Ambiguity and awareness: a coherent multiple priors model.

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Abstract

Ambiguity in the ordinary language sense is that available information is open to multiple interpretations. We model this by assuming that individuals are unaware of some possiblities relevant to the outcome of their decisions and that multiple probabilities may arise over an individual's subjective state space depending on which of these possibilities are realized. We formalize a notion of *coherent* multiple priors and derive a representation result that with full awareness corresponds to the usual unique (Bayesian) prior but with less than full awareness generates multiple priors. We show when information is received with no change in awareness, each element of the set of priors is updated in the standard Bayesian fashion (that is, full Bayesian updating). An increase in awarenss, however, leads to an expansion of the individual's subjective state and (in general) a contraction in the set of priors under consideration.

Keywords ambiguity, unawareness, multiple priors.

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1 Introduction

The idea that choices under uncertainty are subject to problems arising from ambiguity was first put forward by Ellsberg (1961), drawing on the earlier work of Knight (1921, 2006). Like Knight, Ellsberg argued that, in many cases, decisionmakers do not, and could not be expected to, act as if they assigned well-defined numerical probabilities to all the possible outcomes of a given choice. His well-known thought experiments illustrating this argument formed the basis of a large subsequent literature both theoretical and empirical.

In most of this literature, the term 'ambiguity' has been treated as a synonym for what Knight called 'uncertainty' namely the fact that relative likelihoods are not characterized by well-defined numerical probabilities. The standard method of dealing with ambiguity in decision theory is to endow the decisionmaker with multiple priors as in Gilboa and Schmeidler (1989). This approach may be combined with a variety of preference models, notably including the maxmin model of Gilboa and Schmeidler (1989) and the smooth model of Klibanoff, Marinacci, and Mukerji (2005).

For a non-specialist this is puzzling; there is no obvious link to the ordinary meaning (or meanings[1]) of ambiguity as a characteristic of propositions with more than one interpretation. In its normal usage, ambiguity is a linguistic concept, but in decision theory it is typically treated as a property of preferences.

The now-standard usage is quite different from that in Ellsberg's original

[1]Empson (1930) famously distinguished seven types of ambiguity.

article. Ellsberg treated ambiguity, not as a property of preferences or relative likelihoods, but as a property of the information on which judgements of relative likelihoods might be based

> Responses from confessed violators [of the EU axioms] indicate that the difference is not to be found in terms of the two factors commonly used to determine a choice situation, the relative desirability of the possible pay-offs and the relative likelihood of the events affecting them, but in a third dimension of the problem of choice: the nature of one's information concerning the relative likelihood of events. What is at issue might be called the ambiguity of this information, a quality depending on the amount, type, reliability and "unanimity" of information, and giving rise to one's degree of "confidence" in an estimate of relative likelihoods.

In this paper, we argue that informational ambiguity, in the ordinary language sense that the available information is open to multiple interpretation, may be modelled using concepts from the literature on unawareness. When individuals are unaware of some possibilities relevant to the outcome of their decisions, there are multiple probability distributions that may be applicable, depending on whether or not these possibilities are realised.

To represent this idea, we adopt a syntactic representation, in which the state of the world is characterized by the truth values of a finite set of elementary propositions P. The state space Ω is given by the set of all logically possible combinations of truth values, that is, by the truth table for

P. Any sentence ℓ in the language $\mathcal{L}$ generated by P corresponds to an event E in Ω. In particular, any information that arrives at time t can be described by a sentence in $\mathcal{L}$ and the corresponding event (that is, subset) of Ω.

An unboundedly rational decision-maker is aware of all the propositions in P and sentences in $\mathcal{L}$, and therefore has access to a complete and unambigious description of the state space Ω. Assuming the Savage axioms are satisfied, the decision-maker can assign a unique subjective probability π to any event E, and update that probability in line with Bayes rule as new information is received.

We represent a boundedly rational decisionmaker as one who is unaware of at least some propositions in P. For simplicity, consider the case when there is only one such proposition p^*. Now consider any sentence ℓ that does not include p^*, but assigns a truth value to every other p. Such a sentence is potentially ambiguous, since it might correspond to the state associated with $\ell \wedge p^*$ or alternatively to $\ell \wedge \neg p^*$.

The idea may be illustrated by the case of the Ellsberg one-urn problem, as discussed by Billot and Vergopoulos (2018), though the interpretation given here differs from theirs. In the Billot and Vergorpoulos example, the color of the balls in the 'ambiguous' urn is determined by the values of two parameters i and j, where $j \in \{0, 1\}$. In particular, if $i = 3$, a ball added to the urn will be black if $j = 0$ and white if $j = 1$. However, the decisionmaker is unaware of the role of j. Thus, the information contained in the proposition $i = 3$ is ambiguous and incomplete, in a way that the decisionmaker cannot fully understand.

A multiple priors representation of probabilities reflects this ambiguity.

Any complete probability distribution π over the pair (i, j) gives rise to two conditional distributions π_0 corresponding to $\pi(i|j = 0)$ and π_1 corresponding to $\pi(i|j = 1)$. Although the decision-maker, being unaware of j, cannot formulate the full probability distribution π, she may entertain both π_0 and π_1, as well as any convex combination of the two, as prior beliefs about i.

Further difficulties arise when we consider updating in the multiple prior framework. As is well known from the works of Epstein and Breton (1993) and Ghirardato (2002), in a dynamic setting, deviations from additive beliefs require the relaxation of either consequentialism or dynamic consistency, or alternatively, a restriction on the class of preferences. This has lead to three different approaches to updating of multiple priors in the literature. The first approach restricts preferences to be recursive (and thus, dynamically-consistent and consequentialist) as in the case of recursive smooth ambiguity aversion in Klibanoff, Marinacci, and Mukerji (2009), rectangular max-min preferences in Epstein and Schneider (2003), and the generalization of their approach to α-max-min preferences in Beissner, Lin, and Riedel (2016). The second class of models preserves consequentialism, but relaxes dynamic consistency, such as the full-bayesian updating axiomatized by Pires (2002), the maximum-likelihood rule of Gilboa and Schmeidler (1993) and the generalized Bayesian updating rule for the α-max-min preferences by Ghirardato, Maccheroni, and Marinacci (2008). Finally, a third approach consists in preserving dynamic consistency, but relaxing consequentialism as in the updating rule proposed by Hanany and Klibanoff (2007) for the smooth model of ambiguity formulated by Klibanoff, Marinacci, and Mukerji (2005).

In this paper, we formalize this idea to derive a *coherent multiple priors*

(CMP) model. Our goals are twofold. First, we derive a representation theorem for the CMP models and show that, with full awareness, it corresponds to the usual Bayesian model. Second, we consider the problem of updating beliefs, which has proved problematic in the multiple-priors setting. In our setting, updating may arise in response to the receipt of new information or to increased awareness, represented as awareness of new elementary propositions p. We show that when information is received with no change in awareness, each element of the set of priors is updated in the standard Bayesian fashion as in Ghirardato, Maccheroni, and Marinacci (2008). An increase in awareness is represented by an expansion of the state space to which the decision maker has access, and by a corresponding contraction in the set of priors under consideration. We show that, as the decisionmaker approaches full awareness, the set of priors contracts to a singleton $\{\pi\}$. Relative to π the set of priors at any time t is made of conditional probabilities, depending on the truth values of propositions of which the decisionmaker is unaware

The paper is organized as follows. We first set up the description of the decisionmaking problem in both propositional (syntactic) and state-space (semantic) terms. Awareness, information and acts are defined.

Next we consider preferences and ambiguity in a timeless setting. We restate the Ghirardato, Maccheroni, and Marinacci (2004) axioms. The crucial result of this section is to show that preferences satisfying the Ghirardato, Maccheroni, and Marinacci (2004) axioms may be derived from the preferences of a fully aware EU-maximizer, by introducing unawareness. The key idea is that any set of truth values for the propositions of which the individual is unaware, induces a conditional probability distribution over the truth values of the propositions of which she is aware, and therefore over the associated (awareness-constrained) state space.

We next consider updating in response to increases in information and awareness. For changes in information with constant awareness, we show that the preferences we derive display prior-by-prior Bayesian updating, as in Ghirardato, Maccheroni, and Marinacci (2008) and Pires (2002). For changes in awareness, we address the simplest case where the individual becomes aware of a single additional proposition. We show that the result is to expand the state space, dividing each existing state into two new states, one in which the newly discovered proposition is true and the other in which it is false. Conversely, any pair of priors conditioned on events that differ only on the truth value of the new proposition is replaced by a convex combination of the two. In the finite setting we have here, the state space doubles in size, while the set of priors halves. Finally, we offer some concluding comments.

2 Setup.

2.1 The state of the world: propositional and state-space descriptions

Information and awareness evolve over time. However, we will initially consider an individual with fixed information and awareness, suppressing time subscripts.

The world is described by the truth values of a finite set of elementary propositions $P = \left\{p^1, ..., p^N\right\}$. The closure of this set under negation and conjunction defines a language $\mathcal{L}$ with elements ℓ, referred to as sentences.

Individuals have bounded awareness, represented by a set $A \subseteq P$ of elementary propositions which they can express. The restricted language consisting of expressible propositions is denoted $\mathcal{L}^A \subseteq \mathcal{L}$. Information available to the individual is represented by a sentence $\ell^* \in \mathcal{L}^A$ which states the conjunction of all the propositions (not necessarily elementary) they know to be true. Awareness and information are mutually dependent. On the one hand, as will be described in more detail below, the individual's awareness depends on the information they have. On the other hand, that information must be expressed in terms of propositions $p \in A \subseteq P$ expressible by the individual.

This propositional description of the world may be represented equivalently in state space terms more familiar to decision theorists. The state space associated with the truth table for P may be represented by $\Omega = 2^N$ with $\omega \in \{0\,,1\}^N$ a representative element / state. Similarly, for an individual with awareness A the state space of which she is aware can be expressed as

$$S^A = 2^A,$$

with a generic element s^A.

Let $\bar{A} = P \backslash A$ be the set of propositions of which the individual is unaware. The "complementary state space" of which she is *unaware* can be expressed as $\bar{S}^A = 2^{\bar{A}}$, with generic element $\bar{s}^A$. Notice that $\Omega = S^A \times \bar{S}^A$.

Furthermore, each (awareness) state $s^A \in S^A$ corresponds to the event $\left\{s^A\right\} \times \bar{S}^A$ in Ω and each $\bar{s}^A \in \bar{S}^A$ corresponds to the event $S^A \times \left\{\bar{s}^A\right\}$ in Ω. That is, awareness of the form described above leads to a 'coarsening' of the state space (as, for example, in Quiggin (2016)), represented by the projection of Ω onto S^A.

Unawareness in this sense may be distinguished from the case of 'reduction' or 'restriction' of the state space, in which some possible elements of Ω are disregarded or, equivalently, in which some propositions that are possibly true are implicitly assumed to be false. This leads to the possibility of 'surprise' (see, for example, Grant and Quiggin (2015)).

2.2 Acts

Acts will be represented in the usual way as mappings from states to outcomes.[2] We consider only predictions, and thus concentrate on a set comprising just two outcomes $X = \{0, 1\}$. Let Δ denote the set of all lotteries on X, with a generic element denoted by $x \in [0, 1]$, yielding the ('good') outcome 1 with probability x and the ('bad') outcome 0 with complementary probability $1 - x$.

An act α maps Ω to Δ. The set of possible acts is denoted $\mathcal{A}$. Let $\mathcal{C}$ denote the set of all constant acts. Let $\mathcal{B}$ denote the set of 'bets (on events)', that is, $\alpha \in \mathcal{B}$ iff $\alpha(\omega) \in \{0, 1\}$ for all $\omega \in \Omega$.

The outcomes of acts considered by an individual with limited awareness. must be conditional on propositions of which the individual is aware. Hence, for given awareness A, any act α must be measurable with respect to S^A, and we denote by $\mathcal{A}^A$ the subset of such acts. Each act in $\mathcal{A}^A$ induces a mapping of elements of S^A into lotteries on $\{0, 1\}$. In particular, for any s^A an action α specifies the probability of obtaining 1, which with slight abuse of notation

[2] It would be possible to describe actions entirely in propositional terms. If the language includes statements of the form 'receive outcome x', then actions are propositions of the general form 'if proposition p , then 'receive outcome x' obtains. However, this does not appear to be a useful step for the purposes of the current analysis.

we shall denote by $\alpha\left(s^A\right)$.

As is standard, convex mixtures of actions are defined as state-by-state probability mixtures:

$$(\lambda\alpha + (1-\lambda)\,\alpha')\,(\omega) = \lambda\alpha\,(\omega) + (1-\lambda)\,\alpha'\,(\omega)$$

3 Preferences and ambiguity

For any given level of awareness A we define preferences on $\mathcal{A}^A$ by $\succsim^A$.

3.1 The GMM approach

We first impose the Ghirardato, Maccheroni, and Marinacci (2004) (hereafter, GMM) axioms: for any given $\mathcal{A}^A$, we assume the preferences $\succsim^A$ satisfies:

Axiom 1. *(A1) Completeness and transitivity*

Axiom 2. *(A2) Archimedean axiom: if $\alpha \succ^A \alpha' \succ^A \alpha''$, then there are λ and $\mu \in (0;1)$ such that*

$$\lambda\alpha + (1-\lambda)\,\alpha'' \succ^A \alpha' \succ^A \mu\alpha + (1-\mu)\,\alpha''.$$

Axiom 3. *(A3) Certainty independence: if $\bar{\alpha} \in \mathcal{C}$, then $\alpha \succsim^A \alpha'$ iff $\lambda\alpha + (1-\lambda)\,\bar{\alpha} \succsim^A \lambda\alpha' + (1-\lambda)\,\bar{\alpha}$ for all $\lambda \in [0,1]$.*

Axiom 4. *(A4) Monotonicity: if $\alpha\left(s^A\right) \geq \alpha'\left(s^A\right)$ for all $s^A \in S^A$, then $\alpha \succsim^A \alpha'$.*

Axiom 5. *(A5) Non-degeneracy: there are α and α' such that $\alpha \succ^A \alpha'$.*

Following GMM, we have

Lemma 1. *Axioms A1-A5 are equivalent to:*

(i) The existence of a capacity $\rho : s^A \rightarrow [0,1]$ such that for any α, $\alpha' \in \mathcal{B}$, $\alpha \succsim^A \alpha'$ iff $\rho\left(s^A \mid \alpha\left(s^A\right) = 1\right) \geq \rho\left(s^A \mid \alpha'\left(s^A\right) = 1\right)$.

(ii) The existence of a unique convex and (weak*) closed set of priors Π such that for any two acts α and α', Independence (that is, $\alpha \succsim^A \alpha'$ iff $\lambda\alpha + (1-\lambda)\alpha'' \succsim^A \lambda\alpha' + (1-\lambda)\alpha''$ for all $\lambda \in (0,1)$ and all $\alpha'' \in \mathcal{A}$) holds if and only if

$$\sum_{s^A \in S^A} \pi\left(s^A\right)\alpha\left(s^A\right) \geq \sum_{s^A \in S^A} \pi\left(s^A\right)\alpha'\left(s^A\right) \text{ for all } \pi \in \Pi. \tag{1}$$

Whenever (1) is satisfied for two acts α and α', we write $\alpha \succsim^A_* \alpha'$.

Since each constant act can be identified with the probability $x \in [0,1]$, with which it results in the outcome 1 in every state, we write $\alpha_x \in \mathcal{C}$. As in GMM, define:

$$CE^*(\alpha) = \left\{ \begin{array}{c} x \in [0,1] \quad \mid \text{for any } y \in [0,1], \ \alpha_y \succsim^A_* \alpha \text{ implies } y \geq x \text{ and} \\ \alpha \succsim^A_* \alpha_y \text{ implies } x \geq y \end{array} \right\}$$

GMM show that $x \in CE^*(\alpha)$ iff

$$\min_{\pi \in \Pi} \sum_{s^A \in S^A} \pi\left(s^A\right)\alpha\left(s^A\right) \leq x \leq \max_{\pi \in \Pi} \sum_{s^A \in S^A} \pi\left(s^A\right)\alpha\left(s^A\right)$$

3.2 Unawareness and ambiguity

The preferences described in the previous section are normally interpreted in terms of ambiguity. Given our setup, there is a natural interpretation in terms of awareness. Consider any EU preferences over $\mathcal{A}$, $\succsim$, described by a a probability distribution π.[3]

For any $\bar{s}^A \in \bar{S}^A$, that is, for any set of truth values for the propositions of which the individual is unaware, π induces a conditional probability distribution $\pi_{\bar{s}^A} = \pi\left(\cdot|\bar{s}^A\right)$ over S^A. Correspondingly $\succsim$ induces conditional preferences $\succsim_{\bar{s}^A}$ over $\mathcal{A}^A$, given by $\alpha \succsim_{\bar{s}^A} \alpha'$ iff

$$\sum_{s^A \in S^A} \pi_{\bar{s}^A} \alpha\left(s^A\right) \geq \sum_{s^A \in S^A} \pi_{\bar{s}^A} \alpha'\left(s^A\right)$$

Now consider preferences $\succsim^A$ which satisfy axioms *A1–A5* and

Axiom 6. *(A6) Unanimity* $\alpha \succsim_*^A \alpha'$ *if and only if* $\alpha \succsim_{\bar{s}^A} \alpha'$ *for all* $\bar{s}^A \in \bar{S}^A$.

This property may be viewed as a version of the sure-thing principle. If the act induced on Ω by α would be preferred to that induced by α', regardless of which $\bar{s}^A$ obtained, then α must be preferred unconditionally. Recalling that α and α' are measurable wrt S^A, the only effect of $\bar{s}^A$ is to determine the conditional probability distribution $\pi_{\bar{s}^A}$. So, we are evaluating α and α' with respect to a set of probability distributions. A6 says that if α is preferable with respect to each such distribution, then it must be preferred *unambiguously.*

Lemma 2. *Under axioms A1–A6, the set of probabilities* Π *in Lemma 1 is* $\overline{CH}\left\{\pi\left(\cdot|\bar{s}^A\right)|\bar{s}^A \in \bar{S}^A\right\}$.

Note that we do not expect the converse to hold. As we shall show, in the absence of changes in awareness, the probabilities Π derived as conditional

[3]Without loss of generality, we assign utilities 0 and 1 to the payoffs 0 and 1.

distributions based on unawareness follow Bayesian updating in response to the arrival of new information. Axioms *A1–A6* are insufficient to ensure this – we need additional properties as discussed by Ghirardato, Maccheroni, and Marinacci (2008).

4 Time, information and histories.

We now consider changes in *information* as well as awareness *over time*, and the induced changes in beliefs and preferences. Time $t = 0, 1, 2, \ldots, T$ is discrete and finite.

Information is modeled by partitions: where $\mathcal{F}_t$ denotes a partition of Ω at time t. Each $f \in \mathcal{F}_t$ is an event in Ω.[4] The element of $\mathcal{F}_t$ that obtains at time t is denoted $f_t \in \mathcal{F}_t$. The collection $\{\mathcal{F}_t\}_{t=0}^{T}$ constitutes a *filtration*. That is, for each $t = 0, \ldots, T-1$, $\mathcal{F}_{t+1}$ is a refinement of $\mathcal{F}_t$, or equivalently, if $f_{t+1} \in \mathcal{F}_{t+1}$ then $f_{t+1} \subseteq f_t$ for some $f_t \in \mathcal{F}_t$. We write $\mathcal{F}_t(f)$ for the element of the partition at time t which contains $f \subseteq \Omega$ (provided such an element exists). That is,

$$\mathcal{F}_t(f) = \{f_t \in \mathcal{F}_t \mid f \subseteq f_t\}$$

We assume that no uncertainty is resolved at date $t = 0$, that is, $\mathcal{F}_0 = \{\Omega\}$ and all uncertainty is resolved by date T, so that for each $\omega \in \Omega$, $\{\omega\} \in \mathcal{F}_T$.[5]

The evolution of the individual's awareness is modeled as a dated-event process, in which for each time $t = 0, \ldots, T-1$, and each event $f_t \in \mathcal{F}_T$,

[4] Observe that each event f corresponds to a sentence $\ell \in \mathcal{L}$ and vice versa.

[5] Alternatively, we may assume that logical contradictions such as 'the weather is sunny'AND 'the weather is rainy' are ruled out at $t = 0$, and that information received after $t = 0$ relates only to conceivable states, which may therefore be assumed to have non-zero probability.

the set of propositions of which the individual is currently aware is taken to be $A(t, f_t) \subseteq P$. As awareness is (weakly) increasing we require $A(t, f_t) \subseteq A(t', f_{t'})$, for every $t' > t$ and $f_{t'} \in \mathcal{F}_{t'}$ such that $f_{t'} \subseteq f_t$.

Furthermore, to simplify notation and without any essential loss of generality, we will assume that information is updated *only* in odd-numbered periods while awareness may only increase in even numbered periods. That is, for any period $t > 0$ and any event $f_t \in \mathcal{F}_t$, if t is odd then $f_t \in \mathcal{F}_{t+1}$, else if t is even then $A(t+1, f_{t+1}) = A(t, f_t)$, where $f_{t+1} \in \mathcal{F}_{t+1}$ and $f_{t+1} \subseteq f_t$. This means that any new information in an odd-numbered period t must be expressed in terms of propositions expressible in period $t-1$. The sequence of updating is therefore:

(i) At time t (odd) the individual observes information f_t measurable with respect to her level of awareness in period $t-1$.

(ii) At time t (even), she revises her awareness in the light of new information and the passage of time.

Informally, we may interpret this as saying that the individual first observes new information, then considers new propositions suggested by this information. Given her current awareness A, the first stage of the process (odd t) reduces the size of the state space, eliminating as impossible 'states' in S^A corresponding to states in $\Omega - f_t$. The second stage (even t) enriches the state space, by expanding the set of propositions that define possible states that the individual can conceive.

Thus, changes in information and awareness may be described by a sequence of partial histories, where each history $h = \left(t^h, f^h, A^h\right)$ is a triple consisting of a date $t^h \in \{0, 1, 2, \ldots, T\}$ an information set $f^h \in \mathcal{F}_{t^h}$ and an awareness level $A^h = A\left(t^h, f^h\right)$.

As already noted above, awareness and information are interdependent. Awareness A^h at history h depends on the information set f^h, while the information set f^h must be measurable with respect to S^{A^h}.

We shall denote by h_{-1} the (unique) immediate predecessor of a history $h = \left(t^h, f^h, A^h\right)$. Notice that by construction $t^{h_{-1}} = t^h - 1$, $f^{h_{-1}} = \mathcal{F}_{t^{h_{-1}}}\left(f^h\right)$ (with $f^{h_{-1}} = f^h$, if t^h is even) and $A^{h_{-1}} \subseteq A^h$. We define $N^h = A^h - A^{h_{-1}}$, the set of propositions of which the individual becomes newly aware at h, noting that $N^h = \varnothing$ if t^h is odd.

Lemma 3. *Given the alternate updating of awareness and information, for each history* $h = \left(t^h, f^h, A^h\right)$*,* $\mathcal{F}_{t^h+1}$ *is measurable with respect to* S^{A^h}*.*

We may therefore define the set of one-step-ahead (conceivable future) histories $\mathcal{H}^h_{+1}$ considered possible at h as

$$\mathcal{H}^h_{+1} = \left\{h_{+1} = \left(t^h + 1, f, A^h\right) | \mathcal{F}_{t^h}(f) = f^h\right\}.$$

At history h, the set of one-step-ahead actions available to the individual is:

$$\mathcal{A}^h_{+1} = \left\{\alpha : \mathcal{H}^h_{+1} \to [0,1]\right\}.$$

That is, for each one-step-ahead history $h_{+1} \in \mathcal{H}^h_{+1}$ an action specifies the probability of obtaining the good outcome 1, as being $\alpha(h_{+1})$.

Note that at history h, the decision maker is fully aware of all actions in $\mathcal{A}^h_{+1}$. In contrast, actions in $\mathcal{A}^h_{+t}$ for $t > 1$ might only be expressible in terms of higher levels of awareness. For a given history h and a given period $t > 1$, let $\mathcal{F}^h_{+t}$, be the finest coarsening of $\mathcal{F}_{t^h+t}$ measurable w.r.t. S^{A^h}. As above, $\left(\mathcal{F}^h_{+t}\right)_t$ defines a filtration of events.

Define

$$\overline{\mathcal{H}}^h_{+t} = \left\{\tilde{h} = \left(t^{\tilde{h}} + t, f^{\tilde{h}}, A^h\right) \mid f^{\tilde{h}} \in \mathcal{F}^h_t, \mathcal{F}_{t^h}\left(f^{\tilde{h}}\right) = f^h\right\}.$$

as the set of histories that t-step succeed h and which the decisionmaker can express in terms of his awareness at history h. As above, $\left(\overline{\mathcal{H}}^h_{+t}\right)_t$ defines a filtration of events and we denote by $\overline{\mathcal{H}}^h = \cup_t \overline{\mathcal{H}}^h_{+t}$ the set of all such histories.

The actions of which the decisionmaker is aware at h are those measurable w.r.t. to this perceived partition t-steps ahead:

$$\overline{\mathcal{A}}^h_{+t} = \left\{\alpha : \overline{\mathcal{H}}^h_{+t} \to [0,1]\right\}.$$

This is a subset of all possible actions – in particular, the decision maker is not allowed to condition on different changes in awareness in the future and following information revelation.

Convex mixtures, constant acts and binary acts are defined as before.

Non-trivial actions arise only when t is odd, and this will be assumed unless otherwise stated in the sequel.

We assume that the decision maker can form preferences over such actions conditional on history h, i.e. we define $\succsim^h$ over the set of actions $\cup_t \overline{\mathcal{A}}^h_{+t}$ and impose the same set of axioms *A1–A5* as the 'static' preferences above. Hence, we can guarantee the existence of a set of priors Π^h on S^{A^h} with the properties stated in Lemma 1.

Unambiguous preferences $\succsim^h_*$ are defined in analogy to $\succsim^h_*$.

5 Coherent MP model and Bayesian updating

In our setting, updating may arise in response to the receipt of new information or to increased awareness, represented as awareness of new elementary propositions p. We first consider new information, arriving at odd-numbered periods. We will show that, in the absence of changes in awareness, individuals satisfying *A1-A6* will respond to new information by applying Bayesian updating to each of the conditional probability distributions $\pi\left(\cdot|\bar{s}^A\right)$ in Π. Hence, we refer to the model of preferences under bounded awareness as a *coherent* multiple priors model.

5.1 Updating with Constant Awareness

Suppose we move from $h = \left(t^h, f^h, A^h\right)$ with t^h even to $h_{+1} = \left(t^h + 1, f^{h_{+1}}, A^h\right)$ where $f^h = \mathcal{F}_{t^h}\left(f^{h_{+1}}\right)$. That is we refine the information in f^h to $f^{h_{+1}}$ and leave awareness unchanged. A fully aware individual with prior probabilities π replaces the conditional $\pi^h = \pi\left(\cdot|f^h\right)$ with $\pi^{h_{+1}} = \pi\left(\cdot|f^{h_{+1}}\right)$, which is well-defined provided $f^{h_{+1}}$ is not null with respect to the preferences $\succsim^h$. Hence, any conditional probability $\pi^h_{\bar{s}^A} = \pi^h\left(\cdot|\bar{s}^A\right)$ is replaced by $\pi^{h_{+1}}_{\bar{s}^A} = \pi^{h_{+1}}\left(\cdot|\bar{s}^A\right)$. Alternatively, we may replace any $\pi^h_{\bar{s}^A}$ by the conditional probability $\tilde{\pi}^h_{\bar{s}^A} = \pi^h_{\bar{s}^A}\left(\cdot|f^{h_{+1}}\right)$. We observe that $\pi^h_{\bar{s}^A}\left(\cdot|f^{h_{+1}}\right) = \pi^{h_{+1}}\left(\cdot|\bar{s}^A\right)$.

The Coherent MP holder thus has a set of beliefs, each corresponding to what her probability assessment would be conditional on her explicit knowledge and a particular 'state' in the 'unaware state space' obtaining. In order to ensure these set of beliefs are well-defined we require in the following all information events to be non-null.

Axiom 7. *(A7) Non-null (information-)events: For any h and any $\tilde{h} \in \overline{\mathcal{H}}^h_{+t}$, for the bet $\alpha \in \overline{\mathcal{A}}^h_{+t} \cap \mathcal{B}$ defined as $\alpha^{-1}(1) = \tilde{h}$, we have $\alpha \succ^h \alpha_0$.*

As mentioned above, we want to characterize a decisionmaker, who absent changes in awareness behaves as a Bayesian and thus updates each of his priors using the Bayesian rule in view of the new information. We thus adapt the axioms suggested by Ghirardato, Maccheroni, and Marinacci (2008) to our setting, requiring conditional preferences to be consequentialist and to satisfy dynamic consistency whenever awareness remains unchanged.

For a given h such that t^h is even, let $H \subseteq \overline{\mathcal{H}}^h_{+t}$. We define preferences over actions $\mathcal{A}^h_{+t}$ that pay at time $t^h + t$ conditional on an event $H \subseteq \overline{\mathcal{H}}^h_{+t}$ as $\succsim^h_H$.

We assume that conditional preferences $\succsim^h_H$ satisfy the same set of axioms, A1–A5, as the unconditional preferences above. Unambiguous preferences $\succsim^h_{*H}$ are defined in analogy to $\succsim^h_*$.

Axiom 8. *(A8) Consequentialism: For h such that t^h is even, let $H \subseteq \overline{\mathcal{H}}^h_{+t}$. For any actions α and $\alpha' \in \overline{\mathcal{A}}_{+t}$ such that $\alpha\left(\tilde{h}\right) = \alpha'\left(\tilde{h}\right)$ for all $\tilde{h} \in H$, $\alpha \sim^h_H \alpha'$.*

Axiom 9. *(A9) Dynamic consistency of $\succsim^{*h}_H$: Let h be such that t^h — even. Let $H \subseteq \overline{\mathcal{H}}^h_{+t}$. For any actions α and $\alpha' \in \overline{\mathcal{A}}_{+t}$ such that $\alpha\left(\tilde{h}\right) = \alpha'\left(\tilde{h}\right)$ for all $\tilde{h} \notin H$, $\alpha \succsim^h_{*H} \alpha'$ iff $\alpha \succsim^h_* \alpha'$.*

Proposition 1. *Coherent MP preferences satisfy consequentialism and dynamic consistency in relation to changes in information.*

Proof: To see that Consequentialism holds, note that generalized Bayesian updating implies that conditional on H, all $\tilde{h} \notin H$ are assigned 0-probability under all $\pi \in \Pi_H^h$. Furthermore, by the definition of a and a' in *A8*, we have that

$$\min_{\pi \in \Pi_H^h} \sum_{h' \in H} \pi(h') \alpha(h') = \min_{\pi \in \Pi_H^h} \sum_{h' \in H} \pi(h') \alpha'(h') \text{ and}$$
$$\max_{\pi \in \Pi_H^h} \sum_{h' \in H} \pi(h') \alpha(h') = \max_{\pi \in \Pi_H^h} \sum_{h' \in H} \pi(h') \alpha'(h')$$

so that $CE_H^{h*}(\alpha) = CE_H^{h*}(\alpha')$ and thus, $\alpha \sim_{*H}^h \alpha'$ which implies $\alpha \sim_H^h \alpha'$.

As for dynamic consistency of $\succsim_H^{*h}$, note that for any actions a and a' as defined in *A9* and every $\pi \in \Pi^h$

$$\begin{aligned}
&\sum_{h' \in \mathcal{H}_{+1}^h} \pi(h') \alpha(h') - \sum_{h' \in \mathcal{H}_{+1}^h} \pi(h') \alpha'(h') \\
&= \sum_{h' \in H} \pi(h') \alpha(h') - \sum_{h' \in H} \pi(h') \alpha'(h') \\
&= \pi(H) \left[\sum_{h' \in H} \pi(h' \mid H) \alpha(h') - \sum_{h' \in H} \pi(h' \mid H) \alpha'(h') \right].
\end{aligned}$$

$\alpha \succsim_*^h \alpha'$ holds iff the first difference is positive for all $\pi \in \Pi^h$ and this is clearly equivalent to the last difference being positive for all $\pi \in \Pi^h$, or to $\alpha \succsim_{*H}^h \alpha'$.

The following corollary follows from the main result in Ghirardato, Maccheroni, and Marinacci (2008):

Corollary 1. *If* $\left(\succsim_H^h\right)_{\{h,H\}}$ *satisfy Axioms A1–9, then for changes in information, beliefs are updated according to the generalized Bayesian updating rule, i.e., for every* h *with* t^h *– even and* $H \subseteq \mathcal{H}_{+2}^h$ *such that for every*

$h_{+2} \in H$, *awareness remains unchanged,* $A^{h_{+2}} = A^h$, *the set of posteriors conditional on* H, Π^h_H *is given by the generalized Bayesian updating of* Π^h*:*

$$\Pi^h_H = \left\{ \pi \left(\cdot \mid H \right) \mid \pi \in \Pi^h \right\}.$$

5.2 Changes in awareness

We now consider pure changes in awareness. Consider a history $h = (t, f, A)$ with t^h odd. To simplify the exposition, we first consider the case in which the individual becomes aware of a single new proposition p. Suppose the individual's beliefs at h can be represented by a set of coherent probability measures $\Pi^h \subset \Delta \left(S^A \right)$ generated by the prior probability $\pi \in \Delta \left(\Omega \right)$. Now suppose the individual becomes aware of a proposition $p \in \bar{A}$ at $h_{+1} = (t{+}1, f, A') \in \mathcal{H}^h_{+1}$ so that $A' = A \cup \{p\}$.[6] Consider any $s^A \in S^A$ corresponding to a sentence $\ell \in \mathcal{L}^A$. For each such ℓ, the individual at h_{+1} considers two possible sentences $\ell \wedge p$, $\ell \wedge \neg p$ corresponding to the truth or falsity of p.

Noting that $s^A \in \{0, 1\}^{|A|}$ is a binary number, we may define the states $\left(s^A, 1 \right)$ (for p true) and $\left(s^A, 0 \right)$ (for p false) in $S^{A'}$. Similarly, any $\bar{s}^{A'} \in \bar{S}^{A'}$ corresponds to two complementary states $\left(\bar{s}^{A'}, 1 \right)$ (for p true) and $\left(\bar{s}^{A'}, 0 \right)$ (for p false) in $\bar{S}^A$.

Proposition 2. *For changes in awareness involving a single proposition, that is, for every* h *with* t^h *– odd, and* $\left| N^{h_{+1}} \right| = 1$*, the coherent MP representation of preferences at* h_{+1} *is given by the set of priors*

$$\Pi^{h_{+1}} = \overline{CH} \left\{ \pi \left(\cdot | \bar{s}^{A'} \right) | \bar{s}^{A'} \in \bar{S}^{A'} \right\} \tag{2}$$

[6] Given the alternating dates setup the individual at h_{+1} does not learn whether p is true, although this may be resolved by subsequent revelation of information.

where the prior

$$\pi\left(\cdot|\bar{s}^{A'}\right) = k\left(\bar{s}^{A'}\right)\pi\left(\cdot|\left(\bar{s}^{A'},1\right)\right) + \left(1-k\left(\bar{s}^{A'}\right)\right)\pi\left(\cdot|\left(\bar{s}^{A'},0\right)\right) \qquad (3)$$

Proof: Notice that

$$\pi\left(\bar{s}^{A'}\right) = \pi\left(\bar{s}^{A'},1\right) + \pi\left(\bar{s}^{A'},0\right)$$

where π is the probability on Ω and its arguments are considered as events.

If we take

$$k\left(\bar{s}^{A'}\right) = \frac{\pi\left(\bar{s}^{A'},1\right)}{\pi\left(\bar{s}^{A'}\right)}$$

$$\frac{\pi\left(\bar{s}^{A'},0\right)}{\pi\left(\bar{s}^{A'}\right)} = 1-k\left(\bar{s}^{A'}\right)$$

then Π^{h+1} as expressed in the proposition is obtained.

Proposition 3. *If* $\left(\succsim_H^h\right)_{\{h,H\}}$ *satisfy Axioms A1–A9, beliefs are updated according to CMP. In particular, for changes in awareness involving a single proposition, i.e., for every* h *with* t^h *– odd, and* $N^{h+1} = \left\{s^{h+1} \in \{0,1\}\right\}$,

(*i*) $\alpha \succsim_*^{h-1} \alpha'$ *if and only if* $\alpha \succsim_{s^{h+1}}^{h+1} \alpha'$ *for* $s^{h+1} \in \{0,1\}$;

(*ii*) *if* $H_{s^{h+1}} = f^{h-1} \cap \left\{s^{h+1}\right\} \in \overline{\mathcal{H}}^{h+1}$ *for* $s^{h+1} \in \{0,1\}$,

$$\Pi^{h-1} = \overline{CH}\left(\left\{\pi\left(\cdot \mid H_{s^{h+1}}\right) \mid \pi\left(\cdot \mid\right) \in \Pi_{H_{s^{h+1}}}^{h+1}\right\}_{s^{h+1}\in\{0;1\}}\right).$$

(*iii*) Π^{h+1} *satisfies conditions (2) and (3).*

Proof: Parts (*i*) is an immediate consequence of axiom *A6,* whereas (*ii*) follows from Lemma 2. Finally, part (*iii*) follows from (*ii*) combined with the argument given in the proof of Proposition 2.

The Proposition shows that under A6, Unanimity, the set of priors the decisionmaker entertains at h_{-1}, before becoming aware of a certain set of propositions $N^{h_{+1}}$ is given exactly by the convex hull of the posteriors conditional on the truth values of these propositions once he has become aware of them at h_{+1}. Using the fact that under A1-A9, conditional beliefs are formed via Bayesian updating, we obtain the CMP representation of beliefs.

Remark: Note that the assumption $\left|N^{h_{+1}}\right| = 1$ is without loss of generality, since we can have several subsequent periods during which information remains unchanged at each even period, but awareness increases by exactly one proposition at each odd period.

Finally, since for each history $h = (t, f_t, A)$, the extreme points of the set Π^h are derived from conditional probabilities obtained from the (full awareness) prior $\pi \in \Delta(\Omega)$, conditioning on f_t and in turn each $\bar{s}^A$ in $\bar{S}^A$, it follows that $\pi(\cdot|f_t) \in \Pi^h$, since from the iterative law of expectations, we have:

$$\pi(\cdot|f_t) = \sum\nolimits_{\bar{s}^A \in \bar{S}^A} \pi\left(\bar{s}^A|f_t\right) \pi(\cdot|f_t).$$

More generally, for any history $h = (t, f_t, A)$, with t odd, and with an immediate successor $h_{+1} = (t+1, f_t, A')$ embodying a pure increase in awareness (that is, $A \subset A'$), we have from the construction that $\Pi^h \subset \Pi^{h_{+1}}$. That is, the mapping from Π^h to $\Pi^{h_{+1}}$ may be viewed as a contraction with $\pi(\cdot|f_t)$ as a fixed point.

6 Conclusion

Beginning in the late 1970s, alternatives to and generalizations of Expected Utility theory have proliferated in response to behavioral violations of EU predictions and theoretical criticism of the axiomatic foundations of EU. Examples have included probability weighting models for choice under risk (Allais (1953),Kahneman and Tversky (1979), Quiggin (1982), Yaari (1987)), ambiguity models for choice under uncertainty (Gilboa and Schmeidler (1989), Schmeidler (1989), Ghirardato, Maccheroni, and Marinacci (2004), Klibanoff, Marinacci, and Mukerji (2005)) and the rapidly growing literature on unawareness (Schipper (2014)). Kochov (2017) and Piermont (2017) identify behavioral conditions, which distinguish between ambiguity and unawareness in a dynamic axiomatic framework. Dominiak and Tserenjigmid (2017) model a decision maker who upon learning about a new state of the world, but not its probability, might entertain ambiguous beliefs about this state. Thus, in their model, ambiguity can increase as the decision maker becomes more aware.

There have been some attempts to at unification. For example, Mukerji (1997) shows that probability weighting may be derived from a decision-maker's anticipation that her perception of future contingencies is incomplete. Similarly, in this paper, we have shown that the invariant biseparable model of Ghirardato, Maccheroni, and Marinacci (2004) model of choice under ambiguity (which incorporates α-maxmin EU as a special cases), may be derived from the preferences of an EU maximizer with coarse awareness. Updating in response to both new information and refined awareness is well-behaved.

This development raises the possibility of a more general unified theory of EU behavior with bounded awareness that might encompass a wide range of observed behavior as well as being consistent with the fundamental postulate that all humans have bounded cognitive capacity.

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