Dynamic contracting under soft information.

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Abstract

A principal delegates the running of a project to an agent subject to moral hazard over an infinite horizon, and cannot observe any of the outcomes. The agent sends reports at each instant t; naturally reports may be manipulated. Eliciting truthful revelation is necessary to the provision of effort, and is achievable by using audits and penalties. It requires that the continuation value of the agent be kept large enough, and the agent be terminated below a threshold; she receives an *endogenous* information rent. That rent is completely determined by the parameters of the moral hazard problem. The optimal audit trades off the expected audit cost again postponing termination by lowering the information rent. The contract is implemented in standard financial securities. The effect of the governance problem on the cost of capital is subtle: a positive continuation utility at termination implies some recovery by financiers and so decreases the credit spread. But a deterioration in governance increases that spread sharply.

Keywords: dynamic contracts; moral hazard; asymmetric information; compensation.
JEL Classification: D82, D86, G28, L43.

1 Introduction

This paper is concerned with a problem of practical importance: how to design a contract to overcome moral hazard when performance is *not* observed by the principal but instead communicated by the agent, over time. The most obvious example is that of a CEO who undertakes a sequence of payoff-relevant actions, the outcome of which is never observed by

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shareholders; instead s/he sends them accounting reports. This practice, of course, affords the CEO the opportunity to manipulate this information. Indeed, "earnings management" and its economic consequences are well documented, as for example by Kedia and Philippon (2009).

The game I study is simple to lay out. A principal employs a wealthless agent to run a project; the action (effort, diligence) of the agent is not observable and its outcome (drift) may be confounded with some random noise (Brownian process). The outcome of the project is also not observable, but the agent is asked to send a report to the principal at each instant t. Thus a contract can only be conditioned on the history of those reports. An agent has incentives to misreport to enhance payments and to conceal her lack of effort. Mitigating misreporting requires auditing the reports, and imposing penalties upon the detection of misreporting – a standard practice both in the real world and in models of it. Auditing is costly. The penalty upon misreporting is termination *without* payment, which is the harshest penalty that may be imposed on the (wealthless) agent.

In spite of the apparent complexity in eliciting private information from the agent in a dynamic environment, intertemporal incentives help tremendously in solving it. The fundamental intuition is simple: time is helpful to the principal because it allows him to postpone payment to the agent ("back-loading"). This delay introduces a wedge between the instantaneous expected benefit of misreporting (essentially, zero almost surely) and its instantaneous expected cost; if this cost is always positive, lying does not pay. To keep this cost positive, the agent must have something at stake at all times; that is, the principal must pay the agent an information rent. The optimal contract is the one that *i*) induces effort from the agent, *ii*) elicits truthful information revelation and *iii*) minimizes that rent, which is equivalent to maximizing the value to the principal. All three of these characteristics have implications in terms of social welfare. Effort is socially valuable by construction. Truthful information revelation is necessary precisely to maintain the incentives for effort. Absent truthful revelation the agent benefits from a double deviation: shirk and lie (to cover up the shirking). Finally minimizing the information rent is socially valuable as it is tantamount to postponing termination, which is socially wasteful – but necessary to the provision of incentives. An essential step is to find the best auditing policy, which determines the lowest possible information rent.

The main results are twofold. First, truthful information revelation requires the continuation utility of the agent to always remain large enough (in a sense made precise). It is intuitive that if the continuation value is zero, the agent may as well misreport since termination also yields zero. Thus an agent who has "failed" is terminated at a strictly positive continuation value, and is paid out that continuation value; this is a golden parachute that is necessary for truthful information revelation. A similar conclusion has been reached by Inderst and Mueller (2010) in a two-period model with *exogenous* adverse selection (the CEO's type) – see details below. Importantly, here the information of the agent is completely *endogenous*. So is the termination barrier, which is completely determined in equilibrium by the variables of the moral hazard problem. This points directly at the source of the frictions. Moral hazard requires the compensation of the agent to be made contingent on outcomes, which generates the incentives for misreporting. Absent moral hazard the reporting problem is moot. The more acute the moral hazard problem, the larger is the information rent.

Second, the optimal contract trades off the expected cost of auditing the agent with the expected benefit of audit. These marginal quantities are to be understood *over time*. The marginal cost is governed by a "probability" of audit – in fact, the intensity of a Poisson random measure – that depends on the report of the agent. Whether truthful or manipulated, that report is a stochastic process. Truthful revelation requires the intensity to be increasing and convex in the report. The marginal benefit is the expected value of delaying termination. It is socially and privately optimal to delay costly termination as long as possible. Delaying termination requires as low a termination threshold (equivalently, an information rent) as possible, which is achieved by auditing the agent; hence the trade-off between delay and audit cost.

The contract is implemented using standard securities, which enables to connect the cost of funds to frictions. Some results are subtle: the implied credit yield spread (on debt) is *lower* in this model than absent the observability problem. Because the agent is terminated at a positive continuation utility, financiers can also recover some of their investment: a fraction of the debt is de facto secured. The credit spread depends on the parameters of the moral hazard problem, and it increases more sharply than absent the observability problem. This speaks to an increase probability of default that owes to earlier termination. Understanding the underlying frictions allows the analyst to distinguish between the role of recovery and probability of default summarised in a single term: the credit spread.

The problem of ex post audit and moral hazard has been studied in static models. Mookherjee and Png (1989) essentially find first-best solutions by using enough instruments; Roger (2013) provides a second-best solution when the principal cannot rely on all these instruments. In that second-best solution information is systematically distorted for some outcomes and may be pooled for some others, with consequences for the provision of effort. In this dynamic model information is truthfully reported in equilibrium, thanks to the intertemporal incentives provided by the (optimal) contract – however at the cost of "premature" termination (that is, at a continuation utility in excess of the outside option of the agent).

Inderst and Mueller (2010) study a problem of CEO entrenchment over two periods. The critical part is that at the beginning of period 2, the agent holds private exogenous information as to her suitability for the job, hence about the (continuation) value of the firm. To sort types, she is presented with a menu of contracts; the high-type selects the steeper contract and the low type opts out with a golden parachute. There is no such hard, exogenous, private information here but only endogenous private information. However, as in Mueller and Inderst's work, truthful revelation of that information remains essential to the provision of effort incentives.

This paper nests in the growing literature on dynamic contracting, which started with the works of Sannikov (2008), Biais et al. (2007) and DeMarzo and Sannikov (2006). The main departure to all these models is, of course, the non-observability of the outcome process, which requires communication from the agent instead. This gives rise to a new incentive constraint that must be satisfied to elicit information revelation. That incentive constraint also defines an *endogenous* boundary condition of the control problem of the principal. In a working paper, Zhu (2018) studies the reverse problem: performance is observed only by the principal, who must communicate it to the agent and compensate her accordingly. Zhu (2018) shows a monitoring technology with weak statistical power but strong incentive power (e.g. a Brownian process) may be counter-productive. Also related is Felipe Varas and Skrzypacz (2018), who show that firms caring about their reputation are best subject to random inspections, except for those "recently inspected". Inspections reveal the true quality of the firms; that quality is stochastic. The threat of inspection spurs effort, while actual inspections also reveal the true information, which is payoff relevant. Here the threat of termination compels truthful revelation; that in turn is essential to spur effort.

2 Model

2.1 Basics

A principal deals with an agent over an infinite horizon; time is continuous. All parties are risk-neutral; the principal discounts future payments at rate $r > 0$ and the agent is (weakly) more impatient, as her discount rate $\rho \geq r$. We are given a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which a standard Wiener process Z is defined and induces the completed filtration $\mathcal{F}_t$. The agent controls the drift of the operating cashflow

$$dX_t^a = \mu(a_t)dt + \sigma dZ_t, \quad X_0 = x, \tag{2.1}$$

where a is an $(\mathcal{F}_t)$-predictable process taking values in $\{0, A\}$. The function $\mu(a)$ has a very simple structure

$$\mu(a) = \begin{cases} \mu & \text{if} \quad a = 0 \\ \mu + A & \text{if} \quad a = A \end{cases},$$

but whether the agent exerts effort is not observable by the principal. Hence at all times the agent may select the inefficiently low action and mislead the principal. Working costs the agent only $\eta \leq 1$ per unit of effort; thus it is socially efficient for the agent to work. To avoid multiplicity of confusing cases, also suppose $\mu \geq 0$ – so that $\mu + A > 0$, and that σ is not too large – in a sense to be made more precise later. Throughout attention is restricted to processes in $\mathcal{L}^*$, that is, to the class of processes such that

$$\sup_{t>0} \mathbb{E}\left[\int \langle B \rangle_t\right] < \infty, \qquad \text{for any process } B.$$

2.2 Information

The novelty of this model is that the (outcome) process X^a is observed by the agent but *never* by the principal. Instead the agent is asked by the principal to report the process X^a; this report is a process $\widetilde{X}$, which may or may not be X^a. Unlike DS, the process $\widetilde{X}$ is *soft* information; it is a message from the agent to the principal rather than a cash flow that is observable and verifiable. The difficulty then is this: on the one hand, to induce costly effort a contract must feature transfers to the agent that are conditioned on the outcome, that is, on the history of the process X. On the other, since the principal does not observe X, said contract can only be conditioned on the history of reports $\widetilde{X}$. Hence, irrespective of her effort decision, the agent may have incentives to manipulate her report $\widetilde{X}$ to the principal.

To solve this problem the principal may use ex post audits at some strictly positive cost k for each instance. However small, this cost generates a trade-off between auditing and awarding rents, so that the principal never finds it optimal to audit the entire path of the process X on any interval; it k is sufficient to ensure that perfect observation of the path of X can never be restored. Upon auditing the report $\widetilde{X}$ of the agent, the principal discovers exactly whether $X = \widetilde{X}$; if $\widetilde{X}_t \neq X_t$ a penalty P may be extracted.

An audit policy is a non-decreasing, random sequence of audit times $(\zeta_n)_{n\geq 1}$ that may depend on the report of the agent. Let $M_t := \sup\left\{n \middle| \zeta_n \leq t\right\}$ denote the counting process associated with ζ. For completeness the filtration $(\mathcal{F}_t)_{t\geq 0}$ denotes the natural filtration $\sigma\left(\widetilde{X}_s, M_s \middle| s < t\right)$. Thus M is a *Poisson random measure* with intensity ϕ; we suppose this intensity is continuous and smooth in its argument. Given a unique probability measure $\mathbb{P}^a$, this measure exists and is unique.[1]

2.3 Contract and payoffs

The principal maximizes the ex-ante value of the project. The contract is designed at date $t = 0$; all parties can commit to it. A contract $\Xi := (c, \tau, M)$ consists of a consumption process $c(\widetilde{X})$ that depends on the history of reports $\widetilde{X}$, a stochastic termination time τ and the counting process M with associated (random) intensity ϕ. Given the wealth constraint of the agent, admissible transfers consists of non-negative, predictable processes c such that $\sup_t \mathbb{E}\left|c_t\right|^2 < \infty$. For a given contract Ξ the payoff to the agent is

$$U_0(c, a) = \mathbb{E}\left[\int_0^\infty e^{-\rho s}(dc_s - \eta a_s ds)\right],$$

which she maximizes by choosing action a and the report process $\widetilde{X}$. A contract is incentive compatible if the agent finds it optimal to (always) exert effort ($a_t = A$, $\forall t \geq 0$) and to never report anything different from X (i.e. $\{\widetilde{X}_t\}_{t\geq 0} = \{X_t\}_{t\geq 0}$). The payoff of the principal is

$$G_0(\Xi) = \mathbb{E}\left[\int_0^\infty e^{-rs}(dX_t - dc_s) - \sum_{\zeta_n \geq 0}^{\infty} e^{-r(\zeta_n)} k\right],$$

and the principal receives $\pi \geq 0$ upon termination of the project, while the agent has outside option 0. This latter assumption is not always innocuous and is discussed in Section 6.

[1] Details about the Poisson random measure and its construction are relegated to the Appendix, Section A.

3 Incentive compatibility

Characterizing incentive compatibility requires conditions on both effort (action a) and on information revelation (report process $\widetilde{X}$). To address both it is sufficient to rely on properties of the continuation utility of the agent. This is the first order of business; with that in hand one can then tackle incentive compatibility. To do so it is helpful, in Section 3.2, to change the measure $\mathbb{P}$ to an equivalent measure under which the only relevant information is that introduced by the stochastic process Z.

3.1 Continuation utility

Following Spear and Srivastava (1987) and Sannikov (2008), an incentive-compatible contract can be characterized by the stochastic process $W^a = \left\{W_t^a, t \geq 0\right\}$ that describes the continuation payoff to the agent when she chooses a strategy $(a, \widetilde{X})$ and the contract Ξ is executed. Rewrite

$$\begin{aligned} U_t(c,a) &= \int_0^t e^{-\rho s}(dc_s - \eta a_s) + \mathbb{E}\left[\int_t^\infty e^{-\rho(s-t)}(dc_s - \eta a_s)\middle|\mathcal{F}_t\right] \\ &= \int_0^t e^{-\rho s}(dc_s - \eta a_s) + e^{-\rho t}W_t^a, \end{aligned} \tag{3.1}$$

where W_t^a is the continuation payoff under action a. Under standard assumption of integrability the term $\mathbb{E}\left[\int_t^\infty e^{-\rho(s-t)}(dc_s - \eta a_s)\middle|\mathcal{F}_t\right]$ is a martingale; therefore so is $U_t(c,a)$. Then following the work of Sannikov (2008) we can make use of the Martingale Representation Theorem to provide an alternative representation of U_t^a and derive the law of motion of the continuation value W, however with two caveats. First, to provide an explicit representation of the martingale U_t^a that allows us to characterize incentive compatibility, that martingale must include jumps representing the penalty upon lying. A penalty must induce a jump because auditing is not a continuous process; the principal does not audit with certainty at each instant t. The intensity of these jumps is determined in equilibrium by the terms of the contract but are not connected in any direct way to the underlying cash flow process X, which is strictly continuous.[2] Second, the intensity of these jumps may depend on the report of the agent. It may be used to write the martingale representation of U_t^a as jump-diffusion process

$$U_t(c,a) = U_0 + \int_0^t e^{-\rho s}\gamma_s \sigma dZ_s - \int_0^t e^{-\rho s} P_s \left[dM_s - \phi ds\right] \mathbb{I}_{\{\widetilde{X}\neq X\}},$$

[2]See, for example, Bromberg-Silverstein, Moreno-Bromberg and Roger (2019) for such a direct connection.

for some processes γ and P (sensitivities) to be determined, along with the intensity ϕ, which is not completely defined yet. $\mathbb{I}_{\{\widetilde{X} \neq X\}}$ denotes an indicator function that takes value 1 if the condition $\widetilde{X} \neq X$ is met and 0 otherwise. Nonetheless, for any such intensity ϕ,

Lemma 1 *There exist processes γ^a and P such that the process W follows the stochastic differential equation*

$$dW_t^a = (\rho W_t + \eta a_t)\, dt - dc_t + \gamma_t^a \left(d\widetilde{X}_t - \mu(a_t) dt \right) - P_t \left[dM_t - \phi dt \right] \cdot \mathbb{I}_{\{\widetilde{X} \neq X\}}, \quad W_0 = w\,, \tag{3.2}$$

where $\gamma^a \in \mathcal{L}^*$ and $dM_t - \phi dt$ is the compensated Poisson process.

The particulars of this SDE are well understood, up to the detail that M is a random measure. When $d\widetilde{X}_t = dX_t$ the term $d\widetilde{X}_t - \mu(a_t)dt = \sigma dZ_t$. The novelty of this paper is the reliance on information provided by the agent, for which the penalty term is necessary. This reliance on the information of the agent induces a new connection between that information and the effort that can be induced in equilibrium. With this representation of the continuation utility of the agent we can characterize incentive compatibility through conditions on the key processes γ, P and M.

3.2 Information transmission

Fix a process M and recall the term $d\widetilde{X}_t - \mu(a_t)dt$ – and under truthful revelation, $\sigma dZ_t^a = dX_t - \mu(a_t)dt$. Because the drift of X^a and that of W^a are known under the optimal contract, the only information the principal needs to elicit is σdZ_t, rather than dX_t. Let $\mathbb{Q}$ denote the equivalent measure to $\mathbb{P}$ so that $dX_t = \sigma dZ_t$ under $\mathbb{Q}$, and denote by Y the message process (equivalent to $\widetilde{X}$) under the same equivalent measure $\mathbb{Q}$. Note that this change of measure does depend on what action is desired and enforced; that is, action and information transmission interact.[3] Given an action a_t, a history of reported outcomes $dY_s, s < t$ and a true outcome dX_t, Equations (3.1) and (3.2) imply that the agent sends a message dY_t such that, under $\mathbb{Q}$:[4]

$$\sup_{dY_t} \gamma_t^a \sqrt{(dY_t - dX_t)^2} - \phi(Y_t) P_t \cdot \mathbb{I}_{\{Y \neq X\}}, \qquad Y_t = Y_{t^-} + dY_t \tag{3.3}$$

where P_t is the penalty extracted by the principal, and $P \leq W_t$ by limited liability. With a perfect audit technology it is optimal to exert maximal punishment (Baron and Besanko,

[3]That is, $\frac{d\mathbb{P}}{d\mathbb{Q}} = e^{\int \frac{\mu(a_t)}{\sigma} dZ_t - \int \frac{1}{2} \left(\frac{\mu(a_t)}{\sigma} \right)^2 dt}$, which depends on a_t.

[4]The square-root and square arrangement is a normalization: $dY_t \geq dX_t$ even if $dY_t < 0$.

1984); if for any $t \geq 0$, $dY_t \neq dX_t$ the wealthless agent is immediately terminated – so $P_t = W_t$.

The only information the principal has access to is that reported by the agent: the process Y. So the intensity ϕ of M should be conditioned on information contained in $\{Y_t\}_{t\geq 0}$. The principal can also use the information afforded by the Law of Motion (3.2). For any action a_t, any departure from dX_t can only be an upward deviation because reporting $dY_t < dX_t$ $i)$ decreases the continuation utility of the agent and $ii)$ precipitates the time at which termination occurs.[5]

Given the audit technology summarised by the smooth function $\phi(Y_t)$, Problem (3.3) gives rise to an optimality condition

$$\gamma_t^a - \phi_{Y_t}(Y_t) W_t \cdot \mathbb{I}_{\{Y \neq X\}}, \tag{3.4}$$

and an agent prefers reporting dX_t (truthfully) over any other dY_t as long as this expression remains negative for all dY_t. For this condition to either bind or induce $dY_t = dX_t$, one needs $\phi_{Y_t}(Y_t) > 0$; for the solution dY_t^* to be a maximizer, $\phi_{Y_t Y_t}(Y_t^*) \geq 0$ as well – the function $\phi(Y_t)$ must be at least locally convex. It remains to define the domain of ϕ.

The continuation utility W naturally has a lower bound that corresponds either to the outside option of the agent (exogenous) or to a termination barrier that is determined (endogenously) by the terms of the contract. Denote this lower bound by $\underline{W}$; it may be a function of time but the notation should not confuse. Given that the outside option of the agent is 0, $\underline{W} \geq 0$. The continuation utility also must have an upper bound: an unbounded promise W is not credible. Indeed the social surplus of the relationship is finite: $V(W) + W \leq (\mu + A)/r$, and so must be the wealth of the principal. Hence there exists some upper bound $\overline{W}$ as well. For now these exist and need not be optimal values; it is presumed that $\underline{W} < \overline{W}$ so that the project does operate.

Next, on the open interval $\left(\underline{W}, \overline{W}\right)$ the continuation utility follows the dynamics

$$\begin{aligned} dW_t &= (\rho W_t + \eta a_t)dt - dc_t + \gamma_t^a \sigma [dX_t - (\mu + a_t)dt], \quad \text{under } \mathbb{P} \\ &= (\rho W_t + \eta a_t)dt - dc_t + \gamma_t^a \sigma dZ_t, \quad \text{under } \mathbb{P} \\ &= (\rho W_t + \eta a_t)dt - dc_t + \gamma_t^a dX_t, \quad \text{under } \mathbb{Q} \end{aligned}$$

as long as the agent is not terminated – that is, as long as she reveals her information truthfully.[6] Then the largest single variation in the process X (under $\mathbb{Q}$) that remains

[5] The superscript a is dropped where confusion cannot arise.

[6] So far $\overline{W}$ is not set optimally, so dc_t need not be zero for $W < \overline{W}$.

payoff-relevant (so that $W \in [\underline{W}, \overline{W}]$) is given by the condition

$$dW_t = (\rho \underline{W} + \eta a_t)dt - dc_t + \gamma_t^a dX_t = [\overline{W} - \underline{W}]dt.$$

Re-arranging this equation one has

$$d\overline{X}_t := \frac{\left\{\overline{W} - \underline{W}(1+\rho) - \eta a_t\right\} dt + dc_t}{\gamma_t^a}.$$

The term $d\overline{X}_t$ measures the largest possible swing in the continuation utility W starting at $\underline{W}$ and reaching $\overline{W}$. Similarly I can determine a lower bound on the process X:

$$dW_t = (\underline{W} - \overline{W})dt \Longrightarrow d\underline{X}_t := \frac{\left\{\underline{W} - \overline{W}(1+\rho) - \eta a_t\right\} dt + dc_t}{\gamma_t^a},$$

which is the variation in X taking W from $\overline{W}$ down to $\underline{W}$. Because these are bounds for the true innovation X (under $\mathbb{Q}$), they also bound the signal process Y sent by the agent (again, such that the signal remains pay-off relevant): $d\underline{Y}_t = d\underline{X}_t$ and $d\overline{Y}_t = d\overline{X}_t$.

The appropriate intensity ϕ is then defined on the support $\left[\underline{W}/\gamma_t^a, \overline{W}/\gamma_t^a\right]$, which clearly allows for the variations $d\underline{Y}_t, d\overline{Y}_t$ to arise. With this in hand, return to the marginal condition (3.4):

$$\gamma_t^a - \phi_{Y_t}(Y_t)W_t,$$

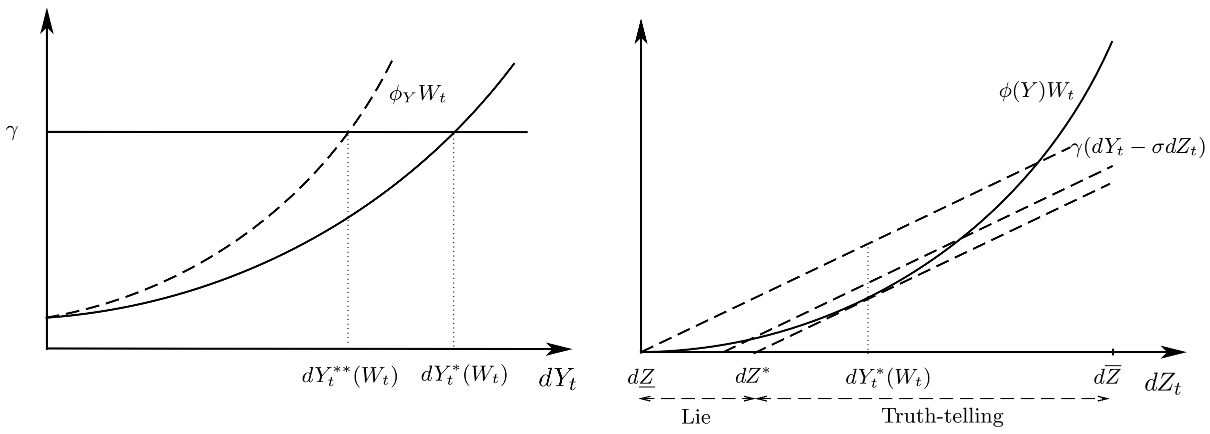
which can only bind at 0 for some positive value of W_t, given the intensity $\phi(\cdot)$ is increasing and convex. Fix some W_t and suppose the condition does bind at 0, when $\phi(\cdot)$ is a convex function, for any true increment dX_t there exists a unique optimal message $dY_t^* := dY_t^*(W_t)$. However, whether the agent optimally reveals the true state or optimally misreports depends on whether, at a solution $dY_t^*(W_t)$ of equation (3.4),

$$\gamma_t^a \sqrt{(dY_t^* - dX_t)^2} > (\leq)\phi(Y_t^*)W_t \cdot \mathbb{I}_{\{Y \neq X\}}, \qquad Y_t^* = Y_{t^-} + dY_t^* \tag{3.5}$$

that is, on whether the LHS and the RHS of (3.5) ever cross. In the first case they do at least once; the agent prefers misreporting.[7] In the second one she truthfully reveals the state.

A problem with Condition (3.5) is that it depends on the true realization dX_t. A large increment dZ_t of the Brownian driver necessarily increases the probability of audit (since only $dY_t \geq dX_t$ is ever reported). Likewise it also increase W_t through the law of motion (3.2) (again, because only $dY_t \geq dX_t$ is ever reported). Thus for large enough an increment dX_t, truthful revelation is optimal. The converse is true for a low realization of the increment dX_t. This is depicted in Figure 1 below. Panel (a) of Figure 1 depicts the marginal conditions of the

[7] The two curves depicted by the LHS and the RHS of (3.5) either cross twice (>), are just tangent (=), or never cross (<).



(a) Marginal conditions

(b) No truthful revelation below dX_t^*.

Figure 1: Optimal message and information revelation

reporting decision of the agent, for two different marginal costs – that is, two different values of the RHS of equation (3.4). The marginal cost functions may differ because of different continuation values W_t. While Panel (a) shows there is a unique optimal message, Panel (b) shows that whether the agent reports this message or the truth depends on the state dX_t: the dotted line is the benefit of misreporting depending on different state realizations. For a low state (below dX_t^*) it is optimal to always report dY_t^*; above dX_t^* the agent reports truthfully. Thus reporting depends on the state because the benefit $\gamma_t^a \sqrt{(dY_t - dX_t)^2}$ depends on the state.

To overcome this problem the principal must put a lower bound on the continuation value of the agent – not only for Condition (3.4) to bind at zero, but also for the resulting message to be truthful. Increasing W_t lifts the expected cost $\phi \cdot W_t$ of misreporting in Figure 1(b). Returning to Figure 1(a), increasing W_t also shifts the optimal message dY_t^* left to dY_t^{**}. The lower bound $\underline{W}$ that is required is a fixed-point that emerges from Condition (3.5).

Proposition 1 *For some W_t, let $dY_t^*(W_t)$ solve*

$$\gamma_t^a - \phi_{Y_t}(Y_t) \cdot W_t = 0.$$

For any true dX_t, the agent reports truthfully ($dY_t^* = dX_t$) as soon as $W_t \geq \underline{W}$, with $\underline{W}$ determined by the fixed-point condition

$$\underline{W} = \frac{\gamma_t^a \left[dY_t^*(\underline{W}) - d\underline{X}_t\right]}{\phi(Y_t^*)} < \infty, \tag{3.6}$$

increasing in γ_t^a and decreasing in ϕ. To enforce truthful reporting the principal must terminate the agent as soon as $W_t \leq \underline{W}$.

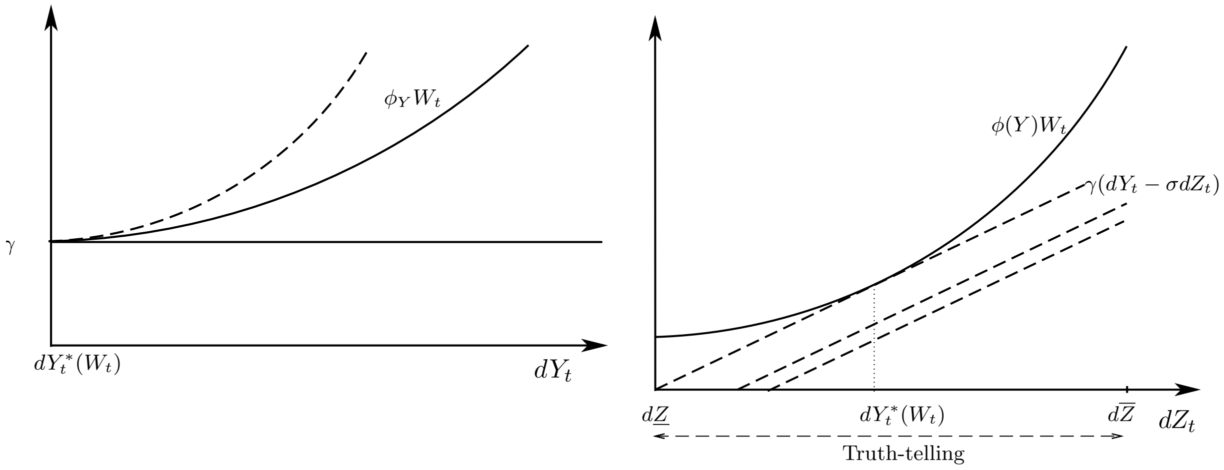
Proposition 1 shows a lot can be accomplished by relying on intertemporal incentives, that is, by simply deferring the compensation of the agent. The intuition is that a contemporaneous lie does not pay immediately; it only improves the continuation utility of the agent. But it does cost immediately as the agent may lose her entire continuation utility if discovered misreporting to the principal.

Of course there is a cost to enforcing truthful revelation. The term $\underline{W}$ is the information rent of the agent; it is what she must be paid to truthfully reveal her information. This is a golden parachute that the agent receives upon termination. Furthermore this golden parachute implies (inefficiently) early termination of the agent by the principal, so incentive compatibility carries a *social* cost. However it is obvious from (3.3) and (3.4) that setting $\underline{W} = 0$ is impossible.

The threshold $\underline{W}$ is a function of the terms of the contract – the endogenous process γ and the intensity ϕ. That is, it is determined by the cost of providing incentives for effort (and the audit technology of course), not by any exogenous private information. The rent is increasing in γ_t^a. As the moral hazard problem worsens (γ increases), so does the information revelation problem. With a worse moral hazard problem, the principal must present the agent with more powerful incentives (larger γ). This generates stronger incentives to misreport information ex post, and so requires a larger information rent.[8] Importantly, $\underline{W}$ is also decreasing in the intensity ϕ: the more likely is an audit at any period of time, the lower the information rent to elicit truthful revelation. This trade-off is central to the optimal contract. The conditions corresponding to Proposition 1 are depicted below.

Figure 2(a) replicates 1(a) however with the optimum choice of the rent $\underline{W}$ (for a fixed intensity $\phi(Y)$). It shows that the marginal benefit of misreporting is always below its marginal cost, even at $d\underline{Y}_t: \ \gamma_t^a < \phi_{Y_t} \cdot \underline{W}$. Likewise with Figure 2(b) replicating Figure 1(b): given outcome $d\underline{X}_t$, at the optimal message $dY_t^*(\underline{W})$ the gain from misreporting is exhausted by its cost – and any other message is strictly dominated. For any other outcome (i.e. $dX_t > d\underline{X}_t$), $\gamma_t^a\sqrt{(dY_t^* - dX_t)^2} < \phi(Y_t^*) \cdot \underline{W}$. That is, while $dY_t^*(\underline{W})$ does maximize the payoff to the agent who decides to misreport, that payoff is weakly dominated by truthful

[8] There is a direct (obvious) effect of increasing γ_t^a; there is also an indirect effect through the intensity $\phi(\cdot)$.



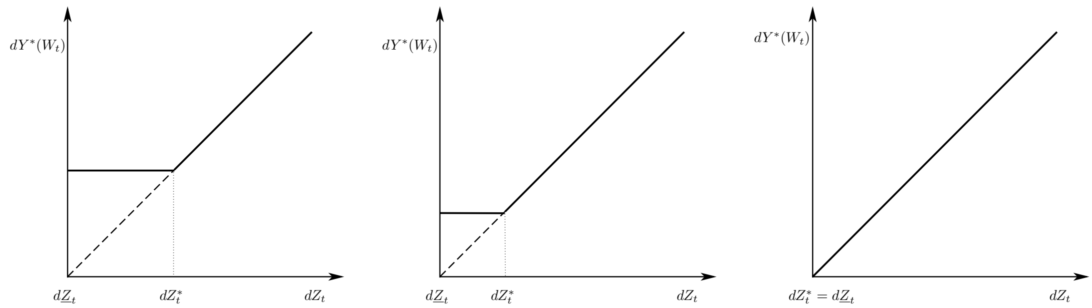
(a) New marginal conditions

(b) Truthful revelation from $d\underline{X}$ on.

Figure 2: Optimal message and truth-telling

revelation given $d\underline{X}_t$ and strictly dominated for all other values of dX_t.

Figure 3 shows the message profile as a function of the true realization dX_t for varying values of the continuation utility W_t. Truthful revelation ($dY_t^* = dX_t$) in all states is only achieved on the right-most panel, for W_t large enough. Panels 3(a) and 3(b) show the optimal



(a) Low continuation.

(b) Higher continuation.

(c) Continuation $W_t = \underline{W}$.

Figure 3: Message profiles.

message dY_t^* is unique for all increments below dZ_t^*.

Remark 1 *It is possible to implement any threshold dX_t^*, not just $dX_t^* = d\underline{X}_t$. However $dX_t^* > d\underline{X}_t$ implies the worst outcomes (below dX_t^*) are never truthfully reported. But these are also precisely the outcomes that matter the most to the principal because decisions – such as termination – have to be made following poor outcomes.*

3.3 Incentive compatibility

In a truth-telling contract the agent never lies. The message process Y therefore is exactly the process X and the law of motion of W follows

$$dW_t = (\rho W_t + \eta a_t)dt - dc_t + \gamma_t^a dX_t, \quad W_0 = w \tag{3.7}$$

under the measure $\mathbb{Q}$, as long as $W_t \geq \underline{W}$. Under $\mathbb{Q}$, $\mathbb{E}[dX_t] = 0$ only if $a_t = A$; the next result is quite intuitive.

Proposition 2 *The agent exerts effort as long as both*

$$\gamma_t^a \geq \eta \quad \textit{and} \quad W_t \geq \underline{W},$$

and $W_t \leq \underline{W}$ triggers termination.

To induce effort the principal must expose the agent to the risky process X, and thus at least to the tune of η, as in DS or in Sannikov (2008). However this relies on observing the process X. In the present model, it requires that the agent truthfully reveal that process X. Truthful revelation is guaranteed (jointly) by the condition $W_t > \underline{W}$ and termination at $\underline{W}$, and effort by $\gamma_t^a \geq \eta$ – provided truthful revelations also holds. That is, the conditions of the Proposition must hold simultaneously to deter a "double deviation". If $W_t < \underline{W}$, not only does the agent have nothing to lose by misreporting, she also has no incentive to work any more – precisely because her lack of effort is optimally confounded by a false report. Hence both truthful revelation and inducing effort require $W_t \geq \underline{W}$.

Remark 2 *Note that $W_t = \underline{W}$ always results in termination by the principal. The intuition is that once W reaches the barrier $\underline{W}$ it crosses it with probability 1. This is formalized as part of the proof of Proposition 1.*

4 Value function and optimal contract

4.1 Overview

It may be helpful to the reader to lay out the steps taken to solve this problem. First one solves the "artificial problem" corresponding to (3.3) where $\mathbb{I}_{\{Y \neq X\}} = \mathbb{I}_{\{Y = X\}} \equiv 1$, which delivers an optimal message $dY_t^*(W_t)$ for any W_t and given an arbitrary intensity $\phi(\cdot)$. From this one deduces that $\phi(\cdot)$ must be a convex function. Then, still given $\phi(\cdot)$

(now restricted to be convex), one selects some $\underline{W}$ such that $i)$ the optimal message $dY_t^*(\underline{W})$ solves Condition (3.3) and $ii)$ Condition (3.5) binds at zero for $dX_t = d\underline{X}_t$. With this, the optimal message $dY_t^*(\underline{W})$ is at least weakly dominated by truthful revelation. This is the construction of Sections 3.1 and 3.2. From there on incentive compatibility is easily characterized in Section 3.3. Now comes optimization over the terms of the contract; the novel parts are the selection of the function ϕ and the optimization over the threshold $\underline{W}$. This is the central trade-off of the optimal contract: the principal may spend more on audit, or award a larger information rent, to achieve information revelation.

4.2 Analysis

As has become standard one may call on dynamic programming to solve this problem, where the continuation utility of the agent is used as a state variable, along with the revealed process Y. This is where working under the measure $\mathbb{Q}$ is helpful: the process $X = \sigma Z$ (also $= Y$ in equilibrium) is confined to the interval $\left[\underline{W}, \overline{W}\right]$ – modulo a known multiplier. There is no need to keep track of the drift of X, as one would under $\mathbb{P}$. To state the payoff to the principal, recall the intensity $\phi(Y_t) = \phi(X_t)$ by Proposition 2. The principal thus maximizes value

$$F(W) := \sup_{\underline{W},c,\beta} \mathbb{E}\left[\int_0^\tau e^{-rs}\left(dX_s - dc_s - \inf_\phi\left[k \cdot \phi(X_s)ds\right]\right)\right], \quad \gamma_t^a \geq \eta,\ W_t > \underline{W}. \tag{4.1}$$

where one notes that the incentive constraint $W_t > \underline{W}$ is endogenous to the problem. The problem can be separated in a first pass. Thus before studying the Bellman equation of this problem it is helpful to focus on the expected cost $\mathbb{E}\left[\int_0^\tau e^{-rs}k \cdot \phi(X)ds\right]$, which the principal seeks to minimize by choice of the function $\phi(\cdot)$ (subject to incentive compatibility). Fix some $\underline{W}$ for now; under the measure $\mathbb{Q}$, Ito's Lemma yields

$$k \cdot r\phi(X)dt = k \cdot \mathbb{E}_{\mathbb{Q}}\left[\phi_X(X)dX + \frac{1}{2}\phi_{XX}(X)dt\right]. \tag{4.2}$$

So the intensity ϕ is a function that is the solution to the boundary value problem

$$r\phi(X) = \frac{1}{2}\phi_{XX}(X), \quad \phi(\underline{X}) = \underline{\phi} > 0,\ \phi(\overline{X}) = \overline{\phi},\ \phi_X(\overline{X}^+) = 0, \tag{4.3}$$

for some boundary conditions $\underline{\phi}$, $\overline{\phi}$ at the boundaries of the domain of X, and subject to the convexity conditions:

$$\phi_X(X) > 0, \quad \phi_{XX}(X) \geq 0.$$

The values of $\underline{\phi}$, $\overline{\phi}$ need to be specified optimally (later) as part of the solution to the cost minimization. The conditions at $\overline{X}$ (and $\overline{X}^{+}$) are simple: there is no need to further increase ϕ past $\overline{\phi}$ since W reflects then. This reflection implies there is no benefit to report anything higher than $\overline{X}$ but there can only be a cost. The function ϕ can be explicitly solved for

$$\phi(X) = Ae^{\lambda_1 X} + Be^{\lambda_2 X}, \quad \lambda_1 = -\lambda_2 = \sqrt{2r}, \tag{4.4}$$

and we know there are constants A, B such that the solution $\phi(\cdot)$ is increasing convex. The boundary conditions of Problem (4.3), together with the definitions $\underline{X} := \underline{W}/\gamma_t^a$ and $\overline{X} := \overline{W}/\gamma_t^a$ are used to determine the constants A, B:

$$\begin{aligned} A &= \frac{1}{1 - e^{2\lambda_1(\overline{X} - \underline{X})}} \left[\frac{\underline{\phi}}{e^{\lambda_1 \underline{X}}} - \frac{\overline{\phi}}{e^{-\lambda_1 \overline{X}}} e^{e^{-2\lambda_1 \underline{X}}} \right] \\ B &= \frac{\overline{\phi}}{e^{-\lambda_1 \overline{X}}} - \frac{e^{2\lambda_1 \overline{X}}}{1 - e^{2\lambda_1(\overline{X} - \underline{X})}} \left[\frac{\underline{\phi}}{e^{\lambda_1 \underline{X}}} - \frac{\overline{\phi}}{e^{-\lambda_1 \overline{X}}} e^{e^{-2\lambda_1 \underline{X}}} \right]. \end{aligned}$$

For a fixed $\underline{W}$ the constants $\underline{\phi}$, $\overline{\phi}$ are the only controls that may be exerted on the function $\phi(\cdot)$. These constants must be chosen to guarantee the function $\phi(\cdot)$ is increasing and convex, which amounts to

$$e^{2\lambda_1 X} \geq \frac{B}{A} \quad \text{and} \quad e^{2\lambda_1 X} \geq -\frac{B}{A}. \tag{4.5}$$

Given $\underline{W}$ the pair $(\underline{\phi}, \overline{\phi})$ satisfying these conditions thus characterizes the cost-minimizing intensity ϕ; let it be denoted $\phi^M(X; \underline{\phi}, \overline{\phi})$.

Of course cost minimization is the not goal of the principal; rather maximizing his payoff is. For any function ϕ solving (4.4), modulo the constant cost k, applying Ito's Lemma to

$$\mathbb{E}\left[\int_0^\tau e^{-rs} k \cdot \phi(X_s) ds\right]$$

yields (4.2), which can therefore be used as an envelope condition on F. In a second step, this leaves the simpler problem

$$V(W) := \sup_{c,\tau,\gamma} \mathbb{E}\left[\int_0^\tau e^{-rs}\left(dX_s - dc_s\right)\right], \quad \gamma_t^a \geq \eta, \; W_t > \underline{W}. \tag{4.6}$$

for some $\underline{W}$. Under the measure $\mathbb{Q}$ this value function satisfies the Bellman equation:

$$rV(W_t)dt = \mathbb{E}\left[dX_t\right] + \sup_{c,\tau}\left\{ -dc_t + \mathbb{E}\Big[V_W(W_t)dW_t + \frac{(\gamma_t^a)^2}{2} V_{WW}(W_t)dt\Big]\right\}, \tag{4.7}$$

subject to the incentive compatibility constraints, and where W follows the Law of Motion (3.7). Auditing does enter this problem, however only through the threshold $\underline{W}$ that the principal still has to select. Since ϕ is parametrized by $\underline{W}$, the choice of $\underline{W}$ affects the cost of audit and also the surplus V generated by the relation with the agent via the termination hurdle. The lower that hurdle, the longer the project may be run, but the higher the audit cost. So the optimal contract hinges on the optimal choice of $\underline{W}$. To characterize the function $V(W)$ I first take $\underline{W}$ fixed and rewrite the preceding Bellman equation as

$$rV(W_t)dt = \sup_{c,\tau}\Big\{ - dc_t + \Big[(\rho W + \eta a_t)dt - dc_t\Big]V_W(W_t) + \frac{(\gamma_t^a)^2}{2}V_{WW}(W_t)dt\Big]\Big\}, \quad (4.8)$$

using $\mathbb{E}\left[dX_t\right] = 0$ and the definition (3.7). Then,

Proposition 3 *Fix γ_t^a and ϕ, and suppose σ is not too large so that $\underline{W} < \overline{W}$. The function $V(W)$ is the unique solution to the ODE*

$$rV(W) = (\rho W + \eta a)V_W(W) + \frac{(\gamma_t^a)^2}{2}V_{WW}(W), \quad a \in \{0, A\} \quad (4.9)$$

on the domain $\left[\underline{W}, \overline{W}\right]$, with boundary conditions

$$V(\underline{W}) = \pi - \underline{W}, \qquad V_W(\overline{W}) = -1.$$

$\underline{W}$ is absorbing and $\overline{W}$ is reflecting. The function $V(W)$ is concave in W over $[\underline{W}, \overline{W}]$. The transfer process c is the local time of W and satisfies

$$dc_t = \max\left\{0, W_t - \overline{W}\right\},$$

where the payment barrier $\overline{W}$ is pinned by the super-contact condition $V_{WW}(\overline{W}) = 0$

The lower boundary of $V(W)$ is $\pi - \underline{W}$: the information rent of the agent is paid out by the principal. It is not just a simple transfer to the agent, for it has consequences on the solution V. To determine the threshold $\underline{W}$ it is helpful to first find the optimal sensitivity γ_t^a and the action a. To this end it is more transparent to revert to the measure $\mathbb{P}$, under which (4.8) rewrites

$$\begin{aligned} rV(W_t)dt &= \sup_{c,\tau}\Big\{\mu + a - dc_t + \Big[\rho W + \eta a_t - dc_t\Big]V_W(W_t) \\ &+ \frac{(\gamma_t^a\sigma)^2}{2}V_{WW}(W_t)dt\Big]\Big\}, \end{aligned} \quad (4.10)$$

Proposition 4 *It is always optimal for the principal to induce effort and to set the cash flow sensitivity as low as possible: $a = A$ and $\gamma_t^a = \eta$.*

Given $\eta \leq 1 < \; a_t = A$ is optimal; it Pareto-dominates and the gain can be efficiently shared through transfers. Conditional on inducing effort, concavity of the value function immediately implies the sensitivity γ_t^a should be set to the lowest level possible – that is, η. To see that inducing effort is optimal, consider the *direct* marginal impact of effort on the payoff of the principal: from the right-hand side of (4.10), $1+\eta V_W$, with $V_W \geq -1$ and $\eta \leq 1$, so clearly $1+\eta V_W \geq 0$ (strictly for $W < \underline{W}$). There is an indirect effect too through $\underline{W}$ that is more difficult to characterize. However it is easy to see that if the principal prefers not inducing effort he can do so by simply offering the agent a flat wage. In this case misreporting has no object, and neither does the information rent $\underline{W}$.

A further consequence of Proposition 4 is that the boundary $\underline{W} := \underline{W}(\gamma^a, \phi)$ is time invariant (since γ^a is constant) – but it does depend on the action the principal chooses to induce. Since $\underline{W}$ is time invariant, so is the upper boundary $\overline{W}$.

Because both γ^a and a are constant, the problem is clearly stationary. With this information about the value function one can turn to the task of determining the optimal termination barrier $\underline{W}$ and the optimal audit intensity ϕ (through the constants $\underline{\phi}, \overline{\phi}$). To do so it is helpful to call on the stochastic representation of the solution V (Feynman-Kac formula) under the measure $\mathbb{Q}$. For any $\underline{W}$, let

$$\tau(\underline{W}) := \inf\left\{t \geq 0 | W_t = \underline{W}\right\},$$

denote the first passage of time at $\underline{W}$ *from above.* Then use the representation

$$V(W) = \frac{1}{r}\mathbb{E}_{\mathbb{Q}}\left[\int_0^\tau e^{-rt}\left(-dc_t + V_W(W)(\rho W - dc_t) + \frac{(\gamma^a)^2\sigma^2}{2}V_{WW}(W)\right)\right], \tag{4.11}$$

to establish that

$$\frac{\partial}{\partial \underline{W}}V(W) = -V_W(\underline{W})\mathbb{E}_{\mathbb{Q}}\left[e^{-r\tau(\underline{W})}\right]. \tag{4.12}$$

On the equilibrium path the Law of Motion (3.7) of W is a transient Ornstein-Uhlenbeck process (since the parameter ρ is positive); the Laplace transform $\mathbb{E}_{\mathbb{Q}}\left[e^{-r\tau(\underline{W})}\right]$ of the stopping time $\tau(\underline{W})$ admits a known representation:

$$\mathbb{E}_{\mathbb{Q}}\left[e^{-r\tau(\underline{W})}\right] = e^{\rho(W_0^2-\overline{W}^2)}\frac{H_{-r/\rho}(W_0\sqrt{\rho})}{H_{-r/\rho}(\underline{W}\sqrt{\rho})},$$

for some initial condition W_0. The function $H_\theta(\cdot)$ is the Hermite function with parameter θ:

$$H_\theta(u) = \frac{2^{1+\theta}}{\Gamma((1-\theta)/2)}\int_0^\infty e^{-s^2}s^{-\theta}(s^2+u^2)^{\theta/2}ds.$$

Then it is immediate that

$$\mathbb{E}_{\mathbb{Q}}\left[e^{-r\tau(\underline{W})}\right] = e^{\rho(W_0^2-\overline{W}^2)}\frac{\int_0^\infty e^{-s^2}s^{r/\rho}(s^2+W_0^2\rho)^{-r/2\rho}ds}{\int_0^\infty e^{-s^2}s^{r/\rho}(s^2+\underline{W}^2\rho)^{-r/2\rho}ds}$$

is decreasing in $\underline{W}$, so that by (4.11) and (4.12) the solution $V(W)$ is decreasing in the termination barrier $\underline{W}$. To pin the solution, one may exploit the fact that the problem is now time-invariant; it reduces to a static parametric optimization problem. Construct the Lagrangean

$$\begin{aligned}\Lambda(\underline{W},\underline{\phi},\overline{\phi}) \quad &:= \quad V(W;\underline{W}) - k\cdot\mathbb{E}\left[\phi^M(X;\underline{W},\underline{\phi},\overline{\phi})\right] \\ &+ \quad \nu_0\left[\phi(Y^*)\underline{W} - \gamma^a[dY^*(\underline{W}) - d\underline{X}]\right] + \nu_1\phi_X^M(X;\underline{\phi},\overline{\phi}) + \nu_2\phi_{XX}^M(X;\underline{\phi},\overline{\phi})\end{aligned} \tag{4.13}$$

to be maximized by choice of $\underline{W},\underline{\phi},\overline{\phi}$, and where ν_0,ν_1,ν_2 are Lagrange multipliers. Clearly one of either ν_1 or ν_2 is zero given the conditions laid out in (4.5). A solution to this problem exists. First the constraint set is never empty; second, the objective function is continuous and defined on $[0,\overline{W}]\times\mathbb{R}\times\mathbb{R}$, where however $\underline{\phi}$ and $\overline{\phi}$ are known to be finite. Last, this objective function is continuous in $\underline{W}$ and differentiable with respects to all the instruments. Denote the solution to Problem (4.13) by $(\underline{W}^*,\underline{\phi}^*,\overline{\phi}^*)$.

Proposition 5 *The optimal contract features random audit with penalties $P_t = W_t$ and termination at $\underline{W}(\gamma^a,\phi^*)$. The intensity of audit ϕ^* and the optimal termination barrier $\underline{W}^*$) are jointly determines as solutions to the maximization problem (4.13) and the fixed-point condition (3.6).*

At the risk of repetition, the termination barrier $\underline{W} := \underline{W}(\gamma^a,\phi^*)$ is *endogenous* in this problem; it is determined as part of the optimal contract and depends on the severity of the moral hazard problem (through $\gamma^a = \eta$). Indeed, in equilibrium

$$\underline{W}(\gamma^a,\phi^*) = \underline{W}(\eta,\phi^*)$$

which makes it transparent that the information revelation problem, and the information rent, are borne out of the effort incentive problem. The use of penalties follows from Proposition 1 and 2.

The solution to (4.13) trades-off the expected cost of audit with the benefit of that audit. The benefit is to decrease the termination threshold $\underline{W}$ (Proposition 1), which (i) limits the transfer $\underline{W}$ to the agent upon termination and (ii) extends the duration of the project. A project that is inherently more valuable triggers more frequent audits, precisely to postpone termination as long as possible. Likewise, a project with a value that is very responsive to the continuation utility of the agent is the subject of more frequent audit; as an imperfect analogy, its option value is large around $\underline{W}$, and so its termination should be postponed. It is immediate from (4.13) that the auditing intensity ϕ must decrease as its cost k increases.

While a more frequent, or cheaper, audit postpones termination, it also precipitates releasing cash to the agent.

Corollary 1 *Let $\underline{W}$ parametrize the function $V(W;\underline{W})$;*

$$\frac{d\overline{W}}{d\underline{W}} > 0$$

Even though delaying termination is socially valuable, having $r < \rho$ still implies that payment cannot be arbitrarily postponed. That is, the time horizon until payment cannot be stretched. Starting from a lower boundary $\underline{W}' < \underline{W}$, it necessarily implies the corresponding $\overline{W}'$ is also lower then the original $\overline{W}$.

5 Implementation and corporate finance

In line with the motivating example the contract may be implemented in the context of executive compensation using standard securities and a severance payment upon termination. The "principal" is a metaphor for a population of diffuse outside investors. Any collective action problem between these investors is abstracted from. Our main result in this section is subtle; it pertains to the default risk. This subtlety shows that understanding frictions in the details matters for correctly assessing and pricing risk. Throughout this section the measure $\mathbb{P}$ is used.

Securities. The agent is awarded a fraction η of the equity of the firm, whereas the balance $1-\eta$ is held by (diffuse) shareholders. The agent cannot sell her shares; this enforces the commitment assumption. Let M_t denote the book value of equity. The law of motion of M is then

$$\begin{aligned} dM_t &= rM_t dt + d\widetilde{X}_t^a - dc_t - dI_t, \quad M_0 = m > 0 \\ &= (rM_t + \mu(a))dt + \sigma dZ_t^a - dc_t - dI_t, \end{aligned} \tag{5.1}$$

where c is the payment stream to the agent and I that to the investors. Already one can see that truthful revelation is important: dM_t is the true law of motion of book equity only if $d\widetilde{X}_t^a = dX_t^a$. Define $W_t = \eta M_t$; this is the stake of the agent, who purchases – either for cash or against a loan – a fraction η of the project (the firm). Combining with the law of motion of W, one has an equivalent representation of dM_t that tracks the law of motion of

W on the equilibrium path:

$$dM_t = \rho M_t dt + \sigma dZ_t^a - \frac{1}{\eta} dc_t, \tag{5.2}$$

Thus the value M of the book value of equity can be used to represent the continuation value of the agent, and decisions can be made using M as an equivalent state variable – as long as the agent reports information truthfully. The payment c is activated when $M_t = \overline{M} := \overline{W}/\eta$, and termination is triggered as soon as $M_t = \underline{M} := \underline{W}/\eta > 0$. The lower bound $\underline{M}$ is strictly positive: a poor-performing CEO is terminated before the firm vanishes, otherwise she cannot be compelled to truthfully inform investors, nor to exert any costly effort. The financiers act prudently and do not allow the continuation utility to enter a territory where the behavior of the CEO can go unchecked. In what follows I set $\pi \equiv 0$ to ease comparisons with other results, in particular those of Biais et al. (2007).

The firm issues debt in the form of bonds at date 0 and continuously pays a coupon $\mu(a) - (\rho - r)M_t$ regardless of the cashflow realization. Equations (5.1) and (5.2) can only hold if distributions to the collection of investors (stockholders and debt-holders) are

$$dI_t = [\mu(a) - (\rho - r)M_t]dt + \frac{1-\eta}{\eta} dc_t. \tag{5.3}$$

The equity only pays when c is activated; its value is the NPV of the dividend stream dc_t

$$S_t = \mathbb{E}\left[\int_t^\tau e^{-r(s-t)} \frac{dc_s}{\eta} \middle| \mathcal{F}_t\right], \tag{5.4}$$

where $S_t := s(m)$ is the solution to the problem

$$rs(m) = \rho m s'(m) + \frac{\sigma^2}{2} s''(m), \quad s(\underline{M}) = 0, \; s'(\overline{M}) = 1$$

on $[\underline{M}, \overline{M}]$ and that of the debt is

$$D_t = \mathbb{E}\left[\int_t^\tau e^{-r(s-t)} \mu(s) - (\rho - r) M_s ds \middle| \mathcal{F}_t\right], \tag{5.5}$$

solving, on the same interval, $D_t := d(m)$

$$rd(m) = \mu(a) - (\rho - r)m + \rho m d'(m) + \frac{\sigma^2}{2} d''(m), \quad d(\underline{M}) = (1-\eta)\underline{M}, \; d'(\overline{M}) = 0$$

The notable aspect of these solutions is that they terminate at $\underline{M} > 0$. Here the debt-holders (optimally) receive the liquidation value at termination; this is captured by the boundary condition $D(\underline{M}) = (1-\eta)\underline{M}$. That liquidation value is their share of the book value of the firm at the termination threshold. In contrast stock holders receive nothing upon termination. This maximises the ex ante value of the debt, and therefore how much may be borrowed in the first place. Note also that $V(\underline{W}) = 0$ here (since $\pi = 0$ in this section), but $d(\underline{M}) > 0$. That is, at $\underline{W}$ the continuation value $V(\underline{W})$ of the firm is zero, but its liquidation value is positive (to debt-holders).

The quantities S, D and M satisfy the equality

$$V(W_t) + (1-\eta)M_t = (1-\eta)S_t + D_t,$$

which differs from Biais et al. (2007) as only a fraction $1-\eta$ of the book value M accrues to financiers, while a fraction η is set aside to meet the promise made to the agent to pay out the continuation $\underline{W}$ at termination.[9]

Default risk. One avenue to measure default risk, or more precisely to assess the expected cost of default risk, is to value the credit risk spread that is implied by holding the debt indefinitely. At each instant $t \in [0, \tau)$, this is implicitly defined by the relation

$$\int_t^\infty e^{-(r+\Delta_t)(s-t)} ds = \mathbb{E}\left[\int_t^\tau e^{-r(s-t)} ds \middle| \mathcal{F}_t\right] + (1-\eta)\underline{M}e^{-r(\tau-t)} \tag{5.6}$$

where $(1-\eta)\underline{M}e^{-r(\tau-t)}$ is the fraction of the book value (i.e. of the break up value) accruing to financiers. By Problems (5.4) and (5.5), all if it is awarded to debtholders. Noting that

$$e^{-r(\tau-t)} = 1 - r\int_t^\tau e^{-r(s-t)} ds,$$

Equation (5.6) implies

Proposition 6 *The credit default spread is given by*

$$\Delta_t = \frac{rL_t - (1-\eta)\underline{M}re^{-r(\tau-t)}}{(1-L_t) + (1-\eta)\underline{M}re^{-r(\tau-t)}}, \quad L_t = \mathbb{E}\left[e^{-r(\tau-t)} ds \middle| \mathcal{F}_t\right], \tag{5.7}$$

and it is lower than under a standard contract where $\underline{M} = 0$. *The credit spread* Δ_t *is a decreasing and convex function of* $\underline{M}$, *and is strictly positive.*

Somewhat paradoxically, a worse contracting environment ($\underline{M} > 0$ to elicit information revelation) renders the debt *less* risky – in the sense of a lower credit spread. The reason is that the contract requires termination at positive values of $\underline{M}$; this implies a positive book value of equity upon liquidation, which can be liquidated for cash and returned to the financiers. Thus by solving its governance problem, the firm lowers its financing cost. This

[9] Most of the results they show also hold here; they are therefore not repeated.

result is borne out in the data. For example Brown et al. (2008), Brown et al. (2012) study governance problems in hedge funds and find that a fund that is perceived by investors to have good governance has a lower cost of funds. This silver lining has limitations though in that

Corollary 2 *For any level of M_t the credit spread Δ_t increases in the effort cost η and in the audit cost k. It increases faster than if $\underline{M} = 0$.*

The first part of this result can also be found in Biais et al. (2007). Here it is worsened by the problem of information revelation; that is, the credit spread Δ_t grows faster in this model than when $\underline{M} = 0$ is sufficient, even though the term $(1 - \eta)\underline{M}re^{-r(\tau - t)}$ initially depresses it (see Equation (5.7)). It is precisely this new term that accelerates the increase in credit spread. While the quantity $\underline{M} := \underline{M}(\eta, k)$ is increasing in η, the share $(1 - \eta)$ left to the financiers decreases, and the termination time τ necessarily arrives sooner. These latter effects countervail the former.

These two results combined seem puzzling. However one should recall that the credit yield spread is a measure of the cost of expected losses. At best it is a summary statistic of the probability of a loss, together with the magnitude of a loss conditional on the event occurring. Here conditional on termination debt holders receive some proceeds; this mitigates the loss upon default and so decreases the spread. However, as the governance problem worsens (higher η and/or higher k), termination is bound to arrive sooner – its probability increases. This increases the spread.

6 Discussion

This part suggests four points for a brief discussion.

The audit ignores history. In this model the audit technology captures contemporaneous departures from X_t; at time t it is blind as to the history of report $\widetilde{X}_s, s < t$. In practice auditors typically look backwards as well as examining contemporaneous information, and so may have access to more information than the technology used here. In extending the auditing horizon one must be careful to not render the problem trivial: if the principal can perfectly observe past history, it is as if observing the process itself. On the other hand, for a finite "memory" of the audit process, the audit decision of the principal becomes complicated. That decision becomes akin to choosing a time interval for review; should it follow a period of high reports? If so, a sequence of high reports may be tempered by some bad ones; so we see there is tremendous freedom for information manipulation. This is an interesting problem that is left for future research. One could conceive of an imperfect audit technology that may be improved at a cost; the trade-off between that cost and the benefit of auditing is essentially already captured by Proposition 4.

Outside option. The outside option (say, $\overline{U}$) of the agent is 0 in this model. As long as $\overline{U} < \underline{W}$ nothing changes. If $\overline{U} \geq \underline{W}$ the very constraint $W \geq \underline{W}$ becomes irrelevant: it is implied by the participation constraint of the agent. A wealthy agent, or an agent with good outside opportunities, quits before the incentive constraint $W_t \geq \underline{W}$ induces termination. In that case one has $\underline{W} \leq \overline{U}$ and the agent no longer receives an information rent.

Contractual forms. Zhu (2013) extends the work of Sannikov (2008) by noting that, depending on the value of effort, the principal may prefer either rewarding (at $\overline{W}$) or punishing (at $\underline{W}$) the agent by suspending the contract at these boundaries. The present paper abstracts from these considerations to focus on the new problem of information revelation. I conjecture that much the same dynamics would arise at both barriers because the point of suspension is to ignore the information provided by the Brownian process. In this model it implies ignoring the reports $\widetilde{X}$ of the agent at the boundaries $\underline{W}$ or $\overline{W}$; these reports therefore do not influence W at these bounds (and so there cannot be any crossing). If the principal ignores the reports, the agent has no incentive to misreport. Thus, if the "Quiet Life" and "Baseline Renegotiation" contracts are ever optimal under observable X, it is reasonable to conjecture they also are when X is not observed.

Sampling for information versus information revelation. In this paper the agent is asked to communicate information to the principal. This premise is in part motivated by practice: a CEO reports information to shareholders, who may choose to verify it. It also maps into the canonical mechanism design construction. Alternatively the principal may not elicit communication from the agent but instead directly sample information. This eliminates the need for an information rent $\underline{W}$ but introduces a new friction instead and new technical considerations.

The new friction stems from the very sampling problem: upon sampling outcomes

X_s, X_t, X_u... the principal wants to infer the action a of the agent. In doing so he necessarily makes mistakes and may incorrectly terminate the agent. The technical difficulties are the mixing of continuous and discrete processes, and the loss of the Markovian property of the value function of the principal.[10] Goldys and Roger (2018) study a (different) model where these problems arise.

7 Conclusion.

In this paper a principal seeks to induce costly effort on the part of an agent, when in addition he does not observe the process resulting from the action of the agent. Instead the principal relies on information that is transmitted by the agent, and that is therefore subject to manipulation. Observability is restored through a combination of audit and penalties. This environment is a stylised version of the relation between a CEO or entrepreneur and her shareholders/investors.

The only feasible penalty is termination, and to have bite it must occur when the agent still has a stake in the project (or the firm). That is, it occurs at a positive value of the continuation of the agent. Then truthful revelation holds for all state realizations. Truthful revelation is also necessary for the provision of effort; without it, the agent can manipulate the information to look as if she were diligent. Truthful revelation is socially costly in that it requires early termination (above zero continuation utility) in equilibrium.

The optimal contract always induces effort, which is socially valuable, always elicits truthful revelation and trades off the cost of an audit with its marginal benefit. That marginal benefit is the value of keeping the project going one more instant; that is, its drift.

The contract is implemented using standard securities: equity, debt and a termination clause. This termination clause has intricate effects on the financing costs. On the one hand, it reduces the implied credit yield spread because the positive termination value implies some recovery by creditors. On the other hand, the credit spread increases more sharply when governance problems worsen. This points to a need to understand what precisely drives the cost of financing.

[10]A time series of observations is required for inference, so one must keep track of history; the current state is not sufficient.

Appendix

A Additional material

A.1 The Poisson random measure.

This section of the Appendix contains details of the construction of the Poisson Random Measure (PRM). It chiefly relies on the following theorem, stated without proof (see, Cinlar (2011) for example). A Radon measure is a generalization of the Lebesgue measure.

Definition 1 *Let $(E, \mathcal{B})$ be a measurable space with $E \subset \mathbb{R}^d$. A Radon measure on $(E, \mathcal{E})$ is a measure ν such that for all compact subsets $\mathcal{B} \in \mathcal{E}$, $\nu(\mathcal{B}) < \infty$.*

Examples of a Radon measure include the Lebesgue measure as a special case; it differs from the Lebesgue measure in that it need not have measure zero on a single point. The Dirac measure is a Radon measure. Importantly for our purposes, a probability measure is a Radon measure.

Theorem 1 *For any Radon measure ν on a measurable space E with σ-algebra $\mathcal{E}$, there exists a unique Poisson Random Measure $M(E)$ with intensity ν.*

To construct the PRM,

1. take iid random variables $\Delta Y_i = Y_t - Y_s$ so that $\Pr(\Delta Y_i \in A) = \frac{\nu(A)}{\nu(E)}$ for $A \subset E \subseteq \mathbb{R}$. Set $\frac{\nu(A)}{\nu(E)} = f(Y_t)$, the Beta density;

2. let $M(E)$ be a Poisson random variable on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with mean $\nu(E)$;

3. define $M(A) := \sum_{i=1}^{M(E)} \mathbb{I}_A(Y_i), \ \forall A \in \mathcal{E}$;

4. scale by a factor λ as required.

The point of the construction is that the properties of the audit probability can be condensed in the PRM, including the dependence of the audit probability to variables of the environment. With this PRM the law of motion of the continuation utility W of the agent can be correctly derived for any probability of audit.

B Proofs

Proof of Lemma 1: The martingale

$$U_t(c,a) = \int_0^t e^{-\rho s}(dc_s - \eta a_s) + \mathbb{E}\left[\int_t^\infty e^{-\rho(s-t)}(dc_s - \eta a_s)\middle|\mathcal{F}_t\right]$$
$$= \int_0^t e^{-\rho s}(dc_s - \eta a_s) + e^{-\rho t}W_t,$$

has equivalent representation

$$U_t(c,a) = U_0 + \int_0^t e^{-\rho s}\gamma_s \sigma dZ_s - \int_0^t e^{-\rho s}P_s\left[dM_s - \phi ds\right],$$

where the processes γ_t and P_t are known to exist (given the measure M) by application of the Martingale Representation Theorem, and M is known to exist by application of Theorem 1. To obtain the law of motion of the process W, equate the two, differentiate w.r.t. t and re-arrange. ■

Proof of Proposition 1: First it must be shown that dealing with the Law of Motion (3.2) is equivalent to solving (3.3); that is, finding the best report dY_t in (3.3) also maximizes the change in W. Fix the action a and the continuation utility $W_t > 0$, and change from measure $\mathbb{P}$ to $\mathbb{Q}$. Let U_t^Y denote the payoff to the agent under message Y_t:

$$U_t^Y = U_t^X + \int_0^t e^{-\rho s}\gamma_s^a(dY_s - dX_s) - \int_0^t e^{-\rho s}P_s\phi_s ds,$$

for some penalty process P; this is tantamount to (3.3).

The incentives at the margin are given by Condition (3.4) binding at zero; denote the solution by $dY_t^*(W_t)$. But this is not quite sufficient; one needs

$$\forall dZ_t, \quad \gamma_t\sqrt{(dY_t^* - dX_t)^2} \leq \phi \cdot W_t.$$

In other words, the benefit function and the cost function may cross; whether they do determines the incentives to misreport. When the above-mentioned condition is satisfied they either are just tangent (at the optimal message dY_t^*) or they do not cross at all. Because $\phi(\cdot)$ must be convex, and $\gamma_t(dY_t - dX_t)$ is linear, if these functions do cross, they cross twice. There are two cases to distinguish, given W_t and dX_t:

1. They cross twice. In this case, there exist two values $dY^0 < dY^1$ of dY_t such that $\gamma_t\sqrt{(dY^i - dX_t)^2} = \phi(Y^i) \cdot W_t,\ i = 0, 1$ and $\gamma_t > \phi(Y^0)_{Y^0} \cdot W_t$ but $\gamma_t < \phi(Y^1)_{Y^1} \cdot W_t$. There also exists a value $dY_t^*(W_t)$ such that (3.4) binds at 0.

2. They never cross. Then either

   (a) they are just tangent. Then there exists the same value $dY_t^*(W_t)$ such that (3.4) binds at 0 and simultaneously $\gamma_t\sqrt{(dY_t^* - dX_t)^2} = \phi(Y_t^*) \cdot W_t$. The agent is indifferent between truth-telling and misreporting, and has no incentives to change her misreporting.

   (b) they are never tangent. Then there exists no value of dY_t such that (3.4) binds; in addition, $\gamma_t\sqrt{(dY_t^* - dX_t)^2} < \phi(Y_t^*) \cdot W_t$.

Case $2a$ is "ideal": the first-order condition (3.4) defines $Y^*(W_t)$ and the condition $\gamma_t\sqrt{(dY_t^* - dX_t)^2} = \phi(Y_t^*) \cdot W_t$ pins $\underline{W}(dZ_t)$ exactly, given dZ_t. It is immediate to see that it is sufficient for this condition to hold at $d\underline{Z}_t$ to deter misreporting for all dZ_t. Let $\underline{W}(\gamma_t^a, \phi)$ be defined by $\gamma_t\sqrt{(dY_t^* - dX_t)^2} = \phi(Y_t^*) \cdot W_t$. Re-arranging, the condition

$$\underline{W} = \frac{\gamma_t^a}{\phi(Y_t^*)}\sqrt{(dY^*(\underline{W}) - \sigma d\underline{Z}_t)^2} =: \varphi(\underline{W})$$

defines $\underline{W}$ as a fixed point of the mapping φ. This fixed point exists and is unique since first,

$$\frac{d\varphi(\underline{W})}{d\underline{W}} = \frac{\gamma_t}{\phi(Y_t^*)}\frac{dY_t^*(\underline{W})}{d\underline{W}}\left[1 + \frac{\phi_Y(Y_t^*)}{\phi(Y_t^*)}\sqrt{(dY^*(\underline{W}) - \sigma d\underline{Z}_t)^2}\right],$$

where $\frac{\phi_Y(Y_t^*)}{\phi(Y_t^*)} \geq 0$ by convexity of $\phi(\cdot)$; hence the sign depends on that of $\frac{dY_t^*}{dW_t}$ only. Second, from the FOC (3.4), the derivative $\frac{dY_t^*}{dW_t}$ exists by the Theorem of the Maximum, which allows the application of the implicit function theorem, and

$$\frac{dY_t^*}{dW_t} = -\frac{1}{W_t}\frac{\phi_{Y_t}}{\phi_{Y_tY_t}} < 0,$$

again, by convexity, so that,

$$\frac{d\varphi(\underline{W})}{d\underline{W}} < 0 < 1$$

necessarily, and one must conclude φ is a contraction.

In the third case one simply has $W_t > \underline{W}$; the condition is simply slack. In the first case $W_t < \underline{W}$. Then the principal needs to enforce $W_t \geq \underline{W}$; since $\underline{W} := \inf W_t$, one then reverts to case 2a. To enforce this condition, the principal terminates the agent at $\underline{W}$. Finally to establish that $\underline{W}$ is necessarily bounded, note that $\phi(\underline{Y}_t) > 0$ and that $\phi(\cdot)$ is increasing. It is also convex, therefore Y_t^* is always finite. Hence the RHS of (3.5) is bounded.

To show termination must occur as soon as $W_t = \underline{W}$, consider the following Lemma.

Lemma 2 *The distributions of the first-visitation and the first-crossing times*

$$\tau_v := \inf \left\{ t \geq 0 | W_t = \underline{W} \right\} \quad \text{and} \quad \tau_c := \inf \left\{ t \geq 0 | W_t < \underline{W} \right\},$$

respectively, are identical.

Proof: Away from the boundary $\underline{W}$, the agent's continuation utility evolves according to

$$dW_t = [\rho W_t + \eta a_t]dt - dc_t + \gamma_t^a dY_t$$

under the measure $\mathbb{Q}$, and where dY_t need not equal dX_t. Let us assume that for some date $\bar{t}$ it holds that $W_{\bar{t}} = \underline{W}$ and that at that point there is no termination. Then, instantaneously the dynamics of W are

$$dW_t = [\rho \underline{W} dt + \eta a_t]dt - dc_t + \gamma_t dY_t, \quad W_{\bar{t}} = \underline{W}.$$

Let us consider the auxiliary process defined via the equation

$$db_t = \rho dt + \gamma_t dY_t, \quad b_{\bar{t}} = 1$$

and define, for $\epsilon > 0$,

$$B(\epsilon) := \inf_{s \in [\bar{t}, \epsilon]} \left\{ b_s \mid W_{\bar{t}} = 1 \right\}.$$

Using the Cameron-Martin theorem we know there is an equivalent measure $\widetilde{\mathbb{Q}}$ under which b is a standard Brownian Motion. From Chesney et al. (2009), page 147, we have that for any $\alpha \leq 1$,

$$\widetilde{\mathbb{Q}}\left\{ B(\epsilon) > \alpha \right\} = \Phi\Big(\frac{-(\alpha - 1) + \rho\epsilon}{\gamma_t \sqrt{\epsilon}} \Big) - e^{2(\alpha-1)\rho} \Phi\Big(\frac{(\alpha - 1) + \rho\epsilon}{\gamma_t \sqrt{\epsilon}} \Big), \tag{B.1}$$

where Φ is the standard normal cumulative distribution function. Letting $\alpha \to 1$ we obtain that for all $\epsilon > 0$

$$\widetilde{\mathbb{Q}}\left\{ B(\epsilon) > 1 \right\} = 0,$$

which, as $\mathbb{Q}$ and $\widetilde{\mathbb{Q}}$ are equivalent, implies $\mathbb{Q}\left\{ B(\epsilon) > 1 \right\} = 0$. Therefore, for all $\epsilon > 0$ it holds that

$$\mathbb{Q}\Big\{ \inf_{[\bar{t}, \bar{t}+\epsilon]} \left\{ W_s | W_{\bar{t}} = \underline{W} \right\} > \underline{W} \Big\} = 0,$$

which concludes the proof of the Lemma. ■

This finally concludes the proof of the Proposition. ■

Proof of Proposition 2: Under measure $\mathbb{Q}$ the law of motion of the continuation utility of the agent write

$$dW_t^a = [\rho W_t + \eta a_t]dt - dc_t + \gamma_t dY_t - W_t[dM_t - \phi dt].$$

Suppose $W_t \geq \underline{W}$, so that $dY_t = dX_t$, then the law of motion becomes

$$dW_t^a = [\rho W_t + \eta a_t]dt - dc_t + \gamma_t dX_t.$$

On the premise that the principal wants to induce effort ($a = A$) – that is, under the measure $\mathbb{Q}$, in this expression $\mathbb{E}[dX_t] = 0$, only if actually $a = A$. If $a = 0$, $\mathbb{E}[dX_t] = -A_t$ and W has motion

$$dW_t^{a=0} = [\rho W_t + (\eta - \gamma_t)a_t]dt - dc_t + \gamma_t dY_t - W_t[dM_t - \phi dt].$$

Therefore, to maximize the expected change in her continuation utility the agent selects $a = A > 0$ if and only if $\gamma_t \geq \eta$.

For the second statement, suppose truthful revelation does not hold: $W_t < \underline{W}$ and there is no termination. The agent can replicate

$$dW_t^a = [\rho W_t + \eta A_t]dt - dc_t + \gamma_t dX_t - W_t[dM_t - \phi dt]$$

by selecting $a = 0$ but by reporting $dY_t = dX_t + \gamma_t A dt$, while $\mathbb{E}_{\mathbb{Q}}[dX_t] = -A_t dt$. ■

Proof of Proposition 3: Fix γ_t^a, λ and a, and suppose $\underline{W} < \overline{W}$; on $\left(\underline{W}, \overline{W}\right)$ there is no termination and $dc_t \equiv 0$, which immediately yields the differential equation (4.9). Under the measure $\mathbb{Q}$ that differential equation is homogenous, so the solution $V(W)$ is also a solution of the homogeneous equation

$$rh(W) = [\rho W + \eta a]h'(W) + \frac{\gamma^2 \gamma_t^2}{2} h''(W). \tag{B.2}$$

Let us denote by h_0 and h_1 the particular solutions to Equation (B.2) that satisfy $h_0(\underline{W}) = 1, h_1(\underline{W}) = 0, h_0'(\underline{W}) = 0$ and $h_1'(\underline{W}) = 1$. Using these *basis functions* we may write

$$V(W) = b_0 h_0(W) + b_1 h_1(W),\ W \in (\underline{W}, \overline{W}),$$

for some $\overline{W} > \underline{W}$ (at this point $\overline{W}$ need not be optimal). To determine b_0 and b_1 I use the boundary conditions $V(\underline{W}) = \pi - \underline{W} \equiv B \geq 0$ (where $\underline{W}$ is fixed for now) and $V_W(\overline{W}) = -1$:

$$b_0 h_0(\underline{W}) + b_1 h_1(\underline{W}) = B \Rightarrow b_0 = B \quad \text{and}$$

$$b_0 h_0'(\overline{W}) + b_1 h_1'(\overline{W}) = -1, \Rightarrow b_1 = -\frac{1}{h_1'(\overline{W})}\left[1 + B h_0'(\overline{W})\right].$$

Therefore, using $\overline{W}$ as a parameter,

$$V(W;\overline{W}) = Bh_0(W) - \left[1 + Bh_0'(\overline{W})\right]\frac{h_1(W)}{h_1'(\overline{W})}, \tag{B.3}$$

which satisfies the boundary condition $V(\underline{W};\overline{W}) = B$. Next one needs to show that for $\underline{W}$ given, there exists a unique $\overline{W} \geq \underline{W}$ and a unique corresponding function $V(\cdot;\overline{W})$ such that $V_{WW}(W,\overline{W};\overline{W}) = 0$. To this end I show that the function $h_1(\cdot)$ is strictly increasing for all $W \geq \underline{W}$. Indeed, if this were not the case, there would exist some $\hat{W}$ such that $h_1'(\hat{W}) = 0$ and $h_1'(W) \leq 0,\ W \in (\hat{W}, \hat{W} + \epsilon)$ for some $\epsilon > 0$. In other words, $\overline{W}$ would be a local maximum of $h_1(\cdot)$; thus, $h_1''(\hat{W}) \leq 0$. From the latter and Equation (B.2) one obtains that $h_1(\hat{W}) \leq 0$. However, by construction $h_1(\cdot)$ is strictly increasing on $[\underline{W}, \hat{w})$, so that $h_1(\hat{W}) > h_1(\underline{W}) = 0$. This is a contradiction so it must be that $h_1'(W) > 0$ for all $W \geq \underline{W}$.

Next I show that the $\overline{W}$ that satisfies $V_{WW}(X,\overline{W};\overline{W}) = 0$ and the corresponding function $V(X,\cdot;\overline{W})$ are jointly unique. Define $\psi(W) := h_0(W)h_1'(W) - h_1(W)h_0'(W)$ and observe that $\psi(\underline{W}) = 1$. Using the boundary condition $V_W(\overline{W};\overline{W}) = -1$,

$$\begin{aligned}
\frac{\gamma^2\sigma^2}{2}V_{WW}(\overline{W}) &= rV(\overline{W}) + [\rho\overline{W} + \eta a] \\
&= [\rho\overline{W} + \eta a] + rB\left(\frac{h_0(\overline{W})h_1(\overline{W}) - h_1(\overline{W})h_0'(\overline{W})}{h_1'(\overline{W})}\right) - r\frac{h_1(\overline{W})}{h_1'(\overline{W})} \\
&= [\rho\overline{W} + \eta a] + rB\frac{\psi(\overline{W})}{h_1'(\overline{W})} - r\frac{h_1(\overline{W})}{h_1'(\overline{W})}
\end{aligned} \tag{B.4}$$

where the second line follows from substituting Equation (B.3) and the third one from a simple rearrangement of terms. Now, the boundary-value problem

$$\begin{aligned}
\psi'(W) &= h_0(W)h_1''(W) - h_1(W)h_0''(W) \\
&= \frac{2[\rho W + \eta a]}{(\gamma\sigma)^2}\left[h_1(W)h_0'(W) - h_1'(W)h_0(W)\right] \\
&= -\frac{2[\rho W + \eta a]}{(\gamma\sigma)^2}\psi(W), \qquad \psi(\underline{W}) \text{ given}
\end{aligned} \tag{B.5}$$

where the second line uses Equation (B.2), together with the boundary condition $\psi(\underline{W}) = 1$ can be solved in closed form:

$$\psi(W) = \exp\left\{ -\frac{1}{(\gamma\sigma)^2}\left(\rho(W^2 - \underline{W}^2) + 2[(\eta - \gamma_t)a](W - \underline{W})\right)\right\}.$$

Next multiply both sides of Equation (B.4) by $h_1'(W)/\psi(W)$ and re-arrange to obtain

$$\frac{\gamma^2\sigma^2}{2}V_{WW}(W)\frac{h_1'(W)}{\psi(W)} = [\rho W + \eta a]\frac{h_1'(W)}{\psi(W)} - r\frac{h_1(W)}{\psi(W)} + rB.$$

We want to show that $V_{WW}(\overline{W}) = 0$, for which we need that the function

$$\varphi(W) := [[\rho W + \eta a] h_1'(W) - r h_1(W)] \frac{1}{\psi(W)}$$

satisfies $\varphi(\overline{W}) = -rB$. Observe that it must hold that $\varphi(\underline{W}) + rB > 0$, otherwise the principal cannot even hire the agent, and $\varphi(\underline{W}) = [\rho \underline{W} + \eta a]$. Hence, it is enough to show that $\varphi(\cdot)$ is strictly decreasing on $[\underline{W}, \infty)$. Differentiating and using Equation (B.2) one has

$$\begin{aligned}
\varphi'(W) &= e^{\frac{1}{(\gamma\sigma)^2}\left[\rho(W^2 - \underline{W}^2) + 2[(\eta - \gamma_t)a](W - \underline{W})\right]} \\
&\times \Bigg[[\rho W + \eta a] h_1''(W) - r h_1'(W) \\
&+ \frac{2[\rho W + \eta a]}{(\gamma\sigma)^2} ([\rho W + \eta a] h_1'(W) - r h_1(W)) \Bigg] \\
&= -e^{\frac{1}{(\gamma\sigma)^2}\left[\rho(W^2 - \underline{W}^2) + 2[(\eta - \gamma_t)a](W - \underline{W})\right]} r h_1'(W) < 0,
\end{aligned}$$

where the last inequality follows from the fact that $h_1' > 0$. Similarly

$$\begin{aligned}
\varphi''(W) &= e^{\frac{1}{(\gamma\sigma)^2}\left[\rho(W^2 - \underline{W}^2) + 2[(\eta - \gamma_t)a](W - \underline{W})\right]} \left[-r h_1''(W) - \frac{2(\rho W + \eta a)}{\gamma^2 \sigma^2} r h_1'(W) \right] \\
&= -e^{\frac{1}{(\gamma\sigma)^2}\left[\rho(W^2 - \underline{W}^2) + 2[(\eta - \gamma_t)a](W - \underline{W})\right]} r \left[h_1''(W) + \frac{2(\rho W + \eta a)}{\gamma^2 \sigma^2} h_1'(W) \right] \\
&= -e^{\frac{1}{(\gamma\sigma)^2}\left[\rho(W^2 - \underline{W}^2) + 2[(\eta - \gamma_t)a](W - \underline{W})\right]} r \frac{2}{\gamma^2 \sigma^2} h_1(W) < 0
\end{aligned}$$

where the last line uses Equation (B.2) again. Hence $\varphi(\cdot)$ is decreasing and strictly concave on $[\underline{W}, \infty)$, so $\overline{W}$ is unique. The condition $V_{WW}(\overline{W}) = 0$ corresponds to the (optimality) super-contact condition in Dumas (1991). ■

Proof of Proposition 4: The use of penalties and of the termination condition follow immediately from Propositions 1 and 2. Since the solution $V(W)$ is concave in W, $V_{WW} < 0$ and the coefficient γ_t^a should be made as small as possible while still satisfying incentive compatibility – hence $\gamma_t^a = \eta$ to induce $a_t = A$. Under the equivalent measure $\mathbb{P}$ the optimality of high effort is immediate from the marginal condition of the principal

$$1 + V_W(W)\eta \geq 0,$$

since $V_W \geq -1$, $W \leq \overline{W}$ and $\eta \leq 1$. Finally to set the optimal audit intensity ϕ I solve the optimization problem

$$\max_{\underline{W}} \; (\pi - \underline{W}) e^{-\underline{W}\sqrt{2r}} - k\mathbb{E}_{\mathbb{Q}}\left[\phi(Y)\right]$$

with the transformation $\phi = \frac{\gamma_t^a}{\underline{W}}(dY_t^* - \sigma d\underline{Z}_t)$. Since ϕ is increasing convex, the truth-telling condition is slack beyond $d\underline{Z}_t$. Differentiation yields the first-order condition of the

Proposition. To check for a maximiser it is enough to notice that

$$\frac{\gamma_t^a}{\underline{W}^2}[dY_t^* - \sigma d\underline{Z}_t] = \frac{\phi}{\underline{W}^2}\frac{\gamma_t^a}{\phi}[dY_t^* - \sigma d\underline{Z}_t] = \frac{\phi}{\underline{W}^2}\varphi(\underline{W}) =: \psi(\underline{W})$$

and

$$\psi_{\underline{W}}(\underline{W}) = \varphi_{\underline{W}} + \frac{\varphi}{\underline{W}^2}\left[\phi_{\underline{W}}\underline{W}^2 - 2\underline{W}\phi\right] < 0.$$

■

Proof of Corollary 1: Let the parametrized value function $V(W;\underline{W})$ for some $\underline{W}$. At $\overline{W}$ this function satisfies

$$V_W(W;\underline{W}) = -1, \; V_{WW}(W;\underline{W}) = 0$$

so that the HJB equation yields $rV(\overline{W};\underline{W}) + \rho\overline{W} = \mu + a(1-\eta)$. Differentiate with respect to $\underline{W}$:

$$r\left(\frac{dV(\overline{W};\underline{W})}{d\underline{W}} + V_W(\overline{W})\right) + \rho\frac{d\overline{W}}{d\underline{W}} = 0$$

re-arrange using the boundary condition $V_W(\overline{W}) = -1$:

$$\frac{d\overline{W}}{d\underline{W}} = -\frac{r}{\rho - r}\frac{dV}{d\underline{W}}, \text{ where } \frac{dV}{d\underline{W}} = -V_W(\underline{W})E\left[e^{-r\tau}|W_0 = w\right] < 0.$$

■

Proof of Proposition 6: The very definition of Δ_t readily establishes it is an increasing, convex function of the quantity L_t, with $\Delta_t \leq 0$ at $L_t = 0$ – the other terms are constant in L_t. Therefore it is sufficient to show $\mathcal{L} > 0, \mathcal{L}' < 0, \mathcal{L}'' > 0$ over the interval $[\underline{M}, \overline{M}]$. Define $L_t := \mathcal{L}(M_t)$ with

$$r\mathcal{L}(m) = \rho m\mathcal{L}'(m) + \frac{\sigma^2}{2}\mathcal{L}''(m), \quad \mathcal{L}(\underline{M}) = 1, \quad \mathcal{L}'(\overline{M}) = 0$$

and consider a candidate solution to this homogenous equation:

$$\mathcal{L}(m) = a_0H_0(m) + a_1H_1(m), \qquad m \in [\underline{M}, \overline{M}],$$

with the conditions $H_0(\underline{M}) = 1, H_0'(\underline{M}) = 0, H_1(\underline{M}) = 0, H_1'(\underline{M}) = 1$. One checks that the Wronksian product $W_{H_0H_1}(\underline{M}) = H_0(\underline{M})H_1'(\underline{M}) - H_1(\underline{M})H_0'(\underline{M}) = 1 > 0$ so that H_0, H_1 are appropriate basis functions. Using the boundary condition $\mathcal{L}(\underline{M}) = 1$ implies $a_0 = 1$, and using $\mathcal{L}'(\overline{M}) = 0$ yields $a_1 = H_0'(\overline{M})/H_1'(\overline{M})$. Then

$$\mathcal{L}(m) = H_0(m) - \frac{H_0'(\overline{M})}{H_1'(\overline{M})}H_1(m).$$

which immediately yields that $\mathcal{L}(\overline{M}) > 0$ – since $W_{H_0,H_1} >$. Then from

$$r\mathcal{L}(\overline{M}) = \rho m\mathcal{L}'(\overline{M}) + \frac{\sigma^2}{2}\mathcal{L}''(\overline{M}) \text{ and } \mathcal{L}'(\overline{M}) = 0$$

$\mathcal{L}''(\overline{M}) > 0$. Now combining $\mathcal{L}'(\overline{M}) = 0$ together with $\mathcal{L}''(\overline{M}) > 0$ and $\mathcal{L}(\overline{M}) > 0$ imply $\mathcal{L}'(m) < 0$ over at least an interval $(\overline{M} - \epsilon, \overline{M})$, $\epsilon > 0$ but small. Otherwise it would have curvature at $\overline{M}$ and one would have $\mathcal{L}'(\overline{M}) > 0$ as well. Suppose $\mathcal{L}'(\cdot) > 0$, $m < \overline{M} - \epsilon$; let

$$\tilde{m} := \sup\left\{m < \overline{M} - \epsilon \middle| \mathcal{L}'(m) \geq 0\right\}.$$

Since $\mathcal{L}(\overline{M}) > 0$ and $\mathcal{L}'(m) < 0$ over $(\tilde{m}, \overline{M})$, one has $\mathcal{L}(\tilde{m}) > 0$ and $\mathcal{L}'(\tilde{m}) = 0$. Therefore $\mathcal{L}''(\tilde{m}) \leq 0$ as $\tilde{m}$ is a turning point of $\mathcal{L}$. Then

$$r\mathcal{L}(\tilde{m}) = \rho m\mathcal{L}'(\tilde{m}) + \frac{\sigma^2}{2}\mathcal{L}''(\tilde{m}) \leq 0,$$

which is a contraction and one must conclude $\mathcal{L}'(m) < 0$ over the whole interval. Next, since $\mathcal{L}$ is everywhere decreasing in $\mathcal{L}(\overline{M}) > 0$, $\mathcal{L}(m) > 0\ \forall m$ over the interval. Finally from

$$r\mathcal{L}(m) = \rho m\mathcal{L}'(m) + \frac{\sigma^2}{2}\mathcal{L}''(m) > 0\ \forall m$$

one must conclude that $\mathcal{L}''(m) > 0$ as well. ■

Proof of Corollary 2: Inspection of the function Δ_t shows it is strictly increasing in L_t, and $L_t = \mathcal{L}(M_t)$. Thus we need to investigate the behavior of the function $\mathcal{L}$ with respect to η and k, and of the terms $(1-\eta)\underline{M}re^{-r(\tau-t)}$. Unlike in Biais et al. (2007), the function $\mathcal{L}$ depends on η, k through both $\overline{M}$ and directly through $\underline{M}$.

Define the basis function h_1 as in the Proof of Proposition 3; we know then that $\overline{W}$ is the solution to

$$[\rho W h_1'(W) - rh_1(W)]e^{\frac{\rho W^2}{\eta^2\sigma^2}} = \mu(a)$$

so equivalently $\overline{M} = \eta\overline{W}$ is the solution to

$$[\rho\eta M h_1'(\eta M) - rh_1(\eta M)]e^{\frac{\rho M^2}{\eta^2\sigma^2}} = \mu(a).$$

Define $H_1(M) := h(\eta M)$; given the properties of $H_1 : H_1(\underline{M}) = 0, H_1'(\underline{M}) = 1$ one has $H_1 = \eta h_1$ and this rewrites

$$[\rho M H_1'(M) - rh_1(M)]e^{\frac{\rho M^2}{\eta^2\sigma^2}} = \frac{\mu(a)}{\eta}.$$

The LHS is an increasing function of M that is independent of $\mu(a), \eta$, so that when the equation holds (at $\overline{M}$) the quantity $\overline{M}$ must be increasing in $\mu(a)$ and decreasing in η. Next define the function $\psi := \partial\mathcal{L}/\partial\overline{M}$; we know this function satisfies the differential equation and boundary conditions

$$r\psi(m) = \rho m\psi'(m) + \frac{\sigma^2}{2}\psi''(m), \quad \psi(\underline{M}) = 0, \;\; \psi'(\overline{M}) = -\mathcal{L}''(\overline{M}).$$

Immediately one has $\psi'(\overline{M}) < 0$; to show $\psi'(m) \leq 0 \; \forall m$, note that if $\psi'(m) > 0$ for at least some $m \in [\underline{M}, \overline{M}]$, ψ must turn at lest once (since $\psi'(\overline{M}) < 0$). That is, there must be at last some $\hat{m}$ such that

$$r\psi(\hat{m}) = \underbrace{\rho m\psi'(\hat{m})}_{=0} + \frac{\sigma^2}{2}\psi''(\hat{m}) \leq 0;$$

which contradicts that $\psi(m)$ increases from $\underline{M}$. Thus $\psi(m) < 0$, $m \in (\underline{M}, \overline{M}]$; that is, the function $\mathcal{L}$ is strictly decreasing in $\overline{M}$. Finally combine with $\overline{M}$ increasing in $\mu(a)$ and decreasing in η.

To show the role of η and k on $\underline{M}$ – which affects both L_t and $\overline{M}$ – I study the behavior of the function $\mathcal{L}$ as $\underline{M}$ changes. First define the function $\zeta := \partial\mathcal{L}/\partial\underline{M}$, as for ψ at $\overline{M}$; it satisfies the equation and boundaries:

$$r\zeta(m) = \rho M\zeta'(m) + \frac{\sigma^2}{2}\zeta''(m), \quad \zeta(\underline{M}) = -\mathcal{L}'(\underline{M}), \quad \zeta'(\overline{M}) = 0.$$

At $\overline{M}$ these conditions imply

$$r\zeta(\overline{M}) = \frac{\sigma^2}{2}\zeta''(\overline{M}),$$

and either $r\zeta(\overline{M}) = \frac{\sigma^2}{2}\zeta''(\overline{M}) < 0$ so that $\zeta(\overline{M}) < 0$ and ζ is locally concave: $\zeta''(\overline{M}) < 0$, and $\overline{M}$ is a local maximizer. Or $r\zeta(\overline{M}) = \frac{\sigma^2}{2}\zeta''(\overline{M}) \geq 0$, therefore with $\zeta(\overline{M}) \geq 0$ and ζ is locally convex. Suppose ζ is locally concave and therefore that $\zeta(\overline{M}) < 0$; because $\zeta(\underline{M}) > 0$, the function ζ must also have a local minimizer $m_0 \in (\underline{M}, \overline{M})$. At that point,

$$\zeta'(m_0) = 0, \zeta''(m_0) \geq 0 \Longrightarrow \zeta(m_0) \geq 0,$$

which contradicts $\zeta < 0$, and therefore contradicts the premise that ζ is locally concave at $\overline{M}$. So ζ must be locally convex and positive around $\bar{(}M)$; if it is locally convex, $\zeta'(\overline{M}) \leq 0$ too. To extend this to the entire interval we must rule out $\zeta' > 0$ anywhere else. Suppose so $\zeta' > 0$ for some $M \in [\underline{M}, \overline{M})$, then it must

1. either start increasing at $\underline{M}$, have at least a local maximum and an inflexion point;

2. or start decreasing at $\underline{M}$, have at least a local minimum followed by a local maximum.

In the first case, take $\zeta'(\underline{M}) > 0$ then $\exists\, m_0$ such that

$$\zeta'(m_0) = 0, \quad \zeta(m_0) > 0, \quad \zeta''(m_0) < 0,$$

but the differential equation

$$r\zeta(m) = \rho M\zeta'(m) + \frac{\sigma^2}{2}\zeta''(m),$$

shows this is a contradiction. Next, for there to be a local minimum with $\zeta > 0$, the following conditions must hold:

$$\zeta'(\underline{M}) < 0, \quad \zeta'(m_0) = 0, \quad \zeta(m_0) > 0, \quad \zeta''(m_0) > 0,$$

and it must be followed by a local maximum at some point $m_1 > m_0$, with conditions:

$$\zeta'(\underline{M}) < 0, \quad \zeta'(m_1) = 0, \quad \zeta(m_1) > 0, \quad \zeta''(m_0) < 0.$$

Again the differential equation

$$r\zeta(m) = \rho M\zeta'(m) + \frac{\sigma^2}{2}\zeta''(m),$$

shows this is another contradiction. Finally one can likewise rule out a local minimum with $\zeta < 0$, for then the conditions

$$\zeta'(\underline{M}) < 0, \quad \zeta'(m_2) = 0, \quad \zeta(m_2) < 0, \quad \zeta''(m_2) > 0,$$

are also contradicted by the differential equation. Hence the function $\zeta(M)$ can only be a monotonically decreasing, convex function over the interval $[\underline{M}, \overline{M}]$, and the function $\mathcal{L}(M)$ is necessarily decreasing in the bound $\underline{M}$. We know from Proposition 1 that $\underline{M}$ increases in both η and k.

The last step verifies that the quantity $(1-\eta)\underline{M}re^{-r(\tau-t)}$ is decreasing in η. Recall that $\underline{M}$ is a construction: $\underline{M} = \underline{W}/\eta$. The quantity $(1-\eta)\underline{W}/\eta$ vanishes at $\eta = 1$, and is not defined at $\eta = 0$ (there is no moral hazard problem then). Using L'Hospital Rule,

$$\frac{d}{d\eta}\left(\frac{d\underline{W}}{\eta}\right)\bigg|_{\eta=0} = \frac{d\underline{W}}{d\eta} > 0, \;\; \frac{d}{d\eta}\left(\frac{1-\eta}{\eta}\right)\bigg|_{\eta=0} = -1$$

and since $(1-\eta)/\eta$ is everywhere decreasing and $\underline{W}$ everywhere increasing, $(1-\eta)\underline{W}/\eta$ is everywhere decreasing. Finally $e^{-r(\tau-t)}$ increase as τ decreases; and τ decreases as $\underline{W}$ increases. Therefore Δ_t decreases in η, k, as claimed. ■

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