Elections and Strategic Voting: Condorcet and Borda *

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Abstract

We show that majority rule is uniquely characterized among voting rules by strategy-proofness, the Pareto principle, anonymity, neutrality, independence of irrelevant alternatives, and decisiveness. Furthermore, there is an extension of majority rule that satisfies these axioms on any preference domain without Condorcet cycles when voters are sufficiently risk-averse. If independence is dropped, then, for the case of three candidates, the remaining axioms characterize exactly *two* voting rules – majority rule and rank-order voting – on rich domains (domains for which no candidate is always extremal, i.e., best or worst).

1. Introduction

How should society choose public officials such as presidents? The obvious answer is to hold elections. However, there are many possible election methods, called *voting rules*, to choose from. Here are a few examples.

In *plurality rule*, each citizen votes for a candidate, and the winner is the candidate with the most votes, even if short of majority.

In *runoff voting* there are two rounds. First, each citizen votes for one candidate. If some candidate gets a majority, she wins. Otherwise, the top two vote-getters face each other in a runoff, which determines the winner.

Under *majority rule* – advocated by the Marquis de Condorcet (Condorcet 1785) – each voter ranks the candidates in order of preference. The winner is then the candidate who, according to the rankings, beats each opponent in a head-to-head contest.

In *rank-order voting* (the Borda Count) – proposed by Condorcet's intellectual archrival Jean-Charles Borda (Borda 1781) – voters again rank the candidates in order of preference. With n candidates, a candidate gets n points for every voter who ranks her first, $n-1$ points for a second-place vote, and so on. The winner is the candidate with the most points.

Each voting rule so far is *ordinal* in the sense that the way a citizen votes can be deduced from his ordinal preferences over candidates (we define ordinality formally in section 3).[2] Next are two voting rules that are cardinal (i.e., a citizen's vote depends on more than just ordinal preferences).

In *approval voting*, each citizen approves as many candidates as he wants. The winner is the candidate with the most approvals. The voting rule fails ordinality because a citizen's

[2] More accurately, the way the citizen votes can be deduced if he is voting non-strategically. We consider *strategic voting* – a major theme of this paper – later in this introduction.

preference ordering doesn't by itself determine the boundary between "approved" and "unapproved" candidates.

In *range voting*, a citizen grades each candidate on, say, a ten-point scale ("1" denotes dreadful and "10" denotes superb). A candidate's points are then summed over citizens, and the candidate with the biggest total score wins.[3]

Faced with all these possibilities, how should society decide what voting rule to adopt? Ever since Arrow (1951), a standard answer is for society to first step back and consider what it wants in a voting rule, i.e., to (i) posit a set of principles or axioms that any good voting rule should satisfy, and (ii) determine which voting rule(s) they are consistent with.

This is our approach here (section 3 gives precise definitions). Specifically, we examine the *Pareto principle* (P) – if all citizens prefer candidate x to y, then y should not be elected; *anonymity* (A) – all citizens' votes should count equally; *neutrality* (N) – all candidates should be treated equally; *decisiveness* (D) – the election should result in one and only winner; *independence of irrelevant alternatives* (IIA) – if x is the winner among a set of candidates Y (the "ballot"), then x must still be the winner if the ballot is reduced from Y to Y' by dropping some losing ("irrelevant") candidates from the ballot;[4] and *ordinality* (O) – the winner should depend only on citizens' *ordinal* rankings and not on preference intensities or other cardinal information.[5]

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[3] Two variants of range voting are (i) *majority judgment* (Balinski and Laraki 2010), which is the same as range voting except that the winner has the biggest *median* (not total) score and (ii) *budget voting*, in which a citizen has a set number of votes that he can allocate in way to the different candidates. The winner is, again, the candidate with the biggest total.

[4] Arrow (1951) and Nash (1950) formulated (nonequivalent) axioms with the name IIA. In this paper and its predecessor (Dasgupta and Maskin 2008), we adopt the Nash formulation, but could have used Arrow's version instead (indeed, an early paper in this line of work, Maskin 1995, does just that.)

[5] Arrow (1951) notes that a citizen's ordinal preference between x and y can be ascertained from a simple eliciting preference intensities. We support this view with Theorem 1 showing that cardinal voting rules can't be strategy proof.

Of these six axioms, IIA is arguably the least "obvious." Moreover, it is the most controversial of these axioms, probably because some well-regarded voting rules (e.g., rank-order voting) violate it. To see this, consider Figure 1, in which there are three candidates *x, y*, and *z* and two groups of voters, one (45% of the electorate) with preferences $x \succ y \succ z$ and the other (55%) with preferences $y \succ z \succ x$. If all 3 candidates run, then *x* wins under rank-order voting. But if *z* drops out, then *y* wins.

Still, IIA has strong appeal, in part because it formalizes the idea that a voting rule should not be vulnerable to vote splitting. Voting splitting arises when candidate *x* would beat *y* in a one-on-one contest, but loses to *y* when *z* runs too (because *z* splits off some of the vote that otherwise would go to *x*). See Figure 2 for an illustration that also shows that plurality rule and runoff voting violate IIA too.

The Arrow Impossibility Theorem establishes that there is *no* voting rule satisfying all of P, A, N, D, IIA, and O with at least three candidates and unrestricted voter preferences (see Theorem A below). In particular, majority rule – although it satisfies the other axioms – fails to be decisive, as Condorcet himself showed in a famous example of a "Condorcet cycle" (see Figure 3).

Thus, in Dasgupta and Maskin (2008), we argue that the natural follow-up question to Arrow is: Which voting rule satisfies these axioms for the widest class of *restricted domains* of preferences? That paper shows that there is a sharp answer to this question: majority rule. Theorem B states that majority rule satisfies the six axioms on a domain if and only that domain does not contain a Condorcet cycle. More strikingly, if some voting rule satisfies the six axioms when preferences are drawn from a particular domain, then majority rule must also satisfy the axioms on that same domain. And, unless the voting rule we started with is itself majority rule, there exists another domain on which majority rule satisfies all 6 axioms and the original voting rule does not (Theorem C).

In this paper, we consider an additional, often-invoked axiom: *strategy proofness* (SP) – a voting rule should induce citizens to vote according to their true preferences, not strategically. There are at least two justifications for SP. First, if citizens vote strategically, then the voting rule in question doesn't produce the outcomes intended; since the rule's inputs are distorted, so are the outputs. Second, strategic voting imposes a burden on citizens. It is hard enough for a conscientious citizen to determine his own preferences: he has to study the candidates carefully. If, on top of this, he must know *other* citizens' preferences in order to react strategically to them, his decision problem becomes several orders of magnitude more difficult.

Our first result (Theorem 1) establishes that any voting rule satisfying SP, D, and A must be ordinal. The proof is straightforward and, in the case of range voting, especially simple: Suppose there are two candidates x and y running and a particular citizen judges them both to be quite good. If he were grading honestly, he would give x a grade of 8 and y a grade of 7. But, in the election, he will want to give x a grade of 10 and y a grade of 1 to maximize x's chance of winning.

Just as we ran into the Arrow Impossibility Theorem in our previous work, we collide with the Gibbard-Satterthwaite Impossibility Theorem (Gibbard 1973, Satterthwaite 1975) once we impose SP. A fortiori (in view of Theorem A), there exists no voting rule that satisfies all seven axioms when voters' preferences are unrestricted. Indeed, there is no voting rule that even satisfies all of A, N, D, and SP (Theorem E).

However, we show that majority rule satisfies the seven axioms on any restricted domain without a Condorcet cycle (Theorem 2).

Implicit in Theorem 2 is the assumption that voters are confined to the restricted domain in question when they misrepresent their preferences. As we argue in section 3, this assumption makes sense in some circumstances, but not all. When voters can misrepresent *freely*, a majority (Condorcet) winner may not exist. Thus, we must extend majority rule so that it always produces a well-defined outcome. Specifically, we use the *Smith set* (Smith 1973, Fishburn 1977), the (unique) minimal subset of candidates that beat any other candidate by a majority. When a majority winner does not exist, we choose a random member of the Smith set as the outcome. Theorem 3 establishes that this extension of majority rule satisfies the seven axioms on any domain without Condorcet cycles, provided that voters are sufficiently risk-averse.

Theorem C from Dasgupta-Maskin (2008) shows that majority rule dominates other voting rules in the sense of satisfying P, A, N, D, IIA, and O more often. When we add SP to the mix, majority rule is, in fact, *uniquely* determined. Theorem 4 establishes that a voting rule satisfying, A, N, D, IIA, and SP (P and O are redundant) on some domain can *only* be majority rule.

Finally, we drop the controversial axiom IIA. Theorem 5 shows that, for the case of three candidates, a voting rule satisfying P, A, N, D, and SP on a *rich* domain (a domain for which no candidate is always top or bottom-ranked) must be either majority rule or rank-order voting (the lengthy proof is relegated to the Appendix). In this sense, Condorcet and Borda are the two heroes of our story.

2. Model

There is a finite set X of potential candidates for a given office[6]. The electorate is a continuum of voters, taken to be the unit interval [0,1] (the continuum makes the probability of a tie negligible, an issue discussed in section 3).

Each voter $i \in [0,1]$ is described by her *utility function* $u_i : X \to \mathbb{R}$. To simplify analysis, we rule indifference out by *assumptio*n. That is, for all $x, y \in X$, if $x \neq y$, then $u_i(x) \neq u_i(y)$. Let $\mathcal{U}_X$ consist of the utility functions on X without indifference. A *profile*, denoted by $u.$, is a specification of a utility function $u_i \in U_X$ for each voter $i \in [0,1]$. A *ballot* is a subset $Y (\subseteq X)$ consisting of the candidates who are *actually* running for the office. Let ΔY consist of the probability distributions over Y.

A *voting rule* is a correspondence that, for each profile $u.$ and each ballot Y, selects a subset of ΔY. That is, $F(u_., Y) \subseteq \Delta Y$, for all $u_.$ and Y. This formulation allows for election methods in which the winner is determined at least partly by chance and there can be more than one winner.

With a continuum voters, we can't literally count the *number* of voters with a particular preference; we must instead consider *proportions* of voters. For that purpose, we can use Lebesgue measure μ on [0,1]. Thus, for profile $u.$, $\mu\left(\left\{i \middle| u_i(x) > u_i(y)\right\}\right)$ is the proportion of voters who prefer candidate x to candidate y.

We can now formally define the voting rules mentioned in the introduction. Here is the definition of plurality rule:

Plurality Rule (First-Past-the-Post):

[6] A "potential" candidate, is one who could conceivably run for the office in question but, in the end, might not.

$$F^{P}(u_{\cdot},Y)=\left\{x\in Y \,\middle|\, \mu\left(\left\{i \,\middle|\, u_i(x)>u_i(y) \text{ for all } y\neq x, y\in Y\right\}\right)\right.$$

$$\left.\geq \mu\left(\left\{i \,\middle|\, u_i(x')>u_i(y) \text{ for all } y\neq x', y\in Y\right\}\right) \text{ for all } x'\in Y\right\}$$

In words, candidate *x* is a winner if at least as high a proportion of voters rank *x* first as they do any other candidate x'.[7]

To deal with potential ties, we focus henceforth on voting rules that are finitely based in the sense that a voter's set of possible utility functions can be partitioned into a finite number of equivalence classes.[8] Formally, *F* is *finitely based* provided there exist a finite set *S* (the base set) and, for each voter *i*, a mapping $h_i\colon \mathcal{U}_X \to S$ such that, for all profiles $u_{\cdot}$ and $u'_{\cdot}$, if $h_i(u_i)=h_i(u'_i)$ for all $i\in[0,1]$, then for all $Y\subseteq X$, $F(u_{\cdot},Y)=F(u'_{\cdot},Y)$. All the voting rules discussed in the introduction are finitely based (e.g., for an ordinal voting rule, *S* is just the set of rankings; for range voting, *S* is the set of possible mappings from candidates to grades between 1 and 10).

3. Axioms

We now define our axioms, which with one exception are standard in the voting literature.[9] We say that a voting rule satisfies a given axiom on domain $\mathcal{U}$ if the axiom holds for all profiles $u_{\cdot}$ drawn from $\mathcal{U}$.

Pareto Principle (P): For all $u_{\cdot}$ on $\mathcal{U}, Y\subseteq X$, and $x\in Y$, if $u_i(x)>u_i(y)$ for all *i*, then $y\notin F(u_{\cdot},Y)$. That is, if everyone prefers *x* to *y* and *x* is on the ballot, then *y* can't be elected.

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[7] Definitions of other rules can be found in the Appendix.

[8] This focus makes it easier to define the concept of a generic profile. See discussion of decisiveness on section 3.

[9] Decisiveness is nonstandard because it explicitly deals with ties (ties are usually ruled out by assumption; for example, in the literature on majority rule the number of the voters is typically assumed to be odd).

Anonymity (A): Suppose $\pi : [0,1] \to [0,1]$ is a measure-preserving[10] permutation. Given $u_{\bullet}$ on $\mathcal{U}$, let $u_{\bullet}^{\pi}$ be the profile such that, for all *i*, $u_i^{\pi} = u_{\pi(i)}$. Then $F(u_{\bullet}^{\pi}, Y) = F(u_{\bullet}, Y)$ for all *Y*. In words, if we permute the profile so that voter *j* gets *i*'s preferences, *k* gets *j*'s preferences, etc., the winner remains the same.

Neutrality (N): Given ballot *Y*, suppose $\rho : Y \to Y$ is a permutation of *Y*. For profile $u_{\bullet}$ on $\mathcal{U}$, if $u_{\bullet}^{\rho,Y}$ is a profile on $\mathcal{U}$ such that $u_i^{\rho,Y}(\rho(x)) = u_i(x)$ for all $x \in Y$, then $F(u_{\bullet}^{\rho,Y}, Y) = \rho(F(u_{\bullet}, Y))$. That is, if we permute the candidates so that candidate *x* becomes *y*, *y* becomes *z*, etc. (and voters' utilities are permuted accordingly), then if *x* won originally, now *y* wins.

P, A, and N are so "natural" that few voting rules used in practice or studied theoretically violate any of them. The same is not true of the next axiom.

Independence of Irrelevant Alternatives (IIA): Given $u_{\bullet}$ on $\mathcal{U}$ and ballot *Y*, if $x \in F(u_{\bullet}, Y)$ and $x \in Y' \subseteq Y$, then $x \in F(u_{\bullet}, Y')$.

As mentioned before, IIA gets at the idea that voting rules shouldn't be vulnerable to vote splitting. However, it rules out plurality rule, runoff voting, and rank-order voting (leaving only majority rule, approval voting, and range voting from the introduction).

We next have:

Ordinality (O): For all $u_{\bullet}$ on $\mathcal{U}$ and $Y \subseteq X$, if profiles $u_{\bullet}$ and $u'_{\bullet}$ satisfying $u_i(x) > u_i(y) \Leftrightarrow u'_i(x) > u'_i(y)$ for all $i \in [0,1]$ and $x, y \in Y$, then $F(u_{\bullet}, Y) = F(u'_{\bullet}, Y)$. That is, only voters' rankings – and not cardinal information about preferences – determines the winner.

[10] "Measure-preserving" means that, for any $C \subseteq [0,1]$, $\mu(C) = \mu(\pi(C))$.

We next turn to decisiveness, the principle that there should be a unique nonstochastic winner. In fact, *none* of the voting rules from the introduction is decisive in this sense; for each ties may occur. For example, with plurality rule, two (or more) candidates might be ranked first by a maximal proportion of voters. Nevertheless, if the number of voters is large, the likelihood of a tie under plurality rule is small. That is why we assume a continuum of voters: the probability of a tie under plurality rule is zero, or, more precisely, ties are nongeneric. To express this formally, it is simplest to restrict attention to finitely based voting rules satisfying A. Fix such a voting rule F with base set S and mappings $h_i\colon \mathcal{U}_X \to S$. Given profile $u_{\cdot}$ and $s \in S$, let

$$m_s = \mu\left(\left\{i \mid h_i(u_i) = s\right\}\right).$$

Clearly, $(m_{s_1}, \ldots, m_{s_{|S|}})$ is a sufficient statistic for u.

Decisiveness (D): Given Y, F results in a unique deterministic winner for generic $(m_{s_1}, \ldots, m_{s_{|S|}})$, i.e., the Lebesgue measure of the set of $|S|$-tuples for which there are ties is zero.

It is easy to verify that all the voting rules in the introduction satisfy D.

We can now state a straightforward version of the Arrow Impossibility Theorem:

Theorem A (Arrow 1951): If $|X| \geq 3$ and $\mathcal{U} = \mathcal{U}_X$, there exists no voting rule satisfying all of P, A, N, IIA, D, and O.

In view of this negative result, we turned in Dasgupta-Maskin (2008) to restricted domains.

When $|X| \geq 3$, majority rule F^C violates D on $\mathcal{U}_X$, as Condorcet's own example (Figure 3) illustrates. However, as long as $\mathcal{U}$ does not contain *Condorcet cycles* (i.e., utility functions

u, u', u'' and candidates *x*, *y*, *z* such that $u(x) > u(y) > u(z)$, $u'(y) > u'(z) > u'(x)$, and $u''(z) > u''(x) > u''(y)$) such a problem can't arise:

Theorem B (Dasgupta-Maskin 2008; see also Sen 1966 and Inada 1969): F^C satisfies P, A, N, IIA, O, and D on $\mathcal{U}$ if and only if $\mathcal{U}$ does not contain Condorcet cycles.

Furthermore, majority rule dominates all other voting rules in the sense that it satisfies the axioms on a wider class of domains than any other:

Theorem C (Dasgupta-Maskin 2008): If *F* satisfies P, A, N, IIA, D, and O on domain $\mathcal{U}$, then F^C also satisfies these axioms on $\mathcal{U}$. Furthermore, if $F(u, Y) \neq F^C(u_{\cdot}, Y)$ for some profile $u_{\cdot}$ on $\mathcal{U}$, then there exists domain $\mathcal{U}'$ on which F^C satisfies all the axioms, but *F* does not.

The current paper's contribution is to add *strategy-proofness* to the mix:

(*Group*) *Strategy-Proofness* (SP): For all coalitions $C \subseteq [0,1]$ and generic profiles $u_{\cdot}$ and $u'_{\cdot}$ (with $u'_i = u_i$ for all $i \notin C$) on $\mathcal{U}$, let $x, y \in Y$ be such that $x = F(u_{\cdot}, Y)$ and $y = F(u'_{\cdot}, Y)$. Then, there exists $i \in C$ such that $u_i(x) > u_i(y)$. That is, if coalition *C* causes the winner to change from *x* to *y* by manipulating preferences from u_C to u'_C, someone in the coalition doesn't gain from the manipulation.

4. Strategy-Proofness and Ordinality

Together with D and A, SP implies that a voting rule is ordinal.

***Theorem 1*:** Suppose that *F* satisfies SP, D and A [11]on $\mathcal{U}$. Then, *F* satisfies O generically on $\mathcal{U}$.

[11] A version of this result can be proved without assuming A, but the argument is more complicated.

Proof: Suppose, to the contrary, that there exist generic profiles $u_\bullet^*$ and $u_\bullet^{**}$ and ballot $Y \subseteq X$ such that $u_i^*(x) > u_i^*(y) \Leftrightarrow u_i^{**}(x) > u_i^{**}(y)$ for all $i \in [0,1]$ and $x, y \in Y$ and yet $x^* \neq x^{**}$, where $x^* = F(u_\bullet^*, Y)$ and $x^{**} = F(u_\bullet^{**}, Y)$. We will show that transforming $u_\bullet^*$ to $u_\bullet^{**}$ one ordering at a time leads to contradiction.

Let $\succ^1$ be an ordering of Y and let $u_\bullet^1$ be the profile such that, for all i,

$$u_i^1 = \begin{cases} u_i^{**}, & \text{if } \succ^1 \text{ is the ordering corresponding to } u_i^{**} \\ u_i^*, & \text{otherwise} \end{cases}$$

Take $x^1 = F(u_\bullet^1, Y)$. If $x^* \succ^1 x^1$, then voters with ordering $\succ^1$ in profile $u_\bullet^1$ (each such voter i has utility function u_i^{**}) are better off manipulating to make the profile $u_\bullet^*$ (i.e., voter i will pretend to have utility function u_i^*). If $x^1 \succ^1 x^*$, then voters with ordering $\succ^1$ in profile $u_\bullet^*$ are better off manipulating to make the profile $u_\bullet^1$. Thus, from SP, we must have $x^1 = x^*$.

Next choose $\succ^2 \neq \succ^1$ and let $u_\bullet^2$ be the profile such that, for all i,

$$u_i^2 = \begin{cases} u_i^{**}, & \text{if } \succ^2 \text{ corresponding to } u_i^{**} \\ u_i^1, & \text{otherwise} \end{cases}$$

By similar argument, $x^2 = x^*$ where $x^2 = F(u_\bullet^2, Y)$. Continuing iteratively, we eventually obtain $u_\bullet^n = u_\bullet^{**}$ and thus $x^n = x^{**}$, a contradiction of $x^* \neq x^{**}$. Q.E.D.

5. Results Invoking IIA

In view of Theorem A, we immediately obtain

Theorem D: *If* $|X| \geq 3$ there exists no voting rule satisfying P, A, N, IIA, D, O, and SP on $\mathcal{U}_X$.[12]

[12] This result also follows directly from Gibbard (1973) and Satterthwaite (1975)

Hence, we show that Theorem B continues to hold if we add SP to the list of axioms:

***Theorem* 2:** F^C satisfies P, A, N, IIA, D and SP on $\mathcal{U}$ if and only if $\mathcal{U}$ does not contain Condorcet cycles.[13]

***Proof*:** If $\mathcal{U}$ contains a Condorcet cycle, then from Figure 3, F^C violates D. For the converse, it suffices – in view of Theorem B – to show that F^C satisfies SP on $\mathcal{U}$. Suppose, to the contrary, there exist $u_\cdot$ and $u'_\cdot$ on $\mathcal{U}$, and coalition C such that

$$(2) \quad x = F^C(u_\cdot, Y)$$

$$(3) \quad u_i(y) > u_i(x) \text{ for all } i \in C$$

and

$$(4) \quad y = F^C(u'_\cdot, Y),$$

where $u'_j = u_j$ for all $j \notin C.$ From (2)

$$(5) \quad \mu\left(\left\{i \middle| u_i(x) > u_i(y)\right\}\right) > \frac{1}{2}$$

Hence, from (3) and (5)

$$\mu(C) < \frac{1}{2},$$

i.e., C constitutes a *minority* of voters, and thus isn't big enough to change the Condorcet winner from x to y, contrary to (4). Q.E.D

Our definition of SP presumes that voters can only manipulate preferences within $\mathcal{U}$. This presumption makes sense in some circumstances. For example, suppose there are two goods – one public and one private – and that a "candidate" x consists of a level p of the public good together with a tax cp levied on each citizen, where c is the per capita cost of the public good in

[13] In view of Theorem 1, O is redundant.

terms of the private good. Consider the mechanism in which citizens vote on *p,* the *median voter's* choice p^* is implemented, and each citizen pays cp^*. If citizens' preferences are convex and increasing in the two goods, then preferences for *p* are single-peaked. Hence, the mechanism results in a Condorcet winner.

Implicitly, the mechanism constrains a citizen to submit single-peaked preference, and thus presumes that the planner knows in advance that preferences *are* single-peaked. While this may be plausible in the public good context, knowing how preferences are restricted for presidential elections seems less likely. In such settings, constraining manipulations to $\mathcal{U}$ seems unreasonable. Thus, the definition of strategy proofness becomes:

*Strategy-Proofness** (SP*): For each $C \subseteq [0,1]$ and generic profiles $u_{\bullet}$ on $\mathcal{U}$ and $u'_{\bullet}$ on $\mathcal{U}_X$ [14] (with $u'_j = u_j$ for all $j \notin C$), let $x = F(u_{\bullet}, Y)$ and $y = F(u'_{\bullet}, Y)$. Then there exists $i \in C$ such that $u_i(x) > u_i(y)$.

Since coalitions now can manipulate outside $\mathcal{U}$, a Condorcet winner may not exist. Following Smith (1973) and Fishburn (1977), define the *Smith set* for profile $u_{\bullet}$ and ballot *Y* to be the smallest set of candidates $Z(u_{\bullet}) \subseteq Y$ such that, for each $x \in Z(u_{\bullet})$ and each $y \notin Y - Z(u_{\bullet})$, a majority of voters prefer *x* to *y*. The Smith set is unique (as Fishburn shows) and reduces to the Condorcet winner if there is one; it is a natural generalization of the majority winner concept. Indeed, Fishburn (1977) argues that the following extension of majority voting rule best preserves the spirit of Condorcet:

$$F^{C^*}(u_{\bullet}, Y) = \begin{cases} x, \text{ if } x \text{ is a Condorcet winner for } u_{\bullet} \text{ and } Y \\ \text{random selection from the Smith set, if a Condorcet winner doesn't exist} \end{cases}$$

[14] Note that u'_i is no longer restricted to $\mathcal{U}$.

Theorem 3: Provided that voters are sufficiently risk-averse, F^{C^*} satisfies P, A, N, IIA, D and SP* on $\mathcal{U}$ if and only if $\mathcal{U}$ contains no Condorcet cycles.

Proof: In view of the proof of Theorem 2, we need show only that if $\mathcal{U}$ contains no Condorcet cycles and *u.* is a generic profile on $\mathcal{U}$, then for sufficiently risk-averse voters, no coalition *C* gains by manipulating. Suppose, to the contrary, that $x = F^{C^*}(u_\bullet, Y)$ and coalition *C* gains from manipulation $u'_\bullet$ (where $u'_i = u_i$ for all $i \notin C$).

If $Z(u'_\bullet)$ is a singleton, i.e. a Condorcet winner, then we obtain the same contradiction as in the proof of Theorem 2. Hence, suppose that $Z(u'_\bullet) = \{x^1, ..., x^m\}$.

Assume first that $x \notin \{x^1, ..., x^m\}$. Because *C* gains from manipulating, then for sufficiently risk averse voters,

(6) $u_i(x^k) > u_i(x)$ for all $i \in C$ for any $k \in \{1, ..., m\}$

Because *x* is a Condorcet winner,

(7) $\mu\left(\left\{i \middle| u_i(x) > u_i(x^k)\right\}\right) > \dfrac{1}{2}$,

And so, from (6)

(8) $\mu(C) < \dfrac{1}{2}$

But by definition of the Smith set

(9) $\mu\left(\left\{i \middle| u'_i(x^k) > u'_i(x)\right\}\right) > \dfrac{1}{2}$,

which contradicts (7) and (8).

Finally, assume that $x \in \{x^1, ..., x^m\}$. Then by definition of the Smith set, there exists $k \in \{1, ..., m\}$ such that (9) holds. Furthermore, if voters are sufficiently risk-averse, (6) must hold. The rest of the argument is as in the preceding paragraph. Q.E.D.

Strikingly, the axioms under discussion, P, A, N, IIA, D and SP, *uniquely* characterize majority rule on any domain that admits a voting rule satisfying these axioms:

Theorem 4: If *F* satisfies A, N, IIA, D, SP on $\mathcal{U}$, then $F = F^C$.[15]

Remark 1: This result provides an alternative to the classic axiomatization of majority rule by May (1952). May's characterization does not impose D and invokes *Positive Association* rather than SP.[16] Also, it focuses on the case $|X| = 2$, which is of limited interest because then plurality rule, runoff voting, rank-order voting and many other rules all coincide with majority rule.

Remark 2: An important difference between Theorem 4 and Theorem C is that, in the latter, P, A, N, IIA, D and O don't *uniquely* characterize F^C.

Proof: The proof is remarkably simple. Suppose *F* satisfies the axioms. From Theorem 1, we can confine attention to ordinal preferences. Assume first that $|Y| = 2,$ specifically, $Y = \{x, y\}$. If, contrary to the theorem,

$$(10)\quad F\begin{pmatrix} \underline{a} & \underline{1-a} \\ x & y \\ y & x \end{pmatrix}, \{x,y\}\Bigg) = y,\ \text{where } a > \frac{1}{2} \text{ for some profile } \begin{matrix} \underline{a} & \underline{1-a} \\ x & y \\ y & x \end{matrix},$$

then from N and A

$$(11)\quad F\begin{pmatrix} \underline{a} & \underline{1-a} \\ y & x \\ x & y \end{pmatrix}, \{x,y\}\Bigg) = x$$

[15] Note that P is redundant.

[16] Positive Responsiveness says that if we alter voters' preferences between *x* and *y* so that all voters like *x* at least as much vis a vis *y* as they did before and some now like *x* strictly more, then (i) if *x* and *y* were both chosen by *F* before, now *x* is uniquely chosen, and (ii) if *x* was uniquely chosen before, it still is.

But, in profile (11), if a coalition of voters with ranking $\begin{smallmatrix} y \\ x \end{smallmatrix}$ pretends to have ranking $\begin{smallmatrix} x \\ y \end{smallmatrix}$ so as to attain profile (10), then they attain outcome *y*, which they prefer to *x*. Hence SP is violated and the theorem is established for $|Y| = 2$.

Assume next that $|Y| > 2$. If

$$F\left(\begin{matrix} \underline{a} & \underline{1-a} \\ | & | \\ x & y \\ | & | \\ y & x \\ | & | \end{matrix}, Y\right) = y, \text{where } a > \frac{1}{2}$$

then from IIA

$F\left(\begin{matrix} \underline{a} & \underline{1-a} \\ x & y \\ y & x \end{matrix}, \{x, y\}\right) = y,$ contradicting the previous paragraph. Q.E.D.

6. Dropping IIA: Condorcet and Borda

Let us now drop IIA, the most controversial axiom. With an unrestricted domain, we continue to get impossibility:

Theorem E: If $|X| \geq 3$, there exists no voting rule satisfying A, N, D, and SP on $\mathcal{U}_x$.[17]

Define a domain $\mathcal{U}$ to be *rich* if for all $x \in X$ there exist $u \in \mathcal{U}$ and $y, z \in X$ such that $u(y) > u(x) > u(z)$, i.e., *x* is not extremal. We now state our final result:

Theorem 5: Suppose $|X| = 3$. If *F* satisfies P, A, N, D, and SP on $\mathcal{U}$ and $\mathcal{U}$ is rich, then $F = F^C$ or $F = F^B$. That is, a voting rule satisfying the axioms on a rich domain must either be majority rule or rank-order voting.

---

[17] This follows from the Gibbard-Satterthwaite Theorem (they impose O, but from Theorem 1, this is unnecessary).

Remark 1: The richness hypothesis is essential for the result. Suppose that $X = \{x, y, z\}$,

$$\mathcal{U} = \left\{ \begin{matrix} x \\ y \\ z \end{matrix}, \begin{matrix} z \\ y \\ x \end{matrix} \right\} \text{ and } F(u_\bullet, \{x, y, z\}) = \begin{cases} x, \text{ if } u_i = \begin{smallmatrix} x \\ y \\ z \end{smallmatrix} \text{ for all } i \\ z, \text{ if } u_i = \begin{smallmatrix} z \\ y \\ x \end{smallmatrix} \text{ for all } i \\ y, \text{ otherwise} \end{cases}$$

It is easy to see that *F* satisfies all the axioms and is neither majority rule nor the Borda count. This doesn't contradict Theorem 5, however, because $\mathcal{U}$ is not rich; *x* and *z* are always extremal. Intuitively, richness allows us to permute alternatives in *x* while remaining in $\mathcal{U}$. Thus, N has bite.

Remark 2: Theorem 5 limits attention to $|X| = 3$, but this is a theoretically important case, since all major voting rules differ when $|X| = 3$ (unlike for $|X| = 2$). Moreover, $|X| = 3$ is also the most significant case *practically* speaking. For example, in 2016, third-place finishers changed the winner in 6% of the U.S. Senate races. But in no case was a fourth-place finisher decisive. [18]

The proof of Theorem 5 is long and complicated, so is relegated to the Appendix. Here is an outline of the argument:

Step 1: We first show that F^B is the unique voting rule satisfying the axioms on the Condorcet cycle domain

$$\mathcal{U}^{CC} = \left\{ \begin{matrix} x \\ y \\ z \end{matrix}, \begin{matrix} y \\ z \\ x \end{matrix}, \begin{matrix} z \\ x \\ y \end{matrix} \right\}.$$

Barbie-Puppe-Tasnad (2006) show that F^B satisfies the axioms on this domain. Thus, it suffices to show that no other voting rule *F* does so. Consider profile

[18] I thank Cecilia Johnson for research assistance on this point.

$$u_{\cdot} = \begin{array}{ccc} \underline{a} & \underline{b} & \underline{1-a-b} \\ x & y & z \\ y & z & x \\ z & x & y \end{array}$$

From symmetry, we can assume without loss of generality that

(12) $F^B(u_{\cdot},\{x,y,z\}) = x$.

Suppose that *y*'s Borda count is higher than *z*'s in $u_{\cdot}$. Then, it can be shown that

(13) $a > \frac{1}{3} > b$ and $a + b > \frac{2}{3}$.

Assume that

(14) $F(u_{\cdot},\{x,y,z\}) = y$

and

(15) $1-a-b > b$ (which is consistent with (13))

Let σ be the permutation such that $\sigma(x) = z,\ \sigma(y) = x,\ \sigma(z) = y$. From (14), A, and N

(16) $F\left(\begin{array}{ccc} \underline{a} & \underline{b} & \underline{1-a-b} \\ z & x & y \\ x & y & z \\ y & z & x \end{array},\ \{x,y,z\} \right) = x$

Thus, some voters with ranking $\begin{array}{c} x \\ y \\ z \end{array}$ in profile $u_{\cdot}$ could pretend to have ranking $\begin{array}{c} y \\ z \\ x \end{array}$ and others could pretend to have $\begin{array}{c} z \\ x \\ y \end{array}$ so as to induce profile (16). Through this manipulation, they change the outcome of *F* from *y* to *x*, which they prefer – a violation of SP. The other cases (when *z*'s Borda count is higher than *y*'s , when $F(u_{\cdot},\{x,y,z\}) = z,$ and when $1-a-b < b$) are similar and handled in the Appendix.

Step 2: We next show that there exists no voting rule satisfying the axioms on any expansion of $\mathcal{U}^{CC}$.

From symmetry it suffices to consider

$$\mathcal{U}^{*}=\left\{\begin{matrix} x & y & z & x \\ y & z & x & z \\ z & x & y & y \end{matrix}\right\}.$$

Suppose to the contrary that F satisfies the axioms on $\mathcal{U}^{*}$. For small $d>0,$ we have

$$(17)\quad F\left(\begin{matrix} \frac{1}{3}-d & \frac{1}{3}+2d & \frac{1}{3}-d \\ \hline x & y & z \\ y & z & x \\ z & x & y \end{matrix},\ \{x,y,z\}\right)=y\,,$$

since for this profile, y is the Borda winner and, from step 1, $F=F^{B}$ on $\mathcal{U}^{CC}$. Assume that

$$(18)\quad F\left(\begin{matrix} \frac{1}{3}-d & \frac{1}{3}+2d & \frac{1}{3}-d \\ \hline x & y & x \\ z & z & z \\ y & x & y \end{matrix},\ \{x,y,z\}\right)=x\,.$$

But then the $\begin{matrix} x \\ y \\ z \end{matrix}$ - and $\begin{matrix} z \\ x \\ y \end{matrix}$ - voters in (17) are better off pretending to have ranking $\begin{matrix} x \\ z \\ y \end{matrix}$ so as to induce profile (18) and outcome x, which they prefer to y. Similar violations of SP can be drawn when the outcome for profile (18) is y or z. The final step of the proof is:

Step 3*:* We show that any voting rule satisfying the axioms on a rich domain without a Condorcet cycle must be majority rule.

The argument is similar to that for Step 1.

Theorem 5 helps resolve a historical tension. Majority rule and rank-order voting are the longest-studied voting rules, and their most famous proponents – Condorcet and Borda – were archrivals. It is satisfying to see that, in a sense, these old adversaries were both right.

45%	**55%**	
x	y	Under rank-order voting,
z	x	x gets 3×45 + 2×55 = 245 points
y	z	y gets 3×55 + 1×45 = 210 points
		z gets 2×45 + 1×55 = 145 points
		so x wins

But if z doesn't run, we have

45%	**55%**	
x	y	Under rank-order voting,
y	x	x gets 2×45 + 1×55 = 145 points
		y gets 2×55 + 1×45 = 155 points
		so y wins

Figure 1 Rank-order Voting

40%	**35%**	**25%**
Trump	Rubio	Cruz
Cruz	Cruz	Rubio
Rubio	Trump	Trump

For the rankings above, Trump wins (with 40%) under plurality rule. However, both Cruz and Rubio defeat Trump in a head-to-head contest, They lose in a 3-way race because they split the anti-Trump vote. Hence, plurality rule violates IIA. Runoff voting does too (Rubio wins in a three-way race, but Cruz wins head-to head with Rubio.

Figure 2 Vote Splitting

35%	33%	32%
x	*y*	*z*
y	*z*	*x*
z	*x*	*y*

Given the rankings above, candidate *z* can't be the winner under majority rule because a majority of voters (68%) prefer *y*. Moreover, *y* can't be the winner because majority (67%) prefer *x*. But *x* can't win because a majority (67%) prefer *y*. The three rankings constitute *Condorcet cycle*.

Figure 3 Condorcet cycles

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Appendix:

Voting Rule Definitions

Majority Rule (Condorcet's method):

$$F^C(u_\bullet, Y) = \left\{ x \in Y \,\middle|\, \mu\left(\left\{ i \,\middle|\, u_i(x) > u_i(y) \right\}\right) \geq \frac{1}{2} \text{ for all } y \neq x, y \in Y \right\}$$

That is, candidate *x* is a winner if, for any other candidate *y*, a majority prefer *x* to *y*.

Rank-Order Voting (Borda count):

$$F^B(u_\bullet, Y) = \left\{ x \in Y \,\middle|\, \int r_{u_i}(x) d\mu(i) \geq \int r_{u_i}(y) d\mu(i) \text{ for all } y \in Y \right\},$$

$$\text{where } r_{u_i}(x) = \left| \left\{ y \in Y \,\middle|\, u_i(x) \geq u_i(y) \right\} \right|$$ [19]

In words, candidate *x*'s point-score $r_{u_i}(x)$ for voter i with preferences u_i is the number of candidates that voter *i* ranks no higher than *x*. Candidate *x* is a winner if the integral of her point-scores over voters is no lower than that of any other candidate *y*.

Approval Voting:

For each voter *i* and each utility function u_i, let $\underline{u}_i$ be the minimum utility that a candidate must generate to gain *i*'s "approval."

Then

$$F^A(u_\bullet, Y) = \left\{ x \in Y \,\middle|\, \mu\left(\left\{ i \,\middle|\, u_i(x) \geq \underline{u}_i \right\}\right) \geq \mu\left(\left\{ i \,\middle|\, u_i(y) \geq \underline{u}_i \right\}\right), \text{for all } y \in Y \right\}$$

[19] For any finite set *S*, $|S|$ denotes the number of elements in *S*.

That is, candidate *x* is a winner if the proportion of voters who approve her is at least as big as that of any other candidate.

Proof of Theorem 5

Step 1: We first show that F^B is the unique voting rule that satisfies A, N, D and SP on

$$\mathcal{U}^{c} = \left\{ \begin{matrix} x \\ y \\ z \end{matrix}, \begin{matrix} y \\ z \\ x \end{matrix}, \begin{matrix} z \\ x \\ y \end{matrix} \right\}.$$

We already noted that the fact that F^B satisfies SP on $\mathcal{U}^{c}$ follows from Barbie et al (2006) (F^B always satisfies A, N, and D). Thus it suffices to show that no other voting rule *F* satisfies the axioms on this domain. Consider profile

$$u_{\bullet} = \begin{array}{ccc} \underline{a} & \underline{b} & \underline{1-a-b} \\ x & y & z \\ y & z & x \\ z & x & y \end{array}$$

where *a* and *b* are the proportions of voters with the corresponding ranking. From symmetry, we can assume without loss of generality that *x* is the Borda winner for this profile.

Case I: Suppose that *y* beats *z* according to the Borda count.

Then because *x* beats *y*, we have $2a+1-a-b > 2b+a > 2(1-a-b)+b$,

and so

(A.1) $a > \dfrac{1}{3} > b$ and $a+b > 2/3$

I.1 Suppose that $F(u_{\cdot}, Y) = y$, i.e.,

$$(\text{A.2})\ \ F\begin{pmatrix} \underline{a} & \underline{b} & \underline{1-a-b} \\ x & y & z \\ y & z & x \\ z & x & y \end{pmatrix} = y \ \begin{pmatrix}\text{henceforth, we suppress the argument } Y, \\ \text{which is always equal to } \{x,y,z\}\end{pmatrix}$$

Let σ be the permutation of $Y = \{x, y, z\}$ such that $\sigma(y) = x$ $\sigma(x) = z$ and $\sigma(z) = y$

Then from A, N and (A.2)

$$(\text{A.3})\ \ F\begin{pmatrix} \underline{a} & \underline{b} & \underline{1-a-b} \\ z & x & y \\ x & y & z \\ y & z & x \end{pmatrix} = x$$

I.1.1 Suppose $1-a-b > b$

Then some voters with ranking $\begin{matrix} x \\ y \\ z \end{matrix}$ in profile *u.* could pretend to have ranking $\begin{matrix} y \\ z \\ x \end{matrix}$ and some

others could pretend to have $\begin{matrix} z \\ x \\ y \end{matrix}$ so as to attain the profile in (A.3). Thus, through this

manipulation, they will change the outcome of *F* from *y* to *x*, which they prefer, a violation of

SP. Hence,

I.1.2 $b > 1-a-b$

I.1.2.1 Suppose

$$F\begin{pmatrix} \underline{1-2b} & \underline{b} & \underline{b} \\ x & y & z \\ y & z & y \\ z & x & x \end{pmatrix} = x$$

Then some $\begin{matrix} x \\ y \\ z \end{matrix}$-voters in profile *u.* could pretend to have ranking $\begin{matrix} z \\ x \\ y \end{matrix}$, and change the outcome

from *y* to *x*, a violation of SP.

I.1.2.2 Suppose

$$(\text{A.4})\ F\begin{pmatrix} \underline{1-2b} & \underline{b} & \underline{b} \\ x & y & z \\ y & z & x \\ z & x & y \end{pmatrix} = y$$

Then applying σ to (A.4), we get from A and N

$$F\begin{pmatrix} \underline{1-2b} & \underline{b} & \underline{b} \\ z & x & y \\ x & y & z \\ y & z & x \end{pmatrix} = x,$$

which implies that $\begin{matrix} x \\ y \\ z \end{matrix}$-voters in *u.* can change the outcome from *y* to *x*, a violation of SP.

Hence,

$$(\text{A.5})\ F\begin{pmatrix} \underline{1-2b} & \underline{b} & \underline{b} \\ z & x & y \\ x & y & z \\ y & z & x \end{pmatrix} = z$$

But from (A.5), $\begin{matrix} z \\ x \\ y \end{matrix}$-voters in (A.3) can pretend to have ranking $\begin{matrix} y \\ z \\ x \end{matrix}$ and change the outcome from

x to *z*, a violation of SP. We conclude that case *I.1* is impossible.

Hence we have

I.2 $F(u.,Y) = z,$ i.e.,

$$(\text{A.7})\ F\begin{pmatrix} \underline{a} & \underline{b} & \underline{1-a-b} \\ x & y & z \\ y & z & x \\ z & x & y \end{pmatrix} = z$$

Apply permutation σ to profile $u_{\bullet}$. From A, N, and (A.7), we obtain

$$\text{(A.8)}\quad F\begin{pmatrix} \underline{a} & \underline{b} & \underline{1-a-b} \\ z & x & y \\ x & y & z \\ y & z & x \end{pmatrix} = y$$

I.2.1 Suppose $1-a-b>b$

Then, analogues to case *I.1*, some $\begin{matrix} x \\ y \\ z \end{matrix}$-voters in *u.* can pretend to have rankings $\begin{matrix} y \\ z \\ x \end{matrix}$ and $\begin{matrix} z \\ x \\ y \end{matrix}$ so as to attain the profile in (A.8). Since this leads to outcome *y*, which they prefer to *z*, this contradicts SP. We conclude that case *I.2.1.* is impossible. Hence,

I.2.2 $1-a-b<b$

Apply permutation σ^{-1} to profile $u_{\bullet}$. From (A), (N), and (A.7), we have

$$\text{(A.9)}\quad F\begin{pmatrix} \underline{a} & \underline{b} & \underline{1-a-b} \\ y & z & x \\ z & x & y \\ x & y & z \end{pmatrix} = x$$

Because $a>b$ and $1-a-b<b$, $\begin{matrix} x \\ y \\ z \end{matrix}$- voters in $u_{\bullet}$ can attain the profile (A.9) by manipulating, a violation of SP. We conclude that case *I.2* is impossible, and, therefore, that Case *I* is impossible.

Case II: Suppose that *z* beats *y* according to the Borda count.

Then because *x* beats *z*, we have $2a+1-a-b>2(1-a-b)+b>2b+a,$

and so

$$\text{(A.10)}\quad a>\frac{1}{3}>b \qquad \frac{2}{3}>a+b$$

II.1 Suppose that $F(u.,Y)=z,$ i.e.,

$$\text{(A.11)}\quad F\begin{pmatrix} \underline{a} & \underline{b} & \underline{1-a-b} \\ x & y & z \\ y & z & x \\ z & x & y \end{pmatrix} = z$$

Apply σ to profile $u_{\cdot}$. From A, N, and (A.11)

we have

$$\text{(A.12)}\quad F\begin{pmatrix} \underline{a} & \underline{b} & \underline{1-a-b} \\ z & x & y \\ x & y & z \\ y & z & x \end{pmatrix} = y$$

II.1.1 Suppose $a > 1-a-b$

Then from (A.10), $1-a-b > b$ and so $\begin{matrix} x \\ y \\ z \end{matrix}$ -voters in (A.11) can pretend to have rankings

$\begin{matrix} y \\ z \\ x \end{matrix}$ and $\begin{matrix} z \\ x \\ y \end{matrix}$ so as to obtain profile (A.12), a contradiction of SP. Hence,

II.1.2 $a < 1-a-b$

II.1.2.1 Suppose

$$\text{(A.13)}\quad F\begin{pmatrix} \underline{a} & \underline{1-2a} & \underline{a} \\ x & y & z \\ y & z & x \\ z & x & y \end{pmatrix} = x$$

Then from $1-a-b > a,$ (A.11) and (A.13), some $\begin{matrix} y \\ z \\ x \end{matrix}$ -voters in profile (A.13) are better off

pretending to be $\begin{matrix} z \\ x \\ y \end{matrix}$ -voters so as to generate profile (A.11), a contradiction of SP.

Hence case *II.1.2.1* is impossible

II.1.2.2 Suppose

$$\text{(A.14)} \quad F\begin{pmatrix} \underline{a} & \underline{1-2a} & \underline{a} \\ x & y & z \\ y & z & x \\ z & x & y \end{pmatrix} = y$$

Apply σ to profile (A.14). From A and N, we have

$$\text{(A.15)} \quad F\begin{pmatrix} \underline{a} & \underline{1-2a} & \underline{a} \\ z & x & y \\ x & y & z \\ y & z & x \end{pmatrix} = x$$

From (A.15), $\begin{matrix} x \\ y \\ z \end{matrix}$ - voters in profile (A.14) can pretend to have ranking $\begin{matrix} y \\ z \\ x \end{matrix}$ and thus improve their outcome from *y* to *x*, a contradiction of SP. Hence case *II.1.2.2* is impossible. We are left with *II.1.2.3*

$$\text{(A.16)} \quad F\begin{pmatrix} \underline{a} & \underline{1-2a} & \underline{a} \\ x & y & z \\ y & z & x \\ z & x & y \end{pmatrix} = z$$

Apply σ to the profile in (A.16). From A and N, we have

$$\text{(A.17)} \quad F\begin{pmatrix} \underline{a} & \underline{1-2a} & \underline{a} \\ z & x & y \\ x & y & z \\ y & z & x \end{pmatrix} = y$$

From (A.17) , $\begin{matrix} x \\ y \\ z \end{matrix}$ -voters in profile (A.16) can pretend to have ranking $\begin{matrix} y \\ z \\ x \end{matrix}$ and thus improve their outcome from *z* to *y*, a contradiction of SP. We conclude that Case *II.1* is impossible.

II.2

$$\text{(A.18)}\quad F\begin{pmatrix} \underline{a} & \underline{b} & \underline{1-a-b} \\ x & y & z \\ y & z & x \\ z & x & y \end{pmatrix} = y$$

Apply σ to the profile in (A.18). From A and N, we have

$$\text{(A.19)}\quad F\begin{pmatrix} \underline{a} & \underline{b} & \underline{1-a-b} \\ z & x & y \\ x & y & z \\ y & z & x \end{pmatrix} = x$$

II.2.1 Suppose $a > 1-a-b$

Then, because $1-a-b > b$, (A.19) implies that $\begin{matrix} x \\ y \\ z \end{matrix}$-votes in (A.18) can pretend to have rankings

$\begin{matrix} y \\ z \\ x \end{matrix}$ and $\begin{matrix} z \\ x \\ y \end{matrix}$ and induce profile (A.19), which improves the outcome from *y* to *x*, contradicting SP.

Hence

II.2.2 $a < 1-a-b$

If

$$\text{(A.20)}\quad F\begin{pmatrix} \underline{a} & \underline{1-2a} & \underline{a} \\ x & y & z \\ y & z & x \\ z & x & y \end{pmatrix} = x \text{ or } z$$

then $\begin{matrix} y \\ z \\ x \end{matrix}$-voters in profile (A.20) can pretend to have ranking $\begin{matrix} z \\ x \\ y \end{matrix}$ and, from (A.18), improve their

outcome to *y*, contradicting SP. Hence,

$$(\text{A.21}) \quad F\begin{pmatrix} \underline{a} & \underline{1-2a} & \underline{a} \\ x & y & z \\ y & z & x \\ z & x & y \end{pmatrix} = y$$

Apply σ to (A.21), and we obtain, from A and N,

$$(\text{A.22}) \quad F\begin{pmatrix} \underline{a} & \underline{1-2a} & \underline{a} \\ z & x & y \\ x & y & z \\ y & z & x \end{pmatrix} = x$$

From (A.22), $\begin{matrix} x \\ y \\ z \end{matrix}$ -voters in (A.21) can pretend to have ranking $\begin{matrix} y \\ z \\ x \end{matrix}$ and improve the outcome from y to x, violating SP. Hence case II is impossible and step 1 is completed.

Step 2: We next show that there exists *no* voting rule satisfying A, N, D, and SP on any expansion of $\mathcal{U}^{CC}$.

From symmetry, consider

$$\mathcal{U}^* = \left\{ \begin{matrix} x \\ y \\ z \end{matrix}, \begin{matrix} y \\ z \\ x \end{matrix}, \begin{matrix} z \\ x \\ y \end{matrix}, \begin{matrix} x \\ z \\ y \end{matrix} \right\}$$

Suppose to the contrary F satisfies the axioms on $\mathcal{U}^*$

For small $d > 0$, we have

$$(\text{A.23}) \quad F\begin{pmatrix} \underline{\frac{1}{3}-d} & \underline{\frac{1}{3}+2d} & \underline{\frac{1}{3}-d} \\ x & y & z \\ y & z & x \\ z & x & y \end{pmatrix} = y$$

because, from step 1, $F = F^B$ on $\mathcal{U}^{CC}$ and y is the Borda winner for the profile in (A.23)

Suppose, first, that we have

Case 1:

$$F\begin{pmatrix} \frac{1}{3}-d & \frac{1}{3}+2d & \frac{1}{3}-d \\ \hline x & y & x \\ z & z & z \\ y & x & y \end{pmatrix} = x$$

But then $\begin{matrix} x \\ y \\ z \end{matrix}$ - and $\begin{matrix} z \\ x \\ y \end{matrix}$ -voters in profile (A.23) are better off pretending to have ranking $\begin{matrix} x \\ z \\ y \end{matrix}$ and

improving the outcome from y to x, a contradiction of SP. Next, suppose

Case II:

$$F\begin{pmatrix} \frac{1}{3}-d & \frac{1}{3}+2d & \frac{1}{3}-d \\ \hline x & y & x \\ z & z & z \\ y & x & y \end{pmatrix} = y, \text{ i.e.(from A),}$$

$$\text{(A.24)} \quad F\begin{pmatrix} \frac{2}{3}-2d & \frac{1}{3}+2d \\ \hline x & y \\ z & z \\ y & x \end{pmatrix} = y$$

From the permutation σ^* with $\sigma^*(x) = y$, $\sigma^*(y) = x$, $\sigma^*(z) = z$, A and N, (A.24) implies

$$\text{(A.25)} \quad F\begin{pmatrix} \frac{2}{3}-2d & \frac{1}{3}+2d \\ \hline y & x \\ z & z \\ x & y \end{pmatrix} = x$$

But $\begin{matrix} y \\ z \\ x \end{matrix}$ - voters in (A.25) can pretend to have ranking $\begin{matrix} x \\ z \\ y \end{matrix}$ and induce (A.24), improving the outcome from x to y and thereby violating SP. We conclude that we must have

Case III:

$$\text{(A.26)} \quad F\begin{pmatrix} \frac{1}{3}-d & \frac{1}{3}+2d & \frac{1}{3}-d \\ \hline x & y & x \\ z & z & z \\ y & x & y \end{pmatrix} = z$$

From (A.26) and A, we have

$$\text{(A.27)} \quad F\begin{pmatrix} \frac{2}{3}-2d & \frac{1}{3}+2d \\ \hline x & z \\ z & x \\ y & y \end{pmatrix} = z,$$

otherwise $\begin{matrix} z \\ x \\ y \end{matrix}$ -voters in (A.27) are better off pretending to have ranking $\begin{matrix} y \\ z \\ x \end{matrix}$ and inducing (A.26), a violation of SP. Let σ^{**} be the permutation for which $\sigma^{**}(x) = z \quad \sigma^{**}(y) = y \quad \sigma^{**}(z) = x$. Then from (A.27) A, and N

$$\text{(A.28)} \quad F\begin{pmatrix} \underline{\frac{2}{3}-2d} & \underline{\frac{1}{3}+2d} \\ z & x \\ x & z \\ y & y \end{pmatrix} = x$$

But $\begin{matrix} z \\ x \\ y \end{matrix}$ -voters in (A.28) can pretend to have ranking $\begin{matrix} x \\ z \\ y \end{matrix}$ and induce profile (A.27) and outcome z, a violation of SP. So case III is also impossible, and step 2 is complete.

Step 3: Finally, we show that any voting rule satisfying A, N, D, and SP on a rich domain without a Condorcet cycle must be majority rule.

There are three such maximal domains:

Case I: $\mathcal{U}^I = \left\{ \begin{matrix} x \\ y \\ z \end{matrix}, \begin{matrix} y \\ z \\ x \end{matrix}, \begin{matrix} y \\ x \\ z \end{matrix}, \begin{matrix} z \\ y \\ x \end{matrix} \right\}$ (single-peaked preferences)

Suppose F satisfies the axioms on $\mathcal{U}^I$ but $F \neq F^C$ on that domain. Then there exists profile

$$u_{\bullet}^I = \begin{matrix} \underline{a} & \underline{b} & \underline{c} & \underline{1-a-b-c} \\ x & y & y & z \\ y & z & x & y \\ z & x & z & x \end{matrix}$$

Such that $F^C(u_{\bullet}^I, \{x, y, z\}) \neq F(u_{\bullet}, \{x, y, z\})$

I.1. $F^C(u_{\bullet}^I, \{x, y, z\}) = x,$, and so $a > \frac{1}{2}$

1.1.1. Suppose

$$\text{(A.29)} \quad F(u_{\bullet}^I, \{x, y, z\}) = y$$

If

$$\text{(A.30)}\quad F\begin{pmatrix} \underline{a} & \underline{1-a} \\ x & y \\ y & x \\ z & z \end{pmatrix} = x \text{ or } z$$

then from (A.29), $\begin{matrix} y \\ x \\ z \end{matrix}$ -voters in profile (A.30) are better off pretending to have rankings $\begin{matrix} y \\ z \\ x \end{matrix}$ and

$\begin{matrix} z \\ x \\ y \end{matrix}$ so as to induce profile $u^I_\bullet$ and outcome *y*, a violation of SP. Hence, we have

$$\text{(A.31)}\quad F\begin{pmatrix} \underline{a} & \underline{1-a} \\ x & y \\ y & x \\ z & z \end{pmatrix} = y$$

Consider permutation σ^* with $\sigma^*(x) = y,\ \sigma^*(y) = x,\ \sigma(z) = z.$ Then, from A, N and (A.31)

$$\text{(A.32)}\quad F\begin{pmatrix} \underline{a} & \underline{1-a} \\ y & x \\ x & y \\ z & z \end{pmatrix} = x$$

From (A.31) and (A.32) and because $a > \frac{1}{2}$, $\begin{matrix} x \\ y \\ z \end{matrix}$ -voters in (A.31) are better off pretending to have

ranking $\begin{matrix} y \\ x \\ z \end{matrix}$ so as to induce profile (A.32) and outcome *x*, violating SP. Hence Case *I.1.1* is

impossible. We next consider

I.1.2

$$\text{(A.33)}\quad F(u^I_\bullet, \{x, y, z\}) = z$$

If

(A.34) $F\begin{pmatrix} \underline{a} & \underline{1-a} \\ x & z \\ y & y \\ z & x \end{pmatrix} = x$ or y

then $\begin{matrix} z \\ y \\ x \end{matrix}$-voters in profile (A.34) are better off pretending to have rankings $\begin{matrix} y \\ z \\ x \end{matrix}$ and $\begin{matrix} y \\ x \\ z \end{matrix}$ so as to induce profile $u_{\bullet}^{I}$ and outcome *z*, a violation of SP. Hence,

(A.35) $F\begin{pmatrix} \underline{a} & \underline{1-a} \\ x & z \\ y & y \\ z & x \end{pmatrix} = z$

Apply permutation σ^{**} (with $\sigma^{**}(x) = z$, $\sigma^{**}(z) = x$, and $\sigma^{**}(y) = y$) to (A.35).

Then, from A and N,

(A.36) $F\begin{pmatrix} \underline{a} & \underline{1-a} \\ z & x \\ y & y \\ x & z \end{pmatrix} = x$

From (A.35) and (A.36) and because $a > \frac{1}{2}$, $\begin{matrix} x \\ y \\ z \end{matrix}$ voters in (A.35) are better off pretending to have ranking $\begin{matrix} z \\ y \\ x \end{matrix}$ so as to induce profile (A.36) and outcome *x*, violating SP. Hence, case *I.1* is impossible. We next consider

1.2 $F^{C}(u_{\bullet}^{I}, \{x, y, z\}) = y$ and so $a < \frac{1}{2}$ and $a + b + c > \frac{1}{2}$

I.2.1 Suppose

(A.37) $F(u_{\bullet}^{I}, \{x, y, z\}) = x$

If

$$\text{(A.38)} \quad F\begin{pmatrix} \underline{a} & \underline{b+c} & \underline{1-a-b-c} \\ x & y & z \\ y & x & y \\ z & z & x \end{pmatrix} = y,$$

then $\begin{matrix} y \\ z \\ x \end{matrix}$ - voters in (A.37) will pretend to have ranking $\begin{matrix} y \\ x \\ z \end{matrix}$ so as to induce profile (A.38) and

outcome y, a violation of SP. If

$$\text{(A.39)} \quad F\begin{pmatrix} \underline{a} & \underline{b+c} & \underline{1-a-b-c} \\ x & y & z \\ y & x & y \\ z & z & x \end{pmatrix} = z,$$

then $\begin{matrix} y \\ x \\ z \end{matrix}$ -voters in profile (A.39) will pretend to have ranking $\begin{matrix} y \\ z \\ x \end{matrix}$ so as to induce profile u. and

outcome x, a violation of SP. Hence,

$$\text{(A. 40)} \quad F\begin{pmatrix} \underline{a} & \underline{b+c} & \underline{1-a-b-c} \\ x & y & z \\ y & x & y \\ z & z & x \end{pmatrix} = x$$

If

$$\text{(A.41)} \quad F\begin{pmatrix} \underline{a} & \underline{1-a} \\ x & y \\ y & x \\ z & z \end{pmatrix} = y,$$

then $\begin{matrix} z \\ y \\ x \end{matrix}$ - voters in (A.40) will pretend to have ranking $\begin{matrix} y \\ x \\ z \end{matrix}$ so as to induce profile (A.41),

contradicting SP. If

$$\text{(A.42)}\quad F\begin{pmatrix} \underline{a} & \underline{1-a} \\ x & y \\ y & x \\ z & z \end{pmatrix} = z,$$

then $\begin{matrix} y \\ x \\ z \end{matrix}$-voters in (A.42) will pretend to have ranking $\begin{matrix} z \\ y \\ x \end{matrix}$ so as to induce profile (A.40) and outcome *x*, contradicting SP. Thus,

$$\text{(A.43)}\quad F\begin{pmatrix} \underline{a} & \underline{1-a} \\ x & y \\ y & x \\ z & z \end{pmatrix} = x$$

Apply permutation σ^* with $\sigma^*(x) = y,\ \sigma^*(y) = x,\ \sigma^*(z) = z$ to (A.43). Then, from A and N

$$\text{(A.44)}\quad F\begin{pmatrix} \underline{a} & \underline{1-a} \\ y & x \\ x & y \\ z & z \end{pmatrix} = y$$

Because $1 - a > \frac{1}{2}$, $\begin{matrix} y \\ x \\ z \end{matrix}$-voters in (A.43) gain from pretending to have ranking $\begin{matrix} x \\ y \\ z \end{matrix}$ so as to induce profile (A.44) and outcome *y*, contradicting SP. Hence Case *I.2.1* is impossible.

By symmetry between *x* and *z*, it is also impossible to have $F(u_\cdot^I, \{x, y, z\}) = z$. Hence, Case *I.2* is impossible, which leaves

I.3 $F^C(u_\cdot^I, \{x, y, z\}) = z$

But by the symmetry between *x* and *z*, this subcase is ruled out by the impossibility of Case *I.1*.

So we turn to

Case II:

$$\mathscr{U}^{II} = \left\{ \begin{matrix} x \\ y \\ z \end{matrix}, \begin{matrix} y \\ z \\ x \end{matrix}, \begin{matrix} y \\ x \\ z \end{matrix} \right\}$$

Suppose F satisfies the axioms on $\mathscr{U}^{II}$ but $F \neq F^C$.

$$u_{\bullet}^{II} = \begin{matrix} \underline{a} & \underline{b} & \underline{1-a-b} \\ x & y & y \\ y & z & x \\ z & x & z \end{matrix}$$

such that $F^C(u_{\bullet}^{II}, \{x, y, z\}) \neq F(u_{\bullet}^{II}, \{x, y, z\})$. Notice that z is Pareto-dominated by y, and so we can't have $F(u_{\bullet}, \{x, y, z\}) = z$ or $F^C(u_{\bullet}, \{x, y, z\}) = z$.

II.1 Suppose

$$\text{(A.45)} \quad F^C \begin{pmatrix} \underline{a} & \underline{b} & \underline{1-a-b} \\ x & y & y \\ y & z & x \\ z & x & z \end{pmatrix} = x, \text{ and so } a > \frac{1}{2}$$

Hence

$$\text{(A.46)} \quad F \begin{pmatrix} \underline{a} & \underline{b} & \underline{1-a-b} \\ x & y & y \\ y & z & x \\ z & x & z \end{pmatrix} = y$$

From (A.46) we have

$$\text{(A.47)} \quad F \begin{pmatrix} \underline{a} & \underline{1-a} \\ x & y \\ y & x \\ z & z \end{pmatrix} = y,$$

otherwise $\begin{matrix} y \\ x \\ z \end{matrix}$-voters in (A.47) would pretend to have ranking $\begin{matrix} y \\ z \\ x \end{matrix}$ so as to induce profile (A.46) and better outcome y. Apply permutation σ^* (with $\sigma^*(x) = y$, $\sigma^*(y) = x$, $\sigma^*(z) = z$ to profile (A.47). From A and N, we obtain

$$\text{(A.48)} \quad F\begin{pmatrix} \underline{a} & \underline{1-a} \\ y & x \\ x & y \\ z & z \end{pmatrix} = x.$$

But some $\begin{matrix} y \\ x \\ z \end{matrix}$-voters in (A.48) are better off pretending to have ranking $\begin{matrix} x \\ y \\ z \end{matrix}$ so as to induce profile (A.47) and better outcome y, a contradiction of SP. Thus case *II.1* is impossible. We have

II.2

$$\text{(A.49)} \quad F^C\begin{pmatrix} \underline{a} & \underline{b} & \underline{1-a-b} \\ x & y & y \\ y & z & x \\ z & x & z \end{pmatrix} = y \text{ and so } a < \frac{1}{2}$$

and

$$\text{(A.50)} \quad F\begin{pmatrix} \underline{a} & \underline{b} & \underline{1-a-b} \\ x & y & y \\ y & z & x \\ z & x & z \end{pmatrix} = x$$

From (A.50)

$$\text{(A.51)} \quad F\begin{pmatrix} \underline{a} & \underline{1-a} \\ x & y \\ y & x \\ z & z \end{pmatrix} = x,$$

otherwise $\begin{matrix} y \\ z \\ x \end{matrix}$ -voters in (A.50) would pretend to have ranking $\begin{matrix} y \\ x \\ z \end{matrix}$ so as to induce profile (A.51) and a more favorable outcome. But (A.51) leads to the same contradictions as in case *II.1*. We conclude that case *II* is impossible.

Case III:

$$\mathcal{U}^{III} = \left\{ \begin{matrix} x \\ y \\ z \end{matrix}, \begin{matrix} x \\ z \\ y \end{matrix}, \begin{matrix} y \\ z \\ x \end{matrix}, \begin{matrix} y \\ x \\ z \end{matrix} \right\}$$

Suppose F satisfies the axioms on $\mathcal{U}^{III}$ but $F \neq F^C$.

Then, there exists profile

$$u_{\bullet}^{III} = \begin{matrix} \underline{a} & \underline{b} & \underline{c} & \underline{1-a-b-c} \\ x & x & y & y \\ y & z & z & x \\ z & y & x & z \end{matrix}$$

for which $F(u_{\bullet}^{III}, \{x,y,z\}) \neq F^C(u_{\bullet}^{III}, \{x,y,z\})$. Notice that z can never be a Condorcet winner.

Hence assume without loss of generality that

(A.52) $F^C(u_{\bullet}^{III}, \{x,y,z\}) = x$ and so $a+b > 1-a-b$

III.1 Suppose

$$\text{(A.53)} \quad F\begin{pmatrix} \underline{a} & \underline{b} & \underline{c} & \underline{1-a-b} \\ x & x & y & y \\ y & z & z & x \\ z & y & x & z \end{pmatrix} = y$$

If

$$\text{(A.54)}\quad F\begin{pmatrix} \underline{a+b} & \underline{c} & \underline{1-a-b-c} \\ x & y & y \\ y & z & x \\ z & x & z \end{pmatrix} = x$$

then $\begin{matrix} x \\ z \\ y \end{matrix}$ -voters in profile (A.53) will pretend to have ranking $\begin{matrix} x \\ y \\ z \end{matrix}$ in order to induce (A.54) and a better outcome *x*. If

$$\text{(A.55)}\quad F\begin{pmatrix} \underline{a+b} & \underline{c} & \underline{1-a-b-c} \\ x & y & y \\ y & z & x \\ z & x & z \end{pmatrix} = z$$

then some $\begin{matrix} x \\ y \\ z \end{matrix}$ -voters in (A.55) will pretend to have ranking $\begin{matrix} x \\ z \\ y \end{matrix}$ in order to induce (A.53) and better outcome *y*. Hence

$$\text{(A.56)}\quad F\begin{pmatrix} \underline{a+b} & \underline{c} & \underline{1-a-b-c} \\ x & y & y \\ y & z & x \\ z & x & z \end{pmatrix} = y$$

Furthermore,

$$\text{(A.57)}\quad F\begin{pmatrix} \underline{a+b} & \underline{1-a-b} \\ x & y \\ y & x \\ z & z \end{pmatrix} = y,$$

otherwise some $\begin{matrix} y \\ x \\ z \end{matrix}$-voters in (A.57) will pretend to have ranking $\begin{matrix} y \\ z \\ x \end{matrix}$ and induce (A.56) leading to

better outcome *y*. But applying σ^* (with $\sigma^*(x) = y,\ \sigma^*(y) = x,\ \sigma^*(z) = z$) to profile (A.57) and

then invoking A and N, we obtain

$$\text{(A.58)}\quad F\begin{pmatrix} \underline{a+b} & \underline{1-a-b} \\ y & x \\ x & y \\ z & z \end{pmatrix} = x,$$

and so $\begin{matrix} y \\ x \\ z \end{matrix}$-voters in (A.58) have the incentive to induce profile (A.57). Hence, Case *III.1* is

impossible. We are left with

$$\text{(A.59)}\quad F\begin{pmatrix} \underline{a} & \underline{b} & \underline{c} & \underline{1-a-b-c} \\ x & x & y & y \\ y & z & z & x \\ z & y & x & z \end{pmatrix} = z$$

If

$$\text{(A.60)}\quad F\begin{pmatrix} \underline{a+b} & \underline{c} & \underline{1-a-b-c} \\ x & y & y \\ y & z & x \\ z & x & z \end{pmatrix} = x,$$

then $\begin{matrix} x \\ z \\ y \end{matrix}$-voters in (A.59) will induce (A.60), a contradiction of SP.

If

$$\text{(A.61)}\quad F\begin{pmatrix} \underline{a+b} & \underline{c} & \underline{1-a-b-c} \\ x & y & y \\ y & z & x \\ z & x & z \end{pmatrix} = y,$$

then we obtain the same contradiction as with (A.56).

Hence,

$$\text{(A.62)}\quad F\begin{pmatrix} \underline{a+b} & \underline{c} & \underline{1-a-b-c} \\ x & y & y \\ y & z & x \\ z & x & z \end{pmatrix} = z$$

But, from P,

$$\text{(A.63)}\quad F\begin{pmatrix} \underline{a+c} & \underline{1-a-c} \\ y & y \\ z & x \\ x & z \end{pmatrix} = y$$

Thus, $\begin{matrix} x \\ y \\ z \end{matrix}$ -voters in profile (A.62) can gain by pretending to have rankings $\begin{matrix} y \\ z \\ x \end{matrix}$ and $\begin{matrix} y \\ x \\ z \end{matrix}$ and inducing profile (A.63). We conclude Case *III* is impossible. Q.E.D.