Competitive Advertising and Pricing

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Abstract

We consider an oligopoly model in which each firm chooses not only its price but also its advertising strategy regarding how much product information to provide. Unlike most previous studies, we impose no structural restriction on feasible advertising content, that is, each firm is endowed with full flexibility in its advertising choice. We provide a general and comprehensive characterization for the unique symmetric pure-price equilibrium and investigate under what conditions the equilibrium exists. Our analysis illuminates how competition shapes firms' advertising incentives and how a firm's advertising decision interacts with its pricing decision.

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1 Introduction

One of the fundamental questions in the economics of advertising is how much, and what, product information a seller should provide for consumers.[1] More product information enables the seller to price-discriminate consumers more effectively or be more aggressive in pricing. However, it comes at the cost of losing some consumers who do not find revealed product characteristics appealing. This essential trade-off has been extensively studied in the monopoly context.[2] A common insight

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[1] In this paper, we restrict attention to this unbiased information transmission role of advertising. However, the literature has identified several other roles of advertising. See Bagwell (2007) and Renault (2015) for thorough overviews of the literature.

[2] Lewis and Sappington (1994) is a seminal contribution in this literature. Subsequent important contributions include Che (1996) (return policies as a way to facilitate consumer experimentation), Ottaviani and Prat (2001) (the value of revealing a signal affiliated to the buyer's private signal), Anderson and Renault (2006) (optimal advertising for search goods), Johnson and Myatt (2006) (U-shaped profit function based on the rotation order), and Roesler and Szentes (2017) (buyer-optimal signal for experience goods).

in that literature is that in the absence of varying advertising costs, a monopolist wishes to provide either no information (on values above the marginal cost) or full information: the former enables her to serve all consumers without charging too low a price, while the latter allows her to extract most from high-value consumers.

We study firms' incentives to provide product information in an oligopoly environment. Specifically, we consider the canonical random utility discrete-choice framework of Perloff and Salop (1985): there are $n(\geq 2)$ firms that compete for consumers in price, and each consumer's value for each product is independently and identically drawn from a common continuous distribution F. We extend the model to allow for strategic advertising by the firms: in our model, each firm chooses how much information to provide about its own product (equivalently, how precise information consumers can have about their values for the product).[3] In other words, we consider a model in which the firms compete not only in price but also through (informative) advertising.

Unlike most classical studies on advertising, but as is common in the recent literature on information design, we endow the firms with full flexibility in advertising. In other words, we impose no structural restriction on the set of feasible advertising strategies and allow each firm to reveal, or hide, no matter what information it wants.[4] As usual, this can be interpreted as each firm having access to numerous advertising channels and/or fine information about various attributes of its product and, therefore, being able to adjust its advertising content without any restriction. As is well-known in the literature on information design (see, e.g., Kolotolin, 2018; Roesler and Szentes, 2017), with risk-neutral consumers, this full flexibility amounts to each firm being able to choose any mean-preserving contraction G_i of F.[5] In other words, fully flexible advertising can be implemented by assuming that the set of feasible advertising strategies coincides with the set of all mean-preserving contractions of F. Given G_i, it is as if each consumer's value for product i is independently and identically drawn according to G_i. In this regard, our model can be interpreted as the one in which the distribution of consumer values, which is exogenously given in Perloff and Salop (1985), is endogenously determined by the firms' advertising choices.

We begin by analyzing the advertising-only game, in which each firm chooses only its advertising strategy given symmetric prices. We show that the advertising-only game always has a unique symmetric equilibrium and, more importantly, the equilibrium advertising strategy, denoted by G^*,

[3]We do not allow for "comparative advertising," in which a firm not only controls its own product information but also can provide information about the rivals' products. See Anderson and Renault (2009) for an important contribution on comparative advertising.

[4]For example, in Lewis and Sappington (1994), the firm can only choose the probability that each consumer fully learns her value. In Johnson and Myatt (2006) and Ivanov (2013), feasible advertising strategies are assumed to be rotation-ordered. Note that in both cases, the order is complete, while there is no such restriction in our model.

[5]In general, given a signal realization s_i, a consumer's updated belief is represented by a conditional distribution $F(\cdot|s_i)$. If the consumer is risk-neutral, however, her purchase decision depends only on her conditional expectation $E[v_i|s_i]$. G_i represents the distribution over those conditional expectations.

has a particular structure: $(G^*)^{n-1}$ is convex over its support and alternates between the full information region (i.e., $G^*(v) = F(v)$) and the $(n-1)$-linear mean-preserving contraction region (in which $(G^*)^{n-1}$ is linear and G^* is a mean-preserving contraction of F over the interval). Our general result can be used to address a variety of specific questions for competitive advertising. For example, each firm reveals all product information if and only if F^{n-1} is convex. If F^{n-1} is single-peaked, then each firm fully reveals low values but obfuscates all high values exactly up to the point where $(G^*)^{n-1}$ is linear.

Our general analysis of the advertising-only game allows us address an important economic question of how the intensity of competition (i.e., the number of firms) affects firms' advertising incentives. We show that if there are sufficiently many firms, then each firm necessarily reveals all product information. Economically, this is because when each firm's demand is sufficiently elastic, due to the existence of many competitors, it is more profitable for a firm to make a few loyal consumers (who find the firm's product particularly attractive) by revealing all product information than to give less dispersed expected values to all consumers by hiding some information. Technically, this is simply because F^{n-1} becomes more convex as n increases, regardless of the shape of F. Note that our general characterization allows us not only to obtain the full information result in the limit, but also to understand how the convergence occurs.

We then analyze the effects of competitive advertising on market prices. Specifically, we derive the unique symmetric price p^* that arises under equilibrium competitive advertising G^* and compare it to the equilibrium price p^F in the full information benchmark (i.e., the equilibrium price in the Perloff-Salop model with F).[6] We show that, despite the fact that G^* is a mean-preserving contraction of F and, therefore, competitive advertising reduces product differentiation relative to fully informative advertising, the relationship between p^* and p^F is ambiguous in general. We explain how different distributional effects of mean-preserving contraction yields this ambiguity.

Finally, we investigate how a firm's pricing decision interacts with its advertising decision and discuss its implications for equilibrium existence. In our model, each firm can deviate both in price (i.e., $p_i \neq p^*$) and in advertising (i.e., $G_i \neq G^*$). In order to determine p^* and G^*, it suffices to consider deviations on each dimension. However, a firm's optimal advertising strategy does depend on its choice of price. In other words, if a firm deviates to $p_i \neq p^*$, then G^* may no longer be its optimal advertising strategy. We provide a general characterization for this interaction between pricing and advertising. In addition, exploiting the result, we illustrate how compound deviations (such that both $p_i \neq p^*$ and $G_i \neq G^*$) affect the existence of equilibrium. In general, a firm's deviation is more profitable when it can adjust both p_i and G_i than when it can change only p_i. Therefore, a pure-price equilibrium is less likely to exist in our model than in the Perloff-Salop

[6] Comparison to the other benchmark where consumers receive no product information (i.e., G_i is degenerate) is trivial and, therefore, omitted.

model.[7] Nevertheless, we show that a common regularity assumption, namely that the density of F is log-concave, is sufficient to guarantee the existence of symmetric pure-price equilibrium.

Related Literature

As explained earlier, the interaction between advertising and pricing has been extensively studied in the monopoly context, but not in the competitive oligopoly setting. To our knowledge, the only exception is Ivanov (2013). He also adopts the Perloff-Salop framework but considers the case where all feasible advertising strategies are rotation-ordered in the sense of Johnson and Myatt (2006). Partly due to the modeling choice, he does not provide a general characterization for equilibrium advertising strategy. Instead, he focuses on showing that a full information equilibrium (in which all the firms choose the highest advertising strategy in rotation order) exists if there are sufficiently many sellers. This paper supplements his work in multiple ways. In particular, we provide more comprehensive equilibrium analysis (covering partial information equilibria), demonstrate that his main economic conclusion holds even if each firm faces no restriction on advertising content, and also study the interaction between a firm's advertising and pricing decisions.

The advertising-only game (in which each firm can choose only its advertising strategy) has been studied by a few recent papers: Boleslavsky and Cotton (2015, 2018), and Au and Kawai (2018). All these papers consider discrete distributions, while we consider the case where F is continuous. Whereas we clearly learn a lot from the insights of the previous work, our analysis has three important advantages. First, continuous F is directly comparable to G^* and, therefore, their relationship is more clear and explicit in our model. For example, if F is discrete then G^* can never coincide with F, while if F is continuous then $G^* = F$ if and only if F^{n-1} is convex (see Corollary 1). Second, it makes the equilibrium structure (i.e., the essential properties of G^*) more transparent. In particular, with continuous F, it follows fairly easily that if $(G^*)^{n-1}$ is not linear around v, then $G^*(v) = F(v)$. With discrete F, this local coincidence is not possible, which greatly complicates the necessary analysis (see Section 4 in Au and Kawai, 2018). Finally, it allows us to link our work to the large discrete-choice literature, most of which consider continuous distributions.

Boleslavsky et al. (2016) also consider a model in which the interaction between advertising and pricing plays a crucial role. They consider an entry game in which consumers' values for the incumbent's product are known, while their idiosyncratic values for the entrant's product are unknown. The two firms engage in Bertrand competition, but the entrant can reveal its product information through "demonstrations" before or after pricing. The main difference is that in their

[7] The existence of pure-price equilibria is a non-trivial problem even in the Perloff-Salop model. The main difficulty lies in the fact that each firm's best response depends on the shape of the entire distribution and, therefore, may not be sufficiently well-behaved unless some strong regularity is introduced into F. See Caplin and Nalebuff (1991) and Quint (2014) for some important contributions.

model, the entrant's incentive to provide product information depends only on the two firms' prices, while in our model, each firm's equilibrium advertising content is fully determined by the other firms' advertising strategies, as shown in Section 3.

We significantly benefit from recent technical developments in the literature on information design.[8] In particular, we make extensive use of a verification technique by Dworczak and Martini (2018) (DM, hereafter).[9] They consider a general programming problem in which the sender's (indirect) payoff depends only on the expected value (state) she induces, represented by a function $u(v)$, and show that in order to evaluate the optimality of a signal (equivalently, a mean-preserving contraction G_i of F), it suffices to check whether there exists a convex function $\phi(v)$ which touches $u(v)$ only on the support of G_i (see Theorem 2 in Section 3). As illustrated by DM with a series of examples and shown by our subsequent analysis, this result permits a simple geometric analysis for a variety of information design problems. Our analysis differs from Dworczak and Martini (2018) mainly in two ways. First, we look for Nash equilibrium in a strategic environment. Therefore, each firm's programming problem, which is exogenously given in DM, is endogenously determined in our model. Second, in our model, an individual firm chooses not only its advertising strategy but also its price. The latter also affects the firm's payoff function $u(v)$ and, therefore, DM's result does not apply directly. We address this problem by finding an optimal signal for each price and verifying whether any of those (compound) deviations is profitable. DM's result applies to the former, but not to the latter.

The remainder of this paper is organized as follows. We introduce the formal model in Section 2. We study the advertising-only game in Section 3. In Section 4, we derive the unique candidate equilibrium price and compare it to that of the Perloff-Salop model (i.e., the full information benchmark). Section 5 provides a sufficient condition for equilibrium existence. Section 6 discusses the effect of buyers' outside option and asymmetric environments. All the missing proofs in the main text are presented in the Appendix, unless noted otherwise.

[8]As is well-known, the elegant con-cavification method in Aumann and Maschler (1995) and Kamenica and Gentzkow (2011), which is powerful for binary-state problems, is less useful when there are many types (in particular, when the state space is continuous). The fundamental problem is that it is not possible to visualize the sender's (information designer's) value function with respect to the posterior belief when the type space is large. Recently, several researchers have developed methods that can be used to solve for an optimal signal in such a problem. As anticipated by Kamenica and Gentzkow (2011), the analysis becomes significantly more tractable if the receiver's optimal action depends only on the expected state (posterior mean), and much progress has been made for this class of problems. As explained in footnote 5, each firm's optimal advertising problem in our model belongs to this class.

[9]Other relevant contributions include Gentzkow and Kamenica (2016), Ivanov (2015), and Kolotilin et al. (2017).

2 The Model

There are $n(\geq 2)$ ex ante homogeneous firms and a unit mass of risk-neutral consumers. Each firm sells a product, which it can produce with marginal cost normalized to zero. The firms' products are horizontally differentiated: each consumer's *true valuation* for each product is drawn according to the distribution function F independently and identically across the products and consumers. We assume that F has finite mean μ_F and continuously differentiable density f with $Supp(F) = [\underline{v}, \overline{v}]$, where both $\underline{v} = -\infty$ and $\overline{v} = \infty$ are allowed. We also make a few mild technical assumptions on F: Both $f(v)$ and $f'(v)$ are bounded, and f has a finite number of peaks and a log-concave tail if $\overline{v} = \infty$.

Each firm simultaneously chooses its price, p_i, and the advertising strategy in order to maximize its profit. Each firm is unrestricted to its choice of advertising content which conveys information to consumers about their true valuations (that is, it is allowed to disclose, or hide, any information it wants). Following the information design approach, we assume that each firm chooses any set of realizations S_i and any joint distribution function H_i over $[\underline{v}, \overline{v}] \times S_i$. Consumers observe a draw from each firm's signal structure before making their purchase decisions.

Exploiting consumers' risk neutrality, we adopt a more tractable formulation for advertising strategy. Each consumer's purchase decision depends only on her *interim valuation*, $E[v_i|S_i = s_i]$, the mean of her posterior distribution. The interim valuation is ex ante a random variable, and only the distribution of the interim valuation is payoff-relevant for a firm. Therefore, we allow each firm to choose the distribution of the consumer's interim valuation directly. Of course, possible distributions for the interim valuations must be consistent with the distribution of the consumers' true value. In particular, it is well-known that $E[v_i|S_i] \sim G_i$ for some signal structure S_i if and only if G_i is a mean-preserving contraction (MPC hereafter) of F_i. That is, the distribution of the consumer's interim valuation must be a MPC of the distribution of the consumers' actual value.

The timing of the game is as follows. The firms simultaneously choose their price p_i and advertising strategy G_i. Then, for each consumer and $i = 1, ..., n$, the interim valuation v_i is drawn according to G_i (independently across the products and consumers), and each consumer decides which product to purchase. We assume that each consumer must purchase one of the products:[10] A consumer purchases product i if and only if $v_i - p_i > v_j - p_j$ for all $j \neq i$.[11]

We analyze symmetric pure-price equilibria of the game. Since consumers' optimal purchase decisions are straightforward, we focus on a Nash equilibrium played by the firms. Let $D(p_i, G_i, p, G)$ denote firm i's demand (that is, the measure of consumers who purchase product i)

[10] This "full market coverage" assumption is common in the literature. See, for example, Perloff and Salop (1985), Caplin and Nalebuff (1991), and Zhou (2017). We explain how to relax this assumption in Section 6.1.

[11] As shown later, in any equilibrium in this paper, the measure of consumers who are indifferent among multiple products is negligible. Therefore, for notational and expositional simplicity, we ignore them throughout the paper.

when firm i's strategy is (p_i, G_i), while all other firms adopt strategy (p, G). Given consumers' optimal choice rule,[12]

$$D(p_i, G_i, p, G) = Pr\{v_i - p_i > v_j - p, \forall j \neq i\} = \int G(v_i - p_i + p^*)^{n-1} dG_i(v_i). \qquad (1)$$

A tuple (p, G) is a symmetric pure-price equilibrium (henceforth, just an equilibrium) if (p, G) is a solution to the following programming problem:

$$\max_{(p_i, G_i)} \pi(p_i, G_i, p, G) \equiv p_i D(p_i, G_i, p, G), \qquad s.t.\ G_i \text{ is a MPC of } F.$$

Structure of analysis. Our analysis proceeds in three steps. First, we analyze a benchmark game in which the firms compete only in their advertising strategies (Section 3). Using techniques recently developed in the information design literature, we show that this game has a unique equilibrium and provide an explicit characterization. In any equilibrium of the full game, the advertising strategy must be identical to the one in this benchmark. Furthermore, this benchmark relates to a literature on competitive information design and is of independent interest.

Second, holding fixed the candidate equilibrium advertising strategy, we derive a candidate for the equilibrium price by following the analysis of the standard discrete choice model *a la* Perloff and Salop (Section 4).

The price-distribution pair derived from the previous two steps is an equilibrium unless the firm has a profitable *compound* deviation in which the firm deviates in both its price and advertising strategy. As a last step, we consider the possibility of such compound deviations, and show that the log-concavity of the density of F provides a simple sufficient condition for the equilibrium existence (Section 5).

3 Competitive Advertising

In this section, we assume that all firms have chosen the same price $p > 0$ and analyze their strategic interaction only in terms of advertising strategy G_i. Since this can be interpreted as an independent game regarding the firms' competitive information revelation, we refer to it as the *advertising-only game*.

Suppose that all the firms other than firm i advertise according to G. Then a consumer with expected value v_i purchases product i with probability $G(v_i)^{n-1}$. Therefore, G is a symmetric

[12] Note that our derivation of $D(p_i, G_i, p, G)$ does not account for the possibility of atoms in G. This significantly simplifies the notation but incurs no loss of generality because in equilibrium, G has no atom.

equilibrium of the advertising-only game if G solves

$$\max_{G_i} \int G(v_i)^{n-1} dG_i(v_i), \qquad s.t.\ G_i \text{ is a MPC of } F. \tag{2}$$

As shown shortly, the advertising-only game has a unique symmetric equilibrium, suggesting a unique candidate for equilibrium advertising strategy of the main game.

3.1 Main Characterization

We begin with definitions that simplify the statement of our main result.

Definition 1 *A distribution function G_1 is an $(n-1)$-*linear MPC *of G_2* on interval *$[v_1, v_2]$ if (a) G_1^{n-1} is linear over $[v_1, v_2] \cap Supp(G_1)$; and (b) G_1 is a MPC of G_2 over the interval $[v_1, v_2]$.*

Definition 2 *G is an* alternating $(n-1)$-linear MPC *of F if there exist a finite partition $\mathcal{P}^* \equiv (w_0, v_0, w_1, v_1, ...w_m, v_m)$ with $\underline{v} = w_0 \leq v_0 < w_1 \leq v_1...w_m \leq v_m = \overline{v}$ such that (a) $G = F$ over intervals $[w_k, v_k]$ $(k = 0, \ldots, m)$; and (b) G is a $(n-1)$-linear MPC of F over intervals $[v_k, w_{k+1}]$ $(k = 0, \ldots, m-1)$.*

An alternating $(n-1)$-linear MPC has a specific structure: It alternates intervals of full disclosure (where $G = F$) with intervals of partial disclosure, in which the maximum interim valuation for the $(n-1)$ other firms' products is uniformly distributed.

The following theorem—the main result of this section—shows that there exists a unique equilibrium in the advertising-only game and, more importantly, identifies the structure of the equilibrium advertising strategy.

Theorem 1 *Let G^* be a distribution such that (i) $(G^*)^{n-1}$ is convex over its support and (ii) G^* is an alternating $(n-1)$-linear MPC of F. Then for any F, there exists a unique equilibrium of the advertising-only game in which the firms advertise according to G^*.*

For a rough intuition behind Theorem 1, observe that each firm's strategy is effectively a choice between spreading and contracting its distribution. Then the demand function in (2) implies that the optimal choice depends on whether G^{n-1} is convex or concave. The global convexity condition of $(G^*)^{n-1}$ is due to the fact that if $(G^*)^{n-1}$ is (locally) strictly concave, each firm' best response is to put an atom, which cannot constitute a symmetric equilibrium. Then, full information revelation is not an equilibrium whenever F^{n-1} has strictly concave intervals. The second condition—alternating $(n-1)$-linear MPC condition—requires that the concave parts of F^{n-1}

must be replaced with linear functions:[13] If the replaced part is strictly convex, then each firm is incentivized to choose full revelation, violating the equilibrium condition. Theorem 1 states that there always exists a unique distribution that satisfies the above two conditions.

In the rest of this subsection, we analyze the advertising-only game with four canonical classes of distributions of F^{n-1}. Specifically, we apply Theorem 1 to the cases in which the density of F^{n-1} is (1) increasing; (2) decreasing; (3) single-peaked; and (4) U-shaped. The examples not only demonstrate the analytical power of Theorem 1, but they also help us to further clarify the equilibrium forces in the competitive advertising. The proofs of the following four corollaries are straightforward from Theorem 1 and, therefore, omitted.

First, consider the case in which F^{n-1} is convex (that is, F^{n-1} has an increasing density).

Corollary 1 (Convex F^{n-1}) *If F^{n-1} is convex, then there exists a unique equilibrium of the advertising–only game in which all sellers provide full information (that is, $G^* = F$).*

To understand this result, let us analyze firm i's best response when all other firms provide full information about their products (i.e., $G_j = F$ for all $j \neq i$). In this case, if firm i advertises according to G_i, the demand for product i is

$$\int F(v_i)^{n-1} dG_i(v_i).$$

Now observe that if F^{n-1} is convex then, by Jensen's inequality, firm i's demand is higher when G_i is more dispersed. It is then straightforward that firm i's demand is maximized when the MPC constraint is binding, that is, $G_i = F$.

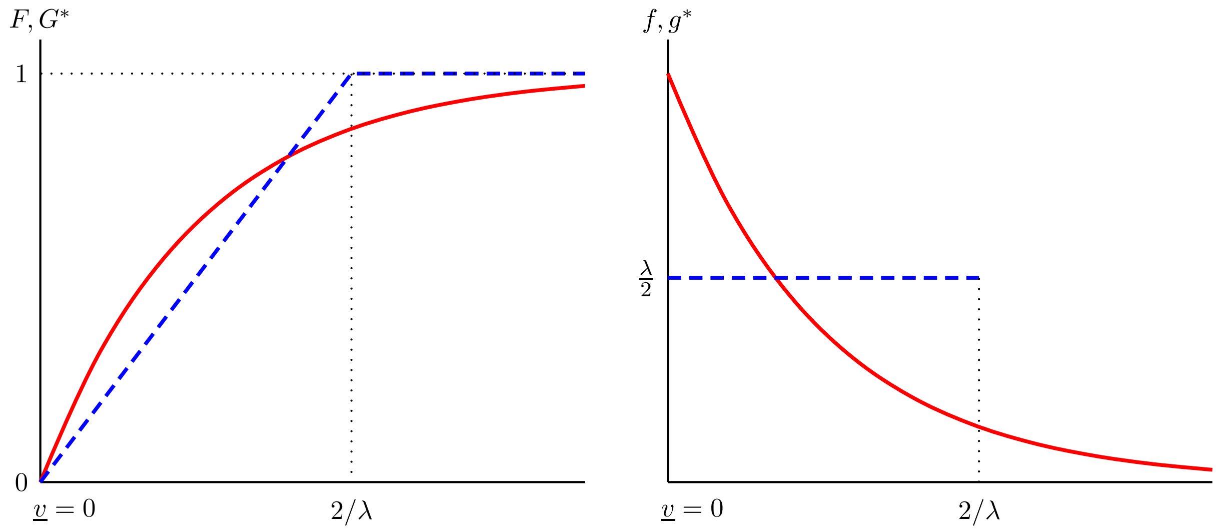
Clearly, full information revelation is no longer an equilibrium under concave distributions (decreasing densities). The next corollary characterizes the equilibrium in this case.

Corollary 2 (Concave F) *If $n = 2$ and F is concave with (normalized) $\underline{v} = 0$,[14] then it is the unique equilibrium in the advertising-only game that both firms advertise according to $G^* = U[0, 2\mu_F]$.*

Suppose that F is concave, and consider firm i's optimal response when firm j provides full information about its product. Opposite to the previous convex case, firm i's profit is higher when G_i is *less* dispersed. In fact, it is optimal for firm i to put all mass on one point μ_F. Given firm i's response, however, firm j has no incentive to provide full information: one profitable deviation is to induce a binary distribution, one point arbitrarily close to $\underline{v}$ and the other slightly above μ_F. This

[13] In general, the linear parts of G^* often include some convex parts of F^{n-1} in order to satisfy the MPC constraint. A complete analysis for the construction of G^* is given in subsection 3.2.3.

[14] The result and the argument apply unchanged even if $n > 2$, provided that F^{n-1} is concave. However, in order for F^{n-1} to be concave, f must be unbounded at $\underline{v}$: $(F(\underline{v})^{n-1})' = (n-1)F(\underline{v})^{n-2}f(\underline{v}) = 0$ for any finite $f(\underline{v})$.



(a) distribution: $F(v)$ (solid) and $G^*(v)$ (dashed) (b) density: $f(v)$ (solid) and $g^*(v)$ (dashed)

Figure 1: An example of Corollary 2 (concave F). This figure depicts for the case where $n = 2$ and $F(v) = 1 - e^{-\lambda v}$ is an exponential distribution with $\mu_F = 1/\lambda$.

in turn induces firm i to change its response, leading to a matching-pennies type of competition. In this case, the equilibrium typically consists of players being indifferent over a set of actions. In our model, this means each firm is indifferent between spreading and contracting its own distribution. Then, Jensen's inequality implies that the equilibrium distribution must be neither strictly convex nor strictly concave, implying that the linear distribution (uniform density) is the only feasible equilibrium.

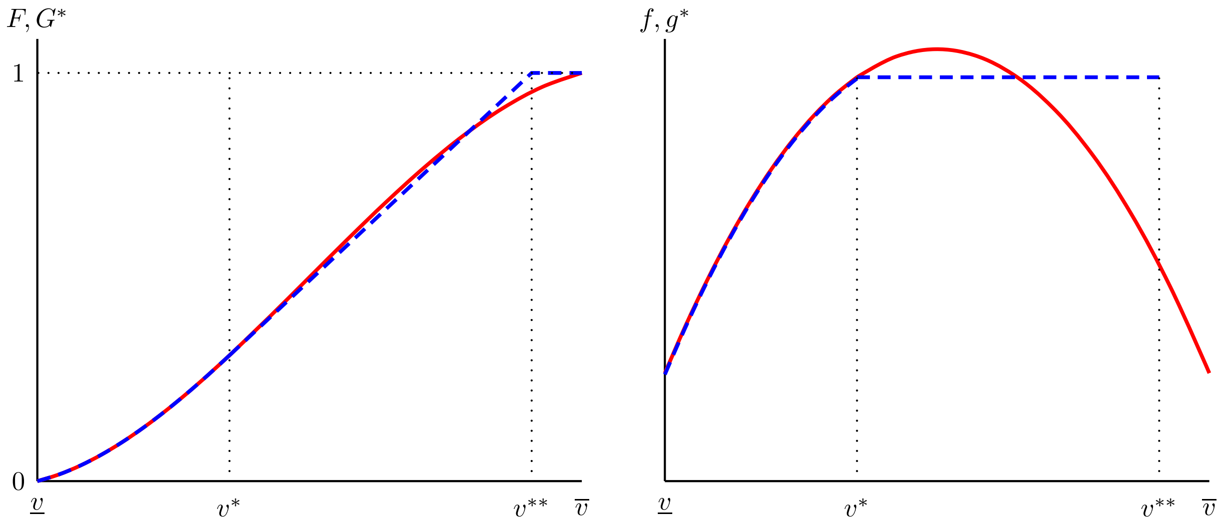
Corollaries 1 and 2 show that if the density function of F^{n-1} is monotone, then G^* takes one of the two structures: $G^* = F$ (under increasing density) or $(n-1)$-linear MPC (under decreasing density). The next two Corollaries demonstrate that if F^{n-1} has a non-monotone density, G^* combines the above two structures in a natural way.

Corollary 3 (Single-peaked density) *Suppose that the density function of F^{n-1} is single-peaked. Then it is the unique equilibrium in the advertising-only game that for some $v^* \in [\underline{v}, \overline{v}]$, each firm fully reveals information for $v < v^*$ and advertises according to the $(n-1)$-linear MPC for $v \geq v^*$. Furthermore, if $v^* > \underline{v}$, G^* is smooth at $v = v^*$.*

Corollary 4 (U-shaped density) *Suppose that $n = 2$ and f is U-shaped.*[15] *Then it is the unique equilibrium in the advertising-only game that for some $v^* \in (\underline{v}, \overline{v}]$, each firm advertises according to the 1-linear MPC for $v < v^*$ and fully reveals information for $v \geq v^*$.*

If F^{n-1} is initially convex and then concave, then G^* initially coincides with F and then takes

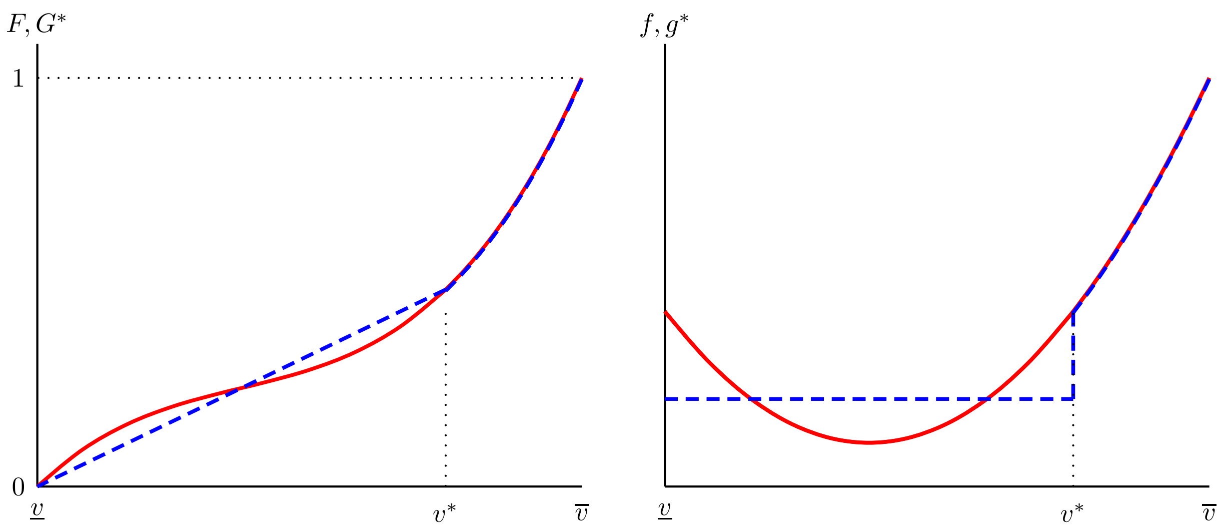
[15] The restriction to the duopoly case (i.e., $n = 2$) is for the same reason as for Corollary 2: See footnote 14.



(a) distribution: $F(v)$ (solid) and $G^*(v)$ (dashed)

(b) density: $f(v)$ (solid) and $g^*(v)$ (dashed)

Figure 2: An example of Corollary 3 (single-peaked density). This figure depicts the case where $n = 2$ and $F(v) = -\frac{4}{3}v^3 + 2v^2 + \frac{1}{3}v$ has a density f which is single-peaked at $1/2$.



(a) distribution: $F(v)$ (solid) and $G^*(v)$ (dashed)

(b) density: $f(v)$ (solid) and $g^*(v)$ (dashed)

Figure 3: An example of Corollary 4 (U-shaped density). This figure depicts the case where $n = 2$ and $F(v) = \frac{16}{7}v^3 - \frac{18}{7}v^2 + \frac{9}{7}v$.

the $(n-1)$-linear structure (see Figure 2). The fact that G^* is smooth at $v = v^*$ is a straightforward implication of Theorem 1: If G^* has upward kink, then it violates the global convexity condition (which induces other firms to deviate to contract their distributions around $v = v^*$). If G^* has downward kink at $v = v^*$, it violates the MPC constraint. Figure 3 depicts the case with u-shaped density, where G^* begins with $(n-1)$-linear structure and then follows F. In this case, G^* may have a downward kink at $v = v^*$.

We note that in both Corollaries 3 and 4, the full information region can be degenerate, in which case the equilibrium advertising strategy has a globally linear structure as in Corollary 2. For example, if we slightly modify the exponential distribution in Figure 1 such that f is single-peaked but $f(\underline{v})$ is still greater than the dashed line in the right panel $(1/2\mu_F)$, then G^* remains globally linear.

If F^{n-1} takes an even more complicated shape, then the above simple structures may not be sufficient. Nevertheless, Theorem 1 states that there always exists a unique equilibrium characterized by G^* which alternates between the full information regions and $(n-1)$-linear MPCs.

3.2 Proof of Theorem 1

We now provide a sketch of our proof of Theorem 1. Recall that we define G^* as a distribution that satisfies the two conditions of Theorem 1. We proceed the proof in three steps. First, we verify that G^* is a symmetric equilibrium strategy in the advertising-only game. We then show that the two conditions of Theorem 1 are necessary for the equilibrium strategy. Finally, given any F, we prove the existence by constructing G^* and show that G^* is unique. All missing details in this subsection can be found in the appendix.

3.2.1 Equilibrium Verification

Suppose that all other firms advertise according to G^*. Then, firm i faces the following constrained maximization problem:

$$\max_{G_i} \int G^*(v_i)^{n-1} dG_i,$$

subject to the constraint that G_i is a MPC of F. Our first goal is to show that $G_i = G^*$ is an optional solution to this problem. This is a non-trivial problem, because firm i's choice set consists of distribution functions subject to MPC constraint which regulates the entire distribution. However, some recent technical developments in the information design literature are applicable to our problem. In particular, the following result by Dworczak and Martini (2018)—which provides a tractable method to verify the optimal information structure of a single-decision maker—significantly simplifies the equilibrium verification of our model.

Theorem 2 (Dworczak and Martini, 2018) *Suppose that F is a distribution function with the support $[\underline{v}, \overline{v}]$. Consider the following programming problem:*

$$\max_{G} \int_{\underline{v}}^{\overline{v}} u(x) dG(x)$$

subject to the constraint that G is an MPC of $F \in \Omega$. G is a solution to the problem if there exists a convex function $\phi : [\underline{v}, \overline{v}] \to \mathcal{R}$ such that (i) $\phi(x) \geq u(x)$ for all $x \in [\underline{v}, \overline{v}]$, (ii) $supp(G) \subset \{x \in [\underline{v}, \overline{v}] : u(x) = \phi(x)\}$, and (iii) $\int_{\underline{v}}^{\overline{v}} \phi(x) dG(x) = \int_{\underline{v}}^{\overline{v}} \phi(x) dF(x)$.

In order to apply this result to our problem, let $\underline{v}^*$ and $\overline{v}^*$ denote the lower and the upper bounds of $supp(G^*)$, respectively. Note that $\underline{v}^* \geq \underline{v}$ and $\overline{v}^* \leq \overline{v}$ since G^* is a MPC of F. Define a function $\phi : [\underline{v}, \overline{v}] \to \mathcal{R}_+$ as

$$\phi(v) = \begin{cases} G^*(v)^{n-1}, & \text{if } v \in [\underline{v}, \overline{v}^*], \\ \alpha(v - \overline{v}^*) + 1, & \text{if } v \in (\overline{v}^*, \overline{v}], \end{cases}$$

where

$$\alpha \equiv \limsup_{v \to \overline{v}^* -} \frac{G^*(\overline{v}^*)^{n-1} - G^*(v)^{n-1}}{v^* - v}. \tag{3}$$

Note that ϕ is well-defined: If $\overline{v}^* = \overline{v}$, then $\phi(v) = G^*(v)^{n-1}$ over the entire domain; if $\overline{v}^* < \overline{v}$, then (by the $(n-1)$-linear MPC condition) $G^*(v)^{n-1}$ is linear for $v \in (\overline{v}^* - \varepsilon, \overline{v}^*]$ (for small $\varepsilon > 0$), and $\phi(v)$ extends that linear portion for $v \in (\overline{v}^*, \overline{v})$.

By definition, $\phi(v) \geq G^*(v)^{n-1}$ for all v, with equality holding for all $v \in supp(G^*) = [\underline{v}^*, \overline{v}^*]$. Since this implies (i) and (ii) in Theorem 2, it suffices to show that

$$\int_{\underline{v}}^{\overline{v}} \phi(v) dG^*(v) = \int_{\underline{v}}^{\overline{v}} \phi(v) dF(v).$$

Recall from Theorem 1 that $G^*(v) = F(v)$ whenever $v \in [w_k, v_k]$ for some $k = 0, ..., m$. Therefore,

$$\int_{\underline{v}}^{\overline{v}} \phi(v) dG^*(v) - \int_{\underline{v}}^{\overline{v}} \phi(v) dF(v) = \sum_{k=0}^{m-1} \left(\int_{v_k}^{w_{k+1}} \phi(v) dG^*(v) - \int_{v_k}^{w_{k+1}} \phi(v) dF(v) \right),$$

The desired result then follows from the fact that for each $k = 0, ..., m-1$,

$$\int_{v_k}^{w_{k+1}} \phi(v) dG^*(v) = \int_{v_k}^{w_{k+1}} \phi(v) dF(v).$$

This crucial equality holds because $\phi(v)$ is linear over $[v_k, w_{k+1}]$ and G^* is a MPC of F over the same interval. Intuitively, the firm is "risk neutral" over $[v_k, w_{k+1}]$ and, therefore, indifferent between F and its MPC G^* over the interval.

3.2.2 Necessity of G^*

Now we show that if G is a symmetric equilibrium in the advertising-only game, then it must satisfy all the properties in Theorem 1. Detailed proofs for the next two lemmas are in the appendix.

The first lemma establishes the global convexity of G^{n-1}.

Lemma 1 *In equilibrium, G^{n-1} must be convex over its support.*

For the intuition, suppose that G^{n-1} is locally strictly concave around v. In this case, just as given in the explanation for Corollary 2, it is optimal for firm i to concentrate all local mass on one point. This, however, violates the symmetric equilibrium requirement that G must be a best response to itself.

One immediate but important corollary of Lemma 1 is that in equilibrium, G must have a convex support, that is, $supp(G) = [\underline{v}^*, \overline{v}^*]$ for some $\underline{v}^*, \overline{v}^* \in [\underline{v}, \overline{v}]$. This is because G^{n-1}, which is also a distribution function, can be convex only when its support is convex and $supp(G) = supp(G^{n-1})$.

It remains to show that any equilibrium G takes the alternating $(n-1)$-linear MPC structure.

Lemma 2 *In equilibrium, G must be an alternating $(n-1)$-linear MPC of F.*

A key to this result is that, given that G^{n-1} is convex, it is weakly beneficial for firm i to spread G_i as much as possible. In particular, since G is an MPC of F, by Jensen's inequality,

$$\int G^{n-1} dG \leq \int G^{n-1} dF,$$

provided that G has the same support as F.[16] However, in equilibrium, firm i should weakly prefer G to F, which is always feasible, when all other firms play G. This implies that the above inequality must hold with equality. Given the convexity of G^{n-1}, the equality can hold either when $G = F$ or when G^{n-1} is linear. This is the fundamental reason why the advertising-only equilibrium takes this particular alternating structure.

3.2.3 Existence and uniqueness of G^*

We complete the proof of Theorem 1 by showing that for each F, there exists a unique G^*. Our existence proof is constructive and, therefore, can also be used to analyze specific examples.

[16] The same-support assumption is only for expositional simplicity and clearly fails in general (see Figure 1). In the proof of Lemma 2 in the appendix, we show that a similar argument applies to the case where $supp(G) \neq supp(F)$.

Preliminaries for existence. For each $\tilde{v} \in [\underline{v}, \overline{v}]$ and $a \in \mathcal{R}_+$, let $H_{\tilde{v},a}$ denote the distribution function that coincides with F below $\tilde{v}$ and is $(n-1)$-linear above $\tilde{v}$ with slope a. Formally,

$$H_{\tilde{v},a}(v)^{n-1} = \begin{cases} F(v)^{n-1}, & \text{if } v \leq \tilde{v}, \\ \min\{a(v-\tilde{v}) + F(\tilde{v})^{n-1}, 1\}, & \text{if } v \in (\tilde{v}, \overline{v}), \\ 1, & \text{if } v = \overline{v}. \end{cases}$$

Note that, unlike F, $H_{\tilde{v},a}$ may put positive mass on $\overline{v}$ (if $a(\overline{v}-\tilde{v}) > 1 - F(\tilde{v})^{n-1}$) or reach 1 before $\overline{v}$ (if $a(\overline{v} - \tilde{v}) < 1 - F(\tilde{v})^{n-1}$). For examples, see the dashed and dotted lines in Figure 4. As we shall see, distributions $H_{\tilde{v},a}$ for different values of $\tilde{v}$ and a serve as building blocks for constructing the alternating $(n-1)$-linear structure of G^*.

Given $H_{\tilde{v},a}$, we define the following function:

$$W_{\tilde{v},a}(v) \equiv \int_{\underline{v}}^{v} (F(x) - H_{\tilde{v},a}(x))dx.$$

As is well known, $H_{\tilde{v},a}$ is an MPC of F over $[v_1, v_2]$ if and only if $W_{\tilde{v},a}(v_1) = W_{\tilde{v},a}(v_2) = 0$ and $W_{\tilde{v},a}(v) \geq 0$ for all $v \in [v_1, v_2]$ (see Section 6.D in Mas-Colell et al., 1995). We exploit this result to check the MPC constraint for G^*.

Consider a distribution $H_{\tilde{v},(F(\tilde{v})^{n-1})'}$, in which its $(n-1)$-linear portion is tangent to F^{n-1} at $v = \tilde{v}$. By construction, $W_{\tilde{v},(F(\tilde{v})^{n-1})'}(v) = 0$ for all $v \leq \tilde{v}$. For $v > \tilde{v}$, we distinguish between the following two cases:

(i) $W_{\tilde{v},(F(\tilde{v})^{n-1})'}(\tilde{v}') \leq 0$ for some $v \in (\tilde{v}, \overline{v}]$.

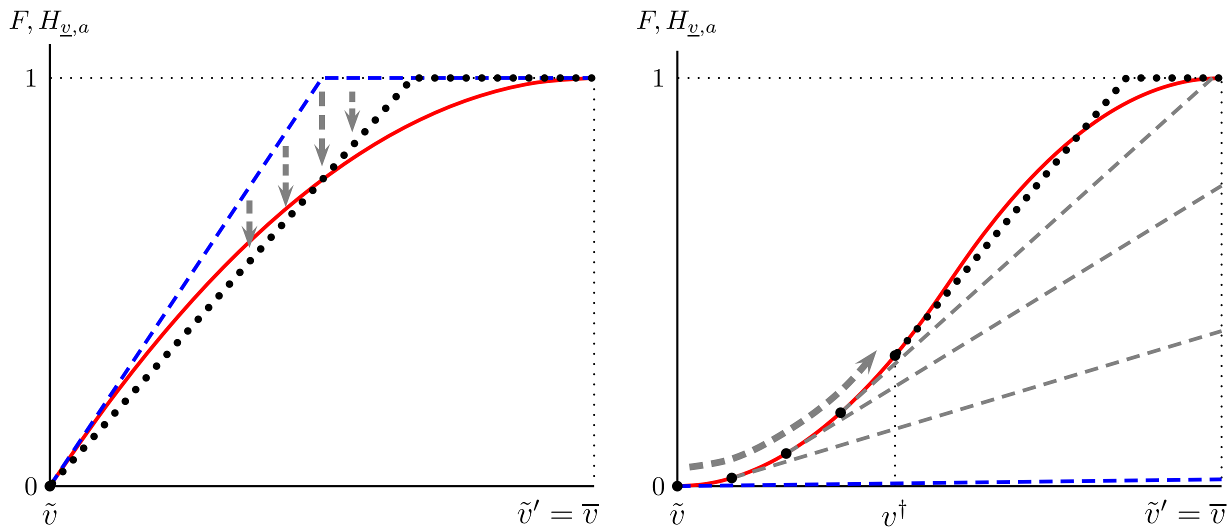
(ii) $W_{\tilde{v},(F(\tilde{v})^{n-1})'}(v) > 0$ for all $v \in (\tilde{v}, \overline{v}]$.

The following lemmas (with proofs in the appendix) illustrate how to find an alternating $(n-1)$-linear MPC of F in each of the two cases.

Lemma 3 *If $W_{\tilde{v},(F(\tilde{v})^{n-1})'}(v) \leq 0$ for some $v \in (\tilde{v}, \overline{v})$, then there exist $a^* \in (0, (F(\tilde{v})^{n-1})']$ and $\tilde{v}' \in (\tilde{v}, \overline{v}]$ such that $H_{\tilde{v},a^*}$ is an MPC of F over $[\tilde{v}, \tilde{v}']$.*

Lemma 4 *If $W_{\tilde{v},(F(\tilde{v})^{n-1})'}(v) > 0$ for all $v \in (\tilde{v}, \overline{v})$, then either F^{n-1} is convex over $[\tilde{v}, \overline{v}]$, or there exist $v^\dagger, \tilde{v}' \in (\tilde{v}, \overline{v}]$ such that $H_{v^\dagger,(F(v^\dagger)^{n-1})'}$ is an MPC of F over $[\tilde{v}, \tilde{v}']$.*

Figure 4 visualizes the arguments in Lemmas 3 and 4 (for the case when $n = 2$). In the left panel, F (solid red) is concave. Therefore, $H_{\tilde{v},(F(\tilde{v})^{n-1})'}$ (dashed blue) is uniformly above F, and thus it is not a MPC of F. In this case, we can find an $(n-1)$-linear MPC of F (dotted black) simply by reducing the slope of $H^{n-1}_{\tilde{v},a}$ (that is, rotating down $H^{n-1}_{\tilde{v},a}$).



(a) an illustration of Lemma 3 (b) an illustration of Lemma 4

Figure 4: This figure visualizes the logic behind Lemmas 3 and 4. In both panels, $\tilde{v} = \underline{v} = 0$ and $\overline{v} = 1$. The distribution function used for the left panel is $F(v) = -v^2 + 2v$, while that for the right panel is $F(v) = 2v^2$ for $v \in [0, 1/2]$ and $F(v) = -2(v-1)^2 + 1$ for $v \in (1/2, 1]$.

In the right panel, $H_{\tilde{v},(F(\tilde{v})^{n-1})'}$ is uniformly below F. Then simply increasing the slope of $H^{n-1}_{\tilde{v},a}$ does not work, as doing so yields $W_{\tilde{v},a}(\tilde{v}+\varepsilon) < 0$ for small ε, violating the MPC condition. In this case, one can move $\tilde{v}$ to the right, which reduces the gap between F and $H_{v^\dagger,(F(v^\dagger)^{n-1})'}$ (see how the dashed gray line shifts as $v^\dagger$ rises). If F^{n-1} is convex above $\tilde{v}$, then $W_{v^\dagger,(F(v^\dagger)^{n-1})'}$ remains bounded above zero for all values of $v^\dagger < \overline{v}$, in which case the only possible alternating $(n-1)$-linear MPC is F itself. Otherwise, Lemma 4 states that there exist $v^\dagger \in (\tilde{v}, \overline{v})$ and $\tilde{v}' \in (v^\dagger, \overline{v}]$ such that $H_{v^\dagger,(F(v^\dagger)^{n-1})'}$ is an alternating $(n-1)$-linear MPC of F over $[\tilde{v}, \tilde{v}']$.

Construction of G^*. Using $H_{\tilde{v},a}$ for different values of $\tilde{v}$ and a as building blocks, we recursively construct G^* in the "forward" direction. We begin with setting $\tilde{v} = \underline{v}$. Then we construct G^* in the following three cases:

(1) If $W_{\tilde{v},(F(\tilde{v})^{n-1})'}(\tilde{v}') \leq 0$ for some $v \in (\tilde{v}, \overline{v}]$: In this case, we set $G^*(v) = H_{\tilde{v},a^*}(v)$ for $v \in [\tilde{v}, \tilde{v}']$, where a^* and $\tilde{v}'$ are defined in Lemma 3. If there exist multiple solutions of $(a^*, \tilde{v}')$, we select the *smallest* a^*.

(2) If $W_{\tilde{v},(F(\tilde{v})^{n-1})'}(v) > 0$ for all $v \in (\tilde{v}, \overline{v})$ and F is not convex over $[\tilde{v}, \overline{v}]$: We set $G^*(v) = H_{v^\dagger,(F(v^\dagger)^{n-1})'}(v)$ for $v \in [\tilde{v}, \tilde{v}']$, where $v^\dagger$ and $\tilde{v}'$ are defined in Lemma 4. If there exist multiple solutions of $(v^\dagger, \tilde{v}')$, then we select the *smallest* $v^\dagger$.

(3) If F is convex over $[\tilde{v}, \overline{v}]$: We set $G^* = F$ for $v \in [\tilde{v}, \overline{v}]$, and the construction is complete.

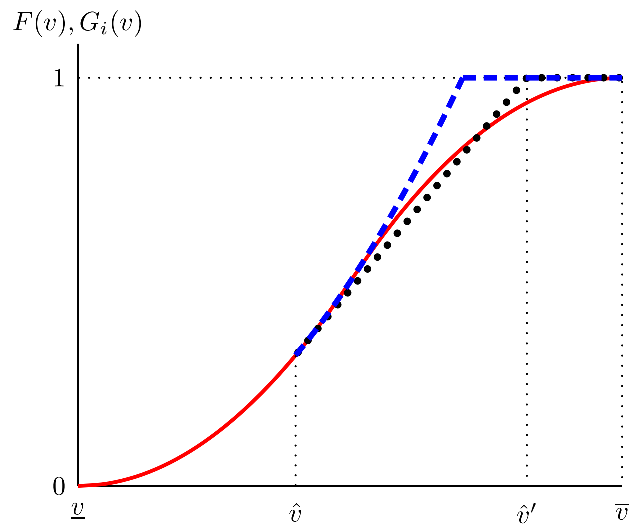


Figure 5: This figure visualizes the argument for uniqueness of G^*. The underlying distribution used for this figure (solid red curve) is $F(v) = 2v^2$ if $v \in [0, 1/2]$ and $F(v) = -2(v-1)^2 + 1$ if $v \in (1/2, 1]$. The black dotted curve represents G_1, while the blue dashed curve depicts G_2, when $G_1(\hat{v} + \varepsilon) < G_2(\hat{v} + \varepsilon)$ for ε sufficiently small.

In cases (1) and (2), the construction is complete if $\tilde{v}' = \overline{v}$. Otherwise, we construct G^* for the next "block" by setting $\tilde{v} = \tilde{v}'$ and repeating the same process.

This construction clearly guarantees that G^* has the alternating $(n-1)$-linear MPC structure. For the convexity of $(G^*)^{n-1}$, notice that we choose the smallest a^* in case (1) and the smallest $v^\dagger$ in case (2). This guarantees that the slope of G^* at the beginning of the next block is no less than the slope at the end of the previous block, leading to the global convexity of $(G^*)^{n-1}$.

Uniqueness of G^*. It remains show that G^* is unique. Suppose that there exist two distribution functions, G_1 and G_2, that satisfy the properties in Theorem 1. Let $\hat{v}$ denote the lowest point at which G_1 and G_2 diverge, that is, $\hat{v} \equiv \inf\{v : G_1(v) \neq G_2(v)\}$ (see Figure 5). Without loss of generality, we assume that $G_1(v) < G_2(v)$ for v sufficiently close to $\hat{v}$. For notational simplicity, let

$$W_i(v) = \int_{\underline{v}}^{v} (F(x) - G_i(x))dx \text{ for } i = 1, 2.$$

Note that, since G_i is an MPC of F, $W_i(v) \geq 0$ for all v and $i = 1, 2$.

We make two observations about G_i's. First, it must be that $G_i(\hat{v}) = F(\hat{v})$: otherwise, G_1^{n-1} and G_2^{n-1} must be linear with different slopes around $\hat{v}$, which violates the definition of $\hat{v}$. Then, obviously, $W_1(\hat{v}) = W_2(\hat{v}) = 0$. Second, G_1^{n-1} must be linear over $(\hat{v}, \hat{v}+\varepsilon)$ for some ε: otherwise, $G_1(v) = F(v)$ for $v \in (\hat{v}, \hat{v} + \varepsilon)$, in which case $G_2(v) > F(v)$ and, therefore, $W_2(\hat{v} + \varepsilon) = \int_{\hat{v}}^{\hat{v}+\varepsilon} (F(x) - G_2(x))dx < 0$, violating the MPC constraint. Let $\hat{v}'$ denote the end point of the

$(n-1)$-linear MPC region (that is, $\hat{v}' \equiv \sup\{v : G_1^{n-1} \text{ is linear over } [\hat{v}, v]\}$).

Now observe that, since G_2^{n-1} is convex over its support, it must be that $G_2(v) > G_1(v)$ for all $v \in (\hat{v}, \hat{v}')$ (see Figure 5). We complete the proof by showing that this dominance leads to $W_2(v) < 0$ for some v. If G_1 does not end on $\hat{v}'$ (that is, $G_1(\hat{v}') < 1$), then G_1 must be an $(n-1)$-linear MPC of F over $[\hat{v}, \hat{v}']$. In this case, $W_1(\hat{v}') = \int_{\hat{v}}^{\hat{v}'}(F(x) - G_1(x))dx = 0$. But then,

$$W_2(\hat{v}') = \int_{\hat{v}}^{\hat{v}'} (F(x) - G_2(x))dx = \int_{\hat{v}}^{\hat{v}'} (G_1 - G_2(x))dx < 0.$$

If G_1 ends on $\hat{v}'$ (that is, $G_1(\hat{v}') = 1$), then G_1 is an $(n-1)$-linear MPC of F over $[\hat{v}, \overline{v}]$. In this case, similarly to the previous case,

$$W_2(\overline{v}) = \int_{\hat{v}}^{\overline{v}} (F(x) - G_2(x))dx = \int_{\hat{v}}^{\overline{v}} (G_1 - G_2(x))dx < 0.$$

3.3 The Effect of Competition

Theorem 1 illustrates how competition influences firms' advertising incentives and, therefore, shapes equilibrium advertising content. A natural and important question in this context is how the intensity of competition affects advertising content, in particular, whether more intense competition makes firms to provide more product information or not. Ivanov (2013) addresses the same economic question, but allows for only a restricted set of advertising strategies that can be ranked in the sense of rotation order (Johnson and Myatt, 2006). He finds that when there are sufficiently many sellers, each seller chooses to provide as much product information as possible. Our results below show that the same economic conclusion can be drawn even with no restriction on feasible advertising content.

Proposition 1 *Let G_n^* be the equilibrium advertising strategy with n firms. For any $v \in (\underline{v}, \overline{v})$, there exists $n(v)$ such that if $n \geq n(v)$ then $G_n^*(v) = F(v)$.*

Technically, this result is due to the fact that F^{n-1} becomes more convex as n increases: in the proof in the appendix, we show that if $\overline{v} < \infty$, then F^{n-1} is necessarily globally convex when n is sufficiently large. The argument for the case where $\overline{v} = \infty$ is more subtle but again relies on the fact that F^{n-1} becomes more convex as n increases. Economically, this is precisely the effect of competition. When there are many competitors, the probability of each firm's winning a consumer is small. In that case, it is better for a firm to provide all product information and, therefore, serve at least a few loyal consumers than to give compromised values to consumers by hiding some product information and, therefore, lose them all.

With some regularity, we can obtain a clearer and stronger result. Specifically, suppose that the density function of F^{n-1} is singled-peaked for any n.[17] In this case, by Corollary 3, $(G^*)^{n-1}$ has at most one linear portion and, therefore, can be characterized by one cutoff v^* such that $G^*(v) = F(v)$ if $v \leq v^*$, while $(G^*)^{n-1}$ is linear over $[v^*, \overline{v}^*]$. The following result shows that in this case, the full information region $[\underline{v}, v^*]$ always expands in n and converges to $[\underline{v}, \overline{v}]$ as n tends to infinity.

Proposition 2 *Let G_n^* be the equilibrium advertising strategy with n firms. Suppose that the density function of F^{n-1} is single-peaked for any $n \geq 2$. For each n, let v_n^* denote the cutoff such that $G_n^*(v) = F(v)$ if $v \leq v_n^*$ and $(G_n^*)^{n-1}$ is linear above v_n^*. Then, $v_n^* \leq v_{n+1}^*$ for any $n \geq 2$ and $\lim_{n\to\infty} v_n^* = \overline{v}$.*

4 Competitive Pricing

In this section, we investigate the effects of competitive advertising on the equilibrium price and welfare. We take the equilibrium advertising strategy G^* characterized in Theorem 1 as given and determine the market price p^*. Also, we compare p^* to the market price in the full information benchmark (that is, the model where $G_i = F$ for all i).

4.1 Equilibrium Price

Given the symmetric advertising strategy G^*, our problem reduces to the random-utility discrete choice model of Perloff and Salop (1985). Assuming that all other firms charge p^*, a consumer purchases product i if and only if $v_i - p_i > v_j - p^*$ for all $j \neq i$. Therefore, an individual firm's optimal pricing problem is given by

$$\max_{p_i} \pi(p_i, G^*, p^*, G^*) = p_i D(p_i, G^*, p^*, G^*) = p_i \int G^*(v_i - p_i + p^*)^{n-1} dG^*(v_i).$$

The firm's first-order condition can be rearranged as follows:

$$\frac{1}{p^*} = -\frac{\partial D(p_i, G^*, p^*, G^*)/\partial p_i|_{p_i=p^*}}{D(p^*, G^*, p^*, G^*)}. \tag{4}$$

[17]This assumption is satisfied by many canonical distribution functions. In particular, this assumption holds whenever f is log-concave: observe that

$$(F^{n-1})'' = (n-1)\{(n-2)F^{n-3}f^2 + F^{n-2}f'\} = (n-1)F^{n-2}f\left((n-2)\frac{f}{F} + \frac{f'}{f}\right).$$

The result follows from the fact that if f is log-concave, then both f/F and f'/f decrease.

This is a well-known optimal pricing formula, which states that the optimal price (markup) is inversely related to the proportion of marginal consumers ($\partial D(p_i, G^*, p^*, G^*)/\partial p_i$) among those who purchase product i ($D(p_i, G^*, p^*, G^*)$). Intuitively, when there are more consumers on the margin, seller i faces a stronger incentive to capture them by lowering her price.

Imposing the symmetry requirement (that p^* must be firm i's optimal price as well) and using the fact that $D(p^*, G^*, p^*, G^*) = 1/n$, we arrive at the following result.

Proposition 3 *In equilibrium, each firm charges*

$$p^* = -\frac{1/n}{\partial D(p_i, G^*, p^*, G^*)/\partial p_i|_{p_i=p^*}} = \frac{1}{n(n-1)\int (G^*)^{n-2} g^* dG^*}. \tag{5}$$

Proposition 3 suggests a unique candidate equilibrium price, in the sense that if a pure-price equilibrium exists, then the market price must be p^* given in (5). However, note that we have not yet discussed the existence of equilibrium: It remains to show that there is no profitable *compound* deviation in which the firm deviates from (p^*, G^*) in *both* its price and advertising strategy. In Section 5, we address this issue and provide a sufficient condition for the equilibrium existence.

The following example illustrates how Proposition 3 can be combined with Theorem 1 to determine the candidate equilibrium outcome (p^*, G^*).

Example 1 (Exponential distribution) *Suppose that $F(v) = 1 - e^{-\lambda v}$, where $\lambda > 0$, $\underline{v} = 0$, and $\overline{v} = \infty$. Note that $\mu_F = 1/\lambda$.*

If $n = 2$, then $F^{n-1} = F$ is concave. Then Corollary 2 implies that $G^* = U[0, 2\mu_F] = U[0, 2/\lambda]$. It is then straightforward to calculate that

$$p^* = \frac{1}{2\int_0^{2/\lambda} \left(\frac{1}{2\lambda}\right)^2 dv} = \frac{1}{\lambda}.$$

If $n > 2$, then the density function of F^{n-1} is single-peaked. Therefore, by Corollary 3, there exists $v^*(> 0)$ such that $G^*(v) = F(v)$ if $v \leq v^*$ and $(G^*)^{n-1}$ is linear above v^*. Since G^* is an MPC of F,

$$E_F[v|v \geq v^*] = E_{G^*}[v|v \geq v^*] = \int_{v^*}^{\overline{v}^*} v \frac{dG^*(v)}{1 - G(v^*)}.$$

Applying the closed-form solution for G^* over $[v^*, \overline{v}^*]$ and $f(v) = \lambda e^{-\lambda v}$, the equality reduces to

$$n(n-1)F(v^*)^{n-2}(1 - F(v^*)) = \frac{1 - F(v^*)^n}{1 - F(v^*)} - nF(v^*)^{n-1}.$$

Combining this with the formula in Proposition 3, it can be shown that

$$\frac{1}{p^*} = \lambda(F(v^*)^n + nF(v^*)^{n-1}(1 - F(v^*)) + n(n-1)F(v^*)^{n-2}(1 - F(v^*))^2) = \lambda.$$

4.2 Price and Welfare under Competitive Advertising

Does the competitive advertising benefit consumers? The answer is not immediately clear: We need to simultaneously consider the change in both the market information and price, as Proposition 3 shows that the market price responds to the amount of revealed (hided) information. To systematically analyze this problem, we compare the equilibrium price p^* to that in the standard Perloff-Salop framework, in which the firms are forced to reveal full product information. Then we discuss the welfare effect of the competitive advertising.

Price Effect of Competitive Advertising Let p^F be the equilibrium price under the advertising strategy $G_i = F$ for all i. By the same logic as for Proposition 3, if the equilibrium exists,[18] then p^F must satisfy the first-order condition:

$$p^F = -\frac{1/n}{\partial D(p_i, F, p^*, F)/\partial p_i|_{p_i=p^*}} = \frac{1}{n(n-1)\int F^{n-2} f dF}. \tag{6}$$

Initial observation suggests that the competitive advertising leads to a lower price, that is, $p^* \le p^F$. Note that in the Perloff-Salop framework, it is product differentiation that enables the firms to charge positive markups.[19] Therefore, under more dispersed consumer preferences, the equilibrium markup is likely to increase as each firm is guaranteed a demand from its "loyal" customers. Since G^* is a MPC of F, it is plausible to expect that $p^* \le p^F$.

The above intuition is not entirely correct, however, because mean-preserving spread (contraction) is not an appropriate measure of preference diversity in this context. Perloff and Salop (1985) point out that p^F may or may not increase when F changes in terms of mean-preserving spread.[20] While their result does not directly apply to our model—the alternating $(n-1)$-linear structure is only a *particular* type of MPC—we also find that the price effect of competitive advertising is ambiguous. The following example demonstrates this ambiguity in a duopoly case.

[18] Similar to Proposition 3, (6) is only a necessary condition for the equilibrium. See Section 5.2 for a complete analysis of the equilibrium existence.

[19] In an extreme example where F is degenerate, the equilibrium price is zero as the problem reduces to the standard Bertrand competition.

[20] Two recent studies, Zhou (2017) and Choi et al. (2018), show that dispersive order (which requires the quantile function of one distribution to be steeper than that of the other distribution at every interior quantile) provides a clear prediction about the change of p^F. However, F and G^* cannot be ranked in terms of dispersive order.

Example 2 *Suppose that* $n = 2$. *We consider two distributions* F_a *and* F_b *that lead to the same equilibrium* $G^* = U[0, 2]$. *In this case,* (5) *and* (6) *imply that the competitive advertising leads to a lower price* ($p^* < p^{F_i}$) *if and only if*

$$\int f_i^2 dv < \int (g^*)^2 dv = \frac{1}{2}.$$

(a) Decreasing density: Suppose that $f_a(v) = \frac{2}{3} - \frac{2}{9}v$ *for* $v \in [0, 3]$. *Then it is straightforward from Corollary 2 that* $G^* = U[0, 2]$. *In this case,* $\int f_a^2 dv = 4/9 < 1/2$. *Therefore, the market price is strictly lower under the competitive advertising.*

(b) U-shaped density: Suppose that $f_b(v) = |v - 1|$ *for* $v \in [0, 2]$. *Again, it is easy to check that* $G^* = U[0, 2]$ *satisfies the conditions in Theorem 1. In this case,* $\int f_b^2 dv = 2/3 > 1/2$, *and thus the market price is strictly higher under the competitive advertising.*

To understand the intuition behind Example 2, observe that the equilibrium G^* changes F in two ways: (1) *reducing its support* (since G^* is a MPC of F), and (2) *flattening its density* (since G^* is linear whenever $G^* \neq F$). Clearly, reducing the support of F implies less product differentiation, leading to a lower market price. Flattening the density f, however, leads to a higher market price. Suppose that G^* is linear and is a MPC of F with the same support $[\underline{v}, \overline{v}]$. Then by the Cauchy-Schwartz inequality,

$$\begin{aligned}\int_{\underline{v}}^{\overline{v}} f(v)^2 dv &= \int_{\underline{v}}^{\overline{v}} f(v)^2 dv \cdot \int_{\underline{v}}^{\overline{v}} \frac{1}{\overline{v} - \underline{v}} dv \\ &\geq \left(\int_{\underline{v}}^{\overline{v}} f(v) \frac{1}{\sqrt{\overline{v} - \underline{v}}} dv \right)^2 = \frac{1}{\overline{v} - \underline{v}} = \int_{\underline{v}}^{\overline{v}} g^*(v)^2 dv,\end{aligned}$$

implying that $p^* \geq p^F$, where the strict inequality holds whenever f is not constant. In case (b) of Example 2, the equilibrium G^* only flattens F without the support reduction, leading to a higher equilibrium price. Case (a) describes a more general situation, in which the change from F to G^* involves both the support reduction and the density flattening.

Welfare Effect of Competitive Advertising Let us now compare welfare under the competitive advertising and under the full information benchmark. Since the consumers are ex ante homogeneous in our model, social surplus is defined as the expected valuation of the product that the representative consumer purchases. Recall that in the symmetric pure-price equilibrium with n firms, a consumer purchases product i if and only if $v_i > v_j$ for all $j \neq i$, where v_1, ..., v_n are independently and identically drawn according to G^*. Therefore, the social surplus and the consumer surplus (the ex ante expected payoff of the representative consumer) are given by

$$SS_n(G^*) \equiv E_{G^*}[\max\{v_1, \ldots, v_n\}] = \int v d(G^*(v)^n),$$
$$CS_n(p^*, G^*) \equiv SS_n(G^*) - p^*,$$

respectively. Similarly, the corresponding measures for the full-information benchmark are given by $SS_n(F)$ and $CS_n(p^F, F)$. The following lemma (proof in the appendix) states the welfare effect of the product information:

Lemma 5 *Suppose that G_1 is a MPC of G_2. Then for any n, $SS_n(G_1) \leq SS_n(G_2)$.*

Lemma 5 implies a clear negative effect of the competitive advertising on social surplus: Whenever $G^* \neq F$, consumers are induced to make a less informed purchase decision, leading to a lower social surplus.

However, the effect of the competitive advertising on consumer surplus is ambiguous, because of ambiguity of the price change (see Example 2). Importantly, the consumers could benefit from the competitive information if p^* is much lower than p^F.[21] Our discussion implies that if a third-party imposes a policy that requires firms to provide full information, then while social surplus unambiguously increases, the change in consumer surplus depends on the shape of the prior distribution F.

5 Combining Advertising and Pricing: Equilibrium Existence

Theorem 1 pins down the equilibrium advertising strategy G^* given p^*, while Proposition 3 determines the equilibrium price given G^*. However, to verify that (p^*, G^*) is indeed an equilibrium, we need to show that the firm has no incentive for compound deviations (that is, the firm chooses $p_i \neq p^*$ *and* $G_i \neq G^*$). This section analyzes such incentives and establishes the equilibrium existence result. Importantly, our analysis describes how a firm's advertisement strategy interacts with its pricing strategy.

5.1 Optimal Advertising for Non-equilibrium Prices

We begin by studying how an individual firm's optimal advertising strategy depends on its price. Specifically, we characterize firm i's optimal advertising strategy G_i^* when the firm chooses p_i and all other firms play (p^*, G^*). In this case, recall from (1) that firm i faces the following advertising

[21] In case (a) of Example 2, it can be shown that consumer surplus is indeed higher under the competitive advertising.

problem:

$$\max_{G_i} \int_{\underline{v}}^{\overline{v}} G^*(v - p_i + p^*)^{n-1} dG_i(v), \qquad s.t.\ G_i \text{ is a MPC of } F. \tag{7}$$

The following proposition provides the full characterization of G_i^*. We note that the optimal advertising strategy reported below is not necessarily unique, but it is one with a particularly simple structure.

Proposition 4 *Suppose that all other firms play (p^*, G^*). For each $p_i > 0$, an optimal advertising strategy G_i^* for firm i is given as follows:*

1. *If $p_i - p^* \geq \overline{v} - \overline{v}^*$, then $G_i^* = F$.*
2. *If $p_i - p^* \leq \mu_F - \overline{v}^*$, then $G_i^* = \delta_{\mu_F}$.*
3. *If $p_i - p^* \in (\mu_F - \overline{v}^*, \overline{v} - \overline{v}^*)$, then*

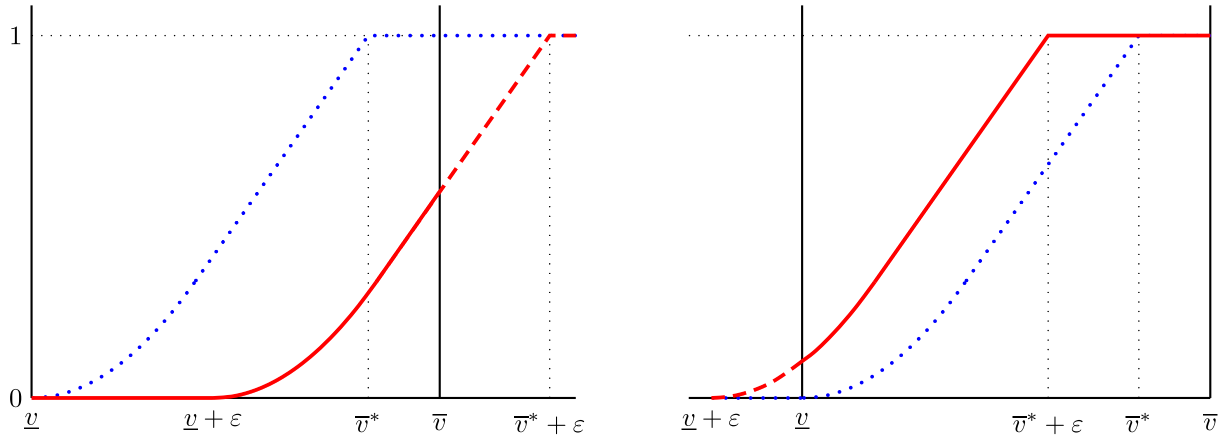
$$G_i^*(v) = \begin{cases} F(v), & \text{if } v \leq \psi, \\ F(\psi), & \text{if } v \in (\psi, \overline{v}^* + p_i - p^*), \\ 1, & \text{if } v \geq \overline{v}^* + p_i - p^*, \end{cases}$$

where ψ is the value such that $E_F[v|v \geq \psi] = \overline{v}^* + p_i - p^*$.

To understand this result, note from (7) that charging $p_i \neq p^*$ effectively shifts firm i's value function $G^*(v)^{n-1}$ leftward (if $p_i < p^*$) or rightward (if $p_i > p^*$). Then the shape of the optimal advertising strategy is determined by the convexity (concavity) of the shifted value function.

Case 1 of Proposition 4 considers a big upward price deviation (p_i sufficiently high). Note that such a deviation leads to a rightward shift of $G^*(v)^{n-1}$. Then as depicted in the left panel of Figure 6, the resulting value function $G^*(v - p_i + p^*)^{n-1}$ becomes convex over $Supp(F) = [\underline{v}, \overline{v}]$ (solid red line). Therefore, by the same argument as for Corollary 1, it is optimal for firm i to reveal all product information. On the contrary, if p_i is sufficiently low (Case 2), then it is optimal for firm i to provide no product information, that is, G_i^* puts all mass on μ_F. Intuitively, for p_i sufficiently low, a consumer with average value μ_F always prefers product i to the other products, regardless of information about others. Therefore, firm i can capture the entire market by providing no information.

When p_i is neither too high nor too low (Case 3), the optimal advertising strategy combines the above two cases: there exists a threshold ψ such that the firm reveals all information below ψ but pools all values above ψ. The right panel of Figure 6 shows that the shifted value function is neither convex nor concave over $[\underline{v}, \overline{v}]$. In this case, firm i first pools all high values at $v = \overline{v}^* + (p_i - p^*)$, guaranteeing that those consumers purchase product i with probability one. Then the firm provides



(a) case 1 ($\varepsilon > \overline{v} - \overline{v}^*$)

(b) case 3 ($\varepsilon \in (\mu_F - \overline{v}^*, \overline{v} - \overline{v}^*)$)

Figure 6: An illustration of Proposition 4. The dotted blue lines represent firm i's value function $G^*(v)^{n-1}$ when firm i posts the equilibrium price $p_i = p^*$. The red lines depict firm i's shifted value function $G^*(v-\epsilon)^{n-1}$, where $\varepsilon = p_i - p^*$. Note that only the values of $G^*(v-\varepsilon)^{n-1}$ for $v \in Supp(F) = [\underline{v}, \overline{v}]$ (solid red) affects the firm's advertising choice.

the full information to consumers with low values, where the corresponding value function is convex.

5.2 Equilibrium Existence

Proposition 4 significantly reduces the technical burden in verifying that (p^*, G^*) is an equilibrium. Given that we derived the optimal advertising strategy G_i^* as a function of p_i, the firm's maximization problem becomes one-dimensional. In this subsection, we provide the equilibrium existence result based on this observation.

Let $D^c(p_i) \equiv D(p_i, G_i^*, p^*, G^*)$ be the firm i's demand function when it posts a price p_i and the corresponding optimal advertising strategy G_i^* derived in Proposition 4 (c stands for *compound* deviation). Then the unique candidate strategy (p^*, G^*) is an equilibrium if

$$p^* \in \arg\max_{p_i} \pi^c(p_i) \equiv p_i D^c(p_i).$$

The following theorem identifies a simple sufficient condition for the equilibrium existence.

Theorem 3 *Suppose that f is log-concave. Then there exists a unique equilibrium where each firm plays (p^*, G^*).*

It is well known that many canonical distributions have log-concave densities.[22] Furthermore,

[22] Examples include uniform, normal, exponential, and logistic distributions; see Bagnoli and Bergstrom (2005) for more detail.

Theorem 3 shows that our existence condition is slightly stronger than that of the standard Perloff-Salop framework. To our knowledge, the only known general existence condition is that both F and $1 - F$ are log-concave (Quint, 2014), which is satisfied if f is log-concave. Since the strategy space is much larger in our model than in the Perloff-Salop framework, we doubt that there exists a more general condition for equilibrium existence. Note also that since our condition guarantees the equilibrium existence in the Perloff-Salop framework, it justifies the price and welfare comparison that we conducted in Section 4.2.

5.2.1 Proof of Theorem 3

Unlike the canonical Perloff-Salop framework, the log-concavity assumption does not directly guarantee well-behaving profit functions in our model: We need to rule out any potential irregularities caused by the compound deviations. Since a similar argument can be applied for other related problems, we present and discuss the proof in the main text. Our proof consists of the following two steps.

Step 1: Defining an imaginary demand function. While Proposition 4 reduces the firm's optimization into a one-dimensional problem, the corresponding demand function $D^c(p_i) = D(p_i, G_i^*, p^*, G^*)$ is still not sufficiently tractable, since the optimal advertising strategy G_i^* has a complex structure. Our solution is to define a tractable *imaginary* demand function that yields the same optimal price as the original problem.

If f is log-concave, then the density of F^{n-1} ($(n-1)F^{n-2}f$) is single-peaked for any n (see footnote 17). Therefore, the equilibrium G^* takes a simple form: there exists $v^* \in [\underline{v}, \overline{v}]$ such that $G^*(v) = F(v)$ for $v \leq v^*$ and $(G^*)^{n-1}$ is linear for $v > v^*$ (see Corollary 3). Recall that $\overline{v}^*$ is the upper bound of $Supp(G^*)$. Define a function $\varphi : \mathbb{R} \to \mathbb{R}_+$ as

$$\varphi(v) = \begin{cases} G^*(v)^{n-1}, & \text{if } v \leq \overline{v}^*, \\ 1 + (n-1)g^*(\overline{v}^*)(v - \overline{v}^*), & \text{if } v > \overline{v}^*. \end{cases}$$

Figure 7 illustrates the relationship between $(G^*)^{n-1}$ and φ. Intuitively, $\varphi(v)$ coincides with $G^*(v)^{n-1}$ for its entire support, then linearly extends $(G^*)^{n-1}$ for $v > \overline{v}^*$. Note that the slope of the extended portion is

$$(G^*(\overline{v}^*)^{n-1})' = (n-1)G^*(\overline{v}^*)^{n-2}g^*(\overline{v}^*) = (n-1)g^*(\overline{v}^*),$$

and thus $\varphi(v)$ is smooth at $v = \overline{v}^*$. Clearly, $\varphi(v)$ is not a distribution function as $\varphi(v) > 1$ for all $v > \overline{v}^*$.

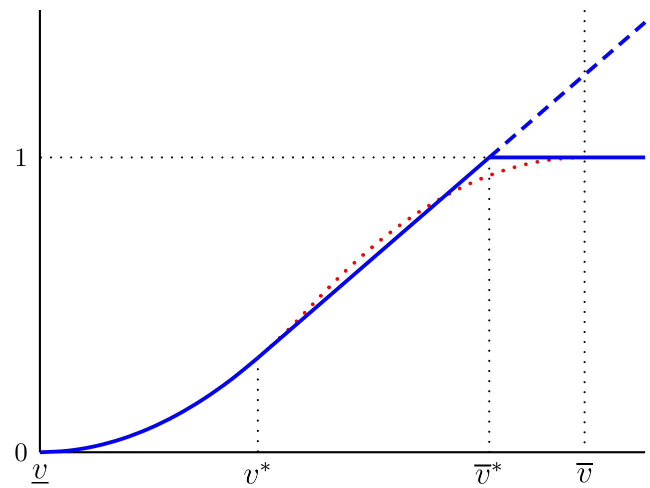


Figure 7: An example of $\varphi(v)$ in a duopoly case ($n = 2$). The figure depicts $F(v)$ (dotted red), $G^*(v)$ (solid blue), and the extended portion of $\varphi(v)$ (dashed blue).

Given $\varphi(v)$, we define an imaginary demand function $\widehat{D}(p_i)$ as

$$\widehat{D}(p_i) \equiv \int \varphi(v - p_i + p^*) dF(v).$$

We interpret $\widehat{D}(p_i)$ as firm i's demand when it faces a "value function" of $\varphi(v - p_i + p^*)$ and chooses the fully informative advertising ($G_i = F$). Note that the firm's advertising strategy is fixed for any p_i. This property of $\widehat{D}(p_i)$ significantly enhances tractability from the original demand function $D^c(p_i)$.

The following lemma shows the relationship between $\widehat{D}(p_i)$ and $D^c(p_i)$.

Lemma 6 $\widehat{D}(p_i) \geq D^c(p_i)$ *for any* p_i*, with equality holding when* $p_i = p^*$.

To understand the first part of the lemma, note that $\varphi(v)$ is convex for all $v \in \mathbb{R}$. Then for any price p_i, firm i's value function ($\varphi(v - p_i + p^*)$) is convex over $[\underline{v}, \overline{v}]$, and thus full information revelation is the optimal advertising strategy. Therefore,

$$\begin{aligned}\widehat{D}(p_i) = \int \varphi(v - p_i + p^*) dF(v) &\geq \int \varphi(v - p_i + p^*) dG^*(v) \\ &\geq \int G^*(v - p_i + p^*)^{n-1} dG^*(v) = D^c(p_i),\end{aligned}$$

where the second inequality holds because $\varphi(v) \geq G^*(v)^{n-1}$ for any v. More importantly, the second part of the lemma comes from the fact that the firm facing a linear value function is indifferent over any distribution of G_i.

Lemma 6 implies that we only need to show that p^* is a unique maximizer of $\hat{\pi}(p_i) \equiv p_i \widehat{D}(p_i)$.

We complete our proof by utilizing the tractability of $\widehat{D}(p_i)$.

Step 2: Proving the optimality of p^*. The following lemma shows the crucial regularity property of $\widehat{D}(p_i)$:

Lemma 7 *If f is log-concave, then $\widehat{D}(p_i)$ is log-concave.*

In the appendix, we prove the lemma by first showing that $\varphi(v)$ is log-concave in v. Then we apply Prékopa's (1971) result, which effectively states that log-concavity is preserved under integration.

Lemma 7 immediately implies that a firm's imaginary profit function $\hat{\pi}(p_i) = p_i\widehat{D}(p_i)$ is log-concave, that is, $\log(p_i) + \log(\widehat{D}(p_i))$ is concave. Therefore, it remains to show that p^* satisfies the first-order condition of $\hat{\pi}(p_i)$, that is,

$$\frac{1}{p^*} = -\frac{\widehat{D}'(p^*)}{\widehat{D}(p^*)}. \tag{8}$$

By Lemma 6, $\widehat{D}(p^*) = D(p^*, G^*, p^*, G^*) = 1/n$. The next lemma provides the final piece of our proof.

Lemma 8 *If f is log-concave, then*

$$\widehat{D}'(p^*) = \frac{\partial D(p_i, G^*, p^*, G^*)}{\partial p_i}|_{p_i=p^*}.$$

Lemma 8 states that in the neighborhood of $p_i = p^*$, deviations in the imaginary problem have the same first-order impact as price-only deviations in the original problem, analyzed in Section 4. Combining Lemmas 6 and 8 show that the right-hand side of (8) is identical to that of (4), which completes our proof.

6 Discussion

We conclude by discussing two important assumptions we have maintained so far.

6.1 Consumer Outside Option

One of the most significant assumptions of our main model is that the consumers do not have an option of not purchasing any product. Clearly, this assumption is innocuous if $\underline{v}$ is sufficiently large (so that $\underline{v} - p^*$ exceeds consumers' outside option). In this subsection, we explain how our main analysis can be modified when the assumption is binding.

In order to make the discussion concise and informative, we focus on the advertising-only game, in which the firms choose G_i in order to maximize their demand given p^*. Other necessary results can be generalized just as in our main model.[23] Without loss of generality, we normalize consumers' outside option to 0 and assume that $\underline{v} < p^* < \overline{v}$: the former inequality implies that consumers' outside option is binding (some consumers do not purchase any product under F), while the latter ensures that it is not too restrictive (nonzero demand under F).

To understand how outside options affect the competitive advertising, suppose that all the firms play G^* in Theorem 1, as if consumers had no outside option. Then the consumers with interim values less than p^* do not purchase any product. This makes an individual firm face a *discontinuous* value function, which stays at zero for $v < p^*$ and then jumps to $G^*(p^*)^{n-1}$ at $v = p^*$. Therefore, firm i's optimal advertising problem is now given as

$$\max_{G_i} \int 1_{\{v \geq p^*\}} G^*(v)^{n-1} dG_i(v), \qquad s.t.\ G_i \text{ is a MPC of } F.$$

Clearly, $G_i = G^*$ does not solve the above problem: Given the discontinuous value function, firm i's optimal advertising strategy must involve pooling information and placing an atom on $v = p^*$. This, however, cannot be part of an equilibrium, because the other firms would have incentives to deviate from G^*. Note that this issue is similar to the one that arises when F^{n-1} is concave (Corollary 2). The solution to this problem is also similar: the equilibrium advertising strategy must have the $(n-1)$-linear MPC structure. However, there are a few crucial differences, as formally reported in the following proposition (with proof in the appendix).

Proposition 5 *Suppose that consumers have an option of not purchasing any product. Given $p^* \in (\underline{v}, \overline{v})$, there exists an equilibrium of the advertising-only game that the firms play G^{**}, where G^{**} is an MPC of F that satisfies the following properties: there exist $v^\dagger \in (\underline{v}, p^*)$, $v^{\dagger\dagger} \in (v^\dagger, \overline{v})$, and $\beta > 0$ such that*

(i) if $v \leq v^{\dagger\dagger}$ then

$$G^{**}(v)^{n-1} = \begin{cases} \min\{F(v)^{n-1}, F(v^\dagger)^{n-1}\}, & \text{if } v \leq p^*, \\ F(v^\dagger)^{n-1} + \beta(v - p^*), & \text{if } v \in (p^*, v^{\dagger\dagger}] \end{cases}$$

where

$$F(v^\dagger)^{n-1} + \beta(v^\dagger - p^*) = 0 \Leftrightarrow \beta \equiv \frac{F(v^\dagger)^{n-1}}{p^* - v^\dagger},$$

[23] One important difference is that G^* and p^* need to be simultaneously determined in the presence of consumer outside option. In the advertising-only game of our main model, each firm's demand is independent of p^*, which allows us to first identify G^* (Section 3) and then p^* (Section 4). With consumer outside option, p^* affects each firm's demand and, therefore, equilibrium advertising G^* as well. This implies that p^* is now characterized a fixed point such that G^*, which can be obtained given p^*, must yield p^* as an equilibrium of the pricing game.

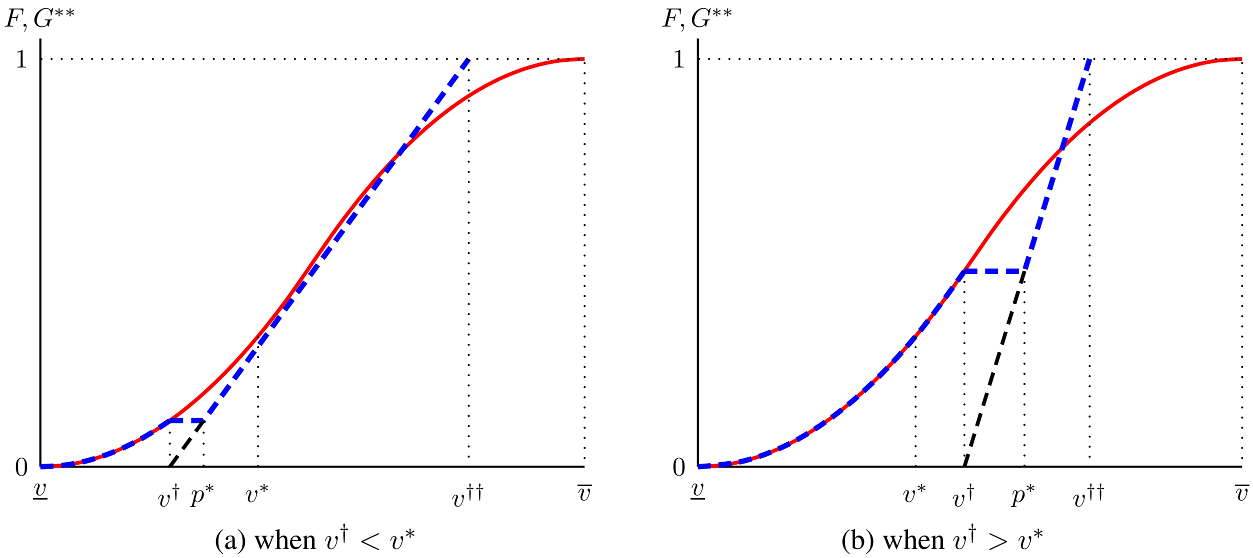


Figure 8: An illustration of Proposition 5. The distribution function F (solid red) is identical to that in the right panel of Figure 4. The blue dashed curve represents G^{**} in each case. v^* denotes the starting point of the $(n-1)$-linear region of G^* in the absence of consumer outside option.

(ii) G^{**} *is an MPC of* F *over* $[v^\dagger, v^{\dagger\dagger}]$*, and*

(iii) G^{**} *features the same properties as* G^* *in Theorem 1 above* $v^{\dagger\dagger}$.

Figure 8 depicts G^{**} for the case where $n = 2$ and F is single-peaked (so that G^* has a full-information region $[\underline{v}, v^*)$ and an $(n-1)$-linear region $[v^*, \overline{v}^*]$). Whether p^* is below v^* (left panel) or above v^* (right panel), G^{**} does not put any mass on $[v^\dagger, p^*]$, which reflects an individual firm's incentive to ensure the interim value of at least p^*. Unlike G^*, $(G^{**})^{n-1}$ is not convex over its support, as clearly shown in Figure 8. However, it is still convex above p^*, which is sufficient for the firms' incentives.

One crucial fact is that $(G^{**})^{n-1}$ has the same (linear) slope at p^* as the line that connects between $(v^\dagger, 0)$ and $(p^*, F(v^\dagger)^{n-1})$, that is, if $(G^{**})^{n-1}$ is linearly extended below p^*, then it must meet $(v^\dagger, 0)$ (see the black dashed line in each panel). In order to see why this is necessary, suppose that $(G^{**})^{n-1}$ is flatter than between $(v^\dagger, 0)$ and $(p^*, F(v^\dagger)^{n-1})$. In this case, just as when $(G^*)^{n-1}$ is concave, it is profitable for a firm to concentrate local mass. To the contrary, if $(G^{**})^{n-1}$ is steeper than between $(v^\dagger, 0)$ and $(p^*, F(v^\dagger)^{n-1})$, then it is profitable for a firm to reveal more product information. The collinearity is necessary and sufficient for neither to be profitable.

6.2 Asymmetric Firms

We have considered the environment where the firms are ex ante symmetric and also focused on symmetric equilibria. Both are clearly restrictive, but it is technically beyond the scope of this paper to relax them. We illustrate the extent to which our analysis so far applies to the case of asymmetric firms, whether they are ex ante asymmetric or behave asymmetrically in the symmetric environment.[24]

Let F_i denote the underlying distribution and G_i^* denote the equilibrium distribution (advertising strategy) for firm i. We also let $\underline{v}_i^*$ and $\overline{v}_i^*$ denote the lower bound and the upper bound, respectively, of the support of G_i^*. Finally, we let p_i^* denote firm i's equilibrium price and $\mathbf{p}^*$ denote the equilibrium price vector (i.e., $\mathbf{p}^* \equiv (p_1^*, ..., p_n^*)$). Given G_j^*'s, $\mathbf{p}^*$ can be derived just as in Section 5. The only difference is that the firms no longer equally divide the market and, therefore, the equilibrium prices cannot take a simple form, as in Proposition 3.

Consider a consumer who has conditional expected value v for product i. She purchases product i if and only if $v - p_i^* > v_j - p_j^*$ and, therefore, with probability

$$Q_i(v, \mathbf{p}^*) \equiv \prod_{j \neq i} G_j^*(v - p_i^* + p_j^*).$$

Note that in the symmetric equilibrium, $Q_i(v, \mathbf{p}^*)$ reduces to $G^*(v)^{n-1}$. Then, by the same argument as for Theorem 1, $Q_i(\cdot, \mathbf{p}^*)$ must be convex over $[\underline{v}_i, \overline{v}_i]$: if $Q_i(\cdot, \mathbf{p}^*)$ is locally concave, then firm i would put mass on one point, which would trigger the other firms to adjust their strategies.

It also holds that firm i's equilibrium advertising strategy takes an alternating structure between full information ($G_i^* = F$) and risk neutrality ($Q_i(\cdot, \mathbf{p}^*)$ is linear): as explained in Section 3.2, this is necessary for firm i to either not be able to adjust its strategy (i.e., binding MPC constraint) or be indifferent over both dispersion and contraction. In symmetric equilibrium, $Q_i(v, \mathbf{p}^*) = G^*(v)^{n-1}$, which enables us to determine G^* when $G^*(v) \neq F(v)$ and, therefore, complete the characterization. With asymmetric sellers, however, this is clearly not sufficient and, unfortunately, we are not aware of how to pin down each G_i^* from this linear result on $Q_i(\cdot, \mathbf{p}^*)$. Note also that, similarly to Section 6.1, $\mathbf{p}^*$ does affect each firm's optimal advertising strategy G_i^*, which introduces further complication.

[24]Even in the standard Perloff-Salop framework, firm asymmetry significantly complicates the analysis. Nevertheless, a lot of progress has been made. See, among others, Quint (2014), who provides a sufficient condition for the existence and uniqueness of pure-price equilibrium for a general (asymmetric) environment and also reports various comparative statics results. A key technical innovation is to recast the pricing problem as a supermodular game. The technique does not apply (at least directly) to our problem where each firm chooses not only its price but also its advertising strategy with full flexibility.

Appendix: Omitted Proofs

Proof of Lemma 1

Suppose that G^{n-1} is not convex. Then, there exist v_1, v_2, v_3, and ε (sufficiently small) such that $v_1 < v_2 < v_3$, $(v_1 - \varepsilon, v_1) \cup (v_3, v_3 + \varepsilon) \subset supp(G)$, and

$$G(v_2)^{n-1} > (1-\alpha)G(v_1)^{n-1} + \alpha G(v_3)^{n-1}, \tag{9}$$

where

$$\alpha \equiv \frac{v_2 - v_1}{v_3 - v_1} \in (0,1).$$

Then, there exist $\delta \in (0, \varepsilon)$ and $\delta' \in (0, \varepsilon)$ such that

$$v_2 = \frac{\int_{v_1-\varepsilon}^{v_1-\varepsilon+\delta} v dG(v) + \int_{v_3+\varepsilon-\delta'}^{v_3+\varepsilon} v dG(v)}{G(v_1-\varepsilon+\delta) - G(v_1-\varepsilon) + G(v_3+\varepsilon) - G(v_3+\varepsilon-\delta')}.$$

For notational convenience, let $\Delta \equiv G(v_1 - \varepsilon + \delta) - G(v_1 - \varepsilon)$ and $\Delta' \equiv G(v_3 + \varepsilon) - G(v_3 + \varepsilon - \delta')$.

Consider the following alternative distribution function G_i:

$$G_i(v) = \begin{cases} G(v), & \text{if } v < v_1 - \varepsilon, \\ G(v_1 - \varepsilon), & \text{if } v \in [v_1 - \varepsilon, v_1 - \varepsilon + \delta], \\ G(v) - \Delta, & \text{if } v \in [v_1 - \varepsilon + \delta, v_2), \\ G(v) + \Delta', & \text{if } v \in [v_2, v_3 + \varepsilon - \delta'), \\ G(v_3 + \varepsilon), & \text{if } v \in [v_3 + \varepsilon - \delta', v_3 + \varepsilon), \\ G(v), & \text{if } v \geq v_3 + \varepsilon. \end{cases}$$

By construction, G_i is a MPC of G and, therefore, a MPC of F. In addition,

$$\int G^{n-1} dG_i - \int G^{n-1} dG = (\Delta + \Delta')G(v_2)^{n-1} - \left(\int_{v_1-\varepsilon}^{v_1-\varepsilon+\delta} G^{n-1} dG + \int_{v_3+\varepsilon-\delta'}^{v_3+\varepsilon} G^{n-1} dG \right) > 0,$$

where the inequality is due to (9). This proves that if G^{n-1} is not convex, then G cannot be an individual seller's best response to G^{n-1}.

Proof of Lemma 2

Define a function $W : [\underline{v}, \overline{v}] \to \mathcal{R}$ as follows:

$$W(v) = \int_{\underline{v}}^{v} (F(v) - G(v)) dv.$$

Since G is a MPC of F, $W(v) \geq 0$ for any v. We show that whenever there is an interval over which $W(v) > 0$, G^{n-1} must be linear over the interval. Since $W(v) = 0$ over an interval only when $F(v) = G(v)$ over the same interval, this immediately proves the alternating structure in the lemma.

Fix an interval $(v_1, v_2) \in [\underline{v}, \overline{v}]$ such that $W(v_1) = W(v_2) = 0$ and $W(v) > 0$ for all $v \in (v_1, v_2)$. There are two cases to consider, depending on whether $v_2 \leq \overline{v}^*$ or not.

(i) Suppose that $v_2 \leq \overline{v}^*$. In this case, consider the following alternative advertising strategy:

$$G_i(v) = \begin{cases} G(v), & \text{if } v < v_1, \\ F(v), & \text{if } v \in [v_1, v_2), \\ G(v), & \text{if } v \geq v_2. \end{cases}$$

In other words, G_i coincides with F on (v_1, v_2) and follows G elsewhere. By construction, G_i is (still) a MPC of F and, therefore, feasible. Since G must be a best response to itself, we must have

$$\int G(v)^{n-1} dG_i(v) \leq \int G(v)^{n-1} dG(v) \Leftrightarrow \int_{v_1}^{v_2} G(v)^{n-1} dG_i(v) \leq \int_{v_1}^{v_2} G(v)^{n-1} dG(v).$$

Now notice that, since $v_2 \leq \overline{v}^*$, G^{n-1} is convex over (v_1, v_2). Combining this with the fact that G_i is a mean-preserving spread of G yields

$$\int G(v)^{n-1} dG_i(v) \geq \int G(v)^{n-1} dG(v) \Leftrightarrow \int_{v_1}^{v_2} G(v)^{n-1} dG_i(v) \geq \int_{v_1}^{v_2} G(v)^{n-1} dG(v).$$

It follows that

$$\int_{v_1}^{v_2} G(v)^{n-1} dG_i(v) = \int_{v_1}^{v_2} G(v)^{n-1} dG(v).$$

The desired result follows from the fact that, since $G_i = F \neq G$ on (v_1, v_2), this equality can hold only when G^{n-1} is linear over (v_1, v_2).

(ii) Now suppose that $v_2 > \overline{v}^*$. Since G is a MPC of F, this case arises only when $v_2 = \overline{v}$ (see Figures 1 and 2).

Let $\tilde{v}$ be the value such that

$$E_F[v|v \geq \tilde{v}] \equiv \int_{\tilde{v}}^{\overline{v}} v \frac{dF(v)}{1 - F(\tilde{v})} = \overline{v}^*.$$

$\tilde{v}$ is well-defined, because $E_F[v|v \geq \tilde{v}]$ is strictly increasing in $\tilde{v}$, $E_F[v|v \geq v_1] = E_G[v|v \geq v_1] < \tilde{v}^*$ (recall that G is an MPC of F), and $E_F[v|v \geq \tilde{v}^*] > \tilde{v}^*$. Then, consider the following alternative advertising strategy:

$$G_i(v) = \begin{cases} G(v), & \text{if } v < v_1, \\ F(v), & \text{if } v \in [v_1, \tilde{v}), \\ F(\tilde{v}), & \text{if } v \in (\tilde{v}, \overline{v}^*), \\ 1, & \text{if } v \geq \overline{v}^*. \end{cases}$$

In other words, G_i coincides with G below v_1, follows F until $\tilde{v}$, and then put all remaining mass on $\tilde{v}^*$. By construction, G_i is an MPC of F and, therefore, feasible. Meanwhile, G_i is a MPS of G: by construction (and since G is an MPC of F above v_1),

$$E_{G_i}[v|v \geq \tilde{v}] = E_F[v|v \geq \tilde{v}] = E_{G_i}[v|v \geq \tilde{v}].$$

In addition, also by construction, G_i crosses G only once from above. Given these (G_i is an MPC of F and a MPS of G), the argument given for the case where $v_2 \leq \overline{v}^*$ applies effectively unchanged: by the optimality of G over G_i and the convexity of G^{n-1}, it must hold that

$$\int_{v_1}^{v_2} G(v)^{n-1} dG_i(v) = \int_{v_1}^{v_2} G(v)^{n-1} dG(v).$$

If $G \neq F$ (which implies $G \neq G_i$), then this equality can hold only when G^{n-1} is linear over $(v_1, \overline{v}^*)$.

Proof of Lemma 3

Suppose that there exists $v \in (\tilde{v}, \overline{v})$ such that $W_{\tilde{v},(F(\tilde{v})^{n-1})'}(v) \leq 0$. If $a = 0$, then F stays uniformly above $H_{\tilde{v},a}$ and, therefore, $W_{\tilde{v},a} > 0$ for all v. In addition, for each $v \in (\tilde{v}, \overline{v}]$, $W_{\tilde{v},a}(v)$ is continuously and strictly increasing in a. Therefore, there always exists $a^* \in (0, (F(\tilde{v})^{n-1})')$ and $\overline{v}' \in (\tilde{v}, \overline{v}]$ such that $W_{\tilde{v},a^*}(v) \geq 0$ for all $v \in (\tilde{v}, \tilde{v}')$ and $W_{\tilde{v},a^*}(\tilde{v}') = 0$. It follows that $H_{\tilde{v},a^*}$ is an $(n-1)$-linear MPC of F over $[\tilde{v}, \tilde{v}']$.

Proof of Lemma 4

Suppose that F^{n-1} is convex over $[\tilde{v}, \overline{v}]$. In this case, for any $v^\dagger \in [\tilde{v}, \overline{v}]$, $H_{v^\dagger,(F(v^\dagger)^{n-1})'}$ is uniformly below F. Therefore, there cannot exist an $(n-1)$-linear MPC of F above $\tilde{v}$. Now suppose that F^{n-1} is not convex. In this case, one can always find $v' \in (\tilde{v}, \overline{v})$ and $v \in (v', \overline{v}]$ such that $H_{v',(F(v')^{n-1})'}(v) < 0$, because if v' belongs to a concave region of F^{n-1} then $H_{v',(F(v')^{n-1})'}$ stays above F around v'. Now let $v^\dagger$ denote the infimum among such v''s. Clearly, $v^\dagger \in (\tilde{v}, \overline{v})$. In addition, since $W_{\tilde{v},(F(\tilde{v})^{n-1})'}(v) > 0$ for all $v \in (\tilde{v}, \overline{v}]$, there must exist $\tilde{v}' \in (v^\dagger, \overline{v}]$ such that

$W_{v^\dagger,(F(v^\dagger)^{n-1})'}(v) \geq 0$ for all $v \in [v^\dagger, \tilde{v}']$, and $W_{v^\dagger,(F(v^\dagger)^{n-1})'}(v^\dagger) = W_{v^\dagger,(F(v^\dagger)^{n-1})'}(\tilde{v}') = 0$. This implies that $H_{v^\dagger,(F(v^\dagger)^{n-1})'}$ is a $(n-1)$-linear MPC of F over $[v^\dagger, \overline{v}']$.

Proof of Proposition 1

We first consider the case where $\overline{v} < \infty$. Let $\varepsilon \equiv \min\{f(v) : v \in [\underline{v}, \overline{v}]\}$ and $M \equiv \max\{|f'(v)| : v \in [\underline{v}, \overline{v}]\}$. Under the maintained technical assumptions on F (namely that $f(v) > 0$ for all $v \in [\underline{v}, \overline{v}]$, and $f'(v)$ is bounded), $\varepsilon > 0$ and $M < \infty$. Then, for any $v \in [\underline{v}, \overline{v}]$,

$$\left(F^{n-1}\right)'' = (n-1)F^{n-3}\left((n-2)f^2 + Ff'\right) \geq (n-1)F^{n-3}((n-2)\varepsilon^2 - M), \tag{10}$$

which is positive everywhere for n sufficiently large. Then, by Corollary 1, $G^* = F$.

Now consider the case where $\overline{v} = \infty$. Given F, let G_n^* denote the equilibrium advertising strategy when there are n firms. We first observe that under our maintained technical assumptions, if n is sufficiently large then the density function of F^{n-1} has exactly one peak (when $\overline{v} = \infty$): let v' denote the point from which f is log-concave. Then, for any n, $(F^{n-1})'$ has only one peak above v' (see footnote 17). In addition, for the same reason as for the case with $\overline{v} < \infty$, F^{n-1} is necessarily convex ($(F^{n-1})'$ is increasing) over $[\underline{v}, v']$. Combining these with the fact that $\overline{v} = \infty$, it follows that $(F^{n-1})'$ has exactly one peak.

Given the above observation, for any n sufficiently large, the equilibrium is characterized by v_n^* such that $G_n^*(v) = F(v)$ if $v \leq v_n^*$ and $(G_n^*)^{n-1}$ is linear above v_n^* (see corollary 3). For each n, we let $\overline{v}_n^*$ denote the upper bound of the support of G_n^* (i.e., $G_n^*(\overline{v}_n^*) = 1$). Then,

$$G_n^*(v)^{n-1} = F(v_n^*)^{n-1} + \left(F(v_n^*)^{n-1}\right)'(v - v_n^*) \text{ whenever } v \in [v_n^*, \overline{v}_n^*].$$

This explicit solution for G_n^* follows from the fact that $v_n^* > \underline{v}$ (because $(F(\underline{v})^{n-1})' = (n-1)F(\underline{v})^{n-2}f(\underline{v}) = 0$ whenever $n \geq 3$) and, therefore, the slope of $(G_n^*)^{n-1}$ must be identical to $(F^{n-1})'$ at v_n^*: MPC implies that $g_n^*(v_n^*) \leq f(v_n^*)$, while convexity of $(G_n^*)^{n-1}$ implies that

$$(n-1)G_n^*(v_n^*)^{n-2}g_n^*(v_n^*) \geq \lim_{v \to v_n^*+}(n-1)G_n^*(v)^{n-2}g_n^*(v) = (n-1)F(v_n^*)^{n-2}f(v_n^*).$$

Now using the explicit $(n-1)$-linear solution for G_n^*, it can be shown that

$$\begin{aligned}
\int_{v_n^*}^{\overline{v}_n^*} v dG_n^*(v) &= \overline{v}_n^* - v_n^* G_n^*(v_n^*) - \frac{1-F(v_n^*)^n}{nF(v_n^*)^{n-2} f(v_n^*)} \\
&= \overline{v}_n^* - v_n^* + v_n^*(1-F(v_n^*)) - \frac{1-F(v_n^*)^{n-1}}{nF(v_n^*)^{n-2} f(v_n^*)} \\
&= \frac{1-F(v_n^*)^{n-1}}{(n-1)F(v_n^*)^{n-2} f(v_n^*)} + v_n^*(1-F(v_n^*)) - \frac{1-F(v_n^*)^n}{nF(v_n^*)^{n-2} f(v_n^*)} \\
&= \frac{1-nF(v_n^*)^{n-1}+(n-1)F(v_n^*)^n}{n(n-1)F(v_n^*)^{n-2} f(v_n^*)} + v_n^*(1-F(v_n^*)).
\end{aligned}$$

For the desired result, it suffices to show that v_n^* grows unboundedly as n tends to infinity. Toward a contradiction, suppose that there exists a subsequent $\{n_k\}$ and $\widehat{v}(<\infty)$ such that for any n_k, $v_n^* \le \widehat{v}$. If so, all $n_k F(v_{n_k})^{n_k-1}$, $(n_k-1)F(v_{n_k})^{n_k}$, and $n(n-1)F(v_{n_k})^{n_k-2}$ converge to 0 as n_k tends to infinity. This implies that for n_k sufficiently large,

$$\int_{v_{n_k}^*}^{\overline{v}_{n_k}^*} v dG_{n_k}^*(v) > \int_{v_{n_k}^*}^{\overline{v}_{n_k}^*} v dF(v),$$

which contradicts the fact that, since G_n^* is a MPC of F, $\int_{v_n^*}^{\overline{v}_n^*} v dG_n^*(v) = \int_{v_n^*}^{\overline{v}_n^*} v dF(v)$ for all n.

Proof of Proposition 2

It suffices to prove the monotonicity result ($v_n^* \le v_{n+1}^*$ for any n), because the asymptotic result ($\lim_{n\to\infty} v_n^* = \overline{v}$) follows from Proposition 1 and the monotonicity result.

Since f is log-concave, for any $n \ge 2$, there exist v_n^* and $\overline{v}_n^*$ such that $G_n^*(\overline{v}_n^*) = 1$ and G_n^* is $(n-1)$-linear over $[v_n^*, \overline{v}_n^*]$. If $v_2^* = \underline{v}$, then it is trivial that $v_2^* \le v_3^*$. In addition, for any $n \ge 3$, $v_n^* > \underline{v}$, because $(F(\underline{v})^{n-1})' = (n-1)F(\underline{v})^{n-2} f(\underline{v}) = 0$. Therefore, without loss of generality, we assume that $v_n^* > \underline{v}$ for any $n \ge 2$. Then, for the same argument as in the proof of Proposition 1, G_n^* is given by

$$G_n^*(v)^{n-1} = \begin{cases} F(v)^{n-1}, & \text{if } v \le v_n^*, \\ F(v_n^*)^{n-1} + (F(v_n^*)^{n-1})'(v - v_n^*), & \text{if } v \in [v_n^*, \overline{v}_n^*] \end{cases}$$

Let $\widehat{v}_n$ denote the point at which $(F^{n-1})' = (n-1)F^{n-2} f$ is maximized. Since $(n-2)f^2 + Ff'$ is increasing in n, $\widehat{v}_n \le \widehat{v}_{n+1}$ for any $n \ge 2$. Since the linear portion can start only in the region where $(n-1)F^{n-2} f$ is increasing (see Figure 2), it is clear that $v_n^* \in (\underline{v}, \widehat{v}_n]$. Now, for each

$\tilde{v} \in [\underline{v}, \widehat{v}_n]$, let $G_{\tilde{v},n}$ denote the distribution function defined as follows:

$$G_{v',n}(v)^{n-1} = \begin{cases} F(v)^{n-1}, & \text{if } v \leq v', \\ \min\{F(v')^{n-1} + (F(v')^{n-1})'(v - v'), 1\}, & \text{if } v > v'. \end{cases}$$

In words, $G_{v',n}(v)$ is a distribution function that is $(n-1)$-linear above v' (with the slope equal to $(F(v')^{n-1})'$. We also let $\overline{v}'$ denote the upper bound of the support of $G_{v',n}$.

We use the following two claims to establish the desired result.

Claim 1 *Given v', $E_{G_{v',n}}[v|v \geq v']$ increases in n.*

Proof. Observe that

$$\begin{aligned} E_{G_{v',n}}[v|v \geq v'] &= \int_{v'}^{\overline{v}'} v \frac{dG_{v',n}(v)}{1 - G_{v',n}(v')} = \frac{\overline{v}' - v' G_{v',n}(v')}{1 - G_{v',n}(v')} - \int_{v'}^{\overline{v}'} G_{v',n}(v) \frac{dv}{1 - G_{v',n}(v')} \\ &= v' + \frac{1}{1 - G_{v',n}(v')} \left(\overline{v}' - v' - \frac{n-1}{n} \frac{1 - G_{v',n}(v')^n}{(F(v')^{n-1})'} \right) \\ &= v' + \frac{1}{1 - G_{v',n}(v')} \left(\frac{1 - G_{v',n}(v')^{n-1}}{(F(v')^{n-1})'} - \frac{n-1}{n} \frac{1 - G_{v',n}(v')^n}{(F(v')^{n-1})'} \right) \\ &= v' + \frac{1}{(F(v')^{n-1})'} \left(\sum_{k=1}^{n-1} G_{v',n}(v')^{k-1} - \frac{n-1}{n} \sum_{k=1}^{n} G_{v',n}(v')^{k-1} \right) \\ &= v' + \frac{\sum_{k=1}^{n-1} F(v')^{k-1} - (n-1) F(v')^{n-1}}{n(n-1) F(v')^{n-2}) f(v')}. \end{aligned}$$

Therefore, it suffices to show that

$$\frac{\sum_{k=1}^{n-1} F(v')^{k-1} - (n-1)F(v')^{n-1}}{n(n-1)F(v')^{n-2} f(v')} < \frac{\sum_{k=1}^{n} F(v')^{k-1} - nF(v')^n}{(n+1)nF(v')^{n-1} f(v')},$$

which is equivalent to

$$(n-1)(1 + F(v')^n) > 2 \sum_{k=1}^{n-1} F(v')^k.$$

Notice that the inequality holds strictly if $v' = \underline{v}$, while it holds with equality if $v' = \overline{v}$. The stated claim then follows from

$$\begin{aligned} \frac{d}{dF(v')} \left((n-1)(1 + F(v')^n) - 2 \sum_{k=1}^{n-1} F(v')^k \right) &= n(n-1)F(v')^{n-2} - 2 \sum_{k=1}^{n-1} kF(v')^{k-1} \\ &\leq F(v')^{n-2} \left(n(n-1) - 2 \sum_{k=1}^{n-2} k \right) = 0. \end{aligned}$$

■

Claim 2 *If* $E_F[v|v \geq v'] \leq E_{G_{v',n}}[v|v \geq v']$*, then* $v_n^* \geq v'$*.*

Proof. Since $(F^{n-1})' = (n-1)F^{n-2}f$ is increasing over $v \in (\underline{v}, \widehat{v}_n)$, if v' increases, then $G_{v',n}$ decreases in the sense of first-order stochastic dominance: notice that for any v, $G_{v',n}(v)$ increases in v'. This implies that $E_F[v] - E_{G_{v',n}}[v]$ is strictly decreasing in v'. Since $G_{v_n^*,n}$ is an MPC of F, it must be that $E_F[v] - E_{G_{v_n^*,n}}[v] = 0$. Therefore, if $E_F[v] \geq E_{G_{v',n}}[v]$, then it is necessarily the case that $v_n^* \geq v'$. The desired result then follows from the fact that, by construction, $G_{v',n}(v) = F(v)$ if $v \leq v'$ and, therefore,

$$E_F[v] - E_{G_{v',n}}[v] = (1 - F(v'))(E_F[v|v \geq v'] - E_{G_{v',n}}[v|v \geq v']).$$

■

Fix n and identify the corresponding equilibrium cutoff v_n^* such that $E_{G_{v_n^*,n}}[v|v \geq v_n^*] = E_F[v|v \geq v_n^*]$. By Claim 1, if n increases, then $E_{G_{v_n^*,n}}[v|v \geq v_n^*] \geq E_F[v|v \geq v_n^*]$. Then, by Claim 2, $v_{n+1}^* \geq v_n^*$.

Proof of Lemma 5

Suppose that $X_{i,1}$, ..., and $X_{i,n}$ are identically and independently drawn according to G_i for each $i = 1, 2$. Then, since $SS_n(G_i) = E[\max\{X_{1,1}, ..., X_{1,n}\}]$, it suffices to show that

$$E[\max\{X_{1,1}, ..., X_{1,n}\}] \leq E[\max\{X_{2,1}, ..., X_{2,n}\}].$$

Since G_1 is a MPC of G_2 and $\max\{y, X_n\}$ is a convex function of X_n for any y,

$$\begin{aligned} E[\max\{X_{2,1}, ..., X_{2,n-1}, X_{1,n}\}] &= E_{X_{2,1},...,X_{2,n-1}}[E[\max\{\max\{X_{2,1}, ..., X_{2,n-1}\}, X_{1,n}\}]] \\ &\leq E_{X_{2,1},...,X_{2,n-1}}[E[\max\{\max\{X_{2,1}, ..., X_{2,n-1}\}, X_{2,n}\}]] \\ &= E[\max\{X_{2,1}, ..., X_{2,n-1}, X_{2,n}\}]. \end{aligned}$$

Repeating the above argument, we have

$$\begin{aligned} E[\max\{X_{1,1}, ..., X_{1,n}\}] &\leq E[\max\{X_{2,1}, X_{1,2}, ..., X_{1,n}\}] \\ &\leq E[\max\{X_{2,1}, X_{2,2}, X_{1,3}, ..., X_{1,n}\}] \\ &\leq ... \\ &\leq E[\max\{X_{2,1}, ..., X_{2,n-1}, X_{1,n}\}] \leq E[\max\{X_{2,1}, ..., X_{2,n-1}, X_{2,n}\}]. \end{aligned}$$

Proof of Proposition 4

For each case, we apply Theorem 2 to establish the optimality of the given advertising strategy. Since p_i is fixed, we assume, without loss of generality, that firm i wishes to maximize its demand $D(p_i, G_i, p^*, G^*)$ by choosing G_i.

(i) $p_i - p^* \geq \overline{v} - \overline{v}^*$.

In this case, $G^*(v - p_i + p^*)^{n-1}$ is convex over $[\underline{v}, \overline{v}]$, because $G^*(v - p_i + p^*)^{n-1} = 0$ if $v \leq \underline{v} + p_i - p^*$ and then follows $(G^*)^{n-1}$ until $\overline{v}(\leq \overline{v}^* + p_i - p^*)$. Then, by Theorem 2, $G_i^* = F$: it suffices to set $\varphi(v) = G^*(v - p_i + p^*)^{n-1}$ for any $v \in [\underline{v}, \overline{v}]$. With $G_i^* = F$, the necessary conditions are trivially satisfied.

(ii) $p_i - p^* \leq \mu_F - \overline{v}^*$.

In this case, it is trivially optimal for firm i not to provide any information, because it makes all consumers purchase product i.

(iii) $p_i - p^* \in (\mu_F - \overline{v}^*, \overline{v} - \overline{v}^*)$.

For this case, consider the following convex function:

$$\varphi(v) = \begin{cases} G^*(v - p_i + p^*)^{n-1}, & \text{if } v \leq \psi, \\ G^*(\psi - p_i + p^*)^{n-1} + \frac{1 - G^*(\psi - p_i + p^*)^{n-1}}{\overline{v}^* - p_i + p^* - \psi}(v - \psi), & \text{if } v > \psi. \end{cases}$$

Since $G^*(v - p_i + p^*)^{n-1}$ is convex over $[\underline{v}, \overline{v}^* + p_i - p^*]$, φ, which is created by extending $G^*(v + p_i - p^*)^{n-1}$ linearly above ψ, is convex over $[\underline{v}, \overline{v}]$. The desired result follows from the fact that the other properties in Theorem 2 also hold: by construction, $\varphi(v) \geq G^*(v + p_i - p^*)^{n-1}$ for any $v \in [\underline{v}, \overline{v}]$ and $supp(G_i^*) = [\underline{v}, \psi] \cup \{\overline{v}^* + p_i - p^*\} = \{v \in [\underline{v}, \overline{v}] : G^*(v + p_i - p^*)^{n-1} = \varphi(v)\}$. In addition,

$$\int_{\underline{v}}^{\overline{v}} \varphi(v) dG_i^*(v) - \int_{\underline{v}}^{\overline{v}} \varphi(v) dF(v) = \int_{\psi}^{\overline{v}} \varphi(v) dG_i^*(v) - \int_{\psi}^{\overline{v}} \varphi(v) dF(v) = 0,$$

where the first equality is because, by construction, $G_i^* = F$ if $v \leq \psi$ and the second equality is because G_i^* is an MPC of F and φ is linear over $[\psi, \overline{v}]$.

Proof of Lemma 6

The first part of the Lemma is proved in the main text. For the second part, observe that by construction, $\varphi(v)$ is linear whenever $G^*(v) \neq F(v)$ (that is, for $v \in [v^*, \overline{v}]$). Therefore,

$$\widehat{D}(p^*) = \int_{\underline{v}}^{\overline{v}} \varphi(v) dF(v) = \int_{\underline{v}}^{\overline{v}} \varphi(v) dG^*(v).$$

Then since $\varphi(v) = G^*(v)^{n-1}$ for any $v \in Supp(G^*) = [\underline{v}, \overline{v}^*]$, we have

$$\int_{\underline{v}}^{\overline{v}} \varphi(v) dG^*(v) = \int_{\underline{v}}^{\overline{v}^*} G^*(v)^{n-1} dG^*(v) = D^c(p^*),$$

which completes the proof.

Proof of Lemma 7

First we show that $\varphi(v)$ is log-concave. If $v < v^*$, then

$$\frac{\varphi'(v)}{\varphi(v)} = \frac{(n-1)F(v)^{n-2}f(v)}{F(v)^{n-1}} = (n-1)\frac{f(v)}{F(v)}.$$

Since F is log-concave (which is implied by log-concavity of f), φ'/φ is decreasing. If $v > v^*$, then $\phi(v)$ is linear and, therefore, clearly log-concave. The desired result then follows from the fact that φ is smooth, which ensures that φ'/φ is continuous around v^*.

Recall that the imaginary demand function $\widehat{D}(p_i)$ is a convolution of φ and f:

$$\widehat{D}(p_i) = \int \varphi(v - p_i + p^*) f(v) dv.$$

We complete our proof by applying Prékopa (1971), which shows that convolution of two log-concave functions is also log-concave.

Proof of Lemma 8

Suppose that firm i posts a price $p_i = p^* + \varepsilon$. Then the imaginary demand and the original demand function (for the price-only deviations) are given by

$$\widehat{D}(p^* + \varepsilon) = \int \varphi(v - \varepsilon) dF(v),$$

$$D(p^* + \varepsilon, G^*, p^*, G^*) = \int G^*(v - \varepsilon)^{n-1} dG^*(v),$$

respectively. Define $\Delta(\varepsilon) \equiv \widehat{D}(p^* + \varepsilon) - D(p^* + \varepsilon, G^*, p^*, G^*)$. Observe that $\Delta(0) = 0$ by Lemma 6. Then it suffices to show that

$$\lim_{\varepsilon \to 0+} \Delta'(\varepsilon) = \lim_{\varepsilon \to 0-} \Delta'(\varepsilon) = 0.$$

Recall that if f is log-concave, then there exists $v^* \in [\underline{v}, \overline{v}]$ such that the equilibrium G^*

coincides with F for $v < v^*$ and $(G^*)^{n-1}$ is linear for $v \geq v^*$. We analyze each of the following three cases:

1. $v^* = \overline{v}$ (which is the case if and only if F^{n-1} is convex).
2. $v^* = \underline{v}$ (which is the case if and only if $n = 2$ and F is concave)
3. $v^* \in (\underline{v}, \overline{v})$.

Case 1: $v^* = \overline{v}$. In this case, F^{n-1} is globally convex, and thus

$$G^*(v) = \begin{cases} F(v) & \text{if } v \leq \overline{v}, \\ 1 & \text{if } v > \overline{v}, \end{cases} \qquad \varphi(v) = \begin{cases} F(v) & \text{if } v \leq \overline{v}, \\ 1 + \hat{\beta}(v - \overline{v}) & \text{if } v > \overline{v}, \end{cases}$$

where $\hat{\beta} = (n-1)f(\overline{v})$. The case with upward price deviations ($\varepsilon > 0$) is trivial: Since $\varphi(v-\varepsilon) = G^*(v-\varepsilon)^{n-1}$ for all $v \in [\underline{v}, \overline{v}]$, $\Delta(\varepsilon) = 0$ for any $\varepsilon > 0$.

Let us consider the case where $\varepsilon < 0$. In this case, the difference from the two demand functions comes only from the interval $[\overline{v} + \varepsilon, \overline{v}]$. Formally,

$$\widehat{D}(p^* + \varepsilon) = \int_{\underline{v}}^{\overline{v}+\varepsilon} F(v-\varepsilon)^{n-1} dF(v) + \int_{\overline{v}+\varepsilon}^{\overline{v}} (1 + \hat{\beta}(v - \varepsilon - \overline{v})) dF(v),$$

$$D(p^* + \varepsilon, G^*, p^*, G^*) = \int_{\underline{v}}^{\overline{v}+\varepsilon} F(v-\varepsilon)^{n-1} dF(v) + \int_{\overline{v}+\varepsilon}^{\overline{v}} 1 dF(v).$$

Therefore,

$$\Delta(\varepsilon) = \int_{\overline{v}+\varepsilon}^{\overline{v}} \hat{\beta}(v - \epsilon - \overline{v}) dF(v),$$

and

$$\Delta'(\varepsilon) = \int_{\overline{v}+\varepsilon}^{\overline{v}} \hat{\beta} dF(v) = -\hat{\beta}(F(\overline{v}) - F(\overline{v} + \varepsilon)),$$

which goes to zero as $\varepsilon \to 0$.

Case 2: $v^* = \underline{v}$. In this case, it must be that $n = 2$ and F is concave (See footnote 14 for a related discussion). Therefore, G^* and φ are given by

$$G^*(v) = \begin{cases} 0 & \text{if } v \leq \underline{v}, \\ \hat{\gamma}(v - \underline{v}) & \text{if } v \in (\underline{v}, \overline{v}^*], \\ 1 & \text{if } v > \overline{v}^*, \end{cases} \qquad \varphi(v) = \begin{cases} 0 & \text{if } v \leq \underline{v}, \\ \hat{\gamma}(v - \underline{v}) & \text{if } v > \underline{v}, \end{cases}$$

where $\hat{\gamma} = 1/(\overline{v}^* - \underline{v})$ and $\overline{v}^* = 2\mu_F - \underline{v}$.

Consider the case with upward price deviations $(\varepsilon > 0)$. Then the demand functions are

$$\begin{aligned}\widehat{D}(p^* + \varepsilon) &= \int_{\underline{v}+\varepsilon}^{\overline{v}} \hat{\gamma}(v - (\underline{v} + \varepsilon))dF(v) \\ &= \hat{\gamma}\left[\int_{\underline{v}+\varepsilon}^{\overline{v}} vf(v)dv - (\underline{v} + \varepsilon)(1 - F(\underline{v} + \varepsilon))\right] \\ &= \hat{\gamma}\left[\mu_F - \int_{\underline{v}}^{\underline{v}+\varepsilon} vf(v)dv - (\underline{v} + \varepsilon)(1 - F(\underline{v} + \varepsilon))\right],\end{aligned}$$

and

$$\begin{aligned}D(p^* + \varepsilon, G^*, p^*, G^*) &= \int_{\underline{v}+\varepsilon}^{\overline{v}^*} \hat{\gamma}(v - (\underline{v} + \varepsilon))dG^*(v) \\ &= \hat{\gamma}^2 \int_{\underline{v}+\varepsilon}^{\overline{v}^*} (v - (\underline{v} + \varepsilon))dv = \frac{\hat{\gamma}^2(\overline{v}^* - (\underline{v} + \varepsilon))^2}{2}.\end{aligned}$$

A straightforward calculation yields

$$\Delta'(\varepsilon) = \hat{\gamma}\left(-(1 - F(\underline{v} + \varepsilon)) + \frac{\overline{v}^* - (\underline{v} + \varepsilon)}{\overline{v}^* - \underline{v}}\right),$$

which converges to zero as $\varepsilon \to 0$.

Next, consider a downward price deviation $(\varepsilon < 0)$. In this case, the two demand functions are given by

$$\widehat{D}(p^* + \varepsilon) = \int_{\underline{v}}^{\overline{v}} \hat{\gamma}(v - (\underline{v} + \varepsilon))dF(v) = \frac{1}{2} - \hat{\gamma}\varepsilon,$$

and

$$\begin{aligned}D(p^* + \varepsilon, G^*, p^*, G^*) &= \int_{\underline{v}}^{\overline{v}^*+\varepsilon} \hat{\gamma}(v - (\underline{v} + \varepsilon))dG^*(v) + \int_{\overline{v}^*+\varepsilon}^{\overline{v}^*} 1dG^*(v) \\ &= \hat{\gamma}^2\left(\frac{(\overline{v}^* + \varepsilon)^2 - \underline{v}^2}{2} - (\underline{v} + \varepsilon)((\overline{v}^* + \varepsilon) - \underline{v})\right) + \hat{\gamma}\varepsilon.\end{aligned}$$

Therefore,

$$\Delta'(\varepsilon) = -\hat{\gamma} + \hat{\gamma}^2((\overline{v}^* + \varepsilon) - (\overline{v}^* + \varepsilon - \underline{v}) - (\underline{v} + \varepsilon)) + \hat{\gamma} = \hat{\gamma}^2\varepsilon,$$

which converges to zero as $\varepsilon \to 0$.

Case 3: $v^* \in (\underline{v}, \overline{v})$**.** For this case, we prove the lemma by showing that

$$\widehat{D}'(p^*) = \frac{dD^c(p_i)}{dp_i}|_{p_i=p^*} = \frac{\partial D(p_i, G^*, p^*, G^*)}{\partial p_i}|_{p_i=p^*}.$$

Note the difference between the second and the third derivatives: $D^c(p_i)$ is firm i's demand function when it deviates to $p_i \neq p^*$ *and* chooses the corresponding optimal advertising strategy G_i^*; $D(p_i, G^*, p^*, G^*)$ is firm i's demand function when it deviates only in its price.

The next two claims prove each of above two equalities.

Claim 3 *There exists* $\delta > 0$ *such that* $\widehat{D}(p_i) = D^c(p_i)$ *for any* $p_i \in (p^* - \delta, p^* + \overline{v} - \overline{v}^*)$. *Therefore,* $\widehat{D}'(p^*) = \frac{dD^c(p_i)}{dp_i}|_{p_i=p^*}$.

Proof. Notice that in this case, $G^*(v - p_i + p^*)^{n-1}$ is convex over $[\underline{v}, v^* + p_i - p^*]$ and flat over $[v^* + p_i - p^*, \overline{v}]$. Then, via Theorem 2, the following advertising strategy is optimal for firm i (note that this is different from the one in Proposition 4 but there may exist multiple optimal advertising strategies):

$$G_i^*(v) = \begin{cases} F(v), & \text{if } v \leq v^* + p_i - p^*, \\ F(v^* + p_i - p^*), & \text{if } v \in [v^* + p_i - p^*, \mu_F(v^* + p_i - p^*)), \\ 1, & \text{if } v \geq \mu_F(v^* + p_i - p^*). \end{cases}$$

$$\begin{aligned}
&D(p_i, G_i^*, p^*, G^*) = \int G^*(v - \varepsilon)^{n-1} dG_i^* \\
&= \int_{\underline{v}}^{v^*+\varepsilon} G^*(v - \varepsilon)^{n-1} dF(v) + (1 - F(v^* + \varepsilon)) G^*(\mu_F(v^* + \varepsilon) - \varepsilon)^{n-1} \\
&= \int_{\underline{v}}^{v^*+\varepsilon} \varphi(v - \varepsilon) dF(v) + (1 - F(v^* + \varepsilon)) \left(F(v^*)^{n-1} + (F(v^*)^{n-1})'(\mu_F(v^* + \varepsilon) - (v^* + \varepsilon)) \right) \\
&= \int_{\underline{v}}^{v^*+\varepsilon} \varphi(v - \varepsilon) dF(v) + \int_{v^*+\varepsilon}^{\overline{v}} \left(F(v^*)^{n-1} + (F(v^*)^{n-1})'(v - (v^* + \varepsilon)) \right) dF(v) \\
&= \int_{\underline{v}}^{v^*+\varepsilon} \varphi(v - \varepsilon) dF(v) + \int_{v^*+\varepsilon}^{\overline{v}} \varphi(v - \varepsilon) dF(v) = \widehat{D}(p_i).
\end{aligned}$$

■

Claim 4 *Let* G_i^* *denote a firm's optimal advertising strategy corresponding to* p_i. *Then,*

$$\frac{dD(p_i, G_i^*, p^*, G^*)}{dp_i}|_{p_i=p^*} = \frac{\partial D(p_i, G^*, p^*, G^*)}{\partial p_i}|_{p_i=p^*},$$

Proof. Recall from the proof of Claim 3 that for p_i around p^*, an optimal advertising strategy is given by

$$G_i^*(v) = \begin{cases} F(v), & \text{if } v \leq v^* + p_i - p^*, \\ F(v^* + p_i - p^*), & \text{if } v \in [v^* + p_i - p^*, \mu_F(v^* + p_i - p^*)), \\ 1, & \text{if } v \geq \mu_F(v^* + p_i - p^*). \end{cases}$$

Therefore, letting $\varepsilon = p_i - p^*$ and using the structure of G^*,

$$\begin{aligned} D(p_i, G_i^*, p^*, G^*) &= \int_{\underline{v}}^{v^*+\varepsilon} G^*(v-\varepsilon)^{n-1} dF(v) + (1 - F(v^*+\varepsilon))G^*(\mu_F(v^*+\varepsilon) - \varepsilon)^{n-1} \\ &= \int_{\underline{v}}^{v^*+\varepsilon} F(v-\varepsilon)^{n-1} dF(v) \\ &\quad + \int_{v^*+\varepsilon}^{\overline{v}} \left(F(v^*)^{n-1} + (F(v^*)^{n-1})'(v - \varepsilon - v^*)\right) dF(v), \end{aligned}$$

while

$$\begin{aligned} &D(p_i, G^*, p^*, G^*) = \int_{\underline{v}}^{v^*} G^*(v-\varepsilon)^{n-1} dF(v) + \int_{v^*}^{\overline{v}^*} G^*(v-\varepsilon)^{n-1} dG^*(v) \\ &= \int_{\underline{v}}^{v^*} F(v-\varepsilon)^{n-1} dF(v) + \int_{v^*}^{v^*+\varepsilon} F(v-\varepsilon)^{n-1} dG^*(v) \\ &\quad + \int_{v^*+\varepsilon}^{\overline{v}^*} \left(F(v^*)^{n-1} + (F(v^*)^{n-1})'(v-\varepsilon-v^*)\right) dG^*(v) \\ &= \int_{\underline{v}}^{v^*} F(v-\varepsilon)^{n-1} dF(v) + \int_{v^*}^{v^*+\varepsilon} F(v-\varepsilon)^{n-1} dG^*(v) \\ &\quad + \int_{v^*}^{\overline{v}} \left(F(v^*)^{n-1} + (F(v^*)^{n-1})'(v-\varepsilon-v^*)\right) dF(v) \\ &\quad - \int_{v^*}^{v^*+\varepsilon} \left(F(v^*)^{n-1} + (F(v^*)^{n-1})'(v-(v^*+\varepsilon))\right) dG^*(v) \\ &= \int_{\underline{v}}^{v^*} F(v-\varepsilon)^{n-1} dF(v) \\ &\quad + (F(v^*)^{n-1})' \int_{v^*}^{v^*+\varepsilon} \left(F(v-\varepsilon)^{n-1} - F(v^*)^{n-1} - (F(v^*)^{n-1})'(v-(v^*+\varepsilon))\right) dv \\ &\quad + \int_{v^*}^{\overline{v}} \left(F(v^*)^{n-1} + (F(v^*)^{n-1})'(v-\varepsilon-v^*)\right) dF(v) \end{aligned}$$

where the second equality is due to the fact that $\int_{v^*}^{\overline{v}^*} v dG^*(v) = \int_{v^*}^{\overline{v}} v dF(v)$. Therefore,

$$\begin{aligned}
&D(p_i, G_i^*, p^*, G^*) - D(p_i, G^*, p^*, G^*) \\
&= \int_{v^*}^{v^*+\varepsilon} F(v-\varepsilon)^{n-1} dF(v) - \int_{v^*}^{v^*+\varepsilon} \left(F(v^*)^{n-1} + (F(v^*)^{n-1})'(v-\varepsilon-v^*)\right) dF(v) \\
&\quad - (F(v^*)^{n-1})' \int_{v^*}^{v^*+\varepsilon} \left(F(v-\varepsilon)^{n-1} - F(v^*)^{n-1} - (F(v^*)^{n-1})'(v-(v^*+\varepsilon))\right) dv
\end{aligned}$$

It is straightforward to see that the derivative of this different with respect to ε at 0 (that is, $p_i = p^*$) is equal to 0. ∎

Proof of Proposition 5

Equilibrium Verification First, we show that G^{**} is a best response to itself and, therefore, indeed an equilibrium. Given that all other firms play G^{**}, an individual firm faces the following advertising problem:

$$\max_G \int_{p^*}^{\overline{v}} G^{**}(v)^{n-1} dG(v), \qquad s.t.\ G \text{ is an MPC of } F.$$

In order to utilize Theorem 2, let $u(v) = 0$ if $v \leq p^*$ and $u(v) = G^{**}(v)^{n-1}$ if $v > p^*$. In addition, define $\phi : [\underline{v}, \overline{v}] \to \mathcal{R}$ as follows:

$$\phi(v) = \begin{cases} 0 & \text{if } v \in [\underline{v}, v^\dagger), \\ F(v^\dagger)^{n-1} + \beta(v-p^*) & \text{if } v \in [v^\dagger, p^*), \\ u(v) = G^{**}(v)^{n-1} & \text{if } v \in [p^*, \overline{v}^*], \\ \alpha(v - \overline{v}^*) + 1 & \text{if } v \in (\overline{v}^*, \overline{v}], \end{cases}$$

where $\overline{v}^*$ is the upper bound of support of G^{**} and α is as defined in equation (3). By the structure of $(G^{**})^{n-1}$, it is clear that ϕ is convex over $[\underline{v}, \overline{v}]$ and $\phi(v) \geq u(v)$ for all v. In addition, $\phi(v) = u(v)$ whenever $v \in supp(G^{**}) = [\underline{v}, v^\dagger] \cup [p^*, \overline{v}^*]$. Therefore, it suffices to show that

$$\int_{\underline{v}}^{\overline{v}} \phi(v) dG^{**}(v) = \int_{\underline{v}}^{\overline{v}} \phi(v) dF(v).$$

Since $G^{**}(v) = F(v)$ if $v \in [\underline{v}, v^\dagger]$,

$$\int_{\underline{v}}^{\overline{v}} \phi(v)dG^{**}(v) - \int_{\underline{v}}^{\overline{v}} \phi(v)dF(v)$$
$$= \left(\int_{v^\dagger}^{v^{\dagger\dagger}} \phi(v)dG^{**}(v) - \int_{v^\dagger}^{v^{\dagger\dagger}} \phi(v)dF(v)\right) + \left(\int_{v^{\dagger\dagger}}^{\overline{v}} \phi(v)dG^{**}(v) - \int_{v^{\dagger\dagger}}^{\overline{v}} \phi(v)dF(v)\right).$$

The first parenthesis is equal to 0, because ϕ is linear and G^{**} is an MPC of F over $[v^\dagger, v^{\dagger\dagger}]$. The second one is also equal to 0, because G^{**} is an MPC of F above $v^{\dagger\dagger}$, with the same alternating structure as in Theorem 1.

Second, since G^{**} features the same properties as G^* in Theorem 1 above $v^{\dagger\dagger}$, the proof of Theorem 2 implies that the second term of the right-hand side must be zero. Then, since $\phi(v)$ is linear over $[v^\dagger, v^{\dagger\dagger}]$ and G^{**} is a MPC over F over the same interval, the first term of the right-hand side must also be zero, leading to the desired result.

Equilibrium Existence Now we prove that there always exists an equilibrium by showing that there exist a unique pair of $v^\dagger \in (\underline{v}, p^*)$ and $\beta > 0$ such that the corresponding G^{**} satisfies the properties in the proposition.

Similar to the existence proof of Theorem 1 (Section 3.2.3) , we use a constructive method to prove the existence. For any $v' \in (\underline{v}, p^*)$ and $b \geq 0$, define a function $H(v; v', b)$ as following:

$$H(v; v', b)^{n-1} = \begin{cases} F(v)^{n-1} & \text{if } v \in [\underline{v}, v'), \\ F(v')^{n-1} & \text{if } v \in [v', p^*), \\ \min\{F(v')^{n-1} + b(v - p^*), 1\} & \text{if } v \in [p^*, \overline{v}]. \end{cases}$$

Also, define

$$W(v; v', b) = \int_{\underline{v}}^{v} (F(v) - H(v; v', b))dv.$$

Recall that $\mu_F(a) = \mathbb{E}_F[v|v \geq a]$, which is continuous in a and strictly increasing over $[\underline{v}, \overline{v}]$. The following lemma states a technical result that we utilize in this proof.

Lemma 9 $\lim_{b\to\infty} W(\overline{v}; v', b) < 0$ *if and only if* $\mu_F(v') > p^*$.

Proof.

$$\begin{aligned}
\lim_{b\to\infty} W(\overline{v}; v', b) &= \lim_{b\to\infty} \int_{v'}^{\overline{v}} (F(v) - H(v; v', b))dv \\
&= \int_{v'}^{\overline{v}} F(v)dv - \left(\int_{v'}^{p^*} F(v')dv + \int_{p^*}^{\overline{v}} 1dv \right) \\
&= (\overline{v} - v'F(v')) - \int_{v'}^{\overline{v}} vdF(v) - (F(v')(p^* - v') + (\overline{v} - p^*)) \\
&= (1 - F(v'))(p^* - \mu_F(v')).
\end{aligned}$$

■

Define $\underline{v}^* = \min\{\nu \in [\underline{v}, p^*) : \mu_F(\nu) \geq p^*\}$. Observe that $\underline{v}^*$ always exists since $\mu_F(p^*) > p^*$. Throughout the next three claims, we show that there exists a unique $v^\dagger \in (\underline{v}^*, p^*)$ and $\beta > 0$ such that $H(v; v^\dagger, \beta)$ is the equilibrium G^{**} for $v \in [\underline{v}, v^{\dagger\dagger}]$.

The first claim states that for any $v' \in (\underline{v}^*, p^*)$, there exists a unique $\tilde{b}(v')$ such that $H(v; v', \tilde{b}(v'))$ can be used for the construction of G^{**}.

Claim 5 *For each $v' \in (\underline{v}^*, p^*)$, there exists a unique $\tilde{b}(v') \in (0, \infty)$ such that (a) $W(v; v', \tilde{b}(v')) \geq 0$ for all $v \in [\underline{v}, \overline{v}]$ and (b) $W(\hat{v}^*; v', \tilde{b}(v')) = 0$ for some $\hat{v}^* \in (p^*, \overline{v}]$.*

Proof. Observe that $W(v; v', b) = 0$ for any $v \in [\underline{v}, v']$ and $W(v; v', b) < 0$ for any $v \in (v', p^*]$. Also, observe that for any $v \in (p^*, \overline{v}]$, $W(v; v', b)$ is continuous and strictly decreases in b. Since $W(v; v', b = 0) > 0$ for any $v \in (p^*, \overline{v}]$ (since F is strictly increasing in v) and $\lim_{b\to\infty} W(\overline{v}; v', b) < 0$ (from Lemma 9 and the fact that $\mu_F(v') > p^*$ for any $v' > \underline{v}^*$), by Intermediate Value Theorem, there must exist a unique $\tilde{b}(v') \in (0, \infty)$ that satisfies the conditions in the claim. ■

We now find the unique $v^\dagger$ such that $H(v; v^\dagger, \tilde{b}(v^\dagger))$ becomes a part of the equilibrium G^{**}. For each $v' \in (\underline{v}^*, p^*)$, define $\kappa(v')$ such that it satisfies

$$F(v')^{n-1} + \tilde{b}(v')(\kappa(v') - p^*) = 0.$$

Observe that if $\kappa(v^\dagger) = v^\dagger$ for some $v^\dagger \in (\underline{v}^*, p^*)$, then $v^\dagger$ and $\beta = \tilde{b}(v^\dagger)$ would satisfy the equation in condition (i) of Proposition 5. Solving for $\kappa(v')$ yields

$$\kappa(v') \equiv p^* - \frac{F(v')^{n-1}}{\tilde{b}(v')}. \tag{11}$$

Using the next two claims, we show that there exists a unique $v^\dagger \in (\underline{v}^*, p^*)$ such that $\kappa(v^\dagger) = v^\dagger$.

Claim 6 *$\kappa(v')$ is continuous and decreasing in v'.*

Proof. Since $F(v')$ is continuous and increasing in v', from (11) it suffices to show that $\tilde{b}(v')$ is continuous and decreasing in v'. Continuity of $\tilde{b}(v')$ is naturally derived from the continuity of $W(v; v', b)$ in both v' and b. For monotonicity, suppose to the contrary that there exists $v', v'' \in (\underline{v}^*, p^*)$ $(v' < v'')$ such that $\tilde{b}(v') < \tilde{b}(v'')$. Let $\hat{\nu} \in (p^*, \overline{v}]$ be such that $W(\hat{\nu}; v', \tilde{b}(v')) = 0$.

However, from the definition of $H(v; v', b)$, it must be that $H(v; v', \tilde{b}(v')) \leq H(v; v'', \tilde{b}(v''))$ for all $v \in [\underline{v}, \hat{\nu})$ with strict inequality holding at least for $v \in (v', p^*)$. Therefore, it must be that $W(\hat{\nu}; v'', \tilde{b}(v'')) < 0$, contradicting to the definition of $\tilde{b}(v'')$. ■

Claim 7 *(a) $\lim_{v' \to \underline{v}^*} \kappa(v') = p^*$ and (b) $\lim_{v' \to p^*} \kappa(v') < p^*$.*

Proof. (a) Suppose that $\mathbb{E}_F[v] \geq p^*$, which implies $\underline{v}^* = \underline{v}$. Then it must be that $\lim_{v' \to \underline{v}^*} \tilde{b}(v') > 0$, since $W(v; \underline{v}, b = 0) > 0$ for any $v > (p^*, \overline{v}]$. Since $\lim_{v' \to \underline{v}} F(v')^{n-1} = 0$, we have the desired result.

Now suppose that $\mathbb{E}_F[v] < p^*$. Then it must be that $\underline{v}^* > \underline{v}$ and $\mu_F(\underline{v}^*) = p^*$. We claim that $\lim_{v' \to \underline{v}^*} \tilde{b}(v') = \infty$. By Lemma 9, $\mu_F(\underline{v}^*) = p^*$ implies that $\lim_{b \to \infty} W(\overline{v}; \underline{v}^*, b) = 0$. Since $\lim_{b \to \infty} H(v; v^*, b) = 1$ for any $v \in (p^*, \overline{v}]$, it follows that for any $v \in (p^*, \overline{v})$, $\lim_{b \to \infty} W(v; \underline{v}^*, b) > 0$. Then for any *finite* b and $v \in (p^*, \overline{v}]$, we have

$$\begin{aligned} \lim_{v' \to \underline{v}^*} W(v; v', b) &= W(v; \underline{v}^*, b) \\ &> \lim_{b \to \infty} W(v; \underline{v}^*, b) > 0, \end{aligned}$$

since $W(\cdot; v', b)$ is continuous in v' and strictly decreasing in b. Therefore, it must be that $\lim_{v' \to \underline{v}^*} \tilde{b}(v') = \infty$, which implies that $\lim_{v' \to \underline{v}^*} \kappa(v') = p^*$.

(b) It must be that $\lim_{v' \to p^*} \tilde{b}(v') > 0$, since $W(v; v', b = 0) > 0$ for any $v \in (p^*, \overline{v}]$. Therefore, from (11), $\lim_{v' \to p^*} \kappa(v') < p^*$. ■

Claims 6 and 7, and the Intermediate Value Theorem together imply that there exists a unique $v^\dagger \in (\underline{v}, p^*)$ such that $\kappa(v^\dagger) = v^\dagger$.

Construction of G^{**}. Let $v^{\dagger\dagger} = \max\{v > p^* : W(v; v^\dagger, \tilde{b}(v^\dagger)) = 0\}$. Observe from Claim 5 that $v^{\dagger\dagger}$ always exists and that $v^{\dagger\dagger} \leq \overline{v}$. For $v \in [\underline{v}, v^{\dagger\dagger}]$, construct G^{**} as

$$G^{**}(v)^{n-1} = \begin{cases} F(v)^{n-1} & \text{if } v < v^\dagger, \\ F(v^\dagger)^{n-1} & \text{if } v \in [v^\dagger, p^*), \\ \min\{F(v^\dagger)^{n-1} + \tilde{b}(v^\dagger)(v - p^*), 1\} & \text{if } v \in [p^*, v^{\dagger\dagger}]. \end{cases}$$

If $v^{\dagger\dagger} = \overline{v}$, then the construction of G^{**} is complete. If $v^{\dagger\dagger} < \overline{v}$, construct G^{**} for $v \in (v^{\dagger\dagger}, \overline{v}]$ using a method identical to the one used in the main model (Subsection 3.2.3). By the construction of $v^{\dagger}$ and $\tilde{b}(v^{\dagger})$, it is straightforward to verify that G^{**} satisfies the properties in Proposition 5.

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