The measurement of resilience*

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Abstract. We provide an axiomatic approach to the measurement of individual resilience. Resilience has been an increasingly important topic in many social sciences and policy circles but, as of now, there does not seem to be much literature on its theoretical foundations. This paper is intended to fill that gap. After an introduction to the notion of resilience and its possible determinants, we introduce a set of intuitively appealing properties that a resilience measure is required to possess. Our result is a characterization of the specific resilience ordering that satisfies these axioms.

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1 Introduction

Resilience has become a highly popular research topic over the last few decades in several disciplines. As Bonanno, Romero and Klein (2015) report, the frequency with which the term 'resilience' or one of its variants appear in the titles of articles published in social-sciences journals has quadrupled from 2000 to 2010, jumping to 800 occurrences. A similarly increasing trend is reported by Hodgson, McDonald and Hosken (2015) for the International Statistical Institute's Web of Science (ISI WoS) where its prevalence as a keyword in peer-reviewed papers in the ecology category has been rising steadily since the early 1970s. Particularly active contributors are psychologists and ecologists who routinely dedicate the first few pages of their writings to a discussion of the definition of the term and mention that it has taken on multiple meanings. The contributions of Ayed, Toner and Priebe (2018), Fletcher and Sarkar (2013), Bonanno (2012), Bonanno, Romero and Klein (2015), among others, are examples within the psychology literature; Hodgson, McDonald and Hosken (2015) or Standish, Hobbs, Mayfield, Bestelmeyer, Suding, Battaglia, Eviner, Hawkes, Temperton, Cramer, Harris, Funk and Thomas (2014), for example, can be consulted in the context of ecology.

The etymology of the term 'resilience' has its roots in the Latin verb *resilire,* meaning 'to jump back' or 'to recoil' and it is defined in the Merriam-Webster dictionary as "the capability of a strained body to recover its size and shape after deformation caused especially by compressive stress" or "an ability to recover from or adjust easily to misfortune or change." The first definition relates to the use of the term in materials science, whereas the second describes it in relation to the social sciences. Both definitions help in visualizing the subject matter of our contribution: resilience captures the response in terms of the functioning of an individual when 'squeezed' by the occurrence of an adverse event such as the death of a spouse, a divorce, a job loss, a terrorist attack, a natural disaster or a severe injury. A resilient individual, once squeezed, is able to move toward the pre-event functioning level quickly. The variable that is mostly used in psychology to capture the functioning of an individual is his or her self-reported health status. Similar observations apply to macro settings, such as an ecosystem whose equilibrium is perturbed by human or natural activities.

An additional distinction in the psychological literature exists depending on the level of functioning reached at the end of the process. Resilience is often associated with a full recovery from the adverse event; the term *thriving* is applied when the person is better off after overcoming adversity as compared to before the event occurred; see, among others, Carver (1998). The latter phenomenon is also known as *growth following adversity* (Linley and Joseph, 2004) or *post-traumatic growth* (Tedeschi and Calhoun, 2004) which can be attributed to newly developed individual skills and a psychological sense of mastery following the negative event.

The confusion with the uses (and abuses, as Bonanno, 2012, one of the leading resilience researchers within psychology, puts it) of the term resilience is rooted in the fact that several contributors attempt to capture the characteristics of a resilient individual or system, rather than focusing on the process described above. In other words, instead of measuring the functioning process following an event, they focus on the predictors of resilient outcomes affecting the process; these predictors may be personal, social, or a notion of system resources. This duality in the approaches to resilience is documented in the systematic review of the mental health literature by Ayed, Toner and Priebe (2018). They identify two broad categories of approaches to resilience, namely, *processes* and *characteristics.* Bonanno, Romero and Klein (2015) offer an integrative framework of this duality by discussing how the process of response to an event is influenced by characteristics, with the process being the subject matter of resilience rather than the characteristics of individuals or societies.

A similar duality of approaches to resilience is present in ecology. The term has been used with different interpretations, leading to the "confusion of resilience" (Hodgson, McDonald and Hosken, 2015, p. 503). In the ecology literature, the majority of contributors follow Holling (1973) in defining resilience as a measure of the ability of ecosystems to absorb disturbances without changing identity. As Scheffer, Carpenter, Foley, Folke and Walker (2001, p. 591) put it, resilience "corresponds to the maximum perturbation that can be taken without causing a shift to an alternative stable state." Pimm (1984, p. 322) proposes an alternative process-oriented approach according to which resilience indicates "how fast the variables return towards their equilibrium following a perturbation." The ecological literature defines Holling's interpretation of resilience as *ecological* resilience and Pimm's as *engineering* resilience (Gunderson, Allen and Holling, 2009), and some contributors (see Standish et al., 2014) propose to relabel Holling's definition by referring to it as 'resilience' and to name Pimm's definition 'recovery' to reduce the confusion about these two important concepts. We note that, in the field of ecology, there seems to be a preference for the characteristics approach.

There are numerous contributions by economists that address resilience in ecology by developing deterministic and stochastic models with regime shifts and estimating the underlying system properties. These models have been used to describe resource-management problems such as those pertaining to coral reefs, lakes, ocean-climate systems, woodland preservation, among others. For an excellent review see Li, Crépin and Folke (2018).

However, there does not seem to be much of a literature within economics when it comes to the measurement of individual resilience. This is somewhat surprising because resilience appears to be linked to individual and social well-being—and to the high economic costs associated with aversive events. Resilience is also at the basis of the new approaches to economic challenges of the OECD (see http://www.oecd.org/naec/projects/resilience/) aiming to better guide policymakers.

To the best of our knowledge, there are only two exceptions in this area of research, firmly based on empirical issues. The first is Etilé, Frijters, Johnston and Shields (2017), who propose an empirical measure of resilience estimating a dynamic finite-mixture model for the Australian population. These authors derive individual-specific values of the parameters that govern individual heterogeneity in the psychological response to ten major adverse events and identify three classes of individuals that differ in their responses to the events. The second proposal is by Cissé and Barrett (2018), who implement an earlier conceptualization of development resilience by Barrett and Constas (2014, p. 14626) as "the capacity over time of a person, household or other aggregate unit to avoid poverty in the face of various stressors and in the wake of myriad shocks." Cissé and Barrett (2018) propose an econometric strategy for estimating multiple conditional moments of a welfare function that enables the computation and forecasting of individual-specific conditional probabilities of being out of poverty.

Measures of individual resilience have been proposed by psychologists in the form of measurement scales, such as the Connor-Davidson Resilience Scale (CD-RISC), that capture characteristics of individuals; for detailed reviews of resilience measurement scales see, for instance, Windle, Bennett and Noyes (2011) and Salisu and Hashim (2017). The CD-RISC scale is based on 25 items evaluated on a five-point Likert scale ranging from 0 (never) to 4 (nearly all of the time). These ratings are added across the 25 categories to arrive at a total score between 0 and 100, and higher scores indicate higher resilience. The scores themselves are obtained from individual answers to questions on the ability to adapt to change, the availability of close and secure relationships, the preference to take the lead in problem solving, and similar attributes.

In this paper, we provide an axiomatic approach to the measurement of resilience, thereby complementing the empirically oriented contributions alluded to above with a theoretical analysis. We are not aware of any earlier work that addresses the basic foundations of measuring resilience and we hope that our observations and analysis provide a substantial step towards filling this gap. Thus, our work plays a role similar to that of Esteban and Ray's (1994) seminal paper on the measurement of polarization. The notion of polarization had been discussed in the literature prior to the publication of their article but Esteban and Ray's (1994) is the first contribution that provides a systematic theoretical examination of the phenomenon.

Our approach to resilience and its measurement relates to the established literature by focusing on the functioning of an individual following an adverse event rather than the characteristics of the individual. Also, we concentrate on the ability to recover from a disturbance rather than the ability to absorb or resist a shock. For these reasons, our measure is an intuitively appealing index that represents an attempt to define and quantify recovery resilience. Its simplicity makes it easy to operationalize in large-scale household survey data where individual resistance to shocks cannot be explicitly separated from the shocks themselves.

The starting point of our analysis is the notion of a stream that describes the functioning of a system or individual over time. For the sake of concreteness, we will apply the term *health stream*, that is, a stream of values of individual health variables over time. These streams could be obtained by means of the mental health component of the Short-Form 12 Health Survey (SF-12), for instance, but our results are applicable to more general settings. In the illustrations of our measure, the health streams we use consist of self-assessed health status and satisfaction with own health. The objective is to establish an ordering defined on these streams that ranks them with respect to their relative resilience. Thus, we propose an ordinal measure of resilience. Ordinal approaches to social index numbers are rather common in the theory of social index numbers. For instance, an ordinal approach to poverty measurement is presented by Sen (1976), and Blackorby and Donaldson (1984) and Ebert (1987) discuss ordinal inequality indices.

We propose a set of intuitively appealing properties of a resilience ordering and it turns out that there is a single specific ordering that satisfies all of them. Although our proposal is ordinal in nature, we suggest how we can use the ranks of individuals to draw conclusions regarding aggregate resilience. The measure is attractive because it can be applied in empirical studies due to the availability of numerous datasets that are suited to our approach, such as the German Socio-Economic Panel Study (SOEP).

In the following section we consider a simple environment which can be used to motivate the properties that we impose on the resilience ordering. In this setting we can also suggest how increased resilience as measured is desirable from the perspective of the individual concerned as well as society at large. We also apply this simple environment as an idealized benchmark for explaining modeling choices we will make when adapting the measure to real-world data sets. In the subsequent Section 3 we then identify the notion of a *down spell* in the context of streams that report an individual's health value in a number of consecutive periods. A down spell is interpreted as a set of time periods during which an adverse event occurs and a subsequent (partial or full) recovery may or may not occur, and this concept forms the foundation for our resilience measure. The identification of such down spells are illustrated in a number of examples. The formal definitions of resilience orderings in general and of our specific proposal are given in Section 4. The axioms (properties) that we impose on a resilience measure are introduced and discussed in Section 5, and our result—a characterization of the resilience ordering that possesses all of our properties—is contained in Section 6. We conclude in Section 7 and establish the independence of our axioms in an appendix.

2 Resilience in a simple environment

Consider a simple (or idealized) environment where the individual has a given level of normal health and experiences a single adverse event, which we will refer to as a *down spell*. Thereafter the individual might return to normal health. In this simple environment we assume that it is impossible for the individual to exceed the level of normal health. Moreover, we assume that the health variable is experienced in continuous time and that the down spell occurs at an instance, with the health variable process being continuous almost everywhere, so that it is Riemann integrable. The assumption that the down spell occurs at an instance justifies an underlying assumption that the individual cannot resist the instantaneous effect and can only react to the down spell through recovering back to normal health. However, as we will assume also throughout the paper, measurement occurs at discrete points in time with a constant time interval between each measurement.

Let $(x(t))_{t=0}^{\infty}$ be the development of the non-negative health variable as a function of continuous time. Assume that the level of normal health equals $x^0 > 0$. Also, let time 0 equal the time at which the down spell occurs. Hence, $\alpha = x^0 - x(0)$ is the *amplitude* of the instantaneous down spell and $\beta = \int_0^{\infty}(x^0 - x(t))dt$ might be interpreted as an additive measure that captures the negative consequences (the *severity* or the *'badness'*) of the down spell, since health would have stayed at the normal level x^0 in the absence of the down spell. This interpretation means that an alternative stream $(x'(t))_{t=0}^{\infty}$ with $x'(t) \geq x(t)$ for all $t \geq 0$, $\alpha' = x^0 - x'(0) = \alpha$, and $\beta' = \int_0^{\infty}(x^0 - x'(t))dt < \beta$ would be preferable to $(x(t))_{t=0}^{\infty}$ as the amplitude is the same but the negative consequences are smaller. This sensitivity will be captured in the general discrete-time environment by the axiom of *recovery monotonicity*.

Furthermore, the interpretation of the integral β as a measure of the severity of the down spell means that varying the timing of the effects of the down spell across time periods does not matter as long as the integral remains the same. Such additivity of the measurement of the severity of the down spell will be captured in the general discrete-time environment by the axioms of *recovery neutrality* and *recovery translation invariance*. Integrating the health variable over time requires that the scale of the health variable is unit comparable between time periods, allowing for *increasing affine* transformations of the health variable but not *arbitrary* (not necessarily affine) increasing transformations. This assumption is needed to ensure that the axioms of recovery neutrality and recovery translation invariance are meaningful. We note that this is a common and largely uncontroversial requirement that is (at least implicitly) made in most approaches to social index numbers, such as inequality or poverty orderings that are based on individual incomes.

The measurement of resilience as the ability to recover from a down spell will depend on the relationship between its amplitude α and its severity β. We will provide structure to the discussion of the question of how to measure resilience by proposing a model of exponential resilience in the context of the simple environment discussed in this section. In particular, we will assume that the ability to recover will be proportional to the difference between normal health and experienced health. Formally, in the case of exponential resilience a stream x is given by

$$x(t) = \begin{cases} x^0 - \alpha e^{-\frac{\rho}{1-\rho}t} & \text{for all } t > 0 \text{ if } \rho \in [0,1), \\ x^0 & \text{for all } t > 0 \text{ if } \rho = 1, \end{cases}$$

where $\rho \in [0,1]$ is a parameter expressing the resilience of the individual. Hence, a more resilient individual has a higher value of ρ, with $\rho = 0$ corresponding to no recovery and $\rho = 1$ corresponding to instantaneous recovery. For any $\rho \in [0,1)$, the absolute rate $\dot{x}(t)$ of recovery is proportional to the difference $x^0 - x(t)$ between normal health and experienced health, that is,

$$\dot{x}(t) = -\frac{\rho}{1-\rho}\left(-\alpha e^{-\frac{\rho}{1-\rho}t}\right) = \frac{\rho}{1-\rho}\left(x^0 - x(t)\right).$$

This implies that

$$\beta = \alpha \int_0^\infty e^{-\frac{\rho}{1-\rho}t} dt = \frac{\alpha}{\frac{\rho}{1-\rho}}[1-0] = \alpha \cdot \frac{1-\rho}{\rho}$$

for $\rho \in (0,1)$, while $\beta = \infty$ if $\rho = 0$ and $\beta = 0$ if $\rho = 1$. Thus, ρ can be measured in continuous time by α and β according to the equality $\rho\beta = \alpha - \rho\alpha$, implying that

$$\rho = \frac{\alpha}{\alpha + \beta}.$$

When recovery is governed by an exponential process with $\rho \in [0,1)$, then if the amplitude α is multiplied by some positive constant λ, so that $x^0 - x'(0) = \alpha' = \lambda\alpha =$

$\lambda\left(x^0 - x(0)\right)$, then the difference between normal health and realized health will be multiplied by the same positive constant λ also at all later times. This follows since

$$x^0 - x'(t) = \alpha' e^{-\frac{\rho}{1-\rho}t} = \lambda \alpha e^{-\frac{\rho}{1-\rho}t} = \lambda\left(x^0 - x(t)\right)$$

for all $t > 0$. The same holds when $\rho = 1$ since $x^0 - x'(t) = 0 = \lambda 0 = \lambda\left(x^0 - x(t)\right)$ for all $t > 0$ in this case. Hence, for all $\rho \in [0,1]$,

$$\beta' = \int_0^\infty \left(x^0 - x'(t)\right) dt = \lambda \int_0^\infty \left(x^0 - x(t)\right) dt = \lambda\beta$$

so that

$$\frac{\alpha'}{\alpha' + \beta'} = \frac{\alpha}{\alpha + \beta} = \rho.$$

Note that the health stream $(x'(t))_{t=0}^\infty$ is derived from the original stream $(x(t))_{t=0}^\infty$ by applying an increasing affine transformation with a multiplicative constant $\lambda > 0$ and an intercept $\mu = (1-\lambda)x^0$ so that

$$x'(t) = \lambda x(t) + (1-\lambda)x^0$$

for all $t \geq 0$. This specific transformation has the property that the level of normal health x^0 is kept unchanged, $x^0 = \lambda x^0 + (1-\lambda)x^0$, which is important in the context of the simple environment considered in this section. The property that applying an increasing affine transformation to the health variable for all t leads to the same resilience will be captured in our general discrete-time environment by the axiom of *affine invariance*.

Exponential resilience has the interesting feature that the resilience parameter ρ can be perfectly measured even if the measurement of the health variable occurs at discrete times with a constant interval between each measurement. Assume that, without loss of generality, that these are unitary intervals with the first post-down-spell measurement at $\Delta \in [0,1)$. Letting

$$\delta = e^{-\frac{\rho}{1-\rho}},$$

this leads to the stream of observations

$$x^0 - \alpha e^{-\frac{\rho}{1-\rho}\Delta},\ x^0 - \delta\alpha e^{-\frac{\rho}{1-\rho}\Delta},\ \ldots,\ x^0 - \delta^t \alpha e^{-\frac{\rho}{1-\rho}\Delta},\ \ldots$$

when $\rho \in [0,1)$, and $x^0 - \alpha, x^0, \ldots, x^0, \ldots$ or the down spell is not detected when $\rho = 1$. Hence, when $\rho \in [0,1)$, the *measured* amplitude a of the down spell is given by

$$a = x^0 - x(\Delta) = \alpha e^{-\frac{\rho}{1-\rho}\Delta}$$

and its *measured* severity b is given by

$$b = \sum_{n=1}^\infty (x^0 - x(\Delta + n)) = (\delta + \delta^2 \ldots + \delta^t + \ldots) \cdot a = \begin{cases} \infty & \text{if } \rho = 0, \\ \frac{\delta}{1-\delta} \cdot a = \frac{e^{-\frac{\rho}{1-\rho}}}{1-e^{-\frac{\rho}{1-\rho}}} \cdot a & \text{if } \rho \in (0,1). \end{cases}$$

Moreover, when $\rho = 1$ and the down spell is detected, measured amplitude a equals α and measured severity b equals 0. Therefore, *measured* resilience r defined by $r = a/(a+b)$ is obtained as

$$r = \frac{a}{a+b} = \begin{cases} 0 & \text{if } \rho = 0, \\ \frac{1}{1+\frac{\delta}{1-\delta}} = 1 - \delta = 1 - e^{-\frac{\rho}{1-\rho}} & \text{if } \rho \in (0,1), \\ 1 & \text{if } \rho = 1. \end{cases}$$

It follows that $r \in [0,1]$ is an increasing and continuous function of ρ and, thus, ordinally equivalent to ρ. The function does not depend on the times of measurement. In view of the natural choice of ρ as the resilience value in this simple exponential model, this is a powerful argument in favor of using r as a measurement of resilience, which is indeed what we propose.

An alternative possibility that we will not pursue is that resilience is arithmetic. In that case, a stream is given by

$$x(t) = \begin{cases} x^0 - \left(\alpha - \frac{\rho^*}{2(1-\rho^*)} t\right) & \text{for all } t > 0 \text{ if } \rho \in [0,1), \\ x^0 & \text{for all } t > 0 \text{ if } \rho^* = 1, \end{cases}$$

where $\rho^* \in [0,1]$ is a different parameter expressing the resilience of the individual. In this alternative formulation, the individual is fully recovered at $t = (2(1-\rho^*)/\rho^*)\,\alpha$ if $\rho^* \in (0,1]$, while the individual never recovers if $\rho^* = 0$. Then

$$\beta = \int_0^{\frac{2(1-\rho^*)}{\rho^*}\alpha} \left(\alpha - \frac{\rho^*}{2(1-\rho^*)} t\right) dt = \alpha^2 \left(\frac{2(1-\rho^*)}{\rho^*} - \frac{1-\rho^*}{\rho^*}\right) = \alpha^2 \cdot \frac{1-\rho^*}{\rho^*}$$

for $\rho^* \in (0,1)$, while $\beta = \infty$ if $\rho^* = 0$ and $\beta = 0$ if $\rho^* = 1$. Thus, also in this alternative can ρ^* be measured in continuous time by α and β according to the relationship $\rho^*\beta = \alpha^2 - \rho^*\alpha^2$, implying that

$$\rho^* = \frac{\alpha^2}{\alpha^2 + \beta}.$$

Arithmetic resilience has the property that increased amplitude leads to a proportional increase in recovery time, which could have been captured by an axiom postulating invariance to proportional changes in amplitude and recovery time. However, under arithmetic resilience, discrete-time measured resilience r^* defined as $r^* = a^2/(a^2 + b)$ will depend not only on the parameter ρ^*, but also on the time of measurement Δ. This undesirable property is one reason why we do not pursue the alternative of arithmetic resilience any further.

When we move beyond this simple environment in the remainder of the paper, we must take into account the possibility that a down spell might last for several periods and that the individual might improve his health status beyond the pre-down-spell level. We must also acknowledge that measurement of the health variable might commence after a down spell has started and stop before full recovery has been achieved. As we will explain in more detail in subsequent sections, we will assume that the individual cannot resist the down spell even when the downward movement lasts for several periods, using measured amplitude a as a measurement of the adverse event. And we will use measured severity b as a measure for the consequences of the adverse event even though a down spell might have negative consequences beyond the time at which full recovery occurs or the observations cease.

3 Down spells in a general discrete-time environment

As mentioned before, our starting point is a socio-economic variable that we want to assess with respect to the notion of resilience. We assume that the data for which the requisite comparisons are to be performed consist of observed health values for a number of consecutive time periods. In order to exclude degenerate cases, we restrict attention to streams of health variables that cover at least three time periods. The finite length of a stream (and, thus, the number of observations) is denoted by T so that $T \in \mathbf{T}$, where $\mathbf{T} = \mathbb{N} \setminus \{1, 2\}$ is the set of positive integers excluding the numbers 1 and 2. For a possible stream length $T \in \mathbf{T}$, a health stream $x = (x_1, \ldots, x_T)$ is composed of T observations, one for each period from 1 to T. We assume that the observed health values are non-negative so that x is an element of $\mathbb{R}^T_+$, the set of all T-dimensional vectors with non-negative components. The length of a stream may vary so that the set from which x is chosen is the union $\cup_{T\in\mathbf{T}} \mathbb{R}^T_+$.

The variables we consider have to be interpreted in a way so that the resilience ordering to be established is invariant with respect to *increasing affine* transformations of the health variable but not with respect to *arbitrary* (not necessarily affine) increasing transformations. This assumption is needed to ensure that some of our axioms are meaningful, as we have already alluded to in Section 2. We will use several figures to illustrate health streams in the remainder of this section. In these diagrammatic examples, the symbols σ^x, σ^x_1 and σ^x_2 are meant to indicate given down spells in the requisite stream x but this piece of notation can be safely ignored for the time being; it will be formally introduced and discussed in detail later on.

Consider a stream x of length T. To identify the down spells that are present in the stream x, we begin by partitioning the full set of time periods $\{1, \ldots, T\}$ into three sets. These three sets represent (i) the periods associated with sustained health; (ii) the time periods in which down spells occur; and (iii) the set of time periods in which (possibly partial) recoveries may or may not occur.

We denote the set of periods in x with sustained health by $\mathbf{S}^x$. The idea is to include, starting from the first period, all time periods among those in $\{1, \ldots, T\}$ that are associated with maximal non-decreasing values in the health variable. Hence, if a decrease in the health level occurs, the individual no longer experiences sustained health. Formally, this set $\mathbf{S}^x$ is defined inductively as follows. The initial period (period 1) always belongs to this set so that we have $1 \in \mathbf{S}^x$. Now let $t \in \{2, \ldots, T\}$ and assume that we have examined each of periods 1 to $t-1$ to determine whether it is a member of $\mathbf{S}^x$. If $x_t \geq x_\tau$ for all $\tau \in \mathbf{S}^x \cap \{1, \ldots, t-1\}$, then $t \in \mathbf{S}^x$. Thus, if we reach period t and the level of health does not drop when considering those periods between 1 and $t-1$ that are already in the set $\mathbf{S}^x$, then period t is added to this set of periods with sustained health; if x_t is below one the values of $x_1, \dots, x_{t-1}$ that have already been added to $\mathbf{S}^x$ in a previous step of the iteration, then t does not belong to the set of periods with sustained health.

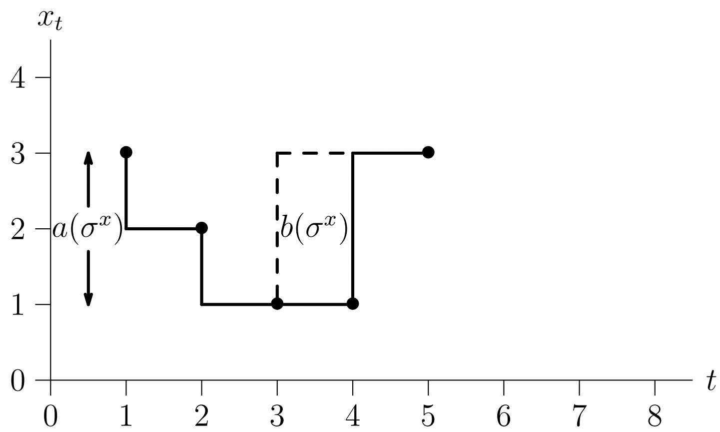

Figure 1: The health stream $x = (3, 2, 1, 1, 3)$.

To illustrate this construction, consider the health stream $x = (3, 2, 1, 1, 3) \in \mathbb{R}^5_+$ of Figure 1. According to the iterative procedure just defined, the initial period 1 is always in the set of periods with sustained health, that is, we have $1 \in \mathbf{S}^x$. To determine whether period $t = 2$ is in $\mathbf{S}^x$, we compare x_2 to the values that correspond to the periods that have already been added to $\mathbf{S}^x$. In this case, the only comparison is that involving x_1 because

$$\mathbf{S}^x \cap \{1, \dots, t-1\} = \mathbf{S}^x \cap \{1\} = \{1\}.$$

We have $x_2 = 2 < 3 = x_1$ and, therefore, $2 \notin \mathbf{S}^x$. The same is true for periods 3 and 4. For $t = 3$, we have

$$\mathbf{S}^x \cap \{1, \dots, t-1\} = \mathbf{S}^x \cap \{1, 2\} = \{1\}$$

and $x_3 = 1 < 3 = x_1$ so that, according to our definition, $3 \notin \mathbf{S}^x$. For $t = 4$, it follows that

$$\mathbf{S}^x \cap \{1, \dots, t-1\} = \mathbf{S}^x \cap \{1, 2, 3\} = \{1\}$$

and $x_4 = 1 < 3 = x_1$ so that, again, $4 \notin \mathbf{S}^x$. The final candidate for membership in $\mathbf{S}^x$ is period $t = T = 5$. It follows that

$$\mathbf{S}^x \cap \{1, \dots, t-1\} = \mathbf{S}^x \cap \{1, \dots, 4\} = \{1\}$$

and $x_5 = 3 \geq 3 = x_1$ so that, by definition, $5 \in \mathbf{S}^x$ and hence $\mathbf{S}^x = \{1, 5\}$.

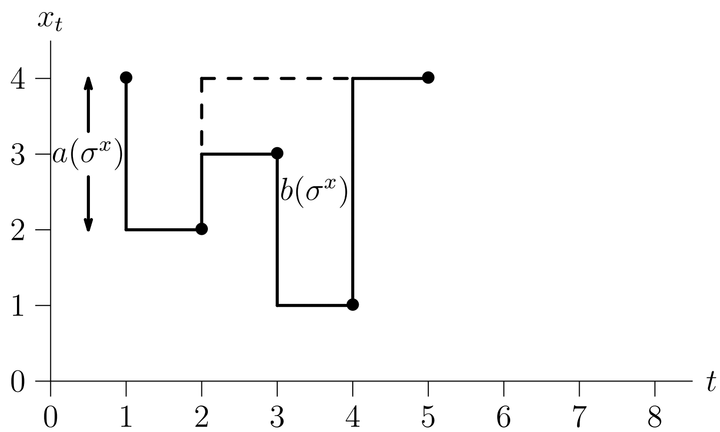

Figure 2: The health stream $x = (4, 2, 3, 1, 4)$.

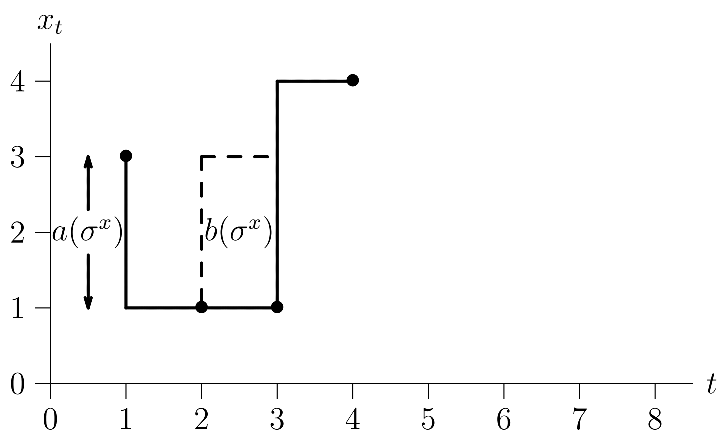

Figure 3: The health stream $x = (3, 1, 1, 4)$.

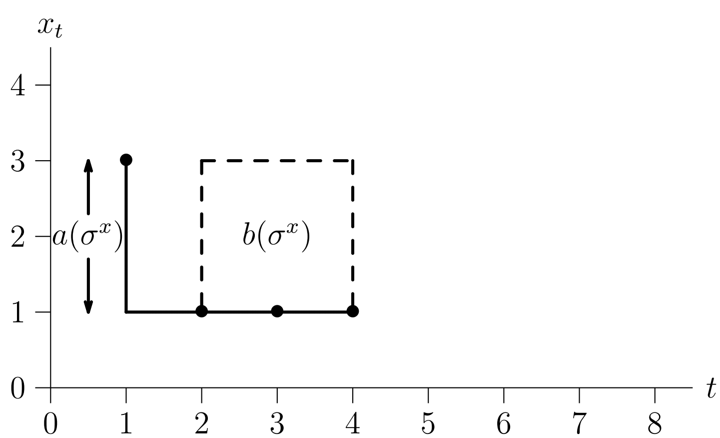

Figure 4: The health stream $x = (3, 1, 1, 1)$.

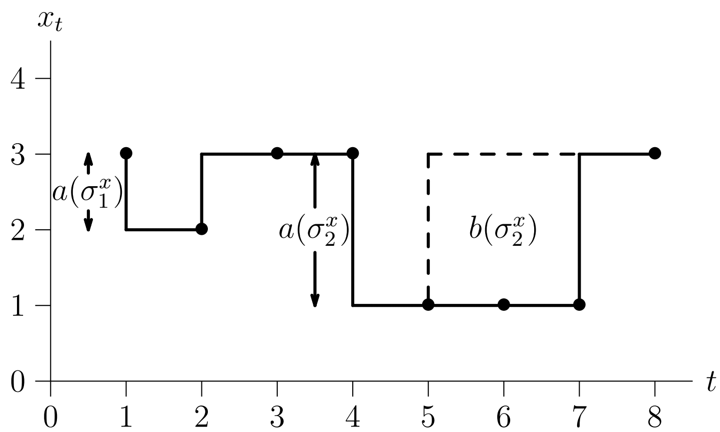

Figure 5: The health stream $x = (3, 2, 3, 3, 1, 1, 1, 3)$.

By using the same iterative procedure, we obtain that $\mathbf{S}^x = \{1, 5\}$ in the health stream $x = (4, 2, 3, 1, 4) \in \mathbb{R}^5_+$ of Figure 2, $\mathbf{S}^x = \{1, 5\}$ in the health stream $x = (3, 1, 1, 4) \in \mathbb{R}^4_+$ of Figure 3, $\mathbf{S}^x = \{1\}$ in the health stream $x = (3, 1, 1, 1) \in \mathbb{R}^4_+$ of Figure 4, and $\mathbf{S}^x = \{1, 3, 4, 8\}$ in the health stream $x = (3, 2, 3, 3, 1, 1, 1, 3) \in \mathbb{R}^8_+$ of Figure 5.

Next, we define the set $\mathbf{D}^x$ of time periods in which down spells occur for a stream x. The construction of this set is intuitive: a period t is part of a down spell if there is an earlier period τ in which there is sustained health such that the value of the health variable is in decline between τ and t. That is, for any $t \in \{1, \ldots, T\}$, $t \in \mathbf{D}^x$ if there exists a period $\tau \in \mathbf{S}^x \cap \{1, \ldots, t-1\}$ such that $x_\tau > \cdots > x_t$. It follows by definition that the set of periods that involve sustained health and the set of periods in which down spells occur must be disjoint.

We use the stream $x = (3, 2, 1, 1, 3)$ of Figure 1 again to provide an illustration of this definition of $\mathbf{D}^x$. For this example, we have $\mathbf{S}^x = \{1, 5\}$. Consider first the period $t = 2$. Because there exists a period $\tau \in \mathbf{S}^x \cap \{1, \ldots, t-1\} = \{1\}$ (namely, period $\tau = 1$) such that

$$x_\tau = x_1 = 3 > 2 = x_2,$$

it follows that period 2 is in $\mathbf{D}^x$. Moreover, because $1 \in \mathbf{S}^x \cap \{1, 2\}$ and

$$x_\tau = x_1 = 3 > 2 = x_2 > 1 = x_3,$$

period $t = 3$ must be a member of $\mathbf{D}^x$ as well. Because $x_3 = 1 \leq 1 = x_4$, the last inequality that defines membership in $\mathbf{D}^x$ is not satisfied for period 4 and, therefore, $4 \notin \mathbf{D}^x$. Thus, we obtain $\mathbf{D}^x = \{2, 3\}$ for this example.

By using the same iterative procedure, we obtain that $\mathbf{D}^x = \{2\}$ in the health stream $x = (4, 2, 3, 1, 4) \in \mathbb{R}^5_+$ of Figure 2, $\mathbf{D}^x = \{2\}$ in the health stream $x = (3, 1, 1, 4) \in \mathbb{R}^4_+$ of Figure 3, $\mathbf{D}^x = \{2\}$ in the health stream $x = (3, 1, 1, 1) \in \mathbb{R}^4_+$ of Figure 4, and $\mathbf{D}^x = \{2, 5\}$ in the health stream $x = (3, 2, 3, 3, 1, 1, 1, 3) \in \mathbb{R}^8_+$ of Figure 5. Note in particular that downward movement from period 3 to 4 for the example of Figure 2 is not part of the down spell since the recovery from period 2 to 3 is only partial. Also, for the example of Figure

4, only period 2 is in $\mathbf{D}^x$ even though no recovery occurs, since no downward movement occurs between period 2 and period 3.

Finally, to complete the description of our partition, the set of periods $\mathbf{U}^x$ is defined as the complement of the union $\mathbf{S}^x \cup \mathbf{D}^x$ in $\{1, \ldots, T\}$, that is,

$$\mathbf{U}^x = \{1, \ldots, T\} \setminus (\mathbf{S}^x \cup \mathbf{D}^x).$$

We refer to the set $\mathbf{U}^x$ as the recovery phase, that is, the set of time periods during which recovery occurs. Note that $\mathbf{U}^x$ may be empty for some streams x. It is immediate that, in the example of Figure 1, we obtain

$$\mathbf{U}^x = \{1, \ldots, T\} \setminus (\mathbf{S}^x \cup \mathbf{D}^x) = \{1, \ldots, 5\} \setminus (\{1, 5\} \cup \{2, 3\}) = \{4\}$$

and, for the example illustrated in Figure 5, it follows that

$$\mathbf{U}^x = \{1, \ldots, T\} \setminus (\mathbf{S}^x \cup \mathbf{D}^x) = \{1, \ldots, 8\} \setminus (\{1, 3, 4, 8\} \cup \{2, 5\}) = \{6, 7\}.$$

In particular, note that there is no recovery phase after the first down spell of this example, as the recovery is immediate.

For the other examples, it follows that $\mathbf{U}^x = \{3, 4\}$ in the health stream $x = (4, 2, 3, 1, 4) \in \mathbb{R}^5_+$ of Figure 2, $\mathbf{U}^x = \{2\}$ in the health stream $x = (3, 1, 1, 4) \in \mathbb{R}^4_+$ of Figure 3, and $\mathbf{U}^x = \{3, 4\}$ in the health stream $x = (3, 1, 1, 1) \in \mathbb{R}^4_+$ of Figure 4.

With the partition $\{\mathbf{S}^x, \mathbf{D}^x, \mathbf{U}^x\}$ of $\{1, \ldots, T\}$ in hand, we can now proceed to a precise definition of a down spell. As seems natural, a down spell in stream x starts in a period in the set $\mathbf{S}^x$ of sustained health if the following period belongs to the set $\mathbf{D}^x$ in which down spells occur. Clearly, the number of down spells and their exact structure are stream-dependent. For the time being, we use the notation σ^x to indicate a generic down spell in x, without explicitly referring to the number of spells in a stream at this stage.

As hinted at above, a down spell σ^x in x starts in period $s(\sigma^x) \in \mathbf{S}^x$ if

$$(s(\sigma^x) + 1) \in \mathbf{D}^x.$$

For example, if $x = (3, 2, 1, 1, 3)$ as in Figure 1, it follows that $s(\sigma^x) = 1$ for the single spell σ^x in x because $(s(\sigma^x) + 1) = 2 \in \mathbf{D}^x$. Analogously, the stream $x = (3, 2, 3, 3, 1, 1, 1, 3)$ of Figure 5 has two down spells that start at $s(\sigma^x_1) = 1$ and at $s(\sigma^x_2) = 4$.

To identify the end of a down spell σ^x, we use the following definition. If

$$\{s(\sigma^x) + 1, \ldots, d(\sigma^x)\} \subseteq \mathbf{D}^x \text{ and } (d(\sigma^x) + 1) \notin \mathbf{D}^x,$$

then the down spell σ^x ends at $d(\sigma^x)$. Note that this includes the possibility that $d(\sigma^x) = T$ if σ^x is the final down spell in the stream x. Thus, the down spell σ^x consists of the time periods in the set

$$D(\sigma^x) = \{s(\sigma^x) + 1, \ldots, d(\sigma^x)\}.$$

As is straightforward to verify, in the case of $x = (3, 2, 1, 1, 3)$, we obtain $d(\sigma^x) = 2$ and, for $x = (3, 2, 3, 3, 1, 1, 1, 3)$, it follows that $d(\sigma^x_1) = 2$ and $d(\sigma^x_2) = 5$.

If there exists $u(\sigma^x) \in \{d(\sigma^x), \ldots, T-1\}$ such that

$$\mathbf{S}^x \cap \{d(\sigma^x)+1, \ldots, u(\sigma^x)\} = \emptyset \text{ and } (u(\sigma^x)+1) \in \mathbf{S}^x,$$

then a full recovery after the down spell σ^x occurs in time period $u(\sigma^x)+1$. The set $U(\sigma^x)$ consists of the time periods after the down spell σ^x has finished and before a full recovery (if any) has occurred, that is,

$$U(\sigma^x) = \{d(\sigma^x)+1, \ldots, u(\sigma^x)\},$$

where $u(\sigma^x) < T$ if a full recovery occurs and $u(\sigma^x) = T$ if a full recovery does not occur. In particular, $U(\sigma^x) = \emptyset$ if $d(\sigma^x) = u(\sigma^x)$ so that recovery is immediate when $u(\sigma^x) < T$ and no recovery is feasible when $u(\sigma^x) = T$ owing to the constraint imposed by reaching the final time period T. In the example of Figure 1, we obtain $u(\sigma^x) = 4$; for Figure 5, the requisite time periods are $u(\sigma_1^x) = 2$ and $u(\sigma_2^x) = 7$. In the example of Figure 4, where no recovery occurs, $u(\sigma^x) = T = 4$.

The size of the down spell σ^x is measured by

$$a(\sigma^x) = x_{s(\sigma^x)} - x_{d(\sigma^x)}$$

where the letter a is associated with the 'a' in *a*mplitude of the down spell. Note that the length of a decline does not matter, only the amplitude. Compared to the simple environment introduced in Section 2, $a(\sigma^x)$ might underestimate the size of the down spell if (i) $s(\sigma) = 1$, but in fact the down spell started before the first time period of the health stream; (ii) $d(\sigma) = s(\sigma)+1$ and partial recovery had started already before the down spell was observed; or (iii) $d(\sigma) > s(\sigma)+1$ and the individual is able to partly resist the forces that cause the down spell.

The severity (or badness) of the consequences of the down spell σ^x is measured by

$$b(\sigma^x) = \sum\nolimits_{t \in U(\sigma^x)} \left(x_{s(\sigma^x)} - x_t\right)$$

where the letter b represents the 'b' in *b*adness. Compared to the simple environment introduced in Section 2, $b(\sigma^x)$ might underestimate the severity of the consequences of the down spell if (i) the relevant counterfactual is that the health variable grows beyond $x(s(\sigma^x))$; or (ii) full recovery has not been achieved by the last period T of the health stream.

We restrict attention to streams with at least one down spell for which (partial) recovery is not made infeasible by the time constraint—that is, after a drop from a sustained level of health, there is at least one time period left before the final period T is reached. Let $T \in \mathbf{T}$ and $x \in \mathbb{R}_+^T$, and consider all down spells σ^x in x for which recovery is not made infeasible so that $d(\sigma^x) < T$. To exclude trivial cases, we only consider streams that contain at least one such down spell.

Now denote the number of such down spells by m^x and the i^{th} of these spells by σ_i^x. Define

$$\Sigma(x) = \{\sigma_1^x, \ldots, \sigma_{m^x}^x\}$$

as the set of all down spells for which recovery is not made infeasible by the time constraint. By assumption, this set is non-empty. To simplify our exposition, we write σ^x instead of σ_1^x if $m^x = 1$, that is, if there is only one permissible down spell in the stream x; this does not create any ambiguity. The set H_T defined by

$$H_T = \{x \in \mathbb{R}_+^T \mid \Sigma(x) \neq \emptyset\}$$

contains all streams of length $T \in \mathbf{T}$ for which the notion of resilience is well-defined in the sense that recovery is not excluded by reaching the end of the sampling period T. Because T may be any integer greater than or equal to three, the set of such streams of any length is given by

$$\Omega = \bigcup_{T \in \mathbf{T}} H_T.$$

Thus, the set Ω constitutes the set of streams that we want to be able to compare by means of what we refer to as a *resilience ordering.*

4 Resilience orderings

A resilience ordering is a complete and transitive binary relation $\succsim$ defined on Ω with the interpretation 'at-least-as-resilient-as.' Thus, for any two streams x and y in Ω, the statement 'x is at least as resilient as y' is expressed by the relational statement $x \succsim y$. The relation $\succsim$ is complete if any two streams x and y in Ω can be compared, that is, if

$$x \succsim y \quad \text{or} \quad y \succsim x$$

for all $x, y \in \Omega$. Transitivity requires that if $x \succsim y$ and $y \succsim z$ for any three streams $x, y, z \in \Omega$, it must also be true that $x \succsim z$. The relation $\succsim$ can be partitioned into a 'more-resilient-than' relation $\succ$ and an 'as-resilient-as' relation $\sim$, defined by letting, for all $x, y \in \Omega$,

$$x \succ y \quad \text{if} \quad [x \succsim y \text{ and not } y \succsim x]$$

and

$$x \sim y \quad \text{if} \quad [x \succsim y \text{ and } y \succsim x]\,.$$

The specific resilience ordering $\succsim^r$ that we propose and characterize in this paper is based on comparing the values of a resilience measure $r\colon \Omega \to (0, 1]$ that is defined in terms of the amplitudes $a(\sigma_i^x)$ and the levels of severity $b(\sigma_i^x)$ associated with the spells that are present in a stream $x \in \Omega$. This resilience measure is defined by letting, for all $x \in \Omega$,

$$r(x) = \frac{\sum_{i=1}^{m^x} a(\sigma_i^x)}{\sum_{i=1}^{m^x} a(\sigma_i^x) + \sum_{i=1}^{m^x} b(\sigma_i^x)}.$$

Hence, the measure depends on the sum of the amplitudes of the down spells, as measured by $a(\sigma_i^x)$ for $i = 1, 2, \ldots, m$, and the sum of the severity levels of the down spells as measured by $b(\sigma_i^x)$ for $i = 1, 2, \ldots, m$. Thus, r increases with the amplitude of a down spell and decreases with its severity during recovery: a *ceteris-paribus* (partial) recovery from a larger drop is associated with higher resilience, and a *ceteris-paribus* increase in the severity of recovery means that resilience is lower. Clearly, $r(x)$ takes on values that are greater than zero and smaller than or equal to one because $a(\sigma_i^x)$ is positive and $b(\sigma_i^x)$ is non-negative. Furthermore, $r(x) = 1$ if recovery is always immediate, that is, if $u(\sigma_i^x) = d(\sigma_i^x)$ for all $i \in \{1, \ldots, m^x\}$.

Our resilience ordering $\succsim^r$ is defined by declaring $x \in \Omega$ to be at least as resilient as $y \in \Omega$ if the value of the resilience measure r at x is greater than or equal to the value of r at y. That is, for all $x, y \in \Omega$,

$$x \succsim^r y \;\Leftrightarrow\; r(x) \geq r(y).$$

We reiterate that the resilience measure r does *not* have any numerical significance—no comparisons other than relative resilience rankings are permissible in the ordinal setting considered throughout this paper.

The notion of *vulnerability* may be defined as the inverse of resilience, that is, as the value of a function $v \colon \Omega \to [1, \infty)$ defined by

$$v(x) = \frac{1}{r(x)}$$

for all $x \in \Omega$. Thus, for our particular measure, we obtain

$$v(x) = \frac{\sum_{i=1}^{m^x} a(\sigma_i^x) + \sum_{i=1}^{m^x} b(\sigma_i^x)}{\sum_{i=1}^{m^x} a(\sigma_i^x)} = \sum_{i=1}^{m^x} \left(\frac{a(\sigma_i^x)}{\sum_{j=1}^{m^x} a(\sigma_j^x)} \cdot \frac{a(\sigma_i^x) + b(\sigma_i^x)}{a(\sigma_i^x)} \right)$$

for all $x \in \Omega$, where $a(\sigma_i^x)/(\sum_{j=1}^{m^x} a(\sigma_j^x))$ is the endogenous weight given to down spell σ_i^x, and $(a(\sigma_i^x) + b(\sigma_i^x))/a(\sigma_i^x)$ is the vulnerability exhibited in down spell σ_i^x. Thus, each spell is weighted according to its fraction of the total amplitude—the sum of the amplitudes over all spells. The vulnerability ordering associated with our resilience ordering $\succsim^r$ is simply its reverse ordering, that is, $x \in \Omega$ is at least as vulnerable as $y \in \Omega$ if $y \succsim^r x$.

We have already provided an intuitive foundation for the resilience measure

$$r(x) = \frac{a(\sigma^x)}{a(\sigma^x) + b(\sigma^x)}$$

in the case of a single spell σ^x in the simple environment of Section 2, using a model of exponential resilience where the ability to recover is proportional to the difference between the pre-down-spell health and experienced health during recovery. In the subsequent sections we provide an axiomatic characterization of this measure in the case of a single down spell by showing that our resilience ordering $\succsim^r$ satisfies a number of properties and that $\succsim^r$ is the sole ordering that satisfies these properties.

Before doing so, we note how the resilience measure r and its inverse v have an intuitive geometric interpretation in the case of a single spell σ^x by returning to our examples. Consider first Figure 1. The stream of health values is given by $x = (3, 2, 1, 1, 3)$. It follows that $\mathbf{S}^x = \{1, 5\}$, $\mathbf{D}^x = \{2, 3\}$ and $\mathbf{U}^x = \{4\}$. There is a singleton set $\Sigma(x) = \{\sigma^x\}$ of down spells with $s(\sigma^x) = 1$, $d(\sigma^x) = 3$ and $u(\sigma^x) = 4$. We obtain $a(\sigma^x) = 2$ and $b(\sigma^x) = 2$ so that, as in the example of Figure 1,

$$r(x) = \frac{2}{2+2} = \frac{1}{2} \text{ and } v(x) = \frac{1}{r(x)} = 2.$$

In Figure 2, we have $x = (4, 2, 3, 1, 4)$. It follows that $\mathbf{S}^x = \{1, 5\}$, $\mathbf{D}^x = \{2\}$ and $\mathbf{U}^x = \{3, 4\}$. There is a singleton set $\Sigma(x) = \{\sigma^x\}$ of down spells with $s(\sigma^x) = 1$, $d(\sigma^x) = 2$ and $u(\sigma^x) = 4$. We obtain $a(\sigma^x) = 2$ and $b(\sigma^x) = 1 + 3 = 4$ so that

$$r(x) = \frac{2}{2+4} = \frac{1}{3} \text{ and } v(x) = \frac{1}{r(x)} = 3.$$

In the example of Figure 3, the requisite stream is $x = (3, 1, 1, 4)$. We obtain $\mathbf{S}^x = \{1, 4\}$, $\mathbf{D}^x = \{2\}$ and $\mathbf{U}^x = \{3\}$. There is a singleton set $\Sigma(x) = \{\sigma^x\}$ of down spells with $s(\sigma^x) = 1$, $d(\sigma^x) = 2$ and $u(\sigma^x) = 3$. Furthermore, we have $a(\sigma^x) = 2$ and $b(\sigma^x) = 2$ so that

$$r(x) = \frac{2}{2+2} = \frac{1}{2} \text{ and } v(x) = \frac{1}{r(x)} = 2.$$

There is no recovery in Figure 4. The stream of health values is $x = (3, 1, 1, 1)$. We have $\mathbf{S}^x = \{1\}$, $\mathbf{D}^x = \{2\}$ and $\mathbf{U}^x = \{3, 4\}$. There is a singleton set $\Sigma(x) = \{\sigma^x\}$ of down spells with $s(\sigma^x) = 1$, $d(\sigma^x) = 2$ and $u(\sigma^x) = 4$. It follows that $a(\sigma^x) = 2$ and $b(\sigma^x) = 2 + 2 = 4$ so that

$$r(x) = \frac{2}{2+4} = \frac{1}{3} \text{ and } v(x) = \frac{1}{r(x)} = 3.$$

In the case of two down spells as depicted in Figure 5, the stream of health values is $x = (3, 2, 3, 3, 1, 1, 1, 3)$. We have $\mathbf{S}^x = \{1, 3, 4, 8\}$, $\mathbf{D}^x = \{2, 5\}$ and $\mathbf{U}^x = \{6, 7\}$. There is a set of two down spells $\Sigma(x) = \{\sigma_1^x, \sigma_2^x\}$. For the first spell, we obtain $s(\sigma_1^x) = 1$, $d(\sigma_1^x) = 2$ and $u(\sigma_1^x) = 2$. It follows that $a(\sigma_1^x) = 1$ and $b(\sigma_1^x) = 0$. The requisite numbers for the second spell are $s(\sigma_2^x) = 4$, $d(\sigma_2^x) = 5$, $u(\sigma_2^x) = 7$, $a(\sigma_2^x) = 2$ and $b(\sigma_2^x) = 2 + 2 = 4$. Thus, we obtain

$$r(x) = \frac{1+2}{1+2+0+4} = \frac{3}{7} \text{ and } v(x) = \frac{1}{r(x)} = \frac{7}{3}.$$

Thus, according to our resilience ordering $\succsim^r$, the most resilient streams are those of Figures 1 and 3 (with a resilience value of 1/2), followed by that of Figure 5 (with a resilience of 3/7). At the bottom, the least resilient (and thus most vulnerable) streams are those in Figures 2 and 4 with a resilience level of 1/3.

5 Properties of a resilience ordering

In our characterization, we focus on the restriction of the resilience ordering to streams with a single down spell. Thus, we define the sets

$$H_T^1 = \{x \in H_T \mid |\Sigma(x)| = 1\}$$

for all $T \in \mathbf{T}$, and the domain considered in our main result is given by

$$\Omega^1 = \{x \in \Omega \mid |\Sigma(x)| = 1\}.$$

The restriction of a resilience ordering to streams with a single down spell is referred to as a single-spell resilience ordering.

For some of our properties, it is convenient to employ the following definition and notation. Two streams $x, y \in H_T^1$ *have the same timing structure* if $\mathbf{S}^x = \mathbf{S}^y$, $\mathbf{D}^x = \mathbf{D}^y$ and $\mathbf{U}^x = \mathbf{U}^y$. If $x, y \in H_T^1$ have the same timing structure, we write $s := s(\sigma^x) = s(\sigma^y)$, $d := d(\sigma^x) = d(\sigma^y)$, $u := u(\sigma^x) = u(\sigma^y)$, and $\mathbf{U} := \mathbf{U}^x = \mathbf{U}^y$. Note that, in this case, $\mathbf{U} = \{d+1, \ldots, u\}$ and $|\mathbf{U}| = u - d$.

Four of the six properties have already been discussed at an informal level in the simple environment introduced in Section 2. One of the two properties that are added here, *amplitude and recovery consistency*, is of crucial importance when we move beyond the simple environment and consider the general discrete-time environment introduced in Section 3; the other, *continuity*, ensures that small changes in the health values do not lead to large changes in resilience.

5.1 Recovery neutrality

We begin with a property that ensures that all periods in the recovery phase are treated equally by our measure. This implies, in particular, that no discounting can be employed. Thus, if the order of the health-variable values that occur during recovery is changed, this is a matter of equal resilience. In other words, the property ensures that our measure treats all time periods in which recovery occurs equally, paying no attention to the order in which the requisite health-variable values appear in a stream.

Recovery neutrality. For all $T \in \mathbf{T}$ and for all $x, y \in H_T^1$ with the same timing structure, if $x_\tau = y_\tau$ for all $\tau \in \{1, \ldots, T\} \setminus \mathbf{U}$ and $(y_\tau)_{\tau \in \mathbf{U}}$ is a permutation of $(x_\tau)_{\tau \in \mathbf{U}}$, then

$$x \sim y.$$

To illustrate this property, consider the streams $x = (5, 1, 2, 4, 3, 5)$ and $y = (5, 1, 3, 2, 4, 5)$. The two streams have the same timing structure with $\mathbf{U} = \{3, 4, 5\}$ and, because $y_3 = 3 = x_5$, $y_4 = 2 = x_3$ and $y_5 = 4 = x_4$, it follows that (y_3, y_4, y_5) is obtained from permuting (x_3, x_4, x_5). Thus, recovery neutrality requires that

$$(5, 1, 2, 4, 3, 5) = x \sim y = (5, 1, 3, 2, 4, 5).$$

5.2 Recovery translation invariance

Translation invariance is a commonly-imposed condition in the design of social index numbers; for example, absolute measures of inequality such as those of Kolm (1976) or Blackorby and Donaldson (1980) are translation invariant. In our setting, the property is defined for pairs of streams with the same timing structure. Although the label translation invariance typically refers to situations in which the same value is added to all components, our version goes beyond that by allowing these additions to be specific to each time period. Nevertheless, we use the term translation invariance because it captures the motivation underlying the axiom. Note that the property is analogous to the axiom of *independence of income source* employed by Weymark (1981) in the context of social welfare functions and inequality measures defined on income distributions. See also Blackorby, Bossert and Donaldson (2005, p. 118) who use a related axiom that they label *incremental equity* in a characterization of utilitarianism.

Recovery translation invariance as defined below demands that adding or subtracting the same vector of health values to the recovery phase of two streams without changing the common timing structure does not affect the relative ranking of the two streams.

Recovery translation invariance. For all $T \in \mathbf{T}$, for all $x, y \in H_T^1$ with the same timing structure and for all $z \in \mathbb{R}^T$ such that $x_\tau = y_\tau$ and $z_\tau = 0$ for all $\tau \in \{1, \ldots, T\} \setminus \mathbf{U}$, if $(x+z), (y+z) \in H_T^1$ and $\mathbf{U}^{(x+z)} = \mathbf{U}^{(y+z)} = \mathbf{U}$, then

$$(x+z) \succsim (y+z) \;\Leftrightarrow\; x \succsim y.$$

Again, we employ an example to illustrate this axiom. Let $x = (5, 1, 2, 4, 3, 5)$ and $y = (5, 1, 3, 3, 3, 5)$. The two streams have the same timing structure with $\mathbf{U} = \{3, 4, 5\}$. Defining $z = (0, 0, -1, -1, 1, 0) \in \mathbb{R}^6$, it follows that $(x+z), (y+z) \in H_T^1$ and $\mathbf{U}^{(x+z)} = \mathbf{U}^{(y+z)} = \mathbf{U}$. Therefore, recovery translation invariance requires that

$$(5, 1, 1, 3, 4, 5) = (x+z) \succsim (y+z) = (5, 1, 2, 2, 4, 5)$$

if and only if

$$(5, 1, 2, 4, 3, 5) = x \succsim y = (5, 1, 3, 3, 3, 5).$$

5.3 Recovery monotonicity

We assume that our resilience ordering possesses a plausible monotonicity property with respect to the health-variable values experienced in the recovery phase. In particular, if all values in the recovery phase increase (while all other health-variable values remain unchanged), resilience increases.

Recovery monotonicity. For all $T \in \mathbf{T}$ and for all $x, y \in H_T^1$ with the same timing structure such that $\mathbf{U} \neq \emptyset$, if $x_\tau > y_\tau$ for all $\tau \in \mathbf{U}$ and $x_\tau = y_\tau$ for all $\tau \in \{1, \ldots, T\} \setminus \mathbf{U}$, then

$$x \succ y.$$

For example, if $x = (5, 2, 3, 5, 4, 6)$ and $y = (5, 1, 2, 4, 3, 6)$, recovery monotonicity requires that $(5, 2, 3, 5, 4, 6) = x \succ y = (5, 1, 2, 4, 3, 6)$ because $x_\tau > y_\tau$ for all $\tau \in \mathbf{U} = \{3, 4, 5\}$ and $x_\tau = y_\tau$ for all $\tau \in \{1, 2, 6\}$.

5.4 Amplitude and recovery consistency

The following axiom requires that certain movements along a stream leave the value of vulnerability unchanged; in particular, only the amplitude and the severity are of importance. The duration of a downwards movement is irrelevant—all that matters is the amplitude of the drop. Furthermore, we do not distinguish between full recovery and excess recovery; any recovery that takes us beyond the pre-drop level is treated in the same way as a recovery to the pre-drop level. Finally, anything that happens prior to the down spell plays no role.

Amplitude and recovery consistency. For all $T, T' \in \mathbf{T}$, for all $x \in H^1_T$ and for all $y \in H^1_{T'}$ such that $|\mathbf{U}^x| = |\mathbf{U}^y|$, if there exists $t \in \mathbb{Z}$ such that $x_{s(\sigma^x)} = y_{s(\sigma^y)}$, $x_{d(\sigma^x)} = y_{d(\sigma^y)}$ and $x_\tau = y_{t+\tau}$ for all $\tau \in \mathbf{U}^x$, then

$$x \sim y.$$

Let $x = (3, 1, 2)$ and $y = (2, 3, 1, 2, 4)$. We have $s(\sigma^x) = 1$, $s(\sigma^y) = 2$, $d(\sigma^x) = 2$, $d(\sigma^y) = 3$, $\mathbf{U}^x = \{3\}$ and $\mathbf{U}^y = \{4\}$. Because

$$x_1 = y_2 = 3 \text{ and } x_2 = y_3 = 1 \text{ and } x_3 = y_4 = 2,$$

amplitude and recovery consistency requires that $x \sim y$.

5.5 Continuity

We employ a mild continuity property that ensures that small changes in the values of the health variables do not lead to large changes in resilience provided that the time period in which down spell ends does not change. This is another well-established condition that is employed throughout the literature concerned with the design of social index numbers.

Continuity. For all $T \in \mathbf{T}$, for all sequences $\langle x^k \rangle_{k \in \mathbb{N}}$, with $x^k \in H^1_T$ for all $k \in \mathbb{N}$ and $\lim_{k\to\infty} x^k = x \in H^1_T$, and for all $y \in H^1_T$,

$$\left[x^k \succsim y \text{ for all } k \in \mathbb{N} \text{ and } \lim_{k\to\infty} d(\sigma^{x^k}) = d(\sigma^x)\right] \Rightarrow x \succsim y$$

and

$$\left[y \succsim x^k \text{ for all } k \in \mathbb{N} \text{ and } \lim_{k\to\infty} d(\sigma^{x^k}) = d(\sigma^x)\right] \Rightarrow y \succsim x.$$

Consider the sequence $\langle x^k \rangle_{k \in \mathbb{N}}$ defined by letting $x^k = (5, 1, 2, 4, 3 - 1/k, 5)$ for all $k \in \mathbb{N}$. Furthermore, let $x = (4, 1, 3, 2, 3, 4)$. It follows that

$$\lim_{k\to\infty} x^k = (5, 1, 2, 4, 3, 5).$$

Continuity demands that if $x^k \succsim (4, 1, 3, 2, 3, 4)$ for all $k \in \mathbb{N}$, then

$$\lim_{k\to\infty} x^k = (5, 1, 2, 4, 3, 5) \succsim (4, 1, 3, 2, 3, 4)$$

and, likewise, if $(4, 1, 3, 2, 3, 4) \succsim x^k$ for all $k \in \mathbb{N}$, then

$$(4, 1, 3, 2, 3, 4) \succsim (5, 1, 2, 4, 3, 5) = \lim_{k\to\infty} x^k.$$

5.6 Affine invariance

Our final axiom is affine invariance. This requirement demands that resilience is invariant with respect to the application of a common increasing affine transformation to all health-variable values. In the present context, this property is of particular appeal because it is a consequence of the conformity of our measure with the case of exponential recovery in the simple continuous-time model described in Section 2. In the following definition, the symbol $\mathbf{1}_T$ is used to denote the T-tuple that consists of T ones.

Affine invariance. For all $T \in \mathbf{T}$, for all $x \in H_T^1$, for all $\lambda \in \mathbb{R}_{++}$ and for all $\mu \in \mathbb{R}$ such that $(\lambda \cdot x + \mu \cdot \mathbf{1}_T) \in H_T^1$,

$$(\lambda \cdot x + \mu \cdot \mathbf{1}_T) \sim x.$$

For $x = (5, 1, 2, 4, 3, 5)$, $\lambda = 1/2$ and $\mu = 1$, affine invariance requires that

$$(7/2, 3/2, 2, 3, 5/2, 7/2) = (\lambda \cdot (5, 1, 2, 4, 3, 5) + \mu \cdot \mathbf{1}_6) \sim (5, 1, 2, 4, 3, 5).$$

6 A characterization

Our main result is the following characterization of the resilience ordering $\succsim^r$.

Theorem 1. *A single-spell resilience ordering $\succsim$ satisfies recovery neutrality, recovery translation invariance, recovery monotonicity, amplitude and recovery consistency, continuity and affine invariance if and only if $\succsim = \succsim^r$.*

Proof. If. To show that $\succsim^r$ satisfies recovery neutrality, assume that $T \in \mathbf{T}$ and $x, y \in H_T^1$ have the same timing structure. If $x_\tau = y_\tau$ for all $\tau \in \{1, \ldots, T\} \setminus \mathbf{U}$ and $(y_\tau)_{\tau \in \mathbf{U}}$ is a permutation of $(x_\tau)_{\tau \in \mathbf{U}}$, it follows immediately that $r(x) = r(y)$ and hence $x \sim^r y$.

Now we establish recovery translation invariance. Let $T \in \mathbf{T}$, $x, y \in H_T^1$ and $z \in \mathbb{R}^T$ be such that x and y have the same timing structure, $x_\tau = y_\tau$ and $z_\tau = 0$ for all $\tau \in \{1, \ldots, T\} \setminus \mathbf{U}$, $(x + z), (y + z) \in H_T^1$ and $\mathbf{U}^{(x+z)} = \mathbf{U}^{(y+z)} = \mathbf{U}$. It follows that

$$\begin{aligned} a(\sigma^{x+z}) = a(\sigma^{y+z}) = a(\sigma^x) = a(\sigma^y), \\ b(\sigma^{x+z}) = b(\sigma^x) - \textstyle\sum_{t \in \mathbf{U}} z_t, \\ b(\sigma^{y+z}) = b(\sigma^y) - \textstyle\sum_{t \in \mathbf{U}} z_t. \end{aligned}$$

Therefore,

$$\begin{aligned} (x + z) \succsim^r (y + z) \quad &\Leftrightarrow \quad \frac{a(\sigma^{x+z})}{a(\sigma^{x+z}) + b(\sigma^{x+z})} \geq \frac{a(\sigma^{y+z})}{a(\sigma^{y+z}) + b(\sigma^{y+z})} \\ &\Leftrightarrow \quad \frac{a(\sigma^x)}{a(\sigma^x) + b(\sigma^x) - \sum_{t \in \mathbf{U}} z_t} \geq \frac{a(\sigma^x)}{a(\sigma^x) + b(\sigma^y) - \sum_{t \in \mathbf{U}} z_t} \\ &\Leftrightarrow \quad b(\sigma^x) \leq b(\sigma^y) \\ &\Leftrightarrow \quad x \succsim^r y. \end{aligned}$$

Next, we prove that $\succsim^r$ satisfies recovery monotonicity. Assume that $T \in \mathbf{T}$ and $x, y \in H_T^1$ have the same timing structure with $\mathbf{U} \neq \emptyset$. Furthermore, assume that $x_\tau > y_\tau$ for all $\tau \in \mathbf{U}$ and $x_\tau = y_\tau$ for all $\tau \in \{1, \ldots, T\} \setminus \mathbf{U}$. This immediately implies that $b(\sigma^x) < b(\sigma^y)$ and, because $\succsim^r$ decreases in the severity, it follows that $x \succ^r y$.

Now consider amplitude and recovery consistency. Let $T, T' \in \mathbf{T}$, $x \in H_T^1$ and $y \in H_{T'}^1$ be such that $|\mathbf{U}^x| = |\mathbf{U}^y|$. Furthermore, assume that there exists $t \in \mathbb{Z}$ such that $x_{s(\sigma^x)} = y_{s(\sigma^y)}$, $x_{d(\sigma^x)} = y_{d(\sigma^y)}$ and $x_\tau = y_{t+\tau}$ for all $\tau \in \mathbf{U}^x$. By definition, this implies that

$$a(\sigma^x) = x_{s(\sigma^x)} - x_{d(\sigma^x)} = y_{s(\sigma^y)} - y_{d(\sigma^y)} = a(\sigma^y)$$

and

$$b(\sigma^x) = \sum_{\tau \in \mathbf{U}} \left(x_{s(\sigma^x)} - x_\tau \right) = \sum_{\tau \in \mathbf{U}} \left(y_{s(\sigma^y)} - y_{t+\tau} \right) = b(\sigma^y)$$

and hence $r(x) = r(y)$, which implies $x \sim^r y$.

That continuity is satisfied follows immediately from the continuity of the restriction of the function r to H_T^1 for all $T \in \mathbf{T}$, provided that we require that $\lim_{k\to\infty} x^k = x \in H_T^1$ and $\lim_{k\to\infty} d(\sigma^{x^k}) = d(\sigma^x)$.

Finally, we prove that $\succsim^r$ satisfies affine invariance. Let $T \in \mathbf{T}$, $x \in H_T^1$, $\lambda \in \mathbb{R}_{++}$ and $\mu \in \mathbb{R}$ be such that $(\lambda \cdot x + \mu \cdot \mathbf{1}_T) \in H_T^1$. It follows that

$$a(\sigma^{\lambda \cdot x + \mu \cdot \mathbf{1}_T}) = \lambda \cdot x_{s(\sigma^x)} - \lambda \cdot x_{d(\sigma^x)} = \lambda \cdot \left(x_{s(\sigma^x)} - x_{d(\sigma^x)} \right) = \lambda \cdot a(\sigma^x)$$

and

$$b(\sigma^{\lambda \cdot x + \mu \cdot \mathbf{1}_T}) = \sum_{t \in \mathbf{U}^\mathbf{x}} \left(\lambda \cdot x_{s(\sigma^x)} - \lambda \cdot x_t \right) = \lambda \cdot \sum_{t \in \mathbf{U}^\mathbf{x}} \left(x_{s(\sigma^x)} - x_t \right) = \lambda \cdot b(\sigma^x).$$

Therefore, $r(\lambda \cdot x + \mu \cdot \mathbf{1}_T) = r(x)$ and hence $(\lambda \cdot x + \mu \cdot \mathbf{1}_T) \sim^r x$. □

In the only-if part of the proof, we proceed in several steps to illustrate how adding one axiom at a time successively narrows down the set of possible orderings until we arrive at the desired conclusion.

We begin by showing that the conjunction of recovery neutrality and recovery translation invariance implies that the criterion is insensitive to the distribution of health values in the recovery phase provided that the sum of the health values in the recovery phase remains unchanged.

Lemma 1. *If a single-spell resilience ordering $\succsim$ satisfies recovery neutrality and recovery translation invariance, then, for all $T \in \mathbf{T}$ and for all $x, y \in H_T^1$ with the same timing structure, if $x_\tau = y_\tau$ for all $\tau \in \{1, \ldots, T\} \setminus \mathbf{U}$ and $\sum_{\tau \in \mathbf{U}} x_\tau = \sum_{\tau \in \mathbf{U}} y_\tau$, then*

$$x \sim y.$$

Proof. Assume that $x, y \in H_T^1$ have the same timing structure, $\sum_{\tau \in \mathbf{U}} x_\tau = \sum_{\tau \in \mathbf{U}} y_\tau$ and $x_\tau = y_\tau$ for all $\tau \in \{1, \ldots, T\} \setminus \mathbf{U}$.

It is trivially true that $x \sim y$ if $|\mathbf{U}| = u - d$ equals 0 or 1.

Now assume that $|\mathbf{U}| = u - d \geq 2$. Define the vectors $x^0, \ldots, x^{u-d-1}$ by $x^0 = x$ and

$$x^k = x^{k-1} + w^k \quad \text{for all } k = 1, \ldots, u-d-1,$$

where, for all $k = 1, \ldots, u-d-1$, $w^k \in \mathbb{R}^T$ is given by $w^k_{u-k+1} = y_{u-k+1} - x^{k-1}_{u-k+1}$, $w^k_{u-k} = -w^k_{u-k+1}$ and $w^k_\tau = 0$ for $\{1, \ldots, T\} \setminus \{u-k, u-k+1\}$. Note that

$$x^{u-d-1}_{d+1} = x_{d+1} - y_{d+2} + x^{u-d-2}_{d+2} = \cdots = x_{d+1} + \sum_{\tau=d+2}^{u} (x_\tau - y_\tau) = y_{d+1}$$

since, by assumption, $\sum_{\tau=d+1}^{u} y_\tau = \sum_{\tau=d+1}^{u} x_\tau$, while, by construction, $x^{u-d-1}_\tau = y_\tau$ for all $\tau \in \{d+2, \ldots, u\}$. Hence, $x^{u-d-1} = y$.

It remains to be shown that $x^k \sim x^{k-1}$ for all $k \in \{1, \ldots, u-d-1\}$. Note that

$$x^{k-1}_{u-k} + x^{k-1}_{u-k+1} = x^k_{u-k} + x^k_{u-k+1}$$

since $w^k_{u-k} = -w^k_{u-k+1}$. Hence, we can define the scalars α^k and β^k as follows.

$$\begin{aligned}
\alpha^k &= \tfrac{1}{2} \cdot \left(x^{k-1}_{u-k} + x^{k-1}_{u-k+1}\right) = \tfrac{1}{2} \cdot \left(x^k_{u-k} + x^k_{u-k+1}\right), \\
\beta^k &= \tfrac{1}{2} \cdot \left(x^{k-1}_{u-k} - x^k_{u-k}\right) = \tfrac{1}{2} \cdot \left(x^k_{u-k+1} - x^{k-1}_{u-k+1}\right).
\end{aligned}$$

Let $z^k \in \mathbb{R}^T$ be given by

$$\begin{aligned}
z^k_{u-k+1} &= \alpha^k - \tfrac{1}{2} \cdot \left(x^{k-1}_{u-k+1} + x^k_{u-k+1}\right), \\
z^k_{u-k} &= \alpha^k - \tfrac{1}{2} \cdot \left(x^{k-1}_{u-k} + x^k_{u-k}\right)
\end{aligned}$$

and $z^k_\tau = 0$ for all $\tau \in \{1, \ldots, T\} \setminus \{u-k, u-k+1\}$. Then

$$\begin{aligned}
x^k_{u-k+1} + z^k_{u-k+1} &= \alpha^k + \tfrac{1}{2} \cdot \left(x^k_{u-k+1} - x^{k-1}_{u-k+1}\right) = \alpha^k + \beta^k \\
&= \alpha^k + \tfrac{1}{2} \cdot \left(x^{k-1}_{u-k} - x^k_{u-k}\right) = x^{k-1}_{u-k} + z^k_{u-k}
\end{aligned}$$

and

$$\begin{aligned}
x^{k-1}_{u-k+1} + z^k_{u-k+1} &= \alpha^k - \tfrac{1}{2} \cdot \left(x^k_{u-k+1} - x^{k-1}_{u-k+1}\right) = \alpha^k - \beta^k \\
&= \alpha^k - \tfrac{1}{2} \cdot \left(x^{k-1}_{u-k} - x^k_{u-k}\right) = x^k_{u-k} + z^k_{u-k}.
\end{aligned}$$

By recovery neutrality and recovery translation invariance, it follows that $x^k \sim x^{k-1}$ for all $k \in \{1, \ldots, u-d-1\}$ and hence $x \sim y$ by transitivity. □

Our next step consists of adding recovery monotonicity to the two axioms of the above lemma. As a consequence, it follows that an additive criterion must be used to compare any two streams with the same timing structure and with identical health-variable values in the periods prior to the recovery phase.

Lemma 2. *If a single-spell resilience ordering $\succsim$ satisfies recovery neutrality, recovery translation invariance and recovery monotonicity, then, for all $T \in \mathbf{T}$ and for all $x, y \in H_T^1$ with the same timing structure, if $x_\tau = y_\tau$ for all $\tau \in \{1, \dots, T\} \setminus \mathbf{U}$, then*

$$x \succeq y \;\Leftrightarrow\; \sum_{\tau \in \mathbf{U}} x_\tau \geq \sum_{\tau \in \mathbf{U}} y_\tau.$$

Proof. Assume that $x, y \in H_T^1$ have the same timing structure and $x_\tau = y_\tau$ for all $\tau \in \{1, \dots, T\} \setminus \mathbf{U}$. Note that the equivalence stated in the lemma is trivially true if $|\mathbf{U}| = u - d = 0$. Thus, we can without loss of generality assume that $|\mathbf{U}| = u - d > 0$. In view of Lemma 1, we only have to prove that, under the assumptions of the lemma statement, the inequality

$$\sum_{\tau \in \mathbf{U}} x_\tau > \sum_{\tau \in \mathbf{U}} y_\tau$$

implies $x \succ y$.

The implication follows directly from recovery monotonicity if $|\mathbf{U}| = u - d = 1$.

Now assume that $\sum_{\tau \in \mathbf{U}} x_\tau > \sum_{\tau \in \mathbf{U}} y_\tau$ and $|\mathbf{U}| = u - d \geq 2$. Define $x', y' \in H_T^1$ as follows. Let $y'_{d+1} = y_{d+1}$ and choose $x'_{d+1} \in (y_{d+1}, x_s)$ such that $x'_{d+1} - y_{d+1} < \sum_{t \in \mathbf{U}} (x_t - y_t)$. Moreover, define

$$x'_\tau = \frac{1}{u - d - 1} \cdot \left(\sum_{t \in \mathbf{U}} x_t - x'_{d+1} \right) \quad \text{and} \quad y'_\tau = \frac{1}{u - d - 1} \cdot \left(\sum_{t \in \mathbf{U}} y_t - y'_{d+1} \right)$$

for all $\tau \in \{d + 2, \dots, u\}$, and let $x'_\tau = y'_\tau = x_\tau = y_\tau$ for all $\tau \in \{1, \dots, T\} \setminus \mathbf{U}$. By definition, x' and y' have the same timing structure as x and y and, moreover, we have

$$\sum_{\tau \in \mathbf{U}} x'_\tau = \sum_{\tau \in \mathbf{U}} x_\tau \quad \text{and} \quad \sum_{\tau \in \mathbf{U}} y'_\tau = \sum_{\tau \in \mathbf{U}} y_\tau$$

as well as $x'_\tau > y'_\tau$ for all $\tau \in \mathbf{U}$. By Lemma 1, $x' \sim x$ and $y' \sim y$. By recovery monotonicity, $x' \succ y'$ and hence $x \succ y$ by transitivity. □

If the axiom of amplitude and recovery consistency is employed in addition to the properties previously imposed, it follows that knowledge of the severity levels $b(\sigma^x)$ and $b(\sigma^y)$ is sufficient to rank the streams x and y, provided that they are associated with recovery phases of the same length and share the same amplitudes.

Lemma 3. *If a single-spell resilience ordering $\succsim$ satisfies recovery neutrality, recovery translation invariance, recovery monotonicity and amplitude and recovery consistency, then, for all $T, T' \in \mathbf{T}$, for all $x \in H_T^1$ and for all $y \in H_{T'}^1$ such that $|\mathbf{U}^x| = |\mathbf{U}^y|$, if $x_{s(\sigma^x)} = y_{s(\sigma^y)}$ and $x_{d(\sigma^x)} = y_{d(\sigma^y)}$, then*

$$x \succsim y \;\Leftrightarrow\; b(\sigma^x) \leq b(\sigma^y).$$

Proof. Assume that $T, T' \in \mathbf{T}$, $x \in H^1_T$ and $y \in H^1_{T'}$ are such that $|\mathbf{U}^x| = |\mathbf{U}^y|$, $x_{s(\sigma^x)} = y_{s(\sigma^y)}$ and $x_{d(\sigma^x)} = y_{d(\sigma^y)}$. Let $u = |\mathbf{U}^x| + 2 = |\mathbf{U}^y| + 2$ and define $x', y' \in H^1_u$ as follows. Let $x'_1 = y'_1 = x_{s(\sigma^x)} = y_{s(\sigma^y)}$, $x'_2 = y'_2 = x_{d(\sigma^x)} = y_{d(\sigma^y)}$, $x'_\tau = x_{d(\sigma^x)-2+\tau} + \max\{0, y_{s(\sigma^x)} - x_{s(\sigma^y)}\}$ and $y'_\tau = y_{d(\sigma^y)-2+\tau} + \max\{0, x_{s(\sigma^x)} - y_{s(\sigma^y)}\}$ for all $\tau \in \{3, \dots, u\}$. Hence, $s(\sigma^{x'}) = s(\sigma^{y'}) = 1$, $d(\sigma^{x'}) = d(\sigma^{y'}) = 2$ and $\mathbf{U}^{x'} = \mathbf{U}^{y'} = \{3, \dots, u\}$. Note that, by construction, $x'_\tau \geq x_\tau \geq 0$ and $y'_\tau \geq y_\tau \geq 0$ for all $\tau \in \{1, \dots, u\}$. By amplitude and recovery consistency, $x' \sim x$ and $y' \sim y$ and, by Lemma 2,

$$x' \succsim y' \;\Leftrightarrow\; b(\sigma^{x'}) \;=\; (u-2) \cdot x'_1 - \sum_{\tau=3}^{u} x'_\tau \;\leq\; (u-2) \cdot y'_1 - \sum_{\tau=3}^{u} y'_\tau \;=\; b(\sigma^{y'})$$

because $x'_1 = y'_1$. The result follows since $b(\sigma^x) = b(\sigma^{x'})$ and $b(\sigma^y) = b(\sigma^{y'})$ and $\succsim$ is transitive. □

The next property to be added is continuity. This axiom allows us to extend the result of the previous lemma to any two streams x and y with the same amplitudes; the recovery phases associated with x and y may now differ in length.

Lemma 4. *If a single-spell resilience ordering $\succsim$ satisfies recovery neutrality, recovery translation invariance, recovery monotonicity, amplitude and recovery consistency and continuity, then, for all $T, T' \in \mathbf{T}$, for all $x \in H^1_T$ and for all $y \in H^1_{T'}$, if $x_{s(\sigma^x)} = y_{s(\sigma^y)}$ and $x_{d(\sigma^x)} = y_{d(\sigma^y)}$, then*

$$x \succsim y \;\Leftrightarrow\; b(\sigma^x) \leq b(\sigma^y).$$

Proof. Assume that $T, T' \in \mathbf{T}$, $x \in H^1_T$ and $y \in H^1_{T'}$ are such that $x_{s(\sigma^x)} = y_{s(\sigma^y)}$ and $x_{d(\sigma^x)} = y_{d(\sigma^y)}$.

If $|\mathbf{U}^x| = |\mathbf{U}^y|$, then the result follows from Lemma 3.

Now assume that $|\mathbf{U}^x| \neq |\mathbf{U}^y|$; without loss of generality, assume that $|\mathbf{U}^x| < |\mathbf{U}^y|$. Let $u = |\mathbf{U}^y| + 2$ and define $x', y' \in H^1_u$ as follows. Let $x'_1 = y'_1 = x_{s(\sigma^x)} = y_{s(\sigma^y)}$, $x'_2 = y'_2 = x_{d(\sigma^x)} = y_{d(\sigma^y)}$, $x'_\tau = x_{d(\sigma^x)-2+\tau} + \max\{0, y_{s(\sigma^x)} - x_{s(\sigma^y)}\}$ for all $\tau \in \{3, \dots, |\mathbf{U}^x| + 2\}$, $x'_\tau = x'_1$ for all $\tau \in \{|\mathbf{U}^x| + 3, \dots, u\}$ and $y'_\tau = y_{d(\sigma^y)-2+\tau}$ for all $\tau \in \{3, \dots, u\}$. Hence, $s(\sigma^{x'}) = s(\sigma^{y'}) = 1$, $d(\sigma^{x'}) = d(\sigma^{y'}) = 2$, $\mathbf{U}^{x'} = \{3, \dots, |\mathbf{U}^x| + 2\}$ and $\mathbf{U}^{y'} = \{3, \dots, u\}$. Note that, by construction, $x'_\tau \geq x_\tau \geq 0$ and $y'_\tau \geq y_\tau \geq 0$ for all $\tau \in \{1, \dots, u\}$. By amplitude and recovery consistency, $x' \sim x$ and $y' \sim y$.

If $|\mathbf{U}^x| = 0$ (and, by assumption, $|\mathbf{U}^y| > 0$), then $b(\sigma^{x'}) = 0 < b(\sigma^{y'})$. Construct the sequence $\langle x^k \rangle_{k \in \mathbb{N}}$ by letting, for all $k \in \mathbb{N}$, x^k be defined by $x^k_1 = x'_1$, $x^k_2 = x'_2$ and $x^k_\tau = x'_1 - \varepsilon/k$ for all $\tau \in \{3, \dots, u\}$, where $(u-2) \cdot \varepsilon < b(\sigma^{y'})$, so that $b(\sigma^{y'}) > b(\sigma^{x^1}) > \cdots > b(\sigma^{x^k}) > b(\sigma^{x^{k+1}}) > \cdots > b(\sigma^{x'}) = 0$. Note that $|\mathbf{U}^{x^k}| = |\mathbf{U}^y|$. By Lemma 3, $\sigma^{y'} \succ \sigma^{x^k}$ for all $k \in \mathbb{N}$ and $\sigma^{x^k} \succ \lim_{\kappa \to \infty} \sigma^{x^\kappa} = \sigma^{x'}$ for all $k \in \mathbb{N}$ by continuity because $\lim_{\kappa \to \infty} x^\kappa = x'$ and $b(\sigma^{x^k})$ is strictly decreasing in k. Hence,

$$x \sim x' \succ y' \sim y$$

because $\succsim$ is transitive.

Finally, assume that $|\mathbf{U}^x| > 0$. Construct the sequence $\langle x^k \rangle_{k\in\mathbb{N}}$ by letting, for all $k \in \mathbb{N}$, x^k be defined by $x_1^k = x_1'$, $x_2^k = x_2'$, $x_3^k = x_3' + (u - |\mathbf{U}^x| - 2) \cdot \varepsilon/k$, $x_\tau^k = x_\tau'$ for all $\tau \in \{4, \ldots, |\mathbf{U}^x| + 2\}$ and $x_\tau^k = x_1' - \varepsilon/k$ for all $\tau \in \{\mathbf{U}^x| + 3, \ldots, u\}$, where $(u - |\mathbf{U}^x| - 2) \cdot \varepsilon < x_1' - x_3'$, so that $x_2^k \leq x_3' < x_3^k < x_1^k$. It follows that $|\mathbf{U}^{x^k}| = |\mathbf{U}^y|$. By construction, $b(\sigma^{x^1}) = \cdots = b(\sigma^{x^k}) = b(\sigma^{x^{k+1}}) = \cdots = b(\sigma^{x'})$. Combined with Lemma 3, it follows that $\sigma^{x^k} \sim \sigma^{x'}$ for all $k \in \mathbb{N}$ by continuity; note that $\lim_{\kappa\to\infty} x^\kappa = x'$. Hence, by Lemma 3 and transitivity,

$$x' \sim x^k \succsim y' \;\Leftrightarrow\; b(\sigma^{x'}) = b(\sigma^{x^k}) \leq b(\sigma^{y'})$$

for all $k \in \mathbb{N}$ so that

$$x \sim x' \succsim y' \sim y \;\Leftrightarrow\; b(\sigma^x) = b(\sigma^{x'}) \leq b(\sigma^{y'}) = b(\sigma^y).$$

□

We are now ready to prove the only-if part of our axiomatization. All that remains to be done is to use affine invariance to arrive at the ordering $\succsim^r$ represented by the resilience measure r.

Proof. Only if. Assume that $\succeq$ is an ordering that satisfies the axioms of the theorem statement. Let $T, T' \in \mathbf{T}$, $x \in H_T^1$ and $y \in H_{T'}^1$.

(i) First, we prove that if $x_{s(\sigma^x)} - x_{d(\sigma^x)} = y_{s(\sigma^y)} - y_{d(\sigma^y)}$, then

$$x \succsim y \;\Leftrightarrow\; b(\sigma^x) \leq b(\sigma^y).$$

Assume that this equality is true and let $\lambda^x = \lambda^y = 1$, $\mu^x = y_{d(\sigma^y)}$ and $\mu^y = x_{d(\sigma^x)}$. By definition, $(\lambda^x \cdot x + \mu^x \cdot \mathbf{1}_T) = (x + y_{d(\sigma^y)} \cdot \mathbf{1}_T) \in H_T^1$ and $(\lambda^y \cdot y + \mu^y \cdot \mathbf{1}_{T'}) = (y + x_{d(\sigma^x)} \cdot \mathbf{1}_{T'}) \in H_{T'}^1$. By affine invariance,

$$x \sim (x + y_{d(\sigma^y)} \cdot \mathbf{1}_T) \;\text{ and }\; y \sim (y + x_{d(\sigma^x)} \cdot \mathbf{1}_{T'}).$$

By construction, writing $x' = (x + y_{d(\sigma^y)} \cdot \mathbf{1}_T)$ and $y' = (y + x_{d(\sigma^x)} \cdot \mathbf{1}_{T'})$, it follows that $x'_{s(\sigma^{x'})} = x_{s(\sigma^x)} + y_{d(\sigma^y)} = y_{s(\sigma^y)} + x_{d(\sigma^x)} = y'_{s(\sigma^{y'})}$ and $x'_{d(\sigma^{x'})} = x_{d(\sigma^x)} + y_{d(\sigma^y)} = y_{d(\sigma^y)} + x_{d(\sigma^x)} = y'_{d(\sigma^{y'})}$. Hence, by Lemma 4 and transitivity, $x \sim y$.

(ii) Now let $x \in H_T^1$ and $y \in H_{T'}^1$ be arbitrary. Let $\mu^x = \mu^y = 0$ and define

$$\lambda^x = \frac{1}{x_{s(\sigma^x)} - x_{d(\sigma^x)}} = \frac{1}{a(\sigma^x)} \quad \text{and} \quad \lambda^y = \frac{1}{y_{s(\sigma^y)} - y_{d(\sigma^y)}} = \frac{1}{a(\sigma^y)}.$$

Let $x' \in H_T^1$ and $y' \in H_{T'}^1$ be defined by $x' = \lambda^x \cdot x$ and $y' = \lambda^y \cdot y$. By affine invariance,

$$x \sim x' \quad \text{and} \quad y \sim y'.$$

We have that

$$x'_{s(\sigma^{x'})} - x'_{d(\sigma^{x'})} = \frac{x_{s(\sigma^x)} - x_{d(\sigma^x)}}{a(\sigma^x)} = 1 = \frac{y_{s(\sigma^y)} - y_{d(\sigma^y)}}{a(\sigma^y)} = y'_{s(\sigma^{y'})} - y'_{d(\sigma^{y'})},$$

thus, by part (i) of the proof,

$$\begin{aligned} x' \succsim y' \quad &\Leftrightarrow \quad \frac{b(\sigma^x)}{a(\sigma^x)} = b(\sigma^{x'}) \leq b(\sigma^{y'}) = \frac{b(\sigma^y)}{a(\sigma^y)} \\ &\Leftrightarrow \quad 1 + \frac{b(\sigma^x)}{a(\sigma^x)} \leq 1 + \frac{b(\sigma^y)}{a(\sigma^y)} \\ &\Leftrightarrow \quad \frac{a(\sigma^x) + b(\sigma^x)}{a(\sigma^x)} \leq \frac{a(\sigma^y) + b(\sigma^y)}{a(\sigma^y)} \\ &\Leftrightarrow \quad \frac{a(\sigma^x)}{a(\sigma^x) + b(\sigma^x)} \geq \frac{a(\sigma^y)}{a(\sigma^y) + b(\sigma^y)}. \end{aligned}$$

As established above, we also have $x \sim x'$ and $y \sim y'$ so that we obtain

$$x \sim x' \succsim y' \sim y \;\Leftrightarrow\; x \succsim^r y,$$

using transitivity and the definition of $\succsim^r$. □

That our axioms are independent is established in the appendix.

7 Concluding remarks

In this paper, we propose and axiomatize a measure of resilience based solely on the properties of the health streams—the fundamental determinants of our notion of resilience. More specifically, our approach treats down spells as the crucial experiences that reflect an individual's ability to recover. Implicit in our definition is the assumption that, in a down spell, the amplitude of the down movement matters but not its duration. Likewise, our properties imply a specific way of identifying the dividing line between a downwards movement and the recovery phase. In particular, we assume that the individual cannot resist the down spell before the recovery phase starts. These features represent modeling choices that we consider attractive in the measurement of resilience. Of course, there are alternative methods of defining the size of a down spell and the transition from a drop to the period in which recovery can occur, and it may be useful to explore some of these in future work. A more general approach is to enrich the framework by taking into consideration information concerning the influencing forces that precipitate a down spell. Similarly, future work could disentangle individual resilience from network and community effects. However, this would require enriched data which might not be as easily available, in the context of large-scale household surveys, as the health streams on which our resilience ordering is based.

Our measure of resilience has a drawback at the level of the individual of not being continuous when a sequence of health streams converges to a health stream for which the down spell ends in a different period. To exemplify, consider Figure 2 and the let the health value in period 3 (instead of having the value 3) approach the health value 2 from below. To be specific, consider the sequence $\langle x^k \rangle_{k\in\mathbb{N}}$ of streams with $x^k = (4, 2, 2{-}1/(k+1), 1, 4)$, so that $x = \lim_{k\to\infty} x_k$ is given by $x = (4, 2, 2, 1, 4)$. For each element of this sequence, $d(\sigma^{x^k}) = 4$, meaning that the down spell ends in period 4 and our measure of resilience $r(x^k)$ equals 1, since full recovery is immediate. However, in the limit, $d(\sigma^x) = 2$ and $r(x) = 2/7$, since full recovery now takes two periods.

By not relying on information at the individual level concerning the forces that precipitate the individual down spells, our measure of resilience can be used for measuring the resilience of large populations of subjects. At such an aggregate level, the issue of whether the measure is continuous at the individual level loses much of its importance. Thus, the measure of resilience that we propose might be well-posed to identify the effects of interventions designed to improve mental-health resilience. This can be accomplished by studying populations of subjects some of whom are treated and some of whom remain untreated. The interventions in question are not meant to avoid, for example, divorces or lay-offs, but to reduce the consequences in terms of negative effects that follow such events.

Although our measure is ordinal in nature (and, thus, statements regarding arithmetic means or similar statistics are not well-defined without further assumptions), we have already suggested in Section 4 how the measure of resilience can be aggregated over different down spells experienced by the same individual. In particular, we proposed to consider a weighted average of the inverse of resilience, vulnerability, with weights corresponding to the amplitudes of the down spells. Implicit in suggesting this method is the assumption that the requisite values are intra-personally unit-comparable. If this procedure were to be extended to down spells experienced by different individuals, these values would have to be not only intra-personally but also inter-personally unit comparable, a considerably stronger assumption. However, it is perfectly legitimate within our ordinal framework to aggregate across individuals by employing criteria that are based on quantiles, for example.

A possible concern is that our ordinal measure of resilience may not allow for a sufficient degree of differentiation across individuals. Nevertheless, this may not pose a serious problem, as demonstrated by the results when applying our measure to the German Socio-Economic Panel. The SOEP is an ongoing panel survey with yearly re-interviews (see http://www.diw.de/gsoep). It is a representative longitudinal micro-level study providing a wide range of demographic and socio-economic information on private households and all household members. The first data was collected in 1984 from a sample of randomly-selected adult respondents in the Federal Republic of Germany. Since then, the same individuals have been surveyed annually. In 1990 the survey was expanded to include the states of the former German Democratic Republic. New samples were included later on to collect information on specific population groups or to boost the sample size. Every year since 1994, individuals are asked to rate their health by responding to the question "How would you describe your current health?" with possible answers on a five-point scale, ranging from "bad" (1) to "very good" (5). They are also asked "How satisfied are you with your health?" where responses are given on an 11-point scale from 0 ("completely dissatisfied") to 10 ("completely satisfied"). We analyzed the years from 1994 to 2016 and restricted the sample to respondents for whom we have at least six consecutive observations, leaving us with 15,015 individuals. A histogram representing the relative frequencies of the attained resilience levels is provided in Figure 6 for self-assessed health status and in Figure 7 for health satisfaction. As is evident from these diagrams, there is considerable variation in both cases.

We conclude by noting that our approach is general enough to accommodate the assessment of resilience in the context of other variables. These could be both at an individual level, such as equivalent household income, and aggregate variables of economic performance, such as unemployment rates and GDP growth rates of countries. In addition, in cases where the resilience measurement is applied to a variable with an underlying positive growth trend, one could allow for recovery back to a level that is not fixed at the pre-downspell level but which increases with time, by first detrending the stream in question and then applying our measure.

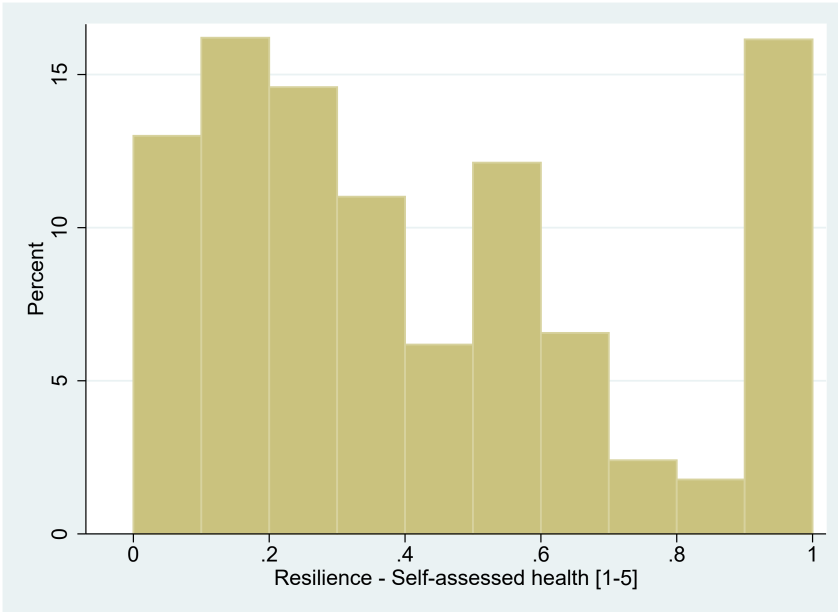

Figure 6: Self-assessed health

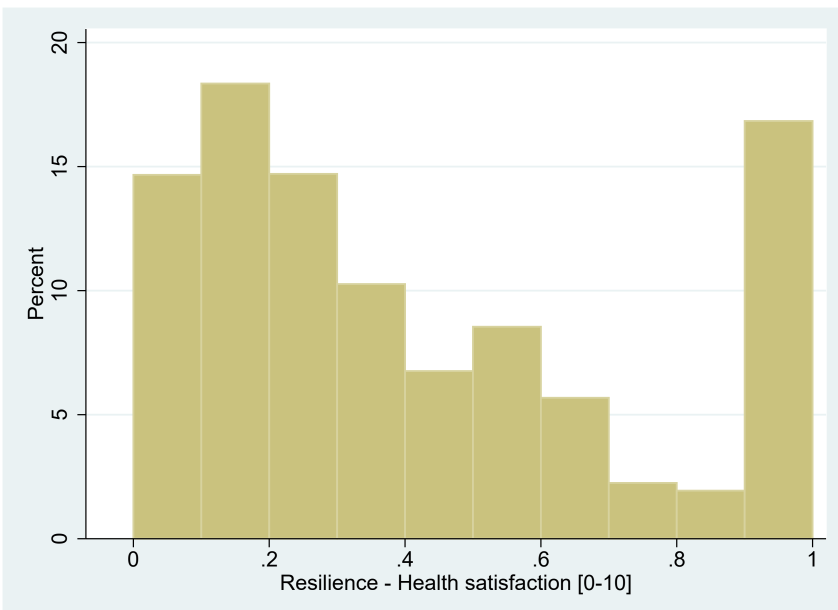

Figure 7: Health satisfaction

A Appendix: Independence of the axioms

For each of the six axioms employed in our characterization, we provide an example that violates the axiom and satisfies the remaining properties. We note that the five examples that satisfy recovery monotonicity also satisfy the following stronger property.

Strong recovery monotonicity. For all $T \in \mathbf{T}$ and for all $x, y \in H_T^1$ with the same timing structure such that $\mathbf{U} \neq \emptyset$, if $x_\tau \geq y_\tau$ for all $\tau \in \mathbf{U}$ with at least one strict inequality and $x_\tau = y_\tau$ for all $\tau \in \{1, \ldots, T\} \setminus \mathbf{U}$, then

$$x \succ y.$$

Thus, the examples also show that strengthening the monotonicity axiom in this way does not affect the independence of the axioms. Note that our ordering $\succsim^r$ possess this stronger property.

A.1 Recovery neutrality

Let $\delta \in (0,1)$ and define, for all $x \in \Omega^1$,

$$r^1(x) = \frac{a(\sigma^x)}{a(\sigma^x) + \sum_{t \in U(\sigma^x)} \delta^{t-d(\sigma^x)} \cdot (x_{s(\sigma^x)} - x_t)}$$

and, for all $x, y \in \Omega^1$, $x \succsim^1 y$ if and only if $r^1(x) \geq r^1(y)$. The ordering $\succsim^1$ satisfies all our axioms except for recovery neutrality.

A.2 Recovery translation invariance

Let $\delta \in (0,1)$ and define, for all $x \in \Omega^1$,

$$r^2(x) = \frac{a(\sigma^x)}{a(\sigma^x) + \sum_{t \in U(\sigma^x)} \delta^{t-d(\sigma^x)} \cdot (x_{s(\sigma^x)} - x_{\pi(t)})},$$

where $\pi\colon U(\sigma^x) \to U(\sigma^x)$ is a bijection satisfying $x_{\pi(t)} \leq x_{\pi(t+1)}$ for all $t \in U(\sigma^x) \setminus \{u(\sigma^x)\}$, and, for all $x, y \in \Omega^1$, $x \succsim^2 y$ if and only if $r^2(x) \geq r^2(y)$. The ordering $\succsim^2$ satisfies all our axioms except for recovery translation invariance.

A.3 Recovery monotonicity

Let $\succsim^3$ be the universal indifference relation, that is, for all $x, y \in \Omega^1$, $x \sim^3 y$. The ordering $\succsim^3$ satisfies all our axioms except for recovery monotonicity.

A.4 Amplitude and recovery consistency

Define, for all $x \in \Omega^1$,

$$r^4(x) = \frac{a(\sigma^x) \cdot (d(\sigma^x) - s(\sigma^x))}{a(\sigma^x) \cdot (d(\sigma^x) - s(\sigma^x)) + b(\sigma^x)}$$

and, for all $x, y \in \Omega^1$, $x \succsim^4 y$ if and only if $r^4(x) \geq r^4(y)$. The ordering $\succsim^4$ satisfies all our axioms except for amplitude and recovery consistency.

A.5 Continuity

Define, for all $x \in \Omega^1$,

$$r^5(x) = \frac{a(\sigma^x)}{a(\sigma^x) + |\mathbf{U}^x| \cdot b(\sigma^x)}$$

and, for all $x, y \in \Omega^1$, $x \succsim^5 y$ if and only if $r^5(x) \geq r^5(y)$. The ordering $\succsim^5$ satisfies all our axioms except for continuity.

A.6 Affine invariance

Define, for all $x, y \in \Omega^1$, $x \succsim^6 y$ if and only if $b(\sigma^x) \leq b(\sigma^y)$. The ordering $\succsim^6$ satisfies all our axioms except for affine invariance.

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