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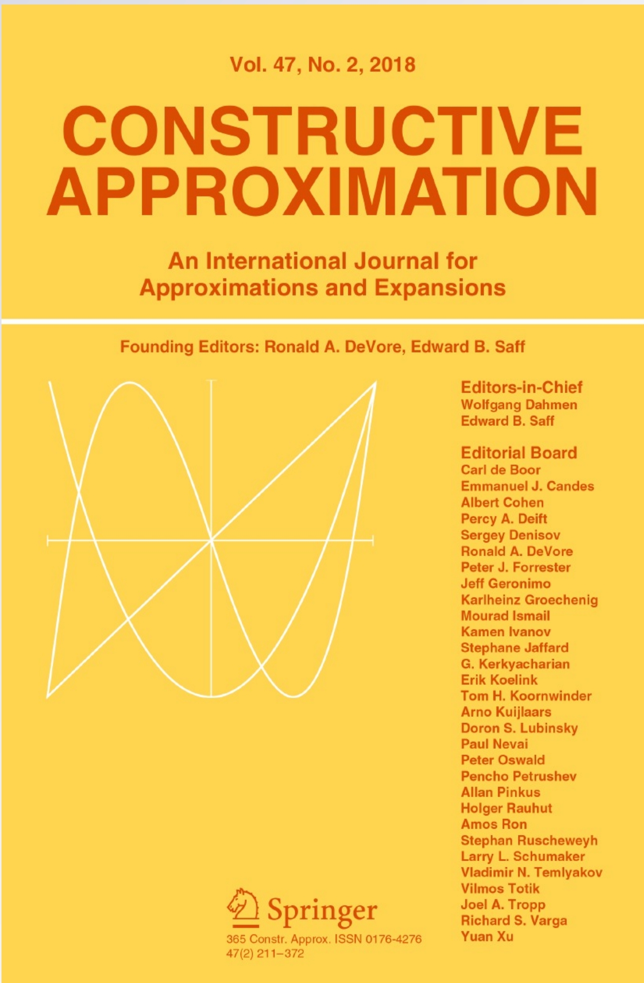

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CONSTRUCTIVE APPROXIMATION

Regularity of Gaussian Processes on Dirichlet Spaces

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Abstract We study the regularity of centered Gaussian processes $(Z_x(\omega))_{x\in M}$, indexed by compact metric spaces (M, ρ). It is shown that the almost everywhere Besov regularity of such a process is (almost) equivalent to the Besov regularity of the covariance $K(x, y) = \mathbb{E}(Z_x Z_y)$ under the assumption that (i) there is an underlying Dirichlet structure on M that determines the Besov regularity, and (ii) the operator K with kernel $K(x, y)$ and the underlying operator A of the Dirichlet structure commute. As an application of this result, we establish the Besov regularity of Gaussian processes indexed by compact homogeneous spaces and, in particular, by the sphere.

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1 Introduction

Gaussian processes have been at the heart of probability theory for a very long time. There is a huge literature about it (see among many others [1,2,31,32,34,36]). They also have been playing a key role in applications for many years and seem to be experiencing an active revival in the recent domains of machine learning (see among others [39,44]) as well as in Bayesian nonparametric statistics (see, for example, [27,56]).

In many areas, it is important to develop regularization procedures or sparse representations. Finding adequate regularizations as well as the quantification of the sparsity play an essential role in the accuracy of the algorithms in statistical theory as well as in approximation theory. A way to regularize or to improve sparsity which is at the same time genuine and easily explainable is to impose regularity conditions.

The regularity of Gaussian processes has also been for a long time in the essentials of probability theory. It goes back to Kolmogorov in the 1930s (see among many others [18,30,33,53,54]).

In applications, an important effort has been put on the construction of Gaussian processes on manifolds or more general domains, with the two especially challenging examples of spaces of matrices and spaces of graphs to contribute to the emerging field of signal processing on graphs and extending high-dimensional data analysis to networks and other irregular domains.

Motivated by these aspects, we explore in this paper the regularity of Gaussian processes indexed by compact metric domains verifying some conditions in such a way that regularity conditions can be identified.

In effect, to prove regularity properties, we need a theory of regularity, compatible with the classical examples: Lipschitz properties and differentiability. At the same time, we want to be able to handle more complicated geometries. For this aspect, we borrow the geometrical framework developed in [14,26].

Many of the constructions for regularity theorems are based on moments bounds for the increments of the process. Our approach here is quite different; it utilizes the spectral properties of the covariance operator. In particular, we use the Littlewood–Paley theory (this point of view was implicit in [12]) to show that the Besov regularity of the sample paths of the process is (almost) equivalent to the Besov regularity of the covariance operator. More precisely, it is shown that the almost everywhere Besov space regularity of such a process is (almost) equivalent to the Besov regularity of the covariance $K(x, y) = \mathbb{E}(Z_x Z_y)$. It is also important to note that unlike many results in the literature, the regularity is expressed using the genuine distance in the domain, not the distance induced by the covariance.

To put our approach to regularity of Gaussian processes in perspective, we next compare it to existing approaches to this problem in the literature. Regularity properties of Gaussian processes are obviously related to some characteristics of the covariance kernel, and there are several ways to tackle this problem, giving rise to different kinds of results. In the first results (see [18] and [53]), the regularity is expressed in terms of a distance directly induced by the covariance $E(Z_x - Z_y)^2$. More recently, following the same approach, it was proved in [41] that local Sobolev regularity can be deduced from mean square differentiability. Other approaches, mostly on $\mathbb{R}$ or $\mathbb{R}^d$, establish Lipschiz regularity using Kolmogorov–Chentsov results (see [45]) that rely on evaluations of $E|Z_x - Z_Y|^p$.

Our method relies heavily on a spectral decomposition coupled with the fact that the covariance kernel (through a commutation property) is related to a geometric setting where regularity spaces (Besov or Sobolev) can easily be expressed in terms of the spectral coefficients. A similar approach has been carried out on the interval, $\mathbb{R}$ and $\mathbb{R}^d$ using wavelet bases, starting from [12], then [40], etc. and more recently applied in [25] to Levy processes. In the recent article [29], an approach in the same spirit is applied in the case of the sphere, except that instead of directly connecting the regularity to the spectrum, a moment condition is derived from the spectrum, and, in turn, a Kolmogorov–Chentsov theorem is proved on the sphere, with regularity properties of the paths as a byproduct. Also, regularity results are obtained in [46] using a Karuhnen Loeve expansion by comparing the smoothness of the paths to the smoothness of the functions contained in the RKHS.

To illustrate our approach, we revisit the Brownian motion as well as the fractional Brownian motion on the interval. We show the standard Besov regularity of these processes but also prove that they can be associated with a genuine geometry that finally appears in a nontrivial way. We also illustrate our main result in the more refined case of two point homogeneous spaces and, in particular, in case of the unit sphere $\mathbb{S}^d$ in $\mathbb{R}^{d+1}$.

In the two subsequent sections, we recall the needed background information about Gaussian processes, the geometrical framework introduced in [14,26], and how it allows the development of a smooth functional calculus as well as a description of regularity. In Sect. 4, we state the main results of the paper: the regularity theorem, the Ito–Nisio representation, and the link with the RKHS. Section 4.3 details the seminal case of the Neumann operator and the standard Brownian motion. In this case, the salient fact is not the regularity result (which is known) but the original geometry corresponding to this process. The proofs of the main results are carried out in Sect. 5. Section 6 recalls some basic (and less basic) facts about positive and negative definite functions on two point homogeneous spaces. Section 7 establishes the Besov regularity of Gaussian processes indexed by the sphere. Section 8 is an appendix where we detail some facts on positive definite and negative definite functions as well as Gaussian probability on separable Banach spaces.

2 Gaussian Processes: Background

In this section, we recall some basic facts about Gaussian processes and establish useful notations.

2.1 General Setting for Gaussian Processes

Let $(\Omega, \mathcal{A}, P)$ be a probability space. Consider a centered Gaussian process on a set M, i.e., a family of real random variables $Z_x(\omega)$ with $x \in M$ and $\omega \in \Omega$ such that for all $n \in \mathbb{N}$, $x_1, \ldots, x_n \in M$, and $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$,

$$\sum_{i=1}^{n} \alpha_i Z_{x_i} \text{ is a centered Gaussian random variable.}$$

The covariance function $K(x, y)$ associated with such a process $(Z_x)_{x \in M}$ is defined by

$$K(x, y) := \mathbb{E}(Z_x Z_y) \quad \text{for} \quad (x, y) \in M \times M.$$

It is readily seen that $K(x, y)$ is real-valued, symmetric, and positive definite (P.D.); i.e.,

$$K(x, y) = K(y, x) \in \mathbb{R}, \quad \text{and}$$
$$\sum_{i,j \leq n} \alpha_i \alpha_j K(x_i, x_j) \geq 0, \quad \forall n \in \mathbb{N},\ \forall\, x_1, \ldots, x_n \in M,\ \forall \alpha_1, \ldots, \alpha_n \in \mathbb{R}.$$

Remark 2.1 In this paper, we only consider real Gaussian variables and real Hilbert spaces.

2.2 Gaussian Processes with a Touch of Topology

We now consider the following more specific setting. Let M be a compact space, and let μ be a Radon measure on $(M, \mathcal{B})$ with support M and $\mathcal{B}$ being the Borel sigma algebra on M. Assuming that $(\Omega, \mathcal{A}, P)$ is a probability space, we let

$$Z : (M, \mathcal{B}) \otimes (\Omega, \mathcal{A}) \mapsto Z_x(\omega) \in \mathbb{R} \text{ be a measurable map}$$

such that $(Z_x)_{x \in M}$ is a Gaussian process. We remember that $K(x, y)$ is continuous, and positive definite function on $M \times M$. Hence obviously the operator K defined by

$$Kf(x) := \int_M K(x, y) f(y) d\mu(y), \quad f \in L^2(M, \mu),$$

is a self-adjoint compact positive operator (even trace-class) on $L^2(M, \mu)$. Moreover, $K(L^2) \subset C(M)$, the Banach space of continuous functions on M. Let $\nu_1 \geq \nu_2 \geq \cdots \geq 0$ be the sequence of eigenvalues of K repeated according to their multiplicities, and let $(u_k)_{k \geq 1}$ be the sequence of respective normalized eigenfunctions:

$$\int_M K(x, y)u_k(y)d\mu(y) = \nu_k u_k(x).$$

The functions u_k are continuous real-valued functions and the sequence $(u_k)_{k\geq 1}$ is an orthonormal basis for $L^2(M, \mu)$. By Mercer's theorem we have the following representation:

$$K(x, y) = \sum_k \nu_k u_k(x)u_k(y),$$

where the convergence is uniform.

Let $\mathcal{H} \subset L^2(\Omega, P)$ be the closed Gaussian space spanned by finite linear combinations of $(Z_x)_{x\in M}$. Clearly, interpreting the following integral as the Bochner integral with value in the Hilbert space $\mathcal{H}$, we can define

$$B_k(\omega) = \frac{1}{\sqrt{\nu_k}} \int_M Z_x(\omega)u_k(x)d\mu(x) \in \mathcal{H}.$$

It is not difficult to prove that B_k is a sequence of independent $N(0, 1)$ variables and that the process

$$\tilde{Z}_x(\omega) := \sum_k \sqrt{\nu_k} u_k(x)B_k(\omega)$$

is a modification of $Z_x(\omega)$, i.e., $P(Z_x = \tilde{Z}_x) = 1, \forall x \in M$.

We are interested in the regularity of the "trajectory" $x \in M \mapsto Z_x(\omega)$ for almost all $\omega \in \Omega$ and for a suitable modification of $Z_x(\omega)$, and for this reason, we will focus on the version $\tilde{Z}_x(\omega)$.

3 Regularity Spaces on Metric Spaces with Dirichlet Structure

On a compact metric space (M, ρ), one has the scale of s-Lipschitz spaces defined by the norm

$$\|f\|_{\mathrm{Lip}_s} := \|f\|_\infty + \sup_{x\neq y} \frac{|f(x) - f(y)|}{\rho(x, y)^s}, \quad 0 < s \leq 1. \tag{3.1}$$

In Euclidean spaces, a function can be much more regular than Lipschitz, for instance differentiable at different order, or belong to some Sobolev space, or even in a more refined way to a Besov space. For this purpose, we consider metric measured spaces with Dirichlet structure. This setting is rich enough to develop a Littlewood–Paley theory in almost complete analogy with the classical case on $\mathbb{R}^d$, see [14,26]. In particular, it allows the development of Besov spaces B^s_{pq} with all sets of indices. At the same time, this framework is sufficiently general to cover a number of interesting cases, as will be shown in what follows.

This setting is quite general. It covers, in particular, the case of compact Riemannian manifolds. It naturally contains the cases of the sphere, interval, ball, and simplex with weights. For more details, see [14, §1.3].

3.1 Metric Spaces with Dirichlet Structure

We assume that (M, μ) is a compact connected measure space, where μ is a Radon measure with support M. Also, assume that A is a self-adjoint non-negative operator mapping real-valued to real-valued functions with dense domain $D(A) \subset L^2(M, \mu)$. Let $P_t = e^{-tA}$, $t > 0$, be the associated self-adjoint semi-group. Furthermore, we assume that A determines a local and regular Dirichlet structure, see [14, §1.2] and for details [11,19,22,49–52]. In fact, we assume that P_t is a Markov semi-group (A verifies the Beurling–Deny condition):

$$0 \le f \le 1 \ \text{ and } \ f \in L^2 \ \text{ imply } \ 0 \le P_t f \le 1,$$

and also $P_t \mathbb{1}_M = \mathbb{1}_M$ (equivalently $A\mathbb{1}_M = 0$). From this it follows that P_t can be extended as a contraction operator on $L^p(M, \mu)$ for $1 \le p \le \infty$; i.e., $\|P_t f\|_p \le \|f\|_p$, and $P_t P_s f = P_{t+s} f$, $t, s > 0$.

The next assumption is that there exists a sufficiently rich subspace $\tilde{D} \subset D(A)$ such that $f \in \tilde{D} \implies f^2 \in D(A)$ (see [11]). Then we define a bilinear operator "square gradient" $\Gamma : \tilde{D} \times \tilde{D} \mapsto L^1$ by

$$\Gamma(f, g) := -\frac{1}{2}[A(fg) - fA(g) - gA(f)].$$

Consequently, $\Gamma(f, f) \ge 0$ and $\int_M A(f)g d\mu = \int_M \Gamma(f, g) d\mu$ (formula for integration by parts).

Main Assumptions

(i) Let

$$\rho(x, y) := \sup_{\Gamma(f,f)\le 1} \big(f(x) - f(y)\big) \quad \text{for } x, y \in M. \tag{3.2}$$

We assume that ρ is a metric on M that generates the original topology on M.
(ii) *The doubling property:* Define $B(x, r) := \{y \in M : \rho(x, y) < r\}$. The assumption is that there exists a constant $d > 0$ such that

$$\mu(B(x, 2r)) \le 2^d \mu(B(x, r)), \quad \forall x \in M, \ \forall r > 0. \tag{3.3}$$

This means that (M, ρ, μ) is a homogeneous space in the sense of Coifman and Weiss [13]. Observe that from (3.3), it follows that

$$\mu(B(x, \lambda r)) \le (2\lambda)^d \mu(B(x, r)) \quad \text{for } x \in M, r > 0, \text{ and } \lambda > 1, \tag{3.4}$$

The constant d plays the role of a dimension.
(iii) *Poincaré inequality:* There exists a constant $c > 0$ such that for all $f \in \tilde{D}, x \in M$, and $r > 0$,

$$\inf_{\lambda \in \mathbb{R}} \int_{B(x,r)} |f - \lambda|^2 d\mu \le cr^2 \int_{B(x,r)} \Gamma(f, f) d\mu.$$

As a consequence, the associated semi-group $P_t = e^{-tA}$, $t > 0$, consists of integral operators of continuous (heat) kernel $p_t(x, y) \ge 0$, with the following properties:
(a) Gaussian localization: for all $x, y \in M$ and $t > 0$,

$$\frac{c_1 \exp\left\{-\frac{\rho^2(x,y)}{c_2 t}\right\}}{\sqrt{\mu(B(x, \sqrt{t}))\mu(B(y, \sqrt{t}))}} \le p_t(x, y) \le \frac{c_3 \exp\left\{-\frac{\rho^2(x,y)}{c_4 t}\right\}}{\sqrt{\mu(B(x, \sqrt{t}))\mu(B(y, \sqrt{t}))}}. \tag{3.5}$$

(b) Hölder continuity: there exists a constant $\kappa > 0$ such that

$$\left|p_t(x, y) - p_t(x, y')\right| \le c_1 \left(\frac{\rho(y, y')}{\sqrt{t}}\right)^{\kappa} \frac{\exp\left\{-\frac{\rho^2(x,y)}{c_2 t}\right\}}{\sqrt{\mu(B(x, \sqrt{t}))\mu(B(y, \sqrt{t}))}} \tag{3.6}$$

for $x, y, y' \in M$ and $t > 0$, whenever $\rho(y, y') \le \sqrt{t}$.
(c) Markov property:

$$\int_M p_t(x, y) d\mu(y) = 1 \quad \text{for } x \in M \text{ and } t > 0. \tag{3.7}$$

Above $c_1, c_2, c_3, c_4 > 0$ are structural constants.

Examples The general setting described above covers a number of particular setups. The simple case when $M = [0, 1]$ and $A = -\frac{d^2}{dx^2}$ is included and described in more detail in Sect. 4.3.1 below. The more general classical case when $M = [-1, 1]$, $d\mu(x) = w(x)dx$ with $w(x) = (1 - x)^{\alpha}(1 + x)^{\beta}$, $\alpha, \beta > -1$, and A is the Jacobi operator

$$Af(x) = -\frac{[w(x)a(x) f'(x)]'}{w(x)}, \quad a(x) = 1 - x^2, \quad D(A) = C^2[-1, 1],$$

is also covered by our setting. This case is described in detail in [14, §7]. The cases of the ball and simplex in $\mathbb{R}^d$ equipped with classical weights and elliptic operators are being developed and are also covered. The sphere in $\mathbb{R}^d$ equipped with the Lebesgue measure and geodesic distance and with A being the Laplace–Beltrami operator is another classical case that is covered by the above setting. This case will play an important role in this article, see Sect. 7 below. A natural generalization of this setup is the case when M is a compact Riemannian manifold equipped with the natural Riemannian measure and geodesic distance and with $A = \operatorname{div} \circ \nabla$, the Laplace operator on M, see, e.g., [22, Chapter 3]. Furthermore, certain cases when M is a compact subset of a Riemannian manifold equipped with a weighted Riemannian measure and geodesic distance, and a weighted Laplace operator A are also included, see [23] for details.

Notation Throughout, we will use the notation $|E| := \mu(E)$, and $\mathbb{1}_E$ will stand for the characteristic function of $E \subset M$. Also $\|\cdot\|_p = \|\cdot\|_{L^p} := \|\cdot\|_{L^p(M,\mu)}$. Positive constants will be denoted by $c, c', c_1, C, C', \dots$, and they may vary at every occurrence. The notation $a \sim b$ will stand for $c_1 \le a/b \le c_2$. As usual we will denote by $\mathbb{N}$ the set of all natural numbers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

From the compactness of M and the fact that A is an essentially self-adjoint non-negative operator, it follows that the spectrum of A is discrete and of the form $0 \le \lambda_1 < \lambda_2 < \cdots$. Furthermore, the respective eigenspaces $\mathcal{H}_{\lambda_k} := \operatorname{Ker}(A - \lambda_k \operatorname{Id})$ are finite dimensional, and

$$L^2(M,\mu) = \bigoplus_{k\ge1} \mathcal{H}_{\lambda_k}.$$

Denoting by $P_{\mathcal{H}_{\lambda_k}}$ the orthogonal projector onto $\mathcal{H}_{\lambda_k}$, the above means that all $f \in L^2(M,\mu)$ can be expressed in the following form: $f = \sum_{k\ge1} P_{\mathcal{H}_{\lambda_k}} f$. In addition,

$$Af = \sum_{k\ge1} \lambda_k P_{\mathcal{H}_{\lambda_k}} f, \quad \forall f \in D(A), \quad \text{and} \quad P_t f = \sum_{k\ge1} e^{-t\lambda_k} P_{\mathcal{H}_{\lambda_k}} f, \quad \forall f \in L^2. \tag{3.8}$$

More generally, for a function $g \in L^\infty(\mathbb{R}_+)$ the operator $g(\sqrt{A})$ is defined by

$$g(\sqrt{A}) f := \sum_{k\ge1} g(\sqrt{\lambda_k}) P_{\mathcal{H}_{\lambda_k}} f, \quad \forall f \in L^2. \tag{3.9}$$

The spectral spaces Σ_λ, $\lambda > 0$, associated with $\sqrt{A}$ are defined by

$$\Sigma_\lambda := \bigoplus_{\sqrt{\lambda_k}\le\lambda} \mathcal{H}_{\lambda_k}.$$

Observe that $\Sigma_\lambda \subset L_\infty$, and hence $\Sigma_\lambda \subset L^p$ for $1 \le p \le \infty$.

From now on we will assume that the eigenvalues $(\lambda_k)_{k\ge1}$ are enumerated with algebraic multiplicities taken into account; i.e., if the algebraic multiplicity of λ is m, then λ is repeated m times in the sequence $0 \le \lambda_1 \le \lambda_2 \le \cdots$. We let $(u_k)_{k\ge1}$ be respective real orthogonal and normalized in L^2 eigenfunctions of A; that is, $Au_k = \lambda_k u_k$.

Let $\Pi_\delta(x,y) := \sum_{\sqrt{\lambda_k}\le\delta^{-1}} u_k(x)u_k(y)$, $\delta > 0$, be the kernel of the orthogonal projector onto $\Sigma_{1/\delta}$. Then as is shown in [14, Lemma 3.19],

$$\Pi_\delta(x,x) \sim |B(x,\delta)|^{-1}. \tag{3.10}$$

A key trait of our setting is that it allows the development of a smooth functional calculus. In particular, if $g \in C^\infty(\mathbb{R})$ is even, compactly supported then the operator

$g(t\sqrt{A})$ defined in (3.9) is an integral operator with kernel $g(t\sqrt{A})(x,y)$ having this localization: For any $\sigma > 0$, there exists a constant $c_\sigma > 0$ such that

$$\left|g(t\sqrt{A})(x,y)\right| \le c_\sigma |B(x,t)|^{-1}\big(1+t^{-1}\rho(x,y)\big)^{-\sigma}, \quad \forall x,y \in M. \tag{3.11}$$

Furthermore, $g(t\sqrt{A})(x,y)$ is Hölder continuous. An immediate consequence of (3.11) is that the operator $g(t\sqrt{A})$ is bounded on $L^p(M)$:

$$\|g(t\sqrt{A})f\|_p \le c\|f\|_p, \quad \forall f \in L^p(M), \quad 1 \le p \le \infty. \tag{3.12}$$

For more details and proofs, see [14, §3.1],[26, §3.1].

For discretization (sampling) purposes, we will use *maximal δ-nets* (sometimes termed *maximal δ-packings*). Recall that a set $\mathcal{X} \subset M$ is a maximal δ-net on M ($\delta > 0$) if $\rho(x,y) \ge \delta$ for all $x,y \in \mathcal{X}$, $x \ne y$, and $\mathcal{X}$ is maximal with this property. It is easily seen that a maximal δ-net on M always exists. Of course, if $\delta > \mathrm{Diam}(M)$, then $\mathcal{X}$ will consist of a single point. The following useful assertion is part of Theorem 4.2 in [14].

Proposition 3.1 *There exists a constant $\gamma > 0$, depending only on the structural constants of our setting, such that for any $\lambda > 0$ and $\delta := \gamma/\lambda$, there exists a δ-net $\mathcal{X}$ obeying*

$$2^{-1}\|g\|_\infty \le \max_{\xi\in\mathcal{X}} |g(\xi)| \le \|g\|_\infty, \quad \forall g \in \Sigma_\lambda. \tag{3.13}$$

Finally, if $N(\delta, M)$ is the covering number of M (or the cardinality of a maximal δ−net), then

$$\dim(\Sigma_{\frac{1}{\sqrt{t}}}) \sim \int_M |B(x,\sqrt{t})|^{-1}d\mu(x) \sim N(\sqrt{t},M) \sim \|e^{-tA}\|_{HS}^2 \le ct^{-d/2}, \quad t > 0. \tag{3.14}$$

Here $\|e^{-tA}\|_{HS}^2 := \int_M\int_M |p_t(x,y)|^2 d\mu(x)d\mu(y)$ is the Hilbert–Schmidt norm.

3.2 Regularity Spaces

In the general setting described above, the full scales of Besov and Tribel–Lizorkin spaces are available [14,26].

The Sobolev spaces $W_p^k = W_p^k(A)$, $k \ge 1$, $1 \le p \le \infty$, are standardly defined by

$$W_p^k := \left\{f \in D(A^{\frac{k}{2}}) : \|f\|_{W_p^k} := \|f\|_p + \|A^{\frac{k}{2}}f\|_p < \infty\right\}. \tag{3.15}$$

The Besov space B^s_{pq}, $s > 0$, $1 \le p,q \le \infty$, is defined by interpolation as in [38]:

$$B^s_{pq} := \big(L^p, W_p^k\big)_{\theta,q}, \quad \theta := s/k, \tag{3.16}$$

where $\big(L^p, W_p^k\big)_{\theta,q}$ is the real interpolation space between L^p and W_p^k, see [14, § 3.1, § 6.1].

The following Littlewood–Paley decomposition of functions will play an important role hereafter. Suppose $\Phi \in C^\infty(\mathbb{R})$ is real-valued, even, and such that $\operatorname{supp}\Phi \subset [-2, 2]$, $0 \le \Phi \le 1$, and $\Phi(\lambda) = 1$ for $\lambda \in [0, 1]$. Let $\Psi(\lambda) := \Phi(\lambda) - \Phi(2\lambda)$. Evidently, $\operatorname{supp}\Psi \cap \mathbb{R}_+ \subset [1/2, 2]$. Set

$$\Psi_0 := \Phi \quad \text{and} \quad \Psi_j(\lambda) := \Psi(2^{-j}\lambda) \quad \text{for } j \ge 1. \tag{3.17}$$

Clearly, $\Psi_0, \Psi \in C^\infty(\mathbb{R})$, Ψ_0 and Ψ are even, $\operatorname{supp}\Psi_j \cap \mathbb{R}_+ \subset [2^{j-1}, 2^{j+1}]$, $j \ge 1$, $\operatorname{supp}\Psi_0 \subset [-2, 2]$, and $\sum_{j\ge 0}\Psi_j(\lambda) = 1$ for $\lambda \in \mathbb{R}_+$. Consequently, for an arbitrary $f \in L^p(M, \mu)$, $1 \le p \le \infty$, one has

$$f = \sum_{j\ge 0} \Psi_j(\sqrt{A}) f \quad \text{in} \quad L^p. \tag{3.18}$$

The Littlewood–Paley characterization of Besov spaces uses the functions Ψ_j from above: If $s > 0$ and $1 \le p, q \le \infty$, then for a function $f \in L^p(M, \mu)$, we have

$$f \in B^s_{p,q} \iff \|\Psi_j(\sqrt{A}) f\|_p = \varepsilon_j 2^{-js}, \quad j \ge 0, \quad \text{with} \quad \{\varepsilon_j\} \in \ell^q. \tag{3.19}$$

Furthermore, if $f \in B^s_{p,q}$, then $\|f\|_{B^s_{p,q}} \sim \|\{\varepsilon_j\}\|_{\ell^q}$. We refer the reader to [14,26] for proofs and more details on Besov spaces in the setting from Sect. 3.1. In particular, the following proposition clarifies the relationship between $B^s_{\infty,\infty}$ and $\operatorname{Lip} s$ (see [14, Proposition 6.4]).

Proposition 3.2 (*a*) *For any* $0 < s \le 1$, *we have* $\operatorname{Lip} s \subset B^s_{\infty,\infty}$.
(*b*) *Assuming that* $\kappa > 0$ *is the constant from* (3.6), *then* $B^s_{\infty,\infty} \subset \operatorname{Lip} s$ *for* $0 < s < \kappa$.

Remark 3.3 In the most interesting cases $\kappa = 1$, Proposition 3.2 implies that $\operatorname{Lip} s = B^s_{\infty,\infty}$ for $0 < s < 1$.

Remark 3.4 It should be pointed out that the Sobolev and Besov spaces defined above originate in the work of J. Petree [38, Chapter 10] and H. Triebel [55, §10.3]. These spaces are the same as the respective Sobolev and Besov spaces in each specific case covered by our general setting. For example, in the case of $M = [-1, 1]$ with Jacobi weight and the Jacobi operator that we alluded to in Sect. 3.1, the above defined Sobolev and Besov spaces coincide with the respective spaces developed in [28]. Likewise, in the case of the unit sphere in $\mathbb{R}^d$ with the Laplace–Beltrami operator mentioned in Sect. 3.1, the Sobolev and Besov spaces from above coincide with the respective spaces developed in [37].

4 Main Results

In this section, we state and discuss our main results. The proofs are carried out in the next section.

We consider a centered Gaussian process $(Z_x)_{x\in M}$ with covariance function $K(x, y)$ as described in Sect. 2.2, indexed by a metric space M with Dirichlet structure just as described in Sect. 3.1. We will adhere to the assumptions and notation from Sect. 3.1.

4.1 Commutation Property

We now make the *fundamental assumption* that the operator K with kernel $K(x, y)$ and the operator A from Sect. 3.1 commute in the following sense:

Definition 4.1 If K is a bounded operator on a Banach space $\mathbb{B}$ ($K \in \mathcal{L}(\mathbb{B})$) and A is an unbounded operator with domain $D(A) \subset \mathbb{B}$, we say that K and A commute if $K(D(A)) \subset D(A)$ and

$$KAf = AKf, \quad \forall f \in D(A).$$

Remark 4.2 Let A be the infinitesimal generator of a contraction semi-group P_t. Then K and A commute in the sense of Definition 4.1 if and only if

$$KP_t = P_t K, \quad \forall t > 0.$$

We refer the reader to [16], Theorem 6.1.27.

We now return to the covariance operator K and the underlying self-adjoint non-negative operator A from our setting. In light of Remark 4.2, our assumption that K and A commute implies that they have the same eigenspaces.

Recall that the eigenvalues of A are ordered in a sequence $0 = \lambda_1 \le \lambda_2 \le \cdots$, where the eigenvalues are repeated according to their multiplicities, and the respective eigenfunctions $(u_k)_{k\ge1}$ are real-valued, orthogonal, and normalized in L^2. Let $(\nu_k)_{k\ge1}$ be the eigenvalues of the covariance operator K. Then

$$Au_k = \lambda_k u_k \quad \text{and} \quad Ku_k = \nu_k u_k, \quad k \ge 1. \tag{4.1}$$

Remark 4.3 As a consequence of the commutation property of K and A, the operator AK is defined everywhere on $L^2(M, \mu)$ and is closed. Therefore, AK is a continuous operator from $L^2(M, \mu)$ to $L^2(M, \mu)$. Clearly,

$$KAf = \sum_{k\ge1} \langle f, u_k\rangle \lambda_k \nu_k u_k, \quad \forall f \in L^2, \text{ and hence } \quad \|KA\|_{\mathcal{L}(L^2)} = \sup_{k\ge1} \lambda_k \nu_k < \infty.$$

Remark 4.4 Assume that we are in the geometric setting described in Sect. 3.1, associated with an operator A. As in Sect. 4.1, suppose $K(x, y)$ is a P.D. kernel such that the associated operator K commutes with A. It is easy to see that

$$A\mathbb{1}_M = 0 \quad \text{and} \quad \dim \operatorname{Ker}(A) = 1. \tag{4.2}$$

Indeed, the Markov property (3.7) yields $A\mathbb{1}_M = 0$. To show that $\dim \operatorname{Ker}(A) = 1$, assume that $Af = 0$, $f \in D(A)$. Then $\Gamma(f, f) = 0$. Assume $f \ne$ constant. Then

$f(x) \neq f(y)$ for some $x, y \in M$, $x \neq y$. Since $\Gamma(f, f) = 0$, we have $\Gamma(af, af) = 0$ for each $a > 0$. Then using (3.2), we get

$$\rho(x, y) \geq a|f(x) - f(f)|, \quad \forall a > 0,$$

implying $\rho(x, y) = \infty$, which contradicts the fact that M is connected (see [14, Proposition 2.1]). Therefore, $Af = 0$ implies $f = \text{const.}$, and hence $\dim \operatorname{Ker}(A) = 1$.

As a consequence of (4.2), we have

$$AK\mathbb{1}_M = KA\mathbb{1}_M = 0.$$

However, as $\dim \operatorname{Ker}(A) = 1$, necessarily $K\mathbb{1}_M = C\mathbb{1}_M$.

Remark 4.5 A priori, on a compact measured space (M, μ), the study of a Gaussian process is only linked with a positive definite kernel $K(x, y)$.

Then, the question is: Given a continuous positive definite function $K(x, y)$ on a compact measure space (M, μ), how does one find a good operator A communications with associated Besov or Sobolev spaces such that one could characterize the regularity of the trajectories of the associated Gaussian process? This is a hard problem. Clearly, a simple answer is given in the case of compact 2-point homogeneous spaces (see Sect. 6), with the sphere being the best known and interesting example.

If $K(x, y) = \sum_k \phi_k(x)\phi_k(y)$, where $K\phi_k = \nu_k\phi_k$, what is the good Dirichlet space associated with an operator A such that $A\phi_k = \lambda_k\phi_k$? The simplest example, the Brownian motion on $([0, 1], dx)$, already shows the difficulties: we have

$$x \wedge y = \sum_k \frac{\sqrt{2}\sin\left(k + \frac{1}{2}\right)\pi x}{\left(k + \frac{1}{2}\right)\pi} \frac{\sqrt{2}\sin\left(k + \frac{1}{2}\right)\pi y}{\left(k + \frac{1}{2}\right)\pi}.$$

The natural associated operator is $A = \frac{d^2}{dx^2}$ with the Neumann–Dirichlet condition: $f(0) = 0$, $f'(1) = 0$. Unfortunately, this is not suitable here since it is not Markovian. This is clear because $\mathbb{1}$ is not an eigenfunction of K. See in Sect. 4.3 how to get around the problem. However, for the moment, we have no solution for this problem in full generality.

4.2 Regularity Theorem, Ito–Nisio Representation and RKHS

We now come to the main results of this paper.

Theorem 4.6 *Let $(Z_x)_{x\in M}$ be a centered Gaussian process with covariance function $K(x, y) := \mathbb{E}(Z_x Z_y)$ indexed by a metric space M with Dirichlet structure induced by a self-adjoint operator A such that K and A commute in the sense of Definition* 4.1. *Then the following assertions hold:*

(*a*) *If the covariance kernel $K(x, y)$ has the regularity described by*

$$\sup_{x\in M} \|K(x, \bullet)\|_{B^s_{\infty,\infty}} < \infty \quad \textit{for some} \quad s > 0, \tag{4.3}$$

then a version of the Gaussian process $Z_x(\omega)$ *has the following regularity: For any* $0 < \alpha < \frac{s}{2}$,

$$Z_x(\omega) \in B^{\alpha}_{\infty,1} \ \textit{for almost all } \omega \quad (B^{\alpha}_{\infty,1} \subset B^{\alpha}_{\infty,\infty}).$$

(*b*) *Conversely, suppose there exists* $\alpha > 0$ *such that* $Z_x(\omega) \in B^{\alpha}_{\infty,\infty}$ *for almost all* ω. *Then*

$$\sup_{x \in M} \|K(x, \bullet)\|_{B^{2\alpha}_{\infty,\infty}} < \infty.$$

Remark 4.7 It is interesting to observe that because of the second part of the theorem, condition (4.3) is necessary.

Another key point is that in the above theorem, the Besov space smoothness parameter $s > 0$ can be arbitrarily large, while $0 < s \leq 1$ in the case when the regularity is characterized in terms of Lipschitz spaces.

Also, note that $B^{\alpha}_{\infty,1}$ is a separable Banach subspace of $B^{\alpha}_{\infty,\infty}$. For all measurability problems and for defining a canonical measure on a functional space, it is simpler to work with separable Banach spaces (see "Appendix II").

4.2.1 Ito–Nisio Representation

The following theorem gives an Ito–Nisio representation of the process.

Theorem 4.8 (Wiener measure) *In the setting from above, if* $K(x, y)$ *is a continuous positive definite function on* M *such that*

$$\sup_{x \in M} \|K(x, \bullet)\|_{B^{s}_{\infty,\infty}} < \infty$$

and the associated kernel operator K *commutes with* A, *then there is a unique probability measure* Q *on the Borel sets of* $B^{\alpha}_{\infty,1}$, $\alpha < \frac{s}{2}$, *such that the family of random variables*

$$\forall x \in M, \ \omega \in B^{\alpha}_{\infty,1} \xrightarrow{\delta_x} \omega(x) \in \mathbb{R}$$

is a centered Gaussian process of covariance $K(x, y)$.

4.2.2 Reproducing Kernel Hilbert Spaces (RKHS)

We finally connect condition (4.3) with the RKHS associated with the process Z_x (see the "Appendix").

As is well known, the covariance kernel K determines a real Hilbert space $\mathbb{H}_K$ of functions for which the evaluation

$$\forall x \in M, \ \delta_x : f \in \mathbb{H}^*_K \mapsto f(x) \text{ is continuous.}$$

Moreover,

$$y \mapsto K(x, y) = K_x(y) \in \mathbb{H}_K, \quad \forall f \in \mathbb{H}_K, \; \delta_x(f) = \langle f, K_x \rangle_{\mathbb{H}_K},$$

and $(K_x)_{x\in M}$ is a total set in $\mathbb{H}_K$. First, one constructs an abstract Hilbert space as the completion of span $\{K(x, \cdot) : x \in M\}$, that is, the completion of the space $\mathbb{H}_K^\circ$ of all functions h of the forms $h(y) = \sum_{i\in F} \alpha_i K(x_i, y)$, where $F \subset \mathbb{N}$ is finite, with a norm defined by

$$\|h\|_{\mathbb{H}}^2 := \sum_{i,j\in F} \alpha_i \alpha_j K(x_i, x_j) = \sum_{j\in F} \alpha_j h(x_j).$$

(It is also well known (see, e.g., [15]) that

$$\|h\|_{\mathbb{H}}^2 = 0 \;\text{ for }\; h \in \mathbb{H}_K^\circ \iff h(y) = 0, \; \forall y \in M.)$$

It can be proved (see, e.g., [47]) that this abstract space can be realized as a space of functions defined everywhere. Furthermore (see [35]),

$$K(x, y) = \sum_{i\in I} g_i(x) g_i(y) \iff g_i \in \mathbb{H}_K, \; \forall i \quad \text{and} \quad (g_i)_{i\in I} \text{ is a tight frame for } \mathbb{H}_K.$$

In our geometric framework, (4.1) entails the following representation of K

$$K(x, y) = \sum_k \nu_k u_k(x) u_k(y) \quad \text{and} \quad \nu_k \geq 0. \tag{4.4}$$

Therefore, $(\sqrt{\nu_k} u_k)_{k\in\mathbb{N}, \nu_k \neq 0}$ is a tight frame of $\mathbb{H}$, and moreover, $(\delta_x)_{x\in M} \subset \mathbb{H}_K^*$ is dense in $\mathbb{H}_K^*$ in the weak $\sigma(\mathbb{H}_K^*, \mathbb{H}_K)$ topology. Actually, by Mercer's theorem, we have (see [47,48]): Let $\mathbb{N}(\nu) := \{k \in \mathbb{N}, \nu_k \neq 0\}$, and define

$$\mathcal{H} := \left\{ f : M \mapsto \mathbb{R} : f(x) = \sum_{k\in\mathbb{N}(\nu)} \alpha_k \sqrt{\nu_k}\, u_k(x), \; (\alpha_k) \in \ell^2 \right\}$$

with inner product

$$\langle f, g \rangle_{\mathcal{H}} = \left\langle \sum_{k\in\mathbb{N}(\nu)} \alpha_k \sqrt{\nu_k}\, u_k(\cdot), \sum_{k\in\mathbb{N}(\nu)} \beta_k \sqrt{\nu_k} u_k(\cdot) \right\rangle_{\mathcal{H}} := \langle (\alpha_k), (\beta_k) \rangle_{\ell^2(\mathbb{N}(\nu))}.$$

Then $\mathcal{H}$ is a Hilbert space of continuous functions and $(\sqrt{\nu_k} u_k)_{k\in\mathbb{N}(\nu)}$ is an orthonormal basis for $\mathcal{H}$, and hence $\mathbb{H}_K = \mathcal{H}$.

In fact, the following theorem holds.

Theorem 4.9 *We have for* $s > 0$*,*

$$\mathbb{H}_K \subseteq B^{\frac{s}{2}}_{\infty,\infty} \iff \sup_{x\in M} \|K(x, \bullet)\|_{B^s_{\infty,\infty}} < \infty.$$

4.3 Seminal Example: The Neumann Operator on [0, 1] and the Brownian Motion

Here we show that the classical Brownian motion on [0, 1] is a particular case of our general theory.

4.3.1 The Neumann Operator on [0, 1]

Let $H^2([0,1])$ be the space of the functions $f \in L^2([0,1])$ twice weakly differentiable and such that $f', f'' \in L^2([0,1])$. Consider the operator

$$Af := -f'', \quad D(A) := \{f \in H^2([0,1]) : f'(0) = f'(1) = 0\}.$$

Clearly,

$$\int_0^1 (Af)g dx = \int_0^1 f'g' dx = \int_0^1 f Ag dx,$$

and hence A is positive and symmetric. In fact, A generates a Dirichlet space, and also

$$\cos k\pi x \in D(A) \quad \text{and} \quad A(\cos k\pi \bullet)(x) = (\pi k)^2 \cos k\pi x, \quad k \geq 1.$$

Therefore, $\{\mathbb{1}, (\sqrt{2}\cos k\pi x)_{k\in\mathbb{N}}\}$ is an orthonormal basis of $L^2([0,1])$ consisting of eigenvectors of A. Write $H^1([0,1]) := \left\{f \in L^2([0,1]) : \int_0^1 |f'(u)|^2 du < \infty\right\}$. This allows us to define a Dirichlet form:

$$A, D(A) = \left\{f \in H^1([0,1]) : \left|\int_0^1 f'(x)\phi'(x)dx\right| \leq c\|\phi\|_2, \ \forall \phi \in H^1([0,1])\right\}.$$

Thus

$$\int_0^1 f'(x)\phi'(x)dx = \int_0^1 Af(x)\phi(x)dx,$$

and the distance is defined by

$$\rho(x,y) = \sup_{\phi\in H^1([0,1]):|\phi'|\leq 1} \big(\phi(x) - \phi(y)\big) = |x-y|.$$

The Poincaré inequality is well known to be valid in this case. Hence we are now in the setting described in Sect. 3.1. We now focus on a Gaussian process that could be analyzed through the regularity structure defined by this operator.

4.3.2 Brownian Motion

Clearly, $\psi(x, y) = |x-y|$ is a negative definite function on [0, 1] (see the "Appendix") as

$$|x-y| = \int_{[0,1]} |1_{[0,x]}(u) - 1_{[0,y]}(u)|^2 du.$$

Therefore, there is a natural positive definite function $\tilde{K}(x, y)$ associated with ψ (see again the "Appendix"):

$$\begin{aligned}\tilde{K}(x, y) &= \frac{1}{2}\Big(\int_0^1 |x-u|du + \int_0^1 |y-u|du - |x-y|\Big)\\ &= \frac{1}{4}\Big[x^2 + (1-x)^2 + y^2 + (1-y)^2 - 2|x-y|\Big]\\ &= x \wedge y + \frac{(1-x)^2 + (1-y)^2 - 1}{2}.\end{aligned}$$

It is easy to verify that $\tilde{K}$ and A commute, as

$$\tilde{K}(\cos k\pi \bullet)(x) = \frac{\cos k\pi x}{(\pi k)^2}, \quad \forall k \in \mathbb{N}, \quad \text{and} \quad \tilde{K}\mathbb{1} = (1/6)\mathbb{1}.$$

(It is easy to see that $\int_0^1 |x-y| \cos k\pi y \, dy = -\frac{2\cos k\pi x}{(\pi k)^2} + \frac{1+(-1)^k}{(\pi k)^2}$.)

$$\text{Thus: } \tilde{K}(x, y) = \frac{1}{6} + 2\sum_{k\geq 1} \frac{\cos k\pi x . \cos k\pi y}{(\pi k)^2}.$$

Also, $\tilde{K}(x, \bullet)$ is uniformly Lip 1. Therefore, Z_x, the centered Gaussian process associated to $\tilde{K}$, is almost surely Lip α, $\alpha < \frac{1}{2}$. The process $Y_x(\omega) = Z_x(\omega) - Z_0(\omega)$ has the same regularity, and

$$\mathbb{E}(Y_x Y_y) = \frac{1}{2}(|x| + |y| - |x-y|) = x \wedge y$$

is the well-known associated kernel. So, $\{Y_x : x \in [0, 1]\}$ is the classical Brownian motion.

5 Proof of the Main Results

The purpose of this section is to prove Theorems 4.6, 4.8, and 4.9. For this we need some preparation.

5.1 Uniform Besov Property of $K(x, y)$ and Discretization

Recalling (4.4), we next represent the Besov norm of $K(x, \bullet)$ in terms of the eigenvalues and eigenfunctions of K and A.

Theorem 5.1 *Let $s > 0$. Then*

$$\sup_{x\in M} \|K(x,\bullet)\|_{B^s_{\infty,\infty}} \sim \max\left\{ \sup_{x\in M} \sum_{k:\sqrt{\lambda_k}\le 1} \nu_k u_k^2(x),\ \sup_{j\ge 1} 2^{js} \sup_{x\in M} \sum_{k:2^{j-1}<\sqrt{\lambda_k}\le 2^j} \nu_k u_k^2(x) \right\}. \tag{5.1}$$

Proof Note first that from (3.19), it follows that (with Ψ_j from (3.17))

$$\sup_x \|K(x,\bullet)\|_{B^s_{\infty,\infty}} \sim \sup_{j\ge 0} 2^{js} \sup_x \|\Psi_j(\sqrt{A})K(x,\bullet)\|_\infty.$$

But, using (4.4), we have $\big(\Psi_j(\sqrt{A})K(x,\bullet)\big)(y) = \sum_k \Psi_j(\sqrt{\lambda_k})\nu_k u_k(x)u_k(y)$, and hence, applying the Cauchy–Schwarz inequality, it follows that

$$\sup_{x,y} \left|\big(\Psi_j(\sqrt{A})K(x,\bullet)\big)(y)\right| = \sup_x \sum_k \Psi_j(\sqrt{\lambda_k})\nu_k u_k^2(x).$$

Consequently,

$$\sup_x \|K(x,\bullet)\|_{B^s_{\infty,\infty}} \sim \sup_j 2^{js} \sup_x \sum_k \Psi_j(\sqrt{\lambda_k})\nu_k u_k^2(x). \tag{5.2}$$

Clearly, from (3.17), we have $0 \le \Psi_j \le 1$, $\operatorname{supp}\Psi_0 \cap \mathbb{R}_+ \subset [0, 2]$, and $\operatorname{supp}\Psi_j \cap \mathbb{R}_+ \subset [2^{j-1}, 2^{j+1}]$ for $j \ge 1$. Therefore,

$$\sup_x \sum_k \Psi_0(\sqrt{\lambda_k})\nu_k u_k^2(x) \le \sup_x \sum_{\sqrt{\lambda_k}<2} \nu_k u_k^2(x) \quad \text{and}$$
$$\sup_x \sum_k \Psi_j(\sqrt{\lambda_k})\nu_k u_k^2(x) \le \sup_x \sum_{2^{j-1}<\sqrt{\lambda_k}<2^{j+1}} \nu_k u_k^2(x), \quad j \ge 1.$$

These estimates and (5.2) readily imply that the left-hand side quantity in (5.1) is dominated by a constant multiple of the right-hand side.

In the other direction, observe that by construction, $\Psi_0(\lambda) = 1$ for $\lambda \in [0, 1]$ and $\Psi_{j-1}(\lambda) + \Psi_j(\lambda) = 1$ for $\lambda \in [2^{j-1}, 2^j]$, $j \geq 1$. Hence

$$\sup_x \sum_{\sqrt{\lambda_k}\leq 1} \nu_k u_k^2(x) \leq \sup_x \sum_k \Psi_0(\sqrt{\lambda_k})\nu_k u_k^2(x) \quad \text{and}$$

$$\sup_x \sum_{2^{j-1}<\sqrt{\lambda_k}\leq 2^j} \nu_k u_k^2(x) \leq \sup_x \sum_k \Psi_{j-1}(\sqrt{\lambda_k})\nu_k u_k^2(x) + \sup_x \sum_k \Psi_j(\sqrt{\lambda_k})\nu_k u_k^2(x), \quad j \geq 1.$$

These inequalities and (5.2) imply that the right-hand side in (5.1) is dominated by a constant multiple of the left-hand side. This completes the proof. □

The following corollary is an indication of how the Besov regularity relates to the "dimension" d of the set M, which appears here through the doubling condition (3.3).

Corollary 5.2 *Let $\gamma > d$ and $s = \gamma - d$. Then*

$$\nu_k = O\big(\sqrt{\lambda_k}\big)^{-\gamma} \implies \sup_x \|K(x, \bullet)\|_{B^s_{\infty,\infty}} \leq c.$$

Proof If $\nu_k \leq c\big(\sqrt{\lambda_k}\big)^{-\gamma}$, then using (3.10) and (3.4), we get for any $j \geq 1$ and $x \in M$,

$$\sum_{k:2^{j-1}\leq\sqrt{\lambda_k}\leq 2^j} \nu_k u_k^2(x) \leq c2^{-\gamma(j+1)} \sum_{k:2^{j-1}\leq\sqrt{\lambda_k}\leq 2^j} u_k^2(x) \leq c2^{-\gamma j} \sum_{k:\sqrt{\lambda_k}\leq 2^j} u_k^2(x)$$
$$= c2^{-\gamma j}\Pi_{2^j}(x, x) \leq c2^{-\gamma j}|B(x, 2^{-j})|^{-1} \leq c2^{-j(\gamma-d)}.$$

A similar estimate with $j = 0$ holds for all k such that $\sqrt{\lambda_k} \leq 1$. Then the corollary follows by Theorem 5.1. □

Remark 5.3 Observe that

$$\sup_x \sum_{k:2^{j-1}\leq\sqrt{\lambda_k}\leq 2^j} \nu_k u_k^2(x) \leq c2^{-js}$$

implies

$$\sum_{k:2^{j-1}\leq\sqrt{\lambda_k}\leq 2^j} \nu_k = \sum_{k:2^{j-1}\leq\sqrt{\lambda_k}\leq 2^j} \int_M \nu_k u_k^2(x)d\mu(x) \leq c2^{-js}|M|.$$

We will utilize maximal δ-nets on M along with Proposition 3.1 for discretization. For any $j \geq 0$, we denote by $\mathcal{X}_j$ the maximal δ-net from Proposition 3.1 with $\delta :=$

$\gamma 2^{-j-1}$ such that

$$2^{-1}\|g\|_\infty \le \max_{\xi\in\mathcal{X}_j} |g(\xi)| \le \|g\|_\infty, \quad \forall g \in \Sigma_{2^{j+1}}. \tag{5.3}$$

The following claim will be instrumental in the proof of Theorem 4.6.

Proposition 5.4 *We have*

$$\sup_{x\in M} \sum_{k:\sqrt{\lambda_k}\le 1} \nu_k u_k^2(x) \sim \max_{\xi\in\mathcal{X}_0} \sum_{k:\sqrt{\lambda_k}\le 1} \nu_k u_k^2(\xi)$$

and, for any $j \ge 1$,

$$\sup_{x\in M} \sum_{k:2^{j-1}<\sqrt{\lambda_k}\le 2^j} \nu_k u_k^2(x) \sim \max_{\xi\in\mathcal{X}_j} \sum_{k:2^{j-1}<\sqrt{\lambda_k}\le 2^j} \nu_k u_k^2(\xi)$$

with absolute constants of equivalence.

This proposition follows readily from the following:

Lemma 5.5 *Let* $\mathcal{X}_j$ *be the maximal* δ*-net from above with* $\delta := \gamma 2^{-j}$, $j \ge 0$, *and let*

$$H(x, y) := \sum_{\sqrt{\lambda_k}\le 2^j} \alpha_k u_k(x) u_k(y), \quad \textit{where} \quad \alpha_k \ge 0.$$

Then

$$\max_{\xi\in\mathcal{X}_j} H(\xi,\xi) \le \sup_{x,y\in M} |H(x,y)| \le 4 \max_{\xi\in\mathcal{X}_j} H(\xi,\xi).$$

Proof Clearly, $H(x, y)$ is a positive definite function, and hence $|H(x, y)| \le \sqrt{H(x,x)H(y,y)}$, implying

$$\max_{\xi,\eta\in\mathcal{X}_j} |H(\xi,\eta)| = \max_{\xi\in\mathcal{X}_j} H(\xi,\xi). \tag{5.4}$$

Evidently, for any fixed $x \in M$, the function $H(x, y) \in \Sigma_{2^j}$ as a function of y and by (5.3),

$$\sup_{y\in M} |H(x,y)| \le 2 \max_{\eta\in\mathcal{X}_j} |H(x,\eta)|.$$

Now, using that $H(x,\eta) \in \Sigma_{2^j}$ as a function of x, we again apply (5.3) to obtain

$$\begin{aligned} \sup_{x,y\in M} |H(x,y)| &\le 2 \sup_{x\in M} \max_{\eta\in\mathcal{X}_j} |H(x,\xi)| = 2 \max_{\eta\in\mathcal{X}_j} \sup_{x\in M} |H(x,\eta)| \\ &\le 4 \max_{\eta\in\mathcal{X}_j} \max_{\xi\in\mathcal{X}_j} |H(\xi,\eta)| = 4 \max_{\xi\in\mathcal{X}_j} H(\xi,\xi). \end{aligned}$$

Here for the last equality we used (5.4). This completes the proof. □

5.2 Proof of Theorem 4.6

(a) Assume $\sup_{x\in M} \|K(x,\bullet)\|_{B^s_{\infty,\infty}} < \infty$. Let $(B_k(\omega))_{k\geq 1}$ be a sequence of independent $N(0,1)$ variables. Then, as alluded to in Sect. 2.2,

$$\tilde{Z}_x(\omega) := \sum_k \sqrt{\nu_k} u_k(x) B_k(\omega)$$

is also a version of $Z_x(\omega)$. Let Ψ_j, $j \geq 0$, be the functions from (3.17), and observe that $f \in B^s_{\infty,1}$ if and only if $\|f\|_{B^s_{\infty,1}} \sim \sum_{j\geq 0} 2^{js} \|\Psi_j(\sqrt{A}) f\|_\infty < \infty$. Clearly,

$$\big(\Psi_j(\sqrt{A})\tilde{Z}_\bullet(\omega)\big)(x) = \sum_k \Psi_j(\sqrt{\lambda_k})\sqrt{\nu_k} u_k(x) B_k(\omega). \tag{5.5}$$

For each $x \in M$, this is a Gaussian variable of variance

$$\sigma_j^2(x) = \sum_k \Psi_j^2(\sqrt{\lambda_k})\nu_k u_k(x)^2 \leq c 2^{-js}.$$

Here we used that $\Psi_j^2(\sqrt{\lambda_k}) \leq 1$, the assumption $\sup_{x\in M} \|K(x,\bullet)\|_{B^s_{\infty,\infty}} < \infty$, and Theorem 5.1.

For any $\alpha > 0$, we have

$$\begin{aligned} \mathbb{E}\Big(\sum_j 2^{j\alpha} \|\Psi_j(\sqrt{A})\tilde{Z}_\bullet(\omega)\|_\infty\Big) &= \sum_j 2^{j\alpha}\mathbb{E}\big(\|\Psi_j(\sqrt{A})\tilde{Z}_\bullet(\omega)\|_\infty\big) \\ &\sim \sum_j 2^{j\alpha}\mathbb{E}\left(\sup_{\xi\in\mathcal{X}_j} |\big(\Psi_j(\sqrt{A})\tilde{Z}_\bullet(\omega)\big)(\xi)|\right) \\ &\leq c\sum_j 2^{j\alpha}2^{-js/2}(1+\log(\mathrm{card}(\mathcal{X}_j)))^{1/2}. \end{aligned}$$

Above, for the equivalence, we used (5.3), and for the last inequality, the following well-known inequality (see, e.g., [21, Lemma 2.3.4] or [34, lemma 10.1]): If $Z_1,\ldots,Z_N$ are centered Gaussian variables (with arbitrary variances), then

$$\mathbb{E}\big(\max_{1\leq k\leq N} |Z_k|\big) \leq c(1+\log N)^{1/2} \max_k \big(\mathbb{E}|Z_k|^2\big)^{1/2}.$$

By (3.14), we have $\mathrm{card}(\mathcal{X}_j) \leq c2^{jd}$. Therefore, if $\alpha < \frac{s}{2}$, then

$$\sum_j 2^{j\alpha}2^{-js/2}(1+\log(\mathrm{card}(\mathcal{X}_j)))^{1/2} \leq c\sum_j 2^{-j(s/2-\alpha)}\big(\log(c2^{jd})\big)^{1/2} < \infty.$$

Consequently, $\mathbb{E}\Big(\sum_j 2^{j\alpha}\|\Psi_j(\sqrt{A})Z_\bullet(\omega)\|_\infty\Big) < \infty$, and hence $x \mapsto \tilde{Z}_x(\omega) \in B^\alpha_{\infty,1}$, $0<\alpha<s/2$, ω-a.s.

(b) Suppose now that $\omega - a.e.$, $x \mapsto Z_x(\omega) \in B^\alpha_{\infty,\infty}$, $\alpha > 0$. Then by (5.5) and (3.19),

$$\sup_j 2^{j\alpha}\Big\|\sum_k \Psi_j(\sqrt{\lambda_k})\sqrt{\nu_k}u_k(x)B_k(\omega)\Big\|_\infty < \infty, \quad \omega - \text{a.s.}$$

By (5.3), this is equivalent to

$$\sup_j 2^{j\alpha}\max_{\xi\in\mathcal{X}_j}\Big|\sum_k \Psi_j(\sqrt{\lambda_k})\sqrt{\nu_k}u_k(\xi)B_k(\omega)\Big| < \infty, \quad \omega - \text{a.s.} \tag{5.6}$$

However, $\{2^{j\alpha}\sum_k \Psi_j(\sqrt{\lambda_k})\sqrt{\nu_k}u_k(\xi)B_k(\omega)\}_{j\in\mathbb{N},\xi\in\mathcal{X}_j}$ is a countable set of Gaussian centered variables. The Borell–Ibragimov–Sudakov–Tsirelson theorem (see, e.g., [31], Theorem 7.1), in particular, asserts that if $(G_t)_{t\in T}$ is a centered Gaussian process indexed by a countable parameter set T and $\sup_{t\in T} G_t < \infty$ almost surely, then $\sup_{t\in T}\mathbb{E}(G_t^2) < \infty$. Consequently, (5.6) implies

$$\sup_{j\in\mathbb{N},\xi\in\mathcal{X}_j}\mathbb{E}\Big(2^{j\alpha}\sum_k \Psi_j(\sqrt{\lambda_k})\sqrt{\nu_k}u_k(\xi)B_k\Big)^2 < \infty.$$

Therefore, there exists a constant $C>0$ such that

$$\max_{\xi\in\mathcal{X}_j}\sum_k \Psi_j^2(\sqrt{\lambda_k})\nu_k u_k^2(\xi) \le C2^{-2j\alpha}.$$

But as before, this yields

$$\max_{\xi\in\mathcal{X}_0}\sum_{k:\sqrt{\lambda_k}\le 1}\nu_k u_k^2(\xi) \le \max_{\xi\in\mathcal{X}_0}\sum_k \Psi_0^2(\sqrt{\lambda_k})\nu_k u_k^2(\xi),$$

and, for $j\ge 1$,

$$\begin{aligned}\max_{\xi\in\mathcal{X}_j}\sum_{k,2^{j-1}\le\sqrt{\lambda_k}\le 2^j}\nu_k u_k^2(\xi) &\le 2\max_{\xi\in\mathcal{X}_j}\sum_k \Psi_{j-1}^2(\sqrt{\lambda_k})\nu_k u_k^2(\xi)\\ &\quad + 2\max_{\xi\in\mathcal{X}_j}\sum_k \Psi_j^2(\sqrt{\lambda_k})\nu_k u_k^2(\xi) \le c2^{-2j\alpha}.\end{aligned}$$

Here we used that $\Psi_{j-1}(\lambda)+\Psi_j(\lambda) = 1$ for $\lambda\in[2^{j-1},2^j]$, implying $\Psi_{j-1}^2(\lambda) + \Psi_j^2(\lambda) \ge 1/2$.

Finally, applying Proposition 5.4, we conclude from above that

$$\sup_{x\in M} \|K(x,\bullet)\|_{B^{2\alpha}_{\infty,\infty}} < \infty.$$

The proof is complete. □

5.3 Proof of Theorem 4.8

We begin with the following:

Lemma 5.6 *Assume $s > 0$ and $1 \le p \le \infty$, and let Ψ_j, $j \ge 0$, be the functions from* (3.17). *Then*

$$f \in B^s_{p,1} \Longleftrightarrow \sum_{j\ge 0} \|\Psi_j(\sqrt{A})f\|_{B^s_{p,1}} < \infty \quad \text{and} \quad \|f\|_{B^s_{p,1}} \sim \sum_{j\ge 0} \|\Psi_j(\sqrt{A})f\|_{B^s_{p,1}}.$$

Proof From (3.18), we have for any $f \in L^p$,

$$f = \sum_{j\ge 0} \Psi_j(\sqrt{A})f, \tag{5.7}$$

implying $\|f\|_{B^s_{p,1}} \le \sum_{j\ge 0} \|\Psi_j(\sqrt{A})f\|_{B^s_{p,1}}$.

For the estimate in the other direction, note that by (3.19),

$$\|\Psi_j(\sqrt{A})f\|_{B^s_{p,1}} \sim \sum_{\ell\ge 0} 2^{\ell s}\|\Psi_\ell(\sqrt{A})\Psi_j(\sqrt{A})f\|_p.$$

However, supp $\Psi_j \cap \mathbb{R}_+ \subset [2^{j-1}, 2^{j+1}]$, $j \ge 1$, and hence $\Psi_\ell(\sqrt{A})\Psi_j(\sqrt{A}) = 0$ if $|\ell - j| > 1$. Therefore,

$$\|\Psi_j(\sqrt{A})f\|_{B^s_{p,1}} \sim \sum_{j-1\le\ell\le j+1} 2^{\ell s}\|\Psi_\ell(\sqrt{A})\Psi_j(\sqrt{A})f\|_p.$$

On the other hand, by estimate (3.12), it follows that $\|\Psi_j(\sqrt{A})g\|_p \le c\|g\|_p, \forall g \in L^p$, and hence $\|\Psi_\ell(\sqrt{A})\Psi_j(\sqrt{A})f\|_p \le c\|\Psi_j(\sqrt{A})f\|_p$, implying

$$\|\Psi_j(\sqrt{A})f\|_{B^s_{p,1}} \le c2^{js}\|\Psi_j(\sqrt{A})f\|_p.$$

This in turn leads to

$$\sum_{j\ge 0}\|\Psi_j(\sqrt{A})f\|_{B^s_{p,1}} \le c\sum_{j\ge 0} 2^{js}\|\Psi_j(\sqrt{A})f\|_p \le c\|f\|_{B^s_{p,1}}.$$

The proof is complete. □

We now complement Theorem 4.6 with the following:

Proposition 5.7 *Under the hypotheses of Theorem 4.6 and with the functions* Ψ_j, $j \geq 0$, *from* (3.17), *if* $\sup_{x\in M} \|K(x,\bullet)\|_{B^s_{\infty,\infty}} < \infty$, *then*

$$\mathbb{E}\Big(\sum_{j\geq 0} \|\Psi_j(\sqrt{A})Z_\bullet(\omega))\|_{B^\alpha_{\infty,1}}\Big) \sim \mathbb{E}\Big(\sum_{j\geq 0} 2^{j\alpha}\|\Psi_j(\sqrt{A})Z_\bullet(\omega)\|_\infty\Big) < \infty, \tag{5.8}$$

the map

$$I : \omega \in \Omega \mapsto \sum_j \psi_j(\sqrt{A})Z_\bullet(\omega)(\cdot) \in B^\alpha_{\infty,1}$$

is measurable, the series converges in the norm of $B^\alpha_{\infty,1}$, *and the image probability* Q *on* $B^\alpha_{\infty,1}$ *satisfies:*

$$\omega \in B^\alpha_{\infty,1} \xrightarrow{\delta_x} \omega(x)$$

is a centered Gaussian process with covariance $K(x,y)$.

Proof The equivalence (5.8) follows from the proof of Theorem 4.6, (a) and Lemma 5.6.

As is well known, for any Banach space B with a measure space $(\Omega,\mathcal{B})$, if G is a finite set of indices $b_i \in B$ and $X_i(\omega)$ are real-valued measurable functions, then $\omega \mapsto \sum_{i\in G} X_i(\omega)b_i$ is measurable from Ω to B. Hence,

$$\omega \in \Omega \mapsto \Psi_j(\sqrt{A})Z_\bullet(\omega) = \sum_k \Psi_j(\sqrt{\lambda_k})\sqrt{\nu_k}u_k(\bullet)B_k(\omega) \in B^\alpha_{\infty,1}$$

is measurable. Consequently, by almost everywhere convergence,

$$I : \omega \in \Omega \mapsto \sum_j \Psi_j(\sqrt{A})Z_\bullet(\omega)(\cdot) \in B^\alpha_{\infty,1}$$

is also measurable, and the image probability measure (of P by I) $I^*(P) = Q$ is a probability measure on the Borel sigma-algebra such that under Q the family of random variables δ_x,

$$\omega \in B^\alpha_{\infty,1} \xrightarrow{\delta_x} \omega(x),$$

is a centered Gaussian process with covariance $K(x,y) = \int_{B^\alpha_{\infty,1}} \omega(x)\omega(y)dQ(\omega)$. □

Finally, Theorem 4.8 holds due to the fact that $B^\alpha_{\infty,1}$ is separable (see "Appendix II"). It also proves Part (b) of Theorem 4.6.

5.4 Proof of Theorem 4.9

Suppose that $\sup_{x\in M} \|K(x,\bullet)\|_{B^s_{\infty,\infty}} < \infty$, and let $f(x) = \sum_{k\in\mathbb{N}(\nu)} \alpha_k \sqrt{\nu_k}\, u_k(x)$, where $(\alpha_k) \in \ell^2$. Then

$$\Psi_j(\sqrt{A}) f(x) = \sum_{k\in\mathbb{N}(\nu)} \Psi_j(\sqrt{\lambda_k})\alpha_k \sqrt{\nu_k}\, u_k(x),$$

implying, for $j \geq 1$,

$$\begin{aligned}|\Psi_j(\sqrt{A}) f(x)| &\leq \Big(\sum_{k\in\mathbb{N}(\nu)} |\alpha_k|^2\Big)^{\frac{1}{2}} \Big(\sum_{k\in\mathbb{N}(\nu)} |\Psi_j(\sqrt{\lambda_k})|^2 \nu_k |u_k(x)|^2\Big)^{\frac{1}{2}}\\ &\leq \|f\|_{\mathbb{H}_K} \Big(\sum_{k:2^{j-1}\leq\lambda_k\leq 2^{j+1}} \nu_k |u_k(x)|^2\Big)^{\frac{1}{2}} \leq c\|f\|_{\mathbb{H}_K} 2^{-js/2},\end{aligned}$$

where for the last inequality, we used the assumption and Theorem 5.1. Similarly, $|\Psi_0(\sqrt{A}) f(x)| \leq c\|f\|_{\mathbb{H}_K}$. Therefore, in light of (3.19),

$$\|f\|_{B^{\frac{s}{2}}_{\infty,\infty}} \leq c\|f\|_{\mathbb{H}_K}. \tag{5.9}$$

Assume that (5.9) holds. Then for every sequence $(\alpha_k) \in \ell^2$ with $\|(\alpha_k)\|_{\ell^2} \leq 1$, we have

$$\Big|\sum_{k\in\mathbb{N}(\nu)} \Psi_j(\sqrt{\lambda_k})\alpha_k \sqrt{\nu_k}\, u_k(x)\Big| \leq c2^{-js/2}, \quad \forall x \in M,$$

which by duality implies

$$\Big(\sum_{k\in\mathbb{N}(\nu)} |\psi_j(\sqrt{\lambda_k})|^2 \nu_k\, |u_k(x)|^2\Big)^{\frac{1}{2}} \leq c2^{-js/2}, \quad j \geq 0.$$

Just as in the proof of Theorem 5.1, we get for $j \geq 1$,

$$\begin{aligned}\sum_{k:2^{j-1}\leq\sqrt{\lambda_k}\leq 2^j} \nu_k u_k^2(x) &\leq \sum_{k\in\mathbb{N}(\nu)} |\Psi_{j-1}(\sqrt{\lambda_k})|^2 \nu_k\, |u_k(x)|^2\\ &\quad + \sum_{k\in\mathbb{N}(\nu)} |\Psi_j(\sqrt{\lambda_k})|^2 \nu_k\, |u_k(x)|^2 \leq c2^{-js},\end{aligned}$$

and similarly, $\sum_{k:\sqrt{\lambda_k}\leq 1} \nu_k u_k^2(x) \leq c$. Consequently, $\sup_{x\in M} \|K(x,\bullet)\|_{B^s_{\infty,\infty}} < \infty$, which completes the proof. □

Remark 5.8 Let $f \in L^2(M, \mu)$. Clearly,

$$\tilde{f}(\omega) : \omega \in W = B^{\alpha}_{\infty,1} \mapsto \int_M f(x)\omega(x)d\mu(x)$$

belongs to W^*. Hence, under Q_α, $\tilde{f}$ is a Gaussian variable and

$$\begin{aligned}
\mathbb{E}(\tilde{f})^2 &= \int_W \Big(\int_M f(x)\omega(x)d\mu(x)\Big)^2 dQ_\alpha(\omega) \\
&= \int_W \int_M f(x)\omega(x)d\mu(x) \int_M f(y)\omega(y)d\mu(y)dQ_\alpha(\omega) \\
&= \int_M \int_M f(x)f(y)\Big(\int_W \omega(x)\omega(y)dQ_\alpha(\omega)\Big)d\mu(x)d\mu(y) = \langle Kf, f\rangle_{L^2(M,\mu)}.
\end{aligned}$$

Consequently,

$$\int_W e^{i\tilde{f}(\omega)}dQ_\alpha(\omega) = e^{-\frac{1}{2}\langle Kf,f\rangle_{L^2(M,\mu)}}.$$

6 Positive and Negative Definite Functions on Compact Homogeneous Spaces

For the reader's convenience, we recall the basics of the general theory of positive definite (P.D.) and negative definite (N.D.) functions in "Appendix I". Here we present some basic facts about positive and negative definite kernels in the general setting of compact two point homogeneous spaces. In the next section, we use these results and Theorem 4.6 to establish the Besov regularity of Gaussian processes indexed by the sphere.

6.1 Group Acting on a Space

Let (M, μ) be a compact space equipped with a positive Radon measure μ. Assume that there exists a group G acting transitively on (M, μ); that is, there exists a map $(g, x) \in G \times M \mapsto g \cdot x \in M$ such that:

1. $h \cdot (g \cdot x) = (hg) \cdot x$, $\forall g, h \in G$,
2. $\exists e \in G$ s.t. $e \cdot x = x$, $\forall x \in M$ (e is the neutral element in G),
3. $\forall x, y \in M$, $\exists g \in G$ s.t. $g \cdot x = y$ (transitivity), and
4. $\int_M (\gamma(g)f)(x)d\mu(x) = \int_M f(g^{-1} \cdot x)d\mu(x) = \int_M f(x)d\mu(x)$ $\forall g \in G, \forall f \in L^1$, where $(\gamma(g)f)(x) := f(g^{-1} \cdot x)$. Hence, $(\gamma(g))_{g\in G}$ is a group of L^1-isometry.

Definition 6.1 A continuous real-valued kernel $K(x, y)$ on $M \times M$ is said to be G-invariant if

$$K(g \cdot x, g \cdot y) = K(x, y), \quad \forall g \in G, \forall x, y \in M.$$

If K is the operator on L^2 with kernel $K(x, y)$, then K is called G-invariant if $\gamma(g)K = K\gamma(g)$, $\forall g \in G$; that is,

$$\int_M K(g^{-1} \cdot x, y) f(y) d\mu(y) = \int_M K(x, y) f(g^{-1} \cdot y) d\mu(y), \quad \forall f \in L^2.$$

Remark 6.2 (*a*) If $K(x, y)$ is a continuous G-invariant kernel, then:
(*i*) $K(x, x) = K(g \cdot x, g \cdot x)$, and hence $K(x, x) \equiv |M|^{-1} \operatorname{Tr}(K)$, and
(*ii*)

$$\int_M K(x, y) d\mu(y) = \int_M K(x, g \cdot y) d\mu(y) = \int_M K(g^{-1} \cdot x, y) d\mu(y), \quad \forall g \in G,$$

and hence $\mathbb{1} := \mathbb{1}_M$ is an eigenfunction of K, i.e.,

$$\int_M K(x, y) \mathbb{1}(y) d\mu(y) = \lambda \mathbb{1}(x), \quad \int_M \int_M K(x, y) d\mu(x) d\mu(y) = \lambda |M|.$$

(*b*) If $K(x, y)$ is a continuous positive G-invariant kernel, then

$$\begin{aligned} \psi_K(x, y) &:= K(x, x) + K(y, y) - 2K(x, y) \\ &= 2(C - K(x, y)) = 2(|M|^{-1} \operatorname{Tr}(K) - K(x, y)) \end{aligned}$$

is G-invariant, and by (8.3) (see "Appendix I"),

$$\tilde{K}(x, y) = K(x, y) + |M|^{-1}(\operatorname{Tr}(K) - 2C').$$

(*c*) Suppose $\psi(x, y)$ is a G-invariant N.D. kernel, and consider the associated P.D. kernel $\tilde{K}$, defined as in (8.2). Then $\tilde{K}(x, y)$ is G-invariant, and

$$x \mapsto \frac{1}{|M|} \int_M \psi(x, u) d\mu(u) \equiv C_0 \ \text{ and } \ \tilde{K}(x, y) = C_0 - \frac{1}{2} \psi(x, y).$$

Thus, in this framework, there is one-to-one correspondence up to a constant between invariant P.D. and N.D. kernels.

6.2 Composition of Operators

Let $K(x, y)$ and $H(x, y)$ be two continuous kernels on $M \times M$ as above, and let K and H be the associated operators. The operator $K \circ H$ is also a kernel operator with kernel $K \circ H(x, y)$:

$$K \circ H(x, y) = \int_M K(x, u) H(u, y) d\mu(u).$$

Observe that:

(1) If $K(x, y) = K(y, x)$, $H(x, y) = H(y, x)$, then

$$K \circ H(x, y) = \int_M K(x, u)H(u, y)d\mu(u) = \int_M H(y, u)K(u, x)d\mu(u) \\ = H \circ K(y, x).$$

(2) If $K(x, y)$ and $H(x, y)$ are G-invariant, then so is $K \circ H$. Indeed,

$$\begin{aligned} K \circ H(g \cdot x, g \cdot y) &= \int_M K(g \cdot x, u)H(u, g \cdot y)d\mu(u) \\ &= \int_M K(g \cdot x, g \cdot u)H(g \cdot u, g \cdot y)d\mu(u) \\ &= \int_M K(x, u)H(u, y)d\mu(u) = K \circ H(x, y). \end{aligned}$$

6.3 Group Action and Metric

Assume that we are in the setting of a Dirichlet space defined through a non-negative self-adjoint operator on $L^2(M, \mu)$ just as in Sect. 3.1. Suppose now that

$$\gamma(g)A = A\gamma(g), \quad \forall g \in G.$$

or equivalently,

$$\gamma(g)P_t = P_t\gamma(g), \quad \forall t > 0,\ \forall g \in G;$$

i.e., $\forall t > 0$, $p_t(x, y)$ is G-invariant. Clearly, $\Gamma(f_1, f_2)$ is also G-invariant: $\Gamma(f_1, f_2) = \Gamma(\gamma(g)f_1, \gamma(g)f_2)$ and the associated metric $\rho(x, y)$ is G-invariant:

$$\rho(g \cdot x, g \cdot y) = \rho(x, y), \quad \forall g \in G.$$

Definition 6.3 In the current framework, (M, μ, A, ρ, G) is said to be a two-point homogeneous space if

$$\forall x, y, x', y' \in M \text{ s.t. } \rho(x, y) = \rho(x', y'),\ \exists g \in G \text{ s.t. } g \cdot x = x',\ g \cdot y = y'.$$

In particular, $\forall (x, y) \in M \times M$, $\exists g \in G$ s.t. $g \cdot x = y$, $g \cdot y = x$.

Theorem 6.4 *Let (M, μ, A, ρ, G) be a compact two-point homogeneous space. Then:*

(1) *Any G-invariant continuous kernel $K(x, y)$ is symmetric.*

(2) *If $K(x, y)$ and $H(x, y)$ are two G-invariant continuous kernels, then $K \circ H = H \circ K$. In particular, if $K(x, y)$ is a G-invariant continuous kernel, then $KA = AK$.*

(3) *Any G-invariant real-valued continuous kernel $K(x, y)$ depends only on the distance $\rho(x, y)$; that is, there exists a continuous function $k : \mathbb{R} \mapsto \mathbb{R}$, such that*

$$K(x, y) = k(\rho(x, y)), \quad \forall x, y \in M.$$

This theorem is a straightforward consequence of the observations from Sect. 6.2 and the definition of two-point homogeneous spaces.

Now let M be a compact Riemannian manifold, and assume that $A := -\Delta_M$ is the Laplacian on M, ρ is the Riemannian metric, and μ is the Riemannian measure. Also, assume that there exists a compact Lie group G of isometries on M such that $(M, \mu, -\Delta_M, \rho, G)$ is a compact two-point homogeneous space. For the connection of the above setting with Gaussian processes, see [6,20].

Let $0 \le \lambda_1 < \lambda_2 < \cdots$ be the spectrum of $-\Delta_M$. Then the eigenspaces $\mathcal{H}_{\lambda_k} := \mathrm{Ker}(\Delta_M + \lambda_k \, \mathrm{Id})$ are finite dimensional, and

$$L^2(M, \mu) = \bigoplus_{k \ge 1} \mathcal{H}_{\lambda_k}.$$

Let $P_{\mathcal{H}_{\lambda_k}}(x, y)$ be the kernel of the orthogonal projector onto $\mathcal{H}_{\lambda_k}$. Then if $K(x, y)$ is a G-invariant positive definite kernel, we have the following decomposition of $K(x, y)$, which follows from the Bochner–Godement theorem [17,24]:

$$K(x, y) = \sum_{k \ge 0} \nu_k P_{\mathcal{H}_{\lambda_k}}(x, y), \quad \nu_k \ge 0.$$

7 Gaussian Process on the Sphere

In this section, we apply our main result (Theorem 4.6) to a Gaussian process parametrized by the unit sphere $\mathbb{S}^d$ in $\mathbb{R}^{d+1}$. This is a Riemannian manifold and a compact two-point homogeneous space. More explicitly,

$$G = SO(d+1), \; H = SO(d), \; G/H = \mathbb{S}^d.$$

The geodesic distance ρ on $\mathbb{S}^d$ is given by

$$\rho(\xi, \eta) = \arccos\langle \xi, \eta \rangle,$$

where $\langle \xi, \eta \rangle$ is the inner product of $\xi, \eta \in \mathbb{R}^{d+1}$. Clearly,

$$\begin{aligned} &\forall \xi, \eta \in \mathbb{S}^d, \; \forall g \in G, \quad \rho(g \cdot \xi, g \cdot \eta) = \rho(\xi, \eta), \quad \text{and} \\ &\forall \xi, \eta \in \mathbb{S}^d, \; \exists g \in G \;\; \text{s.t.} \;\; g \cdot \xi = \eta. \end{aligned}$$

Thus G acts isometrically and transitively on $\mathbb{S}^d$. Furthermore, $\forall \xi, \eta, \xi', \eta' \in \mathbb{S}^d$ s.t. $\rho(\xi, \eta) = \rho(\xi', \eta')$ there exists $g \in G$ s.t. $g \cdot \xi = \xi'$ and $g \cdot \eta = \eta'$. Therefore, $\mathbb{S}^d$ is a compact two-point homogeneous space.

Let $-\Delta_{\mathbb{S}^d}$ be the (positive) Laplace–Beltrami operator on $\mathbb{S}^d$. As is well known, the eigenspaces of $-\Delta_{\mathbb{S}^d}$ are the spaces of spherical harmonics, defined by

$$\mathcal{H}_{\lambda_k} := \mathrm{Ker}(\Delta_{\mathbb{S}^d} + \lambda_k I_d), \quad \lambda_k := k(k+d-1) = k(k+2\nu), \quad k \geq 0 \quad \nu := \frac{d-1}{2}.$$

One has $L^2(\mathbb{S}^d) = \bigoplus_{k\geq 0} \mathcal{H}_{\lambda_k}$, and the kernel of the orthogonal projector $P_{\mathcal{H}_{\lambda_k}}$ onto $\mathcal{H}_{\lambda_k}$ is given by

$$P_{\mathcal{H}_{\lambda_k}}(\xi,\eta) = L_k^d(\langle \xi,\eta\rangle), \quad L_k^d(x) := |\mathbb{S}^d|^{-1}\Big(1+\frac{k}{\nu}\Big)C_k^\nu(x).$$

Here $C_k^\nu(x)$, $k \geq 0$, are the Gegenbauer polynomials defined on $[-1,1]$ by the generating function

$$\frac{1}{(1-2xr+r^2)^\nu} = \sum_{k\geq 0} r^k C_k^\nu(x).$$

Therefore,

$$-\Delta_{\mathbb{S}^d} f = \sum_{k\geq 0} k(k+2\nu) P_{\mathcal{H}_{\lambda_k}} f,$$

and the invariant continuous positive definite functions on $\mathbb{S}^d$ are of the form

$$K(\xi,\eta) = \sum_k \nu_k L_k^d(\langle \xi,\eta\rangle) = \sum_k \nu_k L_k^d(\cos\rho(\xi,\eta)),$$

where

$$\sum_k \nu_k L_k^d(1) = \sum_k \nu_k L_k^d(\langle \xi,\xi\rangle) < \infty.$$

Note that

$$\begin{aligned} L_k^\nu(1)|\mathbb{S}^d| &= \int_{\mathbb{S}^d} L_k^\nu(\langle \xi,\xi\rangle) d\mu(\xi) \\ &= \dim(\mathcal{H}_{\lambda_k}(\mathbb{S}^d)) = \binom{k+d}{d} - \binom{k-2+d}{d} \sim k^{d-1}. \end{aligned}$$

Let

$$W_k^\nu(x) := \frac{L_k^\nu(x)}{L_k^\nu(1)} = \frac{C_k^\nu(x)}{C_k^\nu(1)}. \quad \text{Clearly,} \quad W_k^\nu(1) = \sup_{x\in[-1,1]} |W_k^\nu(x)| = 1.$$

Then (see [8])

$$\lim_{\nu\mapsto 0}\frac{C_k^\nu(x)}{C_k^\nu(1)} = T_k(x) \quad \left(= W_k^0(x) \text{ by convention}\right),$$
$$\lim_{\nu\mapsto \infty}\frac{C_k^\nu(x)}{C_k^\nu(1)} = x^k \quad \left(= W_k^\infty(x) \text{ by convention}\right).$$

Here T_k is the Chebyshev polynomial of the first kind ($T_k(\cos\theta) = \cos k\theta$). The invariant continuous positive definite functions on $\mathbb{S}^d$ are of the form

$$K^\nu(\xi,\eta) = \sum_{k\ge 0} a_k^\nu W_k^\nu(\langle \xi,\eta\rangle) = \sum_{k\ge 0} a_k^\nu W_k^\nu(\cos\rho(\xi,\eta)), \quad a_k^\nu \ge 0, \quad \sum_k a_k^\nu < \infty.$$

Clearly,

$$\sum_k a_k^\nu W_k^\nu(\cos\rho(\xi,\eta)) = \sum_k \frac{a_k^\nu}{L_k^\nu(1)} L_k^\nu(\cos\rho(\xi,\eta)), \quad L_k^\nu(1) \sim k^{d-1}. \tag{7.1}$$

Therefore,

$$\nu_k = |\mathbb{S}^d| \frac{a_k^\nu}{\dim(\mathcal{H}_{\lambda_k})} = O\left(\frac{a_k^\nu}{k^{d-1}}\right).$$

The following **Schoenberg–Bingham result** (see, e.g., [8]) plays a key role here: *If f is a continuous function defined on $[-1,1]$, then $f(\langle\xi,\eta\rangle)$ is a positive definite function on $\mathbb{S}^d$ and invariant with respect to $SO(d+1)$ for all $d \in \mathbb{N}$ if and only if*

$$f(x) = \sum_{n\ge 0} a_n x^n, \quad \textit{where} \quad a_n \ge 0 \quad \textit{and} \quad \sum_{n\ge 0} a_n = f(1) < \infty.$$

Therefore, for such a function f,

$$f(x) = \sum_{k\ge 0} a_k^\nu W_k^\nu(x), \quad a_k^\nu \ge 0, \quad \text{and} \quad \sum_{k\ge 0} a_k^\nu = \sum_{k\ge 0} a_k = f(1),$$

and hence

$$f(\langle\xi,\eta\rangle) = \sum_{k\ge 0} a_k^\nu W_k^\nu(\langle\xi,\eta\rangle) = \sum_{k\ge 0} \frac{a_k^\nu}{L_k^\nu(1)} L_k^\nu(\langle\xi,\eta\rangle) = f(\cos\rho(\xi,\eta)).$$

7.1 Fractional Brownian Process on the Sphere

Theorem 7.1 *For any* $0 < \alpha \le 1$, *the function*

$$\psi(\xi,\eta) = \rho(\xi,\eta)^\alpha, \quad \xi,\eta \in \mathbb{S}^d,$$

is negative definite, and the associated Gaussian process has almost everywhere regularity $B^{\gamma}_{\infty,1}$, $\gamma < \frac{\alpha}{2}$.

Proof Consider first the case when $\alpha = 1$ (*Brownian process*). We will show that for some constant $C > 0$, the function $C - \rho(\xi,\eta)$ is an invariant positive definite function. To this end, by the Schoenberg–Bingham result, we have to prove that there exists a function

$$f(x) = \sum a_n x^n, \quad \text{with } a_n \geq 0, \ \sum_{n\geq 0} a_n < \infty,$$

such that $f(\langle\xi,\eta\rangle) = f(\cos\rho(\xi,\eta)) = C - \rho(\xi,\eta)$. Luckily the function $\frac{\pi}{2} - \arccos x$ does the job. Indeed, it is easy to see that

$$f(x) := \frac{\pi}{2} - \arccos x = \arcsin x = \sum_{j\geq 0} \frac{\left(\frac{1}{2}\right)_j \left(\frac{1}{2}\right)_j}{j!\left(\frac{3}{2}\right)_j} x^{2j+1}$$

and

$$\sum_{j\geq 0} \frac{\left(\frac{1}{2}\right)_j \left(\frac{1}{2}\right)_j}{j!\left(\frac{3}{2}\right)_j} = \frac{\pi}{2} \quad \text{(Gauss)}.$$

Here we use the standard notation $(a)_j := a(a+1)\cdots(a+j-1) = \Gamma(a+j)/\Gamma(a)$. Therefore,

$$f(\langle\xi,\eta\rangle) = \frac{\pi}{2} - \arccos\langle\xi,\eta\rangle = \frac{\pi}{2} - \rho(\xi,\eta).$$

Clearly, $|f(\langle\xi,\eta\rangle) - f(\langle\xi,\eta'\rangle)| \leq \rho(\eta,\eta')$, and by Theorem 4.6, the associated Gaussian process $(Z^d_\xi(\omega))_{\xi\in\mathbb{S}^d}$ is almost surely in $B^s_{\infty,1}(\mathbb{S}^d)$ (hence in $\mathrm{Lip}\, s$) for $0 < s < \frac{1}{2}$. Furthermore,

$$\mathbb{E}(Z^d_\xi - Z^d_\eta)^2 = 2f(1) - 2f(\langle\xi,\eta\rangle) = 2\rho(\xi,\eta).$$

Consider now the general case: $0 < \alpha \leq 1$ (*Fractional Brownian process*). From above, it follows that $\psi(\xi,\eta) := \rho(\xi,\eta)$ is an invariant negative definite kernel. Then the general theory of negative definite kernels yields that for any $0 < \alpha \leq 1$, the kernel $\psi_\alpha(\xi,\eta) = \rho(\xi,\eta)^\alpha$ is invariant and negative definite. Therefore, for a sufficiently large constant $C > 0$,

$$K(\xi,\eta) = C - \frac{1}{2}\rho(\xi,\eta)^\alpha$$

is an invariant positive definite kernel. On the other hand,

$$|K(\xi,\eta) - K(\xi,\eta')| = \frac{1}{2}|\rho(\xi,\eta)^\alpha - \rho(\xi,\eta')^\alpha| \leq \frac{1}{2}\rho(\eta',\eta))^\alpha.$$

By Theorem 4.6, it follows that the associated Gaussian process $(Z_\xi^d(\omega))_{\xi\in\mathbb{S}^d}$ is almost surely in $B^\gamma_{\infty,1}$, $\gamma < \frac{\alpha}{2}$, and hence in Lip s, $s < \frac{\alpha}{2}$, and the proof is complete. □

Remark 7.2 From the definition of the process, we have

$$\mathbb{E}\left(Z_\xi^\alpha - Z_\eta^\alpha\right)^2 = \rho(\xi,\eta)^\alpha.$$

This directly connects to the regularity proof of such a process using a generalization of Kolmogorov–Csensov inequalities. See for instance [3] and [29].

Remark 7.3 If $\alpha > 1$, then $\rho(\xi,\eta)^\alpha$ is no more a negative definite function on the sphere $\mathbb{S}^d$. In fact, to prove such a result, it suffices to prove it for $\mathbb{S}^1$, as the closed geodesic of $\mathbb{S}^d$ are isometric to $\mathbb{S}^1$. As $\mathbb{S}^1$ is a commutative group, one can apply the Bochner theorem: $K(x-y)$ is a positive definite function if and only if the Fourier coefficients of K are non-negative.

Let $\alpha > 0$, and let ϕ be the 2π-periodic function, such that for $x \in [-\pi,\pi]$, $\phi(x) = |x|^\alpha$, so that on $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$, $\phi(x-y) = d_{\mathbb{S}^1}(x,y)^\alpha$. Clearly, for any $k \in \mathbb{Z}$,

$$\hat{\phi}(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} |x|^\alpha e^{-ikx}dx = \frac{1}{\pi}\int_0^\pi x^\alpha \cos kx dx.$$

Integrating by parts, we obtain, for $k \ge 1$,

$$\int_0^\pi x^\alpha \cos kx dx = -\frac{\alpha}{k}\int_0^\pi x^{\alpha-1}\sin kx dx = -\frac{\alpha}{k^{\alpha+1}}\int_0^{k\pi} u^{\alpha-1}\sin u du,$$

and in going further,

$$\int_0^{k\pi} u^{\alpha-1}\sin u du = \sum_{j=0}^{k-1}\int_{j\pi}^{(j+1\pi} u^{\alpha-1}\sin u du = \int_0^\pi \sum_{j=0}^{k-1}(-1)^j(u+j\pi)^{\alpha-1}\sin u du.$$

Now, if $\alpha > 1$, it is easy to see that for $0 < u < \pi$ and $k \ge 1$,

$$\sum_{j=0}^{k-1}(-1)^j(u+j\pi)^{\alpha-1} > 0 \quad \text{if} \quad k \equiv 1 \ (\text{mod } 2)$$

and

$$\sum_{j=0}^{k-1}(-1)^j(u+j\pi)^{\alpha-1} < 0 \quad \text{if} \quad k \equiv 0 \ (\text{mod } 2).$$

Therefore, if $\alpha > 1$, then $K(x-y) = C - d_{\mathbb{S}^1}(x,y)^\alpha$ is never a positive definite function.

7.2 Regularity of Gaussian Processes on the Sphere: General Result

Theorem 7.4 *Let*

$$f(x) = \sum_{n\geq 0} \frac{A_n}{n!} x^n, \quad \textit{where} \quad A_n \geq 0, \quad \textit{and} \quad \frac{A_n}{n!} = O\big(\frac{1}{n^{1+\alpha}}\big), \quad \alpha > 0.$$

Then

$$K(\xi, \eta) := f(\cos\langle \xi, \eta \rangle), \quad \xi, \eta \in \mathbb{S}^d, \quad d \geq 1,$$

is an invariant positive definite function, and the associated Gaussian process $(Z^d_\xi(\omega))_{\xi \in \mathbb{S}^d}$ *is almost surely in* $B^\gamma_{\infty,1}$ *for* $\gamma < \alpha$.

Proof By Corollary 5.2, it suffices to show that $f(x)$ can be represented in the following form (see 7.1):

$$f(x) = \sum_j B_j W^\nu_j(x), \quad 0 \leq B_j = O\Big(\frac{1}{j^{1+2\alpha}}\Big),$$

implying $\nu_j = O\Big(\frac{1}{j^{d+2\alpha}}\Big) = O(\sqrt{\lambda_j})^{2\alpha+d}$. By [8, Lemma 1] and the obvious identity $\Gamma(x+n) = (x)_n \Gamma(x)$, we obtain the representation

$$x^n = \frac{n!}{2^n} \sum_{0\leq 2k \leq n} \frac{n-2k+\nu}{k!(\nu)_{n-k+1}} \frac{(2\nu)_{n-2k}}{(n-2k)!} W^\nu_{n-2k}(x).$$

Substituting this in the definition of $f(x)$, we obtain

$$\begin{aligned} f(x) = \sum_{n\geq 0} \frac{A_n}{n!} x^n &= \sum_{n\geq 0} \frac{A_n}{2^n} \sum_{0\leq 2k\leq n} \frac{n-2k+\nu}{k!(\nu)_{n-k+1}} \frac{(2\nu)_{n-2k}}{(n-2k)!} W^\nu_{n-2k}(x) \\ &= \sum_{j\geq 0} \frac{(j+\nu)(2\nu)_j}{j!} W^\nu_j(x) \sum_{n-2k=j} \frac{A_n}{2^n k!(\nu)_{n-k+1}} \\ &= \sum_{j\geq 0} \frac{(j+\nu)(2\nu)_j}{j!} W^\nu_j(x) \frac{1}{2^j} \sum_{k\geq 0} \frac{A_{j+2k}}{2^{2k} k!(\nu)_{j+k+1}} \\ &=: \sum_{j\geq 0} B_j W^\nu_j(x), \end{aligned}$$

where for the third equality, we applied the substitution $j = n - 2k$ and shifted the order of summation. We also have

$$\begin{aligned} B_j &:= \frac{(j+\nu)(2\nu)_j}{j!2^j} \sum_{k\geq 0} \frac{A_{j+2k}}{2^{2k}k!(\nu)_{j+k+1}} \\ &= \frac{(j+\nu)(2\nu)_j}{j!2^j(\nu)_{j+1}} \sum_{k\geq 0} \frac{A_{j+2k}}{2^{2k}k!(\nu+j+1)_k} \\ &= \frac{(2\nu)_j}{2^j j!(\nu)_j} \sum_{k\geq 0} \frac{A_{j+2k}}{2^{2k}k!(\nu+j+1)_k}. \end{aligned}$$

However, for $n > \alpha$, we have $\frac{c_1(\alpha)}{n^{1+\alpha}} \leq \frac{\Gamma(n-\alpha)}{n!} \leq \frac{c_2(\alpha)}{n^{1+\alpha}}$, and hence

$$\frac{A_n}{n!} = O\Big(\frac{1}{n^{1+\alpha}}\Big) \iff A_n = O(\Gamma(n-\alpha)).$$

We use this to obtain for $j > \alpha$ (with $c = c(\alpha)$),

$$\begin{aligned} \sum_{k\geq 0} \frac{A_{j+2k}}{2^{2k}k!(\nu+j+1)_k} &\leq c \sum_{k\geq 0} \frac{\Gamma(j+2k-\alpha)}{2^{2k}k!(\nu+j+1)_k} \\ &= c\Gamma(j-\alpha) \sum_{k\geq 0} \frac{\Gamma(j+2k-\alpha)}{\Gamma(j-\alpha)} \frac{1}{2^{2k}k!(\nu+j+1)_k} \\ &= c\Gamma(j-\alpha) \sum_{k\geq 0} \frac{(j-\alpha)_{2k}}{2^{2k}} \frac{1}{k!(\nu+j+1)_k} \\ &= c\Gamma(j-\alpha) \sum_{k\geq 0} \Big(\frac{j-\alpha}{2}\Big)_k \Big(\frac{j-\alpha+1}{2}\Big)_k \frac{1}{k!(\nu+j+1)_k}, \end{aligned}$$

where we utilized the Legendre duplication formula (see, e.g., [4]):

$$\frac{(b)_{2k}}{2^{2k}} = \frac{\Gamma(b+2k)}{2^{2k}\Gamma(b)} = \Big(\frac{b}{2}\Big)_k \Big(\frac{b+1}{2}\Big)_k.$$

By the Gaussian identity (see, e.g., [4, Theorem 2.2.2]),

$$\begin{aligned} &\sum_{k\geq 0} \Big(\frac{j-\alpha}{2}\Big)_k \Big(\frac{j-\alpha+1}{2}\Big)_k \frac{1}{k!(\nu+j+1)_k} \\ &\quad = \frac{\Gamma(\nu+j+1)\Gamma\left(\nu+j+1-\frac{j-\alpha}{2}-\frac{j-\alpha+1}{2}\right)}{\Gamma\left(\nu+j+1-\frac{j-\alpha}{2}\right)\Gamma\left(\nu+j+1-\frac{j-\alpha+1}{2}\right)} \\ &\quad = \frac{\Gamma(\nu+j+1)\Gamma\left(\nu+\frac{1}{2}+\alpha\right)}{\Gamma\left(\nu+\frac{j}{2}+1+\frac{\alpha}{2}\right)\Gamma\left(\nu+\frac{j}{2}+\frac{1}{2}+\frac{\alpha}{2}\right)}, \end{aligned}$$

and hence

$$B_j \le c\frac{(2\nu)_j}{j!2^j(\nu)_j}\frac{\Gamma(j-\alpha)\Gamma(\nu+j+1)\Gamma\left(\nu+\frac{1}{2}+\alpha\right)}{\Gamma\left(\nu+\frac{j}{2}+1+\frac{\alpha}{2}\right)\Gamma\left(\nu+\frac{j}{2}+\frac{1}{2}+\frac{\alpha}{2}\right)}.$$

Applying again the Legendre duplication formula, we get

$$\Gamma\Big(\frac{1}{2}\Big)\Gamma(2\nu+j+1+\alpha)=\Gamma\Big(\nu+\frac{j}{2}+1+\frac{\alpha}{2}\Big)\Gamma\Big(\nu+\frac{j}{2}+\frac{1}{2}+\frac{\alpha}{2}\Big)2^{2\nu+j+\alpha}.$$

We use the above to obtain for $j \ge 2\alpha$,

$$\begin{aligned}
B_j &\le c\frac{(2\nu)_j}{j!(\nu)_j}\frac{\Gamma(j-\alpha)\Gamma(\nu+j+1)\Gamma\left(\nu+\frac{1}{2}+\alpha\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(2\nu+j+1+\alpha\right)2^{-2\nu-\alpha}}\\
&= c\frac{\Gamma(2\nu+j)\Gamma(\nu)}{\Gamma(j+1)\Gamma(2\nu)\Gamma(\nu+j)}\frac{\Gamma(j-\alpha)\Gamma(\nu+j+1)\Gamma\left(\nu+\frac{1}{2}+\alpha\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(2\nu+j+1+\alpha\right)2^{-2\nu-\alpha}}\\
&= c2^{\alpha+1}(j+\nu)\frac{\Gamma\left(\nu+\frac{1}{2}+\alpha\right)}{\Gamma\left(\nu+\frac{1}{2}\right)}\frac{\Gamma(j-\alpha)}{\Gamma\left(j-\alpha+1+\alpha\right)}\frac{\Gamma(2\nu+j)}{\Gamma\left(2\nu+j+1+\alpha\right)}\\
&\le c(j+\nu)\frac{1}{(j-\alpha)^{1+\alpha}}\frac{1}{(2\nu+j)^{1+\alpha}} \le \frac{c}{j^{1+2\alpha}}.
\end{aligned}$$

Here we used once again the Legendre duplication formula. It is easy to show that $B_j \le c(\alpha)$, if $j < 2\alpha$. Therefore, $B_j = O\big(\frac{1}{j^{1+2\alpha}}\big)$, and this completes the proof. □

Corollary 7.5 *Let $a > 0$, $b > 0$, $c > a + b$, $\alpha = c - a - b$, and let*

$$F_{a,b;c}(x) := \sum_n \frac{(a)_n(b)_n}{(c)_n}\frac{x^n}{n!}.$$

Then $F_{a,b;c}(\langle \xi, \eta\rangle)$ is an invariant positive definite function on the sphere $\mathbb{S}^d$ and the associated Gaussian process has regularity $B^\gamma_{\infty,1}$, $\gamma < \alpha$, almost everywhere.

8 Appendices

8.1 Appendix I: Positive and Negative Definite Functions

We recall in this appendix some well-known (or not so well-known) facts about positive definite and negative definite functions. For details, we refer the reader to [5,7,9,17,43].

Recall first the definitions of positive and negative definite functions:

Definition 8.1 Given a set M, a real-valued function $K(x, y)$ defined on $M \times M$ is said to be *positive definite* (P.D.) if $K(x, y) = K(y, x)$, $\forall x, y \in M$, and

$$\forall \alpha_1, \ldots, \alpha_n \in \mathbb{R}, \ \forall x_1, \ldots, x_n \in M, \quad \sum_{i,j=1}^{n} \alpha_i \alpha_j K(x_i, x_j) \geq 0.$$

Clearly, if $K(x, y)$ is P.D., then $|K(x, y)| \leq \sqrt{K(x, x)}\sqrt{K(y, y)}$. It is well known that the following characterization is valid:

$$K(x, y) \text{ is P.D.} \quad \Longleftrightarrow \quad K(x, y) = \mathbb{E}(Z_x Z_y),$$

where $(Z_x)_{x \in M}$ is some (centered) Gaussian process.

Definition 8.2 For any $u \in M$, we associate with $K(x, y)$ the following P.D. function:

$$K_u(x, y) := K(x, y) + K(u, u) - K(x, u) - K(y, u) = \mathbb{E}[(Z_x - Z_u)(Z_y - Z_u)],$$

where $(Z_x - Z_u)$ is the process "killed" at the point $u \in M$.

Clearly,

$$K_u \equiv K \Longleftrightarrow K(u, u) = 0.$$

Definition 8.3 Given a set M, a real-valued function $\psi(x, y)$ defined on $M \times M$ is said to be *negative definite* (N.D.) if

$$\psi(x, y) = \psi(y, x), \quad \forall x, y \in M, \quad \psi(x, x) \equiv 0, \quad \text{and}$$
$$\forall \alpha_1, \ldots, \alpha_n \in \mathbb{R} \ \text{ s.t. } \sum_i \alpha_i = 0, \ \ \forall x_1, \ldots, x_n \in M, \quad \sum_{i,j=1}^{n} \alpha_i \alpha_j \psi(x_i, x_j) \leq 0.$$

The following characterization is valid (see, e.g., [7, Proposition 3.2]):

$$\psi(x, y) \ \text{ is N.D.} \quad \Longleftrightarrow \quad \psi(x, y) = \mathbb{E}(Z_x - Z_y)^2,$$

where $(Z_x)_{x \in M}$ is some Gaussian process.

Consequently, if $\psi(x, y)$ is N.D., then $\psi(x, y) \geq 0$, $\forall x, y \in M$, and $\sqrt{\psi(x, y)}$ verifies the triangular inequality.

The following proposition is easy to verify.

Proposition 8.4 (a) *Let $K(x, y)$ be a P.D. kernel on a set M, and set*

$$\psi_K(x, y) := K(x, x) + K(y, y) - 2K(x, y). \tag{8.1}$$

Then ψ_K is negative definite. The function ψ_K will be termed the N.D. function associated with K. In fact, if $K(x, y) = \mathbb{E}(Z_x Z_y)$, then $\psi_K(x, y) = \mathbb{E}(Z_x - Z_y)^2$. Furthermore, $\psi_K \equiv \psi_{K_u}$, $\forall u \in M$.

(b) *Let ψ be a N.D. function, and for any $u \in M$, define*

$$N(u,\psi)(x,y) := \frac{1}{2}[\psi(x,u)+\psi(y,u)-\psi(x,y)].$$

Thus, if $\psi(x,y)=\mathbb{E}(Z_x-Z_y)^2$, then $N(u,\psi)(x,y) := \mathbb{E}\big[(Z_x-Z_u)(Z_y-Z_u)\big]$. Then $N(u,\psi)$ is P.D. Moreover,

$$N(u,\psi_K)=K_u.$$

(c) *If K is P.D., then $K(x,y)\equiv$ constant $\Longleftrightarrow \psi_K \equiv 0$.*

Proposition 8.5 *Let $\psi(x,y)$ be a real-valued continuous N.D. function on the compact space M, μ a positive Radon measure, with support M, and set*

$$\tilde{K}(x,y) := \frac{1}{2|M|}\int_M [\psi(x,u)+\psi(y,u)-\psi(x,y)]d\mu(u).$$

Then

(a) *$\tilde{K}$ is positive definite, and $\psi_{\tilde{K}}=\psi$.*

(b) *$\mathbb{1}$ is an eigenfunction of the operator $\tilde{K}$ with kernel $\tilde{K}(x,y)$; that is,*

$$\int_M \tilde{K}(x,y)d\mu(y)\equiv\tilde{\lambda},\quad \tilde{\lambda}=\frac{1}{2|M|}\int_M\int_M \psi(u,y)d\mu(u)d\mu(y)(\geq 0).$$

(c)

$$\exists z\in M \;\; s.t. \;\; \tilde{K}(z,z)=0 \iff \tilde{K}(x,y)\equiv 0 \iff \psi(x,y)\equiv 0.$$

Proof Parts (a) and (b) are straightforward. For the proof of (c), we first observe the obvious implications:

$$\psi(x,y)\equiv 0 \implies \tilde{K}(x,y)\equiv 0 \implies \tilde{K}(z,z)=0,\ \forall z\in M.$$

Now, let $\tilde{K}(z,z)=0$ for some $z\in M$. Then

$$\frac{1}{2|M|}\int_M [\psi(z,u)+\psi(z,u)-\psi(z,z)]d\mu(u)=0.$$

By definition, $\psi(z,z)=0$, and hence $\int_M \psi(z,u)d\mu(u)=0$. However, $\psi(z,u)$ is continuous, $\psi(z,u)\geq 0$, and supp $(\mu)=M$. Therefore, $\psi(z,u)=0, \forall u\in M$. Now, by the triangle inequality, we obtain for $x,y\in M$,

$$0\leq\sqrt{\psi(x,y)}\leq\sqrt{\psi(x,z)}+\sqrt{\psi(z,y)}=0,$$

and hence $\psi(x,y)\equiv 0$. This completes the proof. □

Remark 8.6 One can verify easily that if $K(x, y)$ is P.D. on M, then

$$K_u(x, y) := K(x, y) + K(u, u) - K(x, u) - K(y, u) \\ = \frac{1}{2}[\psi_K(x, u) + \psi_K(y, u) - \psi_K(x, y)].$$

The proof of the following proposition is straightforward.

Proposition 8.7 *Let M be a compact space, equipped with a Radon measure μ. Assume that $K(x, y)$ is a continuous P.D. kernel, and as previously, let*

$\psi(x, y) := \psi_K(x, y) = K(x, x) + K(y, y) - 2K(x, y)$ *be the associated N.D. kernel,*

$$K_u(x, y) := K(x, y) + K(u, u) - K(x, u) - K(y, u) \\ = \frac{1}{2}[\psi(x, u) + \psi(y, u) - \psi(x, y)], \\ \tilde{K}(x, y) := \frac{1}{2|M|}\int_M [\psi(x, u) + \psi(y, u) - \psi(x, y)]d\mu(u) \\ = \frac{1}{|M|}\int_M K_u(x, y)d\mu(u).$$

Denote by K and $\tilde{K}$ the operators with kernels $K(x, y)$ and $\tilde{K}(x, y)$. Then

$$\tilde{K}(x, y) = K(x, y) + |M|^{-1}\operatorname{Tr}(K) - |M|^{-1}K\mathbb{1}(x) - |M|^{-1}K\mathbb{1}(y). \tag{8.2}$$

Moreover, $\psi_{\tilde{K}} = \psi$, $\tilde{K}_u = K_u$, and $\tilde{K}\mathbb{1} = \tilde{\lambda}\mathbb{1}$, where

$$\tilde{\lambda} = Tr(K) - \frac{1}{|M|}\int_M\int_M K(x, y)d\mu(x)d\mu(y) \\ = \frac{1}{2|M|}\int_M\int_M \psi(u, y)d\mu(u)d\mu(y) \geq 0.$$

In addition,

$$K = \tilde{K} + C \iff K\mathbb{1} = \lambda\mathbb{1}, \tag{8.3}$$

and, if so, $\tilde{\lambda} = (\operatorname{Tr}(K) - \lambda)$, $C = \frac{1}{|M|}(Tr(K) - 2\lambda)$.

Remark 8.8 The following useful assertions can be found in, e.g., [7,9,42,43]. For N.D. functions, there exists a functional calculus that has no equivalent for P.D. functions:

(1) Let F be a bounded completely continuous function, i.e.,

$$\forall z > 0,\ \forall n \in \mathbb{N},\ D^n F(z) \geq 0$$

or equivalently,

$$F(z) = \int_0^\infty e^{-tz} d\mu(t), \quad \mu \geq 0, \quad \mu([0, \infty)) < \infty.$$

Then

$$\psi \text{ is N.D.} \Longrightarrow F(\psi) \text{ is P.D.}$$

(2) If G is a Bernstein function, i.e.,

$$G(z) = az + \int_0^\infty (1 - e^{-tz}) d\mu(t), \quad a \geq 0, \; ; \mu \geq 0, \quad \int_0^\infty \frac{t}{1+t} d\mu(t) < \infty,$$

then

$$\psi \text{ N.D.} \Longrightarrow G(\psi) \text{ is N.D.}$$

For instance, we have:

$$\begin{aligned} \psi \text{ is N.D.} &\Longleftrightarrow \forall t > 0, \; e^{-t\psi} \text{ is P.D.}, \\ \psi \text{ is N.D.} &\Longrightarrow \forall\, 0 < \alpha \leq 1, \; \psi^\alpha \text{ is N.D.}, \\ \psi \text{ is N.D.} &\Longrightarrow \log(1 + \psi) \text{ is N.D.} \end{aligned}$$

8.2 Appendix II: Gaussian Probability on Separable Banach Spaces

For a detailed account of the material in this section, we refer the reader to [10].

Let E be a Banach space, and let $\mathcal{B}(E)$ be the sigma-algebra of Borel sets on E. Let E^* be its topological dual, and assume $\mathcal{F}$ is a vector space of real-valued functions defined on E, and $\gamma(\mathcal{F}, E)$ is the sigma-algebra generated by $\mathcal{F}$.

If $\mathcal{F} = \mathcal{C}_b(E, \mathbb{R})$ is the vector space of continuous bounded functions on E, then $\gamma(\mathcal{C}_b(E, \mathbb{R}), E) = \mathcal{B}(E)$ is the Borel sigma-algebra.

If E is separable, it is well known that the sigma-algebra $\gamma(E^*, E)$ generated by E^* is $\mathcal{B}(E)$.

Proposition 8.9 *Let E be a separable Banach space. Let H be a subspace of E^*, endowed with the $\sigma(E^*, E)$ topology. Then*

$$H \textit{ is closed} \quad \Longleftrightarrow \quad H \textit{ is stable by simple limit.}$$

Proof The implication $\Rightarrow$ is obvious. We now prove $\Leftarrow$. By the Banach–Krein–Smulian theorem, H is $\sigma(E^*, E)$-closed if and only if $\forall R > 0$, $B(0, R) \cap H$ is $\sigma(E^*, E)$-closed. As E is a separable Banach space, we have: For all $R > 0$,

$$B(0, R) = \{f \in E^* : \|f\|_{E^*} \leq R\} \text{ is metrizable (and compact) for } \sigma(E^*, E).$$

Hence we only have to verify that for every sequence $(f_n) \subset B(0, R) \cap H$ such that $\lim_{n \mapsto \infty} f_n = f$ in the $\sigma(E^*, E)$−topology, we have $f \in B(0, R) \cap H$. But clearly, this implies $\forall x \in E,\ \lim_{n \mapsto \infty} f_n(x) = f(x)$, so we have $f \in B(0, R) \cap H$. □

Corollary 8.10 *Let E be a separable Banach space and H a subspace of E^*. Then:*

(1) $\overline{H}^{\sigma(E^*,E)}$ *coincides with the smallest vector space of functions on E, stable by simple limits containing H.*

(2)

$$\gamma(H, E) = \gamma(\overline{H}^{\sigma(E^*,E)}, E).$$

(3) *If H is a subspace of E^* separating E, then*

$$\gamma(H, E) = \gamma(E^*, E) = \mathcal{B}(E).$$

Proof (1) Clearly, as E^* is stable by simple limits (by the Banach–Steinhaus theorem), the smallest vector space of functions on E, stable by simple limits containing H, is contained in E^*; hence, by the preceding proposition, it is $\overline{H}^{\sigma(E^*,E)}$.

(2) Let $\gamma(H, E)$ is the sigma-algebra generated by H. The vector subspace $V = \{u \in E^* : u,\ \gamma(H, E) - \text{measurable}\}$ is stable by simple limits. Hence, $\overline{H}^{\sigma(E^*,E)} \subset V$.

(3) By the Hahn–Banach theorem, if H is separating, $\overline{H}^{\sigma(E^*,E)} = E^*$, and hence

$$\gamma(H, E) = \gamma(E^*, E) = \mathcal{B}(E).$$

□

Lemma 8.11 *Let E be a separable Banach space, and H be a subspace of E^* separating E. There is at most one probability measure P on the Borel sets of E such that, under P, $\gamma \in H$ is a centered Gaussian variable with a given covariance $K(\gamma, \gamma')$:*

$$K(\gamma, \gamma') = \int_E \gamma(\omega)\gamma'(\omega) DP(\omega)$$

on H. Moreover, if such a probability exists, then:

(1) E^* is a Gaussian space, and $\overline{E^*}^{L^2(E,P)}$ is the Gaussian space generated by H.
(2) There exists $\alpha > 0$ such that

$$\int_E e^{\alpha \|x\|_E^2} dP(x) < \infty. \tag{8.4}$$

Proof If $K(\gamma, \gamma')$ is a positive definite function on H, it determines an additive function on the algebra of cylindrical sets related to H:

$$\left\{x \in E : (\gamma_1(x), \ldots, \gamma_n(x)) \in C\right\},\ \gamma_i \in H,\ C \text{ Borel set of } \mathbb{R}^n.$$

Now, the sigma-algebra generated by this algebra is the Borel sigma-algebra of E.

Assume that such a probability P exists. Let $\mathcal{H} = E^* \cap \overline{H}^{L^2(E,P)}$. Clearly, $\overline{H}^{L^2(E,P)}$ is the Gaussian space generated by H, and if $(\gamma_n)_{n\geq 1} \in \mathcal{H}$ is such that $\forall x \in E$, $\lim_{n\mapsto\infty} \gamma_n(x) = \gamma(x)$ exists, then clearly $\gamma \in E^*$ by the Banach–Steinhauss theorem, and $\gamma \in \overline{H}^{L^2(E,P)}$ since a simple limit of random variables in a closed Gaussian space belongs to this Gaussian space. Therefore, $\gamma \in \mathcal{H}$, which by Proposition 8.9 implies that $\mathcal{H}$ is closed. But $H \subset \mathcal{H}$ and $\overline{H}^{\sigma(E^*,E)} = E^*$ leads to $\mathcal{H} = E^*$.

Finally, (8.4) is just the Fernique theorem. □

8.2.1 Cameron–Martin Space

Let us recall that, due to the Fernique theorem and Bochner integration, we have the following map from E^* to E :

$$I : \gamma \in E^* \mapsto \int_E^E \omega\gamma(\omega)dP(\omega) \in E$$

as

$$\left\| \int_E^E \omega\gamma(\omega)dP(\omega) \right\|_E \leq \int^E \|\omega\| |\gamma(\omega)| dP(\omega) \leq \left(\int^E \|\omega\|^2 |dP(\omega) \right)^{\frac{1}{2}} \|\gamma\|_{L^2(P,E)}$$

and

$$\gamma'(I(\gamma)) = \int_E \gamma'(\omega)\gamma(\omega)dP(\omega), \quad \forall \gamma, \gamma' \in E^*.$$

Therefore, I can be extended to $\bar{I} : \overline{E^*}^{L^2(E,P)} \mapsto E$. The subspace

$$\mathbb{H} \subset E, \quad \mathbb{H} = \bar{I}(\overline{E^*}^{L^2(E,P)})$$

with the induced Hilbert structure is the Cameron–Martin space associated with the Gaussian probability space $(E, \mathcal{B}(E), P)$ (see [10]).

Important Special Case

Let M be a set, and let E be a separable Banach space of real-valued functions on M. Let

$$\forall x \in M, \ f \in E \xrightarrow{\delta_x} f(x) \in \mathbb{R}.$$

Suppose $\delta_x \in E^*$. So, $\mathcal{H} = \{\sum_{\text{finite}} \alpha_i \delta_{x_i}\}$ is dense in E^* in the $\sigma(E^*, E)-$ topology.

Let $K(x, y)$ be a positive definite function on $M \times M$. There is at most one probability measure P on the Borel sets of E such that, under P, E^* is a Gaussian space and $(\delta_x)_{x \in M}$ is a centered Gaussian process with covariance

$$K(x, y) = \int_E \delta_x(\omega)\delta_y(\omega)dP(\omega), \quad \text{i.e.,} \quad \int_E e^{-it\delta_x(\omega)}dP(\omega) = e^{-\frac{1}{2}t^2 K(x,x)}, \quad \forall t \in \mathbb{R}.$$

The Cameron–Martin space is identified with the Reproducing Kernel Hilbert Space $\mathbb{H}_K$ associated with K, i.e., the closure of

$$\Big\{ y \in M \mapsto f(y) = \sum_i \lambda_i K(x_i, y) \}; \quad \|f\|^2_{\mathbb{H}_K} = \sum_{i,j} \lambda_i \lambda_j K(x_i, x_j) \Big\}.$$

$\mathbb{H}_K$ is characterized as a Hilbert space of functions on M such that

$$\forall x \in M, \, f \in \mathbb{H}_K \mapsto f(x) = \langle K(x, .), f \rangle_{\mathbb{H}_K} \text{ (is continuous)}.$$

Therefore, if such a P exists on E, then $\mathbb{H}_K \subseteq M$.

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