

Productivity Dispersion, Between-Firm Competition, and the Labor Share

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Emilien Gouin-Bonenfant*

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Abstract: I propose a tractable model of the labor share that emphasizes the interaction between labor market imperfections and productivity dispersion. I bring the model to the data using an administrative dataset covering the universe of firms in Canada. As in the data, most firms have a high labor share, yet the aggregate labor share is low due to the disproportionate effect of a small fraction of large, extremely productive “superstar firms”. I find that a rise in the dispersion of firm productivity leads to a decline of the aggregate labor share in favor of firm profit. The mechanism is that productivity dispersion effectively shields high-productivity firms from wage competition. Reduced-form evidence from cross-country and cross-industry data supports both the prediction and the mechanism. Through the lens of the model, rising productivity dispersion has caused the U.S. labor share to decline starting around 1990.

Keywords: Labor share; productivity dispersion; monopsony; firm dynamics

JEL Codes: E2, J4, L1, O4, D2.

*Department of Economics, University of California San Diego. Email: egouinbo@ucsd.edu. I wish to thank my advisor Jim Hamilton and committee members David Lagakos, Tommaso Porzio, Natalia Ramondo, Allan Timmermann, and Alexis Akira Toda. I am also thankful for helpful comments and suggestions from Julien Champagne, Ying Feng, Fatih Guvenen, Matthias Kehrig, Samuel Krumholz, Pete Klenow, Valerie Ramey, Tara Sullivan, and Nicolas Vincent. Part of this work was conducted while visiting the Bank of Canada as a PhD intern. I also thank Ben Tomlin, Stefano Gnocchi, and Danny Leung for their generous help regarding the data access agreement.

1 Introduction

For most of the twentieth century, worker compensation in the U.S. grew one-for-one with labor productivity. This observation has been enshrined as one of the stylized facts of economic growth (Kaldor [1961]). However, starting around 1990 worker compensation has stagnated relative to labor productivity, leading to a decline of the aggregate labor share (Elsby et al. [2013], Karabarbounis and Neiman [2014]). Over that same period, there has been a dramatic rise in the dispersion of firm productivity (Barth et al. [2016], Andrews et al. [2016], Berlingieri et al. [2017]). In other words, recent productivity gains have been extremely concentrated among some firms rather than being broad-based.

In this paper, I argue that the rise in productivity dispersion is in fact a core part of why the aggregate labor share has declined. In a nutshell, my argument is that since recent productivity gains have been concentrated, firms experiencing high productivity growth have been shielded from wage competition by firms whose productivity has lagged. As a result, the increase in productivity dispersion has caused the aggregate labor share to decline in favor of firm profits.

To formalize this argument, I build a tractable theory of the labor share that aggregates up from the firm level. To guide my modelling choices, I begin by summarizing the distribution of labor shares across firms. I draw on a new dataset covering the universe of firms in Canada and focus on three features of the data. First, within narrow industries, there are large and persistent differences in labor shares across firms. This suggests that it is not simply technology that determines a firm's labor share. Second, in any given year, a large fraction of firms have a labor share above one. This is not just a handful of firms, but close to 20 percent of firms, which is far too large a group to write off as simple measurement error. Third, labor shares at the firm are strongly decreasing in firm output. This means that firms who contribute the most to GDP have the lowest labor shares. These facts are not only a feature of the Canadian economy, they have all been previously documented in some form using U.S. data (Autor et al. [2017], Kehrig and Vincent [2017]). I will argue that any quantitative theory of the labor share aggregating from the firm level must be confronted with these empirical regularities.

My model builds on two ideas. First, there are search frictions in the labor market and firms set wages (as in Burdett and Mortensen [1998]). Labor market imperfections will generate firm-specific gaps between wages and labor productivity, thus implying differences in labor shares between firms operating the same technology. Second, firms face persistent productivity shocks and grow over time by accumulating workers (as in Hopenhayn and Rogerson [1993]). Introducing firm dynamics and an endogenous exit decision to the model will gen-

erate time-series variation in firm-level labor shares and rationalize the existence of negative profits (i.e. labor shares above one).

In the model, firms choose how much capital to rent, what wages to offer, and when to exit. In equilibrium, high productivity firms offer high wages in order to poach workers from lower paying firms and therefore *grow* faster. The labor share of high-productivity firms is low, reflecting the fact that they are able to recruit and retain workers despite paying them wages below their marginal productivity. In other words, these firms exert significant monopsony power in the labor market. In contrast, low-productivity firms offer low wages, grow slower, and have a high labor share. Firm-level productivity is persistent but not permanent, so the model endogenously generates “superstar firms”, who emerge by experiencing extended periods of fast growth. The model thus explains the rise and fall of large firms and, in a stationary equilibrium, generates an extremely skewed firm size distribution that obeys Zipf’s law. While there is no one-to-one relationship between employment and productivity, larger firms are on average more productive. Hence, the aggregate labor share disproportionately reflects the low labor share of large, highly-productive firms.

I then bring the model to the data using a unique firm-level, panel dataset containing employment and tax records for the universe of firms in Canada over the 2000-2015 period. The dataset is ideal as it allows me to measure value-added and worker compensation at the firm-level using the same methodology as in the National Accounts. Despite being very parsimonious, the model matches a number of features of the data that were not targeted. For example, the model generates a large interdecile range of labor shares of 1.06 (0.88 in the data) even though all firms operate the same Cobb-Douglas technology. The aggregate labor share is 0.66 (0.65 in data) while the average (unweighted) firm labor share is 0.97 (0.88 in data). The reason why the average labor share is so high is that 36 percent of firms have labor share above one (17 percent in data). The difference between the average and aggregate labor share is equal to the covariance between labor share and output share of -0.32 (-0.22 in data). Finally, the model predicts that value-added is concentrated within high-productivity, low-labor-share firms: the top decile of firms ranked by labor productivity account for 53 percent of output (42 percent in data) and have a labor share of 0.54 (0.44 in data).

Equipped with the calibrated model, I quantify the effect of an increase in the dispersion of productivity across firms. I feed in an increase in the variance of (log) labor productivity of 0.3, consistent with the measurements from [Barth et al. \[2016\]](#) over the 1977-2007 period. In response to the rise in productivity dispersion, the model predicts a large labor share decline of 2.3%, which is roughly 2/3 of the corporate labor share decline over the 1977-2007 period

(3.5%). The mechanism is that productivity dispersion effectively weakens labor market competition. High productivity firms become increasingly shielded from wage competition as their competitors move further away on the productivity spectrum. As a result, the gap between the wage they pay and their productivity increases, so their profit margins increase. In contrast, the labor share of firms at the bottom of the productivity distribution *increases*. The reason is that the marginal exiting firm is now willing to absorb larger losses as she expects higher profit margins in the future when productivity reverts. Hence, the high productivity dispersion economy exhibits a higher concentration of value-added and a more polarized distribution of firm labor shares (i.e., a “winner takes most” environment).

The micro predictions of the model are almost exactly what has been documented by [Autor et al. \[2017\]](#) and [Kehrig and Vincent \[2017\]](#) using U.S. firm-level data. In particular, the model predicts that in response to a productivity dispersion shock (1) the aggregate labor share decreases, (2) the decline is driven by a reallocation of value-added towards firms with a low and declining labor share, (3) output concentration increases, and (4) employment concentration remains unchanged.

Finally, I test the prediction of the model that an increase in productivity dispersion is associated with a decline of the aggregate labor share. I use cross-country data from 7 OECD countries over the 2001-2011 period as well as cross-industry data from 69 Canadian industries (3-digit NAICS) over the 2000-2015 period. In both cases, I measure productivity dispersion as the interdecile range of labor productivity (value-added per worker) in the cross-section of firms. I estimate cross-sectional and fixed effects regressions models. In all cases, the estimated coefficients range from -0.05 to -0.20 and are statistically significant. Using the Canadian cross-industry data, I also test the mechanism. I estimate the effect of productivity dispersion on the labor share and output share of firms ranked according to their productivity. As predicted by the model, a productivity dispersion shock is associated with a polarization of firm labor shares as well as a reallocation of output shares from low to high productivity firms.

Related Literature. Many studies document a large decline of the labor share across many of countries starting around 1980 ([Blanchard et al. \[1997\]](#), [Elsby et al. \[2013\]](#), [Karabarbounis and Neiman \[2014\]](#), [Dao et al. \[2017\]](#)). The explanation I propose—rising productivity dispersion—does not explain all of the decline but is consistent with both microeconomic evidence and aggregate data. To the best of my knowledge, my paper is the first to link the rise in productivity dispersion to the decline of the labor share.

While it is well-known that there is a large dispersion in firm productivity ([Syverson \[2011\]](#)), a less appreciated fact is that productivity dispersion has increased dramatically over time.

Kehrig [2015] shows that the (cross-sectional) standard deviation of TFP nearly doubled in the U.S. manufacturing sector from the 1970s to the 2000s. Barth et al. [2016] documents a similar trend, where the variance of log revenue per worker has increased within all industries (SIC 1-digit) over 1977-2007. The increase in productivity dispersion is broad-based within the US economy and holds within both young and mature firms as well as within high-tech and non-tech sectors (see Decker et al. [2018]). It also seems to be a global phenomenon. Andrews et al. [2016] use Orbis data covering 24 OECD countries over the period 1997-2014 period and document a strong divergence in productivity growth between firms at the “frontier firms” (defined as the top 5% of firms in terms of productivity) and non-frontier firms. They show that labor productivity grew faster at frontier firms and that the gap reflects divergence in TFP, not capital deepening. Similarly, Berlingieri et al. [2017] use harmonized business register data covering a subset of OECD countries and document a broad-based increase of productivity dispersion, which holds for both labor productivity and TFP.

Existing explanations for the decline of the labor share include a rise of outsourcing along with changes in labor market institutions (Elsby et al. [2013]) as well as capital-labor substitution caused by a decline in the relative price of capital equipment (Karabarbounis and Neiman [2014]). A more recent set of papers argue that product market power has increased, leading to a rising importance of profits in aggregate income. Barkai [2016] uses U.S. National Accounts data to impute aggregate payments to capital equipment and finds that the decline of the labor share was not compensated by a rise of the capital share, but rather a rise of the profit share. Eggertsson et al. [2018] interpret historical data through the lens of DSGE model and argue that increased markups are needed to account for a set of secular trends in the U.S. economy, including the decline of the labor share. De Loecker and Eeckhout [2017] estimate firm-level price markups using Compustat data and find that markups have increased substantially starting around 1980.

Kehrig and Vincent [2017] and Autor et al. [2017] present firm-level evidence that puts restrictions on the set of explanations for the decline of the labor share that are consistent with the data. Kehrig and Vincent [2017] focus on the manufacturing sector over the 1967-2012 period. They find that the aggregate labor share decline was driven by a *reallocation* of value-added towards firms with a low (and declining) labor share, but that there was limited reallocation of inputs (labor and capital). They also show that low-labor-shares firms are mostly firms with high TFP, not firms with high capital intensity. Autor et al. [2017] use data covering most sectors over the 1982-2012 period. They find that the labor share decline was driven by reallocation of market shares, as opposed to a broad-based decline of firm-level labor shares.

They also show that industries that saw the largest increases in sales concentration also saw the largest declines in their labor shares.

Finally, [Hartman-Glaser et al. \[2018\]](#) study the link between the volatility of sales and the aggregate capital share in a firm dynamics model where firms (risk neutral) insure workers (risk averse). They find that an increase in firm-level risk generates an increase in the aggregate capital share. My paper differs in that I study the effect of a rise in productivity dispersion (a cross-sectional moment) while they study the effect of a rise in volatility (a time-series moment). Moreover, the feature that generates a link between productivity dispersion and aggregate labor share in my model is the presence of search frictions—which grants firms monopsony power—as opposed to risk aversion.

My paper also relates to the literature on labor market monopsony. The theoretical underpinning of the *new monopsony literature* is the equilibrium search model developed in [Burdett and Mortensen \[1998\]](#) (henceforth BM).¹ Recent applications of the BM model to study the wage-setting behavior of firms include: [Meghir et al. \[2015\]](#), who study the role of informal labor markets; [Engbom and Moser \[2017\]](#), who study the effect of the large increase of the minimum wage in Brazil, and [Heise and Porzio \[2018\]](#), who study the role of firms and location preferences in shaping spatial wage gaps. My model differs in that it incorporates firm dynamics (endogenous entry, exit, firm growth and productivity shocks). There is a limited amount of work on firm dynamics models with search frictions in the labor market. Notable exceptions include [Elsby and Michaels \[2013\]](#), [Kaas and Kircher \[2015\]](#), [Gavazza et al. \[2018\]](#) and [Coles and Mortensen \[2016\]](#). The main difference with the models in the first three papers is that they do not model job-to-job flows (no on-the-job search). On-the-job search is central element of my model: it is the worker’s threat of quitting to a higher paying firm that forces firms to offer wages above the value of nonemployment.

[Coles and Mortensen \[2016\]](#) (henceforth CM) allow for job-to-job flows, but they do not model endogenous firm entry and exit. They assume that the lower bound of firm productivity is high enough that every firm makes positive profits in every date and state, so firms never wish to exit. In contrast, I do not restrict the firm productivity process and model firm exit explicitly (i.e. firms solve an optimal stopping time problem). Hence, my model generates negative firm profits in equilibrium, which is a pervasive feature of the data. Moreover, CM model the hiring process as costly but unrelated to the wage-setting decision. Instead, I model the matching process exactly as in BM (random matching with exogenous meeting rates), which implies that high-wage firms grow faster due to higher poaching flows. The

¹see [Ashenfelter et al. \[2010\]](#) and [Manning \[2011\]](#) for literature reviews

wage-setting decision of firms is tightly related to this “poaching incentive”.

I leverage the insight from CM that the firm problem in the BM model is tractable out-of-steady-state when the production technology exhibits constant-returns-to-scale (CRS). I build on this result and show that, with exogenous meeting rates and a CRS matching technology, the allocation of workers to firms in a stationary equilibrium can be characterized analytically. In particular, I provide necessary conditions for the existence of a stationary equilibrium (Assumption 3) and a closed-form solution for the employment-weighted productivity distribution and unemployment rate (Proposition 3). These results are crucial for my application, as they allow me to tractably aggregate the model and prove a number of results analytically. Finally, I leverage theoretical results on power laws (Beare and Toda [2017]) and a novel solution method for heterogeneous-agent models with fat-tails (Gouin-Bonenfant and Toda [2018]) to numerically solve the joint-distribution of productivity, employment and labor share.

2 Motivating Facts

The aggregate labor share LS is the sum of firm-level labor shares LS_i weighted by their output share Y_i/Y .

$$LS = \sum_i \underbrace{(Y_i/Y)}_{\text{output share}} \times \underbrace{LS_i}_{\text{labor share}} \quad (2.1)$$

Recent evidence highlights the fact that the decline of the U.S. labor share was driven by a reallocation of output shares toward firms with low labor shares, rather than a uniform decline of firm labor shares (Kehrig and Vincent [2017], Autor et al. [2017]). Hence, to be able to interpret movements of the aggregate labor share through the lens of a model, one needs to work with a model that can speak to the firm-level heterogeneity in the data.

From Equation 2.1, we can see that, to be consistent with both firm-level and aggregate data, a model needs to match (1) the distribution of labor shares, (2) the distribution of output shares, and (3) the dependence between labor shares and output shares. It is well known that output shares are distributed according to Zipf’s law (Axtell [2001]), but little is known regarding the distribution of firm labor shares, especially outside of the manufacturing sector. In this section, I focus on three facts that summarize the distribution of firm-level labor shares and their dependence with output shares.

Data. I use microdata data from the *National Accounts Longitudinal Microdata File* (NALMF), which is produced by Statistics Canada by merging administrative data from different sources. The NALMF contains de-identified data covering the universe of private sector employers in

Canada over the 2000-2015 period. The unit of observation is an enterprise, which is an entity larger than an establishment but smaller than a (consolidated) corporation.² I restrict the main sample to the private corporate sector excluding: Agriculture, Mining, Utilities, Education, and Health (NAICS 11, 21, 22, 61, and 62). Agriculture and Mining are excluded due to data limitations while Utilities, Education, and Health are excluded due to the fact that these sectors are dominated by public entities in Canada. Finally, I restrict the sample to firm-year observations with more than 5 employees. The labor share of firm i in year t is defined as

$$LS_{i,t} = \frac{\text{worker compensation}_{i,t}}{\text{worker compensation}_{i,t} + \text{gross profits}_{i,t}}$$

In Appendix B.1, I describe in detail how I construct the variables and then validate against aggregate data. The final sample contains 3,084,182 firm-year observations, so I do not report standard errors when presenting cross-sectional averages.

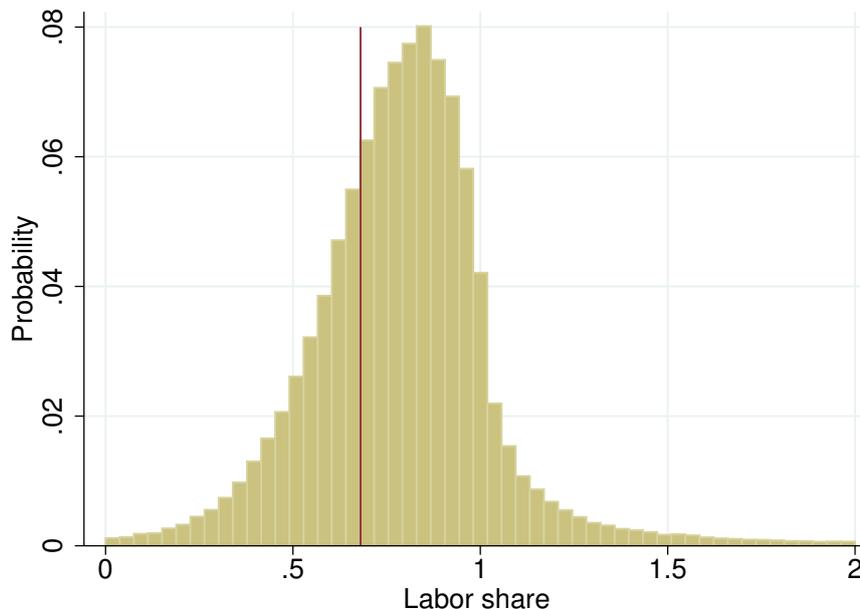


Figure 1: Distribution of firm labor shares (2015). The vertical red line is the aggregate labor share.

Fact #1. There are large and persistent differences in labor shares across firms. Figure 1 presents the distribution of firm labor shares for the year 2015. A striking fact is that behind

²The full definition from *Statistics Canada's* website is "Enterprise refers to the highest level of the Business Register statistical hierarchy and is associated with a complete set of financial statements. The enterprise, as a statistical unit, is defined as the organizational unit of a business that directs and controls the allocation of resources relating to its domestic operations, and for which consolidated financial and balance sheet accounts are maintained from which international transactions, an international investment position and a consolidated financial position for the unit can be derived. It corresponds to the institutional unit as defined for the System of National Accounts."

the aggregate labor share lies a lot of heterogeneity. The vertical red line represents the aggregate labor share of 0.67. Those large labor share differentials also hold within industry. To emphasize this point, Figure 2a presents the average labor share of firms by labor share deciles. I construct the deciles by sorting firms within 2-digit NAICS industry and year bin, and then pooling all industries and years together. The dispersion in firm labor shares within industry is large. For instance, firms in the second decile have a labor share of 0.6 on average while those in the ninth decile have a labor share of almost one. The fact that these differences hold within an industry suggests that there are factors other than technology that generate dispersion in firm labor shares.

These labor share differentials are also quite persistent. Figure 2b plots the empirical transition probability matrix for labor share deciles. For example, the (1,2) entry contains the empirical frequency at which a firm in the first labor share decile moves to the second decile from one year to the next. If differences in labor shares were only due to noise, all entries would be equal to 0.1. I find that the persistence in the data is high, which is exemplified by the fact that values along the diagonal are dark. The probability that a firm's labor share remains within the same or an adjacent decile from one year to the next is between 0.6 and 0.8 depending on the initial decile. For example, the probability that a firm in the fifth decile moves to either the fourth, fifth, or sixth decile in the following year is 0.63. Kehrig and Vincent [2017] have also documented large and persistent labor share differentials in the U.S. manufacturing sector.

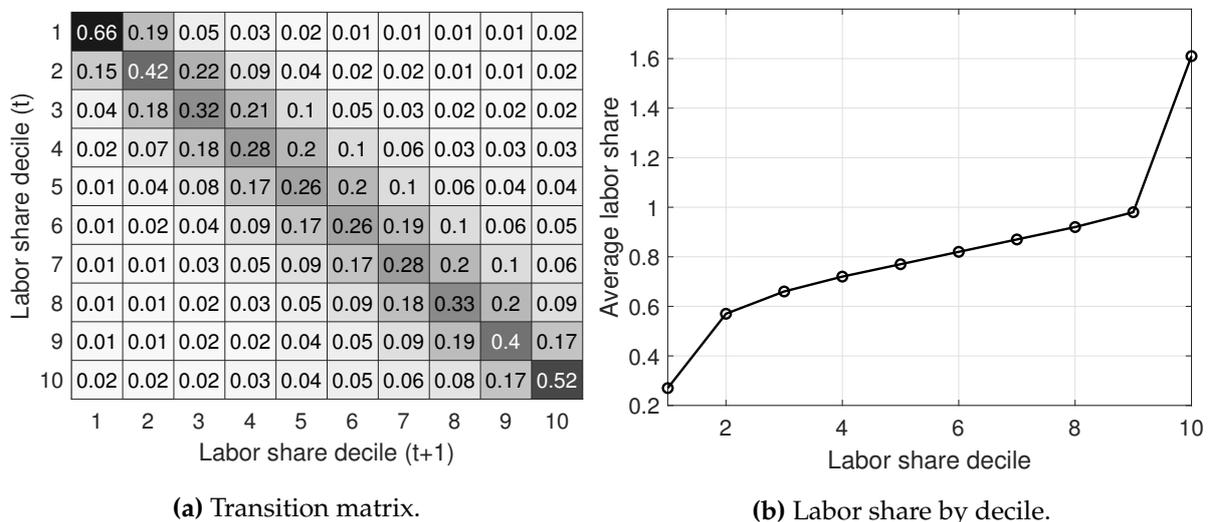


Figure 2: Panel (a) contains the empirical transition probability matrix for firm labor shares (2000-2014); panel (b) contains the average labor share by decile (2000-2015). Labor shares are sorted into deciles within industry-year bins (2-digit NAICS).

Fact #2. In any given year, a large fraction of firms have a labor share above one. Notice from Figure 1 that the distribution of labor shares has a long upper tail. In fact, 17% of firms have a labor share larger than one. A labor share above one means that worker compensation exceeds value-added, and thus implies that gross profits are negative. While the aggregate labor share is always below one, in any given year many firms will make losses. This finding is not only true for the year 2015, Figure 2a) shows that the average labor share for the top two deciles of firms are around 1 and 1.6 respectively. So labor shares above one are a pervasive feature of the data, not just the result measurement errors or a feature of the Canadian data.³ Truncating the labor share distribution at one would imply throwing away nearly one out of five observation and would bias the aggregate labor share downwards.

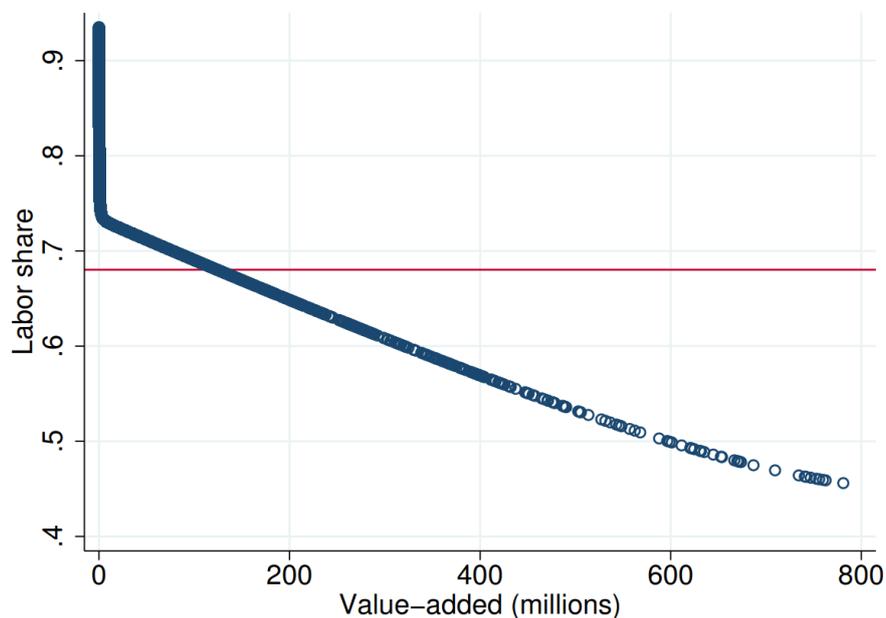


Figure 3: Smoothed scatter plot (LOWESS) of firm-level labor share and value-added (Canada, 2015).

Fact #3. There is a strong negative relationship between firm labor share and firm output. Figure 3 contains a scatterplot of firm-level output (in millions of Canadian dollars) and labor share for the year 2015. The data is smoothed (locally weighted scatterplot smoothing with bandwidth of 0.8) in order to maintain confidentiality. A clear relationship emerges where firms with high output tend to have a lower labor share. To formalize this point, consider the following decomposition of the aggregate labor share in the spirit of Olley and Pakes [1996],

³For instance, Figure 1 in Kehrig and Vincent [2017] shows that the distribution of firm labor shares in the U.S. manufacturing sector has significant mass above one.

which follows directly from Equation 2.1.

$$\underbrace{LS}_{\text{aggregate labor share}} = \underbrace{\bar{LS}}_{\text{average labor share}} + \underbrace{cov(LS_i, Y_i/Y)}_{\text{covariance between labor share and output share}}. \quad (2.2)$$

The aggregate labor share can thus be expressed as the sum of the average (unweighted) firm labor share across and the covariance between labor share and output share. Table 1 presents the results of the decomposition. To account for differences across industries and years, I compute each of the three components in Equation 2.2 separately within industry-year bin (NAICS 2-digit). I then average industries within a year using industry value-added weights and then average over all years. The aggregate labor share is 0.65 yet the average (unweighted) firm labor share is much higher at 0.88. The reason why the gap between average and aggregate labor share is so large is that firms with higher output tend to have a lower labor share. The covariance between firm-level output share and labor share is -0.23. The negative association between firm-level labor share and output share had previously been documented in the case of the U.S. manufacturing sector (Kehrig and Vincent [2017]), the Taiwanese manufacturing sector (Edmond et al. [2015]), and most other sectors of the U.S. economy (Autor et al. [2017]).

Table 1: Aggregate labor share decomposition.

Component	Value
Aggregate labor share	0.65
Average (unweighted) labor share	0.88
Covariance between labor share and output share	-0.23

Notes. The three components are defined in Equation 2.2. Each component is computed within each 2-digit NAICS industry and then averaged using value-added weights. The results are 2000-2015 averages.

3 Model

3.1 Environment

Preferences and production. Time is continuous and an interval $[t, t + 1)$ represents a year. The economy is populated by an endogenous measure F of heterogeneous firms and a unit measure of identical, infinitely-lived workers. Firms and workers are risk neutral and discount the future at rate $r > 0$. Firms compete for workers by posting wage contracts w and rent

capital in a perfectly competitive market at user cost $R \equiv r + \delta$, where δ is the depreciation rate of capital. Firms can change the wage policy at any time and do not precommit to future wages. For simplicity, I assume that firms are required to pay all of their workers the same wage w . Workers can be either employed or unemployed and are available to receive job offers in both states. The flow value of unemployment is exogenous and given by $b > 0$. Firms produce an homogeneous good and differ only in their total factor productivity (TFP) level $z \geq 0$. The production function is Cobb-Douglas with constant-returns-to-scale $zK^\alpha N^{1-\alpha}$, where $N \geq 0$ denotes the measure of workers a firm employs and $K \geq 0$ denotes its stock of capital.

Matching technology. At exogenous rate $\mu \geq 0$, an unemployed worker meets an entrepreneur (a potential firm). The entrepreneur then draws a productivity level z from the distribution Γ_0 and decides whether or not to enter. There is free entry, so entrepreneurs enter whenever expected discounted profits are positive. Without loss of generality, entry will follow a threshold rule where the entrepreneur enters if and only if $z \geq z_l$. The threshold z_l is an equilibrium object which I will characterize later. I now make the following assumption regarding the source distribution (CDF) Γ_0 .

Assumption 1. *The cumulative distribution function $\Gamma_0 : [0, +\infty) \rightarrow [0, 1]$ is differentiable and satisfies*

$$\Gamma'_0(z) > 0, \quad \int_0^\infty z\Gamma'_0(z)dz = 1, \quad \int_0^\infty z^{\frac{1}{1-\alpha}}\Gamma'_0(z)dz < \infty.$$

The first condition ensures that there is positive mass everywhere on $[0, \infty)$, the second condition is merely a normalization, and the third condition ensures that aggregate output is finite.

Firm productivity changes over time. At rate χ , continuing firms draw a new productivity level from Γ_0 . This particular process implies that firm productivity z remains constant for a stochastic duration, and then jumps to a new level that is independent from the previous one. The optimal behavior of firms will be characterized later, but the decision rule will be to exit whenever z falls below an endogenous threshold z_l . Notice that I am using the same notation z_l for the entry and exit thresholds. I will later show (Lemma 2) that those two objects are equal in equilibrium. Firms decide when to exit. When they exit, their workers are sent into unemployment. To simplify notation, define the distribution Γ_0 truncated at z_l by $\Gamma(z)$, the rate at which an unemployed workers meets an entering firm χ_e , the firm exit rate by χ_x , and the arrival rate of productivity shocks χ_s by

$$\Gamma(z) \equiv \frac{\Gamma_0(z) - \Gamma_0(z_l)}{1 - \Gamma_0(z_l)}, \quad \chi_e \equiv \mu(1 - \Gamma_0(z_l)), \quad \chi_x \equiv \chi\Gamma_0(z_l), \quad \chi_s \equiv \chi(1 - \Gamma_0(z_l)). \quad (3.1)$$

The law of motion for the measure of active firms F is thus given by

$$\dot{F} = \underbrace{\chi_e u}_{\text{firm entry}} - \underbrace{\chi_x F}_{\text{firm exit}}. \quad (3.2)$$

I now describe how continuing firms meet new workers and grow. A firm with a measure N of workers meets a new worker at rate λN , where $\lambda > 0$ is a technological parameter. I assume random matching, meaning that the identity of the workers a firm meets is randomly drawn from the population, independently of their current status (employed or unemployed).⁴ Of those workers, a fraction u will be unemployed while the remainder $1 - u$ will be employed. Diagram 1 summarizes the possible events when unemployed (left) and employed (right) over a short time interval $[t, t + \Delta]$.

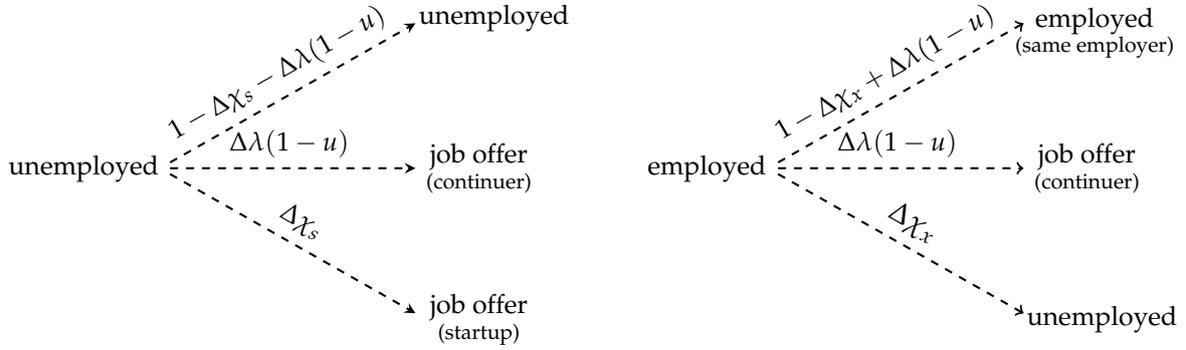


Diagram 1. Possible events when unemployed (left) and employed (right) over a short time interval $[t, t + \Delta]$.

Notice that the rate at which workers (employed or unemployed) receive a job offer from a continuing firm is given by $\lambda(1 - u)$. This is because each continuing firm meets λN workers at every instant, so the measure of meetings is given by λ times aggregate employment $1 - u$. When workers meet firms, I assume that they do not observe the firm's productivity z .

Assumption 2. *The wage paid by a firm is public information but its productivity z is private.*

Solution concept. The solution that I study is a *Pure Strategy Bayesian Equilibrium*. The equilibrium is *Bayesian* given that workers need to form expectations about the future path of wages at different firms in order to compute continuation values. The *pure strategy* part prevents agents from randomizing their actions. I will focus on size-invariant equilibria in which wages

⁴To give some context, in [Burdett and Mortensen \[1998\]](#) a firm meets a worker at a *constant* rate λ . Since the meeting rate does not grow with a firm's size N , it means that the matching technology exhibits decreasing-returns-to-scale. The only departure from *BM* in the labor market module of my model is that I assume that the matching technology exhibits constant-returns-to-scale.

are increasing in firm productivity (as in [Coles and Mortensen \[2016\]](#)). The focus on size-invariant equilibria means that a firm with productivity z and size N offers a wage $w(z)$ that does not depend on its size N . This is not very restrictive since both the production technology and matching technology exhibit constant-returns-to-scale. The only reason why firms would offer size-dependent wages would be if workers' beliefs induced them to behave in that manner. The focus on equilibria in which wages are increasing in firm productivity means that the equilibrium policy function of the firm satisfies $w(z') > w(z)$ whenever $z' > z$. This is not restrictive, since the case $w(z') \leq w(z)$ would not be consistent with an equilibrium.

To simplify the construction of the equilibrium, I will impose from now on that the workers' beliefs are consistent with a size-invariant equilibrium in which wages are increasing in firm productivity. I Denote the belief function, which maps productivity to wages, by $\hat{w}(z)$. The interpretation is that a workers believe that firms with productivity z will pay $\hat{w}(z)$ with probability one. Given the focus on equilibria where $w(z)$ is monotone, the equilibrium belief function must satisfy $z' > z \implies \hat{w}(z') > \hat{w}(z)$.

3.2 Worker and firm problems

Worker Problem. The only decision made by workers is whether or not to accept a job offer. To do so, workers need to compute the present value of accepting and rejecting a job offer paying. The value of unemployment U and the value of working at a firm paying $W(w)$ are the solutions to the following system of equations.

$$rU = b + \underbrace{\chi_e \int \max \{W(\hat{w}(z)) - U, 0\} d\Gamma(z)}_{\text{job offer from entrant}} + \underbrace{\lambda(1-u) \int \max \{W(w') - U, 0\} d\tilde{P}(w')}_{\text{job offer from continuing firm}} \quad (3.3)$$

$$rW(w) = w + \underbrace{\chi_s \int (W(\hat{w}(z)) - W(w)) d\Gamma(z)}_{\text{wage change due to productivity shock}} + \underbrace{\lambda(1-u) \int \max \{W(w') - W(w), 0\} d\tilde{P}(w')}_{\text{job offer from continuing firm}} + \underbrace{\chi_x (U - W(w))}_{\text{job destruction}} \quad (3.4)$$

$$W(w) \geq U \quad (3.5)$$

Given that workers can quit to unemployment, the value of being employed $W(w)$ must weakly exceed the value of unemployment U (Equation 3.5). The distribution \tilde{P} represents the distribution of wages in the population of workers. Since firms meet workers proportionally to their employment, the distribution of wage offers coming from continuing firms is precisely \tilde{P} . The optimal job acceptance rule is characterized in the following lemma.

Proposition 1. *Employed workers accept any wage-increasing job offer. Unemployed workers accept any job offer above an endogenously-determined reservation wage w_r given by*

$$w_r = b + (\mu - \chi) \int_{z_l}^{\infty} \max \{W(\hat{w}(z)) - U, 0\} \Gamma'_0(z) dz \quad (3.6)$$

Proof. See Appendix A.1 □

By definition, active firms pay wages above w_r , otherwise all their workers would have quit to unemployment. Therefore, the law of motion for the unemployment rate can be expressed as

$$\dot{u} = \underbrace{(1-u)\chi_x}_{\text{Unemployment inflows}} - u \underbrace{(\chi_e + \lambda(1-u))}_{\text{Unemployment outflows}}. \quad (3.7)$$

Firm growth. Equipped with the worker's optimal job offer acceptance rule, I now characterize the link between firm-level wage and employment growth. The instantaneous change in employment at a firm of size N paying $w \geq b$ is given by

$$\dot{N} = \tilde{g}(w)N, \quad (3.8)$$

where the employment growth function $\tilde{g}(w)$ is an endogenous object which depends on the wage distribution $\tilde{P}(w)$. Notice that \tilde{g} depends on the current wage policy w but not on size N . This result is due to the fact that meeting rates are linear in size and that the worker's acceptance rule depends only on the current wage (Proposition 1). A simple expression can be obtained for the employment growth function \tilde{g} conditional on survival

$$\tilde{g}(w) \equiv \underbrace{\lambda u + \lambda(1-u)\tilde{P}(w)}_{\text{hiring rate}} - \underbrace{\lambda(1-u)(1-\tilde{P}(w))}_{\text{separation rate}} \quad \forall w \geq w_r \quad (3.9)$$

Since meeting rates are exogenous, the hiring rate depends only on the fraction of job offers that are accepted. As a result of the random matching assumption, a fraction u of meetings are with unemployed workers and a fraction $1-u$ are with employed workers. Unemployed workers accept all job offers above w_r , while employed workers require a wage increase to quit (Proposition 1). The measure of employed workers at firms paying less than w is equal to $\tilde{P}(w)$, so the hiring rate from employment is simply given by $\lambda(1-u)\tilde{P}(w)$. Firms paying higher wages thus have a higher hiring rate from employment as they are able to poach workers higher up in the wage distribution.

Separations occur only due to quits. At rate $\lambda(1-u)$, a worker receives a competing job offer. Since firms meet workers proportionally to their size, the distribution of offered wages is equal to the distribution of wages in the population $\tilde{P}(w)$. The rate at which a firm paying w

loses its workers to competitors is thus given by $\lambda(1-u)(1-\tilde{P}(w))$. Hence, high-wage firms face a lower separation rate due to the fact that a smaller fraction of their workers get poached by competing firms. The following Lemma characterizes the employment growth function.

Lemma 1. *The growth rate function $\tilde{g}(w)$ is weakly increasing and bounded from above by λ .*

Proof. The results are obtained directly from Equation 3.9, using the fact that $\tilde{P}(w) \rightarrow 1$ \square

Firm problem. The firm problem is to choose a sequence of wage rates and levels of capital to rent $\{w_s, K_s\}_{s=0}^T$ as well as an “exit time” T to maximize the expected flow of discounted profits.⁵ Let $v(z, N)$ denote the value of a firm with productivity z and employment N .

$$v(z, N) = \max_{\{w_s, K_s\}_{s=0}^T, T \geq 0} \mathbb{E}_0 \int_0^T e^{-rs} \left(z_s K_s^\alpha N_s^{1-\alpha} - w_s N_s - R K_s \right) ds \quad (3.10)$$

$$N_0 = N, \quad z_0 = z \quad (\text{initial condition})$$

$$w_s \geq w_r \quad (\text{participation constraint})$$

$$dN_s = \tilde{g}(w_s) N_s ds \quad (\text{law of motion for labor})$$

$$dz_s = d\mathcal{J}_s(z'_s - z_s) \quad (\text{productivity process})$$

$\{\mathcal{J}_s\}_{s=0}^\infty$ denotes a Poisson process with intensity χ and $\{z'_s\}_{s=0}^\infty$ is a sequence of independent draws from Γ_0 . Due to the exit decision, the value function does not obey a standard *Hamilton-Jacobi-Bellman* (HJB) equation. It is obtained as the solution to the following variational inequality of the obstacle type.⁶

$$\min \left\{ rv(z, N) - \max_{\substack{w \geq b \\ K \geq 0}} \left\{ zK^\alpha N^{1-\alpha} - wN - RK + \frac{\partial}{\partial N} v(z, N) \tilde{g}(w) N \right\} \right. \\ \left. - \chi \left(\int v(z', N) \Gamma_0(dz') - v(z, N) \right), v(z, N) \right\} = 0. \quad (3.11)$$

Basically, Equation 3.11 says that either the value function obeys a regular HJB equation or it is equal to zero. Before going further, I make a parametric assumption to ensure that the value function is well-defined.

Assumption 3. *The meeting rate λ and rate of productivity resets χ satisfy the following inequality.*

$$0 < \lambda < \chi$$

⁵The notation can be confusing but the exit time T is not chosen at time zero, but rather is a function of the full history of the idiosyncratic shocks.

⁶For a discussion on variational inequalities in economics, and in particular to solve optimal stopping time problems in firm dynamics models, I refer the reader to [Achdou et al. \[2014\]](#).

The requirement that $\lambda > 0$ ensures that at least some firms grow over time (see Lemma 1), while $\lambda < \chi$ ensures that firms do not grow too fast. The case $\lambda \geq \chi$ would imply that the most productive firm would “take over” the whole economy.

Lemma 2. *The value function homogeneous of degree one in N , which means that $v(z, N) = v(z, 1)N$. Firm entry and exit follows a threshold rule, so that firms exit whenever $z < z_l$ and entrepreneurs enter whenever $z \geq z_l$.*

Proof. See Appendix A.2 □

From now on, I denote the value of a firm with productivity z and a unit measure of worker by $v(z) \equiv v(z, 1)$ and the stock of capital per worker by $k \equiv K/N$. Over the range $z \geq z_l$, the value function $v(z)$ obeys the following HJB equation

$$rv(z) = \max_{\substack{w \geq b \\ k \geq 0}} \left\{ \underbrace{zk^\alpha - w - Rk}_{\text{profits}} + \underbrace{v(z)\tilde{g}(w)}_{\text{growth}} \right\} + \underbrace{\chi \left(\int v(z')\Gamma_0(dz') - v(z) \right)}_{\text{productivity shocks}}. \quad (3.12)$$

Over the range $z < z_l$, we have that $v(z) = 0$.⁷ From Equation 3.12, we can see that the annuity value of a firm $rv(z)$ depends not only on current profits $zk^\alpha - w - Rk$ but also on future growth prospects and accounts for the risk of productivity shocks. A corollary of the homogeneity result (Lemma 2) is that the wage function $w(z)$ and the optimal the stock of capital per worker $k(z)$ are also size-independent. From the first-order conditions, we have

$$\underbrace{z\alpha k(z)^{\alpha-1}}_{\text{marginal product of capital}} = \underbrace{R}_{\text{User cost of capital}} \quad (3.16)$$

$$\underbrace{1}_{\text{marginal increase in wage bill}} = \underbrace{v(z)\tilde{g}'(w(z))}_{\text{marginal increase in value of the firm}}. \quad (3.17)$$

The first-order condition for capital is standard and implies that firms rent capital up to the point where the marginal product of capital is equal to its user cost. The first-order condition for wages captures the core trade-off faced by a wage-setting firm in a dynamic environment. When a firm offers a high wage, it makes lower per-employee profit but grows faster, which

⁷More generally, the value function can be characterized as the solution to a *Linear Complementary Problem*.

$$v(z) \left(rv(z) - \max \left\{ zk^\alpha - w - Rk + v(z)\tilde{g}(w) \right\} \right) - \chi \left(\int v(z')\Gamma_0(dz') - v(z) \right) = 0 \quad (3.13)$$

$$rv(z) - \max \left\{ zk^\alpha - w - Rk + v(z)\tilde{g}(w) \right\} - \chi \left(\int v(z')\Gamma_0(dz') - v(z) \right) \geq 0 \quad (3.14)$$

$$v(z) \geq 0 \quad (3.15)$$

The exit region is characterized by a set $\mathcal{X} = \{z : v(z) = 0\}$. I use this representation of the problem for the numerical solution (see Appendix A.10).

increases the future value of the firm. I denote the equilibrium growth rate of employment by $g(z) \equiv \tilde{g}(w(z))$. The following proposition provides closed-form expressions for the policy functions.

Proposition 2. *Capital intensity $k(z)$, labor productivity $LP(z)$ and the wage schedule $w(z)$ are increasing in z and are given by*

$$k(z) = (\alpha/R)^{\frac{1}{1-\alpha}} z^{\frac{1}{1-\alpha}} \quad (3.18)$$

$$LP(z) = (\alpha/R)^{\frac{\alpha}{1-\alpha}} z^{\frac{1}{1-\alpha}} \quad (3.19)$$

$$w(z) = w_r + \int_{z_l}^z v(\zeta) g'(\zeta) d\zeta. \quad (3.20)$$

Proof. The expressions for $k(z)$ and $LP(z)$ are obtained directly from the first-order condition 3.16. For the derivation of $w(z)$, see Appendix A.3. \square

Given that employment is predetermined while capital per worker is flexible, high z firms rent more capital per worker (see Equation 3.18). Equation 3.20 highlights the fact that workers earn a wage above their reservations wage and that the premium $w(z) - w_r$ is higher at more productive firms. Finally, the firm-level labor share $LS(z)$ is given by

$$LS(z) \equiv \frac{w(z)}{LP(z)} = \frac{w_r + \int_{z_l}^z v(\zeta) g'(\zeta) d\zeta}{(\alpha/R)^{\frac{\alpha}{1-\alpha}} z^{\frac{1}{1-\alpha}}}. \quad (3.21)$$

3.3 Equilibrium and characterizations

Distribution. The analysis thus far is in partial equilibrium since the employment growth function \tilde{g} depends on the endogenously-determined wage distribution \tilde{P} through Equation 3.9. The wage distribution is central to my analysis. It fully summarizes the amount of wage competition that firms face. Instead of deriving a law of motion of the wage distribution, I focus on the employment-weighted productivity distribution $P(z) \equiv \tilde{P}(w(z))$, as it will be easier to characterize. The law of motion for $P(z)$ is given by

$$\dot{P}(z) = \underbrace{(1-u)\lambda P(z)(P(z)-1)}_{\text{Job-to-job flows}} + \underbrace{\frac{u}{1-u}\chi_e \Gamma(z) + u\lambda P(z)}_{\text{Employment inflows}} - \underbrace{\chi_x P(z)}_{\text{Employment outflows}} + \underbrace{\chi_s(\Gamma(z) - P(z))}_{\text{Productivity shocks}}. \quad (3.22)$$

In Appendix A.4, I provide a derivation of the law of motion. Conditional on $P(z)$, the equilibrium employment growth function $g(z)$ can be computed using the definition of $\tilde{g}(w)$ (Equation 3.9).

$$g(z) = \lambda u + \lambda(1-u)P(z) - \lambda(1-u)(1-P(z)) \quad (3.23)$$

I now define the equilibrium concept.

Equilibrium definition. A *Stationary Bayesian Equilibrium* consists of a productivity distribution $\Gamma(z)$, rates (χ_e, χ_s, χ_x) , a value function $v(z)$, a productivity threshold z_l , policy functions $w(z)$, $k(z)$, $g(z)$, a reservation wage w_r , an unemployment rate u , and an employment-weighted productivity distribution $P(z)$ that satisfy the following conditions:

1. The truncated productivity distribution $\Gamma(z)$ and rates χ_e, χ_s, χ_x are determined by the optimal entry and exit of firms (Equation 3.1);
2. The value function $v(z)$ and productivity threshold z_l solve the optimal stopping time problem of the firm (Equations 3.12);
3. The policy functions $k(z)$, $w(z)$, and $g(z)$ solve the firm problem (Equations 3.18, 3.20, and 3.23);
4. The reservation wage solve the worker problem (Equation 3.6);
5. The unemployment rate and employment-weighted productivity distribution $P(z)$ are stationary solutions to their respective laws of motion (Equations 3.7 and 3.22);
6. Workers beliefs are correct, meaning that $\hat{w}(z) = w(z)$.

In a stationary equilibrium the measure of active firms is $F = \frac{\chi_e}{\chi_x} u$ (follows from Equation 3.2). Notice that F does not appear in the equilibrium definition. The reason is that, due to constant-returns-to-scale, large firms are merely scaled replicas of small firms. Hence, the measure of firms is irrelevant. What matters is the allocation of workers to firms. The following proposition provides closed-form expressions for the equilibrium unemployment rate u and the employment-weighted productivity distribution $P(z)$.

Proposition 3. *Conditional on the productivity threshold z_l , there exists a unique pair of unemployment rate u and employment-weighted distribution $P(z)$ that are stationary solutions to their respective laws of motion (Equations 3.7 and 3.22). They are given by*

$$u = \frac{\lambda + \chi_e + \chi_x - \sqrt{(\lambda + \chi_e + \chi_x)^2 - 4\lambda\chi_x}}{2\lambda} \quad (3.24)$$

$$P(z) = \frac{\lambda(1-u) + \chi - \lambda u - \sqrt{(\lambda(1-u) + \chi - \lambda u)^2 - 4\lambda(1-u)(\chi - \lambda u)\Gamma(z)}}{2(1-u)\lambda}. \quad (3.25)$$

Proof. See Appendix A.4. The existence of a solution is guaranteed by Assumption 3. \square

A corollary of Proposition 3 is that the distribution of workers across firm productivity ranks is invariant to the underlying productivity distribution Γ . For example, the measure of workers employed at firms below the median productivity level $\Gamma^{-1}(1/2)$ is given

by $P(\Gamma^{-1}(1/2))$. Notice from Equation 3.25 that $P(\Gamma^{-1}(1/2))$ depends only on search frictions (χ, λ, u) , and in particular *does not* depend on the productivity distribution Γ . Before taking the model to the data, I discuss the theoretical mechanisms that shape the equilibrium relationship between firm-level productivity z , size N , and wages w .

Pass-through of productivity to wages. How do wages offered relate to firm productivity? Recall that the only incentive for firms to offer wages above the reservation wage w_r is to increase the growth rate of employment. This incentive is captured by the first-order condition for wages evaluated along the equilibrium path (Equation 3.17)

$$w'(z) = v(z)g'(z). \quad (3.26)$$

Equation 3.26 states that the pass-through of firm productivity to wages is equal to the increase in the growth rate $g'(z)$ multiplied by the present value of rents accruing to the firm $v(z)$. The interpretation of $w'(z)$ as a pass-through comes from the fact that if the productivity of a firm increases from z to $z + \Delta$, its wage will increase from $w(z)$ to approximately $w(z) + w'(z)\Delta$. I now use the expression for $g(z)$ (Equation 3.23) to unpack the growth effect $g'(z)$.

$$w'(z) = \underbrace{v(z)}_{\text{value effect}} \times \underbrace{2\lambda(1-u)}_{\text{search frictions}} \times \underbrace{P'(z)}_{\text{local competition}} \quad (3.27)$$

Equation 3.27 highlights the importance of wage competition in shaping the pass-through of productivity to wages. The “local competition” term $P'(z)$ represents the density of employment at z . If the density of employment at z is high, a firm faces strong incentives to increase its wage so that it can poach workers from firms with similar productivity. If, on the other hand, a firm operates at a productivity level where the density of employment is low, it faces weak incentives to increase its wage. Optimal wage-setting behavior in the model is thus intimately tied to between-firm competition for workers. Notice that the pass-through of productivity to wages is stronger when search frictions are less severe (λ is high) and when the labor market is tight (u is low). Using this characterization of $w'(z)$, I now establish two properties of the pass-through of labor productivity $LP(z)$ to wages.

Proposition 4. *The pass-through of labor productivity to wages $\frac{dw}{dLP}(z)$ satisfies two properties:*

(i) *It is non-negative*

$$\frac{dw}{dLP}(z) \geq 0 \quad \forall z \geq 0$$

(ii) *It converges to zero*

$$\lim_{z \rightarrow \infty} \frac{dw}{dLP}(z) = 0$$

Proof. See Appendix A.5 □

The main takeaway from Proposition 4 is that very productive firms face weak incentives to pass productivity gains to their workers in the form of higher wages—a prediction shared by standard wage posting models as Burdett and Mortensen [1998]. The reason is that they face very few competitors with similar levels of productivity. Increasing wages does not increase employment growth much as they already are the highest paying firms, so they face no incentive to do so. In the language of monopsony theory, high-wage firms face a (locally) inelastic labor supply curve, so their wage mark-down (i.e. gap between wage and marginal productivity) is high.

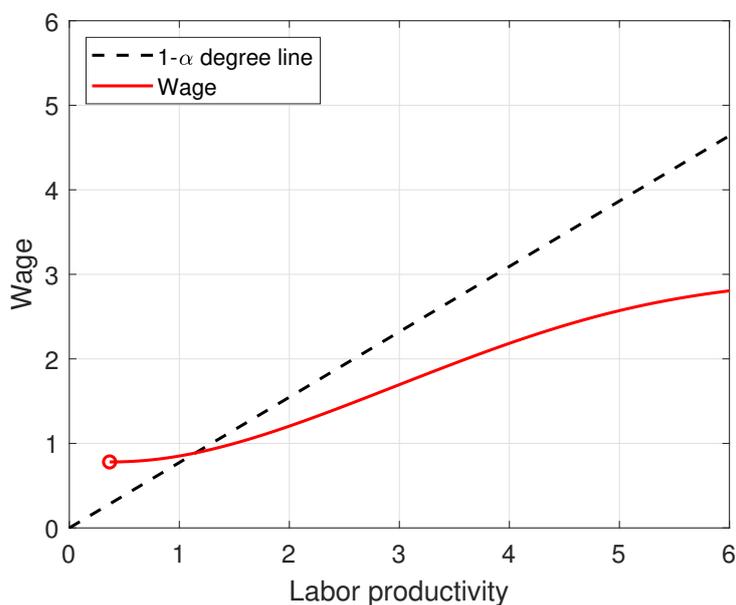


Figure 4: Equilibrium wage schedule in the calibrated model.

Figure 4 illustrates the relationship between labor productivity and wages in the calibrated model (see Section 4.1 for a description of the calibration strategy). We can see that wages increase with labor productivity, but the relationship becomes very weak in the upper range. The dashed line represents a share $1 - \alpha$ of labor productivity, which would be the wage in a frictionless, competitive model.⁸ Figure 5 contains the resulting relationship between labor share and labor productivity. Notice that some firms have a higher labor share than $1 - \alpha$, which means that after payments to capital are taken into account, they make negative profits. In contrast, high-productivity firms have a labor share below $1 - \alpha$.

In the calibrated model, the labor share is everywhere decreasing. This is not always the case, yet I can prove the following result.

⁸With perfect competition, wages are equal to the marginal product of labor $w = (1 - \alpha)LP$, which implies that the labor should be $1 - \alpha$ at every firm.

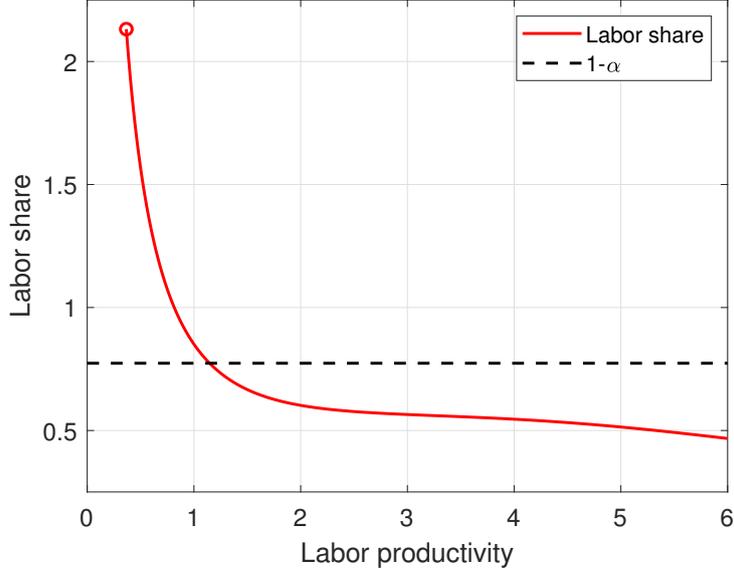


Figure 5: Equilibrium labor share schedule in the calibrated model.

Proposition 5. *There exists a productivity threshold \tilde{z} such that the labor share becomes decreasing in firm productivity.*

$$\exists \tilde{z} : \forall z > \tilde{z}, \frac{d}{dz} LS(z) < 0$$

Proof. See Appendix A.5. □

Joint-distribution of productivity and employment. In the model, firm decisions are size-invariant. Yet, in a stationary equilibrium firm productivity and employment size are *dependent* random variables. While there is no one-to-one relationship between productivity z and size N , I prove that *on average*, more productive firms are larger.

Proposition 6. *The expected value of firm employment N conditional on productivity z is an increasing function of z . It is given by*

$$\mathbb{E}(N|z) = \frac{1-u}{u} \frac{\chi_x}{\chi_e} \frac{\chi - \lambda u}{\sqrt{(\lambda(1-u) + \chi - \lambda u)^2 - 4\lambda(1-u)(\chi - \lambda u)\Gamma(z)}}.$$

Proof. See Appendix A.6. □

The reason why productivity and size are contemporaneously correlated is that more productive firms pay higher wages and thus grow faster, and thus tend to be larger. The dependence between size and productivity in a stationary equilibrium is fully summarized by the joint density $\varphi(z, N)$, which can be obtained as the solution to the following *Kolmogorov*

Forward Equation

$$0 = \underbrace{\chi_s \left(\frac{\partial}{\partial z} \Gamma(z) \int_{z_l}^{\infty} \varphi(z', N) dz' - \varphi(z, N) \right)}_{\text{productivity shocks}} - \underbrace{g(z) \frac{\partial}{\partial N} (\varphi(z, N) N)}_{\text{firm growth}} + \underbrace{\chi_x \left(\frac{\partial}{\partial z} \Gamma(z) \psi(N-1) - \varphi(z, N) \right)}_{\text{firm turnover}}, \quad (3.28)$$

where $\psi(N)$ denotes the *Dirac Delta* function. The following proposition characterizes the upper tail of the firm size distribution.

Proposition 7. *The firm size distribution has a Pareto upper tail, meaning that there exists a unique ζ such that*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P}(N > n) = -\zeta,$$

Moreover, the Pareto exponent satisfies $\eta > 1$.

Proof. The equilibrium firm growth process falls into the class of models covered by Theorem 3.1 in [Beare and Toda \[2017\]](#). The result that $\zeta > 1$ follows from Assumption 3. \square

The stationary density $\varphi(z, N)$ does not admit a closed-form solution, so I use the ‘‘Pareto Extrapolation’’ method developed in [Gouin-Bonenfant and Toda \[2018\]](#) to numerically solve φ over a finite grid and extrapolate the tails off the grid using the analytical characterization of the Pareto exponent ζ (see Appendix A.10). One well-known empirical regularity regarding the firm size distribution is Zipf’s law, which states that the size distribution of firms obeys a power law with Pareto exponent close to (but above) one (see [Gabaix \[1999\]](#) and [Axtell \[2001\]](#)). Proposition 7 states that the stationary firm size distribution has a Pareto upper tail with exponent *above* one. Whether or not the model delivers Zipf’s law is ultimately a quantitative question.

Aggregating the model. The aggregate labor share is obtained by integrating firm-level labor shares weighted by their output share against the stationary distribution.

$$LS = \int \int \underbrace{\omega(z, N)}_{\text{output share}} \times \underbrace{LS(z)}_{\text{labor share}} \times \underbrace{\varphi(dz, dN)}_{\text{stationary distribution}} \quad (3.29)$$

I denote by $\omega(z, N)$ the output share of a firm with employment N and productivity z

$$\omega(z, N) = \frac{N \times LP(z)}{Y},$$

where Y is aggregate output.⁹ Equation 3.29 highlights the fact that the aggregate labor share is shaped by rich interactions between firms in the labor market. In general, changes in the

⁹Using the employment-weighted productivity distribution $P(z)$, the labor share can be expressed as

$$LS = \int \omega(z, 1) \times LS(z) \times P(dz).$$

distribution of TFP draws Γ_0 will affect the aggregate labor share LS by changing the relative productivity differentials of firms throughout the productivity distribution, and therefore their incentives.

Proposition 8. *If TFP increases by a factor $\pi > 0$ for all firms and the value of unemployment b increases by a factor $\pi^{\frac{1}{1-\alpha}}$, then the aggregate labor share LS remains unchanged.*

Proof. See Appendix A.7 □

3.4 Interpreting differences in firm-level TFP

Before I go to the data, I want to emphasize the fact that I am being agnostic about the reasons why firms differ in their TFP. In the data, I can not observe prices separately from quantities, so the measure of labor productivity that I will construct is revenue-based. Hence, differences in labor productivity can be due to differences in prices and/or differences in physical productivity.

I now give an example of a micro foundation of TFP in the model that arises purely from differences in prices. So far, I have assumed that firms produce an homogeneous good whose price is normalized to one. I now relax this assumption. Suppose that firms all have the same productivity $z = 1$ but produce differentiated goods. Let firms be indexed by $\omega \in [0, F]$. A representative consumer derives utility from a bundle of goods according to the following utility function

$$\mathcal{U}(c) = \int_0^F \beta(\omega)c(\omega)d\omega,$$

where c and β are infinite-dimensional non-negative vectors that represent respectively the amount of each variety that is being consumed and the preference weights.

Suppose that the price of each variety $p(\omega)$ is determined in a centralized market and that the representative consumer maximizes utility. For the market to clear, it must be the case that $p(\omega) = \beta(\omega)$ for all $\omega \in [0, F]$. The revenue production function of firm ω is thus given by $zK^\alpha N^{1-\alpha}$, where $z = \beta(\omega)$. Hence, we can reinterpret TFP differences in the model—both between firms and within firm over time—as arising purely from changing consumer preferences rather than from physical productivity. The ultimate question remains the same: when does firm (revenue) productivity translates into wages?

4 Quantifying the model

4.1 Calibration strategy

First, I model the distribution of TFP draws Γ_0 as a Gamma distribution with mean normalized to one. The distribution is thus characterized by a single shape parameter $\eta > 0$ which determines the thickness of the upper tail of distribution. The resulting PDF is defined over $[0, \infty)$ and given by

$$\Gamma'_0(z) = \frac{\eta^\eta}{\Gamma(\eta)} z^{\eta-1} e^{-z\eta}, \quad (4.1)$$

where $\Gamma(\eta)$ is the Gamma function. Lower values of $\eta > 0$ are thus associated with both higher variance and skewness. Second, I set the rate at which an unemployed worker meets an entrepreneur to $\mu = \chi$. Since the measure of active firms F is irrelevant for the labor share, the difference μ and χ only affects the reservations wage and the unemployment rate. By setting $\mu = \chi$, the reservation wage w_r becomes equal to the value of unemployment b (see Equation 3.6). I will therefore identify b directly. The model has seven parameters. I assign values to the first three (r, δ, α) and jointly calibrate the last four (λ, χ, η, b) .

Assigned parameters. I set the net interest rate r to 3%, which is the difference between the effective business borrowing rate in Canada over the 2000-2008 period of as measured by the Bank of Canada¹⁰ (5.3%) and the average realized CPI inflation over that same period (2.3%). I obtain the depreciation rate δ by using industry-level data from Statistics Canada. For the set of industries covered in the main sample, the average depreciation rate over the 2000-2015 is 17.6% (Table 031-0006). The Cobb-Douglas exponent α is set to 0.185 as to imply, conditional on $R \equiv r + \delta$, a capital-to-output ratio of 1.10 as measured in the main sample. I find a similar capital-output ratio of 1.02 using public data from Canadian National Accounts for the covered industries (Statistics Canada Tables 031-0006 and 383-0063). Table 2 summarizes these choices.

Calibrated parameters. The remaining parameters $\theta \equiv (\lambda, \chi, \eta, b)'$ are jointly calibrated. I target moments from the data and choose the set of parameters that minimizes the distance

¹⁰“The effective interest rate for businesses is a weighted-average borrowing rate for new lending to non-financial businesses, estimated as a function of bank and market interest rates. The weights are derived from business credit data. The business effective rate is a function of: short-term commercial paper and bankers’ acceptance rates, with terms of one and three months; the bank prime business lending rate, which is adjusted for movements in bank funding costs, so as to estimate the effective bank prime lending rate faced by new borrowers; and longer term borrowing rates, approximated using Merrill Lynch bond indices, which include both investment and non-investment grade companies (non-financial).”

Table 2: Assigned parameters.

Parameter	Symbol	Value	Target	Source
Discount factor	r	0.030	Real interest rate of 3%	Bank of Canada
Depreciation rate	δ	0.176	Depreciation rate of 17.6%	Statistics Canada
Capital share	α	0.227	Capital output ratio of 1.10	Statistics Canada

between the model-implied moments and the data.¹¹ To ensure that Assumption 3 is satisfied, I restrict the parameters space to Θ , which is defined by

$$\Theta = \mathbb{R}_+^4 \cap \left\{ \theta \mid 0 < \lambda < \chi \right\}. \quad (4.2)$$

I choose 4 moments (statistics to be precise): the job reallocation rate, the autocorrelation of labor productivity, the interdecile range of labor productivity, and the unemployment rate. While there is no one-to-one mapping between the parameters and the moments, I now provide an informal mapping. The meeting rate λ determines the pace at which firms meet new workers, and therefore directly affects the job reallocation rate. The rate of productivity resets χ directly affects the autocorrelation of labor productivity at the firm. The shape parameter η , which determines the variance and skewness of the distribution of TFP draws, will determine the interdecile range of labor productivity. Finally, the flow value of nonemployment b will determine the productivity threshold z_l , and therefore the firm exit rate χ_x . Since all unemployment inflows comes from firm exit, b will determine the unemployment rate. Table 3 summarizes these relationships.

Table 3: Informal mapping between calibrated parameters and moments.

Parameter	Symbol	Moment
Meeting rate	λ	Job reallocation rate
Rate of productivity resets	χ	Autocorrelation of labor productivity
Shape parameter of TFP distribution	η	Interdecile range of labor productivity
Flow value of nonemployment	b	Unemployment rate

I take the annual unemployment rate and job reallocation rate directly from Canadian public data and are average them over the 2000-2015 period (Tables 282-0087 and 527-0001). Job reallocation is defined as the average of the job creation rate and the job destruction rate. The autocorrelation of labor productivity is obtained by estimating ρ in the following regression

¹¹See [Gourieroux et al. \[1993\]](#) for a theoretical treatment of the indirect inference method

model using the Canadian microdata

$$\log LP_{i,t} = c + \mu_t + \gamma_j + \rho \log LP_{i,t-1} + \eta_{i,t}, \quad \mathbb{E}_{i,t} \eta_{i,t} = 0, \quad (4.3)$$

where $LP_{i,t}$ represent the labor productivity (value-added per worker) at firm i in period t and μ_t, γ_j are year and 3-digit NAICS industry fixed-effects. I obtain a coefficient of 0.816.

The interdecile range of labor productivity is also computed using the microdata. First, I remove year and industry (2-digit NAICS) fixed effects to firm-level log labor productivity. I then compute the difference the 90th and 10th percentiles of the (residualized) log labor productivity in the pooled 2000-2015 sample.

Denote the vector of statistics in the data by $\hat{\Lambda}$ and the vector of the statistics in the model evaluated at θ by $\Lambda(\theta)$. For each value of $\theta \in \Theta$, I solve the model numerically (see Appendix A.10 for a description of the solution method) and compute the model-implied statistics $\Lambda(\theta)$. I then minimize the following quadratic form

$$\min_{\theta \in \Theta} (\Lambda(\theta) - \hat{\Lambda})' (\Lambda(\theta) - \hat{\Lambda}). \quad (4.4)$$

Since Θ is not compact, I implement the numerical optimization problem by using its closure $\bar{\Theta} \equiv \Theta \cup \{\theta \mid 0 \leq \lambda \leq \chi\}$. I then verify that the minimizer θ^* is in Θ . I obtain the parameter estimate $(\lambda, \chi, \eta, b)' = (0.2095, 0.2295, 2.9518, 0.7806)'$. Even though the parameter b is identified mainly through the unemployment rate, which arguably is not very informative about the value of nonemployment, its estimated value is fairly close to standard values in the literature. The average wage in the economy is 1.2186, which means that b is equal to 64% of the average wage which is close (but higher) than the widely used value of 0.4 in the labor search literature.¹²

Table 4 contains the targeted statistics in the model and in the data. One thing to notice is that, even though there are as many parameters as moments, the fit is not perfect. This can be explained by the fact that the equations mapping the structural parameters to the statistics are non-linear, so that an exact solution $\beta(\theta^*) = \hat{\beta}$ need not exist. Nevertheless, the only statistic that the model does not match satisfactorily is the job reallocation, and the value in the model is 7.5% which is not too far from the 10.5% found in the data.

Burdett-Mortensen model. To contrast the predictions of my model with those of the classic model of [Burdett and Mortensen \[1998\]](#) (BM), I calibrate a plain-vanilla BM model augmented with a capital as a factor of production. Since the model is well-known, I relegate the derivation to the Appendix (See Appendix A.8). For the calibration, I use the two free parameters of the

¹²Note that [Hagedorn and Manovskii \[2008\]](#) propose an calibration strategy that yields a value of $b = 0.955$.

Table 4: Targeted moments.

Statistic	Data	Model	BM
Job reallocation rate	0.105	0.075	0
Autocorrelation of labor productivity	0.816	0.807	1
Interdecile range of labor productivity	1.544	1.551	1.551
Unemployment rate	0.071	0.086	0.086

Notes. The “Model” column present the model-implied statistics while the “BM” column presents the model-implied statistics in a [Burdett and Mortensen \[1998\]](#) model.

model to exactly match the unemployment rate and interdecile range of labor productivity in the full model. From [Table 4](#), we can see that the static nature of the *BM* model restricts the productivity process to be fully persistent (firm productivity does not change through time) and job reallocation to be zero (firms do not grow or shrink over time).

4.2 Non-targeted moments

I now assess the ability of the calibrated model to match the data along several non-targeted dimensions, paying particular attention to the facts documented in [Section 2](#).

Aggregate labor share decomposition. Consider the decomposition of the aggregate labor share LS described in [Equation 2.2](#).

$$\underbrace{LS}_{\text{aggregate labor share}} = \underbrace{\bar{LS}}_{\text{average labor share}} + \underbrace{cov(LS_i, Y_i/Y)}_{\text{covariance between labor share and output share}}.$$

Table 5: Aggregate labor share decomposition (medvel versus data).

Component	Data	Model	BM
Aggregate labor share	0.653	0.655	0.538
Average (unweighted) labor share	0.884	0.970	0.542
Covariance between labor share and output share	-0.231	-0.315	-0.004

Notes. The “Data” columns is taken from [Table 1](#); the “Model” column present the model-implied statistics using the calibrated model; the “BM” column presents the model-implied statistics in the calibrated [Burdett and Mortensen \[1998\]](#) model.

I now reproduce Table 1 and include model-implied moments (see Table 5). The model generates an aggregate labor share of 0.66, which is nearly identical to the one in the data. The average (unweighted) firm-level labor share in the model is 0.97, which is close but higher than in the data (0.88). The resulting covariance between labor share and output share is negative (fact #3), and stands at -0.32 in the model (-0.23 in the data). Overall, the fit is very good, which is surprising given that the calibration procedure did not use data on the labor share. In contrast, the BM model generates a counterfactually low aggregate labor share (0.54) and a near zero correlation between firm-level labor share and output share.

Unpacking the aggregate labor share. Both in the model and in the data, the average firm has a high labor share, yet the most important firms—those with high output shares—tend to have a lower labor share. I now provide a non parametric decomposition of the aggregate labor share. Denote by LS_d and ω_d the labor share and output share within labor productivity decile d . The aggregate labor share can be expressed as

$$LS = \sum_{d=1}^{10} \omega_d LS_d. \quad (4.5)$$

Figure 6 plots the labor share and output shares within each labor productivity decile, both in the model and in the data.¹³ To provide a benchmark, I include an horizontal line at $1 - \alpha$ in Figure 6a, which represents the labor share that would prevail in a frictionless, competitive model.

First, notice that there are large differences in labor shares across firms (fact #1). In the model, the difference between the labor share at the 10th decile and the 1st decile is 1.28 (1.14 in the data). A large fraction of firms have a labor share above one (fact #2). In the model, the bottom 4 deciles of firms have a labor share above one (bottom 2 deciles in the data) while the bottom 5 deciles have a labor share above $1 - \alpha$ (bottom 5 deciles in the data). In contrast, the top decile of firms have a low labor share of roughly 54% (44% in the data).

The reason why the aggregate labor share is disproportionately determined by the low labor share of top firms can be understood from Figure 6b. Both in the model and in the data, value-added is extremely concentrated within high-productivity firms. In the model, the top

¹³To construct the productivity deciles in the data, I first sort firm-year observations by labor productivity within industry-year bins (2-digit NAICS) and then group them into 10 deciles. I then compute the measure of interest (labor share, output share, etc.) within each decile and average over industries and years (see Appendix B.3 for details). The advantage of doing things in this manner is that (1) each productivity decile contains the same industry mix as the overall economy and (2) the aggregate labor share LS has an exact decomposition, both in the model and in the data.

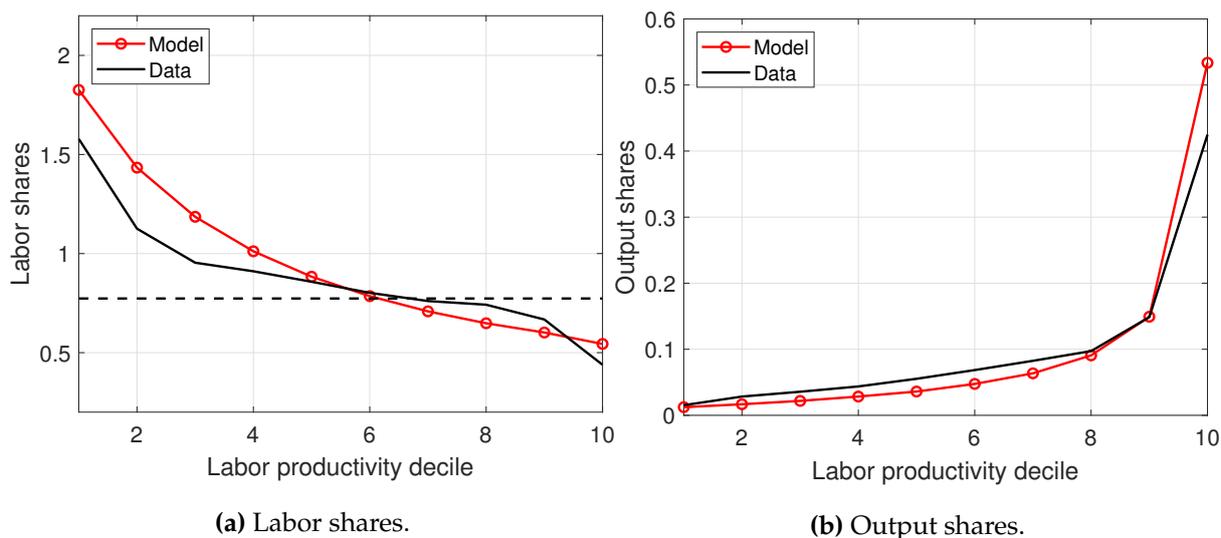


Figure 6: Decomposition in the model and in the data. Panel (a) contains labor shares by decile; panel (b) contains output shares.

decile of firms account for 53% of value-added (43% in the data), while the bottom 5 deciles accounts for a mere 12% (18% in the data). The negative covariance between labor share and output share documented earlier can thus explained by the fact that, in a stationary equilibrium, labor shares are decreasing in firm productivity while output shares are increasing in firm productivity. Firms in the top deciles achieve high output shares through a combination of employing more workers (Proposition 6) and being more productive.

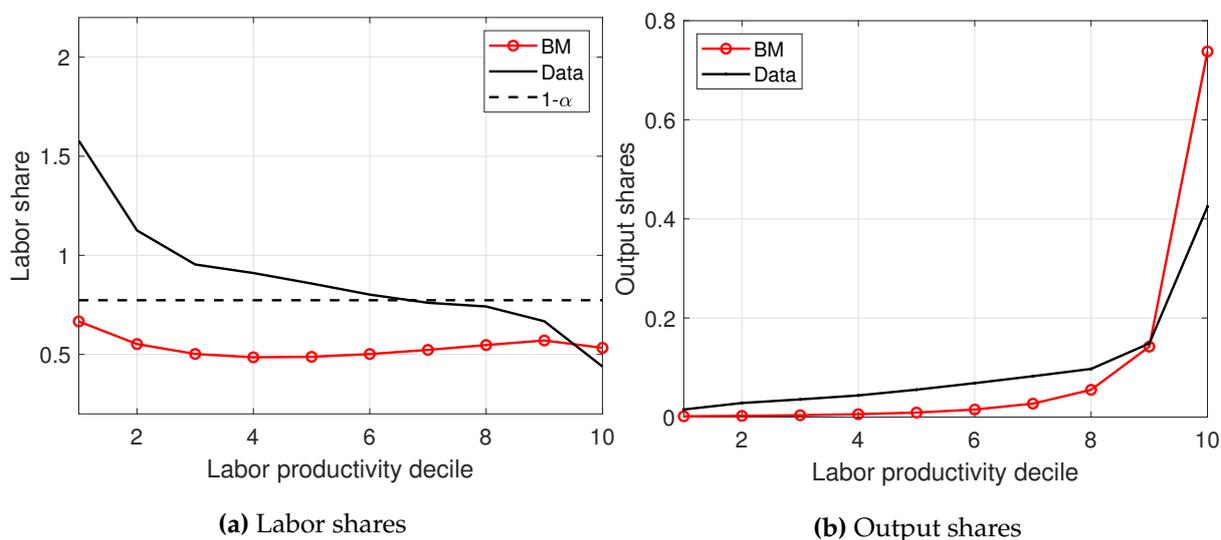


Figure 7: Decomposition in the [Burdett and Mortensen \[1998\]](#) model and in the data. Panel (a) contains labor shares by decile; panel (b) contains output shares.

Table 5 highlights the fact that the *BM* model—which is essentially a static version of my model—generates predictions that are not in line with the data. Most importantly, the aggre-

gate labor share is too low (0.54 versus 0.65 in the data) due to the fact that firm profits are too high. In both model, the share of value-added going to payments to capital is α , which implies that the share of firm profits is 11% in my model and 23% in the BM model. For comparison, Eggertsson et al. [2018] and Barkai [2016] estimate a profit share of roughly 0.12 in the U.S. over the 2000-2015 period.

Figure 7 reproduces the previous analysis in the context of the BM model. Notice that firm-level labor shares are uniformly below $1 - \alpha$. This is because firms that select into entry only if they make positive profits. In my model, flow profits can be either positive or negative. The reason is that firm productivity changes over time and some firms may choose to continue operating despite being temporarily unprofitable in order to maintain they scale. Hence, the *selection margin* accounts for the fact that the BM model predicts such a low labor share. Another counterfactual prediction of the BM model is that value-added is too concentrated (see Figure 7b). The top decile of firms account for 74% of aggregate value-added (43% in the model). This is because firm productivity does not change over time, so high productivity firms end up hiring most workers.

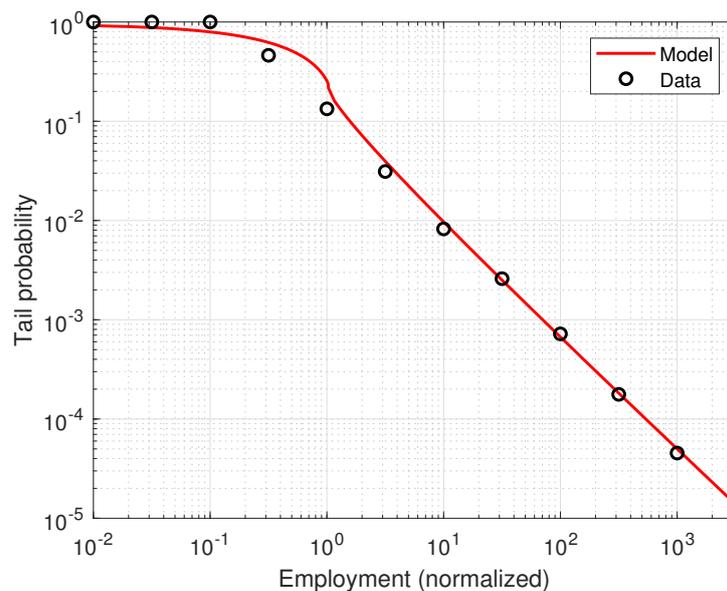


Figure 8: Complementary CDF of normalized firm size (firm employment over average firm employment). The data is computed for every year over 2000-2015 and then averaged.

Firm size distribution. In the data, it is well-known that the firm size distribution is well approximated by Zipf’s law (i.e. a Pareto distribution with exponent slightly above one). Figure 8 contains the complementary CDF of normalized firm size (firm employment over average firm employment) in the model and in the data. Despite being a non targeted moment, the fit

is exceptional. The reason is that the model generates a Pareto upper tail (Proposition 7) and in the calibrated model the Pareto exponent is 1.08 (see Appendix A.10 for details), which is close to the value of 1.06 estimated by Dixon and Rollin [2012] using Canadian data, and Axtell [2001] using U.S. data. The fact that the model endogenously generate a Pareto exponent close to one is consistent with the findings in Toda [2016], who shows that Zipf’s law robustly arises in models with a constant-returns-to-scale technology and a fixed supply of labor.

An advantage of working with a constant-returns-to-scale firm dynamics model is that it can easily replicate the extreme concentration of employment and value-added in the data without having to compromise on the fit of the productivity distribution. In fact, the firm-size distribution is invariant to the productivity distribution (see remark after Proposition 3), so the model would generate Zipf’s law for any value of η . The model endogenously generates “superstar firms”, who emerge by experiencing extended periods of fast growth. Since firm-level productivity is persistent but not permanent, the model provides a natural explanation for the rise and fall of large firms.

4.3 Model validation

In the model, the reason why high-output-share firms tend to have a lower labor share is entirely due to the fact that they tend to be more productive and pay wages significantly below labor productivity. I now test this prediction. First, I estimate the relationship between firm-level labor share and output in a regression framework. I estimate the following model

$$\log LS_{i,t} = \mu_t + \gamma_j + \beta_Y \log Y_{i,t} + \varepsilon_{i,t},$$

where μ_t and γ_j are year and industry (3-digit NAICS) fixed effects. The estimated coefficient $\hat{\beta}_Y = -0.112$ is negative and highly statistically significant (column 1 of Table 6). Is the covariance between output share and labor share driven by the fact that the capital share $R^{\frac{K}{Y}}$ is higher at larger firms? Recall that in the model, the capital share is equal to α at every firm, so cross-sectional differences in the capital-output ratio play no role in driving the relationship between firm-level output and the labor share. This needs not be true in the data, so I add the capital-output ratio as a separate regressor in the regression

$$\log LS_{i,t} = \mu_t + \gamma_j + \beta_Y \log Y_{i,t} + \beta_{K/Y} \log(K/Y)_{i,t} + \varepsilon_{i,t}.$$

Column 2 of Table 6 reports the estimated coefficients. The estimated relationship between labor share and output barely changes after controlling for K/Y (-0.113 versus -0.112). Hence, high-output firms tend to have a lower labor share, irrespective of their capital-output ratio. Consistently with economic theory, the coefficient on the capital-output ratio is negative

($\hat{\beta}_{K/Y} = -0.018$), meaning that more capital-intensive firms tend to have a lower labor share. But notice that adding the capital-output ratio as a regressor barely improves the fraction of variance explained by the model; the R^2 increases from 0.120 to 0.125.

I now decompose the effect of output Y on the labor share. Notice that the logarithm of firm output is the sum of number of the logarithms of employment N and labor productivity LP . I now include (log) employment and labor productivity separately

$$\log LS_{i,t} = \mu_t + \gamma_j + \beta_{LP} \log LP_{i,t} + \beta_N \log N_{i,t} + \varepsilon_{i,t}.$$

Column 3 of Table 6 reports the results. In the model, the labor share is size-invariant conditional on firm productivity (Proposition 2). Consistently with this prediction, the estimated coefficient on labor productivity is large significant ($\hat{\beta}_{LP} = -0.314$) but the coefficient on size is very small ($\hat{\beta}_N = 0.009$). The interpretation is that high-productivity firms tend to have a lower labor share, but after controlling for labor productivity, size is mostly irrelevant and if anything predicts a *higher* labor share.

Table 6: Labor share and output at the firm-level.

Labor share ($\log LS_{i,t}$)	(1)	(2)	(3)
Value-added ($\log Y_{i,t}$)	-0.112*** (0.001)	-0.113*** (0.001)	
Capital-output ratio ($\log (K/Y)_{i,t}$)		-0.018*** (0.000)	
Size ($\log N_{i,t}$)			0.009*** (0.001)
Labor productivity ($\log LP_{i,t}$)			-0.313*** (0.002)
Industry fixed effects	✓	✓	✓
Year fixed effects	✓	✓	✓
Sample size	3,084,182	3,084,182	3,084,182
R^2	0.120	0.125	0.273

Notes. Standard errors in parentheses clustered at the firm-level (*** $p < 0.01$, ** $p < 0.05$, * $p < 0.1$).

Wages and monopsony power. The market imperfection that allows firms in the model to earn profits is the presence of search frictions. It is well known that a wage-setting firm will exert monopsony power in the labor market when there are search frictions (Bontemps et al. [2000]). The key difference between competitive and monopsonistic labor market models relates to the elasticity of the labor supply curve faced by a firm. In a competitive environment,

firms face a perfectly elastic labor supply curve, which means that they can hire any amount of labor at the going wage rate. In contrast, monopsonists face an upward-sloping labor supply curve, which means that they must increase wages in order to recruit additional workers. In my model, the relationship is captured by Equation 3.7, which implies that firm-level employment growth over a time period $[t, t + 1)$ is related to the offered wage through

$$\log N_{t+1} - \log N_t = \int_t^{t+1} \tilde{g}(w_s) ds,$$

where the employment growth function $\tilde{g}(w)$ is increasing in w (Lemma 1). I now test this prediction in the data by estimating the following equation

$$\log N_{i,t+1} - \log N_{i,t} = \mu_t + \gamma_i + \theta \log w_{i,t} + \varepsilon_{i,t},$$

where $N_{i,t}$ and $w_{i,t}$ denote the employment and average wage of firm i in year t and the terms μ_t and γ_i are year and firm fixed effects. I estimate a slope of $\hat{\theta} = 0.213$ (see Table 7). In the calibrated model, the population regression coefficient is $\theta = 0.368$ which is larger but of the same order of magnitude.¹⁴ The interpretation is that when a firm increases its wage by 10%, its annual growth rate of employment increase by 2.1 percentage points (3.7 in the model).

Table 7: Employment growth and wage at the firm-level.

Employment growth ($\log N_{i,t+1} - \log N_{i,t}$)	
Logarithm of average wage ($\log w_{i,t}$)	0.212*** (0.001)
Firm fixed effects	✓
Year fixed effects	✓
Sample size	1,167,691
R^2	0.275

Notes. Standard errors in parentheses (*** $p < 0.01$, ** $p < 0.05$, * $p < 0.1$). The sample, which contains annual data, is restricted to observations for which there is no missing values at time $t + 1$ up to $t + 5$.

Firm size wage premium. To further validate the model, I estimate the “firm size wage premium”—in the tradition of [Brown and Medoff \[1989\]](#)—by estimating the following cross-sectional regression

$$\log w_{i,t} = \mu_t + \gamma_j + \beta_N \log N_{i,t} + \varepsilon_{i,t},$$

¹⁴I compute the model-implied regression coefficient as $\theta = \frac{\text{cov}(g(z), \log w(z))}{\text{var}(\log(w(z)))}$, where $z \sim \Gamma$.

where $w_{i,t}$ and $N_{i,t}$ denote the average wage and employment at firm i in year t . The terms μ_t and γ_j are year and industry (3-digit NAICS) fixed effects. The point I want to make is that, despite the fact that the wage schedule is size-invariant, the model still generates a cross-sectional relationship between wage and size. I estimate a firm size wage premium $\hat{\beta}$ of 0.030 (see Table 8), which is close to the model-implied regression coefficient of 0.042.¹⁵ The interpretation is that a 10% increase in a firm’s size is associated with a 0.3% (0.42% in the model) increase in the average wage. The positive relationship between size and wage in the model is explained by the fact that high productivity firms tend to be larger (Proposition 6) and pay higher wages (Proposition 4).

Table 8: Firm size wage premium.

Logarithm of average wage ($\log w_t$)	
Logarithm of firm size ($\log N_t$)	0.030*** (0.001)
Year fixed effects	✓
Industry fixed effects	✓
Sample size	3,084,182
R^2	0.390

Notes. Standard errors clustered at the firm level in parentheses (*** $p < 0.01$, ** $p < 0.05$, * $p < 0.1$)

5 Productivity dispersion and the labor share

5.1 Quantifying the effect of productivity dispersion

I now quantify the effect of an exogenous and permanent increase in productivity dispersion using the calibrated model. To do so, I feed in an exogenous increase in productivity dispersion of the same magnitude as the U.S. economy experienced over the 1977-2007 period. I use the measure of productivity dispersion constructed by Barth et al. [2016] (their Table 3). They use data from the Census Bureau’s Economic Census files and compute the variance of log revenue per worker in the cross-section of establishments.¹⁶ The data is based on quinquennial

¹⁵I compute the model-implied regression coefficient as $\beta_N = \frac{cov(\log N, \log w(z))}{var(\log N)}$, where $(z, N) \sim \varphi$.

¹⁶The authors caution that “The Economic Census expanded in scope over the 1977–2002 period, but the Business Register and LBD covered all industries throughout. As a check, we calculated the variance of revenues per worker restricted to industries where in each year total industry employment in the economic census is greater or equal to 90% of total industry employment in the LBD. The variance trend is very similar for 1977, the variance is

censuses of establishments, so they report values for the years 1977, 1982, 1987, 1992, 1997, 2002, and 2007. They find an increase of 0.311 over the full sample, almost all of which occurs in the post 2000 period. While the measure does not account for industry differences, they provide measures for 8 broad industries, and the average increase within industries is nearly identical and stands at 0.297.

To generate a “productivity dispersion shock” in the model, I decrease the parameter η , which determines the shape of the distribution of TFP draws Γ_0 . Recall that the distribution was parametrized in a way to ensure that the mean is equal to one (Equation 4.1), so the change in η represents a *mean-preserving spread* (see Figure 11). I choose $\eta = 1.6176$, which implies a 0.3 increase in the cross-sectional variance of (log) revenue. I also increase the flow value of unemployment to $b = 0.8304$ in order to keep the firm exit rate constant before and after the shock. If we think of b as an unemployment benefit, then increasing b is consistent with unemployment benefits increasing with GDP.

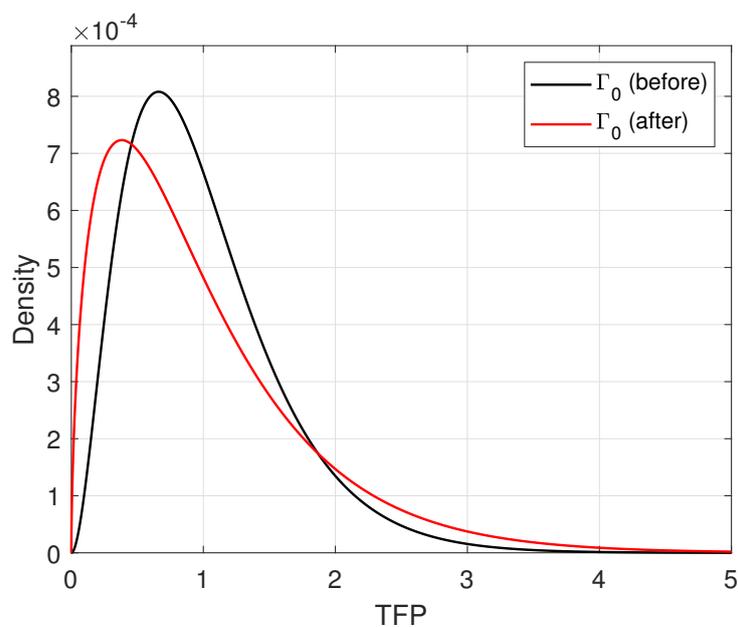


Figure 9: Distribution of productivity dispersion Γ_0 before and after the shock.

The model experiment consists in comparing the stationary equilibrium of the model before and after the shock. Table 9 contains the response of several variables to the productivity dispersion shock. First, the aggregate labor share declines by 2.26 percentage point, from 0.655 to 0.632. Interestingly, the average (unweighted) labor share moves in the opposite direction, increasing from 0.970 to 1.265. The employment share of the top decile of firms does not change. This is not surprising in light of the corollary of Proposition 3, which states that the allocation of workers to firm productivity ranks is invariant to the underlying distribution Γ .

0.945; for 1982, it is 0.965; for 1987, it is 0.991; for 1992, it is 1.036; and for 1997, it is 1.111”

Given that the productivity gap between the bottom and top decile of firms increases and the employment shares remain unchanged, the output share of the top decile of firms increases by 6.3 percentage points from 0.540 to 0.597.

Table 9: Model response to the productivity dispersion shock.

Variable	Before	After	Change (%)
Aggregate labor share	0.655	0.632	-2.26
Average (unweighted) labor share	0.970	1.265	+29.59
Output share of the top decile	0.540	0.597	+3.10
Employment share of the top decile	0.285	0.285	0

Notes. The employment/output shares of the top decile are defined as the share of employment/output of the top 10% of firms ranked by productivity.

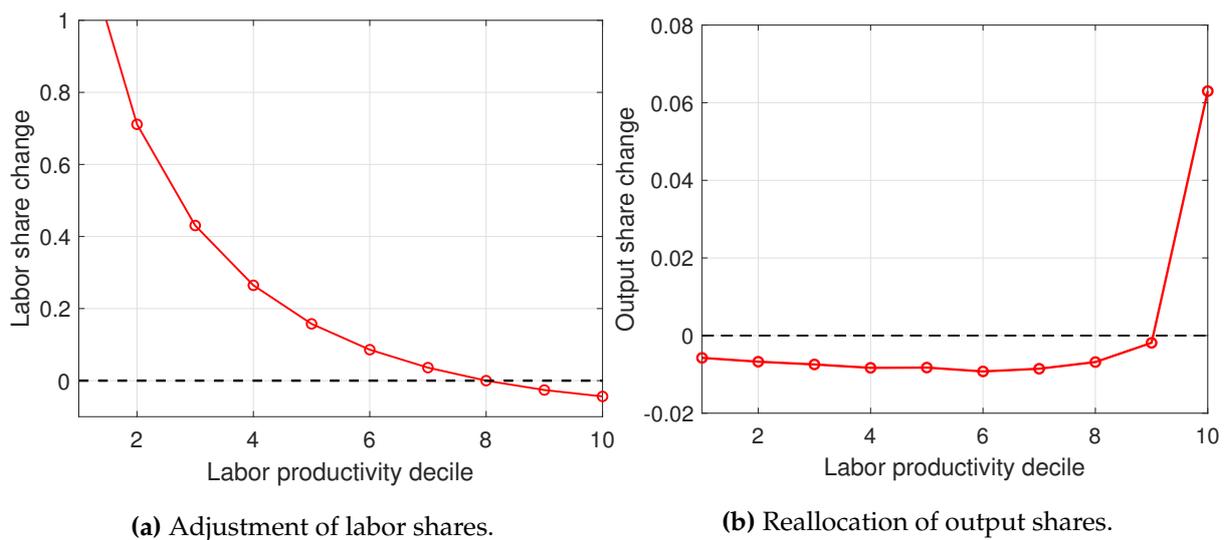


Figure 10: Model response to a permanent increase in productivity dispersion. Panel (a) contains the change in the labor shares within each productivity decile. Panel (b) contains the change in the output shares within each productivity decile.

To highlight the forces at play, I now study the impact of the productivity dispersion shock amongst different segments of firms. Figure 10 plots the response of labor shares and output shares in the model for all productivity deciles. The positive response of the average (unweighted) labor share can be explained by the fact that, for all but the top two deciles, firm labor shares increase. The *adjustment* of firm labor shares is more pronounced for low productivity firms, which leads to an increase in the dispersion of firm labor shares. Turning to Figure 10b, we see that the bottom 9 deciles of firms lose output shares at the expense of the

top decile. Having established that high productivity firms have lower labor shares, the *reallocation* of output shares towards low-labor-share firms mechanically decreases the aggregate labor share.

Shift-share decomposition. To quantify the relative importance of the “adjustment” and “re-allocation” components of the decline of the aggregate labor share implied by the productivity dispersion shock, I use a traditional shift-share decomposition as in Baily et al. [1992].¹⁷ Denote by x_d the value of a variable within productivity decile d . Using the exact decomposition from Equation 4.5, the change in the aggregate labor share can be expressed as

$$\Delta LS = \underbrace{\sum \omega_d \Delta LS_d}_{\text{adjustment}} + \underbrace{\sum \Delta \omega_d LS_d}_{\text{reallocation}} + \underbrace{\sum \Delta \omega_d \Delta LS_d}_{\text{directed reallocation}}. \quad (5.1)$$

The adjustment measures the effect of changing the labor shares LS_d while keeping the labor shares ω_d constant. The reallocation component measures the effect of changing the output shares ω_d while keeping the labor shares LS_d constant. The third term, which I interpret as “directed reallocation”, accounts for the comovements between *changes* in labor shares and output shares. For example, if there is reallocation of output shares towards firms with a *declining* labor share, then the directed reallocation term will be negative. Table 10 decomposes the model-implied aggregate labor share change into the contribution of three components Equation 5.1. The contribution of the “reallocation” and “directed reallocation” components are the largest at -1.15 and -0.26 percentage points respectively. The adjustment component works in the opposite direction and accounts for a positive contribution of 0.52 percentage points.

Table 10: Shift-share decomposition of the aggregate labor share response in the model.

Component	Contribution (%)
Adjustment	+2.91
Reallocation	-2.93
Directed reallocation	-2.25
Total	-2.26

Notes. The three components are defined in Equation 5.1.

Comparison with U.S. data. The model predicts that, in response to a productivity dispersion of the same magnitude as what has been estimated in the U.S. over 1977-2007, the aggre-

¹⁷Baily et al. [1992] decompose the data at the establishment-level while I decompose it at the productivity decile level. Since the ranking of firms in terms of productivity remains unchanged after the productivity dispersion shock, these two approaches are equivalent.

gate labor share declines by 2.26 percentage points. To compare this number with the data, I first compute the U.S. corporate sector labor share as the ratio of “compensation of employees” to “gross value-added” using BEA Table 1.14. The labor share exhibits cyclical behavior (Nekarda and Ramey [2013]), so I also estimate a cubic trend over the full sample (1929-2017). Over 1977-2007, the trend labor share decline has been 3.62 percentage points. Hence, through the lens of the model, rising productivity dispersion explains $2.26/3.62 \approx 63\%$ of the labor share decline, making it a primary contributor.

Figure 5.1 contains the time series of productivity dispersion and the labor share. As we can see, in the first half of the sample (1977-1992), the increase in productivity dispersion—the variance of (log) revenue per worker—has been limited (+0.07) and the trend labor decline has been only 0.4 percentage point. Over the second half of the sample (1992-2007), the increase in productivity dispersion has been large (+0.24) and the labor share declined by 3.2 percentage points.

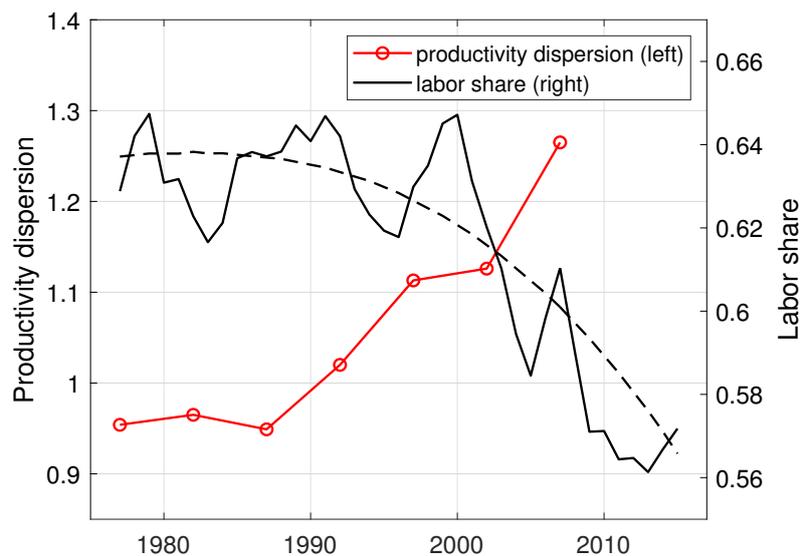


Figure 11: The corporate sector labor share is the ratio of “compensation of employees” to “gross value added”. The trend is obtained by estimating a cubic function of time over 1929-2017. Productivity dispersion is the variance of (log) revenue per worker taken from Barth et al. [2016].

In addition to a decline of the aggregate labor share, the model generates micro-level predictions that can be confronted with the data. First, the decline of the labor share in the model is overwhelmingly driven by the reallocation of output shares towards firms with a low (and declining) labor share, and is partially offset by an *increase* in the labor share of most firms (Table 10). This is precisely what Kehrig and Vincent [2017] document in the U.S. manufacturing sector. They show that over the 1967-2012 period, the aggregate labor share declined by 4.5 percentage point per decade, while the median labor share increased by 0.7 percentage point

per decade. They reconcile this divergence by showing that low-labor-share firms have gained output shares over time. Autor et al. [2017] document a similar pattern within most industries.

Second, the model predicts that, in response to a productivity dispersion shock, the concentration of output increases but employment concentration remains constant (Table 9). Consistent with this prediction, Kehrig and Vincent [2017] find that there has been a massive reallocation of output towards low-labor-share firms, but that there was limited reallocation of employment. At the aggregate level, there has been a secular increase in sales concentration in the U.S. over the past two decades (Grullon et al. [2017]), which coincided with the decline of the labor share. The negative comovement between labor share and concentration is often interpreted as evidence in favor of rising monopoly power (Barkai [2016]), but in my model the two series move in opposite direction as the result of an increase in productivity dispersion.

5.2 Empirical evidence

I now leverage cross-country and cross-industry data to test the prediction that there is a negative relationship between productivity dispersion and the labor share. The model predicts that the *level* of productivity dispersion should be negatively correlated with the *level* labor share. But it also predicts that *changes* in productivity dispersion should translate in *changes* in the labor share. I will use different data sources—which I will describe shortly—and estimate the following regressions models.

$$LS_{j,t} = \mu_t + \beta_{OLS} PD_{j,t} + \varepsilon_{j,t} \quad (5.2)$$

$$LS_{j,t} = \mu_t + \gamma_j + \beta_{FE} PD_{j,t} + \varepsilon_{j,t} \quad (5.3)$$

$$\Delta^k LS_{j,t} = \mu_t + \gamma_j + \beta_{LD} \Delta^k PD_{j,t} + \varepsilon_{j,t+k} \quad (5.4)$$

PD denotes the interdecile range of labor productivity across firms, LS is the labor share, μ_t are year fixed-effects, and γ_j is an industry and/or country fixed effects. The first specification (Equation 5.2), which I call “OLS”, uses cross-sectional variation in the *levels* for identification. The second specification (Equation 5.3), from now on fixed-effects or “FE”, uses variation in the *level* of LS and PD within a unit j over time for identification. For the last specification (Equation 5.4), from now on long differences or “LD”, I use non-overlapping k -year periods to calculate the changes Δ , so the identification comes from the low frequency comovement of LS and PD for identification.

Cross-country data from OECD countries. The OECD has developed a project called *Mul-tiProd* that seeks to provide harmonized cross-country data on productivity and wage dis-

persion. The data they collect is obtained by running a standardized routine on individual country's production surveys and business registers. The variables that I will use (productivity dispersion) is taken directly from the replication material of [Berlingieri et al. \[2017\]](#). As I did with the Canadian microdata, they measure productivity dispersion as the logarithm of the interdecile range of labor productivity in the cross-section of firms. The measures are computed at the NACE two-digit industry level and then averaged with employment weights to provide, for each country-year, a measure for the manufacturing sector and for non-financial services sectors. I merge the productivity dispersion data with publicly-available data on the labor share.¹⁸ I obtain a balanced panel of 7 countries (Denmark, Finland, France, Italy, Japan, Norway and New Zealand) and two industry groups (manufacturing and services) for the 2001-2011 period. One year of data was missing for France and Japan, so I imputed values using linear interpolation.

Figure [12a](#) plots the cross-sectional relationship between productivity dispersion and the labor share at the beginning of the sample in 2001. There is a clear negative relationship, where country-industry observations with higher productivity dispersion tend to have a lower labor share. Figure [12b](#) plots the same variables, but in 10 year differences from 2001 to 2011. Again, the relationship is negative. In [Table 11](#), present the results in regression form for specifications [5.2](#) through [5.4](#). Given the small cross-section ($N = 14$), I conduct hypothesis ($\beta = 0$) by doing a permutation tests. Usual methods such as cluster-robust standard errors would not be valid. For specifications OLS and FE, the data is "block permuted" at the country-industry level to account for the autocorrelation of errors within a country-industry over time. In all specification, I obtain negative slope coefficients ranging from -0.077 (OLS) to -0.203 (10-year differences). The results are statistically significant at the 5% and 10% level for the FE and 10-year differences specifications respectively.

Cross-industry data from Canada. I now aggregate the Canadian microdata to construct a panel dataset covering most 3-digit NAICS industries over the 2000-2015 period (see [Appendix B.4](#) for details regarding the construction of the dataset) and repeat the previous exercise. [Figures 13a](#) and [13b](#) plot the relationship between productivity dispersion and the labor share for the year 2001 and the period 2001-2011 respectively. Again, there is a clear negative relationship, where industries with higher (increasing) productivity dispersion tend to have a lower

¹⁸The data on the labor share by sector is obtained from EUKLEMS project (Denmark, Finland, France and Italy), WORLDKLEMS (Japan), and directly from national statistics institutes (Norway and New Zealand). For a description of the methodology underlying the EUKLEMS and WORLDKLEMS data, see respectively [O'Mahony and Timmer \[2009\]](#) and [Jorgenson \[2012\]](#).

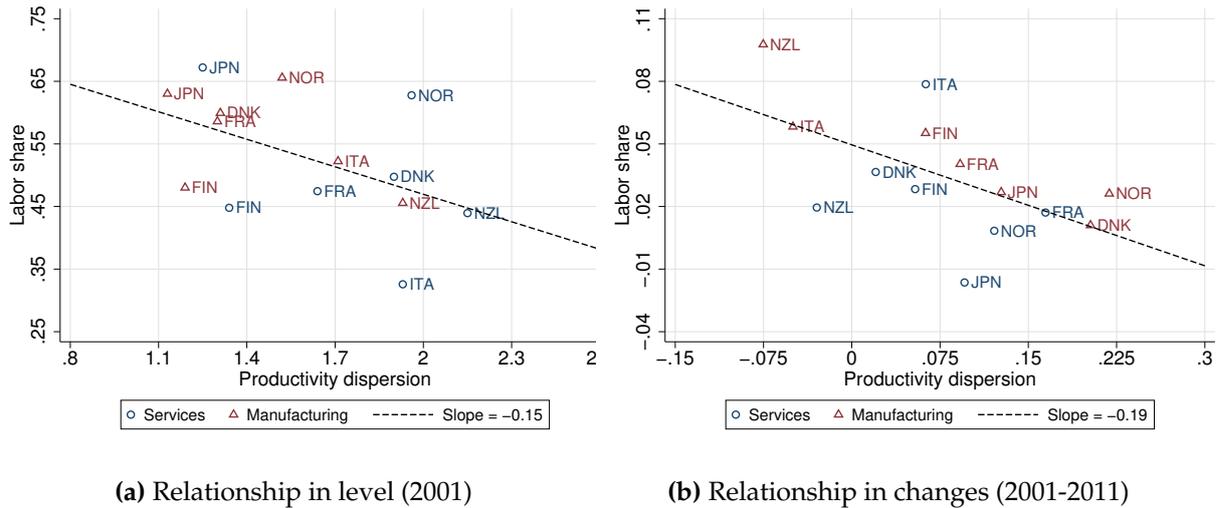


Figure 12: Cross-country relationship between the labor share and productivity dispersion. Productivity dispersion is defined as the logarithm of the interdecile range of labor productivity in the cross-section of firms. Panel (a) present the data in level for the year 2001. Panel (b) contain 10-year differences of both variables over the 2001-2011 period.

Table 11: Productivity dispersion and the labor share: cross-country regressions.

	(1) OLS	(2) FE	(3) 5y diff.	(4) 10y diff.
Productivity dispersion	-0.077 ($p = 0.360$)	-0.199* ($p = 0.096$)	-0.120 ($p = 0.130$)	-0.203** ($p = 0.029$)
Year FE	✓	✓		
Industry FE	✓	✓	✓	✓
Country FE		✓		
N	154	154	28	14
R^2	0.250	0.860	0.136	0.498

Notes. The p -values are obtained by permutation test ($*** p < 0.01$, $** p < 0.05$, $* p < 0.1$). For specifications (1) and (2), the data is permuted at the country-industry. Each column contains an estimated coefficient from a separate regression of the labor share LS on productivity dispersion. Productivity dispersion is defined as the logarithm of the interdecile range of labor productivity in the cross-section of firms. The four specifications used are described in Equations 5.2, 5.3 and 5.4.

(decreasing) labor share. Table 12 presents the regressions results. For the OLS and FE specification, I use standard errors clustered at the industry level to account for possible time-series dependence of the errors within an industry. In all specification, I obtain negative slope coefficients ranging from -0.063 (5-year differences) to -0.208 (10-year differences). The results are statistically significant at the 5% level for all specifications except for the 5-year differences. The results are remarkably similar to the cross-country exercise despite the fact that completely

different datasets are used.

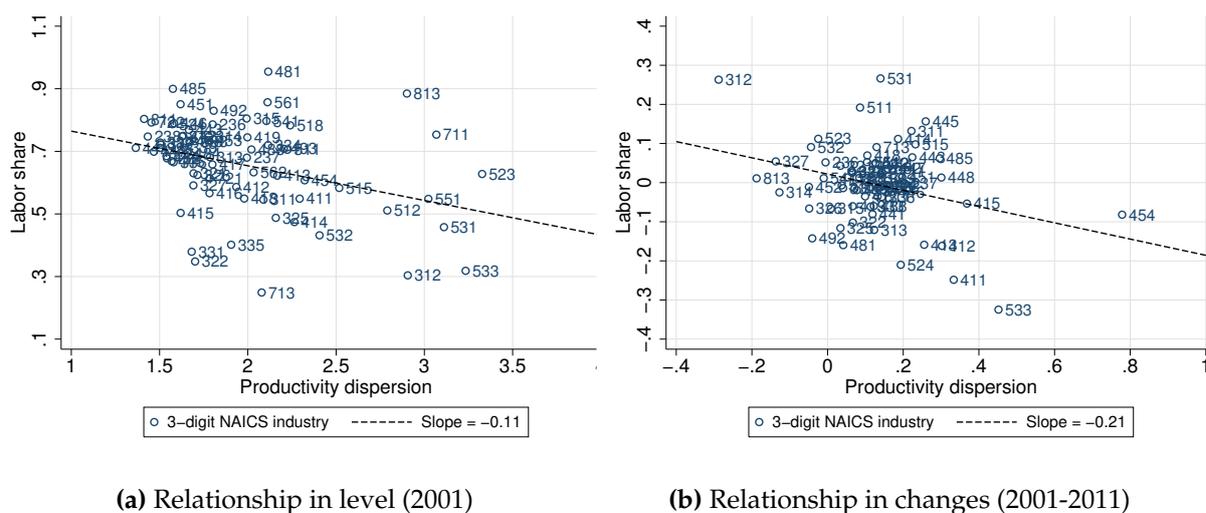


Figure 13: Cross-industry relationship between the labor share and productivity dispersion. Productivity dispersion is defined as the logarithm of the interdecile range of labor productivity in the cross-section of firms. Panel (a) present the data in level for the year 2001. Panel (b) contain 10-year differences of both variables over the 2001-2011 period.

Table 12: Productivity dispersion and the aggregate labor share: cross-industry regressions.

	(1) OLS	(2) FE	(3) 5y diff.	(4) 10y diff.
Productivity dispersion	-0.112*** (0.029)	-0.118** (0.047)	-0.063* (0.034)	-0.208** (0.079)
Year FE	✓	✓		
Industry FE		✓	✓	
<i>N</i>	1104	1104	207	69
<i>R</i> ²	0.154	0.736	0.228	0.092

Notes. Standard errors for specification (1) and (2) are clustered at the industry level (*** $p < 0.01$, ** $p < 0.05$, * $p < 0.1$). Each column contains an estimated coefficient from a separate regression of the labor share *LS* on productivity dispersion. Productivity dispersion is defined as the logarithm of the interdecile range of labor productivity in the cross-section of firms. The four specifications used are described in Equations 5.2, 5.3 and 5.4.

So far, I have established that the labor share comoves negatively with the interdecile range (90/10 percentile ratio) of the firm productivity dispersion. I now use the cross-industry data to quantify the relative importance of right-tail dispersion (90/50 percentile ratio) and left-tail dispersion (50/10 percentile ratio). To do so, I estimate Equations 5.2, 5.3 and 5.4 but this time adding both left-tail and right-tail dispersion as two separate regressors (the results

are reported in Table 13). A clear pattern emerges, where right-tail dispersion is associated with a lower labor share, but left-tail dispersion seems unrelated with the labor share. These findings are consistent with my narrative, which says that it is the existence of extremely high-productivity firms that weakens labor market competition, not the existence of extremely low-productivity firms.

Table 13: Right/left-tail productivity dispersion and the aggregate labor share: cross-industry regressions.

	(1) OLS	(2) FE	(3) 5y diff.	(4) 10y diff.
Right-tail productivity dispersion	-0.445*** (0.081)	-0.290*** (0.100)	-0.203** (0.083)	-0.316*** (0.107)
Left-tail productivity dispersion	0.201** (0.087)	-0.031 (0.078)	0.006 (0.051)	-0.103 (0.105)
Year FE	✓	✓		
Industry FE		✓		
<i>N</i>	1104	1104	207	69
<i>R</i> ²	0.308	0.743	0.247	0.122

Notes. For specifications (1) and (2), the standard errors in parentheses are clustered at the industry level (*** $p < 0.01$, ** $p < 0.05$, * $p < 0.1$). Each column contains an estimated coefficient from a separate regression of the aggregate labor share *LS* on left-tail and right-tail productivity dispersion. Right-tail (left-tail) productivity dispersion is measured as the logarithm of the 90/50 (50/10) percentile ratio of labor productivity in the cross-section of firms. The four specifications used are described in Equations 5.2, 5.3 and 5.4.

Testing the mechanism In the model, a rise in productivity dispersion leads to a decline of the aggregate labor share that operates through a reallocation of value-added towards high-productivity firms, as opposed to a broad-based decline of firm labor shares. In contrast, the model predicts an *increase* the average (unweighted) firm labor share (Table 9). I now test this prediction using the Canadian cross-industry data. First, table 14 presents the estimated relationship between productivity dispersion and the average firm labor share. In all specification, I obtain positive slope coefficients ranging from 0.044 (5-year differences) to 0.168 (FE). The results are statistically significant at the 5% level for all specifications except for the 5-year differences.

These results are consistent with the model experiment, in which the labor share of the bottom 80% of firms (in terms of productivity) increased while the labor share of the top 20% of firms decreased. In addition, the top 10% of firms gained output shares at the expense of the bottom 90% (see Figure 10). A more stringent test of the model mechanism is to estimate the

Table 14: Productivity dispersion and the average labor share: cross-industry regressions.

	(1)	(2)	(3)	(4)
	OLS	FE	5y diff.	10y diff.
Productivity dispersion	0.094*** (0.031)	0.168*** (0.027)	0.044* (0.024)	0.140** (0.058)
Year FE	✓	✓		
Industry FE		✓		
<i>N</i>	1104	1104	207	69
<i>R</i> ²	0.177	0.826	0.136	0.077

Notes. For specifications (1) and (2), the standard errors in parentheses are clustered at the industry level (*** $p < 0.01$, ** $p < 0.05$, * $p < 0.1$). Each column contains an estimated coefficient from a separate regression of the average (unweighted) firm labor share $\bar{L}S$ on productivity dispersion. Productivity dispersion is measured as the logarithm of the 90/10 percentile ratio of labor productivity in the cross-section of firms. The four specifications used are described in Equations 5.2, 5.3 and 5.4.

response of labor shares and output shares along the productivity distribution. Specifically, for each productivity quintile $q \in \{1, \dots, 5\}$, I estimate the following fixed-effects regression

$$Y_{q,j,t} = \mu_t + \gamma_j + \beta_q PD_{j,t} + \varepsilon_{q,j,t}, \quad (5.5)$$

where μ_t is a year fixed-effect and γ_j is a 3-digit NAICS industry fixed effect. $Y_{q,j,t}$ represents the variable of interest (labor share or output share) in productivity quintile q , industry j , and year t . $PD_{j,t}$ represents the interdecile range of labor productivity within industry j in year t . I use clustered standard errors at the industry-level. The productivity quintiles are constructed by sorting firms within industry-years. Figure 14 plots the estimated coefficients with their 95% confidence intervals. As predicted by the model, the labor share of low-productivity firms increases with productivity dispersion while the opposite is true for high-productivity firms (Figure 14a). The coefficients are imprecisely estimated, but for the two top quintiles they are negative and significant at the 5% level. Turning to the coefficients for output shares, we have that the coefficient for the top quintile of firms is positive while it is negative or near-zero for all the other quintiles. Only the top quintile is significant at the 5% level. Taken as a whole, these findings are remarkably consistent with the transmission mechanism of productivity dispersion to aggregate labor share in the model (see Figure 10 for comparison)

6 Concluding Remarks

I present a new theory of the labor share that emphasizes the role of firm competition for workers. In my model, heterogeneous firms grow by accumulating workers and compete

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A Proofs

A.1 Proof of Proposition 1

First, I show that the value of unemployment $W(w)$ is strictly increasing in the current wage w . From Equation 3.4, we have that

$$rW'(w) = 1 - \chi_s W'(w) + \lambda(1 - u) \int \frac{\partial}{\partial w} \max \{W(w') - W(w), 0\} d\tilde{P}(w') - \chi_x W'(w).$$

I then bound the derivative from below

$$rW'(w) \geq 1 - \chi_s W'(w) - \lambda(1 - u)W'(w) - \chi_x W'(w).$$

Re-arranging, I obtain the desired result

$$W'(w) \geq \frac{1}{r + \chi_s + \chi_x + \lambda(1 - u)} > 0.$$

Second, I derive the expression for the reservation wage w_r . Setting $W(w_r) = U$ in Equation 3.4, I obtain

$$rU = w_r + \chi_s \int (W(\hat{w}(z)) - U) d\Gamma(z) + \lambda(1 - u) \int \max \{W(w'), U\} d\tilde{P}(w').$$

Using the the inequality $W(w) \geq U$,

$$rU = w_r + \chi_s \int \max \{W(\hat{w}(z)) - U, 0\} d\Gamma(z) + \lambda(1 - u) \int \max \{W(w'), U\} d\tilde{P}(w').$$

and combining with Equation 3.3, I obtain

$$w_r = b + (\chi_e - \chi_s) \int \max \{W(\hat{w}(z)) - U, 0\} d\Gamma(z).$$

Finally, using the definitions of χ_e and χ_s (Equation 3.1), I obtain the desired expression

$$w_r = b + (\mu - \chi)(1 - \Gamma_0(z_l)) \int \max \{W(\hat{w}(z)) - U, 0\} d\Gamma(z).$$

Since $\Gamma(z)$ is the truncation of Γ_0 at z_l (Equation 3.1), then the expressions is equivalent to

$$w_r = b + (\mu - \chi) \int_{z_l}^{\infty} \max \{W(\hat{w}(z)) - U, 0\} d\Gamma_0(z).$$

[Go back to Proposition 1](#)

A.2 Proof of Lemma 2

First, I show that the value function is homogeneous. Dividing by N on both sides of Equation 3.10 and defining $n_s \equiv N_s/N_0$ and $k_s \equiv K_s/N_s$, we obtain

$$\frac{v(z, N)}{N} = \max_{T, \{w_s, k_s\}_{s=0}^T} \mathbb{E}_0 \int_0^T e^{-rs} (z_s k_s^\alpha - w_s - Rk_s) n_s ds$$

$$n_0 = 1, \quad z_0 = z$$

$$w_s \geq w_r, \quad dn_s = \tilde{g}(w_s)n_s ds, \quad dz_s = d\mathcal{J}_s(z'_s - z_s)$$

The expression for $\frac{v(z,N)}{N}$ is equal to Equation 3.10 when $N = 1$, so we obtain the desired result that $v(z,N) = v(z,1)N$. I now prove that the optimal stopping time is a threshold rule. Given that z is the only state variable, we now only need to show that $v'(z) \geq 0$. Using the homogeneity result, Equation 3.11 reduces to

$$rv'(z) = \min \left\{ \max_{w,k} \left\{ zk^\alpha - w - Rk + v(z)\tilde{g}(w) \right\} + \chi \left(\int v(\zeta)\Gamma_0(d\zeta) - v(z) \right), 0 \right\}.$$

Using the Envelope theorem, we have

$$rv'(z) \in \{k(z)^\alpha + v'(z)g(z) - \chi v(z), 0\} \implies v'(z) \in \left\{ \frac{k(z)^\alpha}{r + \chi - g(z)}, 0 \right\}.$$

Assumption 3 combined with Lemma 1 implies that $\chi - g(z) > 0$, so we obtain the desired result that $v'(z) \geq 0$. The fact that the exit threshold z_l coincides with the entry threshold follows directly from homogeneity and free entry: $v(z)N_0 > 0 \iff v(z) > 0 \iff z > z_l$.

[Go back to Lemma 2](#)

A.3 Proof of Proposition 2

From the first-order condition (Equation 3.17), we have that

$$1 = v(z)\tilde{g}'(w(z)).$$

Using the definition $g(z) \equiv \tilde{g}(w(z))$, it follows from the chain rule that $g'(z) = \tilde{g}'(w(z))w'(z)$. Plugging back in the first-order condition, I obtain

$$w'(z) = v(z)g'(z).$$

To solve for $w(z)$, I need a boundary condition. I now use the fact that the marginal exiting firm is constrained by the worker's reservation wage w_r . Without the constraint $w \geq w_r$, firms would never exit as they would be able to achieve zero flow profit by setting $w = 0$ and $k = 0$. Using the fact that $w(z_l) = w_r$, I obtain the desired result

$$w(z) = w_r + \int_{z_l}^z v(\zeta)g'(\zeta)d\zeta.$$

I now verify that the second order condition $v(z)\tilde{g}''(w(z)) < 0$ holds for interior solutions (i.e. for all $z > z_l$). Since $z > z_l \implies v(z) > 0$, I only need to verify that $\tilde{g}''(w(z)) \leq 0$. From the definition of $g(z)$, we have that

$$g'(z) = \tilde{g}'(w(z))w'(z) \implies g''(z) = \tilde{g}''(w(z))(w'(z))^2 + \tilde{g}'(w(z))w''(z).$$

And from the first-order condition, we have that $w'(z) = g'(z)v(z) \implies w''(z) = g''(z)v(z) + g'(z)v'(z)$. Putting together,

$$\begin{aligned} g''(z) &= \tilde{g}''(w(z))(w'(z))^2 + \tilde{g}'(w(z))(g''(z)v(z) + g'(z)v'(z)) \\ \implies g''(z)(1 - \tilde{g}'(w(z))v(z)) &= \tilde{g}''(w(z))(w'(z))^2 + \tilde{g}'(w(z))g'(z)v'(z) \end{aligned}$$

But from the first-order condition, we have that $1 - \tilde{g}'(w(z))v(z) = 0$, so

$$\tilde{g}''(w(z)) = -\frac{\tilde{g}'(w(z))g'(z)v'(z)}{(w'(z))^2} = -\frac{(g'(z))^2v'(z)}{w'(z)}$$

The sign of $\tilde{g}''(w(z))$ therefore depends on the sign of $w'(z)$. Since I restrict the analysis to equilibria where wages are increasing in productivity, we have that $\tilde{g}''(w(z)) < 0$.

[Go back to Proposition 2](#)

A.4 Proof of Proposition 3

First, the steady-state unemployment can be solved independently of $P(z)$. The law of motion (Equation 3.7) is

$$\dot{u} = (1 - u)\chi_x - u(\chi_e + \lambda(1 - u))$$

Setting $\dot{u} = 0$, we have a quadratic equation of the form $au^2 + bu + c = 0$ with constants given by

$$a = \lambda, \quad b = -(\lambda + \chi_e + \chi_x), \quad c = \chi_x.$$

First, I verify that the discriminant $(\lambda + \chi_e + \chi_x)^2 - 4\lambda\chi_x$ is non-negative.

$$(\lambda + \chi_e + \chi_x)^2 - 4\lambda\chi_x \geq (\lambda + \chi_x)^2 - 4\lambda\chi_x = (\lambda - \chi_x)^2 \geq 0$$

There are therefore two candidate solutions

$$u^{(1)} = \frac{\lambda + \chi_e + \chi_x - \sqrt{(\lambda + \chi_e + \chi_x)^2 - 4\lambda\chi_x}}{2\lambda}, \quad u^{(2)} = \frac{\lambda + \chi_e + \chi_x + \sqrt{(\lambda + \chi_e + \chi_x)^2 - 4\lambda\chi_x}}{2\lambda}$$

First, I show that $u^{(2)}$ is not a valid solution since it is greater than one.

$$u^{(2)} > 1 \iff \chi - \lambda + \sqrt{(\lambda + \chi_e + \chi_x)^2 - 4\lambda\chi_x} > 0$$

Under Assumption 3, the first term is positive which concludes the proof. Then, I show that $u^{(1)}$ is a valid solution (i.e. it satisfies $0 \leq u^{(1)} \leq 1$). Using the fact that $\chi_x \geq 0$, we have that

$$u^{(1)} \geq \frac{\lambda + \chi_e + \chi_x - (\lambda + \chi_e + \chi_x)}{2\lambda} = 0$$

Now, I show that $u^{(1)} \leq 1$.

$$u^{(1)} \leq 1 \iff \lambda + \chi_e + \chi_x - \sqrt{(\lambda + \chi_e + \chi_x)^2 - 4\lambda\chi_x} \leq 2\lambda$$

$$\iff \chi_e + \chi_x - \lambda \leq \sqrt{(\lambda + \chi_e + \chi_x)^2 - 4\lambda\chi_x}$$

So, if $\chi_e + \chi_x - \lambda < 0$, the proof is done. Otherwise, I square on both sides.

$$(\chi_e + \chi_x - \lambda)^2 \leq (\lambda + \chi_e + \chi_x)^2 - 4\lambda\chi_x \iff \chi_e \geq 0$$

Which concludes the proof.

I now provide a derivation of each term of *Kolmogorov Forward Equation* for the employment-weighted productivity distribution $P(z)$.

$$\dot{P}(z) = \underbrace{(1-u)\lambda P(z)(P(z)-1)}_{\text{Job-to-job flows}} + \underbrace{\frac{u}{1-u}\chi_e\Gamma(z) + u\lambda P(z)}_{\text{Employment inflows}} - \underbrace{\chi_x P(z)}_{\text{Employment outflows}} + \underbrace{\chi_s(\Gamma(z) - P(z))}_{\text{Productivity shocks}}.$$

At rate $\lambda(1-u)$, an employed worker receives a competing job offer. The distribution of productivity (CDF) of such workers, after they have made their decision whether or not to accept the competing job offer, is P^2 . The reason is that the distribution of z in the population of workers is precisely the employment-weighted productivity distribution P while the distribution of job offers z' is also P , since firms meet worker proportionally to their size. The CDF of $\max\{z, z'\}$ is therefore P^2 .¹⁹ Using the KFE formula for jump processes, the term that accounts for job-to-job flows is therefore

$$\lambda(1-u)(P^2(z) - P(z)) = \lambda(1-u)P(z)(P(z) - 1).$$

At rate χ_e , an unemployed worker meets an entrepreneur and enters the workforce with productivity distributed according to $\Gamma(z)$. At rate $\lambda(1-u)$, an unemployed worker meets a firm and enters the workforce with productivity distributed according to $P(z)$. Since the ratio of unemployment to employment is $\frac{u}{1-u}$, the term that accounts for unemployment inflows is

$$\frac{u}{1-u}(\chi_e\Gamma(z) + \lambda(1-u)P(z)) = \frac{u}{1-u}\chi_e\Gamma(z) + u\lambda P(z)$$

At rate χ_x , a worker with productivity distributed according to $P(z)$ is sent to unemployment due to firm exit so that the term which accounts for employment outflows is $-\chi_x P(z)$. At rate χ_s , an employed worker's productivity resets to a draw from Γ , so that the term which accounts for productivity shocks is $\chi_s(\Gamma(z) - P(z))$.

I now compute the stationary solution of the the law of motion for $P(z)$. Setting $\dot{P}(z) = 0$, we obtain a quadratic equation in $P(z)$ of the form $aP^2(z) + bP(z) + c = 0$, with coefficients given by

$$a = (1-u)\lambda, \quad b = -(\lambda(1-2u) + \chi), \quad c(z) = \left(\frac{u}{1-u}\chi_e + \chi_s\right)\Gamma(z),$$

¹⁹If X and Y are two independent random variables with CDFs F, G , then $\mathbb{P}(\max\{X, Y\} \leq z) = F(z)G(z)$.

I will now use the fact that, in steady-state, $\chi_e u = (1 - u)\chi_x - \lambda(1 - u)u$ (Equation 3.7). The expression for $c(z)$ thus simplifies to $c(z) = (\chi - \lambda u)\Gamma(z)$, so the discriminant is

$$\Delta(z) \equiv (\lambda(1 - u) - \chi - \lambda u)^2 - 4\lambda(1 - u)(\chi - \lambda u)\Gamma(z).$$

I will repeatedly use the fact that $\sqrt{\Delta(z)} \geq |\chi - \lambda|$. Under Assumption 3, the discriminant is strictly positive, which implies that there are two real roots.

$$P^{(1)}(z) = \frac{\lambda(1 - 2u) + \chi - \sqrt{\Delta(z)}}{2(1 - u)\lambda}, \quad P^{(2)}(z) = \frac{\lambda(1 - 2u) + \chi + \sqrt{\Delta(z)}}{2(1 - u)\lambda}.$$

First, I show that $P^{(2)}(z) > 1$ for all $z > z_l$, which implies that $P^{(2)}$ it is not a valid CDF. Using the fact that $\sqrt{\Delta(z)} \geq \chi - \lambda$, we have that

$$P^{(2)}(z) \geq \frac{\lambda(1 - 2u) + \chi - \chi + \lambda}{\lambda(1 - u)} = 1$$

To conclude, I show that $P^{(1)}$ is a valid CDF, meaning that (1) $P^{(1)'}(z) \geq 0$ and (2) $P^{(1)}(z_l) = 0$, and (3) $\lim_{z \rightarrow \infty} P^{(1)}(z) = 1$. First,

$$P^{(1)'}(z) = \frac{\chi - \lambda u}{\sqrt{\Delta(z)}} \Gamma'(z) \geq 0.$$

Second,

$$P^{(1)}(z_l) = \frac{\lambda(1 - 2u) + \chi - (\lambda(1 - 2u) + \chi)}{2(1 - u)\lambda} = 0.$$

Third,

$$\lim_{z \rightarrow \infty} P^{(1)}(z) = \frac{\lambda(1 - 2u) + \chi - (\chi - \lambda)}{2\lambda(1 - u)} = 1$$

Going from the second to last equation to the last one involves basic algebra as well.

[Go back to Proposition 3](#)

A.5 Proof of Propositions 4 and 5

First, I will prove 3 lemmas.

Lemma 3. *There exists constants $\bar{v}_1, \bar{v}_2, \bar{g}' < \infty$ such that, for all $z \geq z_l$*

$$v(z) \leq \bar{v}_1 + \bar{v}_2 z^{\frac{1}{1-\alpha}}$$

$$g'(z) \leq \bar{g}' \Gamma_0(z)$$

$$w(z) \leq \bar{w}$$

Proof. From the Equation 3.12 and optimizing over k , we have

$$rv(z) = \max_{w \geq w_r} \left\{ (1 - \alpha)(\alpha/R)^{\frac{\alpha}{1-\alpha}} z^{\frac{1}{1-\alpha}} - w - v(z)\tilde{g}(w) \right\} + \chi \left(\int v(z')\Gamma_0(dz') - v(z) \right)$$

Since $w \geq w_r$ and $\tilde{g}(w) \leq \lambda$, we have

$$\begin{aligned} rv(z) &\leq (1 - \alpha)(\alpha/R)^{\frac{\alpha}{1-\alpha}} z^{\frac{1}{1-\alpha}} - w_r - v(z)\lambda + \chi \int v(z')\Gamma_0(dz') - \chi v(z) \\ \implies v(z) &\leq \underbrace{\frac{\chi \int v(z')\Gamma_0(dz') - w_r}{r + \chi - \lambda}}_{\equiv \bar{v}_1} + \underbrace{\frac{(1 - \alpha)(\alpha/R)^{\frac{\alpha}{1-\alpha}}}{r + \chi - \lambda}}_{\equiv \bar{v}_1} z^{\frac{1}{1-\alpha}} \end{aligned}$$

From Equation 3.23, we have

$$g'(z) = 2\lambda(1 - u)P'(z).$$

From Equation 3.22, we have

$$P'(z) = \Delta(z)^{-\frac{1}{2}}(\chi - \lambda u)(1 - \Gamma(z_1))\Gamma_0(z),$$

where $\Delta(z) = (\lambda(1 - u) - \chi - \lambda u)^2 - 4\lambda(1 - u)(\chi - \lambda u)\Gamma(z)$. Putting together and using the fact that $\Delta(z) \geq \chi - \lambda > 0$

$$g'(z) \leq \underbrace{\frac{2\lambda(1 - u)(\chi - \lambda u)(1 - \Gamma(z_1))}{\chi - \lambda}}_{\equiv g'} \Gamma_0(z)$$

□

Lemma 4. *There exists a constant $\bar{w} < \infty$ such that, for all $z \geq z_1$*

$$w(z) \leq \bar{w}$$

Proof. Since $w'(z) > 0$, then

$$w(z) \leq \lim_{z \rightarrow \infty} w(z) = w_r + \int_0^\infty g'(x)v(x)dx$$

Using the results from Lemma 3,

$$w(z) \leq w_r + \underbrace{\kappa \int_0^\infty z^{\frac{1}{1-\alpha}} \Gamma'_0(z) dz}_{\equiv \bar{w}} < \infty.$$

The last inequality comes from Assumption 1. □

Lemma 5. *The expression $w'(z)z$ converges to zero as $z \rightarrow \infty$*

Proof. From Assumption 1, we have that $\int z^{\frac{1}{1-\alpha}} \Gamma'_0(z) < \infty$, which implies that $z^{1+\frac{1}{1-\alpha}} \Gamma'_0(z) \rightarrow 0$ Using the results from Lemma 3,

$$w'(z)z = g'(z)v(z)z \leq \bar{v}_0 \bar{g}' z \Gamma'_0(z) + \bar{v}_1 \bar{g}' z^{1+\frac{1}{1-\alpha}} \Gamma'_0(z) \rightarrow 0$$

□

I now prove Proposition 4 that $\frac{dw}{dLP}(z) \rightarrow 0$.

$$\frac{dw}{dLP}(z) = \frac{w'(z)}{LP'(z)} = \frac{w'(z)z}{(1-\alpha)(\alpha/R)^{\frac{\alpha}{1-\alpha}} z^{\frac{1}{1-\alpha}}} \rightarrow 0$$

The last step follows from $w'(z)z \rightarrow 0$ (Lemma 5) and $z^{\frac{1}{1-\alpha}} \rightarrow \infty$.

I now prove Proposition 5. I will show that there exists a z' such that, for every $z > z'$, $\frac{dLS}{dz}(z) < 0$, which is equivalent as showing that $\frac{d \log LS}{dz}(z) < 0$. Notice that

$$\log LS(z) = \log w(z) - \log LP(z) \implies \frac{d \log LS}{dz}(z) = \frac{w'(z)}{w(z)} - (1-\alpha) \frac{1}{z}$$

So, we have that

$$\frac{d \log LS}{dz}(z) < 0 \iff \frac{w'(z)z}{w(z)} < 1 - \alpha$$

The right-hand-side is positive for all $\alpha < 1$, while the left hand side converges to zero, since $w'(z)z \rightarrow 0$ and $w(z) \rightarrow \bar{w}$. By continuity, there exists a z' such that, for every $z > z'$, $\frac{w'(z)z}{w(z)} < 1 - \alpha$.

[Go back to Proposition 4](#)

A.6 Proof of Proposition 6

First, the average firm size is given by the measure of employed workers $1 - u$ over the measure of firm F . The law of motion for the measure of firms is given by

$$\dot{F} = \chi_e u - \chi_x F,$$

which implies that, in a stationary equilibrium, $F = u \frac{\chi_e}{\chi_x}$ and the average firm size is thus given by

$$\mathbb{E}(N) = \frac{1 - u}{u} \frac{\chi_x}{\chi_e}$$

To compute the conditional expectation $\mathbb{E}(N|z)$, I will use the employment-weighted distribution P . For a set $S \in \mathbb{R}_+$, the measure of employment at firms with $z \in S$ is given by $(1 - u) \int_S dP(\zeta)$ while the measure of firms is given by $F \int d\Gamma(\zeta)$. It follows that the average firm size for firms with $z \in S$ is given by

$$\mathbb{E}(N|z \in S) = \frac{(1 - u) \int_S dP(\zeta)}{F \int_S d\Gamma(\zeta)}$$

In the limit, when $S = \{z\}$, I obtain the following formula by using the expression

$$\mathbb{E}(N|z) = \frac{1 - u}{F} \frac{P'(z)}{\Gamma'(z)} = \frac{1 - u}{u} \frac{\chi_x}{\chi_e} \frac{\chi - \lambda u}{\sqrt{(\lambda(1 - u) + \chi - \lambda u)^2 - 4\lambda(1 - u)(\chi - \lambda u)\Gamma(z)}},$$

which is increasing in z .

[Go back to Proposition 6](#)

A.7 Proof of Proposition 8 (Incomplete)

Lemma 6. Let U^* , $W^*(w)$, w_r^* the worker's value functions and reservation wage after the broad-based increase in productivity $z \rightarrow \pi z$ and $b \rightarrow \pi^{\frac{1}{1-\alpha}} b$.

$$\pi^{-\frac{1}{1-\alpha}} U^* = U$$

$$W^*(\pi^{-\frac{1}{1-\alpha}} w) = W(w)$$

Lemma 7. Let $v^*(z)$ and $w^*(z)$ the firm's value function and reservation wage after the broad-based increase in productivity $z \rightarrow \pi z$ and $b \rightarrow \pi^{\frac{1}{1-\alpha}} b$.

$$\pi^{-\frac{1}{1-\alpha}} v^*(z) = v(z)$$

$$\pi^{-\frac{1}{1-\alpha}} w^*(z) = w(z)$$

A.8 Burdett and Mortensen [1998] with capital

Derivation. I now present a derivation of Burdett and Mortensen [1998], the only extension being the addition of capital as a factor of production. The laws of motion for the wage distribution $P(w, t)$ ²⁰ and firm-level employment $N(w, t)$ are respectively given by

$$\begin{aligned} \frac{\partial}{\partial t} N(w, t) &= \underbrace{\lambda P(w)}_{\text{Hires}} - \underbrace{\delta N(w, t)}_{\text{Layoffs}} - \underbrace{\lambda \bar{F}(w) N(w, t)}_{\text{Quits}} \\ \frac{\partial}{\partial t} P(w, t) &= \underbrace{\delta (H(w) - P(w, t))}_{\text{Job destruction}} + \underbrace{\lambda (F(w) P(w, t) - P(w, t))}_{\text{Job creation + Churn}}. \end{aligned}$$

The stationary solution for employment is given by

$$N(w) = \frac{\lambda \delta}{(\delta + \lambda \bar{F}(w))^2}.$$

Firms chose the wage policy $w \geq b$ and capital stock per worker $k \geq 0$ as to maximize long-run profits

$$\max_{w, k} z k^\alpha N(w) - R k N(w) - w N(w).$$

The optimal capital stock is $k(z) = \left(\frac{z\alpha}{R}\right)^{\frac{1}{1-\alpha}}$. Substituting it in the objective, I obtain

$$\max_w c(z) N(w) - w N(w),$$

²⁰ $P(0, t)$ is, by convention, the unemployment rate (i.e. the measure of workers working at firms which pay $w = 0$).

where $c(z) \equiv (1 - \alpha)zk(z)^\alpha$. The first-order condition given by

$$c(z)N'(w(z)) = N(w(z)) + w(z)N'(w(z)).$$

Now define $N(z) \equiv N(w(z))$ which means that $N'(z) = N'(w(z))w'(z)$, which implies that

$$c(z)N'(z) = w'(z)N(z) + w(z)N'(z) \implies \left(w(z)N(z) \right)' = c(z)N'(z).$$

This is an ODE. Combined with the initial condition $w(z_l) = b$ —i.e. the worst operating firm is constrained by the worker's participation constraint—I obtain a unique solution.

$$w(z)N(z) - w(z_l)N(z_l) = N(z)c(z) - N(z_l)c(z_l) - \int_{z_l}^z c'(\zeta)N(\zeta)d\zeta$$

$$w(z) = c(z) + (b - c(z_l)) \frac{N(z_l)}{N(z)} - \int_{z_l}^z c'(\zeta) \frac{N(\zeta)}{N(z)} d\zeta$$

The entry threshold z_l satisfies $c(z_l) = b$. The formula simplifies to

$$w(z) = c(z) - \int_{z_l}^z c'(\zeta) \frac{N(\zeta)}{N(z)} d\zeta$$

Using the fact that the wage schedule is increasing in z , we have that

$$N(z) = \frac{\lambda\delta}{(\delta + \lambda\Gamma(z))^2}, \quad P(z) = \frac{\delta}{\delta + \lambda\bar{\Gamma}(z)}$$

where Γ is the distribution of productivity at active firms and Γ_0 is the distribution of productivity of potential entrants

$$\Gamma(z) = \frac{\Gamma_0(z) - \Gamma_0(z_l)}{1 - \Gamma_0(z_l)}$$

The threshold z_l is given by

$$z_l = \frac{b^{1-\alpha}R^\alpha}{(1-\alpha)^{1-\alpha}\alpha^\alpha}.$$

Calibration. Notice that the equilibrium functions N and P only depend on (λ, δ) through their ratio $\psi \equiv \lambda/\delta$. Since the wage schedule (and therefore the labor share) depend only on N , it also only depends on ψ . Hence, I normalize $\delta = 1$. The free parameters of the model are therefore $(\alpha, R, \lambda, b, \eta)$. First, I calibrate (α, R, η) exactly as in Section 4.1. Then, I choose λ as to match exactly the unemployment in the full model. Finally, b is chosen such that the entry threshold in the *BM* model coincides with the entry threshold in the full model. This last step implies that the distribution of productivity amongst active firms Γ coincides in both models.

Firm size distribution. I now provide a characterization of the firm size distribution. Using the definition of $N(z)$, we have

$$\mathbb{P}(N(z) \leq n) = \mathbb{P}\left(\Gamma(z) \leq 1 + \frac{\delta}{\lambda} - \sqrt{\frac{\delta}{\lambda}} n^{-\frac{1}{2}}\right) = 1 + \frac{\delta}{\lambda} - \sqrt{\frac{\delta}{\lambda}} n^{-\frac{1}{2}}.$$

the last equation uses the fact that $\Gamma(z) \sim U[0,1]$. So that the firm size distribution obeys a power law with Pareto exponent $\zeta = 1/2$. But the distribution is bounded

$$\frac{\lambda\delta}{(\delta + \lambda)^2} \leq N(z) \leq \frac{\lambda}{\delta},$$

So the firm size distribution is

$$\mathbb{P}(N \leq n) = \begin{cases} 0 & \text{if } n \leq \frac{\lambda\delta}{(\delta + \lambda)^2} \\ 1 + \frac{\delta}{\lambda} - \sqrt{\frac{\delta}{\lambda}} n^{-\frac{1}{2}} & \text{if } \frac{\lambda\delta}{(\delta + \lambda)^2} \leq n \leq \frac{\lambda}{\delta} \\ 1 & \text{if } n \geq \frac{\lambda}{\delta} \end{cases}$$

Closed-form expression for aggregate labor share. I now provide a closed-form solution for the aggregate labor share in the special case where there is no capital ($\alpha = 0$) and the distribution of firm productivity is Pareto ($\Gamma(z) = 1 - (z/b)^{-\eta}$ over $z \geq b$). I require $\eta > 1$ to ensure that output is finite. The formula for the wage is

$$w(z) = z - \int_b^z N(x) dx$$

Aggregating, I obtain

$$\underbrace{\int_b^\infty w(x) N(x) d\Gamma(x)}_{wN} = \underbrace{\int_b^\infty x N(x) d\Gamma(x)}_Y - \underbrace{\int_b^\infty \int_b^z N(y) dy d\Gamma(x)}_\Pi$$

where wN is worker compensation, Y is output, and Π is firm profits. I now focus on the last term and will use the fact that $(1 - \Gamma(z)) = \frac{1}{\eta} z \frac{d\Gamma(z)}{dz}$

$$\int_b^\infty \int_b^z N(y) dy d\Gamma(x) = \int_b^\infty N(y) \left(\int_b^\infty 1\{y \leq x\} d\Gamma(x) \right) dy = \int_b^\infty N(y) (1 - \Gamma(y)) dy = \frac{1}{\eta} Y$$

The labor share $LS = wN/Y$ is therefore given by

$$LS = 1 - \frac{1}{\eta}$$

When productivity dispersion is high (η is low), then the labor share is low.

A.9 Extension with imitation

Suppose that, at rate ψ a firm has the opportunity to imitate a competing firm randomly selected from the population of active firms. Let Γ_0 be the source distribution (as before) and $\Gamma(z)$

be the truncated source distribution. The KFE (defined over $z > z_l$) for the firm productivity distribution $\Lambda(z)$ is now given by

$$0 = \chi \left(\Gamma(z) - \Lambda(z) \right) + \psi \left(\Lambda(z)^2 - \Lambda(z) \right).$$

The unique solution is given by

$$\Lambda(z) = \frac{\chi + \psi - \sqrt{(\chi + \psi)^2 - 4\chi\psi\Gamma(z)}}{2\psi}.$$

Notice that high values of ψ imply (1) less productivity dispersion and (2) more persistence.

$$\dot{P}(z) = \underbrace{(1-u)\lambda P(z)(P(z)-1)}_{\text{Job-to-job flows}} + \underbrace{\psi P(z)(\Lambda(z)-1)}_{\text{Imitation}} + \underbrace{\frac{u}{1-u}\chi_e \Gamma(z) + u\lambda P(z)}_{\text{Employment inflows}} - \underbrace{\chi_x P(z)}_{\text{Employment outflows}} + \underbrace{\chi_s (\Gamma(z) - P(z))}_{\text{Productivity shocks}}.$$

A.10 Solution method for $\mu = \chi$ case

I now present a solution method for the case $\mu = \chi$ that I use to solve for the firm value function and threshold (v, z_l) as well as the productivity and size distribution φ .

Firm value function. First, I combine the HJB (Equation 3.12) with the expressions for the wage schedule $w(z)$ as a function of $v(\cdot)$ (Equation 3.20), labor productivity $LP(z)$ (Equation 3.19) and the employment growth function $g(z)$ (Equation 3.23). I obtain the following equations.

$$rv(z) = (1-\alpha)LP(z) - b - \int_{z_l}^z v(\zeta)g'(\zeta)d\zeta + v(z)g(z) + \chi \int v(\zeta)\Gamma_0(d\zeta) - \chi v(z) \quad \forall z \geq z_l$$

$$v(z) = 0 \quad \forall z < z_l$$

To simplify notation, I define the the linear operators $\mathcal{A}, \mathcal{B}, \mathcal{C}$ respectively defined by the actions

$$\mathcal{A}f(z) = \int_{z_l}^z f(\zeta)g'(\zeta)d\zeta, \quad \mathcal{B}f(z) = f(z)g(z), \quad \mathcal{C}f(z) = \int v(\zeta)\Gamma_0(d\zeta), \quad \mathcal{I}f(z) = f(z)$$

The HJB for $v(z)$ can be rewritten as

$$rv(z) = (1-\alpha)LP(z) - b - \mathcal{A}v(z) + \mathcal{B}v(z) + \chi\mathcal{C}v(z) - \chi\mathcal{I}v(z) \quad \forall z \geq z_l$$

$$v(z) = 0 \quad \forall z < z_l.$$

The system can be equivalently written as

$$v(z) \left(\mathcal{M}v(z) + q(z) \right) = 0, \quad \mathcal{M}v(z) + q(z) \geq 0, \quad v(z) \geq 0$$

where $\mathcal{M} \equiv (r + \chi)\mathcal{I} + \mathcal{A} - \mathcal{B} - \chi\mathcal{C}$ and $q(z) \equiv b - (1 - \alpha)LP(z)$. The threshold z_l does not appear anywhere but can be recovered from the solution $v(z)$ as $z_l \equiv \inf \{z : v(z) > 0\}$. I now consider a discrete approximation of the system of equations over a grid $G_z = \{z_1, \dots, z_{N_z}\}$. Let $v \equiv (v(z_1), \dots, v(z_{N_z}))'$ and $LP \equiv (LP(z_1), \dots, LP(z_{N_z}))'$. I approximate the operators $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{I}$ by the finite difference method and obtain $N_z \times N_z$ matrices A, B, C, I and M . The resulting system of equations is given by

$$v^T (Mv + q) = 0, \quad Mv + q \geq 0, \quad v \geq 0,$$

which is a plain-vanilla *Linear Complementarity Problem* (LCP) that can be solved with standard routines! I recover the threshold as $z_l \equiv \min_i \{i \mid v_i > 0\}$

Iteration. To solve for the equilibrium, I need to iterate over the LCP. The reason is that the firm growth function $g(z)$ depends on z_l . First, I start with a guess for z_l , compute g and solve the LCP. I then use the new threshold z'_l to compute g and solve the LCP again. I stop when the difference between the productivity threshold in between iterations satisfies $|z'_l - z_l| < \epsilon$ where $\epsilon > 0$ is a pre-set tolerance level.

Productivity and size distribution. To compute the joint-distribution of firm size N and productivity z , I use the Pareto Extrapolation method developed in [Gouin-Bonenfant and Toda \[2018\]](#). Since the method applies to discrete-time model, I “time discretize” the model over short time intervals of one week. The first step of the Pareto extrapolation algorithm computes a value for the Pareto exponent of the size distribution ζ . The second step computes a discrete approximation of the distribution φ over a finite grid accounting for movements of particles in and out of the grid. In order to compute moments, I use the truncated distribution φ and apply the correction terms provided in [Gouin-Bonenfant and Toda \[2018\]](#), which involve the Pareto exponent ζ .

B Data

B.1 Variables construction

I now describe the methodology used to construct the main firm-level variables: value-added, employment, wage and capital stock. First, I use four variables which are constructed by summing line items from the corporate tax returns (T2 forms): gross profits, wages and salaries, tangible capital assets, and intangible capital assets. The last two variables (tangible capital

assets and intangible capital assets) represent book values of assets net of accumulated depreciation. The line items used to construct the variables are summarized in Table 16. First, I construct a measure of gross value added at the firm which is consistent with the income approach in the *System of National Accounts*. Denote by $Y_{i,t}$ the output of firm i in year t by

$$Y_{i,t} \equiv \text{gross profits}_{i,t} + \text{wages and salaries}_{i,t}.$$

This approach is consistent with the income approach to measuring GDP which, in the corporate sector, sums the income which accrues to firm owners (gross operating income) and the income which accrues to workers (worker compensation). Note that the following accounting identity

$$\text{gross profits} \equiv \text{revenue}_{i,t} - \text{cost of goods sold}_{i,t} - \text{operating expense}_{i,t} - \text{wages and salaries}_{i,t}$$

implies that the value-added approach to computing value-added also holds at the firm-level:

$$Y_{i,t} \equiv \text{revenue}_{i,t} - \text{cost of goods sold}_{i,t} - \text{operating expense}_{i,t}$$

The labor share at the firm LS (i.e. wages, salaries and benefits as a share of output) is computed as

$$LS_{i,t} \equiv \text{wages and salaries}_{i,t} / Y_{i,t}.$$

I winsorize the labor share at the 1% level in the upper tail. To be precise, I replace the values of $LS_{i,t}$ above the 99th percentile by the value of the 99th percentile (approximately 20).

Finally, I remove from the main sample firm-year observations that either have (1) negative value-added, and (3) missing values in employment, value-added, tangible capital assets, intangible capital assets or industry code. The resulting extract (from now on the *main sample*) covers roughly approximately 60% of employment and GDP amongst the industries covered and 40% of economy-wide employment and GDP over the 2000-2015 period.

Employment $N_{i,t}$ is obtained by averaging the number of employees declared on the monthly PD7 forms filed throughout the year. This procedure avoids attributing larger employment levels to firms with high turnover. If for instance, we had summed the number workers who worked at a firm throughout the year, we would be imputing a large workforce to firms with a high labor turnover (i.e. a high share of employees who only during part of the year). The average wage $w_{i,t}$ and labor productivity $LP_{i,t}$ are thus given by

$$w_{i,t} = \text{wages and salaries}_{i,t} / N_{i,t}, \quad LP_{i,t} = Y_{i,t} / N_{i,t}.$$

The value of the stocks of tangible and intangible capital assets are directly measured at book value net of accumulated depreciation on the firm's balance sheet. The indicator variable for-

eign takes value one if the country of residence of the ultimate shareholder or group of shareholders is not Canada and takes the value zero otherwise. The indicator variable subsidiary takes the value one if and only if the firm is owned by another firm (i.e. the ownership threshold is 50%). Similarly, the indicator variable parent takes the value one if and only if the firm is the highest legal operating entity in an ownership structure. In order to be “the highest legal operating entity in an ownership structure”, a firm must (1) have at least one subsidiary and (2) not be the subsidiary of another firm. Finally, the indicator variable fdi indicates the presence of inward or outward direct investment flows. The threshold is 10% meaning that the variable fdi takes value one if (1) the firm has at least 10% of the voting equity in a foreign firm or (2) the firm has 10% of its voting equity owned by a foreign firm.

Table 15: Summary statistics (Main sample, 2000-2015)

Variable	Observations	Mean	Std. Dev.
revenue	3084182	1.01e+07	1.94e+08
cost of goods sold	3084182	5684233	1.38e+08
wages and salaries	3084182	1842980	3.16e+07
operating expenses	3084182	1562873	5.92e+07
intangible assets	3084182	600173.7	2.64e+07
tangible assets	3084182	2500054	9.15e+07
employment	3084182	38.63308	571.1065
wage	3084182	42845.83	59672.26
labor productivity	3084182	60014.69	215973.2
value-added	3084182	2809567	5.18e+07
R&D expenditures	3084182	53235.86	2778371
foreign	3084182	.0149553	.1213742
subsidiary	3084182	.0508939	.2197811
fdi	3084182	.0333246	.1794827

B.2 Sample Validation

I compute the aggregate labor share, I use data from the *National Accounts* (Statistics Canada table 380-0063) and apply the same methodology as Koh et al. [2016], which assumes that components of income which are ambiguous (i.e. taxes net of subsidies and net mixed income) have a labor share equal to the aggregate labor share. This methodology implies the following formula:

$$LS_{NAC} \equiv \frac{\text{worker compensation}}{\text{worker compensation} + \text{gross operating surplus}}.$$

Figure 15 plots the labor share in the National Accounts and in the main sample. A few remarks are in order. First, the Canadian labor share in the National Accounts has sustained a large decline over the course of the 1990s and early-2000s but has then somewhat recovered

Variable	Line items (GIFI)	Variable	Line items (GIFI)
Revenue	Trade sales of goods and sv. (8000)	Cost of goods sold	Opening inventory (8300)
	Sales to related parties (8020)		Cost of materials (8320)
	Sales from resource prop. (8040)		Trades and subcontracts (8360)
	Commission revenue (8120)		Production costs (8370)
	Rental revenue (8140)		Resource prod. costs (8400)
	Vehicle leasing (8150)		Other direct costs (8450)
	NPO amounts received (8220)		Closing inventory (8500)
	Other revenue (8230)		
Operating expenses	Advertising and promotion (8520)	Wages and salaries	Direct wages (8340)
	Insurance (8690)		Benefits on direct wages (8350)
	Bank charges (8715)		Employee benefits (8620)
	Credit card charges (8716)		Salaries and wages (9060)
	Collection and credit costs (8717)	Tangible capital assets	Land (1600)
	Memberships (8761)		Depletable assets (1620)
	Franchise fees (8763)		Buildings (1680)
	Government fees (8764)		Machinery and equip. (1740)
	Office expenses (8810)		Other tangible assets (1900)
	Professional fees (8860)	Intangible capital assets	Goodwill (2012)
	Rental (8910)		Quota (2014)
	Repairs and maintenance (8960)		Licenses (2016)
	Other repairs and maint. (9010)		Incorporation costs (2018)
	Sub-contracts (9110)		Trademarks/patents (2020)
	Supplies (9130)		Customer lists (2022)
	Travel expenses (9200)		Rights (2024)
	Utilities (9220)	Research and dev. (2026)	
Other expenses (9270)			

Table 16: List of line items used to construct revenue, expenses, and capital assets. The codes reported follow the *General Index of Financial Information (GIFI)* standard used in Canada.

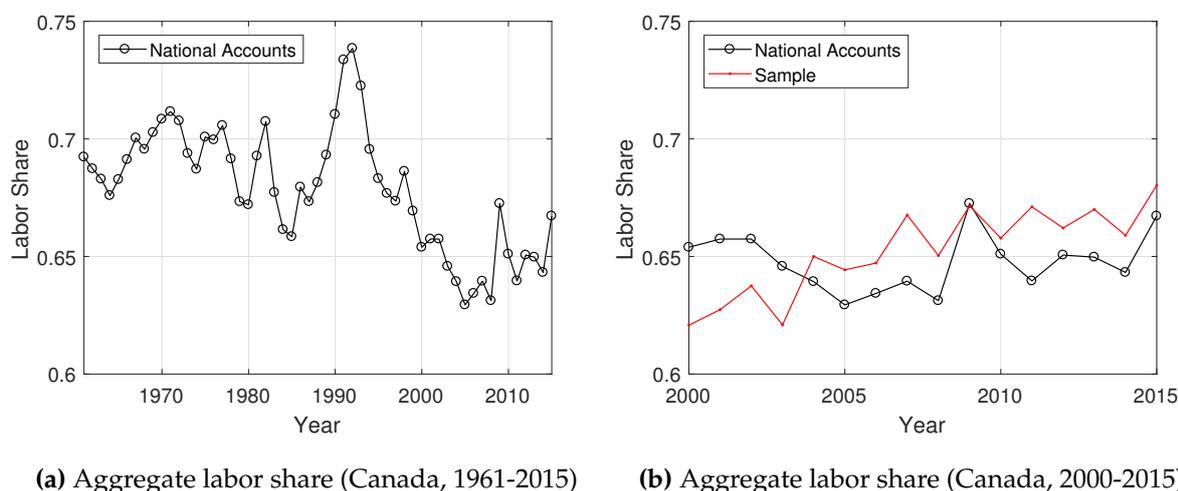


Figure 15: Labor share in the National Accounts and in the main sample. The labor share is computed using data from Statistic Canada's Table 380-0063.

over the course of the 2005-2015 period. Second, the aggregate labor share in the main sample has a similar level and trend as the one in the National account over the period where both datasets are available (2000-2015). Third, the labor share is a bit lower in the main main sample

over the 2000-2004 period. Overall, the level and dynamics of the labor share is consistent with the aggregate data.²¹

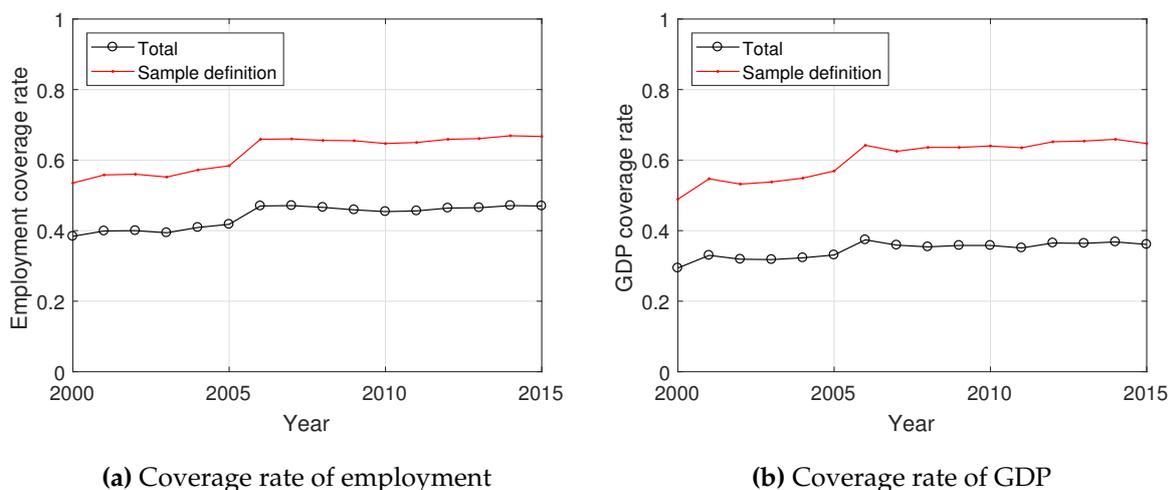


Figure 16: Coverage rate of the main sample (Canada, 2000-2015). Employment and GDP data is taken from Statistic Canada’s Table 383-0033. The “total” uses the aggregate data for all industries while the “sample definition” line corresponds to the aggregate data obtained by excluding the some industries, exactly as in the main sample

Figure 15 plots the coverage rate of the main sample, in terms of both employment and gross value added, benchmarked against aggregate data in the National Accounts. The “total” uses the aggregate data for all industries while the “sample definition” line corresponds to the aggregate data obtained by excluding the some industries (Agriculture, Mining, Utilities, Education and Health - NAICS 11, 21, 22, 61, 62, 92), exactly as in the main sample. The “sample definition” is fairly high for both employment and GDP, at around 60%. As expected, the “total” coverage is lower, at around 40%.

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B.3 Averages by labor productivity deciles

Within each industry-year (where industries are defined as 2-digit NAICS sectors), I sort firms by labor productivity (i.e. value-added per worker). I then compute 9 thresholds and assign a “labor productivity decile” to each firm-year observation, so that each decile contains 10% of the firms. Within each decile, I compute the wage, labor productivity and capital per worker by computing an employment-weighted average of firm-level wages, labor productivity and

²¹Unfortunately, Table 383-0033 does not allow for the computation of the labor share at the industry level.

capital per worker. Within each industry-year bin, I then normalize the average wage and labor productivity by their value in the first decile. Finally, I average over all industries within a year by applying employment weights and then compute a simple average of each variable over 2000-2015.

To compute the labor share and output share within each decile. I simply sum wages and value-added within each industry-year-decile bin. The labor share is constructed as the ratio of wages to value-added. The output share of a decile is compute as the share of a decile's value-added within its industry-year bin. Finally, I average over all industries within a year by applying value-added weights and compute a simple average of each variable over 2000-2015. Table 17 contains the resulting data.

Table 17: Moments by labor productivity deciles

Variable	Labor productivity deciles									
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
Wage	0.3971	0.6085	0.6366	0.7244	0.7868	0.8508	0.9135	1.1062	1.3056	1.9426
Labor productivity	0.1641	0.3538	0.4345	0.5178	0.5963	0.6922	0.783	0.9731	1.2781	2.8894
Capital stock	0.2955	0.2869	0.3017	0.3217	0.412	0.4803	0.5865	0.7774	1.1722	3.6385
Output share	0.0153	0.0284	0.0357	0.0437	0.0553	0.0685	0.0825	0.0972	0.1486	0.4248
Employment share	0.0933	0.0801	0.0818	0.0841	0.0925	0.099	0.1056	0.0998	0.1162	0.1476
labor share	1.5784	1.1257	0.9539	0.9107	0.8578	0.8019	0.7606	0.742	0.6674	0.4389

Notes. See text for details regarding the construction of the variables.

B.4 Industry-level dataset

To construct the industry-level dataset, I use the main sample which covers most industries and most firms in Canada over the 2000-2015 and sort firms by labor productivity within industry-year bins and assign to each firm a productivity quintile. For example, the 20% most productive firms are in the 5th quintile while the bottom 20% least productive firms are in the 1st quintile. An industry is defined according to 3-digit NAICS definitions. I then compute measures of wage, labor productivity and output share within each decile exactly as in described in Appendix B.3. Within each industry-year bin, I also compute the labor share LS , the average (unweighted) firm labor share \bar{LS} as well as the employment and output shares. I restrict the sample to industry-year observations which have at least 100 firms in every year and censor the observations of \bar{LS} . I am left with a balanced panel dataset covering 69 industries over the period 2000-2015