A Mean Field Game of Optimal Portfolio Liquidation *

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Abstract

We consider a mean field game (MFG) of optimal portfolio liquidation under asymmetric information. We prove that the solution to the MFG can be characterized in terms of a FBSDE with possibly singular terminal condition on the backward component or, equivalently, in terms of a FBSDE with finite terminal value, yet singular driver. Extending the method of continuation to linear-quadratic FBSDE with singular driver we prove that the MFG has a unique solution. Our existence and uniqueness result allows to prove that the MFG with possibly singular terminal condition can be approximated by a sequence of MFGs with finite terminal values.

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1 Introduction and overview

Mean field games (MFGs) are a powerful tool to analyse strategic interactions in large populations when each individual player has only a small impact on the behavior of other players. In the economics literature, mean field games (or anonymous games) were first considered by Jovanovic and Rosenthal [19].[1] In the mathematical literature they were independently introduced by Huang, Malhamé and Caines [17] and Lasry and Lions [22].

In a standard MFG, each player $i \in \{1, ..., N\}$ chooses an action from a given set of admissible controls that minimizes a cost functional of the form

$$J^i(u) = \mathbb{E}\left[\int_0^T f(t, X_t^i, \bar{\mu}_t^N, u_t^i)dt + g(X_T^i, \bar{\mu}_T^N)\right] \tag{1.1}$$

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[1] Anonymous and mean field type games were subsequently analyzed in the economics literature by many authors including Blonski [4, 5], Daskalakis and Papadimitriou [10], Horst [15], and Rath [26].

subject to the state dynamics

$$\begin{cases} dX_t^i = b(t, X_t^i, \bar{\mu}_t^N, u_t^i)\,dt + \sigma(t, X_t^i, \bar{\mu}_t^N, u_t^i)\,dW_t^i, \\ X_0^i = x_0 \end{cases}. \tag{1.2}$$

Here $W^1, \cdots, W^N$ are independent Brownian motions defined on some underlying filtered probability space, $X^i \in \mathbb{R}^d$ is the *state* of player i, $u = (u^1, \cdots, u^N)$, $u^i = (u_t^i)_{t\in[0,T]}$ is an adapted stochastic process, the *action* of player i, and $\bar{\mu}_t^N := \frac{1}{N}\sum_{j=1}^N \delta_{X_t^j}$ denotes the empirical distribution of the individual players' states at time $t \in [0, T]$.

The existence of *approximate Nash equilibria* in the above game for large populations has been established in [7, 17] using a representative agent approach; a corresponding result for anonymous games can be found in [10]. The idea is to approximate the dynamics of the empirical distribution of the states by a deterministic measure-valued process, and then to consider the optimization problem of a representative player subject to the equilibrium constraint that the distribution of the representative player's state process X under her optimal strategy coincides with the pre-specified measure-valued process. The resulting MFG can then be formally described as follows:

$$\begin{cases} 1. & \text{fix a deterministic function } t \in [0,T] \mapsto \mu_t \in \mathcal{P}(\mathbb{R}^d); \\ 2. & \text{solve the corresponding stochastic control problem :} \\ & \inf_u \mathbb{E}\left[\int_0^T f(t, X_t, \mu_t, u_t)\,dt + g(X_T, \mu_T)\right], \\ & \text{subject to} \\ & dX_t = b(t, X_t, \mu_t, u_t)\,dt + \sigma(t, X_t, \mu_t, u_t)\,dW_t \\ & X_0 = x_0, \\ 3. & \text{solve } \; Law(X) = \mu \text{ where } X \text{ is the optimal state process from 2,} \end{cases} \tag{1.3}$$

where $\mathcal{P}(\mathbb{R}^d)$ is the space of probability measures on $\mathbb{R}^d$ and $Law(X)$ denotes the law of the process X.

In this paper we analyze a novel class of MFGs arising in models of optimal portfolio liquidation under market impact. Our MFGs can be characterized, equivalently, in terms of a forward-backward stochastic differential equation (FBSDE) with a possibly singular terminal condition on the backward component, or in terms of a FBSDE with finite terminal condition yet singular driver. We prove an existence and uniqueness of solutions result for such games by establishing a generalization of the method of continuation introduced in [16, 23] to linear-quadratic FBSDE systems with singular driver. Our existence result allows us to prove that the representative agent's value functions resulting from from a sequence of unconstrained optimization problems with increasing penalization of open positions at the terminal time converge to the value function of the constrained problem under additional assumptions on the market impact term.

1.1 Single player models of optimal portfolio liquidation

Single-player portfolio liquidation models have been extensively analyzed in recent years. Their main characteristic is a singularity at the terminal time of the Hamilton-Jacobi-Bellmann equation. In such models the controlled state sequence typically follows a dynamic of the form

$$x_t = x - \int_0^t \xi_s\,ds,$$

where $x \in \mathbb{R}$ is the initial portfolio, and ξ is the trading rate. The set of admissible controls is confined to those processes ξ that satisfy almost surely the liquidation constraint

$$x_T = 0.$$

Furthermore, it is often assumed that the unaffected benchmark price process follows a Brownian motion W (or some Brownian martingale) and that the trader's transaction price is given by

$$S_t = \sigma W_t - \int_0^t \kappa_s \xi_s \, ds - \eta_t \xi_t.$$

The integral term accounts for permanent price impact, i.e. the impact of past trades on current prices, while the term $\eta_t \xi_t$ accounts for the instantaneous impact that does not affect future transactions. The resulting expected cost functional is then of the linear-quadratic form

$$\mathbb{E}\left[\int_0^T \Big(\kappa_s \xi_s X_s + \eta_s |\xi_s|^2 + \lambda_s |x_s|^2\Big)\, ds\right]$$

where κ, η and λ are bounded adapted processes. The process λ describes the trader's degree of risk aversion; it penalizes slow liquidation. The process η describes the degree of market illiquidity; it penalizes fast liquidation. The process κ describes the impact of past trades on current transaction prices.

There are basically two approaches to overcome the challenges resulting from the singular terminal state constraint. The majority of the literature, including Ankirchner et al. [2], Graewe et al. [13], Kruse and Popier [20] and Popier [24, 25] considers finite approximations of the singular terminal value, and then shows that the minimal solution to the value function with singular terminal condition can be obtained by a monotone convergence argument. A second approach, originally introduced in Graewe et al. [14] and further generalised in Graewe and Horst [12] is to determine the precise asymptotic behaviour of a potential solution to the HJB equation at the terminal time, and to characterize the value function in terms of a PDE or BSDE with finite terminal value yet singular driver, for which the existence of a solution in a suitable space can be proved using standard fixed point arguments.

1.2 MFGs of optimal portfolio liquidation

Let $(\Omega, \mathcal{G}, \{\mathcal{G}_t, t \geq 0\}, \mathbb{P})$ be a probability space that carries independent standard Brownian motions $W^0, W^1, ..., W^N$. We consider a game of optimal portfolio liquidation with asymmetric information between a large number N of players. Following [6] we assume that the transaction price for each player $i = 1, ..., N$ is

$$S_t^i = \sigma W_t^0 - \int_0^t \frac{\kappa_s^i}{N} \sum_{j=1}^N \xi_s^j \, ds - \eta_t^i \xi_t^i.$$

In particular, the permanent price impact depends on the players' average trading rate. The optimization problem of player $i = 1, ..., N$ is thus to minimize the cost functional

$$J^{N,i}\left(\vec{\xi}\right) = \mathbb{E}\int_0^T \left(\frac{\kappa_t^i}{N}\sum_{j=1}^N \xi_t^j X_t^i + \eta_t^i(\xi_t^i)^2 + \lambda_t^i (X_t^i)^2\right) dt \tag{1.4}$$

subject to the state dynamics

$$dX_t^i = -\xi_t^i \, dt, \; X_0^i = x^i \text{ and } X_T^i = 0. \tag{1.5}$$

Here, $\vec{\xi} = (\xi^1, \cdots, \xi^N)$ is the vector of strategies of each player. Let

$$\mathbb{F}^i := (\mathcal{F}_t^i, 0 \leq t \leq T), \quad \text{with} \quad \mathcal{F}_t^i := \sigma(W_s^0, W_s^i, 0 \leq s \leq t). \tag{1.6}$$

We assume that the processes $(\kappa^i, \eta^i, \lambda^i)$ are progressively measurable with respect to the augmented σ-field which we still denote by $\mathbb{F}^i$ and conditionally independent and identically distributed, given W^0. Our MFG is different from standard MFGs in at least three important respects. First, the players interact through the impact of their strategies rather than states on the other players' payoff functions (see also [9]). Second, the players have private information about their instantaneous market impact, risk aversion and impact of the other players' actions on their own payoff functions. In fact, while each player's transaction price is driven by a common Brownian motion W^0, their cost coefficients are measurable functions of both the common factor W^0 and an independent idiosynchratic factor W^i. As a result, ours is a MFGs with common noise (see [8]). Third, and most importantly, the individual state dynamics are subject to the terminal state constraint arising from the liquidation requirement. Hence, the MFG associated with the N player game (1.4) and (1.5) is given by:

$$\begin{cases} \text{1. fix a } \mathbb{F}^0 \text{ progressively measurable process } \mu \text{ (in some suitable space);} \\ \text{2. solve the corresponding parameterized constrained optimization problem :} \\ \inf_\xi \mathbb{E}\left[\int_0^T \left(\kappa_s\mu_s X_s + \eta_s \xi_s^2 + \lambda_s X_s^2\right)\, ds\right] \\ \ \text{s.t. } dX_t = -\xi_t\, dt,\ X_0 = x \text{ and } X_T = 0; \\ \text{3. search for the fixed point } \mu_t = \mathbb{E}[\xi_t^* | \mathcal{F}_t^0], \text{ for } a.e.\ t \in [0,T], \\ \ \text{where } \xi^* \text{ is the optimal strategy from 2.} \end{cases} \tag{1.7}$$

Here, $\mathbb{F}^0 := (\mathcal{F}_t^0, 0 \le t \le T)$ with $\mathcal{F}_t^0 = \sigma(W_s^0, 0 \le s \le t)$, and κ, η and λ are $\mathbb{F} := (\mathcal{F}_t, 0 \le t \le T)$ progressively measurable with respect to $\mathcal{F}_t := \sigma(W_s^0, W_s, 0 \le s \le t)$, where W^0 and W are independent Brownian motions of dimension 1 and $m-1$, respectively, defined on some filtered probability space $(\Omega, \mathcal{G}, \mathbb{P})$. All filtrations are assumed to be augmented by the null sets.

The three papers closest to ours are Cardaliague and Lehalle [6], Carmona and Lacker [9], and Huang, Jaimungal and Nourin [18]. In [9], the authors propose a benchmark model as a motivation to their general result. They apply a weak formulation approach to solve the problem and assume the action space to be compact. Furthermore, each player's portfolio process is subject to random fluctuations, described by independent Brownian motions. As a result, their model is much closer to a standard MFG, but no liquidation constraint is possible in their framework. The papers [6] and [18] consider mean field models parameterized by different preferences and with major-minor players, respectively. Again, no liquidation constraint is allowed. To the best of our knowledge, ours is the first paper to consider MFGs with terminal state constraint.

There are three approaches to solve mean field games. In their original paper [22], Lasry and Lions followed an analytic approach. They analyzed a coupled forward-backward PDE system, where the backward component is the Hamiltion-Jacobi-Bellman equation arising from the representative agent's optimization problem, and the forward component is a Kolmogorov-Fokker-Planck equation that characterizes the dynamics of the state process. A more probabilistic approach was introduced by Carmona and Delarue in [7]. Using a maximum principle of Pontryagin type, they showed that the fixed point problem reduces to solving a McKean-Vlasov FBSDEs. A *relaxed solution* concept to MFGs was introduced by Lacker in [21] for MFGs with regular controls and later in Fu and Horst [11] for MFGs with singular controls.

We apply the probabilistic method to solve the MFG with terminal constraint (1.7). In a first step we show how the analysis of our MFG can be reduced to the analysis of a conditional mean-field type FBSDE. The forward component describes the optimal portfolio process; hence both its initial and terminal condition are known. The backward component describes the optimal trading rate; its terminal value is unknown. Making an affine ansatz, the mean-field type FBSDE with unkown terminal condition can be replaced by a coupled FBSDE with known initial and terminal condition, yet singular driver. Proving the existence of a small time solution to this FBSDE by a fixed point argument is not hard. The challenge is to prove the existence of a global solution on the whole time interval. This is achieved by a generalization of the method of continuation established in [16, 23] to linear-quadratic FBSDE systems with singular driver.

The benchmark case of constant cost coefficients can be solved in closed form. For this case we show that when the strength of interaction is large, the players initially trade very fast in equilibrium to avoid the negative drift generated by the mean field interaction. As such, our model provides a possible explanation for large price drops in markets with many strategically interacting investors.

Under additional assumptions on the market impact parameter we further prove that the solution to the MFG can be approximated by the solutions to a sequence of MFGs where the liquidation constraint is replaced by an increasing penalization of open positions at the terminal time. The convergence result can be viewed as a consistency result for both, the unconstrained and the constrained problem.

The remainder of the paper is organized as follows. In Section 2 we state and prove our existence and uniqueness of solutions result for the MFG (1.7). In a first step we prove that the adjoint equation associated with the MFG (1.7) has a unique solution. Then, we verify that the adjoint equation does indeed yield the optimal solution. Subsequently we establish additional results on the equilibrium trading strategies and portfolio processes if all the players share the same information and provide an explicit solution to a deterministic benchmark model. In Section 3 we prove that the solution to the MFG yields an ϵ-Nash equilibrium in a game with finitely many players. In Section 4 we prove that the MFG with singular terminal condition can be approximated by MFGs that penalize open positions at the terminal time under additional assumptions on the market impact term.

Notation. Throughout, we adopt the convention that C denotes a constant which may vary from line to line. Moreover, for a filtration $\mathbb{G}$, $\mathrm{Prog}(\mathbb{G})$ denotes the sigma-field of progressive subsets of $[0,T]\times\Omega$ and we consider the set of progressively measurable processes w.r.t. $\mathbb{G}$:

$$\mathcal{P}_{\mathbb{G}}([0,T]\times\Omega;\mathbb{I}) = \{u : [0,T]\times\Omega \to \mathbb{I} \mid u \text{ is } \mathrm{Prog}(\mathbb{G}) - \text{measurable}\}.$$

We define the following subspaces of $\mathcal{P}_{\mathbb{G}}([0,T]\times\Omega;\mathbb{I})$:

$$L^{\infty}_{\mathbb{G}}([0,T]\times\Omega;\mathbb{I}) = \left\{ u \in \mathcal{P}_{\mathbb{G}}([0,T]\times\Omega;\mathbb{I});\ \|u\| := \operatorname*{ess\,sup}_{t,\omega} |u(t,\omega)| < \infty \right\} ;$$

$$L^{p}_{\mathbb{G}}([0,T]\times\Omega;\mathbb{I}) = \left\{ u \in \mathcal{P}_{\mathbb{G}}([0,T]\times\Omega;\mathbb{I});\ \mathbb{E}\left(\int_0^T |u(t,\omega)|^2 dt\right)^{p/2} < \infty \right\};$$

$$S^{p}_{\mathbb{G}}([0,T]\times\Omega;\mathbb{I}) = \left\{ u \in \mathcal{P}_{\mathbb{G}}([0,T]\times\Omega;\mathbb{I});\ \mathbb{E}\left(\sup_{0\le t\le T} |u(t,\omega)|^p\right) < \infty \right\}.$$

Whenever the notation $T-$ appears in the definition of a function space we mean the set of all functions whose restriction satisfy the respective property on $[0,\tau]$ for any $\tau < T$, e.g., by $\psi \in L^2([0,T-]\times\Omega;\mathbb{R})$, we mean $\psi \in L^2([0,\tau]\times\Omega;\mathbb{R})$ for any $\tau < T$. For notational convenience, we put

$$D^2_{\mathbb{F}}([0,T]\times\Omega;\mathbb{R}) := L^2_{\mathbb{F}}([0,T]\times\Omega;\mathbb{R}) \cap S^2_{\mathbb{F}}([0,T-]\times\Omega;\mathbb{R}).$$

2 Probabilistic approach to MFGs with state constraint

In this section, we state and prove an existence and uniqueness of solutions result for the MFG (1.7). A control ξ is admissible in that game if $\xi \in \mathcal{A}_{\mathbb{F}}(t,x)$ with

$$\mathcal{A}_{\mathbb{F}}(t,x) := \left\{ \xi \in L^2_{\mathbb{F}}([t,T]\times\Omega),\ \int_t^T \xi_s\, ds = x \right\}.$$

For a given $\mu \in L^2_{\mathbb{F}^0}([0,T]\times\Omega;\mathbb{R})$ we denote the value function of the corresponding optimization problem by

$$V(t,x;\mu) := \inf_{\xi\in\mathcal{A}_{\mathbb{F}}(t,x)} \mathbb{E}\left[\int_t^T \left(\kappa_s X_s \mu_s + \eta_s \xi_s^2 + \lambda_s X_s^2\right) ds \middle| \mathcal{F}_t \right].$$

By Y we denote the adjoint process to X. The Hamiltonian corresponding to the above optimization problem in (1.7) is

$$H(t,\xi,X,Y;\mu) = -\xi Y + \kappa_t \mu X + \eta_t \xi^2 + \lambda_t X^2.$$

Thus, the stochastic maximum principle suggests that the solution to the optimization problem can be characterised in terms of the FBSDE

$$\begin{cases} dX_t = & -\, \xi_t \, dt, \\ -dY_t = & (\kappa_t \mu_t + 2\lambda_t X_t)\, dt - Z_t \, d\widetilde{W}_t, \\ X_0 = & x \\ X_T = & 0, \end{cases} \tag{2.1}$$

where $\widetilde{W} = (W^0, W)$ is a m-dimensional Brownian motion. The liquidation constraint $X_T = 0$ results in a singularity of the value function at liquidation time; see [14]. As a result, the terminal condition for Y cannot be determined a priori. It is implicitly encoded in the FBSDE (2.1). In particular, the first equation holds on $[0,T]$ while the second equation holds on $[0,T)$.

A standard approach yields the candidate optimal control

$$\xi_t^* = \frac{Y_t}{2\eta_t}. \tag{2.2}$$

The analysis of the MFG hence reduces to that of the following conditional mean-field type FBSDE

$$\begin{cases} dX_t = & -\, \dfrac{Y_t}{2\eta_t}\, dt, \\ -dY_t = & \left(\kappa_t \mathbb{E}\left[\dfrac{Y_t}{2\eta_t} \middle| \mathcal{F}_t^0 \right] + 2\lambda_t X_t \right) dt - Z_t \, d\widetilde{W}_t, \\ X_0 = & x \\ X_T = & 0. \end{cases} \tag{2.3}$$

In order to construct a solution to the problem (2.3), we define the following spaces of weighted stochastic processes.

Definition 2.1. For $\nu \in \mathbb{R}$, the space

$$\mathcal{H}_\nu := \{ Y \in \mathcal{P}_{\mathbb{F}}([0,T]\times\Omega; \mathbb{R}\cup\{\infty\}) : \ (T-.)^{-\nu} Y. \in S^2_{\mathbb{F}}([0,T]\times\Omega;\mathbb{R}\cup\{\infty\}) \}$$

is endowed with the norm

$$\|Y\|_{\mathcal{H}_\nu} := \|Y\|_\nu := \left(\mathbb{E}\left[\sup_{0\le t\le T} \left| \frac{Y_t}{(T-t)^\nu} \right|^2 \right] \right)^{\frac{1}{2}},$$

and the space

$$\mathcal{M}_\nu := \{ Y \in \mathcal{P}_{\mathbb{F}}([0,T]\times\Omega; \mathbb{R}\cup\{\infty\}) : \ (T-.)^{-\nu} Y. \in L^\infty_{\mathbb{F}}([0,T]\times\Omega;\mathbb{R}\cup\{\infty\}) \}$$

is endowed with the norm

$$\|Y\|_{\mathcal{M}_\nu} := \operatorname*{ess\,sup}_{(t,\omega)\in[0,T]\times\Omega} \frac{|Y_t|}{(T-t)^\nu}.$$

Fact 2.2. *The following facts are readily verified:*

- $\mathcal{H}_\nu \subset \mathcal{H}_{-1+\nu}$ *with* $\|\cdot\|_{\mathcal{H}_{-1+\nu}} \le T \|\cdot\|_{\mathcal{H}_\nu}$.

- *If* $K_1 \in \mathcal{M}_{-1}$ *and* $K_2 \in \mathcal{H}_\nu$*, then* $K_1K_2 \in \mathcal{H}_{-1+\nu}$.
- *If* $K \in \mathcal{H}_\nu$*, with* $\nu > 0$*, then* $K_T = 0$ *a.s.*

The same properties hold for the space $\mathcal{M}_\nu$.

We assume throughout that the cost coefficients are bounded and that the dependence of an individual player's cost function on the average action is weak enough. The weak interaction condition is consistent with the game theory literature on mean-field type games where some form of moderate dependence condition is usually required to prove the existence of Nash equilibria; see, e.g., [15] and references therein. It is also consistent with the monotonicity condition for FBSDE systems originally proposed by [16, 23] and slightly weaker than the generalizations to mean-field type FBSDEs established in [3, 7]. Specifically, we assume that the following condition is satisfied.

Assumption 2.3. The processes κ, λ, $1/\lambda$, η and $1/\eta$ belong to $L^\infty_{\mathbb{F}}([0,T]\times\Omega;[0,\infty))$. We denote by $\|\lambda\|$, $\|\kappa\|$, $\|\eta\|$ the bounds of the respective cost coefficients, and by $\lambda_\star$ and $\eta_\star$ the lower bounds of λ and η, respectively and assume that there exists a $\theta > 0$ such that

$$2\eta_\star - \frac{\|\kappa\|}{2\theta} > 0, \qquad 2\lambda_\star - \frac{\|\kappa\|\theta}{2} > 0. \tag{2.4}$$

The following quantity will be important in our subsequent analysis:

$$\alpha := \eta_\star/\|\eta\| \in (0,1]. \tag{2.5}$$

We are now ready to sate our first major result. Its proof is given in the next subsection.

Theorem 2.4. *Under Assumption 2.3 there exists a unique solution*

$$(X,Y,Z) \in \mathcal{H}_\alpha \times L^2_{\mathbb{F}}([0,T]\times\Omega;\mathbb{R}) \times L^2_{\mathbb{F}}([0,T-]\times\Omega;\mathbb{R}^m)$$

to the FBSDE (2.3)*. Moreover, the MFG* (1.7) *admits a unique equilibrium* μ^**. The equilibrium is given by*

$$\mu^*_t = \mathbb{E}\left[\frac{Y_t}{2\eta_t}\middle|\mathcal{F}^0_t\right], \quad t \in [0,T).$$

2.1 Proof of existence

In this section we prove our existence and uniqueness of equilibrium result. Decoupling the FBSDE (2.3) by $Y = AX + B$ yields the following system of Riccati type equations:

$$\begin{cases} -dA_t = \left(2\lambda_t - \dfrac{A_t^2}{2\eta_t}\right)dt - Z^A_t\,d\widetilde{W}_t, \\ -dB_t = \left(\kappa_t\mathbb{E}\left[\dfrac{1}{2\eta_t}(A_tX_t+B_t)\middle|\mathcal{F}^0_t\right] - \dfrac{A_tB_t}{2\eta_t}\right)dt - Z^B_t\,d\widetilde{W}_t, \\ A_T = \infty \\ B_T = 0. \end{cases} \tag{2.6}$$

The existence of a unique solution $A \in \mathcal{M}_{-1}$ to the first equation is established in Lemma A.1. From that lemma it also follows that

$$\exp\left(-\int_r^s \frac{A_u}{2\eta_u}du\right) \le \left(\frac{T-s}{T-r}\right)^\alpha \tag{2.7}$$

for any $0 \le r \le s < T$. Hence we need to solve the following FBSDE:

$$\begin{cases} dX_t = -\dfrac{1}{2\eta_t}(A_t X_t + B_t)\,dt, \\ -dB_t = \left(\kappa_t \mathbb{E}\left[\dfrac{1}{2\eta_t}(A_t X_t + B_t)\middle|\mathcal{F}_t^0\right] - \dfrac{A_t B_t}{2\eta_t}\right)dt - Z_t^B\,d\widetilde{W}_t, \\ X_0 = x \\ B_T = 0. \end{cases} \tag{2.8}$$

Our approach is based on an extension of the method of continuation that accounts for the singularity of the process A at the terminal time and hence for the singularity in the driver of the FBSDE. We apply to the method of continuation to the triple (X, B, Y) rather than the pair (X, B), where $Y = AX + B$ is treated as a bridge, and search for solutions

$$(X, B, Y = AX + B) \in \mathcal{H}_\alpha \times \mathcal{H}_\gamma \times L^2_{\mathbb{F}}([0,T] \times \Omega; \mathbb{R}),$$

where α was defined in (2.5) and γ is any constant

$$0 < \gamma < \alpha \wedge 1/2.$$

Specifically, the method of continuation will be applied to the FBSDE

$$\begin{cases} dX_t = -\dfrac{1}{2\eta_t}(A_t X_t + B_t)\,dt, \\ -dB_t = \left(\kappa_t \mathfrak{p}\mathbb{E}\left[\dfrac{1}{2\eta_t}(A_t X_t + B_t)\middle|\mathcal{F}_t^0\right] + f_t - \dfrac{A_t B_t}{2\eta_t}\right)dt - Z_t^B\,d\widetilde{W}_t, \\ dY_t = \left(-2\lambda_t X_t - \kappa_t \mathfrak{p}\mathbb{E}\left[\dfrac{A_t X_t + B_t}{2\eta_t}\middle|\mathcal{F}_t^0\right] - f_t\right)dt + Z_t^Y\,d\widetilde{W}_t, \\ X_0 = x, \\ B_T = 0, \end{cases} \tag{2.9}$$

where $\mathfrak{p} \in [0,1]$ and $f \in L^2_{\mathbb{F}}([0,T] \times \Omega; \mathbb{R})$. We emphasise that the first two equations hold on $[0,T]$, while the third equation holds on $[0,T)$.

In a first step, we provide an a priori estimate for the processes Z^B and Z^Y.

Lemma 2.5. *Assume that $f \in L^2_{\mathbb{F}}([0,T] \times \Omega; \mathbb{R})$ and that there exists a solution (X, B, Y, Z^B, Z^Y) to (2.9) such that*

$$(X, B, Y) \in \mathcal{H}_\alpha \times \mathcal{H}_\gamma \times S^2_{\mathbb{F}}([0,T-] \times \Omega, \mathbb{R}).$$

Then

$$(Z^B, Z^Y) \in L^2_{\mathbb{F}}([0,T] \times \Omega; \mathbb{R}^m) \times L^2_{\mathbb{F}}([0,T-] \times \Omega; \mathbb{R}^m)$$

and there exists a constant $C > 0$ such that

$$\mathbb{E}\left[\int_0^T |Z_t^B|^2\,dt\right] \le C\left(\|B\|_\gamma^2 + \|X\|_\alpha^2 + \mathbb{E}\left[\int_0^T |f_t|^2\,dt\right]\right)$$

and such that for each $\tau < T$

$$\mathbb{E}\left[\left|\int_0^\tau |Z_s^Y|^2\,ds\right|^2\right] \le C\left(\mathbb{E}\left[\sup_{0\le t\le\tau} |Y_t|^2\right] + \|X\|_\alpha^2 + \|B\|_\gamma^2 + \mathbb{E}\left[\int_0^T |f_t|^2\,dt\right]\right).$$

In particular, $\int_0^\cdot Z_s^B\,d\widetilde{W}_s$ is a true martingale on $[0,T]$ and $\int_0^\cdot Z_s^Y\,d\widetilde{W}_s$ is a true martingale on $[0,\tau]$, for each $\tau < T$.

Proof. Since $A \in \mathcal{M}_{-1}$ and $\eta_\star > 0$ there exists a constant $C > 0$ that is independent of $s \in [0,T]$ such that

$$\left| \frac{A_s B_s}{2\eta_s} - \kappa_s \mathbb{E}\left[\frac{A_s X_s + B_s}{2\eta_s} \middle| \mathcal{F}_s^0 \right] - f_s \right| \le C \left[\frac{|B_s|}{T-s} + \mathbb{E}\left(\frac{|X_s|}{T-s} + |B_s| \middle| \mathcal{F}_s^0 \right) + |f_s| \right].$$

Since $(X,B) \in \mathcal{H}_\alpha \times \mathcal{H}_\gamma$ this implies,

$$\begin{aligned} \int_t^T Z_s^B d\widetilde{W}_s &= B_t + \int_t^T \left\{ \frac{A_s B_s}{2\eta_s} - \kappa_s \mathbb{E}\left[\frac{A_s X_s + B_s}{2\eta_s} \middle| \mathcal{F}_s^0 \right] - f_s \right\} ds \\ &\le C \sup_{0\le t\le T} \frac{|B_t|}{(T-t)^\gamma} + C \sup_{0\le s\le T} \mathbb{E}\left[\sup_{0\le t\le T} \frac{|X_t|}{(T-t)^\alpha} \middle| \mathcal{F}_s^0 \right] \\ &\quad + C \sup_{0\le s\le T} \mathbb{E}\left[\sup_{0\le t\le T} \frac{|B_t|}{(T-t)^\gamma} \middle| \mathcal{F}_s^0 \right] + \int_0^T |f_t| \, dt. \end{aligned}$$

Thus, by Doob's maximal inequality,

$$\mathbb{E}\left[\sup_{0\le t\le T} \left| \int_t^T Z_s^B d\widetilde{W}_s \right|^2 \right] \le C \left(\|B\|_\gamma^2 + \|X\|_\alpha^2 + \mathbb{E}\left[\int_0^T |f_t|^2 \, dt \right] \right).$$

Similarly, for each $0 < \tau < T$,

$$\mathbb{E}\left[\sup_{0\le t\le \tau} \left| \int_t^\tau Z_s^Y \, d\widetilde{W}_s \right|^2 \right] \le C \left(\mathbb{E}\left[\sup_{0\le t\le \tau} |Y_t|^2 \right] + \|X\|_\alpha^2 + \|B\|_\gamma^2 + \mathbb{E}\left[\int_0^T |f_t|^2 \, dt \right] \right) < \infty.$$

□

In a second step, we now prove an existence of solutions result for the FBSDE (2.9) with $\mathfrak{p} = 0$.

Lemma 2.6. *For $\mathfrak{p} = 0$ there exists for every given data $f \in L^2_{\mathbb{F}}([0,T] \times \Omega; \mathbb{R})$ a unique solution $(X, B, Y, Z^B, Z^Y) \in \mathcal{H}_\alpha \times \mathcal{H}_\gamma \times D^2_{\mathbb{F}}([0,T] \times \Omega; \mathbb{R}) \times L^2_{\mathbb{F}}([0,T] \times \Omega; \mathbb{R}^m) \times L^2_{\mathbb{F}}([0,T-] \times \Omega; \mathbb{R}^m)$ to (2.9). It is given by*

$$\begin{cases} B_t = \mathbb{E}\left[\int_t^T f_s e^{-\int_t^s (2\eta_r)^{-1} A_r \, dr} \, ds \middle| \mathcal{F}_t \right], \quad t \in [0,T] \\ X_t = x e^{-\int_0^t (2\eta_r)^{-1} A_r \, dr} - \int_0^t (2\eta_s)^{-1} B_s e^{-\int_s^t (2\eta_r)^{-1} A_r \, dr} \, ds, \quad t \in [0,T] \\ Y_t = A_t X_t + B_t, \quad t \in [0,T), \end{cases}$$

and $Z^B \in L^2_{\mathbb{F}}([0,T] \times \Omega; \mathbb{R}^m)$ and $Z^Y \in L^2_{\mathbb{F}}([0,T-] \times \Omega; \mathbb{R}^m)$ are given by the martingale representation theorem.

Proof. For $\mathfrak{p} = 0$ the process X solves a linear ODE and the pair (B, Z^B) solves a linear BSDE. Hence, the explicit representations follow from the respective solution formulas. It remains to establish the desired integration properties. To this end, we first apply Hölder's inequality in order to obtain,

$$\frac{|B_t|}{(T-t)^\gamma} \le \frac{1}{(T-t)^\gamma} \mathbb{E}\left[\int_t^T |f_s| \, ds \middle| \mathcal{F}_t \right] \le \left(\mathbb{E}\left[\int_t^T |f_s|^{\frac{1}{1-\gamma}} \, ds \middle| \mathcal{F}_t \right] \right)^{1-\gamma} < \infty.$$

Using Doob's maximal inequality, Jensen's inequality and the fact that $\gamma < \frac{1}{2}$ we conclude that,

$$\mathbb{E}\left[\sup_{0\le t\le T} \left| \frac{B_t}{(T-t)^\gamma} \right|^2 \right] \le \mathbb{E}\left[\sup_{0\le t\le T} \left(\mathbb{E}\left[\int_0^T |f_s|^{\frac{1}{1-\gamma}} \, ds \middle| \mathcal{F}_t \right] \right)^{2(1-\gamma)} \right] \le C \mathbb{E}\left[\int_0^T |f_s|^2 \, ds \right].$$

From (2.7) and the solution formula for X and using that $\gamma < \alpha$ we obtain that $X \in \mathcal{H}_\alpha$ because,

$$
\begin{aligned}
|X_t| &\leq \frac{x(T-t)^\alpha}{T^\alpha} + C\int_0^t |B_s| \left(\frac{T-t}{T-s}\right)^\alpha ds \\
&\leq \frac{x(T-t)^\alpha}{T^\alpha} + C\left(\sup_{0\leq s\leq T} \frac{|B_s|}{(T-s)^\gamma}\right)\left(\int_0^t (T-s)^{\gamma-\alpha}\, ds\right)(T-t)^\alpha \\
&\leq (T-t)^\alpha \left\{\frac{x}{T^\alpha} + \frac{CT^{1+\gamma-\alpha}}{1+\gamma-\alpha}\left(\sup_{0\leq s\leq T} \frac{|B_s|}{(T-s)^\gamma}\right)\right\}.
\end{aligned}
$$

The previously established properties of A, X and B yield $Y \in S^2_{\mathbb{F}}([0,T-]\times\Omega;\mathbb{R})$ with

$$
\mathbb{E}\left[\sup_{t\in[0,\tau]} Y_t^2\right] \leq \frac{C}{(T-\tau)^{2(1-\alpha)}}\|X\|_\alpha^2 + (T-\tau)^{2\gamma}\|B\|_\gamma^2. \tag{2.10}
$$

For any $\epsilon > 0$, integration by part implies that

$$
\begin{aligned}
X_{T-\epsilon}Y_{T-\epsilon} - X_0Y_0 &= \int_0^{T-\epsilon} X_t\, dY_t + \int_0^{T-\epsilon} Y_t\, dX_t \\
&= -\int_0^{T-\epsilon} X_t(2\lambda_t X_t + f_t)\, dt - \int_0^{T-\epsilon} \frac{Y_t^2}{2\eta_t}\, dt + \int_0^{T-\epsilon} X_t Z_t^Y\, d\widetilde{W}_t.
\end{aligned}
$$

The positivity of the process A along with the definition of the process Y yields $X_{T-\epsilon}Y_{T-\epsilon} \geq X_{T-\epsilon}B_{T-\epsilon}$. Thus, taking expectations on both sides of the above equation, letting $\epsilon \to 0$ and using $X_T = B_T = 0$ yields,

$$
-\mathbb{E}[X_0Y_0] \leq -\mathbb{E}\left[\int_0^T 2\lambda_t X_t^2\, dt\right] - \mathbb{E}\left[\int_0^T X_t f_t\, dt\right] - \mathbb{E}\left[\int_0^T \frac{Y_t^2}{2\eta_t}\, dt\right].
$$

Together with the inequality (2.10) for $\tau = 0$ this shows that

$$
\mathbb{E}\left[\int_0^T Y_t^2\, dt\right] \leq C\mathbb{E}\left[\int_0^T X_t^2\, dt\right] + C\mathbb{E}\left[\int_0^T f_t^2\, dt\right] + C\|X\|_\alpha^2 + C\|B\|_\gamma^2 < \infty.
$$

□

In a third step we now establish the continuation result for the FBSDE (2.9) from which we shall then deduce the existence of a unique global solution to our original MFG.

Lemma 2.7. *If for some $\mathfrak{p} \in [0,1]$ the FBSDE (2.9) is for every data $f \in L^2_{\mathbb{F}}([0,T]\times\Omega;\mathbb{R})$ uniquely solvable in $\mathcal{H}_\alpha \times \mathcal{H}_\gamma \times D^2_{\mathbb{F}}([0,T]\times\Omega;\mathbb{R}) \times L^2_{\mathbb{F}}([0,T]\times\Omega;\mathbb{R}^m) \times L^2_{\mathbb{F}}([0,T-]\times\Omega;\mathbb{R}^m)$, then this holds also for $\mathfrak{p}+\mathfrak{d}$ with $\mathfrak{d} > 0$ small enough (independent of $\mathfrak{p}$ and f).*

Proof. Let us fix $\mathfrak{d} > 0$, $Y \in L^2_{\mathbb{F}}([0,T]\times\Omega;\mathbb{R})$ and $f \in L^2_{\mathbb{F}}([0,T]\times\Omega;\mathbb{R})$ and consider the following system:

$$
\begin{cases}
d\widetilde{X}_t = -\dfrac{1}{2\eta_t}(A_t\widetilde{X}_t + \widetilde{B}_t)\, dt, \\
-d\widetilde{B}_t = \left(\kappa_t\mathfrak{p}\mathbb{E}\left[\dfrac{1}{2\eta_t}\left(A_t\widetilde{X}_t + \widetilde{B}_t\right)\middle|\mathcal{F}_t^0\right] + \kappa_t\mathfrak{d}\mathbb{E}\left[\dfrac{Y_t}{2\eta_t}\middle|\mathcal{F}_t^0\right] + f_t - \dfrac{A_t\widetilde{B}_t}{2\eta_t}\right) dt - Z_t^{\widetilde{B}}\, d\widetilde{W}_t, \\
d\widetilde{Y}_t = \left(-2\lambda_t\widetilde{X}_t - \kappa_t\mathfrak{p}\mathbb{E}\left[\dfrac{A_t\widetilde{X}_t + \widetilde{B}_t}{2\eta_t}\middle|\mathcal{F}_t^0\right] - \kappa_t\mathfrak{d}\mathbb{E}\left[\dfrac{Y_t}{2\eta_t}\middle|\mathcal{F}_t^0\right] - f_t\right) dt + Z_t^{\widetilde{Y}}\, d\widetilde{W}_t, \\
X_0 = x \\
B_T = 0.
\end{cases} \tag{2.11}
$$

Then

$$f(Y) := \kappa\mathfrak{d}\mathbb{E}\left[\frac{Y}{2\eta}\middle|\mathcal{F}^0\right] + f \in L^2_{\mathbb{F}}([0,T]\times\Omega;\mathbb{R}).$$

Thus, by assumption there exists a unique solution

$$(\widetilde{X},\widetilde{B},\widetilde{Y},Z^{\widetilde{B}},Z^{\widetilde{Y}}) \in \mathcal{H}_\alpha \times \mathcal{H}_\gamma \times D^2_{\mathbb{F}}([0,T]\times\Omega;\mathbb{R}) \times L^2_{\mathbb{F}}([0,T]\times\Omega;\mathbb{R}^m) \times L^2_{\mathbb{F}}([0,T-]\times\Omega;\mathbb{R}^m)$$

to (2.11), and $\widetilde{Y} = A\widetilde{X} + \widetilde{B}$. This defines a mapping $Y \mapsto (\widetilde{X},\widetilde{B},\widetilde{Y})$ from $L^2_{\mathbb{F}}([0,T]\times\Omega;\mathbb{R})$ to $\mathcal{H}_\alpha\times\mathcal{H}_\gamma\times L^2_{\mathbb{F}}([0,T]\times\Omega;\mathbb{R})$ and hence also a mapping $(X,B,Y)\mapsto(\widetilde{X},\widetilde{B},\widetilde{Y})$ on $\mathcal{H}_\alpha\times\mathcal{H}_\gamma\times L^2_{\mathbb{F}}([0,T]\times\Omega;\mathbb{R})$. In what follows we prove that this second mapping is a contraction for some $\mathfrak{d}>0$. For the unique fixed point the system (2.11) reduces to the system (2.9) with $\mathfrak{p}$ replaced by $\mathfrak{p}+\mathfrak{d}$. This then yields the desired result.

In order to establish the contraction property, we denote for two processes $Y, Y' \in L^2_{\mathbb{F}}([0,T]\times\Omega;\mathbb{R})$ by $(\widetilde{X},\widetilde{B},\widetilde{Y})$ and $(\widetilde{X}',\widetilde{B}',\widetilde{Y}')$ the corresponding processes defined by (2.11) and put

$$\widetilde{\xi}_t = \frac{\widetilde{Y}_t}{2\eta_t}, \quad \widetilde{\xi}'_t = \frac{\widetilde{Y}'_t}{2\eta_t}, \quad \widetilde{\mu}_t = \mathbb{E}\left[\widetilde{\xi}_t\middle|\mathcal{F}^0_t\right], \quad \widetilde{\mu}'_t = \mathbb{E}\left[\widetilde{\xi}'_t\middle|\mathcal{F}^0_t\right].$$

For any $\varepsilon>0$ integration by part yields that

$$\begin{aligned}(\widetilde{X}'_{T-\varepsilon}-\widetilde{X}_{T-\varepsilon})\widetilde{Y}_{T-\varepsilon} &= \int_0^{T-\varepsilon}(\widetilde{X}'_s-\widetilde{X}_s)\,d\widetilde{Y}_s + \int_0^{T-\varepsilon}\widetilde{Y}_s\,d(\widetilde{X}'_s-\widetilde{X}_s)\\ &= -\int_0^{T-\varepsilon}(\widetilde{X}'_s-\widetilde{X}_s)(\mathfrak{p}\kappa_s\widetilde{\mu}_s+2\lambda_s\widetilde{X}_s)\,ds - \int_0^{T-\varepsilon}\widetilde{Y}_s(\widetilde{\xi}'_s-\widetilde{\xi}_s)\,ds\\ &\quad -\int_0^{T-\varepsilon}(\widetilde{X}'_s-\widetilde{X}_s)f(Y_s)\,ds + \int_0^{T-\varepsilon}(\widetilde{X}'_s-\widetilde{X}_s)Z^{\widetilde{Y}}_s\,d\widetilde{W}_s\end{aligned}$$

and

$$\begin{aligned}(\widetilde{X}_{T-\varepsilon}-\widetilde{X}'_{T-\varepsilon})\widetilde{Y}'_{T-\varepsilon} &= -\int_0^{T-\varepsilon}(\widetilde{X}_s-\widetilde{X}'_s)(\mathfrak{p}\kappa_s\widetilde{\mu}'_s+2\lambda_s\widetilde{X}'_s)\,ds - \int_0^{T-\varepsilon}\widetilde{Y}'_s(\widetilde{\xi}_s-\widetilde{\xi}'_s)\,ds\\ &\quad -\int_0^{T-\varepsilon}(\widetilde{X}_s-\widetilde{X}'_s)f(Y'_s)\,ds + \int_0^{T-\varepsilon}(\widetilde{X}_s-\widetilde{X}'_s)Z^{\widetilde{Y}'}_s\,d\widetilde{W}_s.\end{aligned}$$

Taking the sum of these two equations and using that

$$\begin{aligned}(\widetilde{X}_{T-\varepsilon}-\widetilde{X}'_{T-\varepsilon})(\widetilde{Y}'_{T-\varepsilon}-\widetilde{Y}_{T-\varepsilon}) &= -A_{T-\varepsilon}(\widetilde{X}_{T-\varepsilon}-\widetilde{X}'_{T-\varepsilon})^2 - (\widetilde{X}_{T-\varepsilon}-\widetilde{X}'_{T-\varepsilon})(\widetilde{B}_{T-\varepsilon}-\widetilde{B}'_{T-\varepsilon})\\ &\le -(\widetilde{X}_{T-\varepsilon}-\widetilde{X}'_{T-\varepsilon})(\widetilde{B}_{T-\varepsilon}-\widetilde{B}'_{T-\varepsilon})\end{aligned}$$

yields

$$\begin{aligned}&2\int_0^{T-\varepsilon}\eta_s(\widetilde{\xi}'_s-\widetilde{\xi}_s)^2\,ds + 2\int_0^{T-\varepsilon}\lambda_s(\widetilde{X}'_s-\widetilde{X}_s)^2\,ds\\ &+\int_0^{T-\varepsilon}(\widetilde{X}'_s-\widetilde{X}_s)(f(Y'_s)-f(Y_s))\,ds + \int_0^{T-\varepsilon}(\widetilde{X}_s-\widetilde{X}'_s)(\widetilde{Z}^{Y'}_s-\widetilde{Z}^{Y}_s)\,d\widetilde{W}_s\\ \le\ &-(\widetilde{X}_{T-\varepsilon}-\widetilde{X}'_{T-\varepsilon})(\widetilde{B}_{T-\varepsilon}-\widetilde{B}'_{T-\varepsilon}) + \int_0^{T-\varepsilon}\left[\mathfrak{p}\kappa_s(\widetilde{\mu}_s-\widetilde{\mu}'_s)(\widetilde{X}'_s-\widetilde{X}_s)\right]ds.\end{aligned}$$

Taking expectations on both sides drops the martingale part. Then we can pass to the limit as $\varepsilon\to0$ because $\widetilde{X}$, $\widetilde{X}'\in\mathcal{H}_\alpha$ and $\widetilde{B}$, $\widetilde{B}'\in\mathcal{H}_\gamma$ in order to obtain a constant C such that for any $\theta>0$

$$\begin{aligned}&\left(2\eta_\star-\frac{\|\kappa\|}{2\theta}\right)\mathbb{E}\left[\int_0^T(\widetilde{\xi}'_s-\widetilde{\xi}_s)^2\,ds\right] + \left(2\lambda_\star-\frac{\|\kappa\|}{2}\theta\right)\mathbb{E}\left[\int_0^T(\widetilde{X}'_s-\widetilde{X}_s)^2\,ds\right]\\ \le\ &C\mathfrak{d}\mathbb{E}\left[\int_0^T(\widetilde{X}'_s-\widetilde{X}_s)^2\,ds\right] + C\mathfrak{d}\mathbb{E}\left[\int_0^T(Y'_s-Y_s)^2\,ds\right].\end{aligned}$$

In view of Assumption 2.3 we can choose a $\theta > 0$ such that

$$2\eta_\star - \frac{\|\kappa\|}{2\theta} > 0, \qquad 2\lambda_\star - \frac{\|\kappa\|\theta}{2} > 0,$$

which implies that

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T (\widetilde{\xi}'_s - \widetilde{\xi}_s)^2\, ds\right] + \mathbb{E}\left[\int_0^T (\widetilde{X}'_s - \widetilde{X}_s)^2\, ds\right] \\
&\le C\mathfrak{d}\mathbb{E}\left[\int_0^T (\widetilde{X}'_s - \widetilde{X}_s)^2\, ds\right] + C\mathfrak{d}\mathbb{E}\left[\int_0^T (Y'_s - Y_s)^2\, ds\right].
\end{aligned}$$

Thus, when $\mathfrak{d}$ is small enough,

$$\mathbb{E}\left[\int_0^T |\widetilde{Y}_t - \widetilde{Y}'_t|^2\, dt\right] \le \mathfrak{a}\mathbb{E}\left[\int_0^T |Y_t - Y'_t|^2\, dt\right]$$

for some $\mathfrak{a} < 1$. We notice that the bound on $\mathfrak{d}$ only depends on T, κ, η and λ.

Now using the definition of $\widetilde{\xi}$ and $\widetilde{\xi}'$ the solution formula for linear BSDEs yields

$$|\widetilde{B}_t - \widetilde{B}'_t| \le \|\kappa\|\mathbb{E}\left[\int_t^T \left\{\mathfrak{p}\mathbb{E}\left[|\widetilde{\xi}_s - \widetilde{\xi}'_s|\middle|\mathcal{F}^0_s\right] + \mathfrak{d}\mathbb{E}\left[|Y_s - Y'_s|\middle|\mathcal{F}^0_s\right]\right\} ds\middle|\mathcal{F}_t\right].$$

Thus

$$\begin{aligned}
|\widetilde{B}_t - \widetilde{B}'_t| &\le C(T-t)^\gamma \mathbb{E}\left[\int_t^T \mathbb{E}\left[|\widetilde{\xi}_s - \widetilde{\xi}'_s|^{\frac{1}{1-\gamma}}\middle|\mathcal{F}^0_s\right] ds\middle|\mathcal{F}_t\right]^{1-\gamma} \\
&\quad + c\mathfrak{d}(T-t)^\gamma \mathbb{E}\left[\int_t^T \mathbb{E}\left[|Y_s - Y'_s|^{\frac{1}{1-\gamma}}\middle|\mathcal{F}^0_s\right] ds\middle|\mathcal{F}_t\right]^{1-\gamma}.
\end{aligned}$$

Since $2\gamma < 1$, Doob's maximal inequality along with the previously established L^2 bounds yields

$$\mathbb{E}\left[\sup_{t\in[0,T]} \frac{|\widetilde{B}_t - \widetilde{B}'_t|^2}{(T-t)^{2\gamma}}\right] \le C\mathbb{E}\left[\int_0^T |\widetilde{\xi}_s - \widetilde{\xi}'_s|^2 ds\right] + C\mathfrak{d}^2\mathbb{E}\left[\int_0^T |Y_s - Y'_s|^2 ds\right].$$

Now using the dynamics of $\widetilde{X}$ we obtain

$$\begin{aligned}
|\widetilde{X}_t - \widetilde{X}'_t| &= \left|\int_0^t -\{\mathfrak{p}(2\eta_s)^{-1}(\widetilde{B}_s - \widetilde{B}'_s)\} e^{-\int_s^t (2\eta_r)^{-1}A_r\, dr}\, ds\right| \\
&\le C\int_0^t \{|\widetilde{B}_s - \widetilde{B}'_s|\}\left(\frac{T-t}{T-s}\right)^\alpha ds \\
&\le C\frac{T^{1+\gamma-\alpha}}{1+\gamma-\alpha}(T-t)^\alpha \sup_{0\le s\le T} \frac{|\widetilde{B}_s - \widetilde{B}'_s|}{(T-s)^\gamma}.
\end{aligned}$$

Hence this leads to

$$\mathbb{E}\left[\sup_{t\in[0,T]} \frac{|\widetilde{X}_t - \widetilde{X}'_t|^2}{(T-t)^{2\alpha}}\right] \le C\left[\|\widetilde{B}_s - \widetilde{B}'_s\|^2_\gamma\right].$$

To summarize, we obtain a constant $\mathfrak{d}$ such that $(X, B, Y) \to (\widetilde{X}, \widetilde{B}, \widetilde{Y})$ is a contraction in $\mathcal{H}_\alpha \times \mathcal{H}_\gamma \times L^2_{\mathbb{F}}([0,T] \times \Omega; \mathbb{R})$. Since $\widetilde{Y} = A\widetilde{X} + \widetilde{B}$, $\widetilde{Y} \in D^2_{\mathbb{F}}([0,T] \times \Omega; \mathbb{R})$ and using Lemma 2.5, we see that the following system admits a unique solution $(\widetilde{X}, \widetilde{B}, \widetilde{Y}, Z^{\widetilde{B}}, Z^{\widetilde{Y}}) \in \mathcal{H}_\alpha \times \mathcal{H}_\gamma \times D^2_{\mathbb{F}}([0,T]\times\Omega;\mathbb{R}) \times L^2_{\mathbb{F}}([0,T]\times\Omega;\mathbb{R}^m) \times L^2_{\mathbb{F}}([0,T-]\times\Omega;\mathbb{R}^m)$:

$$\begin{cases} d\widetilde{X}_t = -\dfrac{1}{2\eta_t}(A_t\widetilde{X}_t + \widetilde{B}_t)\,dt, \\ -d\widetilde{B}_t = \left(\kappa_t\mathfrak{p}\mathbb{E}\left[\dfrac{1}{2\eta_t}\left(A_t\widetilde{X}_t + \widetilde{B}_t\right)\middle|\mathcal{F}^0_t\right] + \kappa_t\mathfrak{d}\mathbb{E}\left[\dfrac{\widetilde{Y}_t}{2\eta_t}\middle|\mathcal{F}^0_t\right] + f_t - \dfrac{A_t\widetilde{B}_t}{2\eta_t}\right)dt - Z^{\widetilde{B}}_t\,d\widetilde{W}_t, \\ d\widetilde{Y}_t = \left(-2\lambda_t\widetilde{X}_t - \kappa_t\mathfrak{p}\mathbb{E}\left[\dfrac{A_t\widetilde{X}_t + \widetilde{B}_t}{2\eta_t}\middle|\mathcal{F}^0_t\right] - \kappa_t\mathfrak{d}\mathbb{E}\left[\dfrac{\widetilde{Y}_t}{2\eta_t}\middle|\mathcal{F}^0_t\right] - f_t\right)dt + Z^{\widetilde{Y}}_t\,d\widetilde{W}_t, \\ X_0 = x \\ B_T = 0. \end{cases}$$

Using again the relation $\widetilde{Y} = A\widetilde{X} + \widetilde{B}$, the above system is equivalent to (2.9) with $\mathfrak{p}$ replaced by $\mathfrak{p}+\mathfrak{d}$. This proves the assertion. □

Using Lemmata 2.5, 2.6 and 2.7 and by induction on $\mathfrak{p}$, we obtain the following result.

Theorem 2.8. *There exists a unique solution $(X^*, B^*, Y^*, Z^{B^*}, Z^{Y^*}) \in \mathcal{H}_\alpha \times \mathcal{H}_\gamma \times D^2_{\mathbb{F}}([0,T]\times\Omega;\mathbb{R}) \times L^2_{\mathbb{F}}([0,T]\times\Omega;\mathbb{R}^m)\times L^2_{\mathbb{F}}([0,T-]\times\Omega;\mathbb{R}^m)$ to the FBSDEs* (2.3) *and* (2.8)*. Moreover, there exists a constant $C>0$ depending on η, λ, κ, T and x, such that*

$$\|X^*\|_\alpha + \|B^*\|_\gamma + \mathbb{E}\left[\int_0^T |Y^*_t|^2\,dt\right] \le C.$$

From the equation (2.2) we obtain the following candidates for the optimal portfolio process and the optimal trading strategy:

$$\begin{aligned} X^*_t &= xe^{-\int_0^t \frac{A_r}{2\eta_r}\,dr} - \int_0^t \frac{B_s}{2\eta_s}e^{-\int_s^t \frac{A_r}{2\eta_r}\,dr}\,ds, \\ \xi^*_t &= xe^{-\int_0^t \frac{A_r}{2\eta_r}\,dr}\frac{A_t}{2\eta_t} + \frac{B_t}{2\eta_t} - \frac{A_t}{2\eta_t}\int_0^t \frac{B_s}{2\eta_s}e^{-\int_s^t \frac{A_r}{2\eta_r}\,dr}\,ds. \end{aligned} \tag{2.12}$$

By construction, $X^*_T = 0$ and hence ξ^* is an admissible liquidation strategy. The following theorem shows that it is indeed the optimal liquidation strategy and that its conditional expectation defines the desired equilibrium for our MFG.

Theorem 2.9. *The process ξ^* is an optimal control. Hence $\mu^* = \mathbb{E}[\xi^*|\mathcal{F}^0]$ is the solution to the MFG. Moreover, the value function is given by*

$$V(t,x;\mu^*) = \frac{1}{2}A_t x^2 + \frac{1}{2}B_t x + \frac{1}{2}\mathbb{E}\left[\int_t^T \kappa_s X^*_s\xi^*_s\,ds\middle|\mathcal{F}_t\right] \tag{2.13}$$

Proof. For any $\xi \in \mathcal{A}_{\mathbb{F}}(t,x)$, let X^ξ be the corresponding state process. Then it holds that,

$$\lim_{s\nearrow T}\mathbb{E}\left[X^\xi_s Y_s\middle|\mathcal{F}_t\right] = 0. \tag{2.14}$$

Indeed, since $A \in \mathcal{M}_{-1}$, for any $0 \le t \le s < T$

$$
\begin{aligned}
\left|\mathbb{E}\left[X_s^\xi Y_s^* \middle| \mathcal{F}_t\right]\right| &= \left|\mathbb{E}\left[X_s^\xi (X_s^* A_s + B_s^*) \middle| \mathcal{F}_t\right]\right| \\
&\le \frac{C}{T-s}\mathbb{E}\left[(X_s^\xi)^2 + (X_s^*)^2 \middle| \mathcal{F}_t\right] + \mathbb{E}\left[|X_s^\xi B_s^*| \middle| \mathcal{F}_t\right] \\
&= \frac{C}{T-s}\mathbb{E}\left[\left(\int_s^T \xi_u\,du\right)^2 + \left(\int_s^T \xi_u^*\,du\right)^2 \middle| \mathcal{F}_t\right] + \mathbb{E}\left[|X_s^\xi B_s^*| \middle| \mathcal{F}_t\right] \\
&\le C\mathbb{E}\left[\int_s^T \xi_u^2\,du + \int_s^T (\xi_u^*)^2\,du \middle| \mathcal{F}_t\right] + \mathbb{E}\left[|X_s^\xi B_s^*| \middle| \mathcal{F}_t\right] \xrightarrow{s\nearrow T} 0.
\end{aligned}
$$

With this, we can now show that ξ^* is a best response against μ^*. In fact, for each $\varepsilon > 0$ and each $t \in [0, T-\varepsilon]$ the convexity of the Hamiltonian yields,

$$
\begin{aligned}
&\mathbb{E}\left[\int_t^{T-\varepsilon} \left(\kappa_s\mu_s^* X_s^\xi + \eta_s\xi_s^2 + \lambda_s(X_s^\xi)^2\right) ds \middle| \mathcal{F}_t\right] - \mathbb{E}\left[\int_t^{T-\varepsilon} \left(\kappa_s\mu_s^* X_s^* + \eta_s(\xi_s^*)^2 + \lambda_s(X_s^*)^2\right) ds \middle| \mathcal{F}_t\right] \\
&= \mathbb{E}\left[\int_t^{T-\varepsilon} \left(H(s,\xi_s,X_s^\xi,Y_s^*;\mu^*) - H(s,\xi_s^*,X_s^*,Y_s^*;\mu^*) + (\xi_s-\xi_s^*)Y_s\right) ds \middle| \mathcal{F}_t\right] \\
&\ge \mathbb{E}\left[\int_t^{T-\varepsilon} \left(\partial_\xi H(s,\xi_s^*,X_s^*,Y_s^*;\mu^*)(\xi_s-\xi_s^*) + \partial_x H(s,\xi_s^*,X_s^*,Y_s^*;\mu^*)(X_s^\xi - X_s^*) + (\xi_s-\xi_s^*)Y_s^*\right) ds \middle| \mathcal{F}_t\right] \\
&= \mathbb{E}\left[\int_t^{T-\varepsilon} \left((\kappa_s\mu_s^* + 2\lambda_s X_s^*)(X_s^\xi - X_s^*) + (\xi_s-\xi_s^*)Y_s^*\right) ds \middle| \mathcal{F}_t\right].
\end{aligned}
$$

Furthermore, integration by part implies that for any $\varepsilon > 0$,

$$
\begin{aligned}
&Y_{T-\varepsilon}^*(X_{T-\varepsilon}^* - X_{T-\varepsilon}^\xi) \\
&= Y_t^*(X_t^* - X_t^\xi) + \int_t^{T-\varepsilon} (X_s^* - X_s^\xi)\,dY_s^* + \int_t^{T-\varepsilon} Y_s^*\,d(X_s^* - X_s^\xi) \\
&= -\int_t^{T-\varepsilon} (\kappa_s\mu_s^* + 2\lambda_s X_s^*)(X_s^* - X_s^\xi)\,ds + \int_t^{T-\varepsilon} Z_s(X_s^* - X_s^\xi)\,d\widetilde{W}_s \\
&\quad - \int_t^{T-\varepsilon} Y_s^*(\xi_s^* - \xi_s)\,ds.
\end{aligned}
\tag{2.15}
$$

Therefore,

$$
\begin{aligned}
&\mathbb{E}\left[\int_t^{T-\varepsilon} \left(\kappa_s\mu_s^* X_s + \eta_s\xi_s^2 + \lambda_s X_s^2\right) ds \middle| \mathcal{F}_t\right] - \mathbb{E}\left[\int_t^{T-\varepsilon} \left(\kappa_s\mu_s^* X_s^* + \eta_s(\xi_s^*)^2 + \lambda_s(X_s^*)^2\right) ds \middle| \mathcal{F}_t\right] \\
&\ge \mathbb{E}\left[Y_{T-\varepsilon}^*(X_{T-\varepsilon}^* - X_{T-\varepsilon}) \middle| \mathcal{F}_t\right].
\end{aligned}
$$

Taking $\varepsilon \to 0$, the equation (2.14) does indeed yield,

$$
J(t,x,\xi;\mu^*) - J(t,x,\xi^*;\mu^*) \ge 0.
$$

In view of (2.15) and using (2.14) again yields (2.13) as,

$$
\begin{aligned}
&\mathbb{E}\left[\int_t^T (\kappa_s\xi_s^* X_s^* + \lambda_s(X_s^*)^2 + \eta_s(\xi_s^*)^2)\,ds \middle| \mathcal{F}_t\right] \\
&= \frac{1}{2}A_t(X_t^*)^2 + \frac{1}{2}B_t X_t^* + \frac{1}{2}\mathbb{E}\left[\int_t^T \kappa_s X_s^* \xi_s^*\,ds \middle| \mathcal{F}_t\right].
\end{aligned}
$$

□

Remark 2.10. Since $(X^*, B^*) \in \mathcal{H}_\alpha \times \mathcal{H}_\gamma$ and $\xi^* \in \mathcal{A}_{\mathcal{F}}(t,x)$,

$$\begin{aligned} &B_t^* x + \mathbb{E}\left[\int_t^T \kappa_s \xi_s^* X_s^* \, ds \middle| \mathcal{F}_t\right] \\ &\leq x(T-t)^\gamma \sup_{0\leq t\leq T}\left|\frac{B_t^*}{(T-t)^\gamma}\right| + \|\kappa\|(T-t)^\alpha \mathbb{E}\left[\int_0^T |\xi_s^*|\, ds \sup_{0\leq t\leq T}\left|\frac{X_t}{(T-t)^\alpha}\right| \middle| \mathcal{F}_t\right] \xrightarrow{t\nearrow T} 0. \end{aligned}$$

As a result, we get the following terminal condition for the value function:

$$\lim_{t\uparrow T} V(t,x;\mu) = \begin{cases} 0, x = 0; \\ \infty, x \neq 0. \end{cases}$$

2.2 Common information environment

In this section, we consider the benchmark case where all the randomness is generated by the Brownian motion W^0 that drives the benchmark price process. In particular, all players share the same information. In this case it turns out that both the optimal strategy and the optimal position are non-negative throughout the liquidation interval. We can not prove (and do not expect) a similar result under asymmetric information.

Assumption 2.11. The processes κ, λ, η and $1/\eta$ belong to $L^\infty_{\mathbb{F}^0}([0,T]\times\Omega;[0,\infty))$.

Under the above assumption the consistency condition reduces to

$$\mu = \xi^* \tag{2.16}$$

and the conditional mean-field FBSDE reduces to the following FBSDE

$$\begin{cases} dX_t = -\dfrac{Y_t}{2\eta_t}\, dt, \\ -dY_t = \left(\dfrac{\kappa_t Y_t}{2\eta_t} + 2\lambda_t X_t\right) dt - Z_t \, dW_t^0, \\ X_0 = x, \\ X_T = 0. \end{cases} \tag{2.17}$$

In particular, the optimization problem is now a time consistent one. The linear ansatz $Y = AX$ yields,

$$-dA_t = \left(2\lambda_t + \frac{\kappa_t A_t}{2\eta_t} - \frac{A_t^2}{2\eta_t}\right) dt - Z_t^A dW_t^0, \quad A_T = \infty. \tag{2.18}$$

This singular terminal condition on A is necessary to satisfy the constraint $X_T = 0$. This equation has a unique solution, due to Corollary A.2. By (2.17),

$$X_t^* = x e^{-\int_0^t \frac{A_r}{2\eta_r}\, dr}.$$

Lemma 2.12. *Under Assumption 2.11, the processes A, X^*, $Y = AX^*$ and $\xi^* = \mu = \frac{Y}{2\eta}$ are all non negative and*

$$A \in \mathcal{M}_{-1}, \; X^* \in \mathcal{M}_\alpha, \; Y \in \mathcal{M}_{\alpha-1}, \; \xi^* \in \mathcal{M}_{\alpha-1}.$$

Proof. Due to Lemma A.1, the following estimate holds for any $0 \leq t < T$:

$$\frac{1}{\mathbb{E}\left[\int_t^T \frac{1}{2\eta_s} e^{-\int_0^s \frac{\kappa_r}{2\eta_r}\, dr}\, ds \middle| \mathcal{F}_t^0\right]} \leq \widetilde{A}_t$$

Hence the process A_t is bounded from below by:

$$A_t \geq \frac{e^{-\int_0^t \frac{\kappa_r}{2\eta_r}\,dr}}{\mathbb{E}\left[\int_t^T \frac{1}{2\eta_s} e^{-\int_0^s \frac{\kappa_r}{2\eta_r}\,dr}\,ds \middle| \mathcal{F}_t^0\right]} = \frac{1}{\mathbb{E}\left[\int_t^T \frac{1}{2\eta_s} e^{-\int_t^s \frac{\kappa_r}{2\eta_r}\,dr}\,ds \middle| \mathcal{F}_t^0\right]}$$
$$\geq \frac{1}{\mathbb{E}\left[\int_t^T \frac{1}{2\eta_s}\,ds \middle| \mathcal{F}_t^0\right]} \geq 2\eta_\star \frac{1}{(T-t)}.$$

Hence,

$$e^{-\int_0^t \frac{A_r}{2\eta_r}\,dr} \leq \exp\left(-2\eta_\star \int_0^t \frac{1}{2\eta_r(T-r)}\,dr\right) \leq \left(\frac{T-t}{T}\right)^\alpha. \tag{2.19}$$

The conclusion on X^* can be deduced immediately. Again from Lemma A.1, $\widetilde{A}$ is bounded from above:

$$\widetilde{A}_t \leq \frac{1}{(T-t)^2}\mathbb{E}\left[\int_t^T \left(2\eta_s e^{\int_0^s \frac{\kappa_r}{2\eta_r}\,dr} + 2(T-s)^2\lambda_s e^{\int_0^s \frac{\kappa_r}{2\eta_r}\,dr}\right) ds \middle| \mathcal{F}_t^0\right].$$

Thus we get an upper bound on A:

$$A_t \leq \frac{e^{-\int_0^t \frac{\kappa_r}{2\eta_r}\,dr}}{(T-t)^2}\mathbb{E}\left[\int_t^T \left(2\eta_s e^{\int_0^s \frac{\kappa_r}{2\eta_r}\,dr} + 2(T-s)^2\lambda_s e^{\int_0^s \frac{\kappa_r}{2\eta_r}\,dr}\right) ds \middle| \mathcal{F}_t^0\right]$$
$$\leq \frac{2}{(T-t)^2}\left[\|\eta\| e^{\int_0^T \frac{\kappa_r}{2\eta_r}\,dr}(T-t) + \frac{1}{3}\|\lambda\| e^{\int_0^T \frac{\kappa_r}{2\eta_r}\,dr}(T-t)^3\right]$$
$$\leq \frac{2}{(T-t)} e^{\frac{\|\kappa\|T}{2\eta_\star}}\left[\|\eta\| + \frac{\|\lambda\|T^2}{3}\right].$$

Collecting all inequalities we get that $A \in \mathcal{M}_{-1}$ and

$$|\xi_t^*| = \frac{A_t|X_t^*|}{2\eta_t} = |x|\frac{A_t e^{-\int_0^t \frac{A_r}{2\eta_r}\,dr}}{2\eta_t}$$
$$\leq \frac{|x|}{\eta_\star T^\alpha}\left[\|\eta\| + \frac{\|\lambda\|T^2}{3}\right] e^{\frac{\|\kappa\|T}{2\eta_\star}} (T-t)^{\alpha-1}.$$

A similar inequality holds for Y. □

From the representation (2.17), we deduce that Y is a non negative supermartingale. In particular, the limit at the terminal time T of Y exists and is finite. Since $X^* \in \mathcal{M}_\alpha$, we deduce that $\lim_{t\nearrow T} Y_t X_t^* = 0$. Moreover the process Z belongs to $L^p_{\mathbb{F}^0}([0,T-]\times\Omega;[0,+\infty))$ for any p.

The following verification theorem shows that ξ^* is optimal. The proof is similar to that of Theorem 2.9.

Theorem 2.13. *Under Assumption 2.11, $\xi^*(=\mu^*)$ is an admissible optimal control as well as the equilibrium to MFG. Moreover the value function is given by:*

$$V(t,x;\mu^*) = \frac{1}{2}A_t x^2 + \frac{1}{2}\mathbb{E}\left[\int_t^T \kappa_s \mu_s^* X_s^*\,ds \middle| \mathcal{F}_t^0\right]. \tag{2.20}$$

2.3 An example

In this section, we consider a deterministic benchmark example that can be solved explicitly. We assume that the following assumption holds.

Assumption 2.14. The processes σ, λ, κ, η are positive constants.

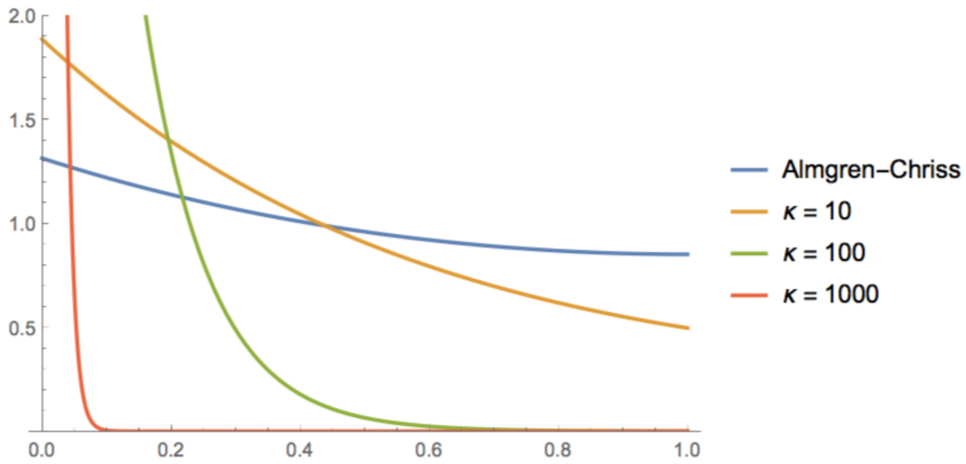

Figure 1: Optimal liquidation rate ξ^* corresponding to parameters $T = 1$, $X = 1$, $\lambda = 5$ and $\eta = 5$. The dashed line corresponds to $\kappa = 0$, that is the Almgren-Chriss model with temporary impact.

Under the preceding assumption, the Riccati equation (A.2) reduces to

$$-dA_t = \left(2\lambda + \frac{\kappa A_t}{2\eta} - \frac{A_t^2}{2\eta}\right) dt, \quad A_T = \infty,$$

whose explicit solution is

$$A_t = \frac{2\eta\left(\alpha_+ e^{\alpha_+ T} e^{\alpha_- t} - \alpha_- e^{\alpha_- T} e^{\alpha_+ t}\right)}{e^{\alpha_+ T} e^{\alpha_- t} - e^{\alpha_- T} e^{\alpha_+ t}},$$

where

$$\alpha_+ = \frac{\kappa + \sqrt{\kappa^2 + 16\eta\lambda}}{4\eta}, \quad \alpha_- = \frac{\kappa - \sqrt{\kappa^2 + 16\eta\lambda}}{4\eta}.$$

For the forward component of (2.17), we have

$$X_t^* = \frac{e^{\alpha_+(T-t)} - e^{\alpha_-(T-t)}}{e^{\alpha_+ T} - e^{\alpha_- T}} X.$$

Finally, we have the optimal liquidation rate as follows,

$$\xi_t^* = \frac{\alpha_+ e^{\alpha_+(T-t)} - \alpha_- e^{\alpha_-(T-t)}}{e^{\alpha_+ T} - e^{\alpha_- T}} X. \tag{2.21}$$

When $\kappa \to 0$, $\xi_t^* \to \frac{\gamma \cosh(\gamma(T-t))}{\sinh(\gamma T)} X$ with $\gamma = \sqrt{\frac{\lambda}{\eta}}$. This corresponds to the benchmark model in [1]. This convergence can also be seen from Figure 1 and Figure 2. Furthermore, we see that - as in the corresponding single player models - the optimal liquidation rate is always positive, i.e. round trips are no beneficial.

We also see that when the impact of interaction is strong, then the players trade very fast initially and slowly afterwards. The intuitive reason is that, when the interaction is strong, an individual player would benefit from trading fast slightly before his competitors start trading in order to avoid the negative drift generated by the mean-field interaction. As all the players are statistically identical, they "coordinate" on an equilibrium trading strategy as depicted in Figure 1. Thus, our model provides a possible explanation for large price increases or decreases in markets with strategically interacting players with similar preferences.

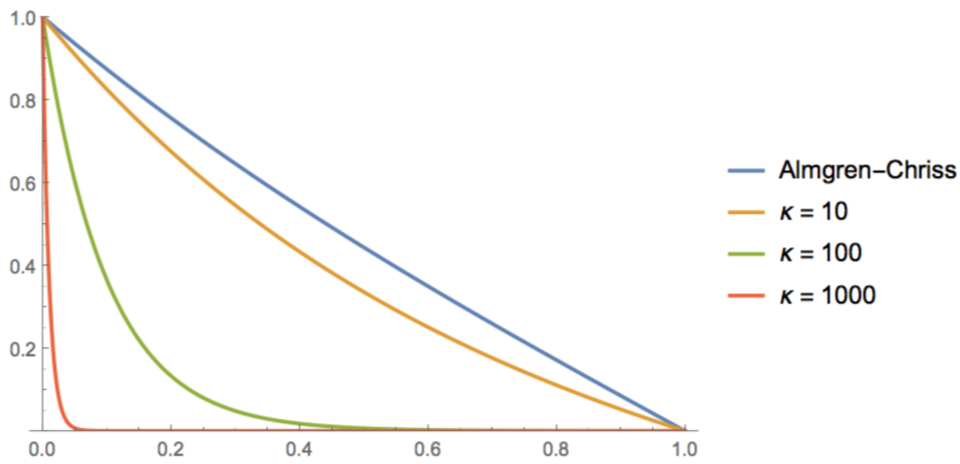

Figure 2: Current state X^* corresponding to parameters $T = 1$, $X = 1$, $\lambda = 5$ and $\eta = 5$. The dashed line corresponds to $\kappa = 0$, that is the Almgren-Chriss model with temporary impact.

3 Approximate Nash Equilibrium

In this section we show that a particular ϵ-Nash equilibrium for the N player game can be constructed using the solution to the MFG (1.7) when the number of players is large. As it is often the case in the MFG literature (see, e.g. [9]), the construction of approximate Nash equilibria requires additional assumptions. We shall assume that the market depth and the risk aversion parameter depend only on the common noise.

Assumption 3.1. The processes κ^i satisfy

$$\kappa^i \in L^\infty_{\mathbb{F}^i}([0,T] \times \Omega; [0,\infty)), \quad i = 1, .., N \text{ and } \{\kappa^i\} \text{ admit a common upper bound } \|\kappa\|$$

while

$$\eta^i, \lambda^i \in L^\infty_{\mathbb{F}^0}([0,T] \times \Omega; [0,\infty)).$$

In order to prove that under the preceding assumption the individual best responses against the mean-field equilibrium form an ϵ-Nash equilibrium in the original N player game when N is large enough, we introduce the benchmark cost functionals

$$J^i(\xi;\mu) := \mathbb{E}\left[\int_0^T \kappa^i_t \mu_t X^i_t + \eta^i_t (\xi^i_t)^2 + \lambda^i_t (X^i_t)^2 \, dt\right].$$

The result of last section yields that

$$J^i(\xi;\mu^{*,i}) \geq J^i(\xi^{*,i};\mu^{*,i}), \tag{3.1}$$

for any $\xi \in L^2_{\mathbb{F}^i}([0,T] \times \Omega; \mathbb{R})$, where

$$\begin{aligned} \xi^{*,i} &:= \frac{A^i X^{*,i} + B^{*,i}}{2\eta^i} \in D^2_{\mathbb{F}^i}([0,T] \times \Omega; \mathbb{R}) \\ \mu^{*,i}_t &:= \mathbb{E}\left[\xi^{*,i}_t \middle| \mathcal{F}^0_t\right], \ t \in [0,T) \end{aligned}$$

and $(X^{*,i}, B^{*,i}, A^i)$ are the solutions to the system (2.6) and (2.8), with κ, η, λ and W replaced by κ^i, η^i, λ^i and W^i, respectively. The following lemma shows that all the conditional expected actions $\mu^{*,i}$ coincide with the MFG equilibrium constructed in the previous sections.

Lemma 3.2. *Under Assumption 3.1 it holds for each $i = 1, ..., N$ that*

$$\mu_t^{*,i} = \mu_t^*, \qquad a.s.\ a.e.$$

where μ^* is given by Theorem 2.4. Moreover,

$$\mu_t^{*,i} = \frac{A_t \widetilde{X}_t}{2\eta_t} + \frac{\widetilde{B}_t}{2\eta_t},$$

where $\widetilde{X}$ and $\widetilde{B}$ are given by (3.5) *and* (3.6), *respectively.*

Proof. Under Assumption 3.1 the FBSDE system reduces to,

$$\begin{cases} -dA_t = \left(2\lambda_t - \frac{A_t^2}{2\eta_t}\right) dt - Z_t^A\, dW_t^0, \\ dX_t^i = -\frac{A_t X_t^i + B_t^i}{2\eta_t}\, dt \\ -dB_t^i = \left(\frac{\kappa_t^i A_t}{2\eta_t}\mathbb{E}\left[X_t^i | \mathcal{F}_t^0\right] + \frac{\kappa_t^i}{2\eta_t}\mathbb{E}\left[B_t^i | \mathcal{F}_t^0\right] - \frac{A_t B_t^i}{2\eta_t}\right) dt - Z_t^{B^i}\, d\widetilde{W}_t^i, \\ A_T = \infty, \\ X_0^i = x, \\ B_T^i = 0, \end{cases} \tag{3.2}$$

where $\widetilde{W}_t^i = (W^0, W^i)$. Let

$$\widetilde{X}_t^i := \mathbb{E}\left[X_t^i | \mathcal{F}_t^0\right], \qquad \widetilde{B}_t^i := \mathbb{E}\left[B_t^i | \mathcal{F}_t^0\right].$$

Then,

$$B_t^i = \mathbb{E}\left[\int_t^T \left(\frac{\kappa_s^i A_s}{2\eta_s}\widetilde{X}_s^i + \frac{\kappa_s^i}{2\eta_s}\widetilde{B}_s^i - \frac{A_s B_s^i}{2\eta_s}\right) ds \middle| \mathcal{F}_t\right].$$

Taking the conditional expectation $\mathbb{E}[\cdot|\mathcal{F}_t^0]$,

$$\begin{aligned} \widetilde{B}_t^i &= \mathbb{E}\left[\int_t^T \left(\frac{\kappa_s^i A_s}{2\eta_s}\widetilde{X}_s^i + \frac{\kappa_s^i}{2\eta_s}\widetilde{B}_s^i - \frac{A_s B_s^i}{2\eta_s}\right) ds \middle| \mathcal{F}_t^0\right] \\ &= \mathbb{E}\left[\int_0^T \left(\frac{\kappa_s^i A_s}{2\eta_s}\widetilde{X}_s^i + \frac{\kappa_s^i}{2\eta_s}\widetilde{B}_s^i - \frac{A_s B_s^i}{2\eta_s}\right) ds \middle| \mathcal{F}_t^0\right] - \mathbb{E}\left[\int_0^t \left(\frac{\kappa_s^i A_s}{2\eta_s}\widetilde{X}_s^i + \frac{\kappa_s^i}{2\eta_s}\widetilde{B}_s^i - \frac{A_s B_s^i}{2\eta_s}\right) ds \middle| \mathcal{F}_t^0\right] \\ &= \mathbb{E}\left[\int_0^T \left(\frac{\kappa_s^i A_s}{2\eta_s}\widetilde{X}_s^i + \frac{\kappa_s^i}{2\eta_s}\widetilde{B}_s^i - \frac{A_s B_s^i}{2\eta_s}\right) ds \middle| \mathcal{F}_t^0\right] - \int_0^t \left(\frac{\widetilde{\kappa}_s A_s}{2\eta_s}\widetilde{X}_s^i + \frac{\widetilde{\kappa}_s}{2\eta_s}\widetilde{B}_s^i - \frac{A_s \widetilde{B}_s^i}{2\eta_s}\right) ds, \end{aligned}$$

where

$$\widetilde{\kappa}_t = \mathbb{E}[\kappa_t^i | \mathcal{F}_t^0] = \mathbb{E}[\kappa_t^j | \mathcal{F}_t^0].$$

Hence, $\widetilde{B}_t^i + \int_0^t \left(\frac{\widetilde{\kappa}_s A_s}{2\eta_s}\widetilde{X}_s^i + \frac{\widetilde{\kappa}_s}{2\eta_s}\widetilde{B}_s^i - \frac{A_s \widetilde{B}_s^i}{2\eta_s}\right) ds$ is an $\mathbb{F}^0$-martingale. The martingale representation theorem yields the existence of some ζ^i such that

$$\widetilde{B}_t^i + \int_0^t \left(\frac{\widetilde{\kappa}_s A_s}{2\eta_s}\widetilde{X}_s^i + \frac{\widetilde{\kappa}_s}{2\eta_s}\widetilde{B}_s^i - \frac{A_s \widetilde{B}_s^i}{2\eta_s}\right) ds = \int_0^t \zeta_s^i\, dW_s^0. \tag{3.3}$$

Moreover, taking conditional expectation on both sides of the expression for X^i, we have

$$\widetilde{X}_t^i = x - \mathbb{E}\left[\int_0^t \frac{A_s X_s^i + B_s^i}{2\eta_s}\, ds \middle| \mathcal{F}_t^0\right] = x - \int_0^t \frac{A_s \widetilde{X}_s^i + \widetilde{B}_s^i}{2\eta_s}\, ds. \tag{3.4}$$

Thus $(\widetilde{X}^i, \widetilde{B}^i, \zeta^i)$ is the solution of the system

$$\begin{cases} -dA_t = \left(2\lambda_t - \frac{A_t^2}{2\eta_t}\right) dt - Z_t^A \, dW_t^0, \\ d\widetilde{X}_t^i = -\frac{A_t \widetilde{X}_t^i + \widetilde{B}_t^i}{2\eta_t} \, dt \\ -d\widetilde{B}_t^i = \left(\frac{\widetilde{\kappa}_t A_t}{2\eta_t} \widetilde{X}_t^i + \frac{\widetilde{\kappa}_t}{2\eta_t} \widetilde{B}_t^i - \frac{A_t \widetilde{B}_t^i}{2\eta_t}\right) dt - \zeta_t^i \, dW_t^0, \\ A_T = \infty, \\ \widetilde{X}_T^i = x, \\ \widetilde{B}_T^i = 0. \end{cases}$$

Let us now prove that the solution does not depend on i. By making the ansatz $\widetilde{B}^i = \widetilde{C}^i \widetilde{X}^i$, we have

$$-d\widetilde{C}^i = \left(\frac{\widetilde{\kappa}_t (A_t + \widetilde{C}_t^i)}{2\eta_t} - \frac{A_t \widetilde{C}_t^i}{\eta_t} - \frac{(\widetilde{C}^i)_t^2}{2\eta_t}\right) dt - Z_t^{\widetilde{C}^i} \, dW_t^0, \quad t \in [0, T).$$

Let $D^i := A + \widetilde{C}^i$. Then $d\widetilde{X}_t^i = -\frac{D_t^i}{2\eta_t} \widetilde{X}_t^i \, dt$, and D^i satisfies

$$-dD_t^i = \left(2\lambda_t + \frac{\widetilde{\kappa}_t D_t^i}{2\eta_t} - \frac{(D_t^i)^2}{2\eta_t}\right) dt - Z_t \, dW_t^0, \quad D_T = \infty.$$

The singular terminal condition is necessary to satisfy the liquidation constraint. By Corollary A.2 this equation has a unique solution. In particular, D^i and hence $\widetilde{X}^i$ are independent of i:

$$\widetilde{X}_t^i = \widetilde{X}_t = x e^{-\int_0^t \frac{D_s}{2\eta_s} \, ds}. \tag{3.5}$$

From (3.3) we have

$$\widetilde{B}_t^i = \widetilde{B}_t = \mathbb{E}\left[\int_t^T \frac{\widetilde{\kappa}_s A_s \widetilde{X}_s}{2\eta_s} \exp\left(-\int_t^s \frac{A_r - \widetilde{\kappa}_r}{2\eta_r} \, dr\right) ds \,\middle|\, \mathcal{F}_t^0\right]. \tag{3.6}$$

Hence $\widetilde{B}^i$ is independent of i as well. Hence,

$$\mu_t^{*,i} = \mathbb{E}\left[\frac{A_t X_t^i + B_t^i}{2\eta_t} \,\middle|\, \mathcal{F}_t^0\right] = \frac{A_t \widetilde{X}_t^i}{2\eta_t} + \frac{\widetilde{B}_t^i}{2\eta_t},$$

is independent of i, too. □

We are now ready to state and prove the main result of this section.

Theorem 3.3. *Assume that Assumption 3.1 is satisfied and that the admissible control space for each player $i = 1, ..., N$ is given by*

$$\mathcal{A}^i := \left\{\xi \in \mathcal{A}_{\mathbb{F}^i}(0, x) : \mathbb{E}\left[\int_0^T |\xi_t|^2 \, dt\right] \leq M\right\}$$

for some fixed positive constant M large enough. Then it holds for each $1 \leq i \leq N$ and each $\xi^i \in \mathcal{A}^i$ that

$$J^{N,i}\left(\vec{\xi^*}\right) \leq J^{N,i}(\xi^{*,-i}, \xi^i) + O\left(\frac{1}{\sqrt{N}}\right),$$

where $(\xi^{*,-i}, \xi^i) = (\xi^{*,1}, \cdots, \xi^{*,i-1}, \xi^i, \xi^{*,i+1}, \cdots, \xi^{*,N})$. In particular, the strategy profile $\vec{\xi^*}$ forms an ϵ-Nash equilibrium.

Proof. Let us compute

$$\mathbb{E}\left[\int_0^T \left(\mu_t^* - \frac{1}{N}\sum_{j=1}^N \xi_t^{*,j}\right)^2 dt\right] = \frac{1}{N^2}\mathbb{E}\left[\int_0^T \left(\sum_{j=1}^N \left(\mu_t^* - \xi_t^{*,j}\right)\right)^2 dt\right]$$

$$= \frac{1}{N^2}\mathbb{E}\left[\int_0^T \sum_{i\neq j}\left(\mu_t^* - \xi_t^{*,i}\right)\left(\mu_t^* - \xi_t^{*,j}\right) dt\right] + \frac{1}{N^2}\mathbb{E}\left[\int_0^T \sum_{j=1}^N \left(\mu_t^* - \xi_t^{*,j}\right)^2 dt\right]$$

$$\leq \frac{1}{N^2}\mathbb{E}\left[\int_0^T \sum_{i\neq j}\mathbb{E}\left[\left(\mu_t^* - \xi_t^{*,i}\right)\left(\mu_t^* - \xi_t^{*,j}\right)\Big|\mathcal{F}_t^0\right] dt\right] + \frac{2M}{N}.$$

Since $\xi^{*,i}$ and $\xi^{*,j}$ are conditionally independent given W^0 for $i \neq j$, the first term is equal to zero. Hence we obtain

$$\mathbb{E}\left[\int_0^T \left(\mu_t^* - \frac{1}{N}\sum_{j=1}^N \xi_t^{*,j}\right)^2 dt\right] \leq \frac{2M}{N}.$$

By the symmetry of the N player game, it is sufficient to show the result for Player 1. First, notice that $\xi^{*,1} \in \mathcal{A}^1$ if M is large enough. For each admissible strategy $\xi \in \mathcal{A}^1$, let X be the corresponding state process. By (3.1) we have that

$$\begin{aligned}
&J^{N,1}(\xi, \xi^{*,2}, \cdots, \xi^{*,N}) - J^{N,1}(\xi^{*,1}, \cdots, \xi^{*,N}) \\
&= J^{N,1}(\xi, \xi^{*,2}, \cdots, \xi^{*,N}) - J^1(\xi; \mu^*) + J^1(\xi; \mu^*) - J^1(\xi^{*,1}; \mu^*) + J^1(\xi^{*,1}; \mu^*) - J^{N,1}(\xi^{*,1}, \cdots, \xi^{*,N}) \\
&\geq \mathbb{E}\int_0^T \left[\kappa_t^1 \left(\frac{1}{N}\sum_{j=2}^N \xi_t^{*,j} + \frac{1}{N}\xi_t\right) X_t + \eta_t^1 \xi_t^2 + \lambda_t^1 X_t^2\right] dt \\
&\quad - \mathbb{E}\left[\int_0^T \left(\kappa_t^1 \mu_t^* X_t + \eta_t^1 \xi_t^2 + \lambda_t^1 X_t^2\right) dt\right] \\
&\quad + \mathbb{E}\left[\int_0^T \left(\kappa_t^1 \mu_t^* X_t^{*,1} + \eta_t^1 (\xi_t^{*,1})^2 + \lambda_t^1 (X_t^{*,1})^2\right) dt\right] \\
&\quad - \mathbb{E}\left[\int_0^T \left(\kappa_t^1 \frac{1}{N}\sum_{j=1}^N \xi_t^{*,j} X_t^{*,1} + \eta_t^1 (\xi_t^{*,1})^2 + \lambda_t^1 (X_t^{*,1})^2\right) dt\right] \\
&:= I_1 + I_2.
\end{aligned}$$

For the first difference I_1 in the above inequality, we have that

$$\begin{aligned}
\sup_{\xi\in\mathcal{A}^1} |I_1| &\leq \frac{\|\kappa\|}{N}\sup_{\xi\in\mathcal{A}^1}\mathbb{E}\left[\int_0^T |X_t||\xi_t|\, dt\right] + \|\kappa\|\sup_{\xi\in\mathcal{A}^1}\mathbb{E}\left[\int_0^T |X_t|\left|\frac{1}{N}\sum_{j=2}^N \xi_t^{*,j} - \mu_t^*\right| dt\right] \\
&\leq \frac{\|\kappa\|}{N}\sup_{\xi\in\mathcal{A}^1}\left(\mathbb{E}\left[\int_0^T |X_t|^2\, dt\right]\right)^{\frac{1}{2}}\sup_{\xi\in\mathcal{A}^1}\left(\mathbb{E}\left[\int_0^T |\xi_t|^2\, dt\right]\right)^{\frac{1}{2}} \\
&\quad + \|\kappa\|\sup_{\xi\in\mathcal{A}^1}\left(\mathbb{E}\left[\int_0^T |X_t|^2\, dt\right]\right)^{\frac{1}{2}}\left(\mathbb{E}\left[\int_0^T \left|\frac{1}{N}\sum_{j=1}^N \xi_t^{*,j} - \mu_t^* - \frac{1}{N}\xi_t^{*,1}\right| dt\right]\right)^{\frac{1}{2}} \\
&\leq \frac{M\|\kappa\|T}{N} + \frac{3\|\kappa\|TM}{\sqrt{N}}.
\end{aligned}$$

For the second difference I_2, we have that

$$I_2 \leq \|\kappa\| \left(\mathbb{E}\left[\int_0^T |X_t^{*,1}|^2 \, dt \right] \right)^{\frac{1}{2}} \left(\mathbb{E}\left[\int_0^T \left| \mu_t^* - \frac{1}{N} \sum_{j=1}^N \xi_t^{*,j} \right| \, dt \right] \right)^{\frac{1}{2}}$$
$$\leq \frac{\sqrt{2}\|\kappa\| T M}{\sqrt{N}}$$

This proves the assertion. □

Remark 3.4. When searching for the approximate Nash equilibria, we may as well assume that the individual players have full information. That is to say, we may assume that the admissible control space for each player is

$$\mathcal{A} := \left\{ \xi \in \mathcal{A}_{\overline{\mathbb{F}}^N}(0,x) : \mathbb{E}\left[\int_0^T |\xi_t|^2 \, dt \right] \leq M \right\},$$

where $\overline{\mathbb{F}}^N = (\overline{\mathcal{F}}_t^N, 0 \leq t \leq T)$ with $\overline{\mathcal{F}}_t^N := \sigma(W_t^0, W_t^1, \cdots, W_t^N)$. By the same argument as in Section 2.1, we have

$$J^i(\xi; \mu^*) \geq J^i(\xi^{*,i}; \mu^*),$$

for all $\xi \in \mathcal{A}_{\overline{\mathbb{F}}^N}(0,x)$. Thus, the same analysis as in Theorem 3.3 implies that for all $\xi^i \in \mathcal{A}_{\overline{\mathbb{F}}^N}(0,x)$

$$J^{N,i}\left(\vec{\xi^*}\right) \leq J^{N,i}(\xi^{*,-i}, \xi^i) + O\left(\frac{1}{\sqrt{N}}\right).$$

4 Approximation by unconstrained MFGs

In this section, we prove that the solution to our singular MFG can be approximated by the solutions to non-singular MFGs under additional assumptions on the market impact parameter. Specifically, we consider the following unconstrained MFGs:

$$\begin{cases} \text{1. fix a process } \mu; \\ \text{2. solve the standard optimization problem: minimize} \\ \quad J^n(\xi;\mu) = \mathbb{E}\left[\int_0^T \left(\kappa_t \mu_t X_t + \eta_t \xi_t^2 + \lambda_t X_t^2 \right) dt + n X_T^2 \right] \\ \quad \text{such that } dX_t = -\xi_t \, dt \quad X_0 = x; \\ \text{3. solve the fixed point equation } \mu_t^* = \mathbb{E}[\xi_t^* | \mathcal{F}_t^0] \ a.e.\ t \in [0,T], \\ \quad \text{where } \xi^* \text{ is the optimal strategy from 2.} \end{cases} \tag{4.1}$$

We will need the following assumption. It implies in particular that $X^* \in \mathcal{H}_1$.

Assumption 4.1. There exists a constant C such that for any $0 \leq r \leq s < T$

$$\exp\left(- \int_r^s \frac{A_u}{2\eta_u} du \right) \leq C \left(\frac{T-s}{T-r} \right).$$

The following result is proven in the appendix.

Lemma 4.2. *Assumption 4.1 holds under each of the following conditions:*

- η *is deterministic ;*

- $1/\eta$ *is a positive martingale ;*
- $1/\eta$ *has uncorrelated multiplicative increments, namely for any* $0 \le s \le t$

$$\mathbb{E}\left[\frac{\eta_s}{\eta_t}\middle|\mathcal{F}_s\right] = \mathbb{E}\left[\frac{\eta_s}{\eta_t}\right].$$

Using the same arguments as in Section 2, the unconstrained control problem leads to the following conditional mean field FBSDE

$$:\begin{cases} dX_t^n = \left(-\dfrac{A_t^n X_t^n + B_t^n}{2\eta_t}\right) dt, \\ -dB_t^n = \left(-\dfrac{A_t^n B_t^n}{2\eta_t} + \kappa_t \mathbb{E}\left[\dfrac{A_t^n X_t^n + B_t^n}{2\eta_t}\middle|\mathcal{F}_t^0\right]\right) dt - Z_t^{B^n} d\widetilde{W}_t, \\ dY_t^n = \left(-2\lambda_t X_t^n - \kappa_t \mathbb{E}\left[\dfrac{A_t^n X_t^n + B_t^n}{2\eta_t}\middle|\mathcal{F}_t^0\right]\right) dt + Z_t^{Y^n} d\widetilde{W}_t, \\ X_0^n = x, \\ B_T^n = 0, \\ Y_T^n = 2nX_T^n, \end{cases} \tag{4.2}$$

where

$$\begin{cases} -dA_t^n = \left\{2\lambda_t - \dfrac{(A_t^n)^2}{2\eta_t}\right\} dt - Z_t^{A^n} d\widetilde{W}_t, \\ A_T^n = 2n. \end{cases} \tag{4.3}$$

The existence of a solution (A^n, Z^{A^n}) to the BSDE (4.3) can be deduced from Lemma A.3. By the same lemma the sequence $\{A^n\}$ is a non decreasing sequence converging pointwise to A and there exists a constant $\mathfrak{C} > 0$ such for any n,

$$\|A^n\|_{\mathcal{M}_{-1}} + \|A^n\|_{\mathcal{M}_{-1}^n} \le \mathfrak{C},$$

where the space $\mathcal{M}_{n,\nu}$ is defined as

$$\mathcal{M}_\nu^n := \left\{U \in \mathcal{P}_\mathbb{F}([0,T]\times\Omega;\mathbb{R}\cup\{\infty\}) :\ \left(T - . + \frac{\eta_\star}{n}\right)^{-\nu} U_. \in L_\mathbb{F}^\infty([0,T]\times\Omega;\mathbb{R}\cup\{\infty\})\right\},$$

and endowed with the norm

$$\|U\|_{\mathcal{M}_\nu^n} := \operatorname*{ess\,sup}_{(t,\omega)\times[0,T]\times\Omega} \frac{|U_t|}{\left(T - t + \frac{\eta_\star}{n}\right)^\nu}.$$

We shall also need the following analogs to the space $\mathcal{H}_\nu$:

$$\mathcal{H}_\nu^n := \left\{U \in \mathcal{P}_\mathbb{F}([0,T]\times\Omega;\mathbb{R}\cup\{\infty\}) :\ \left(T - . + \frac{\eta_\star}{n}\right)^{-\nu} U_. \in S_\mathbb{F}^2([0,T]\times\Omega;\mathbb{R}\cup\{\infty\})\right\},$$

endowed with the norm

$$\|U\|_{n,\nu} := \left(\mathbb{E}\left[\sup_{0\le t\le T}\left|\frac{U_t}{\left(T - t + \frac{\eta_\star}{n}\right)^\nu}\right|^2\right]\right)^{\frac{1}{2}}.$$

The next result can be obtained using similar arguments as in the proof of Theorem 2.8. In fact, we have a slightly stronger result.

Theorem 4.3. *Assume Assumption 2.3 holds. For any fixed* $\mathfrak{p} \in [0,1]$ *and* $f \in L_\mathbb{F}^2([0,T]\times\Omega;\mathbb{R})$*, there exists a unique solution* $(X^n, B^n, Y^n, Z^{B^n}, Z^{Y^n}) \in \mathcal{H}_\alpha^n \times \mathcal{H}_\gamma^n \times S_\mathbb{F}^2([0,T]\times\Omega;\mathbb{R}) \times L_\mathbb{F}^2([0,T]\times\Omega;\mathbb{R}^m) \times$

$L^2_{\mathbb{F}}([0,T]\times\Omega;\mathbb{R}^m)$ *to the following system*

$$\begin{cases} dX^n_t = -\dfrac{1}{2\eta_t}(A^n_t X^n_t + B^n_t)\,dt, \\ -dB^n_t = \left(\kappa_t \mathfrak{p}\mathbb{E}\left[\dfrac{1}{2\eta_t}(A^n_t X^n_t + B^n_t)\middle|\mathcal{F}^0_t\right] + f_t - \dfrac{A^n_t B^n_t}{2\eta_t}\right)dt - Z^{B^n}_t\,d\widetilde{W}_t, \\ dY^n_t = \left(-2\lambda_t X^n_t - \kappa_t\mathfrak{p}\mathbb{E}\left[\dfrac{A^n_t X^n_t + B^n_t}{2\eta_t}\middle|\mathcal{F}^0_t\right] - f_t\right)dt + Z^{Y^n}_t\,d\widetilde{W}_t, \\ X^n_0 = x \\ B^n_T = 0, \\ Y^n_T = 2nX^n_T. \end{cases} \tag{4.4}$$

Proof. The proof is similar to that of Theorem 2.8. We only need to note that by Lemma A.3,

$$e^{-\int_s^t \frac{A^n_r}{2\eta_r}\,dr} \le \left(\frac{T-t+\frac{\eta_\star}{n}}{T-s+\frac{\eta_\star}{n}}\right)^\alpha.$$

□

In order to establish the convergence of the value functions of the unconstrained problems to the value function of the constrained problem we need a uniform norm estimate for the sequence (X^n, B^n, Y^n).

Lemma 4.4. *Let Assumption 2.3 hold. If $f \in L^2_{\mathbb{F}}([0,T]\times\Omega;\mathbb{R})$, there exists a constant $\overline{\mathfrak{C}} > 0$ such that*

$$\|X^n\|_{n,\alpha} + \|B^n\|_{n,\gamma} + \mathbb{E}\left[\int_0^T |Y^n_t|^2\,dt\right] \le \overline{\mathfrak{C}}, \tag{4.5}$$

for any n where (X^n, B^n, Y^n) is the unique solution to (4.2).

Proof. The proof is split into three steps.

Step 1. When $\mathfrak{p} = 0$ in (4.4), there exists $R \in \mathbb{R}$ independent of n such that

$$\|X^n\|_{n,\alpha} + \|B^n\|_{n,\gamma} + \left(\mathbb{E}\left[\int_0^T |Y^n_t|^2\,dt\right]\right)^{\frac{1}{2}} \le R.$$

This bound follows from modifications of arguments given in the proof of Lemma 2.6. In fact,

$$\|B^n\|_{n,\gamma} \le \|B^n\|_\gamma \le C\|f\|_{L^2} \le R_1.$$

Moreover,

$$|X^n_t| \le \frac{x(T-t+\frac{\eta_\star}{n})^\alpha}{(T+\frac{\eta_\star}{n})^\alpha} + C\int_0^t |B^n_s|\left(\frac{T-t+\frac{\eta_\star}{n}}{T-s+\frac{\eta_\star}{n}}\right)^\alpha ds.$$

This implies $\|X^n\|_{n,\alpha} \le R_2$. Finally, by analogy to the proof of Lemma 2.7, doing integration by part for $X^n Y^n$, we have

$$\mathbb{E}\left[\int_0^T (Y^n_t)^2\,dt\right] \le C\mathbb{E}\left[\int_0^T (X^n_t)^2\,dt\right] + C\mathbb{E}\left[\int_0^T f_t^2\,dt\right] \le R_3.$$

Step 2. Suppose that for some $\mathfrak{p} \in [0,1]$, the solution to (4.4) satisfies

$$\|X^n\|_{n,\alpha} + \|B^n\|_{n,\gamma} + \left(\mathbb{E}\left[\int_0^T |Y_t^n|^2\,dt\right]\right)^{\frac{1}{2}} \le kR,$$

for some $k \ge 1$ independent of n. Then there exists $\mathfrak{d} > 0$ independent of $\mathfrak{p}$ such that the solution $(\widetilde{X}^n, \widetilde{B}^n, \widetilde{Y}^n)$ to (4.4) with $\mathfrak{p}$ replaced by $\mathfrak{p} + \mathfrak{d}$ satisfies the same estimate for some $K > k$:

$$\|\widetilde{X}^n\|_{n,\alpha} + \|\widetilde{B}^n\|_{n,\gamma} + \left(\mathbb{E}\left[\int_0^T |\widetilde{Y}_t^n|^2\,dt\right]\right)^{\frac{1}{2}} \le KR. \tag{4.6}$$

To prove this assertion, we introduce for any given $Y^n, f \in L^2_{\mathbb{F}}([0,T]\times\Omega;\mathbb{R})$ the FBSDE system

$$\begin{cases} d\widetilde{X}_t^n = -\dfrac{1}{2\eta_t}(A_t^n \widetilde{X}_t^n + \widetilde{B}_t^n)\,dt, \\ -d\widetilde{B}_t^n = \left(\kappa_t \mathfrak{p}\mathbb{E}\left[\dfrac{1}{2\eta_t}\left(A_t^n\widetilde{X}_t^n + \widetilde{B}_t^n\right)\middle|\mathcal{F}_t^0\right] + \kappa_t\mathfrak{d}\mathbb{E}\left[\dfrac{Y_t^n}{2\eta_t}\middle|\mathcal{F}_t^0\right] + f_t - \dfrac{A_t^n\widetilde{B}_t^n}{2\eta_t}\right)dt - Z_t^{\widetilde{B}^n}\,d\widetilde{W}_t, \\ d\widetilde{Y}_t^n = \left(-2\lambda_t\widetilde{X}_t^n - \kappa_t\mathfrak{p}\mathbb{E}\left[\dfrac{A_t^n\widetilde{X}_t^n + \widetilde{B}_t^n}{2\eta_t}\middle|\mathcal{F}_t^0\right] - \kappa_t\mathfrak{d}\mathbb{E}\left[\dfrac{Y_t^n}{2\eta_t}\middle|\mathcal{F}_t^0\right] - f_t\right)dt + Z_t^{\widetilde{Y}^n}\,d\widetilde{W}_t, \\ \widetilde{X}_0^n = x \\ \widetilde{B}_T^n = 0, \\ \widetilde{Y}_T^n = 2n\widetilde{X}_T^n. \end{cases} \tag{4.7}$$

By Theorem 4.3, there exists a unique solution to (4.7). This defines a mapping

$$\Gamma : Y^n \to \widetilde{Y}^n.$$

on $L^2_{\mathbb{F}}([0,T]\times\Omega;\mathbb{R})$. We now show the Γ has a unique fixed point and that this fixed point belongs to $B^{L^2}_{2kR}(0)$, the subset of $L^2_{\mathbb{F}}([0,T]\times\Omega;\mathbb{R})$ such that the L^2-norm is bounded by $2kR$.

By the same arguments as in the proof as Lemma 2.7 we have,

$$\mathbb{E}\left[\int_0^T |\Gamma(Y^n)(t) - \Gamma((\overline{Y}^n))(t)|^2\,dt\right] \le C\mathfrak{d}\mathbb{E}\left[\int_0^T |Y_t^n - \overline{Y}_t^n|^2\,dt\right] \le \frac{1}{4}\mathbb{E}\left[\int_0^T |Y_t^n - \overline{Y}_t^n|^2\,dt\right],$$

where $\mathfrak{d}$ is small enough but independent of $\mathfrak{p}$. Taking $\overline{Y}^n = 0$, we have

$$\left(\mathbb{E}\left[\int_0^T |\widetilde{Y}_t^n|^2\,dt\right]\right)^{\frac{1}{2}} \le \frac{1}{2}\left(\mathbb{E}\left[\int_0^T |Y_t^n|^2\,dt\right]\right)^{\frac{1}{2}} + \left(\mathbb{E}\left[\int_0^T |\Gamma(0)(t)|^2\,dt\right]\right)^{\frac{1}{2}}.$$

Note that $\Gamma(0)$ corresponds to the solution to (4.4) with $\mathfrak{p}$. By assumption,

$$\left(\mathbb{E}\left[\int_0^T |\Gamma(0)(t)|^2\,dt\right]\right)^{\frac{1}{2}} \le kR.$$

Thus, if we assume $Y^n \in B^{L^2}_{2kR}$,

$$\left(\mathbb{E}\left[\int_0^T |\widetilde{Y}_t^n|^2\,dt\right]\right)^{\frac{1}{2}} \le 2kR.$$

This implies that Γ is a mapping from $B^{L^2}_{2kR}(0)$ to itself. Since $B^{L^2}_{2kR}(0)$ is a Banach space the unique fixed point belongs to $B^{L^2}_{2kR}(0)$. This yields the desired L^2 estimate fro $\widetilde{Y}$.

Let $(\widetilde{X}^n, \widetilde{B}^n)$ be the solution corresponding to $\widetilde{Y}^n$ and $\mathfrak{p}+\mathfrak{d}$. Then, by Hölder's inequality,

$$\frac{\left|\widetilde{B}^n_t\right|}{(T-t)^\gamma} \le \frac{1}{(T-t)^\gamma}\mathbb{E}\left[\int_t^T \kappa_s(\mathfrak{p}+\mathfrak{d})\mathbb{E}\left[\frac{|\widetilde{Y}^n_s|}{2\eta_s}\middle|\mathcal{F}^0_s\right] ds\middle|\mathcal{F}_t\right]$$
$$\le \frac{\|\kappa\|}{2\eta_\star}\left(\mathbb{E}\left[\int_t^T \mathbb{E}\left[|\widetilde{Y}^n_s|^{\frac{1}{1-\gamma}}\middle|\mathcal{F}^0_s\right] ds\middle|\mathcal{F}_t\right]\right)^{1-\gamma}.$$

Doob's maximal inequality yields that

$$\mathbb{E}\left[\sup_{0\le t\le T}\left|\frac{\widetilde{B}^n_t}{(T-t)^\gamma}\right|^2\right]$$
$$\le \frac{\mathfrak{d}\|\kappa\|^2}{4\eta_\star^2}\mathbb{E}\left[\sup_{0\le t\le T}\left(\mathbb{E}\left[\int_t^T \mathbb{E}\left[|\widetilde{Y}^n_s|^{\frac{1}{1-\gamma}}\middle|\mathcal{F}^0_s\right] ds\middle|\mathcal{F}_t\right]\right)^{2(1-\gamma)}\right]$$
$$\le C\mathbb{E}\left[\int_0^T |\widetilde{Y}^n_t|^2\,dt\right].$$

Hence,

$$\|\widetilde{B}^n\|_{n,\gamma} \le C\left(\mathbb{E}\left[\int_0^T |\widetilde{Y}^n_t|^2\,dt\right]\right)^{\frac{1}{2}} \le CR$$

and

$$\|\widetilde{X}^n\|_{n,\alpha} \le C\|\widetilde{B}^n\|_{n,\gamma} \le CR.$$

Step 3. Since $\mathfrak{d}$ is independent of $\mathfrak{p}$, by iteration for only finitely many times, we have the solution for (4.2) with $\mathfrak{p}=1$ and $f=0$ with the uniform estimate (4.5). □

We see that under Assumption 4.1, we may take $\alpha = 1$ in the estimate of Theorem 2.8. That is

$$\|X^*\|_1 < \infty. \tag{4.8}$$

This allows us to prove the convergence of the optimal position and control.

Lemma 4.5. *Under Assumption 2.3 and Assumption 4.1,*

$$\lim_{n\to+\infty}\left\{\mathbb{E}\left[\int_0^T |X^n_t - X^*_t|^2\,dt\right] + \mathbb{E}\left[\int_0^T |B^n_t - B^*_t|^2\,dt\right] + \mathbb{E}\left[\int_0^T |Y^n_t - Y^*_t|^2\,dt\right]\right\} = 0.$$

Proof. Using the same arguments as in the proof of Lemma 2.7, we get for each $\epsilon > 0$

$$\begin{aligned}&\mathbb{E}\left[\int_0^{T-\epsilon} |Y^n_t - Y^*_t|^2\,dt\right] + \mathbb{E}\left[\int_0^{T-\epsilon} |X^n_t - X^*_t|^2\,dt\right]\\ &\le C\mathbb{E}\left[|(B^n_{T-\epsilon} - B^*_{T-\epsilon})(X^n_{T-\epsilon} - X^*_{T-\epsilon})|\right] + C\mathbb{E}\left[|(A^n_{T-\epsilon} - A_{T-\epsilon})X^*_{T-\epsilon}(X^n_{T-\epsilon} - X^*_{T-\epsilon})|\right].\end{aligned} \tag{4.9}$$

The two terms in the above summation admit the following estimates

$$\mathbb{E}[|(B^n_{T-\epsilon} - B^*_{T-\epsilon})(X^n_{T-\epsilon} - X^*_{T-\epsilon})|]$$
$$\le C\mathbb{E}[|B^n_{T-\epsilon}|^2] + C\mathbb{E}[|B^*_{T-\epsilon}|^2] + C\mathbb{E}[|X^n_{T-\epsilon}|^2] + C\mathbb{E}[|X^*_{T-\epsilon}|^2]$$
$$\le C\left(\epsilon + \frac{1}{n}\right)^{2\gamma}\|B^n\|^2_{n,\gamma} + C\left(\epsilon + \frac{1}{n}\right)^{2\alpha}\|X^n\|^2_{n,\alpha} + C\epsilon^{2\gamma}\|B^*\|^2_\gamma + C\epsilon^2\|X^*\|^2_1,$$

respectively,

$$
\begin{aligned}
&\mathbb{E}[|(A^n_{T-\epsilon} - A_{T-\epsilon})X^*_{T-\epsilon}(X^n_{T-\epsilon} - X^*_{T-\epsilon})|] \\
&\le C\mathbb{E}\left[\sup_{0\le t\le T}\left|\frac{X^*_t}{T-t}\right|\left(\sup_{0\le t\le T}\frac{|X^n_t|}{(T-t+\frac{\eta_\star}{n})^\alpha}\right)\left(\epsilon+\frac{1}{n}\right)^\alpha + \epsilon\sup_{0\le t\le T}\frac{|X^*_t|}{T-t}\right] \quad \text{(by Lemma } A.1 \text{ and Lemma } A.3\text{)} \\
&\le C\left[\left(\epsilon+\frac{1}{n}\right)^\alpha + \epsilon\right](\|X^n\|^2_{n,\alpha} + \|X^*\|^2_1) \\
&\le C\left[\left(\epsilon+\frac{1}{n}\right)^\alpha + \epsilon\right] \quad \text{(by Lemma 4.4 and (4.8)).}
\end{aligned}
$$

Letting ϵ go to zero in (4.9), by Theorem 2.8 and Lemma 4.4 we get

$$
\mathbb{E}\left[\int_0^T |Y^n_t - Y^*_t|^2\,dt\right] + \mathbb{E}\left[\int_0^T |X^n_t - X^*_t|^2\,dt\right] \le C\left(\frac{1}{n}\right)^{2\gamma} + C\left(\frac{1}{n}\right)^2 + \frac{C}{n^\alpha}.
$$

Hence we obtain the desired limit for $(Y^n - Y^*)$ and $(X^n - X^*)$. By the expression for B, we have

$$
\begin{aligned}
|B^n_t - B^*_t| \le{}& \mathbb{E}\left[\int_t^T e^{-\int_t^s \frac{A^n_r}{2\eta_t}\,dr}\kappa_s\mathbb{E}\left[\frac{|Y^n_s - Y^*_s|}{2\eta_s}\middle|\mathcal{F}^0_s\right]ds\middle|\mathcal{F}_t\right] \\
&+ \mathbb{E}\left[\int_t^T \left|1 - e^{-\int_t^s \frac{(A_r - A^n_r)}{2\eta_t}\,dr}\right|\kappa_s\mathbb{E}\left[\frac{|Y^*_s|}{2\eta_s}\middle|\mathcal{F}^0_s\right]ds\middle|\mathcal{F}_t\right].
\end{aligned}
$$

Let us recall that $\{A^n\}$ is a non decreasing sequence converging to A. This leads to

$$
\mathbb{E}\left[\int_0^T |B^n_t - B^*_t|^2\,dt\right] \to 0.
$$

□

Let us denote by $V^n(t,x;\mu^n)$ the value function associated with the penalized problem (4.1). The next theorem shows the convergence of $V^n(0,x;\mu^n) := V^n(x)$ to the value function $V(0,x;\mu) := V(x)$ associated with the constrained MFG.

Theorem 4.6. *Under Assumption 2.3 and Assumption 4.1, the value function $V^n(x)$ converges to $V(x)$.*

Proof. Any admissible control ξ of the original problem is admissible for this penalized setting. Hence we have immediately that $V^n \le V$. By Lemma 4.5, we have

$$
\begin{aligned}
V(x) &= \mathbb{E}\left[\int_0^T \kappa_s\mathbb{E}\left[\frac{Y^*_s}{2\eta_s}\middle|\mathcal{F}^0_s\right]X^*_s + \eta_s(\xi^*_s)^2 + \lambda_s(X^*_s)^2\,ds\right] \\
&\ge \mathbb{E}\left[\int_0^T \left(\kappa_s\mathbb{E}\left[\frac{Y^n_s}{2\eta_s}\middle|\mathcal{F}^0_s\right]X^n_s + \eta_s(\xi^n_s)^2 + \lambda_s(X^n_s)^2\right)ds + n(X^n_T)^2\right] \\
&= V^n(x) \qquad (4.10) \\
&\ge \mathbb{E}\left[\int_0^T \left(\kappa_s\mathbb{E}\left[\frac{Y^n_s}{2\eta_s}\middle|\mathcal{F}^0_s\right]X^n_s + \eta_s(\xi^n_s)^2 + \lambda_s(X^n_s)^2\right)ds\right] \\
&\to \mathbb{E}\left[\int_0^T \left(\kappa_s\mathbb{E}\left[\frac{Y^*_s}{2\eta_s}\middle|\mathcal{F}^0_s\right]X^*_s + \eta_s(\xi^*_s)^2 + \lambda_s(X^*_s)^2\right)ds\right].
\end{aligned}
$$

□

Remark 4.7. As a by-product of the proof, we get that $\lim_{n\to+\infty} \mathbb{E}\left[n(X_T^n)^2\right] = 0$. Moreover,

$$|X_T^n| \le \frac{C}{n}\left(x + \sup_{0\le t\le T} \frac{|B_t^n|}{\left(T - t + \frac{\eta_\star}{n}\right)^\gamma}\right) \to 0 \quad \text{a.s..}$$

The proof of convergence of the value function simplifies substantially under the common information assumption (Section 2.2). In this case, $Y^n = A^n X^n$ where

$$-dA_t^n = \left(2\lambda_t + \frac{\kappa_t A_t^n}{2\eta_t} - \frac{(A_t^n)^2}{2\eta_t}\right) dt - Z_t^{A^n} dW_t^0, \quad A_T^n = 2n$$

and

$$dX_t^n = -\frac{A_t^n X_t^n}{2\eta_t}\, dt, \; X_0 = x.$$

The optimal strategy and the resulting portfolio process are given by, respectively,

$$\xi_t^{n,*} = \mu_t^{n,*} = \frac{A_t^n X_t^n}{2\eta_t}, \quad X_t^{n,*} = x e^{-\int_0^t \frac{A_r^n}{2\eta_r}\, dr} \qquad t \in [0,T].$$

Since the sequence A^n is non decreasing and converges to A, we deduce that $X^{n,*}$ converges to X^* a.s. and that $\xi^{n,*}$ converges to ξ^* a.e. a.s.. For any admissible strategy $\xi \in \mathcal{A}_{\mathbb{F}^0}(t,x)$ with associated portfolio process X,

$$\begin{aligned}
&\mathbb{E}\left[\int_t^T \left(\kappa_s \xi_s X_s + \eta_s \xi_s^2 + \lambda_s X_s^2\right) ds \,\middle|\, \mathcal{F}_t^0\right] \\
&\ge \mathbb{E}\left[\int_t^T \left(\kappa_s \xi_s^{n,*} X_s^{n,*} + \eta_s (\xi_s^{n,*})^2 + \lambda_s (X_s^{n,*})^2\right) ds + n(X_T^{n,*})^2 \,\middle|\, \mathcal{F}_t^0\right] \\
&\ge \mathbb{E}\left[\int_t^T \left(\kappa_s \xi_s^{n,*} X_s^{n,*} + \eta_s (\xi_s^{n,*})^2 + \lambda_s (X_s^{n,*})^2\right) ds \,\middle|\, \mathcal{F}_t^0\right].
\end{aligned}$$

For any $\varepsilon > 0$,

$$\begin{aligned}
&\lim_{n\to+\infty} \mathbb{E}\left[\int_t^{T-\varepsilon} \left(\kappa_s \xi_s^{n,*} X_s^{n,*} + \eta_s (\xi_s^{n,*})^2 + \lambda_s (X_s^{n,*})^2\right) ds \,\middle|\, \mathcal{F}_t^0\right] \\
&= \mathbb{E}\left[\int_t^{T-\varepsilon} \left(\kappa_s \xi_s^* X_s^* + \eta_s (\xi_s^*)^2 + \lambda_s (X_s^*)^2\right) ds \,\middle|\, \mathcal{F}_t^0\right].
\end{aligned}$$

Hence, the monotone convergence theorem gives the desired convergence.

A Appendix

In this appendix we recall an existence of solutions result for a stochastic Riccati equation with singular terminal condition and prove Lemma 4.2. We assume throughout that λ, η and $1/\eta$ are bounded.

A.1 Stochastic Riccati equations with singular terminal value

Lemma A.1. *[2, Theorem 2.2][14, Theorem 6.1, Theorem 6.3] In $L^2_{\mathbb{F}}(\Omega; C[0,T-]) \times L^2_{\mathbb{F}}([0,T-];\mathbb{R}^m)$ there exists a unique solution to*

$$\begin{cases} -dA_t = \left(2\lambda_t - \dfrac{A_t^2}{2\eta_t}\right) dt - Z_t^A\, dW_t, \\ A_T = \infty. \end{cases}$$

Moreover, there holds the following estimate

$$\frac{1}{\mathbb{E}\left[\int_t^T \frac{1}{2\eta_s}\, ds \middle| \mathcal{F}_t\right]} \leq A_t \leq \frac{1}{(T-t)^2}\mathbb{E}\left[\int_t^T 2\eta_s + 2(T-s)^2\lambda_s\, ds \middle| \mathcal{F}_t\right]. \tag{A.1}$$

Corollary A.2. *The BSDE*

$$\begin{cases} -dA_t = & \left(2\lambda_t + \dfrac{\kappa_t A_t}{2\eta_t} - \dfrac{A_t^2}{2\eta_t}\right) dt - Z_t^A dW_t^0, \\ A_T = & \infty. \end{cases} \tag{A.2}$$

has a unique solution.

Proof. Let $\widetilde{A}_t = A_t e^{\int_0^t \frac{\kappa_s}{2\eta_s}\, ds}$. Then,

$$\begin{cases} -d\widetilde{A}_t = \left[2\lambda_t e^{\int_0^t \frac{\kappa_s}{2\eta_s}\, ds} - \dfrac{\widetilde{A}_t^2}{2\eta_t e^{\int_0^t \frac{\kappa_s}{2\eta_s}\, ds}}\right] dt - \widetilde{Z}_t\, dW_t^0, \\ \widetilde{A}_T = \infty. \end{cases} \tag{A.3}$$

Hence, the assertion follows from the preceding lemma. □

Lemma A.3. *For each n, there exists a unique solution A^n to the BSDE*

$$\begin{cases} -dA_t^n = & \left(2\lambda_t - \dfrac{(A_t^n)^2}{2\eta_t}\right) dt - Z_t^{A^n}\, dW_t, \\ A_T^n = & 2n. \end{cases} \tag{A.4}$$

The solution admits the following estimate:

$$A_t^n \geq \frac{1}{\frac{1}{2n} + \mathbb{E}\left[\int_t^T \frac{1}{2\eta_s}\, ds \middle| \mathcal{F}_t\right]}.$$

Moreover, the sequence A^n is non decreasing and converges to A. There exists a constant $\mathfrak{C}$ such that for any n:

$$\|A^n\|_{\mathcal{M}_{-1}} + \|A^n\|_{\mathcal{M}_{-1}^n} \leq \mathfrak{C}.$$

Proof. The first and second assertions are results of [2, Proposition 3.1,Theorem 3.2], respectively. For any t, n and a, we have

$$2\lambda_t - \frac{a^2}{2\eta_t} \leq 2\lambda_t - \frac{2}{\left(T-t+\frac{\eta_\star}{n}\right)}a + \frac{2\eta_t}{\left(T-t+\frac{\eta_\star}{n}\right)^2} = g(t,a).$$

Let us denote by Ψ^n the solution of the BSDE with generator g and terminal condition $2n$. By the comparison principle for BSDEs, we have $A_t^n \leq \Psi_t^n$ and by the solution formula for linear BSDEs,

$$\begin{aligned} \Psi_t^n &= \left(\frac{T+\frac{\eta_\star}{n}}{T-t+\frac{\eta_\star}{n}}\right)^2 \mathbb{E}\left[\left(\frac{\frac{\eta_\star}{n}}{T+\frac{\eta_\star}{n}}\right)^2 2n + \int_t^T \left(\frac{T-s+\frac{\eta_\star}{n}}{T+\frac{\eta_\star}{n}}\right)^2 \left(\frac{2\eta_s}{\left(T-s+\frac{\eta_\star}{n}\right)^2} + 2\lambda_s\right) \middle| \mathcal{F}_t\right] \\ &= \frac{2\eta_\star^2}{n}\frac{1}{\left(T-t+\frac{\eta_\star}{n}\right)^2} + \frac{1}{\left(T-t+\frac{\eta_\star}{n}\right)^2}\mathbb{E}\left[\int_t^T \left(2\eta_s + 2\left(T-s+\frac{\eta_\star}{n}\right)^2 \lambda_s\right) \middle| \mathcal{F}_t\right]. \end{aligned}$$

Hence

$$\begin{aligned}
&\left(T-t+\frac{\eta_\star}{n}\right)\Psi_t^n \\
&\le \frac{2\eta_\star^2}{\eta_\star+n(T-t)} + \frac{1}{(T-t+\frac{\eta_\star}{n})}\mathbb{E}\left[\int_t^T \left(2\eta_s + 2\left(T-s+\frac{\eta_\star}{n}\right)^2\lambda_s\right)\bigg|\mathcal{F}_t\right] \\
&\le 2\eta_\star + \frac{1}{T-t}\mathbb{E}\left[\int_t^T \left(2\eta_s + 2\left(T-s+\frac{\eta_\star}{n}\right)^2\lambda_s\right)\bigg|\mathcal{F}_t\right] = \mathfrak{C}.
\end{aligned}$$

Thus $\left(T-t+\frac{\eta_\star}{n}\right)A_t^n \le \mathfrak{C}$, that is $\|A^n\|_{\mathcal{M}_{-1}^n} \le \mathfrak{C}$. □

A.2 On Assumption 4.1

Assumption 4.1 states that there exists a constant C such that a.s. for any $0 \le r \le s < T$

$$\exp\left(-\int_r^s \frac{A_u}{2\eta_u}du\right) \le C\left(\frac{T-s}{T-r}\right).$$

The left-hand side is equal to the optimal state process χ of the optimal control problem studied in [2, 14] with initial value equal to 1 at time r. In particular from the proof of [2, Theorem 4.2], the process M defined on $[r,T)$ by

$$M_s = \frac{1}{A_r}\left[A_s\chi_s + 2\int_r^s \lambda_u\chi_u du\right]$$

is a non-negative local martingale with $M_r = 1$. Hence for any $s \in [r,T)$

$$\exp\left(-\int_r^s \frac{A_u}{2\eta_u}du\right) = \chi_s \le \frac{A_r}{A_s}M_s \le \frac{\|\eta\|+T\|\lambda\|}{\eta_\star}\left(\frac{T-s}{T-r}\right)M_s = C\left(\frac{T-s}{T-r}\right)M_s.$$

Since M is also a non-negative supermartingale M_t converges almost surely as t goes to T and the limit M_T satisfies $\mathbb{E}(M_T) \le 1$. Therefore Assumption 4.1 does not strike us as overly restrictive.

Proof of Lemma 4.2. From (A.1)

$$-\frac{A_u}{2\eta_u} \le -\frac{1}{\mathbb{E}\left[\int_u^T \frac{\eta_u}{\eta_s}ds\Big|\mathcal{F}_u\right]} = -\frac{1}{\int_u^T \mathbb{E}\left[\frac{\eta_u}{\eta_s}\Big|\mathcal{F}_u\right]ds}.$$

By the very definition of uncorrelated multiplicative increments for $1/\eta$ and from [2, Lemma 5.1]

$$-\frac{A_u}{2\eta_u} \le -\frac{1}{\int_u^T \mathbb{E}\left[\frac{\eta_u}{\eta_s}\right]ds} = -\frac{1}{\int_u^T \frac{\mathbb{E}[1/\eta_s]}{\mathbb{E}[1/\eta_u]}ds} = -\frac{\mathbb{E}[1/\eta_u]}{\int_u^T \mathbb{E}[1/\eta_s]\,ds} = \frac{1}{N_u}dN_u$$

with $N_u := \int_u^T \mathbb{E}[1/\eta_s]\,ds$. Hence

$$\exp\left(-\int_r^s \frac{A_u}{2\eta_u}du\right) = \exp\left(\int_r^s \frac{1}{N_u}dN_u\right) = \frac{N_s}{N_r} = \frac{\int_s^T \mathbb{E}[1/\eta_v]\,dv}{\int_r^T \mathbb{E}[1/\eta_v]\,dv} \le \frac{\|\eta\|}{\eta_\star}\left(\frac{T-s}{T-r}\right).$$

If $1/\eta$ is a positive martingale, then again from [2, Lemma 5.1], we get that $1/\eta$ has uncorrelated multiplicative increments. If η is deterministic, we have directly that

$$-\frac{A_u}{2\eta_u} \le -\frac{1}{\eta_u\int_u^T \frac{1}{\eta_s}ds} = \frac{1}{N_u}\frac{dN_u}{du}$$

with again $N_u = \int_u^T 1/\eta_s\,ds$.

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