal{T} \times \mathcal{T}_R^\theta$ is a Nash equilibrium such that*

$$(31) \quad \mathcal{J}^0(\tau, 0) \leq \mathcal{J}^0(\tau, \Gamma^{*,0})$$

for all $\tau \in \mathcal{T}$. Then $(\tau^*, 0, \Gamma^{*,1})$ is a Nash equilibrium.

Proof. First note that (29) holds since $(\tau^*, \Gamma^{*,0}, \Gamma^{*,1})$ is a Nash equilibrium. Moreover, for any $\Gamma^0 \in \mathcal{T}_R$,

$$(32) \quad \mathcal{J}_{x,\varphi}^0(\tau^*, 0) \leq \mathcal{J}_{x,\varphi}^0(\tau^*, \Gamma^{*,0}) \leq \mathcal{J}_{x,\varphi}^0(\tau^*, \Gamma^0),$$

where the first inequality comes from (31) and the second from (28). Thus (28) holds for the candidate equilibrium $(\tau^*, 0, \Gamma^{*,1})$.

It remains to show that (30) holds for the candidate equilibrium. To do that, note first that inserting $\Gamma^0 = 0$ in (32) yields

$$\mathcal{J}_{x,\varphi}^0(\tau^*, 0) = \mathcal{J}_{x,\varphi}^0(\tau^*, \Gamma^{*,0}).$$

Consequently,

$$\widehat{\mathcal{J}}_{x,\varphi}(\tau^*, 0, \Gamma^{*,1}) = \widehat{\mathcal{J}}_{x,\varphi}(\tau^*, \Gamma^{*,0}, \Gamma^{*,1}),$$

so

$$\begin{aligned} \widehat{\mathcal{J}}_{x,\varphi}(\tau, 0, \Gamma^{*,1}) &\leq \widehat{\mathcal{J}}_{x,\varphi}(\tau, \Gamma^{*,0}, \Gamma^{*,1}) \\ &\leq \widehat{\mathcal{J}}_{x,\varphi}(\tau^*, \Gamma^{*,0}, \Gamma^{*,1}) = \widehat{\mathcal{J}}_{x,\varphi}(\tau^*, 0, \Gamma^{*,1}) \end{aligned}$$

for $\tau \in \mathcal{T}$, where the first inequality follows from (31) and the second one from $(\tau^*, \Gamma^{*,0}, \Gamma^{*,1})$ being a Nash equilibrium. This completes the proof. \square

4. ADJUSTED BELIEFS AND NASH EQUILIBRIA

Since we assume fully rational players, if an equilibrium exists both players are able to compute it in the sense that they both know the stopping time τ^* and the increasing processes $\Gamma^{*,0}$ and $\Gamma^{*,1}$ that are used to generate γ_θ^* . Assume that the players agree on an equilibrium $(\tau^*, \gamma_\theta^*)$. Given the generating processes $\Gamma^{*,0}$ and $\Gamma^{*,1}$, the uninformed player calculates what may be referred to as the *adjusted posterior probability*

$$(33) \quad \Pi_t^* := \mathbb{P}(\theta = 1 | \mathcal{F}_t^X, \gamma_\theta^* > t), \quad t \geq 0.$$

Using properties of conditional expectations we can write

$$(34) \quad \begin{aligned} \Pi_t^* &= \frac{\mathbb{P}(\theta = 1, \gamma_\theta^* > t | \mathcal{F}_t^X)}{\mathbb{P}(\gamma_\theta^* > t | \mathcal{F}_t^X)} \\ &= \frac{\mathbb{P}(\gamma_1^* > t | \mathcal{F}_t^X, \theta = 1) \mathbb{P}(\theta = 1 | \mathcal{F}_t^X)}{\mathbb{P}(\gamma_\theta^* > t | \mathcal{F}_t^X)} \\ &= \frac{(1 - \Gamma_t^{*,1}) \Pi_t}{\mathbb{P}(\gamma_\theta^* > t | \mathcal{F}_t^X)}, \end{aligned}$$

where the last equality is obtained using the same arguments as those used in the proof of Proposition 3.1. Similarly, for the denominator we have

$$(35) \quad \begin{aligned} \mathbb{P}(\gamma_\theta^* > t | \mathcal{F}_t^X) &= \mathbb{P}(\gamma_0^* > t | \mathcal{F}_t^X, \theta = 0) \mathbb{P}(\theta = 0 | \mathcal{F}_t^X) \\ &\quad + \mathbb{P}(\gamma_1^* > t | \mathcal{F}_t^X, \theta = 1) \mathbb{P}(\theta = 1 | \mathcal{F}_t^X) \\ &= (1 - \Gamma_t^{*,0})(1 - \Pi_t) + (1 - \Gamma_t^{*,1}) \Pi_t. \end{aligned}$$

Combining (34)–(35) gives

$$(36) \quad \Pi_t^* = \frac{(1 - \Gamma_t^{*,1})\Phi_t}{1 - \Gamma_t^{*,0} + (1 - \Gamma_t^{*,1})\Phi_t},$$

and then it becomes straightforward to see that the adjusted posterior probability satisfies

$$(37) \quad \Phi_t^* := \frac{\Pi_t^*}{1 - \Pi_t^*} = \Phi_t \frac{1 - \Gamma_t^{*,1}}{1 - \Gamma_t^{*,0}}, \quad t \geq 0.$$

Thus Φ_t^* is the likelihood ratio of the adjusted posterior probability.

Before proceeding we would like to emphasize a subtle point whose understanding is key to the proof of Theorem 5.1 below. When constructing Nash equilibria we will need to verify conditions (28) and (29); so the question arises as to what τ^* should depend on. To this end we recall that processes $\Gamma \in \mathcal{A}^\theta$ are not observable by Player 1 (Remark 3.6) because the two players do not communicate (they only see their opponent stop at some point). Therefore, if Player 2 plays a non-equilibrium pair (Γ^0, Γ^1) the stopping time τ^* is not affected by this choice. In fact, since both players are equally clever and know the equilibrium pairs (τ^*, Γ^*) , Player 1's choice of τ^* should depend only on the (equilibrium) adjusted belief process associated with Γ^* . That is, one should expect that τ^* is a stopping time for the paths of the process (X, Φ^*) . This fact will be key in the second step of the proof of Theorem 5.1.

5. A VERIFICATION RESULT

In this section we provide a verification result (Theorem 5.1) which addresses the question of existence of a Nash equilibrium from the point of view of PDE theory. In particular we show that a triple of functions (u, u^0, u^1) with $u := u_0 + \varphi u_1$ that solves an appropriate quasi-variational inequality provides the equilibrium payoffs for the game as in (28), (29) and (30). This is done by identifying a Nash equilibrium from the candidate functions (u, u^0, u^1) . The formulation in terms of a quasi-variational inequality bridges the probabilistic formulation of our problem to PDE theory and will be used in the next section to construct a full solution to a specific example with linear payoffs.

Denote by $W_{loc}^{2,\infty}(\mathcal{I} \times (0, +\infty))$ the usual Sobolev space of functions in L_{loc}^∞ whose first and second derivatives are also functions in L_{loc}^∞ (recall also that letting C_K^1 be the space of C^1 functions on a compact K , by Sobolev embedding $W_{loc}^{2,\infty} \subset C_K^1$ for any compact K , [1, Thm. 4.12]). In what follows, for $i = 0, 1$, denote by \mathcal{L}^i the second order differential operator associated

with the dynamics of (X, Φ) under the measure \mathbb{P}^i , that is

$$(38) \quad \mathcal{L}^0 := \frac{1}{2} \left(\omega^2(x) \varphi^2 \partial_{\varphi\varphi} + \sigma^2(x) \partial_{xx} + 2(\sigma\omega)(x) \varphi \partial_{x\varphi} \right) + \mu_0(x) \partial_x,$$

$$(39) \quad \mathcal{L}^1 := \frac{1}{2} \left(\omega^2(x) \varphi^2 \partial_{\varphi\varphi} + \sigma^2(x) \partial_{xx} + 2(\sigma\omega)(x) \varphi \partial_{x\varphi} \right) + \mu_1(x) \partial_x + \omega^2(x) \varphi \partial_{\varphi}.$$

In the next theorem we will use the following localising sequences of stopping times: for a C^1 function h , let

$$I(h)_t := \int_0^t \left(\sigma^2(X_s) (\partial_x h)^2(X_s, \Phi_{s-}^*) + \omega^2(X_s) \Phi_{s-}^* (\partial_{\varphi} h)^2(X_s, \Phi_{s-}^*) \right) ds,$$

with Φ^* as in (37), then we set

$$(40) \quad \tau_n(h) := \inf \{ t \geq 0 : I(h)_t \geq n \} \wedge n.$$

Before stating the theorem we also notice that given a set $U \subset \mathcal{I} \times (0, +\infty)$, its closure should be understood relatively to $\mathcal{I} \times (0, +\infty)$, in the sense that \overline{U} does not include the boundary of the state-space, i.e. $\overline{U} \cap \partial(\mathcal{I} \times (0, +\infty)) = \emptyset$.

Theorem 5.1 (Quasi-variational inequality). *Let Assumption 2.1 hold. Let $u, u^0, u^1 : \mathcal{I} \times (0, +\infty) \rightarrow [0, \infty)$ be continuous functions with $u := u^0 + \varphi u^1$. Denote*

$$\mathcal{C} := \{ (x, \varphi) \in \mathcal{I} \times (0, +\infty) : u(x, \varphi) > (1 + \varphi)f(x) \},$$

$$\mathcal{C}^i := \{ (x, \varphi) \in \mathcal{I} \times (0, +\infty) : u^i(x, \varphi) < g(x) \},$$

and $\mathcal{S} := (\mathcal{I} \times (0, +\infty)) \setminus \mathcal{C}$, $\mathcal{S}^i := (\mathcal{I} \times (0, +\infty)) \setminus \mathcal{C}^i$ for $i = 0, 1$.

For $i = 0, 1$, assume that

$$u \in W_{loc}^{2,\infty}(\mathcal{C}^0 \cap \mathcal{C}^1) \cap C^1(\overline{\mathcal{C}^0 \cap \mathcal{C}^1}) \cap C^2(\overline{\mathcal{C} \cap \mathcal{C}^0 \cap \mathcal{C}^1}),$$

and

$$u^i \in C^2(\overline{\mathcal{C} \cap \mathcal{C}^0 \cap \mathcal{C}^1}),$$

and that (u, u^0, u^1) solve the quasi-variational inequality

$$(41) \quad \max\{\mathcal{L}^0 u(x, \varphi), (1 + \varphi)f(x) - u(x, \varphi)\} = 0, \quad \text{a.e. } (x, \varphi) \in \mathcal{C}^0 \cap \mathcal{C}^1,$$

$$(42) \quad \mathcal{L}^i u^i(x, \varphi) = 0, \quad \text{for all } (x, \varphi) \in \mathcal{C} \cap \mathcal{C}^0 \cap \mathcal{C}^1 \text{ and for } i = 0, 1,$$

with the additional conditions

$$(43) \quad u^i(x, \varphi) = f(x), \quad \text{for } (x, \varphi) \in \mathcal{S},$$

$$(44) \quad u_{\varphi}^i(x, \varphi) = 0, \quad \text{for } (x, \varphi) \in \mathcal{S}^0 \cup \mathcal{S}^1.$$

Assume also that there exists $\Gamma^* \in \mathcal{A}^{\theta}$, with $\mathbb{P}^i(\Gamma_t^{*,0} < 1) = 1$ and $\mathbb{P}^i(\Gamma_t^{*,1} < 1) = 1$, for all $t \geq 0$ and $i = 0, 1$, such that, recalling (37),

we have: \mathbb{P}^0 and \mathbb{P}^1 -a.s.,

$$(45) \quad \Delta\Gamma_t^{*,0} \cdot \Delta\Gamma_t^{*,1} = 0, \text{ for all } t \geq 0,$$

$$(46) \quad (X_t, \Phi_t^*) \in \overline{\mathcal{C}^0 \cap \mathcal{C}^1}, \text{ for all } t \geq 0,$$

$$(47) \quad \begin{cases} \text{for } i = 0, 1 \text{ and for all } t \geq 0, \\ d\Gamma_t^{i,*} = \mathbb{1}_{\{(X_t, \Phi_{t-}^*) \in \mathcal{S}^i\}} d\Gamma_t^{i,*} \text{ and} \\ \int_{\Phi_{t-}^*}^{\Phi_t^*} \mathbb{1}_{\{(X_t, z) \notin \mathcal{S}^i\}} dz = 0. \end{cases}$$

Moreover, assume that $\tau^* := \inf\{t \geq 0 : (X_t, \Phi_t^*) \notin \mathcal{C}\}$ is finite \mathbb{P} -a.s., and that the transversality conditions

$$(48) \quad \lim_{n \rightarrow +\infty} \mathbb{E}_{x,\varphi}^i \left[\mathbb{1}_{\{\tau^* > \tau_n\}} u^i(X_{\tau_n}, \Phi_{\tau_n}^*) \right] = 0, \quad i = 0, 1,$$

hold for $\tau_n = \tau_n(u^i)$ and $\tau_n = \tau_n(u)$ as in (40), and for all $(x, \varphi) \in \mathcal{I} \times (0, +\infty)$.

Then, letting $\gamma_\theta^* \in \mathcal{A}^\theta$ be the randomised stopping time generated by Γ^* , we have that $(\tau^*, \gamma_\theta^*)$ forms a Nash equilibrium. Consequently, a value v exists, and the equilibrium payoffs are given by

$$(49) \quad v = u(x, \varphi) = \widehat{\mathcal{J}}_{x,\varphi}(\tau^*, \gamma_\theta^*) \text{ and } u^i(x, \varphi) = \mathcal{J}_{x,\varphi}^i(\tau^*, \gamma_\theta^*), \text{ for } i = 0, 1.$$

Proof. We start by observing that under our assumptions the stopping times $\tau_n(u^i)$, $i = 0, 1$, and $\tau_n(u)$ are such that $\tau_n(u^i), \tau_n(u) \rightarrow \infty$ as $n \rightarrow \infty$, \mathbb{P}^0 and \mathbb{P}^1 -a.s. (for this result we need $\Gamma_t^{*,i} < 1$ for all $t \geq 0$).

Optimality of τ^* . Let $\tau \in \mathcal{T}$. Thanks to the assumed regularity of u we can apply a version of Itô's formula for functions in $W_{loc}^{2,\infty}$ (see, e.g., [20, Thm. 4.1, Ch. VIII]) to $u(X, \Phi^*)$, upon noticing that (X, Φ^*) only takes values in $\overline{\mathcal{C}^0 \cap \mathcal{C}^1}$ as per (46). Letting $\{\tau_n\}_{n=1}^\infty$ be the localizing sequence of stopping times $\tau_n = \tau_n(u)$ and using that $\mathcal{L}^0 u \leq 0$ almost everywhere on $\mathcal{C}^0 \cap \mathcal{C}^1$, we obtain

$$(50) \quad \begin{aligned} & \mathbb{E}_{x,\varphi}^0 \left[(1 - \Gamma_{\tau \wedge \tau_n}^{*,0}) u(X_{\tau \wedge \tau_n}, \Phi_{\tau \wedge \tau_n}^*) \right] \\ & \leq u(x, \varphi) - \mathbb{E}_{x,\varphi}^0 \left[\int_0^{\tau \wedge \tau_n} u(X_t, \Phi_{t-}^*) d\Gamma_t^{*,0,c} \right] \\ & \quad + \mathbb{E}_{x,\varphi}^0 \left[\int_0^{\tau \wedge \tau_n} u_\varphi(X_t, \Phi_{t-}^*) \left(\Phi_{t-}^* d\Gamma_t^{*,0,c} - \Phi_t d\Gamma_t^{*,1,c} \right) \right] \\ & \quad + \mathbb{E}_{x,\varphi}^0 \left[\sum_{t \leq \tau \wedge \tau_n} \left((1 - \Gamma_t^{*,0}) u(X_t, \Phi_t^*) - (1 - \Gamma_{t-}^{*,0}) u(X_t, \Phi_{t-}^*) \right) \right], \end{aligned}$$

where $\Gamma^{*,i,c}$ denotes the continuous part of $\Gamma^{*,i}$, $i = 0, 1$.

Since $u_\varphi(x, \varphi) = u_\varphi^0(x, \varphi) + \varphi u_\varphi^1(x, \varphi) + u^1(x, \varphi)$ and recalling (47) we see that (44) implies

$$(51) \quad \begin{aligned} u_\varphi(X_t, \Phi_{t-}^*) & \left(\Phi_{t-}^* d\Gamma_t^{*,0,c} - \Phi_t d\Gamma_t^{*,1,c} \right) \\ & = u^1(X_t, \Phi_{t-}^*) \Phi_{t-}^* d\Gamma_t^{*,0,c} - g(X_t) \Phi_t d\Gamma_t^{*,1,c}. \end{aligned}$$

Then combining the integrals with respect to the continuous parts of the increasing processes one finds

$$(52) \quad \begin{aligned} \mathbb{E}_{x,\varphi}^0 & \left[\int_0^{\tau \wedge \tau_n} u_\varphi(X_t, \Phi_{t-}^*) \left(\Phi_{t-}^* d\Gamma_t^{*,0,c} - \Phi_t d\Gamma_t^{*,1,c} \right) - \int_0^{\tau \wedge \tau_n} u(X_t, \Phi_{t-}^*) d\Gamma_t^{*,0,c} \right] \\ & = -\mathbb{E}_{x,\varphi}^0 \left[\int_0^{\tau \wedge \tau_n} g(X_t) (d\Gamma_t^{*,0,c} + \Phi_t d\Gamma_t^{*,1,c}) \right]. \end{aligned}$$

Next, we compute the contributions from jumps and recall (45). On the event $\{\Delta\Gamma_t^{*,0} > 0\}$ we have, recalling (44) and (47),

$$\begin{aligned} u^0(X_t, \Phi_t^*) & = u^0(X_t, \Phi_{t-}^*) = g(X_t) \\ u^1(X_t, \Phi_t^*) & = u^1(X_t, \Phi_{t-}^*). \end{aligned}$$

Consequently, using (37) and that $\Delta\Gamma_t^{*,1} = 0$, we get

$$(53) \quad \begin{aligned} (1 - \Gamma_t^{*,0})u(X_t, \Phi_t^*) - (1 - \Gamma_{t-}^{*,0})u(X_t, \Phi_{t-}^*) \\ & = (1 - \Gamma_t^{*,0})(g(X_t) + \Phi_t^* u^1(X_t, \Phi_t^*)) \\ & \quad - (1 - \Gamma_{t-}^{*,0})(g(X_t) + \Phi_{t-}^* u^1(X_t, \Phi_{t-}^*)) = -\Delta\Gamma_t^{*,0} g(X_t). \end{aligned}$$

Similarly, on the event $\{\Delta\Gamma_t^{*,1} > 0\}$ we have

$$(54) \quad (1 - \Gamma_t^{*,0}) \left(u(X_t, \Phi_t^*) - u(X_t, \Phi_{t-}^*) \right) = -\Delta\Gamma_t^{*,1} \Phi_t g(X_t).$$

By combining (50), (52), (53) and (54) we obtain

$$(55) \quad \begin{aligned} \mathbb{E}_{x,\varphi}^0 & \left[(1 - \Gamma_{\tau \wedge \tau_n}^{*,0}) u(X_{\tau \wedge \tau_n}, \Phi_{\tau \wedge \tau_n}^*) \right] \\ & \leq u(x, \varphi) - \mathbb{E}_{x,\varphi}^0 \left[\int_0^{\tau \wedge \tau_n} g(X_t) (d\Gamma_t^{*,0} + \Phi_t d\Gamma_t^{*,1}) \right], \end{aligned}$$

where we notice that the integral with respect to the increasing processes now includes the jump part as well. Rearranging terms and using that $u(x, \varphi) \geq (1 + \varphi)f(x)$ for $(x, \varphi) \in \mathcal{C}^0 \cap \mathcal{C}^1$ we get

$$(56) \quad \begin{aligned} u(x, \varphi) & \geq \mathbb{E}_{x,\varphi}^0 \left[(1 - \Gamma_{\tau \wedge \tau_n}^{*,0}) f(X_{\tau \wedge \tau_n}) (1 + \Phi_{\tau \wedge \tau_n}^*) \right. \\ & \quad \left. + \int_0^{\tau \wedge \tau_n} g(X_t) (d\Gamma_t^{*,0} + \Phi_t d\Gamma_t^{*,1}) \right]. \end{aligned}$$

Passing to the limit as $n \rightarrow \infty$ and using Fatou's lemma gives

$$u(x, \varphi) \geq \sup_{\tau \in \mathcal{T}} \widehat{\mathcal{J}}_{x,\varphi}(\tau, \gamma_\theta^*).$$

To obtain the reverse inequality we repeat the steps above with $\tau^* \wedge \tau_n$ in place of τ , where $\tau_n = \tau_n(u)$ as in (40). In this case we can use standard Itô's formula because $u \in C^2(\overline{\mathcal{C} \cap \mathcal{C}^0 \cap \mathcal{C}^1})$ and $(X_{t \wedge \tau^*}, \Phi_{t \wedge \tau^*})_{t \geq 0}$ is bound to evolve in $\overline{\mathcal{C} \cap \mathcal{C}^0 \cap \mathcal{C}^1}$. Then the inequality in (50) is an equality, so (55) becomes

$$\begin{aligned} u(x, \varphi) &= \mathbb{E}_{x, \varphi}^0 \left[(1 - \Gamma_{\tau^* \wedge \tau_n}^{*,0}) u(X_{\tau^* \wedge \tau_n}, \Phi_{\tau^* \wedge \tau_n}^*) + \int_0^{\tau^* \wedge \tau_n} g(X_t) (d\Gamma_t^{*,0} + \Phi_t d\Gamma_t^{*,1}) \right] \\ &= \mathbb{E}_{x, \varphi}^0 \left[(1 - \Gamma_{\tau^*}^{*,0}) f(X_{\tau^*}) (1 + \Phi_{\tau^*}^*) \mathbb{1}_{\{\tau^* \leq \tau_n\}} \right] \\ &\quad + \mathbb{E}_{x, \varphi}^0 \left[(1 - \Gamma_{\tau_n}^{*,0}) u(X_{\tau_n}, \Phi_{\tau_n}^*) \mathbb{1}_{\{\tau_n < \tau^*\}} \right] \\ &\quad + \mathbb{E}_{x, \varphi}^0 \left[\int_0^{\tau^* \wedge \tau_n} g(X_t) (d\Gamma_t^{*,0} + \Phi_t d\Gamma_t^{*,1}) \right], \end{aligned}$$

where we have used that $u(X_{\tau^*}, \Phi_{\tau^*}^*) = f(X_{\tau^*}) (1 + \Phi_{\tau^*}^*)$. From $u(x, \varphi) = u^0(x, \varphi) + \varphi u^1(x, \varphi)$ and (48) we obtain

$$\lim_{n \rightarrow +\infty} \mathbb{E}_{x, \varphi}^0 \left[(1 - \Gamma_{\tau_n}^{*,0}) u(X_{\tau_n}, \Phi_{\tau_n}^*) \mathbb{1}_{\{\tau_n < \tau^*\}} \right] = 0,$$

so using monotone convergence we take limits as $n \rightarrow \infty$ to conclude that

$$u(x, \varphi) = \sup_{\tau \in \mathcal{T}} \widehat{\mathcal{J}}_{x, \varphi}(\tau, \gamma_\theta^*) = \widehat{\mathcal{J}}_{x, \varphi}(\tau^*, \gamma_\theta^*).$$

Optimality of Γ^* . Pick $\Gamma \in \mathcal{A}^\theta$ and note that $(X_{t \wedge \tau^*}, \Phi_{t \wedge \tau^*}^*)_{t \geq 0} \in \overline{\mathcal{C} \cap \mathcal{C}^0 \cap \mathcal{C}^1}$. Since $u^i \in C^2(\overline{\mathcal{C} \cap \mathcal{C}^0 \cap \mathcal{C}^1})$ for $i = 0, 1$, we can apply standard Itô's formula to $u^i(X, \Phi^*)$ and use that $\mathcal{L}^i u^i = 0$ on $\mathcal{C} \cap \mathcal{C}^0 \cap \mathcal{C}^1$. This gives

$$\begin{aligned} (57) \quad &\mathbb{E}_{x, \varphi}^i \left[(1 - \Gamma_{\tau^* \wedge \tau_n}^i) u^i(X_{\tau^* \wedge \tau_n}, \Phi_{\tau^* \wedge \tau_n}^*) \right] \\ &= u^i(x, \varphi) - \mathbb{E}_{x, \varphi}^i \left[\int_0^{\tau^* \wedge \tau_n} u^i(X_t, \Phi_{t-}^*) d\Gamma_t^{i,c} \right] \\ &\quad + \mathbb{E}_{x, \varphi}^i \left[\int_0^{\tau^* \wedge \tau_n} \frac{1 - \Gamma_{t-}^i}{1 - \Gamma_{t-}^{*,0}} u_\varphi^i(X_t, \Phi_{t-}^*) (\Phi_{t-}^* d\Gamma_t^{*,0,c} - \Phi_t d\Gamma_t^{*,1,c}) \right] \\ &\quad + \mathbb{E}_{x, \varphi}^i \left[\sum_{t \leq \tau^* \wedge \tau_n} ((1 - \Gamma_t^i) u^i(X_t, \Phi_t^*) - (1 - \Gamma_{t-}^i) u^i(X_t, \Phi_{t-}^*)) \right], \end{aligned}$$

where $\{\tau_n\}_{n=1}^\infty$ is the localizing sequence of stopping times $\tau_n = \tau_n(u^i)$. Recalling that $u_\varphi^i = 0$ on the support of $t \mapsto d\Gamma_t^{i,*}$ (cf. (47)) we immediately see that

$$(58) \quad \mathbb{E}_{x, \varphi}^i \left[\int_0^{\tau^* \wedge \tau_n} \frac{1 - \Gamma_{t-}^i}{1 - \Gamma_{t-}^{*,0}} u_\varphi^i(X_t, \Phi_{t-}^*) (\Phi_{t-}^* d\Gamma_t^{*,0,c} - \Phi_t d\Gamma_t^{*,1,c}) \right] = 0.$$

Moreover, (47) guarantees

$$u^i(X_t, \Phi_t^*) - u^i(X_t, \Phi_{t-}^*) = 0, \quad \mathbb{P}_{x, \varphi}^i\text{-a.s.}$$

so that by simply adding and subtracting $(1 - \Gamma_{t-}^i)u^i(X_t, \Phi_t^*)$ in the sum of jumps in (57) we obtain

$$\begin{aligned}
 (59) \quad & \mathbb{E}_{x,\varphi}^i \left[\sum_{t \leq \tau^* \wedge \tau_n} ((1 - \Gamma_t^i)u^i(X_t, \Phi_t^*) - (1 - \Gamma_{t-}^i)u^i(X_t, \Phi_{t-}^*)) \right] \\
 & = -\mathbb{E}_{x,\varphi}^i \left[\sum_{t \leq \tau^* \wedge \tau_n} u^i(X_t, \Phi_t^*) \Delta \Gamma_t^i \right] \\
 & \geq -\mathbb{E}_{x,\varphi}^i \left[\sum_{t \leq \tau^* \wedge \tau_n} g(X_t) \Delta \Gamma_t^i \right],
 \end{aligned}$$

where the final inequality uses that $u^i \leq g$ on $\mathcal{C} \cap \mathcal{C}^{1-i}$.

Next, plugging (58) and (59) in (57), and using again that $u^i \leq g$ on $\mathcal{C} \cap \mathcal{C}^{1-i}$, we arrive at

$$(60) \quad \mathbb{E}_{x,\varphi}^i \left[(1 - \Gamma_{\tau^* \wedge \tau_n}^i) u^i(X_{\tau^* \wedge \tau_n}, \Phi_{\tau^* \wedge \tau_n}^*) \right] \geq u^i(x, \varphi) - \mathbb{E}_{x,\varphi}^i \left[\int_0^{\tau^* \wedge \tau_n} g(X_t) d\Gamma_t^i \right],$$

where the integral now includes both the continuous part and the jump part of the increasing process. Using (43) we see that

$$\begin{aligned}
 u^i(x, \varphi) & \leq \mathbb{E}_{x,\varphi}^i \left[(1 - \Gamma_{\tau^*}^i) f(X_{\tau^*}) \mathbf{1}_{\{\tau^* \leq \tau_n\}} \right] \\
 & \quad + \mathbb{E}_{x,\varphi}^i \left[(1 - \Gamma_{\tau_n}^i) u^i(X_{\tau_n}, \Phi_{\tau_n}^*) \mathbf{1}_{\{\tau_n < \tau^*\}} \right] + \mathbb{E}_{x,\varphi}^i \left[\int_0^{\tau^* \wedge \tau_n} g(X_t) d\Gamma_t^i \right].
 \end{aligned}$$

Passing to the limit as $n \rightarrow \infty$, using the transversality condition (48) and monotone convergence we obtain

$$u^i(x, \varphi) \leq \mathbb{E}_{x,\varphi}^i \left[(1 - \Gamma_{\tau^*}^i) f(X_{\tau^*}) + \int_0^{\tau^*} g(X_t) d\Gamma_t^i \right].$$

Consequently,

$$u^i(x, \varphi) \leq \inf_{\Gamma \in \mathcal{A}^\theta} \mathcal{J}_{x,\varphi}^i(\tau^*, \gamma_\theta), \quad \text{for } i = 0, 1.$$

The reverse inequality is obtained by taking $\Gamma = \Gamma^*$ in the proof above and observing that in doing so the inequalities in (59) and (60) become equalities. We thus obtain

$$u^i(x, \varphi) = \inf_{\Gamma \in \mathcal{A}^\theta} \mathcal{J}_{x,\varphi}^i(\tau^*, \gamma_\theta) = \mathcal{J}_{x,\varphi}^i(\tau^*, \gamma_\theta^*),$$

for $i = 0, 1$, which completes the proof. \square

Remark 5.2. *The assumption $u \in C^2(\overline{\mathcal{C} \cap \mathcal{C}^0 \cap \mathcal{C}^1})$ is needed in the generality of the theorem because a priori the law of (X, Φ^*) may have atoms on the boundary of the domain, i.e. on $\partial(\overline{\mathcal{C} \cap \mathcal{C}^0 \cap \mathcal{C}^1})$. However, in practical examples where something is known about the geometry of $\mathcal{C}^0 \cap \mathcal{C}^1$ one may be able to rule out the existence of such atoms and the assumption may be relaxed to $u \in C^1(\overline{\mathcal{C} \cap \mathcal{C}^0 \cap \mathcal{C}^1})$ with bounded second derivatives.*

Remark 5.3. *The assumption that $\mathbb{P}^i(\Gamma_t^{*,0} < 1) = 1$ and $\mathbb{P}^i(\Gamma_t^{*,1} < 1) = 1$ for all $t \geq 0$ is useful for the localisation of the stochastic integrals in the proof, and to avoid that the process Φ^* reaches the endpoints of its state-space (where u and u^i are not properly defined). However, this is also a natural assumption as we now explain: if for example, for some $t \geq 0$ one has $\mathbb{P}^i(\Gamma_t^{*,0} = 1) > 0$, then full information is revealed at time t , for all $\omega \in \{\Gamma_t^{*,0} = 1\}$. This is an undesirable situation for Player 2 and one may expect that such strategies should not be optimal.*

We are not aware of any standard PDE results that guarantee the solvability of the quasi-variational inequality above. Nevertheless, the structure of (41)–(42) resembles that of quasi-variational inequalities for nonzero-sum Dynkin games (see, e.g., [5] and more recently [11]), as we should expect from Proposition 3.5 and Remark 3.6. Hence one may hope that general existence of solutions can be found following ideas from that literature.

We will show in the next section that the assumptions in Theorem 5.1 hold in an example with a linear payoff structure.

6. AN EXAMPLE WITH LINEAR PAYOFFS

In this section we study an example where the underlying diffusion is a geometric Brownian motion and the payoff functions are linear. More explicitly, let

$$dX_t = \mu X_t dt + \sigma X_t dW_t,$$

where $\mu = \mu_0(1 - \theta) + \mu_1\theta$ and (with a small abuse of notation) μ_0 and μ_1 now are constants satisfying $\mu_0 < \mu_1$. In this case, the signal-to-noise ratio $\omega = (\mu_1 - \mu_0)/\sigma$ is also a constant. Furthermore, let

$$(61) \quad f(x) = x \quad \text{and} \quad g(x) = (1 + \epsilon)x,$$

where $\epsilon > 0$. Given a randomised stopping pair $(\tau, \gamma_\theta) \in \mathcal{T} \times \mathcal{T}_R^\theta$, the stopping game with asymmetric information has a payoff

$$R(\tau, \gamma_\theta) = X_\tau \mathbf{1}_{\{\tau < \gamma_\theta\}} + (1 + \epsilon)X_{\gamma_\theta} \mathbf{1}_{\{\tau \geq \gamma_\theta\}},$$

where we also recall that under \mathbb{P}^0 we have

$$\begin{cases} dX_t = \mu_0 X_t dt + \sigma X_t dW_t^0, \\ d\Phi_t = \omega \Phi_t dW_t^0, \end{cases}$$

and under \mathbb{P}^1 we have

$$\begin{cases} dX_t = \mu_1 X_t dt + \sigma X_t dW_t^1, \\ d\Phi_t = \omega^2 \Phi_t dt + \omega \Phi_t dW_t^1. \end{cases}$$

Remark 6.1. *Clearly, the case $\mu = \mu_0$ is advantageous for Player 2, while the case $\mu = \mu_1$ would be preferred by Player 1. Furthermore, if $\mu_0 < \mu_1 < 0$, then the inf-player (Player 2) would never stop, whereas if $0 < \mu_0 < \mu_1$ the sup-player (Player 1) would never stop.*

In light of the above remark, in the rest of this section we make the following standing assumption.

Assumption 6.2. *We have $\mu_0 < 0 < \mu_1$.*

One key advantage of the linear structure of our example is that we can effectively reduce the problem to one state variable, hence simplifying the rest of the analysis. In particular we will see below that Φ is the only relevant dynamic in the optimisation. For $i = 0, 1$ let $\tilde{\mathbb{P}}^i$ be defined by

$$(62) \quad \frac{d\tilde{\mathbb{P}}^i}{d\mathbb{P}^i} \Big|_{\mathcal{F}_t^X} = \exp \left\{ -\frac{\sigma^2}{2}t + \sigma W_t^i \right\},$$

and notice that $\tilde{W}_t^i := -\sigma t + W_t^i$ is a $\tilde{\mathbb{P}}^i$ -Brownian motion. For future reference we note that

$$(63) \quad dX_t = (\mu_i + \sigma^2)X_t dt + \sigma X_t d\tilde{W}_t^i, \quad \text{under } \tilde{\mathbb{P}}^i$$

and

$$(64) \quad d\Phi_t = \sigma\omega\Phi_t dt + \omega\Phi_t d\tilde{W}_t^0, \quad \text{under } \tilde{\mathbb{P}}^0,$$

$$(65) \quad d\Phi_t = (\sigma\omega + \omega^2)\Phi_t dt + \omega\Phi_t d\tilde{W}_t^1, \quad \text{under } \tilde{\mathbb{P}}^1.$$

It is also easy to verify that Φ and X are effectively linked by direct proportionality, that is

$$(66) \quad \varphi^{-1}\Phi_t = x^{-\omega/\sigma}(X_t)^{\omega/\sigma} e^{((i-1/2)\omega^2 - \frac{\omega}{\sigma}(\mu_i - \frac{\sigma^2}{2}))t}, \quad \tilde{\mathbb{P}}^i\text{-a.s.}$$

Lemma 6.3. *For $(x, \varphi) \in \mathbb{R}_+ \times \mathbb{R}_+$ and $(\tau, \gamma_\theta) \in \mathcal{T} \times \mathcal{T}_R^\theta$ our game payoff can be rewritten as*

$$(67) \quad \hat{\mathcal{J}}_{x,\varphi}(\tau, \gamma_\theta) = x \left(\tilde{\mathbb{E}}_\varphi^0 \left[e^{\mu_0\tau} (1 - \Gamma_\tau^0) + (1 + \epsilon) \int_0^\tau e^{\mu_0 t} d\Gamma_t^0 \right] \right. \\ \left. + \tilde{\mathbb{E}}_\varphi^0 \left[e^{\mu_0\tau} \Phi_\tau (1 - \Gamma_\tau^1) + (1 + \epsilon) \int_0^\tau e^{\mu_0 t} \Phi_t d\Gamma_t^1 \right] \right).$$

Moreover, we have $\hat{\mathcal{J}}_{x,\varphi}(\tau, \gamma_\theta) = \mathcal{J}_{x,\varphi}^0(\tau, \Gamma^0) + \varphi \mathcal{J}_{x,\varphi}^1(\tau, \Gamma^1)$, where

$$(68) \quad \mathcal{J}_{x,\varphi}^0(\tau, \Gamma^0) = x \tilde{\mathbb{E}}_\varphi^0 \left[e^{\mu_0\tau} (1 - \Gamma_\tau^0) + (1 + \epsilon) \int_0^\tau e^{\mu_0 t} d\Gamma_t^0 \right]$$

and

$$(69) \quad \mathcal{J}_{x,\varphi}^1(\tau, \Gamma^1) = x \tilde{\mathbb{E}}_\varphi^1 \left[e^{\mu_1\tau} (1 - \Gamma_\tau^1) + (1 + \epsilon) \int_0^\tau e^{\mu_1 t} d\Gamma_t^1 \right].$$

Proof. The expression in (67) follows from (22) and (26), upon noticing that

$$X_t = x e^{\mu_0 t} \frac{d\tilde{\mathbb{P}}^0}{d\mathbb{P}^0} \Big|_{\mathcal{F}_t^X}, \quad \mathbb{P}^0\text{-a.s.},$$

and arguing as in the proof of Corollary 3.2. Likewise, (68) and (69) follow from (24) and (25). \square

It is intuitively clear that Player 2 should never stop in the case $\mu = \mu_0$. Note that, for any $\Gamma^0 \in \mathcal{A}$, integration by parts allows us to rewrite (68) as

$$\mathcal{J}_{x,\varphi}^0(\tau, \Gamma^0) = x \left(1 + \tilde{\mathbb{E}}_\varphi^0 \left[\mu_0 \int_0^\tau e^{\mu_0 t} (1 - \Gamma_t^0) dt + \epsilon \int_0^\tau e^{\mu_0 t} d\Gamma_t^0 \right] \right).$$

Using that $\mu_0 < 0$, we immediately obtain

$$\mathcal{J}_{x,\varphi}^0(\tau, \Gamma^0) \geq x \left(1 + \tilde{\mathbb{E}}_\varphi^0 \left[\mu_0 \int_0^\tau e^{\mu_0 t} dt \right] \right) = \mathcal{J}_{x,\varphi}^0(\tau, 0),$$

i.e. (31) holds. Consequently, Proposition 3.7 shows that it is sufficient to look for a Nash equilibrium in the subclass of $\gamma \in \mathcal{T}_R^\theta$ for which $\gamma_0 = +\infty$ (equivalently, $\Gamma^0 = 0$).

Now we need to work out the remaining equilibrium control $\Gamma^{*,1}$ and the stopping time τ^* . We first formulate an educated guess on the structure of τ^* and $\Gamma^{*,1}$, and subsequently we verify that using such a guess we can produce a solution of the quasi-variational inequality from Theorem 5.1.

6.1. Candidate adjusted likelihood ratio. In the case $\mu = \mu_1$, the existence of asymmetric information creates an incentive for the informed player not to stop immediately in order to ‘fool’ the uninformed player. Indeed, if the uninformed player is made to believe that the drift is low (i.e., $\mu = \mu_0$), then the uninformed player may choose to stop early, which is beneficial for the informed player since then only the smaller payoff has to be paid. Thus it is natural that Player 2 will only want to stop when Φ becomes too high (i.e., the uninformed player has a strong belief that the drift is μ_1).

Including the idea of randomisation in the reasoning above, we expect that the informed player will stop at some upper threshold according to some ‘intensity’. The effect of randomisation is to generate an adjusted likelihood ratio Φ^* , which can be interpreted as the belief of the uninformed player after manipulation performed by the informed one. For Player 2 it is therefore a question of finding the optimal trade-off between manipulating Player 1’s beliefs and stopping not too late.

Following the heuristics above we conjecture that Player 2 will construct $\Gamma^{*,1}$ in a way that reflects the process $\Phi^* = \Phi(1 - \Gamma^{*,1})$ at an upper threshold. With this idea in mind, let $B \in (0, \infty)$ and an initial belief $\varphi \in (0, \infty)$ be given. It is well known that there exists a unique pair of processes (Y, L) such that $\tilde{\mathbb{P}}_\varphi^0$ -a.s. one has

$$(70) \quad (L)_{t \geq 0} \text{ is continuous and non-decreasing with } L_0 = 0,$$

$$(71) \quad Y_{0-} = \varphi, Y_0 = \varphi \wedge B \text{ and } Y_t \in (0, B] \text{ for } t \geq 0,$$

$$(72) \quad (Y, L) \text{ solves } \begin{cases} dY_t = \sigma \omega Y_t dt + \omega Y_t d\tilde{W}_t^0 - dL_t, \\ \int_0^t 1_{\{Y_s < B\}} dL_s = 0. \end{cases}$$

Then Y is a diffusion process with reflection at B . Define the process $\Gamma^B \in \mathcal{A}$ by $\Gamma_{0-}^B = 0$, $\Gamma_0^B = \max\{0, 1 - B/\varphi\}$ and

$$(73) \quad \Gamma_t^B = 1 - (1 - \Gamma_0^B)e^{-L_t/B}, \quad \tilde{\mathbb{P}}_\varphi^0\text{-a.s.}$$

Next we show that the adjusted likelihood ratio corresponding to the pair $\Gamma = (0, \Gamma^B)$ is given by the reflected process Y .

Proposition 6.4. *Fix $B \in (0, \infty)$, and consider the processes (Y, L) and Γ^B as above. Then for any $\varphi \in (0, \infty)$ we have*

$$(74) \quad \Phi_t^B := \Phi_t(1 - \Gamma_t^B) = Y_t, \quad \text{for all } t \geq 0, \quad \tilde{\mathbb{P}}_\varphi^0\text{-a.s.}$$

Proof. Noticing that (72) implies $dL_t = \mathbf{1}_{\{Y_t=B\}}dL_t$ we can write the first equation in (72) as

$$dY_t = \sigma\omega Y_t dt + \omega Y_t d\tilde{W}_t^0 - B^{-1}Y_t dL_t.$$

Recalling now (70)–(71) and thanks to the above equation we can write Y explicitly under $\tilde{\mathbb{P}}_\varphi^0$ as

$$Y_t = (\varphi \wedge B) \exp\left(\omega\tilde{W}_t^0 + \left(\sigma\omega - \frac{\omega^2}{2}\right)t - B^{-1}L_t\right).$$

A direct comparison of the expression above with Γ^B in (73) and Φ in (64) gives (74). \square

Below we formulate and solve a variational problem based on the conjecture mentioned at the beginning of the section: Player 2 will select a threshold $B \in \mathbb{R}_+$ and adopt the randomised stopping time generated by the couple $(0, \Gamma^B) \in \mathcal{A} \times \mathcal{A}$. Player 1 will instead choose a threshold $A \in (0, B)$ and stop at

$$(75) \quad \tau_A := \inf\{t \geq 0 : \Phi_t^B \leq A\}.$$

6.2. Quasi-variational inequality for the problem with linear payoff. Here we use a constructive approach to obtain the candidate quasi-variational inequality for the game, which we will then test against the requirements of Theorem 5.1 in the next section.

As mentioned above, we look for an equilibrium with $\Gamma^{*,0} \equiv 0$. If Player 2 plays $\Gamma = (0, \Gamma^B)$ and Player 1 plays τ_A , we obtain from (68)

$$(76) \quad \mathcal{J}_{x,\varphi}^0(\tau_A, \Gamma^{*,0}) = x\tilde{\mathbb{E}}_\varphi^0[e^{\mu_0\tau_A}] =: xV_0(\varphi).$$

The idea is that we should verify that $u^0(x, \varphi) = xV_0(\varphi)$ with u^0 as in Theorem 5.1. This will be done in Theorem 6.10, using facts collected in this section.

It is easy to check (see, e.g., [36]) that V_0 satisfies

$$(77) \quad \begin{cases} \frac{\omega^2\varphi^2}{2}V_0''(\varphi) + \sigma\omega\varphi V_0'(\varphi) + \mu_0V_0(\varphi) = 0, & \varphi \in (A, B) \\ V_0(\varphi) = 1, & \varphi \in (0, A] \\ V_0'(B-) = 0. \end{cases}$$

Notice that the condition at B is the usual normal reflection condition. Moreover, observing that $\tilde{\mathbb{P}}_\varphi^0(\Phi_0^B = B) = 1$ for $\varphi \geq B$, it follows that $\tilde{\mathbb{E}}_\varphi^0[e^{\mu_0\tau_A}] = \tilde{\mathbb{E}}_B^0[e^{\mu_0\tau_A}]$ for $\varphi \geq B$, and therefore

$$(78) \quad V_0'(\varphi) = 0, \quad \varphi \geq B.$$

Moreover, we also note that $\mu_0 < 0$ implies that

$$(79) \quad V_0(\varphi) \leq 1, \quad \varphi \geq A.$$

Conversely, an application of Itô's formula gives

Lemma 6.5. *Assume there exists $\bar{V}_0 \in C^1([A, +\infty))$ with $\bar{V}_0 \in C^2([A, B])$ that solves (77)–(78). Then $\bar{V}_0(\varphi) = \tilde{\mathbb{E}}_\varphi^0[e^{\mu_0\tau_A}] = V^0(\varphi)$.*

Next we introduce a function $xV_1(\varphi)$ which we want to associate with $\mathcal{J}_{x,\varphi}^1(\tau_A, \Gamma^B)$ from (69). We cast a boundary-value problem for V_1 according to the following logic:

- (i) In the interval (A, B) neither of the two players should stop, so the function V_1 should be harmonic for the process Φ^B with creation at rate μ_1 ;
- (ii) The informed player will only stop when the process Φ^B exceeds B (although not necessarily at the first hitting time of B). Then we expect $V_1(B) = 1 + \epsilon$;
- (iii) For the choice of B to be optimal for Player 2 (given that Player 1 uses τ_A), the classical smooth-fit condition should hold, that is we expect $V_1'(B-) = 0$;
- (iv) If the uninformed player stops first (according to τ_A) then the cost for Player 2 is $V_1(A) = 1$.

Combining the four items above gives us the boundary value problem

$$(80) \quad \begin{cases} \frac{\omega^2\varphi^2}{2}V_1''(\varphi) + (\omega^2 + \sigma\omega)\varphi V_1'(\varphi) + \mu_1 V_1(\varphi) = 0, & \varphi \in (A, B) \\ V_1(\varphi) = 1, & \varphi \in (0, A] \\ V_1(\varphi) = 1 + \epsilon, & \varphi \in [B, \infty) \\ V_1'(B) = 0. \end{cases}$$

Notice that if $V_1 \in C^1([A, +\infty)) \cap C^2([A, B])$ solves the above system, then it is easy to verify

$$(81) \quad xV_1(\varphi) = \mathcal{J}_{x,\varphi}^1(\tau_A, \Gamma^B)$$

thanks to an application of Itô calculus. Further we can establish monotonicity of V_1 , which will be useful later in this section.

Lemma 6.6. *Assume that $0 < A < B$ and $V_1 \in C^2([A, B])$ solves (80). Then $V_1'(\varphi) \geq 0$ for $\varphi \in [A, B]$ and $1 \leq V_1(\varphi) < 1 + \epsilon$ for $\varphi \in [A, B)$.*

Proof. Let us start by observing that the first, third and fourth equations in (80) imply

$$\frac{\omega^2 B^2}{2}V_1''(B-) = -\mu_1(1 + \epsilon) < 0.$$

The above and $V_1'(B) = 0$ imply that there exists $\lambda_0 > 0$ (with $B - \lambda_0 \geq A$) such that

$$(82) \quad V_1'(\varphi) > 0, \quad \text{for } \varphi \in (B - \lambda_0, B).$$

With the aim of reaching a contradiction, assume that there exists $\varphi \in (A, B)$ such that $V_1'(\varphi) < 0$. Then we can also define

$$c := \sup\{\varphi \in (A, B) : V_1'(\varphi) < 0\},$$

and clearly $c \in (A, B - \lambda_0]$. Due to continuity of V_1' it must be $V_1'(c) = 0$. Since the ODE is of Euler type, its solution is a linear combination of power functions; thus $V_1 \in C^\infty(A, B)$. Then, setting $v_1 := V_1'$ and differentiating the first equation in (80) we get that v_1 must solve the boundary value problem

$$(83) \quad \begin{cases} \frac{\omega^2 \varphi^2}{2} v_1''(\varphi) + (2\omega^2 + \sigma\omega)\varphi v_1'(\varphi) + \alpha v_1(\varphi) = 0, & \varphi \in (c, B), \\ v_1(c+) = v_1(B-) = 0, \end{cases}$$

with $\alpha := \omega^2 + \sigma\omega + \mu_1$. It then follows, e.g. by the Feynman-Kac formula, that $v_1(\varphi) = 0$ for $\varphi \in (c, B)$, which contradicts (82).

Then $V_1'(\varphi) \geq 0$ in $[A, B]$ as claimed. Moreover, $1 \leq V_1(\varphi) < 1 + \epsilon$ for $\varphi \in [A, B)$, by the second and third equation in (80). \square

Hereafter, when referring to V_1 we will implicitly assume that it solves (80) (we show in the next subsection that (80) and (84) below can be solved simultaneously in a unique way).

Recalling (67) it is now natural to associate $\widehat{\mathcal{J}}_{x,\varphi}(\tau_A, 0, \Gamma^{1,B})$ to the function

$$xV(\varphi) := x(V_0(\varphi) + \varphi V_1(\varphi)).$$

Using the second equations in (77) and (80), we see immediately that $V(A) = 1 + A$. Moreover, recalling also (26), the function $\Xi(x, \varphi) := xV(\varphi)/(1 + \varphi)$ should represent the equilibrium payoff for the uninformed player (notice that indeed $\Xi(x, A) = x$). Thus, by optimality, we expect that the classical smooth-fit condition holds at the boundary A , i.e. $\Xi_\varphi(x, A+) = 0$ or, equivalently, $V'(A+) = 1$. Hence, using (77) and (80) we obtain that V should solve the boundary value problem

$$(84) \quad \begin{cases} \frac{\omega^2 \varphi^2}{2} V''(\varphi) + \sigma\omega\varphi V'(\varphi) + \mu_0 V(\varphi) = 0, & \varphi \in (A, B) \\ V(\varphi) = 1 + \varphi, & \varphi \in (0, A] \\ V'(A+) = 1, \\ V'(B-) = 1 + \epsilon. \end{cases}$$

Moreover,

$$(85) \quad V'(\varphi) = 1 + \epsilon, \quad \varphi \geq B$$

by (78) and (80). Before closing this section we provide some useful properties of V .

Lemma 6.7. *Assume that $V \in C^2([A, B])$ solves (84). Then $V(\varphi) > 1 + \varphi$ and $V'(\varphi) > 1$ for $\varphi \in (A, B)$.*

Proof. First note that the ODE is of Euler type, so $V \in C^\infty(A, B)$. Differentiating the first equation in (84) and imposing the boundary conditions for V' at A and B , we find that $v := V'$ solves

$$(86) \quad \begin{cases} \frac{\omega^2 \varphi^2}{2} v''(\varphi) + (\omega^2 + \sigma\omega)\varphi v'(\varphi) + \mu_1 v(\varphi) = 0, & \varphi \in (A, B) \\ v(A+) = 1, \\ v(B-) = 1 + \epsilon. \end{cases}$$

Recalling the dynamics of Φ under $\tilde{\mathbb{P}}^1$ (see (65)), setting

$$\rho_A := \inf\{t \geq 0 : \Phi_t \leq A\} \quad \text{and} \quad \rho_B := \inf\{t \geq 0 : \Phi_t \geq B\},$$

and using Itô's formula, we easily obtain

$$(87) \quad \begin{aligned} v(\varphi) &= \tilde{\mathbb{E}}_\varphi^1 \left[e^{\mu_1(\rho_A \wedge \rho_B)} v(\Phi_{\rho_A \wedge \rho_B}) \right] \\ &= \tilde{\mathbb{E}}_\varphi^1 \left[e^{\mu_1(\rho_A \wedge \rho_B)} \right] + \epsilon \tilde{\mathbb{E}}_\varphi^1 \left[e^{\mu_1 \rho_B} \mathbf{1}_{\{\rho_B < \rho_A\}} \right]. \end{aligned}$$

Now both claims in the lemma follow from (87), due to $\mu_1 > 0$ and recalling that $V(\varphi) = 1 + \varphi$ for $\varphi \leq A$. \square

Finally, we notice that Lemma 6.7 and the fact that $V'(A+) = 1$ imply that $V''(A+) \geq 0$. Then plugging the second and third equation of (84) into the first one and using $V''(A+) \geq 0$ we obtain

$$\frac{\omega^2 A^2}{2} V''(A+) + \sigma\omega A + \mu_0(1 + A) = 0 \implies \mu_1 A + \mu_0 \leq 0.$$

Corollary 6.8. *Assume that $V \in C^2([A, B])$ solves (84). Then it must be $A \leq -\mu_0/\mu_1$.*

6.3. Solution of the variational problem and Nash equilibrium. We now show that (80) and (84) can be solved simultaneously in a unique way. Note that once the functions V and V_1 and the boundary points A and B are found, the function V_0 is automatically determined from the relation $V_0(\varphi) := V(\varphi) - \varphi V_1(\varphi)$.

The general solution of the ODE for V_1 in (80) is

$$(88) \quad V_1(\varphi) = C_1 \varphi^{\beta_1 - 1} + C_2 \varphi^{\beta_2 - 1},$$

where C_1 and C_2 are constants and $\beta_1 \in (0, 1)$ and $\beta_2 < 0$ are solutions of the quadratic equation

$$\frac{1}{2} \omega^2 \beta(\beta - 1) + \sigma\omega\beta + \mu_0 = 0.$$

The third and fourth boundary conditions in (80) can be used to determine C_1 and C_2 as

$$C_1 = \frac{(1 - \beta_2)(1 + \epsilon)}{\beta_1 - \beta_2} B^{1 - \beta_1} \quad \text{and} \quad C_2 = \frac{(\beta_1 - 1)(1 + \epsilon)}{\beta_1 - \beta_2} B^{1 - \beta_2}.$$

From the condition $V_1(A) = 1$ and the derived expressions for C_1 and C_2 , we arrive at the equation

$$(89) \quad (1 - \beta_2) \left(\frac{A}{B}\right)^{\beta_1 - 1} + (\beta_1 - 1) \left(\frac{A}{B}\right)^{\beta_2 - 1} = \frac{\beta_1 - \beta_2}{1 + \epsilon}.$$

Lemma 6.9. *There exists a unique value of $A/B \in (0, 1)$ satisfying (89).*

Proof. Letting $h(z) = (1 - \beta_2)z^{\beta_1 - 1} + (\beta_1 - 1)z^{\beta_2 - 1} - (\beta_1 - \beta_2)/(1 + \epsilon)$ we can see that $h'(z) = (1 - \beta_2)(\beta_1 - 1)[z^{\beta_1 - 2} - z^{\beta_2 - 2}] > 0$ for $z \in (0, 1)$ since $\beta_1 \in (0, 1)$ and $\beta_2 < 0$. Furthermore, $\lim_{z \downarrow 0} h(z) = -\infty$ and $h(1) = \epsilon(\beta_1 - \beta_2)/(1 + \epsilon) > 0$. Hence we conclude that there is a unique root of $h(z) = 0$ in $(0, 1)$. \square

Next, the general solution of the ODE for V in (84) is

$$(90) \quad V(\varphi) = D_1 \varphi^{\beta_1} + D_2 \varphi^{\beta_2}$$

for constants D_1 and D_2 . The second and third boundary conditions in (84) can be used to determine D_1 and D_2 as

$$D_1 = \frac{A^{-\beta_1}}{\beta_1 - \beta_2} [-\beta_2 + (1 - \beta_2)A] \quad \text{and} \quad D_2 = \frac{A^{-\beta_2}}{\beta_1 - \beta_2} [\beta_1 + (\beta_1 - 1)A].$$

From the boundary condition $V'(B-) = 1 + \epsilon$ and the derived expressions for D_1 and D_2 , we arrive at the equation

$$(1 + \epsilon)(\beta_1 - \beta_2)B = (A/B)^{-\beta_2} \beta_2 [\beta_1 + (\beta_1 - 1)A] \\ - (A/B)^{-\beta_1} \beta_1 [\beta_2 + (\beta_2 - 1)A].$$

Denoting the unique root of (89) as $\delta = A/B \in (0, 1)$ we set $A = \delta B$ to obtain

$$(91) \quad (1 + \epsilon)(\beta_1 - \beta_2)B = \delta^{-\beta_2} \beta_2 [\beta_1 + (\beta_1 - 1)\delta B] \\ - \delta^{-\beta_1} \beta_1 [\beta_2 + (\beta_2 - 1)\delta B].$$

The linear equation (91) has the unique solution

$$(92) \quad B = \frac{\beta_1 \beta_2 (\delta^{-\beta_2} - \delta^{-\beta_1})}{(1 + \epsilon)(\beta_1 - \beta_2) - \beta_2 (\beta_1 - 1) \delta^{1 - \beta_2} + \beta_1 (\beta_2 - 1) \delta^{1 - \beta_1}},$$

and it is straightforward to check that $B > 0$, using that $\delta \in (0, 1)$ in both the numerator and denominator.

From the above we see that A and B are uniquely determined by (89) and (92), and the corresponding candidate values V_1 and V are given by (88) and (90), respectively. Notice that in order to define V on $(0, +\infty)$ we simply extend V constructed above, in a C^1 way, by taking

$$(93) \quad V(\varphi) = V(B) + (1 + \epsilon)(\varphi - B), \quad \text{for } \varphi \geq B,$$

$$(94) \quad V(\varphi) = 1 + \varphi, \quad \text{for } \varphi \leq A.$$

Moreover, we extend V_1 to $[B, +\infty)$ in a C^1 way and to $(0, A]$ in a continuous way by taking

$$(95) \quad V_1(\varphi) = 1 + \epsilon, \quad \text{for } \varphi \geq B,$$

$$(96) \quad V_1(\varphi) = 1, \quad \text{for } \varphi \leq A.$$

Theorem 6.10. *Let $A < B$ be the unique solution of (89) and (92), and let V_1 and V be constructed as in (88) and (90) with (93)–(96). Denote $V_0(\varphi) := V(\varphi) - \varphi V_1(\varphi)$ and recall Φ^B and τ_A from (74) and (75). Let $\Gamma^* = (0, \Gamma^B)$, let γ_θ^* be the randomised stopping time generated by Γ^* , and set $\tau^* := \tau_A$. Then the randomised stopping pair $(\tau^*, \gamma_\theta^*)$ is a Nash equilibrium for the game with linear payoffs as in (61). Moreover, for all $(x, \varphi) \in \mathbb{R}_+ \times \mathbb{R}_+$ we have*

$$\begin{aligned} \widehat{\mathcal{J}}_{x,\varphi}(\tau^*, \gamma_\theta^*) &= xV(\varphi), \\ \mathcal{J}_{x,\varphi}^0(\tau^*, 0) &= xV_0(\varphi), \\ \mathcal{J}_{x,\varphi}^1(\tau^*, \Gamma^B) &= xV_1(\varphi). \end{aligned}$$

Proof. The proof relies on showing that $u(x, \varphi) := xV(\varphi)$ and $u^i(x, \varphi) := xV_i(\varphi)$, $i = 0, 1$, fulfill all conditions in Theorem 5.1.

Let us start by setting, for $i = 0, 1$,

$$\mathcal{C} := \{(x, \varphi) : u(x, \varphi) > (1+\varphi)x\} \quad \text{and} \quad \mathcal{C}^i := \{(x, \varphi) : u^i(x, \varphi) < (1+\epsilon)x\}$$

and $\mathcal{S} := \mathbb{R}_+^2 \setminus \mathcal{C}$, $\mathcal{S}^i := \mathbb{R}_+^2 \setminus \mathcal{C}^i$. From Lemma 6.6 and the second equation in (80) we obtain $\mathcal{C}^1 = \mathbb{R}_+ \times (0, B)$ and $\mathcal{S}^1 = \mathbb{R}_+ \times [B, +\infty)$. Similarly, from (85) and Lemma 6.7 we get $\mathcal{C} = \mathbb{R}_+ \times (A, +\infty)$ and $\mathcal{S} = \mathbb{R}_+ \times (0, A]$.

Since V and V_1 solve (84)–(85) and (80), respectively, it is immediate to check that V_0 solves (77)–(78). Moreover, Lemma 6.5 guarantees that V_0 also satisfies (76). Then (79) holds as well, implying $\mathcal{C}^0 = \mathbb{R}_+^2$ and $\mathcal{S}^0 = \emptyset$.

Now that \mathcal{C} , \mathcal{C}^i , \mathcal{S} , \mathcal{S}^i are specified, it is easy to check that on \mathcal{S} we have

$$(97) \quad \mathcal{L}^0 u(x, \varphi) = \mathcal{L}^0 [x(1 + \varphi)] = x(\mu_0 + \mu_1 \varphi) \leq 0,$$

where the last inequality follows from Corollary 6.8 (recall that \mathcal{L}^i is the infinitesimal generator of (X, Φ) under the measure \mathbb{P}^i). Therefore, (97) and (84) imply (41). Moreover, (77) and (80) imply (42). Furthermore, the second equations in (77) and in (80) imply (43), and (78) and the third equation of (80) imply (44).

It is clear that $(X_t, \Phi_t^B)_{t \geq 0}$ meets conditions (45)–(47) by construction since all probability measures we consider are equivalent on \mathcal{F}_t , for each $t < \infty$. Moreover, $\mathbb{P}^i(\tau_A < \infty) = 1$ since τ_A is the first hitting time of a constant level for a reflected diffusion, so it follows that $\mathbb{P}(\tau_A < \infty) = (1 - \pi)\mathbb{P}^0(\tau_A < \infty) + \pi\mathbb{P}^1(\tau_A < \infty) = 1$.

It only remains to check the transversality condition (48). First we notice that $\tau_n(u)$ and $\tau_n(u^i)$, $i = 0, 1$, defined as in (40) converge to infinity as $n \rightarrow \infty$ under \mathbb{P}^i and $\widetilde{\mathbb{P}}^i$, $i = 0, 1$, thanks to the regularity of u and u^i .

Using that $\Phi_t^B \in [0, B]$ for all $t \geq 0$, $\mathbb{P}_{x,\varphi}^0$ -a.s. and that V_0 is bounded by one, we obtain

$$0 \leq \lim_{n \rightarrow +\infty} \mathbb{E}_{x,\varphi}^0 \left[X_{\tau_n} V_0(\Phi_{\tau_n}^B) \mathbf{1}_{\{\tau_A > \tau_n\}} \right] \leq \lim_{n \rightarrow +\infty} \mathbb{E}_{x,\varphi}^0 \left[X_{\tau_n} \mathbf{1}_{\{\tau_A > \tau_n\}} \right] = 0$$

since the \mathbb{P}^0 -geometric Brownian $\{X_t, t \geq 0\}$ is uniformly integrable. Thus (48) holds for $i = 0$.

To prove (48) for $i = 1$ we see that it follows from (80) and an application of Ito's formula that $Z_t := e^{\mu_1(t \wedge \tau_A)} V_1(\Phi_{t \wedge \tau_A}^B)$ is a $\tilde{\mathbb{P}}_\varphi^1$ -martingale. By Fatou's lemma,

$$\tilde{\mathbb{E}}_\varphi^1 \left[e^{\mu_1 \tau_A} \right] \leq \lim_{t \rightarrow \infty} \tilde{\mathbb{E}}_\varphi^1 \left[e^{\mu_1(t \wedge \tau_A)} V_1(\Phi_{t \wedge \tau_A}^B) \right] \leq V_1(\varphi) < \infty,$$

which also implies $\tilde{\mathbb{P}}^1(\tau_A < +\infty) = 1$, as needed below.

Finally, we have

$$\begin{aligned} 0 &\leq \lim_{n \rightarrow +\infty} \mathbb{E}_{x,\varphi}^1 \left[X_{\tau_n} V_1(\Phi_{\tau_n}^B) \mathbf{1}_{\{\tau_A > \tau_n\}} \right] \\ &\leq (1 + \epsilon) \lim_{n \rightarrow +\infty} \lim_{t \rightarrow +\infty} \mathbb{E}_{x,\varphi}^1 \left[X_{\tau_n} \mathbf{1}_{\{\tau_A \wedge t > \tau_n\}} \right] \\ &= x(1 + \epsilon) \lim_{n \rightarrow +\infty} \lim_{t \rightarrow +\infty} \tilde{\mathbb{E}}_\varphi^1 \left[e^{\mu_1 \tau_n} \mathbf{1}_{\{\tau_A \wedge t > \tau_n\}} \right] \\ &\leq x(1 + \epsilon) \lim_{n \rightarrow +\infty} \tilde{\mathbb{E}}_\varphi^1 \left[e^{\mu_1 \tau_n} \mathbf{1}_{\{\tau_A > \tau_n\}} \right] = 0, \end{aligned}$$

where we used that $\Phi_t^B \in [0, B]$ for all $t \geq 0$, $\tilde{\mathbb{P}}_\varphi^1$ -a.s. and that V_1 is bounded by $1 + \epsilon$ on $(0, B]$, and the last equality is due to dominated convergence. \square

7. NUMERICAL RESULTS

To illustrate the Nash equilibrium found in Section 6 we consider a base-case set of parameters with $\mu_0 = -1$, $\mu_1 = 1$, $\sigma = 0.5$ and $\epsilon = 0.1$. For these parameters, the boundaries defined by (89) and (92) are found to be $A = 0.329$ and $B = 0.868$. For ease of interpretation, however, in the following we will return to the posterior probability process Π^* , where we denote the lower boundary as $a := A/(1 + A)$ and the upper (reflecting) boundary as $b := B/(1 + B)$. For our base case this corresponds to $a = 0.248$ and $b = 0.465$. Furthermore, we let $x = 1$ and note that $u(1, \varphi) = V(\varphi)$ and $u^i(1, \varphi) = V_i(\varphi)$; accordingly, we refer to V and V_i as value functions.

Firstly, Figure 1 demonstrates a typical sample path of the Π^* -process and its associated $\Gamma^{*,1}$ -process. Note that in this particular example, Player 1 stops at $\tau^* \approx 0.06$, and that $\Gamma_{\tau^*}^{*,1} \approx 0.13$. Consequently, Player 1 stops before Player 2 if either $\mu = \mu_0$ or if $\mu = \mu_1$ and the uniformly distributed randomisation device \mathcal{U} takes a value larger than 0.13.

Next, Figure 2 shows the value functions for Player 1 and Player 2 corresponding to our base case. Note that V and V_1 satisfy smooth fit conditions at a and b respectively, and V_0 satisfies the reflection condition at b . We also

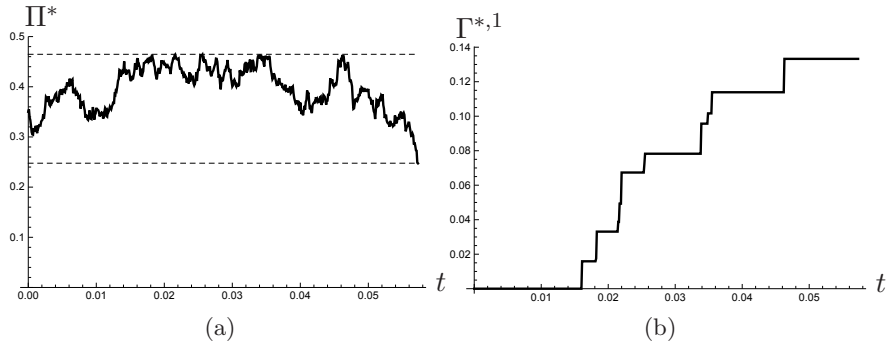


FIGURE 1. A typical sample path of the Π^* -process (left) and its associated $\Gamma^{*,1}$ -process (right) for our base-case parameters. Note that the dashed lines on the left represent the optimal boundaries $a = 0.248$ and $b = 0.465$ and that we have chosen $\pi = 0.35$.

observe the properties of V_0 described in (78) and (79), along with the properties of V_1 and V described in Lemmas 5.5 and 5.6, respectively. When the true drift is μ_1 , the informed player expects to pay out considerably more than Player 1 has reason to believe, and when the true drift is μ_0 , the informed player expects to pay out less. When $\mu = \mu_1$, the gap between $1 + \epsilon$ and the value of the game to Player 2 can be seen to represent the reduction in Player 2's expected cost due to Player 1 being uninformed. Similarly, when $\mu = \mu_0$, the gap between 1 and the value of the game to Player 2 represents the reduction in Player 2's expected cost due to Player 1 being uninformed.

Figure 3 shows comparative static results for the changing of all four parameters ($\mu_0, \mu_1, \sigma, \epsilon$) with the base case used above. We first note that the signal-to-noise ratio, $\omega = (\mu_1 - \mu_0)/\sigma$, plays a crucial role in understanding these results since a higher ω will result in faster learning by the uninformed player. In this sense, changes in the parameters μ_0, μ_1 and σ will affect the signal-to-noise ratio and hence the speed of learning, which will ultimately have an impact on the equilibrium outcome. Furthermore, changing μ_0, μ_1 and σ will not only have an effect on the speed of learning (through the signal-to-noise ratio) but also on the expected payoff of the game, potentially resulting in non-monotone dependencies due to these competing effects. Finally, we note that ϵ only influences the problem through the payoff structure of the game and has no impact on the rate at which Player 1 is able to learn about the drift. With this understanding in mind, we now proceed to describe the comparative statics results observed in Figure 3.

We first consider the effect of changing μ_1 on the equilibrium outcome. As μ_1 increases (all else being equal), the good scenario for Player 1 gets better, both due to a larger drift, and also due to an increased signal-to-noise ratio which speeds up the learning process. This indicates that the threshold a

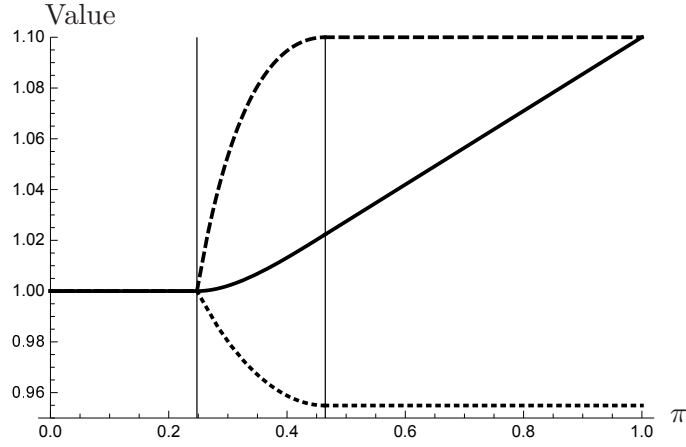


FIGURE 2. The value of the game to Player 1 (solid line; $(1 - \pi)V_0 + \pi V_1$) along with the value of the game to Player 2 when $\mu = \mu_0$ (dotted line; V_0) and $\mu = \mu_1$ (dashed line; V_1). The base-case parameters are $\mu_0 = -1$, $\mu_1 = 1$, $\sigma = 0.5$ and $\epsilon = 0.1$; therefore $a = 0.248$ and $b = 0.465$ (represented by the two vertical lines).

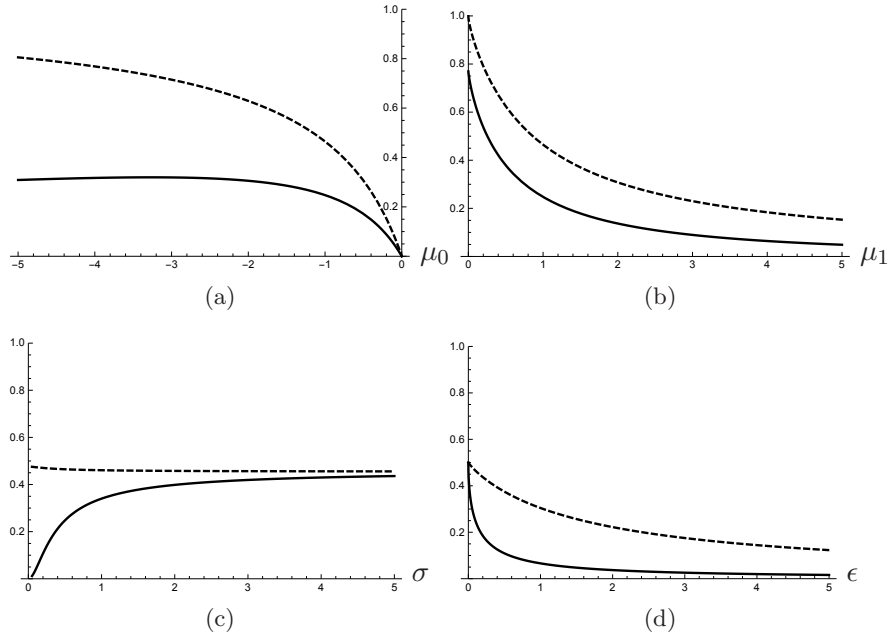


FIGURE 3. The optimal boundaries ($a =$ solid line and $b =$ dashed line) for the base-case parameters ($\mu_0 = -1$, $\mu_1 = 1$, $\sigma = 0.5$ and $\epsilon = 0.1$) as we vary μ_0 , μ_1 , σ and ϵ , respectively.

should be decreasing in the drift μ_1 , which is also confirmed numerically, see Figure 3(b). Likewise, if $\mu = \mu_1$ and μ_1 is large, then continuing is costly for Player 2, and at the same time, the advantage of having additional information about the drift is smaller (because of the increased signal-to-noise ratio). Consequently, the threshold b should be decreasing in μ_1 , which is also confirmed numerically.

When considering a change in μ_0 , there are two competing effects on both players. On one hand, a decreasing μ_0 is bad for Player 1 (the sup-player), and hence has an increasing effect on the threshold a . On the other hand, a decreasing μ_0 increases the signal-to-noise ratio, which speeds up the learning process, and hence decreases a . Figure 3(a) confirms the suspicion that there is no monotone dependence of a on μ_0 . For the same reasons as above, the effect of a change in μ_0 on the upper threshold b is ambiguous. However, this potential ambiguity is not visible in Figure 3(a) for our base-case parameters.

From Figure 3(c) we see that as σ increases the optimal threshold is increasing for Player 1 and decreasing for Player 2. The intuition behind this is that, as σ increases, the signal-to-noise ratio decreases, resulting in slower learning and hence a smaller value function for Player 1 and hence an increased a . For Player 2, however, while an increased σ means that they are better able to hide their information from Player 1 (an incentive to increase b), the reduced variance of the Π -process also means that first hitting time of a given threshold is larger for an increased σ . Since $\mu_1 > 0$, a longer expected time to stop would ultimately result in an increased expected cost for Player 2 (an incentive to decrease b). By the numerics, the net result for our base-case parameters is that Player 2 reduces their threshold b as σ increases.

Lastly, we consider the effect of a change in ϵ . Since the value of ϵ does not impact the ability of Player 1 to learn about the drift, its effect on the equilibrium can only be through the payoff structure of the game. Therefore, all value functions clearly increase in ϵ ; for Player 1 this means that the continuation region is increasing in ϵ , so that the threshold a is decreasing. However, no easy monotonicity for b can be deduced as there is no obvious effect on the continuation region for Player 2 (since also the obstacle depends on ϵ). From Figure 3(d) we observe the anticipated monotonic dependence of a on ϵ and, for our base-case parameters at least, b is also seen to be monotonic decreasing in ϵ .

Finally, to calculate the *value of information* for the game, we end the article with an informal discussion on the case with symmetric and incomplete information. Assume that both players have the same initial prior distribution for μ , that is they agree on π as the initial probability that the drift is μ_1 , and $1 - \pi$ as the probability that the drift is μ_0 . Then randomisation is not needed for either player and a Nash equilibrium in stopping times

(τ_1, τ_2) can be obtained. In fact, the game with linear payoffs reduces to

$$U(x, \varphi) = \frac{x}{1 + \varphi} \sup_{\tau_1} \inf_{\tau_2} \tilde{\mathbb{E}}_{\varphi}^0 \left[e^{\mu_0 \tau_1} (1 + \Phi_{\tau_1}) 1_{\{\tau_1 < \tau_2\}} \right. \\ \left. + (1 + \epsilon) e^{\mu_0 \tau_2} (1 + \Phi_{\tau_2}) 1_{\{\tau_2 \leq \tau_1\}} \right]$$

where

$$d\Phi_t = \sigma \omega \Phi_t dt + \omega \Phi_t d\tilde{W}_t^0$$

under $\tilde{\mathbb{P}}^0$. It is then straightforward to check that one can find $A, B \in (0, \infty)$ with $A < B$ and a function \hat{V} with $1 + \varphi \leq \hat{V} \leq (1 + \epsilon)(1 + \varphi)$ such that

$$\begin{cases} \frac{\omega^2 \varphi^2}{2} \hat{V}''(\varphi) + \sigma \omega \varphi \hat{V}'(\varphi) + \mu_0 \hat{V}(\varphi) = 0, & \text{for } \varphi \in (A, B) \\ \hat{V}(\varphi) = 1 + A, & \text{for } \varphi \in (0, A] \\ \hat{V}'(A+) = 1, & \\ \hat{V}(\varphi) = (1 + \epsilon)(1 + B), & \text{for } \varphi \in [B, \infty) \\ \hat{V}'(B-) = 1 + \epsilon. & \end{cases}$$

Using standard verification arguments, $(\tau_1^*, \tau_2^*) := (\tau_A, \tau_B)$ is a Nash equilibrium of stopping times, and the corresponding value function is given by $U(x, \varphi) = x \hat{V}(\varphi) / (1 + \varphi)$.

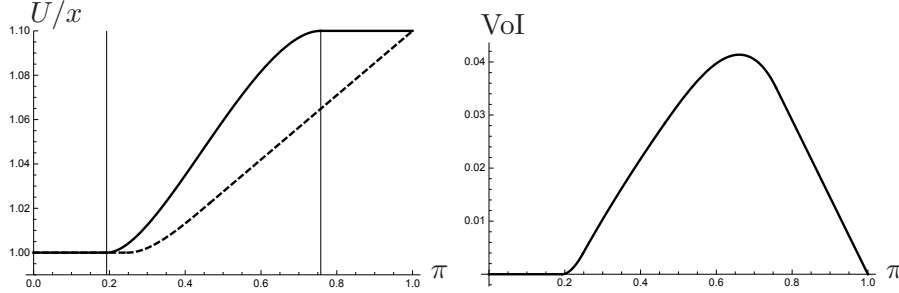


FIGURE 4. On the left: The common value function for both players in the symmetric incomplete information case (solid line) in comparison to the value function in the asymmetric case (dashed line). The two vertical lines correspond to the values $a := A/(1 + A)$ and $b := B/(1 + B)$ (for the symmetric case). On the right: The difference between these values, which represents the value of information in our game. Base-case parameters: $\mu_0 = -1$, $\mu_1 = 1$, $\sigma = 0.5$ and $\epsilon = 0.1$, which yields $a = 0.193$ and $b = 0.758$ (for the symmetric case).

Figure 4 plots the value function $U(1, \pi) = U(x, \pi)/x$ for our base-case parameters, along with the value function of the uninformed player for the asymmetric case for comparison. The difference between the asymmetric value function and the symmetric one is also plotted and can be interpreted as the value of information in this setting.

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