Quantile-based risk sharing with heterogeneous beliefs

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Abstract

We study risk sharing games with quantile-based risk measures and heterogeneous beliefs, motivated by the use of internal models in finance and insurance. Explicit forms of Pareto-optimal allocations and competitive equilibria are obtained by solving various optimization problems. For Expected Shortfall (ES) agents, Pareto-optimal allocations are shown to be equivalent to equilibrium allocations, and the equilibrium price is unique. For Value-at-Risk (VaR) agents or mixed VaR and ES agents, a competitive equilibrium does not exist. Our results generalize existing ones on risk sharing games with risk measures and belief homogeneity, and draw an interesting connection to early work on optimization properties of ES and VaR.

Key-words: Risk sharing, competitive equilibrium, belief heterogeneity, quantiles, non-convexity, risk measures

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1 Introduction

The main focus of this paper is risk sharing games with quantile-based risk measures and heterogeneous beliefs, where various optimization problems naturally appear. Quantile-based risk measures, including Value-at-Risk (VaR) and Expected Shortfall (ES)[1], are the standard risk metrics used in current banking and insurance regulation, such as Basel II, III, Solvency II, and the Swiss Solvency Test. Risk sharing games via VaR or ES are studied in the context of capital optimization; see Embrechts et al. (2017) and the references therein[2].

In the current regulatory frameworks (e.g. BCBS (2016)), internal models are extensively used, naturally leading to model heterogeneity, that is, firms use different models for the same future events. See Embrechts (2017) for a recent discussion on the use of internal models in banking and insurance. Heterogeneous beliefs are typically represented by a collection of probability measures to reflect the divergence of agents' viewpoints[3] on the distributions of risks. In this model landscape, the various agents may not be fully informed on the internal models used by competitors and hence the search for a competitive equilibrium becomes relevant (see Section 2 for definitions). For a discussion on heterogeneous beliefs in finance, see e.g. Xiong (2013) and the references therein. Technically, quantile-based risk sharing games with heterogeneous beliefs are essentially different from these with homogeneous beliefs or these based on expected utilities. For instance, the risk sharing problem is straightforward for ES agents if all agents use the same probability measure as in Embrechts et al. (2017), but highly non-trivial in the setting of heterogeneous beliefs. Moreover, an expected utility is linear with respect to the underlying probability measure, whereas quantile-based risk measures are not.

In this paper, we concentrate mainly on the mathematical results and provide only brief discussions on their economic relevance. Our main contributions are summarized

[1]ES is also called CVaR, AVaR or TVaR in various contexts. In particular, CVaR is common in the optimization literature, e.g. Pflug (2000) and Rockafellar and Uryasev (2000, 2002). In this paper, we stick to the term ES following the risk management literature, e.g. McNeil et al. (2015) and Embrechts et al. (2017).

[2]Amongst others, Barrieu and El Karoui (2005), Acciaio and Svindland (2009), Jouini et al. (2008), Dana and Le Van (2010), Rüschendorf (2013) and Anthropelos and Kardaras (2017) studied risk sharing games with convex risk measures and expected utilities, different from the setting of quantile-based risk measures in this paper.

[3]Following the tradition in game theory, we refer to a participant in the risk sharing games, such as an investor or a firm, as an agent.

as follows. Explicit formulas of Pareto-optimal allocations and competitive equilibria are obtained for ES agents, and the Fundamental Theorems of Welfare Economics (see e.g. Starr (2011)) are established. For the case of VaR agents and that of mixed VaR, ES and RVaR (see Section 6 for a definition) agents, Pareto-optimal allocations share a similar form as in the case of ES agents, but competitive equilibria do not exist. In all cases, we find a Pareto-optimal allocation $(X_1^*, \dots, X_n^*)$ of the general (but not unique) form

$$X_i^* = (X - x^*)\mathrm{I}_{A_i^*} + \frac{x^*}{n}, \quad i = 1, \dots, n, \tag{1.1}$$

where X is the total risk to share, $(A_1^*, \dots, A_n^*)$ is a partition of the sample space, and x^* is a constant. Nevertheless, the determination of $(x^*, A_1^*, \dots, A_n^*)$ for ES agents is computationally very different from that for VaR agents. As an interesting consequence of our main results, we obtain a multiple-measure version of the optimization formula of ES of Rockafellar and Uryasev (2000, 2002) and Pflug (2000). Thanks to the convexity of ES, results in Barrieu and El Karoui (2005) on convex risk measures become helpful in deriving the Pareto-optimal allocations for ES agents; in the case of VaR, which is not convex, optimization problems become more involved. Furthermore, the dependence structure of the Pareto-optimal allocation in (1.1) can be described as *mutual exclusivity* (see Puccetti and Wang (2015)); this is in sharp contrast to *comonotonicity* in the classic setting of risk sharing with expected utilities or convex risk measures (see Rüschendorf (2013)).

2 Preliminaries

2.1 Risk sharing games

Let $(\Omega, \mathcal{F})$ be a measurable space and $\mathcal{P}$ be the set of all probability measures on $(\Omega, \mathcal{F})$. Let $\mathcal{X}$ be the set of bounded random variables on $(\Omega, \mathcal{F})$. Given a random variable $X \in \mathcal{X}$, we define the set of *allocations* of X as

$$\mathbb{A}_n(X) = \left\{ (X_1, \dots, X_n) \in \mathcal{X}^n : \sum_{i=1}^n X_i = X \right\}. \tag{2.1}$$

There are n agents in the risk sharing game. For $i = 1, \dots, n$, agent i is equipped with a risk measure $\rho_i : \mathcal{X} \to \mathbb{R}$, which is the agent's objective to minimize. The risk measures $\rho_1, \ldots, \rho_n$ used in this paper shall later be specified as VaR and ES under different probability measures.

We consider two classic notions of risk sharing: Pareto optimality and competitive equilibria. First, a Pareto-optimal allocation is one that cannot be strictly improved.

Definition 2.1 (Pareto-optimal allocations)**.** Fix the risk measures $\rho_1, \ldots, \rho_n$ and the total risk $X \in \mathcal{X}$. An allocation $(X_1, \ldots, X_n) \in \mathbb{A}_n(X)$ is *Pareto-optimal* if for any allocation $(Y_1, \ldots, Y_n) \in \mathbb{A}_n(X)$, $\rho_i(Y_i) \leqslant \rho_i(X_i)$ for all $i = 1, \ldots, n$ implies $\rho_i(Y_i) = \rho_i(X_i)$ for all $i = 1, \ldots, n$.

Next we formulate competitive equilibria for a one-period exchange market in the classic sense of Arrow-Debreu as in Föllmer and Schied (2016) and Embrechts et al. (2017). To reach a competitive equilibrium, agents in the market minimize their own risk measures by trading with each other. Assume that agent i has an initial risk (random loss/wealth) $\xi_i \in \mathcal{X}$ for $i = 1, \ldots, n$. Let $X = \sum_{i=1}^n \xi_i$ be the total risk. A probability measure $Q \in \mathcal{P}$ represents the pricing rule (risk-neutral probability measure) for the microeconomic market among the agents, that is, by taking a risk Y in this market, one receives a (discounted) monetary payment of $\mathbb{E}^Q[Y]$.

For each $i = 1, \ldots, n$, agent i may trade the initial risk ξ_i for a new position $X_i \in \mathcal{X}$, and this under the budget constraint $\mathbb{E}^Q[X_i] \geqslant \mathbb{E}^Q[\xi_i]$. In general, the budget constraint will be binding (equality is attained) as the admissible set $\mathcal{X}$ is rich enough. In this setting, each agent's target is

$$\begin{array}{lll} \text{to minimize} & \rho_i(X_i) \quad \text{over} \quad X_i \in \mathcal{X} & \\ \text{subject to} & \mathbb{E}^Q[X_i] \geqslant \mathbb{E}^Q[\xi_i], & \end{array} \qquad i = 1, \ldots, n. \tag{2.2}$$

To reach an equilibrium, the market clearing equation

$$\sum_{i=1}^n X_i^* = X = \sum_{i=1}^n \xi_i$$

needs to be satisfied, where X_i^* solves (2.2), $i = 1, \ldots, n$.

Definition 2.2 (Competitive equilibria)**.** Fix the risk measures $\rho_1, \ldots, \rho_n$, the initial risks $\xi_1, \ldots, \xi_n \in \mathcal{X}$ and the total risk $X = \sum_{i=1}^n \xi_i$. A pair $(Q, (X_1^*, \ldots, X_n^*)) \in \mathcal{P} \times \mathbb{A}_n(X)$ is *a competitive equilibrium* if

$$X_i^* \in \underset{X_i \in \mathcal{X}}{\arg\min} \left\{\rho_i(X_i) : \mathbb{E}^Q[X_i] \geqslant \mathbb{E}^Q[\xi_i]\right\}, \ i = 1, \dots, n. \tag{2.3}$$

The probability measure Q in a competitive equilibrium is called an *equilibrium price*, and the allocation $(X_1^*, \dots, X_n^*)$ in a competitive equilibrium is called an *equilibrium allocation*.

It is well known that, through the classic Fundamental Theorems of Welfare Economics (e.g. Starr (2011)), Pareto-optimal allocations and equilibrium allocations are closely related. This relationship will become clear in our setting through the main results of the paper.

2.2 VaR, ES, and agents with heterogeneous beliefs

The key feature of this paper is belief heterogeneity among agents. The heterogeneity of probability measures means that the agents hold possibly different beliefs (models) about the future of the market. Following the setup of homogeneous beliefs in Embrechts et al. (2017), we mainly consider two popular risk measures, the Value-at-Risk (VaR) and the Expected Shortfall (ES), both widely used in modern banking and insurance regulation. For a random loss $X \in \mathcal{X}$ and a given level $\alpha \in [0, 1)$, its VaR under a probability measure $Q \in \mathcal{P}$ is defined as

$$\mathrm{VaR}_\alpha^Q(X) = \inf\{x \in \mathbb{R} : Q(X > x) \leqslant \alpha\}. \tag{2.4}$$

Note that $\mathrm{VaR}_\alpha^Q(X)$ is the left end-point of the interval of $(1-\alpha)$-quantiles of X under Q. For $X \in \mathcal{X}$, the Expected Shortfall (ES) at level $\alpha \in (0, 1)$ under the probability measure $Q \in \mathcal{P}$ is defined as

$$\mathrm{ES}_\alpha^Q(X) = \frac{1}{\alpha}\int_0^\alpha \mathrm{VaR}_u^Q(X)\mathrm{d}u. \tag{2.5}$$

A well-known optimization property linking VaR and ES is established in Rockafellar and Uryasev (2000) and Pflug (2000), namely,

$$\mathrm{ES}_\alpha^Q(X) = \min\left\{\frac{1}{\alpha}\mathbb{E}^Q[(X-x)_+] + x : x \in \mathbb{R}\right\}, \tag{2.6}$$

and

$$\mathrm{VaR}_\alpha^Q(X) \in \arg\min\left\{\frac{1}{\alpha}\mathbb{E}^Q[(X-x)_+] + x : x \in \mathbb{R}\right\}. \tag{2.7}$$

In this paper, we will generalize the above result in a multiple-measure game-theoretic framework.

For $i=1,\dots,n$, let agent i be equipped with a probability measure $Q_i \in \mathcal{P}$ representing her belief about the future randomness. This agent's objective is to minimize a VaR or an ES, and she shall be referred to as a *VaR agent* or an *ES agent*, respectively. We will also consider *RVaR agents* as in Embrechts et al. (2017); see Section 6.

To study risk sharing games for risk measures, define the *inf-convolution* of risk measures (see Rüschendorf (2013)) as

$$\mathop{\square}_{i=1}^{n} \rho_i(X) = \inf\left\{\sum_{i=1}^{n} \rho_i(X_i) : (X_1,\dots,X_n) \in \mathbb{A}_n(X)\right\}, \quad X \in \mathcal{X}. \tag{2.8}$$

It is well-known that for *monetary risk measures* (Artzner et al. (1999)) including VaR and ES, Pareto optimality is equivalent to optimality with respect to the sum (see Proposition 1 of Embrechts et al. (2017)). More precisely, $(X_1,\dots,X_n)$ is a Pareto-optimal allocation of X if and only if

$$\sum_{i=1}^{n} \rho(X_i) = \mathop{\square}_{i=1}^{n} \rho_i(X). \tag{2.9}$$

Therefore, the following two optimization problems are of crucial importance in our study of risk sharing games, namely,

$$\mathop{\square}_{i=1}^{n} \mathrm{VaR}_{\alpha_i}^{Q_i}(X) = \inf\left\{\sum_{i=1}^{n} \mathrm{VaR}_{\alpha_i}^{Q_i}(X_i) : (X_1,\dots,X_n) \in \mathbb{A}_n(X)\right\}, \tag{2.10}$$

and

$$\square_{i=1}^{n} \mathrm{ES}_{\alpha_i}^{Q_i}(X) = \inf\left\{\sum_{i=1}^{n} \mathrm{ES}_{\alpha_i}^{Q_i}(X_i) : (X_1,\dots,X_n) \in \mathbb{A}_n(X)\right\}, \tag{2.11}$$

where $\alpha_i \in (0,1)$, $i=1,\dots,n$.

Notation. Throughout the paper, we use I_A to represent the indicator function of the event $A \in \mathcal{F}$, and let $\pi_n(A)$ be the set of n-partitions of $(A, \mathcal{F}|_A)$. For real numbers $x_1,\dots,x_n$, write $\bigwedge_{i=1}^n x_i = \min\{x_1,\dots,x_n\}$ and $\bigvee_{i=1}^n x_i = \max\{x_1,\dots,x_n\}$.

3 Pareto-optimal allocations for ES agents

In this section, we investigate Pareto-optimal allocations for ES agents. Throughout this section, $\alpha_1, \dots, \alpha_n \in (0,1)$, $Q_1, \dots, Q_n \in \mathcal{P}$, and the risk measure of agent i is $\mathrm{ES}^{Q_i}_{\alpha_i}$, $i = 1, \dots, n$. We first give the sufficient and necessary condition for the existence of a Pareto-optimal allocation. In the proposition below, $\sup(\varnothing)$ is set to $-\infty$ by convention.

Proposition 3.1. *For $X \in \mathcal{X}$, the following hold.*

(i) $\square_{i=1}^n \mathrm{ES}^{Q_i}_{\alpha_i}(X) = \sup\{\mathbb{E}^Q[X] : Q \in \overline{\mathcal{Q}}\}$, *where*

$$\overline{\mathcal{Q}} = \left\{ Q \in \mathcal{P} : \frac{\mathrm{d}Q}{\mathrm{d}Q_i} \leqslant \frac{1}{\alpha_i},\ i = 1, \dots, n \right\}. \tag{3.1}$$

(ii) A Pareto-optimal allocation of X exists if and only if

$$\sum_{i=1}^n \frac{1}{\alpha_i} Q_i(A_i) \geqslant 1 \quad \textit{for all } (A_1, \dots, A_n) \in \pi_n(\Omega). \tag{3.2}$$

Proof. Note that each $\mathrm{ES}^{Q_i}_{\alpha_i}$ is a convex risk measure. The part (i) follows immediately from Barrieu and El Karoui (2005). Moreover, the "if" part of (ii) follows from Theorem 11.3 of Rüschendorf (2013). It suffices to show the "only if" part of (ii). Note that a Pareto-optimal allocation of X exists only if $\overline{\mathcal{Q}}$ is non-empty. We assert that this is in turn equivalent to (3.2). Indeed, for any $A \in \mathcal{F}$, define

$$Q'(A) = \min\left\{ \sum_{i=1}^n \frac{Q_i(A \cap A_i)}{\alpha_i} : (A_1, \dots, A_n) \in \pi_n(\Omega) \right\}.$$

It can be verified that Q' satisfies monotonicity and σ-additivity with $Q'(\varnothing) = 0$, that is, Q' is a measure on $(\Omega, \mathcal{F})$. On the other hand, for a probability measure Q, $Q \leqslant Q'$ if and only if $Q \in \overline{\mathcal{Q}}$. To see this, first note that if $Q \leqslant Q'$, then for $A \in \mathcal{F}$, letting $A_i = A$ yields $Q(A) \leqslant Q'(A) \leqslant Q_i(A)/\alpha_i$, $i = 1, \dots, n$. This implies $\frac{\mathrm{d}Q}{\mathrm{d}Q_i} \leqslant \frac{1}{\alpha_i}$ and thus, $Q \in \overline{\mathcal{Q}}$. On the other hand, $Q \in \overline{\mathcal{Q}}$ implies

$$Q(A) \leqslant \frac{Q_i(A)}{\alpha_i} \quad \text{for any } A \in \mathcal{F}, \quad i = 1, \dots, n,$$

and hence, for any $(A_1, \dots, A_n) \in \pi_n(\Omega)$,

$$Q(A) = \sum_{i=1}^n Q(A \cap A_i) \leqslant \sum_{i=1}^n \frac{Q_i(A \cap A_i)}{\alpha_i} \quad \text{for any } A \in \mathcal{F},$$

so that $Q \leqslant Q'$. That is, $Q \leqslant Q'$ if and only if $Q \in \overline{\mathcal{Q}}$. Hence, $\overline{\mathcal{Q}}$ is non-empty if and only if $Q'(\Omega) \geqslant 1$, that is, (3.2) holds. $\square$

Remark 3.1. From Proposition 3.1 (ii), the existence of a Pareto-optimal allocation only depends on $(\alpha_1, \dots, \alpha_n)$ and $(Q_1, \dots, Q_n)$, but not on the total risk X.

Next we explicitly describe Pareto-optimal allocations for the ES agents. First we translate the inf-convolution of ES into another optimization problem. For given $Q_1, \dots, Q_n \in \mathcal{P}$, we let Q be a measure dominating $Q_1, \dots, Q_n$, and

$$B_j = \left\{ \frac{1}{\alpha_j} \frac{\mathrm{d}Q_j}{\mathrm{d}Q} = \bigwedge_{i=1}^n \frac{1}{\alpha_i} \frac{\mathrm{d}Q_i}{\mathrm{d}Q} \right\} \setminus \left(\bigcup_{k=1}^{j-1} B_k \right), \; j = 1, \dots, n. \tag{3.3}$$

We shall fix $B = (B_1, \dots, B_n)$ as in (3.3) throughout the rest of Sections 3-4. Apparently, the choice of Q is irrelevant in the definition of $B_1, \dots, B_n$, and one can safely choose $Q = \frac{1}{n}\sum_{i=1}^n Q_i$. Roughly speaking, B_j is the set of points on which $\mathrm{d}Q_j/\alpha_j$ is the smallest among $\mathrm{d}Q_i/\alpha_i$, $i = 1, \dots, n$, and we only count once if there is a tie for the minimum. From the definition of $B_1, \dots, B_n$, it is straightforward to verify

$$\min \left\{ \sum_{i=1}^n \frac{1}{\alpha_i} Q_i(A_i) : (A_1, \dots, A_n) \in \pi_n(\Omega) \right\} = \sum_{i=1}^n \frac{1}{\alpha_i} Q_i(B_i),$$

and therefore by Proposition 3.1 (ii), a Pareto-optimal allocation exists if and only if $\sum_{i=1}^n \frac{1}{\alpha_i} Q_i(B_i) \geqslant 1$.

Proposition 3.2. *Assume $\overline{\mathcal{Q}}$ in (3.1) is non-empty. Then for $X \in \mathcal{X}$,*

$$\begin{aligned}
&\mathop{\square}_{i=1}^n \mathrm{ES}_{\alpha_i}^{Q_i}(X) \\
&= \min \left\{ \sum_{i=1}^n \frac{1}{\alpha_i} \mathbb{E}^{Q_i}[(X - x)_+ \mathrm{I}_{A_i}] + x : (A_1, \dots, A_n) \in \pi_n(\Omega), \; x \in \mathbb{R} \right\} \\
&= \min \left\{ \sum_{i=1}^n \frac{1}{\alpha_i} \mathbb{E}^{Q_i}[(X - x)_+ \mathrm{I}_{B_i}] + x : x \in \mathbb{R} \right\}.
\end{aligned}$$

Proof. Fix $X \in \mathcal{X}$. As $\overline{\mathcal{Q}}$ is non-empty, or equivalently (3.2) holds, we have $\square_{i=1}^{n} \mathrm{ES}_{\alpha_i}^{Q_i}(X) > -\infty$. Define

$$V(X) = \inf\left\{\sum_{i=1}^{n} \frac{1}{\alpha_i}\mathbb{E}^{Q_i}[(X-x)_+ \mathrm{I}_{A_i}] + x : (A_1,\dots,A_n) \in \pi_n(\Omega),\ x \in \mathbb{R}\right\}. \tag{3.4}$$

We first show

$$V(X) \geqslant \mathop{\square}_{i=1}^{n} \mathrm{ES}_{\alpha_i}^{Q_i}(X). \tag{3.5}$$

For any $(A_1,\dots,A_n) \in \pi_n(\Omega)$ and $x \in \mathbb{R}$, let $X_i = (X-x)\mathrm{I}_{A_i} + \frac{x}{n}$, $i = 1,\dots,n$. Clearly, $X_1 + \dots + X_n = X$. Moreover, for $i = 1,\dots,n$,

$$\begin{aligned}
\mathrm{ES}_{\alpha_i}^{Q_i}(X_i) &= \mathrm{ES}_{\alpha_i}^{Q_i}((X-x)\mathrm{I}_{A_i}) + \frac{x}{n} \\
&\leqslant \mathrm{ES}_{\alpha_i}^{Q_i}((X-x)_+\mathrm{I}_{A_i}) + \frac{x}{n} \leqslant \frac{1}{\alpha_i}\mathbb{E}^{Q_i}[(X-x)_+\mathrm{I}_{A_i}] + \frac{x}{n}.
\end{aligned}$$

Therefore, for all $x \in \mathbb{R}$ and $(A_1,\dots,A_n) \in \pi_n(\Omega)$, there exists $(X_1,\dots,X_n) \in \mathbb{A}_n(X)$ such that

$$\sum_{i=1}^{n} \mathrm{ES}_{\alpha_i}^{Q_i}(X_i) \leqslant \sum_{i=1}^{n} \frac{1}{\alpha_i}\mathbb{E}^{Q_i}[(X-x)_+\mathrm{I}_{A_i}] + x.$$

It follows that

$$\square_{i=1}^{n} \mathrm{ES}_{i}^{Q_i}(X) \leqslant V(X).$$

Thus (3.5) holds.

Next we need to show $\square_{i=1}^{n} \mathrm{ES}_{i}^{Q_i}(X) \geqslant V(X)$. For $A := (A_1,\dots,A_n) \in \pi_n(\Omega)$, write

$$v_A(x) = \sum_{i=1}^{n} \frac{1}{\alpha_i}\mathbb{E}^{Q_i}[(X-x)_+\mathrm{I}_{A_i}] + x, \quad x \in \mathbb{R}. \tag{3.6}$$

Clearly, v_A is differentiable, and

$$v_A'(x) = -\sum_{i=1}^{n} \frac{1}{\alpha_i} Q_i(X > x, A_i) + 1.$$

Therefore, v_A' is an increasing function of x, with $v_A'(\infty) = 1$ and

$$v_A'(-\infty) = 1 - \sum_{i=1}^{n} \frac{1}{\alpha_i} Q_i(A_i) \leqslant 0$$

as a result of the condition (3.2). Let $x_A^* = \inf\{x \in \mathbb{R} : v_A'(x) \geqslant 0\}$. Obviously x_A^* minimizes v_A. Moreover, noting that $Q_i(X > x, A_i)$ is right-continuous in x for $i = 1, \dots, n$, v_A' is a right-continuous function. Therefore, $v_A'(x_A^*) \geqslant 0$, and equivalently,

$$\sum_{i=1}^{n} \frac{1}{\alpha_i} Q_i(X > x_A^*, A_i) \leqslant 1. \tag{3.7}$$

Next, let

$$Q_A^*(C) = \sum_{i=1}^{n} \frac{1}{\alpha_i} Q_i(C \cap A_i \cap \{X > x_A^*\}), \quad C \in \mathcal{F}.$$

Let us verify

1. Q_A^* is σ-additive, because it is the sum of n measures.

2. $Q_A^*(\Omega) \leqslant 1$ by (3.7).

Now we make some adjustment to Q_A^* so that $Q_A^*(\Omega) = 1$. Note that by a symmetric argument,

$$\sum_{i=1}^{n} \frac{1}{\alpha_i} Q_i(X \geqslant x_A^*, A_i) \geqslant 1. \tag{3.8}$$

Therefore, if $Q_A^*(\Omega) \leqslant 1$, we can replace Q_A^* by Q_A^{**}, which is a linear combination of Q_A^* and Q_A', defined as

$$Q_A'(C) = \sum_{i=1}^{n} \frac{1}{\alpha_i} Q_i(C \cap A_i \cap \{X \geqslant x_A^*\}), \quad C \in \mathcal{F},$$

so that $Q_A^{**}(\Omega) = 1$. In the following we safely assume $Q_A^*(\Omega) = 1$ (otherwise we just replace it by Q_A^{**}), that is, Q_A^* is a probability measure.

We can verify

$$\begin{aligned} \mathbb{E}^{Q_A^*}[X] &= \sum_{i=1}^{n} \frac{1}{\alpha_i} \mathbb{E}^{Q_i}[X \mathrm{I}_{\{X > x_A^*\}} \mathrm{I}_{A_i}] \\ &= \sum_{i=1}^{n} \frac{1}{\alpha_i} \mathbb{E}^{Q_i}[(X - x_A^* + x_A^*) \mathrm{I}_{\{X > x_A^*\}} \mathrm{I}_{A_i}] \\ &= \sum_{i=1}^{n} \frac{1}{\alpha_i} \mathbb{E}^{Q_i}[(X - x_A^*)_+ \mathrm{I}_{A_i}] + x_A^* Q_A^*(\Omega) = v_A(x_A^*). \end{aligned} \tag{3.9}$$

Therefore,

$$V(X) \leqslant \mathbb{E}^{Q_A^*}[X] \quad \text{for all } (A_1, \ldots, A_n) \in \pi_n(\Omega). \tag{3.10}$$

Let B_j, $j = 1, \ldots, n$ be defined as in (3.3). Clearly $B = (B_1, \ldots, B_n) \in \pi_n(\Omega)$. It follows that, for $C \in \mathcal{F}$,

$$\begin{aligned} Q_B^*(C) &= \sum_{i=1}^n \frac{1}{\alpha_i} Q_i(C \cap B_i \cap \{X > x_B^*\}) \\ &\leqslant \sum_{i=1}^n \frac{1}{\alpha_j} Q_j(C \cap B_i) \leqslant \frac{1}{\alpha_j} Q_j(C), \quad j = 1, \ldots, n. \end{aligned}$$

As a consequence, we have $Q_B^* \in \overline{\mathcal{Q}}$. It follows that

$$\mathbb{E}^{Q_B^*}[X] \leqslant \sup_{Q \in \overline{\mathcal{Q}}} \mathbb{E}^Q[X] = \mathop{\square}_{i=1}^n \mathrm{ES}_{\alpha_i}^{Q_i}(X).$$

Together with (3.5) and (3.10), we have

$$V(X) \leqslant \mathbb{E}^{Q_B^*}[X] \leqslant \mathop{\square}_{i=1}^n \mathrm{ES}_{\alpha_i}^{Q_i}(X) \leqslant V(X). \tag{3.11}$$

This completes the proof. $\square$

With the help of Proposition 3.2, we are ready to present an explicit form of Pareto-optimal allocations for the ES agents. Define

$$x_B^* = \inf\left\{x \in \mathbb{R} : \sum_{i=1}^n \frac{1}{\alpha_i} Q_i(X > x, B_i) \leqslant 1\right\}, \tag{3.12}$$

and

$$y_B^* = \inf\left\{x \in \mathbb{R} : \sum_{i=1}^n \frac{1}{\alpha_i} Q_i(X > x, B_i) < 1\right\}. \tag{3.13}$$

The quantities x_B^* and y_B^* will be used repeatedly later in the paper. Note that, if $Q_1 = \cdots = Q_n = Q$, then by definition of $(B_1, \ldots, B_n)$,

$$x_B^* = \inf\left\{x \in \mathbb{R} : \frac{1}{\alpha} Q(X > x) \leqslant 1\right\} = \mathrm{VaR}_\alpha^Q(X),$$

where $\alpha = \bigvee_{i=1}^n \alpha_i$. Thus, x_B^* can be seen as a generalized left-quantile (VaR) of X in the multi-measure framework, whereas y_B^* is a generalized right-quantile of X. By definition, $Q_i(x_B^* < X < y_B^*,\ B_i) = 0$ for $i = 1, \ldots, n$. Similarly to the left/right-quantiles, x_B^* and y_B^* are often identical for practical settings.

Theorem 3.3. *Assume $\overline{\mathcal{Q}}$ in (3.1) is non-empty. A Pareto-optimal allocation $(X_1^*, \ldots, X_n^*)$ of $X \in \mathcal{X}$ is given by*

$$X_i^* = (X - x^*)\mathrm{I}_{B_i} + \frac{x^*}{n}, \quad i = 1, \ldots, n, \tag{3.14}$$

for $x^* \in [x_B^*, y_B^*]$, where $(B_1, \ldots, B_n)$, x_B^* and y_B^* are in (3.3), (3.12) and (3.13).

Proof. For $i = 1, \ldots, n$, using (2.6) with $x = \frac{x^*}{n}$, we have

$$\mathrm{ES}_{\alpha_i}^{Q_i}(X_i^*) \leqslant \frac{1}{\alpha_i}\mathbb{E}^{Q_i}[(X - x^*)_+\mathrm{I}_{B_i}] + \frac{x^*}{n}.$$

By taking a derivative of v_B defined by (3.6) with A_i replaced by B_i, $i = 1, \ldots, n$, we have, for $x^* \in [x_B^*, y_B^*]$,

$$\sum_{i=1}^{n} \frac{1}{\alpha_i}\mathbb{E}^{Q_i}[(X - x^*)_+\mathrm{I}_{B_i}] + x^* = \min\left\{\sum_{i=1}^{n} \frac{1}{\alpha_i}\mathbb{E}^{Q_i}[(X - x)_+\mathrm{I}_{B_i}] + x : x \in \mathbb{R}\right\}.$$

Therefore, by Proposition 3.2, we have $\sum_{i=1}^n \mathrm{ES}_{\alpha_i}^{Q_i}(X_i^*) \leqslant \square_{i=1}^n \mathrm{ES}_{\alpha_i}^{Q_i}(X)$. This implies the Pareto optimality of $(X_1^*, \ldots, X_n^*)$. $\square$

The economic interpretation of the Pareto-optimal allocation in (3.14) is very simple. For each $i = 1, \ldots, n$, agent i takes the risk $(X - x^*)\mathrm{I}_{B_i}$ plus a constant (side-payment). Looking at the definition of B_i, it is clear that agent i thinks the event B_i is the least likely to happen, compared to other agents' beliefs on the same event. The rest of the risk, which is more likely to happen according to agent i (relative to other agents), is taken by others. This intuitively implies, quoting Chen et al. (2012), "When agents disagree about disaster risk, they will insure each other against the types of disasters they fear most".

We make some technical observations about Theorem 3.3.

(i) A constant shift (side-payment) among $X_1^*, \ldots, X_n^*$ defined in (3.14) does not compromise the optimality; hence, $(X_1^*+c_1, \ldots, X_n^*+c_n)$ is also a Pareto-optimal allocation, where $c_1, \ldots, c_n$ are constants and $\sum_{i=1}^n c_i = 0$. Later we shall see in Proposition 3.5

that, under an extra condition, the Pareto-optimal allocation is unique on the set $\{X > y_B^*\}$ up to constant shifts.

(ii) The dependence structure of the Pareto-optimal allocation $(X_1^*, \dots, X_n^*)$ in (3.14) is worth noting. On the set $\{X > x^*\}$, $X_1^*, \dots, X_n^*$ are *mutually exclusive*, a form of extremal negative dependence (see Puccetti and Wang (2015)). This is in sharp contrast to the case of homogeneous beliefs, where a Pareto-optimal allocation for strictly convex functionals is always comonotonic (see Rüschendorf (2013)), a form of extremal positive dependence.

(iii) As an immediate consequence of Theorem 3.3,

$$\mathop{\square}_{i=1}^{n} \mathrm{ES}_{\alpha_i}^{Q_i}(X) = \sum_{i=1}^{n} \frac{1}{\alpha_i} \mathbb{E}^{Q_i}[(X - x_B^*)_+ \mathrm{I}_{B_i}] + x_B^*. \tag{3.15}$$

We can easily see that in the case of $n = 1$, Proposition 3.2 gives, for any $\alpha \in (0, 1)$, $Q \in \mathcal{P}$ and $X \in \mathcal{X}$,

$$\mathrm{ES}_\alpha^Q(X) = \min\left\{ \frac{1}{\alpha}\mathbb{E}^Q[(X - x)_+] + x : x \in \mathbb{R} \right\}, \tag{3.16}$$

and Theorem 3.3 (setting $n = 1$) implies that the above minimum is achieved by $x^* = \mathrm{VaR}_\alpha^Q(X)$, a celebrated result (see (2.6) and (2.7)) established by Rockafellar and Uryasev (2000). In other words, Theorem 3.3 can be regarded as a generalization of the result of Rockafellar and Uryasev (2000) in a multiple-measure framework. Using (3.16), we obtain the following corollary of Theorem 3.3, giving a solution to an optimization problem similar to (2.6) and (2.7).

Corollary 3.4. *The optimization problem*

$$\begin{aligned} &\textit{to minimize} \quad \sum_{i=1}^{n} \frac{1}{\alpha_i} \mathbb{E}^{Q_i}[(X_i - x_i)_+] + \sum_{i=1}^{n} x_i \\ &\textit{over} \qquad x_1, \dots, x_n \in \mathbb{R},\ (X_1, \dots, X_n) \in \mathbb{A}_n(X) \end{aligned}$$

admits a solution $(x_1^*, \dots, x_n^*, X_1^*, \dots, X_n^*)$ *where* $x_1^* = \dots = x_n^* = x_B^*/n$, *and* x_B^* *and* $X_1^*, \dots, X_n^*$ *are given in Theorem 3.3.*

Remark 3.2. If $Q_1 = \cdots = Q_n = \mathbb{P}$, Proposition 3.2 reduces to the classic result

$$\mathop{\square}_{i=1}^{n} \mathrm{ES}^{\mathbb{P}}_{\alpha_i}(X) = \min\left\{ \frac{1}{\bigvee_{i=1}^n \alpha_i} \mathbb{E}^{\mathbb{P}}[(X-x)_+] + x : \ x \in \mathbb{R} \right\} = \mathrm{ES}^{\mathbb{P}}_{\bigvee_{i=1}^n \alpha_i}(X).$$

In this case, according to Theorem 3.3, the Pareto-optimal allocation is one where all the risk is taken by one agent with the largest α_i value, and the other agents make side-payments to this agent. This is a special case of Theorem 2 of Embrechts et al. (2017).

Next we study the uniqueness of the form of Pareto-optimal allocations. Since an ES only depends on the tail part of a risk, it is natural that uniqueness can only be established on the set $\{X > y^*_B\}$. Moreover, it is straightforward to verify that the allocation can be very flexible on the set $\{\mathrm{d}Q_i/\mathrm{d}Q = 0\}$ for each $i = 1, \dots, n$, where $Q = \frac{1}{n}\sum_{i=1}^n Q_i$. Hence, we focus our discussion on the case in which $Q_1, \dots, Q_n$ are equivalent.

The following proposition characterizes the form of Pareto-optimal allocations, which requires an intuitive condition

$$\text{the sets } \left\{ \frac{1}{\alpha_j}\frac{\mathrm{d}Q_j}{\mathrm{d}Q} = \bigwedge_{i=1}^n \frac{1}{\alpha_i}\frac{\mathrm{d}Q_i}{\mathrm{d}Q} \right\}, \ j = 1, \dots, n, \ \text{are disjoint,} \tag{3.17}$$

so that the sets on which $\frac{1}{\alpha_i}\frac{\mathrm{d}Q_i}{\mathrm{d}Q}$ is the smallest, $i = 1, \dots, n$, are distinguishable.

Proposition 3.5. *Suppose that $Q_1, \dots, Q_n$ are equivalent to $Q \in \mathcal{P}$ and (3.17) holds. Any Pareto-optimal allocation $(X^*_1, \dots, X^*_n)$ of $X \in \mathcal{X}$ satisfies, for some constants $c_1, \dots, c_n$ with $\sum_{i=1}^n c_i = y^*_B$,*

$$(X^*_i - c_i)_+ = (X - y^*_B)_+ \mathrm{I}_{B_i} \quad Q\text{-a.s.}, \quad i = 1, \dots, n,$$

where y^*_B is defined in (3.13).

Proof. Assume $\overline{\mathcal{Q}}$ in (3.1) is non-empty so that a Pareto-optimal allocation exists; otherwise there is nothing to show. Let $(X^*_1, \dots, X^*_n)$ be a Pareto-optimal allocation and $y_i = \mathrm{VaR}^{Q_i}_{\alpha_i}(X^*_i)$, $i = 1, \dots, n$. Fix $i = 1, \dots, n$. We assert that $X^*_i > y_i$ implies $X^*_j \geqslant y_j$ for any $j \neq i$, Q_i-almost surely. To see this, assume that there exist i and j such that $Q_i(X^*_i > y_i, \ X^*_j < y_j) > 0$. Then there exists $\delta > 0$ such that $Q_i(X^*_i > y_i, \ X^*_j < y_j - \delta) > 0$. Let $A = \{X^*_i > y_i, \ X^*_j < y_j - \delta\}$. It follows that $\mathrm{ES}^{Q_i}_{\alpha_i}(X^*_i - \delta \mathrm{I}_A) < \mathrm{ES}^{Q_i}_{\alpha_i}(X^*_i)$ whereas $\mathrm{ES}^{Q_j}_{\alpha_j}(X^*_j + \delta \mathrm{I}_A) = \mathrm{ES}^{Q_j}_{\alpha_j}(X^*_j)$ since $\mathrm{VaR}^{Q_i}_{\alpha_i}(X^*_i) = y_i$ and $\mathrm{VaR}^{Q_j}_{\alpha_j}(X^*_j) = y_j$. This contradicts the Pareto optimality of $(X_1^*, \dots, X_n^*)$. Hence, we have

$$Q_i(X_i^* > y_i,\ X_j^* < y_j) = 0 \quad \text{for all } i, j = 1, \dots, n.$$

Since $Q_1, \dots, Q_n$ are equivalent, it follows that

$$\sum_{i=1}^n (X_i^* - y_i)_+ = \left(\sum_{i=1}^n X_i^* - \sum_{i=1}^n y_i\right)_+ = \left(X - \sum_{i=1}^n y_i\right)_+ \quad Q\text{-a.s.} \tag{3.18}$$

Define $Z_i = \frac{1}{\alpha_i}\frac{\mathrm{d}Q_i}{\mathrm{d}Q}$, $i = 1, \dots, n$. By (3.16), the minimization problem in (2.11) is equivalent to

$$\begin{array}{ll} \text{to minimize} & \sum_{i=1}^n \mathbb{E}^Q[Z_i(X_i - \mathrm{VaR}_{\alpha_i}^{Q_i}(X_i))_+] + \sum_{i=1}^n \mathrm{VaR}_{\alpha_i}^{Q_i}(X_i) \\ \text{over} & (X_1, \dots, X_n) \in \mathbb{A}_n(X). \end{array} \tag{3.19}$$

From (2.6) and (3.18), we know that an optimizer $(X_1^*, \dots, X_n^*)$ of (3.19) satisfies $\sum_{i=1}^n (X_i^* - y_i)_+ = (X - \sum_{i=1}^n y_i)_+$ Q-almost surely, where $y_i = \mathrm{VaR}_{\alpha_i}^{Q_i}(X_i^*)$, $i = 1, \dots, n$. Consider the optimization problem

$$\begin{array}{ll} \text{to minimize} & \sum_{i=1}^n \mathbb{E}^Q[Z_i W_i] + y \\ \text{over} & (W_1, \dots, W_n) \in \mathbb{A}_n((X - y)_+),\ y \in \mathbb{R} \\ \text{subject to} & W_i \geqslant 0,\ i = 1, \dots, n. \end{array} \tag{3.20}$$

Note that the constraints in (3.19) are replaced by weaker constraints, and we allow to choose $y \in \mathbb{R}$ in (3.20) which is fixed as $y = \sum_{i=1}^n \mathrm{VaR}_{\alpha_i}^{Q_i}(X_i)$ in (3.19). From there, it is clear that the minimum value of (3.20) is no larger than that of (3.19). We shall later see that (3.19) and (3.20) are indeed equivalent. Recall $B_j = \{Z_j = \bigwedge_{i=1}^n Z_i\}$, $j = 1, \dots, n$, and $B_1, \dots, B_n$ are disjoint. For fixed $y \in \mathbb{R}$, writing $W = (X - y)_+$, the optimization problem

$$\begin{array}{ll} \text{to minimize} & \sum_{i=1}^n \mathbb{E}^Q[Z_i W_i] + y \\ \text{over} & (W_1, \dots, W_n) \in \mathbb{A}_n(W) \\ \text{subject to} & W_i \geqslant 0,\ i = 1, \dots, n \end{array} \tag{3.21}$$

admits a unique optimizer via point-wise optimization,

$$W_i^* = W \mathrm{I}_{B_i}, \quad i = 1, \dots, n.$$

Next, we consider the second-step optimization of (3.20),

$$\text{to minimize} \qquad \sum_{i=1}^n \frac{1}{\alpha_i} \mathbb{E}^{Q_i}[(X-y)_+ \mathrm{I}_{B_i}] + y \qquad \text{over } y \in \mathbb{R}. \tag{3.22}$$

By taking a derivative with respect to y, the set of optimizers of the problem (3.22) is the interval $[x_B^*, y_B^*]$. Let $y^* \in [x_B^*, y_B^*]$. For $x_1, \dots, x_n \in \mathbb{R}$ with $\sum_{i=1}^n x_i = y^*$, define $X_i^* = (X - y^*)\mathrm{I}_{B_i} + x_i$, $i = 1, \dots, n$. We can verify $(X_1^*, \dots, X_n^*) \in \mathbb{A}_n(X)$ and $(X_i^* - x_i)_+ = W_i^*$, $i = 1, \dots, n$. Thus, optimization problems (3.19) and (3.20) have the same minimum objective values, and an optimizer $(X_1^*, \dots, X_n^*)$ of (3.19) necessarily satisfies $(X_i^* - x_i)_+ = W\mathrm{I}_{B_i} = (X - y^*)_+ \mathrm{I}_{B_i}$ for some $x_1, \dots, x_n \in \mathbb{R}$ with $\sum_{i=1}^n x_i = y^*$.

In summary, for any Pareto-optimal allocation $(X_1^*, \dots, X_n^*)$, there exist $(x_1, \dots, x_n) \in \mathbb{R}$ and $y^* \in [x_B^*, y_B^*]$ satisfying $\sum_{i=1}^n x_i = y^*$, such that for $i = 1, \dots, n$,

$$(X_i^* - x_i)_+ = (X - y^*)_+ \mathrm{I}_{B_i}.$$

Finally, noting that $\{X > y^*\} = \{X > y_B^*\}$ Q-almost surely and $y_B^* \geqslant y^*$, by letting $c_i = x_i - y^* + y_B^*$, we have

$$(X_i^* - c_i)_+ = (X - y_B^*)_+ \mathrm{I}_{B_i}, \quad i = 1, \dots, n.$$

This completes the proof. □

4 Competitive equilibria for ES agents

In this section, we study competitive equilibria as in Definition 2.2 for ES agents. Similarly to Section 3, throughout this section, $\alpha_1, \dots, \alpha_n \in (0,1)$, $Q_1, \dots, Q_n \in \mathcal{P}$, and the risk measure of agent i is $\mathrm{ES}_{\alpha_i}^{Q_i}$, $i = 1, \dots, n$. Each agent's objective is

$$\begin{array}{lll} \text{to minimize} & \mathrm{ES}_{\alpha_i}^{Q_i}(X_i) \quad \text{over} \quad X_i \in \mathcal{X} & \\ \text{subject to} & \mathbb{E}^Q[X_i] \geqslant \mathbb{E}^Q[\xi_i] & \end{array} \qquad i = 1, \dots, n, \tag{4.1}$$

where $(\xi_1, \ldots, \xi_n) \in \mathbb{A}_n(X)$ is the vector of initial risks.

With the Pareto optimality problem solved explicitly in Section 3, we establish in this section that for ES agents, a Pareto-optimal allocation is equivalent to an equilibrium allocation. Thus, the two Fundamental Theorems of Welfare Economics (FTWE) hold for ES agents[4]. As in the proof of Proposition 3.2, throughout this section we define the probability measure Q^*_B via

$$Q^*_B(C) = \sum_{i=1}^{n} \frac{1}{\alpha_i} Q_i(C \cap B_i \cap \{X > x^*_B\}), \quad C \in \mathcal{F}, \tag{4.2}$$

where $B = (B_1, \ldots, B_n)$ and x^*_B are defined by (3.3) and (3.12). We can verify that Q^*_B does not depend on the order of $Q_1, \ldots, Q_n$, although the choice of B in (3.3) does. Later we shall see that the probability measure Q^*_B turns out to be the unique equilibrium price for the ES agents. We first present the FTWE for ES agents.

Theorem 4.1. *An allocation of $X \in \mathcal{X}$ is Pareto-optimal if and only if it is an equilibrium allocation for some initial risks $(\xi_1, \ldots \xi_n) \in \mathbb{A}_n(X)$.*

Proof. First, an equilibrium allocation is necessarily Pareto-optimal, as the so-called *non-satiation* condition (see Starr (2011)) holds for the ES agents. For the reader who is not familiar with the FTWE, we provide a self-contained simple proof. Suppose that $(Q, (X^*_1, \ldots, X^*_n))$ is a competitive equilibrium, and $(X^*_1, \ldots, X^*_n)$ is not Pareto-optimal. Then, there exists $(Y_1, \ldots, Y_n) \in \mathbb{A}_n(X)$ such that $\mathrm{ES}^{Q_i}_{\alpha_i}(Y_i) \leqslant \mathrm{ES}^{Q_i}_{\alpha_i}(X^*_i)$ for all $i = 1, \ldots, n$ and there exists $j \in \{1, \ldots, n\}$ such that $\mathrm{ES}^{Q_j}_{\alpha_j}(Y_j) < \mathrm{ES}^{Q_j}_{\alpha_j}(X^*_j)$. If $\mathbb{E}^Q[Y_j] \geqslant \mathbb{E}^Q[\xi_j]$, then X^*_j is not optimal for (4.1) since it is strictly dominated by Y_j, and thus a contradiction. If $\mathbb{E}^Q[Y_j] < \mathbb{E}^Q[\xi_j]$, then there exists $k \in \{1, \ldots, n\}$ such that $\mathbb{E}^Q[Y_k] > \mathbb{E}^Q[\xi_k]$. Similarly, X^*_k is not optimal for (4.1) since it is strictly dominated by $Y_k - \mathbb{E}^Q[Y_k] + \mathbb{E}^Q[\xi_k]$, and thus a contradiction.

Next we show that a Pareto-optimal allocation is necessarily an equilibrium allocation. Let $(X^*_1, \ldots, X^*_n)$ be a Pareto-optimal allocation. For $i = 1, \ldots, n$, consider the individual optimization problem in (4.1) with the initial risks $\xi_i = X^*_i$ and the pricing

[4] Roughly speaking, the first FTWE states that, under some conditions, an equilibrium allocation is Pareto-optimal, and the second FTWE states that, under some conditions, a Pareto-optimal allocation is an equilibrium allocation.

measure Q_B^*, namely

$$\min_{X_i\in\mathcal{X}} \mathrm{ES}_{\alpha_i}^{Q_i}(X_i) \text{ subject to } \mathbb{E}^{Q_B^*}[X_i]\geqslant \mathbb{E}^{Q_B^*}[X_i^*]. \tag{4.3}$$

Note that for any $X_i\in\mathcal{X}$ with $\mathbb{E}^{Q_B^*}[X_i]\geqslant \mathbb{E}^{Q_B^*}[X_i^*]$, we have

$$\mathrm{ES}_{\alpha_i}^{Q_i}(X_i) = \sup_{Q\in\mathcal{P},\,\mathrm{d}Q/\mathrm{d}Q_i\leqslant 1/\alpha_i} \mathbb{E}^{Q}[X_i]\geqslant \mathbb{E}^{Q_B^*}[X_i]\geqslant \mathbb{E}^{Q_B^*}[X_i^*],$$

where the first inequality follows from $Q_B^*\in\overline{\mathcal{Q}}$ with $\overline{\mathcal{Q}}$ defined in (3.1). Therefore, the minimum value of the objective in the optimization problem (4.3) is at least $\mathbb{E}^{Q_B^*}[X_i^*]$. As a consequence, $\mathrm{ES}_{\alpha_i}^{Q_i}(X_i^*)\geqslant \mathbb{E}^{Q_B^*}[X_i^*]$. Noting that $(X_1^*,\dots,X_n^*)$ is a Pareto-optimal allocation, from (3.11) we have

$$\sum_{i=1}^{n}\mathrm{ES}_{\alpha_i}^{Q_i}(X_i^*) = \square_{i=1}^{n}\mathrm{ES}_{\alpha_i}^{Q_i}(X) = \mathbb{E}^{Q_B^*}[X]. \tag{4.4}$$

Combined with the fact that $\mathrm{ES}_{\alpha_i}^{Q_i}(X_i^*)\geqslant \mathbb{E}^{Q_B^*}[X_i^*]$, we have

$$\mathrm{ES}_{\alpha_i}^{Q_i}(X_i^*) = \mathbb{E}^{Q_B^*}[X_i^*],\quad i=1,\dots,n.$$

That is, X_i^* is an optimizer of the optimization problem (4.3). By definition, $(Q_B^*,(X_1^*,\dots,X_n^*))$ is a competitive equilibrium. □

Remark 4.1. From Proposition 3.1 and Theorem 4.1, Pareto-optimal allocations and equilibria may exist for ES agents even if their beliefs are not equivalent. This is in sharp contrast to the classic setting of expected utility agents, where generally no Pareto-optimal allocations or equilibria exist if beliefs are not equivalent.

In the proof of Theorem 4.1, we have already seen that Q_B^* is an equilibrium price for the ES agents. The next theorem verifies that Q_B^* is indeed the unique equilibrium price.

Theorem 4.2. *For a given $X\in\mathcal{X}$, the equilibrium price is uniquely given by Q_B^*.*

Proof. Let $(Q,(X_1^*,\dots,X_n^*))$ be a competitive equilibrium. We show the uniqueness of the equilibrium price in two steps.

(i) Assume for the purpose of contradiction that there exists $A\in\mathcal{F}$ with $A\subset\{X>x_B^*\}$ such that $Q(A)>Q_B^*(A)$. Since $Q_B^*(A)=\sum_{i=1}^n\frac{1}{\alpha_i}Q_i(A\cap B_i)$, we know that

$Q(A\cap B_j) > \frac{1}{\alpha_j} Q_j(A\cap B_j)$ for some $j\in\{1,\dots,n\}$. For a positive constant m, take $Y_j = X_j^* + m\mathrm{I}_{A\cap B_j} - mQ(A\cap B_j)$. Obviously, $\mathbb{E}^Q[Y_j] = \mathbb{E}^Q[X_i^*]$. We can verify that

$$\begin{aligned}
\mathrm{ES}^{Q_j}_{\alpha_j}(Y_j) &= \mathrm{ES}^{Q_j}_{\alpha_j}(X_j^* + m\mathrm{I}_{A\cap B_j} - mQ(A\cap B_j)) \\
&= \mathrm{ES}^{Q_j}_{\alpha_j}(X_j^* + m(\mathrm{I}_{A\cap B_j})) - mQ(A\cap B_j) \\
&\leqslant \mathrm{ES}^{Q_j}_{\alpha_j}(X_j^*) + m\mathrm{ES}^{Q_j}_{\alpha_j}(\mathrm{I}_{A\cap B_j}) - mQ(A\cap B_j) \\
&\leqslant \mathrm{ES}^{Q_j}_{\alpha_j}(X_j^*) + m\frac{1}{\alpha_j}Q_j(A\cap B_j) - mQ(A\cap B_j) < \mathrm{ES}^{Q_j}_{\alpha_j}(X_j^*),
\end{aligned}$$

where the first inequality is due to the subadditivity of ES (see e.g. Embrechts and Wang (2015)). This contradicts the fact that $(Q,(X_1^*,\dots,X_n^*))$ is a competitive equilibrium, since Y_j strictly dominates X_j^* in the individual optimization (4.1). Therefore, we conclude that $Q(A)\leqslant Q_B^*(A)$ for all $A\in\mathcal{F}$ with $A\subset\{X>x_B^*\}$.

(ii) By Theorem 4.1, $(X_1^*,\dots,X_n^*)$ is Pareto-optimal, and hence (4.4) holds. Since $(Q,(X_1^*,\dots,X_n^*))$ is a competitive equilibrium, for $i=1,\dots,n$, we have $\mathrm{ES}^{Q_i}_{\alpha_i}(X_i^*)\leqslant \mathbb{E}^Q[X_i^*]$, otherwise X_i^* would have been strictly dominated by $Y_i=\mathbb{E}^Q[X_i^*]$. By (4.4), we have

$$\mathbb{E}^{Q_B^*}[X] = \sum_{i=1}^n \mathrm{ES}^{Q_i}_{\alpha_i}(X_i^*) \leqslant \sum_{i=1}^n \mathbb{E}^Q[X_i^*] = \mathbb{E}^Q[X]. \tag{4.5}$$

From part (i), we know that Q is dominated by Q_B^* on $\{X>x_B^*\}$. Assume $Q(X\leqslant x_B^*)>0$. It follows that

$$\begin{aligned}
\int_{\{X>x_B^*\}} X\mathrm{d}(Q_B^*-Q) &> x_B^*(Q_B^*(X>x_B^*) - Q(X>x_B^*)) \\
&= x_B^* Q(X\leqslant x_B^*).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbb{E}^{Q_B^*}[X] - \mathbb{E}^Q[X] &= \int_{\{X>x_B^*\}} X\mathrm{d}(Q_B^*-Q) + \int_{\{X\leqslant x_B^*\}} X\mathrm{d}(Q_B^*-Q) \\
&> x_B^* Q(X\leqslant x_B^*) - \int_{\{X\leqslant x_B^*\}} X\mathrm{d}Q \\
&\geqslant x_B^* Q(X\leqslant x_B^*) - x_B^* Q(X\leqslant x_B^*) = 0,
\end{aligned}$$

contradicting (4.5). From there, we conclude that $Q(X\leqslant x_B^*)=0$.

Combining (i) and (ii), we have $Q(A) \leqslant Q_B^*(A)$ for all $A \in \mathcal{F}$. Since both Q and Q_B^* are probability measures, we conclude that $Q = Q_B^*$. $\square$

As a consequence of Theorems 4.1 and 4.2, we have the following corollary characterizing all equilibria for given initial risks.

Corollary 4.3. *For any choice of initial risks* $(\xi_1, \dots \xi_n) \in \mathbb{A}_n(X)$*, a competitive equilibrium is necessarily and sufficiently given by* $(Q_B^*, (X_1^*, \dots, X_n^*))$*, where* $(X_1^*, \dots, X_n^*) \in \mathbb{A}_n(X)$ *is a Pareto-optimal allocation such that* $\mathbb{E}^{Q_B^*}[X_i^*] = \mathbb{E}^{Q_B^*}[\xi_i]$*,* $i = 1, \dots, n$*.*

Recalling Theorem 3.3, an explicit form of Pareto-optimal allocations is given by

$$X_i^* = (X - x^*)\mathrm{I}_{B_i} + c_i, \quad i = 1, \dots, n, \tag{4.6}$$

for any $x^* \in [x_B^*, y_B^*]$, where $\sum_{i=1}^n c_i = x^*$, $B = (B_1, \dots, B_n)$, x_B^* and y_B^* are defined by (3.3), (3.12) and (3.13).

Corollary 4.4. *Assume* $\overline{\mathcal{Q}}$ *in* (3.1) *is non-empty. Then* $(Q_B^*, (X_1^*, \dots, X_n^*))$ *given in* (4.2) *and* (4.6) *is a competitive equilibrium.*

5 Risk sharing games for VaR agents

In this section, we investigate risk-sharing games for VaR agents. Throughout this section, $\alpha_1, \dots, \alpha_n \in (0, 1)$, $Q_1, \dots, Q_n \in \mathcal{P}$, and the risk measure of agent i is $\mathrm{VaR}_{\alpha_i}^{Q_i}$, $i = 1, \dots, n$. The main difference between VaR and ES is the non-convexity of VaR, and hence the classic approach based on convex analysis cannot be used.

We introduce a few key quantities in our analysis for VaR agents. For $X \in \mathcal{X}$ and $x \in [-\infty, \infty)$, define the set

$$\Gamma(x) = \{(Q_1(X > x, A_1), \dots, Q_n(X > x, A_n)) : (A_1, \dots, A_n) \in \pi_n(\Omega)\} + \mathbb{R}_+^n, \tag{5.1}$$

where $\mathbb{R}_+ = [0, \infty)$. Note that for fixed $A \in \mathcal{F}$ and $i = 1, \dots, n$, $Q_i(X > x, A)$ is right-continuous and decreasing in x, with $Q_i(X > \infty, A) = 0$. Therefore, for each $(\alpha_1, \dots, \alpha_n) \in (0, 1)^n$, there exists a smallest number $x^* \in [-\infty, \infty)$ such that $(\alpha_1, \dots, \alpha_n) \in \Gamma(x^*)$. That is,

$$x^* = \min\{x \in [-\infty, \infty) : (\alpha_1, \dots, \alpha_n) \in \Gamma(x)\}.$$

It follows that there exists $(A_1^*,\dots,A_n^*)\in\pi_n(\Omega)$ such that

$$(\alpha_1,\dots,\alpha_n)\geqslant(Q_1(X>x^*,A_1^*),\dots,Q_n(X>x^*,A_n^*)).\tag{5.2}$$

One can verify that for $X\in\mathcal{X}$, $x^*>-\infty$ if and only if

$$\bigvee_{i=1}^{n}\frac{Q_i(A_i)}{\alpha_i}>1\quad\text{for all }(A_1,\dots,A_n)\in\pi_n(\Omega).\tag{5.3}$$

To show this, note that $x^*>-\infty$ if and only if there exists $x\in\mathbb{R}$ such that for all $(A_1,\dots,A_n)\in\pi_n(\Omega)$, $Q_i(X>x,A_i)\geqslant\alpha_i$ for some $i=1,\dots,n$. This is in turn equivalent to (5.3).

Now we present the Pareto-optimal allocations for VaR agents.

Theorem 5.1. *For $X\in\mathcal{X}$, the following hold.*

(i) We have

$$\mathop{\square}_{i=1}^{n}\mathrm{VaR}_{\alpha_i}^{Q_i}(X)=\min\{x\in[-\infty,\infty):(\alpha_1,\dots,\alpha_n)\in\Gamma(x)\}=x^*,$$

where Γ is given by (5.1).

(ii) If (5.3) holds, that is, $x^*>-\infty$, a Pareto-optimal allocation $(X_1^*,\dots,X_n^*)$ of X is given by

$$X_i^*=(X-x^*)\mathrm{I}_{A_i^*}+\frac{x^*}{n},\quad i=1,\dots,n,$$

where $(A_1^*,\dots,A_n^*)$ satisfies (5.2).

Proof. (i) First we show $\square_{i=1}^n\mathrm{VaR}_{\alpha_i}^{Q_i}(X)\geqslant x^*$. For $(X_1,\dots,X_n)\in\mathbb{A}_n(X)$, let $D_i=\{X_i>\mathrm{VaR}_{\alpha_i}^{Q_i}(X_i)\}$, $i=1,\dots,n$. Clearly, $Q_i(D_i)\leqslant\alpha_i$. Let $C_i=D_i\cup(\bigcup_{j=1}^n D_j)^c$ for $i=1,\dots,n$. We have $\bigcup_{i=1}^n C_i=\Omega$, and hence there exists $(A_1,\dots,A_n)\in\pi_n(\Omega)$ such that $A_i\subset C_i$. Write $x=\sum_{i=1}^n\mathrm{VaR}_{\alpha_i}^{Q_i}(X_i)$. We have

$$\{X>x\}=\left\{\sum_{i=1}^n X_i>\sum_{i=1}^n\mathrm{VaR}_{\alpha_i}^{Q_i}(X_i)\right\}\subset\bigcup_{i=1}^n D_i.$$

Therefore, for $i=1,\dots,n$,

$$Q_i(X>x,A_i)\leqslant Q_i(X>x,C_i)=Q_i(X>x,D_i)\leqslant Q_i(D_i)\leqslant\alpha_i.$$

This shows $(\alpha_1, \dots, \alpha_n) \in \Gamma(x)$. As a consequence,

$$x^* = \min\{x \in [-\infty, \infty) : (\alpha_1, \dots, \alpha_n) \in \Gamma(x)\} \leqslant x = \sum_{i=1}^n \mathrm{VaR}_{\alpha_i}^{Q_i}(X_i).$$

From there we obtain $\square_{i=1}^n \mathrm{VaR}_{\alpha_i}^{Q_i}(X) \geqslant x^*$.

Next we show $\square_{i=1}^n \mathrm{VaR}_{\alpha_i}^{Q_i}(X) \leqslant x^*$. Take any $x \in \mathbb{R}$ such that $(\alpha_1, \dots, \alpha_n) \in \Gamma(x)$. By definition, there exists $(A_1, \dots, A_n) \in \pi_n(\Omega)$ such that

$$Q_i((X - x)\mathrm{I}_{A_i} > 0) = Q_i(X > x, A_i) \leqslant \alpha_i, \quad i = 1, \dots, n. \tag{5.4}$$

Note that (5.4) implies $\mathrm{VaR}_{\alpha_i}^{Q_i}((X - x)\mathrm{I}_{A_i}) \leqslant 0$ for $i = 1, \dots, n$. Let

$$X_i = (X - x)\mathrm{I}_{A_i} + \frac{x}{n}, \quad i = 1, \dots, n. \tag{5.5}$$

We have $(X_1, \dots, X_n) \in \mathbb{A}_n(X)$ and

$$\sum_{i=1}^n \mathrm{VaR}_{\alpha_i}^{Q_i}(X_i) = \sum_{i=1}^n \mathrm{VaR}_{\alpha_i}^{Q_i}((X - x)\mathrm{I}_{A_i}) + x \leqslant x.$$

This shows $\square_{i=1}^n \mathrm{VaR}_{\alpha_i}^{Q_i}(X) \leqslant x$ for all real numbers $x \geqslant x^*$. Therefore, $\square_{i=1}^n \mathrm{VaR}_{\alpha_i}^{Q_i}(X) \leqslant x^*$. In summary, we have $\square_{i=1}^n \mathrm{VaR}_{\alpha_i}^{Q_i}(X) = x^*$.

(ii) Suppose $x^* > -\infty$. Similarly to (5.4) and (5.5), from the definition of x^* and A_i^*, we have

$$\sum_{i=1}^n \mathrm{VaR}_{\alpha_i}^{Q_i}(X_i^*) = \sum_{i=1}^n \mathrm{VaR}_{\alpha_i}^{Q_i}((X - x^*)\mathrm{I}_{A_i^*}) + x^* \leqslant x^*.$$

Together with (i), we conclude that $(X_1^*, \dots, X_n^*)$ is a Pareto-optimal allocation of X. □

As an immediate consequence of Theorem 5.1 (ii), a Pareto-optimal allocation exists if and only if (5.3) holds. Similarly to the case of ES agents, the existence of a Pareto-optimal allocation only depends on $(\alpha_1, \dots, \alpha_n)$ and $(Q_1, \dots, Q_n)$, but not on the total risk X.

The Pareto-optimal allocation for VaR agents in Theorem 5.1,

$$X_i^{\mathrm{VaR}} = (X - x^*)\mathrm{I}_{A_i^*} + \frac{x^*}{n}, \quad i = 1, \dots, n,$$

and that for ES agents in Theorem 3.3,

$$X_i^{\mathrm{ES}} = (X - y^*)\mathrm{I}_{B_i} + \frac{y^*}{n}, \quad i = 1, \dots, n,$$

share amazing similarity in their forms. Nevertheless, we should clarify that the calculation of $(A_1^*, \dots, A_n^*, x^*)$ and $(B_1^*, \dots, B_n^*, y^*)$ above are completely different, and these two risk sharing problems have essentially distinct features. We remark two significant differences. First, the optimization problem for VaR agents is a non-convex one, whereas that for ES agents is convex. Second, $(B_1^*, \dots, B_n^*, y^*)$ has explicit forms, but $(A_1^*, \dots, A_n^*, x^*)$ does not; an efficient way to compute $(A_1^*, \dots, A_n^*, x^*)$ seems unavailable at the moment.

Remark 5.1. If $Q_1 = \cdots = Q_n = \mathbb{P}$, we have

$$\begin{aligned}
&\mathop{\square}_{i=1}^{n} \mathrm{VaR}_{\alpha_i}^{\mathbb{P}}(X) \\
&= \inf\{x \in \mathbb{R} \mid \mathbb{P}(A_i^x) = \alpha_i, A_i^x = A_i \cap \{X > x\}, (A_1, \dots, A_n) \in \pi_n(\Omega)\} \\
&= \inf\{x \in \mathbb{R} \mid \mathbb{P}(X > x) = \alpha_1 + \cdots + \alpha_n\} = \mathrm{VaR}_{\sum_{i=1}^n \alpha_i}^{\mathbb{P}}(X).
\end{aligned}$$

This is a special case of Theorem 2 of Embrechts et al. (2017).

Next, we observe that a competitive equilibrium for VaR agents does not exist. In this setting, each agent's objective is

$$\begin{array}{lll} \text{to minimize} & \mathrm{VaR}_{\alpha_i}^{Q_i}(X_i) \quad \text{over} \quad X_i \in \mathcal{X} & \\ \text{subject to} & \mathbb{E}^Q[X_i] \geqslant \mathbb{E}^Q[\xi_i], & \end{array} \qquad i = 1, \dots, n, \tag{5.6}$$

Proposition 5.2. *For any choice of initial risks, a competitive equilibrium for the VaR agents does not exist.*

Proof. First note that there is a correspondence between the optimizer of (5.6) and that of the following optimization problem

$$\text{to minimize} \quad \mathcal{V}_i(Y_i) = \mathrm{VaR}_{\alpha_i}^{Q_i}(Y_i) - \mathbb{E}^Q[Y_i] \quad \text{over} \quad Y_i \in \mathcal{X}, \qquad i = 1, \dots, n, \tag{5.7}$$

by choosing $X_i = Y_i - \mathbb{E}^Q[Y_i] + \mathbb{E}^Q[\xi_i]$. Note that for any probability measure Q, one can easily find a random variable X_i such that $\mathrm{VaR}^{Q_i}_{\alpha_i}(Y_i) < \mathbb{E}^Q[Y_i]$. Then by the positive homogeneity of $\mathcal{V}_i$, we have that the infimum of the objective function is $-\infty$, and hence (5.6) admits no optimizer. $\square$

A possible alternative setting to study competitive equilibria for VaR agents is to restrict the set of admissible positions for each agent and to slightly relax the definition of pricing measures; see Embrechts et al. (2017) for the case allowing only $0 \leqslant X_i \leqslant X$. In the latter paper, for VaR agents with homogeneous beliefs, the equilibrium price Q is shown to be a zero measure instead of a probability measure, and this is beyond the framework of this paper.

6 Risk sharing games for mixed VaR and ES agents

In this section, we consider the risk sharing problem in which some agents are VaR agents and the others are ES agents. The result naturally generalizes to RVaR agents, which shall be defined later. Define the index sets $I = \{1, \dots, m\}$ and $J = \{m+1, \dots, n\}$, $0 \leqslant m < n$. Without loss of generality, assume that for $i \in I$ and $j \in J$, the objective of agent i is $\mathrm{VaR}^{Q_i}_{\alpha_i}$ and that of agent j is $\mathrm{ES}^{Q_j}_{\beta_j}$, where $\alpha_i, \beta_j \in (0,1)$ and $Q_i, Q_j \in \mathcal{P}$. Note that here we allow I to be empty but J is assumed non-empty, i.e. there is at least one ES agent. For notional simplicity, in this section we write, for $X \in \mathcal{X}$,

$$V(X) = \inf\left\{\sum_{i\in I} \mathrm{VaR}^{Q_i}_{\alpha_i}(X_i) + \sum_{j\in J} \mathrm{ES}^{Q_j}_{\beta_j}(X_j) : (X_1, \dots, X_n) \in \mathbb{A}_n(X)\right\}. \tag{6.1}$$

We first verify that $V(X) > -\infty$ if and only if

$$\bigvee_{i=1}^{n} \frac{Q_i(A_i)}{\alpha_i} > 1 \ \text{ for all } (A_1, \dots, A_m) \in \pi_m(\Omega), \tag{6.2}$$

and

$$\sum_{j\in J} \frac{Q_j(B_j)}{\beta_j} \geqslant 1 \ \text{ for all } (B_1, \dots, B_n) \in \pi_n(\Omega) \text{ with } Q_i(B_i) \leqslant \alpha_i, i \in I. \tag{6.3}$$

To see this, first note that (6.2) is a necessary condition for $V(X) > -\infty$ by (5.3). Hence, we only need to show that when (6.2) holds, $V(X) > -\infty$ if and only if (6.3) holds. To show the necessity, assume (6.3) does not hold. There exists $(B_1,\dots,B_n)\in\pi_n(\Omega)$ with $Q_i(B_i)\leqslant\alpha_i$, $i\in I$, such that $\sum_{j\in J}Q_j(B_j)/\beta_j<1$. Define

$$X_i=(X+x)\mathrm{I}_{B_i}-\frac{x}{n},\quad i=1,\dots,n,$$

where $x>0$. Then

$$\begin{aligned}\sum_{i\in I}\mathrm{VaR}_{\alpha_i}^{Q_i}(X_i)+\sum_{j\in J}\mathrm{ES}_{\beta_j}^{Q_j}(X_j)&\leqslant\sum_{j\in J}\mathrm{ES}_{\beta_j}^{Q_j}((X+x)\mathrm{I}_{B_j})-x\\&=\sum_{j\in J}\frac{1}{\beta_j}\mathbb{E}^{Q_j}[X\mathrm{I}_{B_j}]+\sum_{j\in J}\frac{Q_j(B_j)}{\beta_j}x-x.\end{aligned}$$

Letting $x\to\infty$, we have that the right hand side of the above equation converges to $-\infty$, and hence, $V(X)=-\infty$. The sufficiency is implied by the following theorem.

Theorem 6.1. *Assume that* (6.2) *and* (6.3) *hold. Then for* $X\in\mathcal{X}$*,*

$$V(X)=\min\left\{\sum_{j\in J}\frac{1}{\beta_j}\mathbb{E}^{Q_j}[(X-x)_+\mathrm{I}_{B_j}]+x\;\middle|\;\begin{array}{l}(B_1,\dots,B_n)\in\pi_n(\Omega)\\Q_i(B_i)\leqslant\alpha_i,\ i\in I,\ x\in\mathbb{R}\end{array}\right\}.\tag{6.4}$$

Proof. Note that X is bounded and both sides of (6.4) are translation invariant[5]. Without loss of generality, we assume $X\geqslant 0$. Denote by $R(X)$ the right hand side of (6.4); we will show $V(X)=R(X)$. For any $(B_1,\dots,B_n)\in\pi_n(\Omega)$ such that $Q_i(B_i)\leqslant\alpha_i$ for $i\in I$, take $X_i=X\mathrm{I}_{B_i}$, $i=1,\dots,n$. Then, $\mathrm{VaR}_{\alpha_i}^{Q_i}(X_i)=0$, $i\in I$, and for $j\in J$,

$$\begin{aligned}\mathrm{ES}_{\beta_j}^{Q_j}(X_j)&=\min\left\{\frac{1}{\beta_j}\mathbb{E}^{Q_j}[(X\mathrm{I}_{B_j}-x)_+]+x:x\in\mathbb{R}\right\}\\&=\min\left\{\frac{1}{\beta_j}\mathbb{E}^{Q_j}[(X\mathrm{I}_{B_j}-x)_+]+x:x\in\mathbb{R}_+\right\}\\&=\min\left\{\frac{1}{\beta_j}\mathbb{E}^{Q_j}[(X-x)_+\mathrm{I}_{B_j}]+x:x\in\mathbb{R}\right\}.\end{aligned}$$

Therefore, for all $(B_1,\dots,B_n)\in\pi_n(\Omega)$ such that $Q_i(B_i)\leqslant\alpha_i$, $i\in I$, we have

$$V(X)\leqslant\sum_{j\in J}\min\left\{\frac{1}{\beta_j}\mathbb{E}^{Q_j}[(X-x)_+\mathrm{I}_{B_j}]+x:x\in\mathbb{R}\right\}$$

[5]Following the risk management literature, a functional $f:\mathcal{X}\to\mathbb{R}$ is called *translation invariant* if $f(X+c)=f(X)+c$ for all $X\in\mathcal{X}$ and $c\in\mathbb{R}$.

$$\leqslant \min\left\{\sum_{j\in J}\frac{1}{\beta_j}\mathbb{E}^{Q_j}[(X-x)_+\mathrm{I}_{B_j}]+x : x\in\mathbb{R}\right\}.$$

Hence, $V(X)\leqslant R(X)$. To show $V(X)\geqslant R(X)$, we need to prove that for any $(X_1,\dots,X_n)\in\mathbb{A}_n(X)$, there exists $(B_1,\dots,B_n)\in\pi_n(\Omega)$ with $Q_i(B_i)\leqslant\alpha_i$, $i\in I$ such that

$$\sum_{i\in I}\mathrm{VaR}_{\alpha_i}^{Q_i}(X_i)+\sum_{j\in J}\mathrm{ES}_{\beta_j}^{Q_j}(X_j)\geqslant\min_{x\in\mathbb{R}}\left\{\sum_{j\in J}\frac{1}{\beta_j}\mathbb{E}^{Q_j}[(X-x)_+\mathrm{I}_{B_j}]+x\right\}. \tag{6.5}$$

Because of translation invariance of VaR and ES, without loss of generality, assume $\mathrm{VaR}_{\alpha_i}^{Q_i}(X_i)=0$, $i\in I$. As $Q_i(\{X_i>0\})\leqslant\alpha_i$, $i\in I$, there exists a set $B_1\in\mathcal{F}$ such that $\{X_1>0\}\subset B_1$ and $Q_1(B_1)\leqslant\alpha_1$. Similarly, for $i\in I$, let B_i be a set such that $\{X_i>0\}\setminus\cup_{k=1}^{i-1}B_k\subset B_i$ and $Q_i(B_i)\leqslant\alpha_i$. Let $B=\cup_{i\in I}B_i$,

$$X_i^*=X_i\mathrm{I}_B+d\,\mathrm{I}_{B_i},\ i\in I,\ \text{and}\ X_j^*=(X_j-d/(n-m))\mathrm{I}_B+X_j\mathrm{I}_{B^c},\ j\in J,$$

where $d>0$ is large enough such that

$$\text{ess-sup}(X_j-d/(n-m))\mathrm{I}_B<\min\{X_j\mathrm{I}_{B^c}\},\quad j\in J.$$

Clearly, $X_j^*\leqslant X_j$ for $j\in J$, and $\mathrm{VaR}_{\alpha_i}^{Q_i}(X_i^*)=\mathrm{VaR}_{\alpha_i}^{Q_i}(X_i)=0$. By (6.3), we have $Q_j(B^c)\geqslant\beta_j$, implying $\mathrm{ES}_{\beta_j}^{Q_j}(X_j^*)=\mathrm{ES}_{\beta_j}^{Q_j}(X_j\mathrm{I}_{B^c})$. Also note that $\cup_{i\in I}\{X_i>0\}\subset B$, implying $\sum_{j\in J}X_j\mathrm{I}_{B^c}=\left(X-\sum_{i\in I}X_i\right)\mathrm{I}_{B^c}\geqslant X\mathrm{I}_{B^c}$. Using the above facts, we have

$$\begin{aligned}\sum_{i\in I}\mathrm{VaR}_{\alpha_i}^{Q_i}(X_i)+\sum_{j\in J}\mathrm{ES}_{\beta_j}^{Q_j}(X_j)&\geqslant\sum_{i\in I}\mathrm{VaR}_{\alpha_i}^{Q_i}(X_i^*)+\sum_{j\in J}\mathrm{ES}_{\beta_j}^{Q_j}(X_j^*)\\&=\sum_{j\in J}\mathrm{ES}_{\beta_j}^{Q_j}(X_j\mathrm{I}_{B^c})\\&\geqslant\mathop{\square}_{j\in J}\mathrm{ES}_{\beta_j}^{Q_j}\left(\sum_{j\in J}X_j\mathrm{I}_{B^c}\right)\geqslant\mathop{\square}_{j\in J}\mathrm{ES}_{\beta_j}^{Q_j}(X\mathrm{I}_{B^c}).\end{aligned}$$

By Proposition 3.2, we have

$$\mathop{\square}_{j\in J}\mathrm{ES}_{\beta_j}^{Q_j}(X\mathrm{I}_{B^c})$$

$$= \min \left\{ \sum_{j \in J} \frac{1}{\beta_j} \mathbb{E}^{Q_j}[(X \mathrm{I}_{C_j} - x)_+] + x : x \in \mathbb{R},\ (C_j)_{j \in J} \in \pi_{n-m}(B^c) \right\}$$

$$\geqslant R(X).$$

Thus, $V(X) \geqslant R(X)$, and this completes the proof. □

If $I = \varnothing$, Theorem 6.1 reduces to Proposition 3.2. The case of $J = \varnothing$ (see Theorem 5.1) is not included in Theorem 6.1 since the expression in (6.4) involves a sum of expectations over J. By Theorem 6.1, there exist $x^* \in \mathbb{R}$ and $(B_1, \ldots, B_n) \in \pi_n(\Omega)$ such that $Q_i(B_i) \leqslant \alpha_i$, $i \in I$, and

$$V(X) = \sum_{j \in J} \frac{1}{\beta_j} \mathbb{E}^{Q_j}[(X - x^*)_+ \mathrm{I}_{B_j}] + x^*.$$

A Pareto-optimal allocation $(X_1^*, \ldots, X_n^*)$ of X is given by

$$X_i^* = (X - x^*)\mathrm{I}_{B_i} + \frac{x^*}{n}, \quad i = 1, \ldots, n. \tag{6.6}$$

Nevertheless, analytical formulas of the above $x^* \in \mathbb{R}$ and $(B_1, \ldots, B_n) \in \pi_n(\Omega)$ are not available. A necessary condition for $(B_1, \ldots, B_n) \in \pi_n(\Omega)$ above is

$$B_j \subset \left\{ \frac{1}{\alpha_j} \frac{\mathrm{d}Q_j}{\mathrm{d}Q} = \bigwedge_{i \in J} \frac{1}{\alpha_i} \frac{\mathrm{d}Q_i}{\mathrm{d}Q} \right\}, \quad j \in J,$$

but to determine $(B_1, \ldots, B_n)$ seems a very complicated task, even computationally.

Remark 6.1. When there are mixed VaR and ES agents, as the VaR agents do not care about the the risk above a certain quantile, an intuitive idea to find the Pareto-optimal allocation is to first allocate risks for the VaR agents as in Theorem 5.1, and then allocate risks for the ES agents as in Theorem 3.3. Such a technical treatment turns out to give an optimal allocation in the setting of homogeneous beliefs in Embrechts et al. (2017). Unfortunately, it does not necessarily lead to a Pareto-optimal allocation in our setting of heterogeneous beliefs, and hence yields a sharp contrast to the case of homogeneous beliefs treated in Embrechts et al. (2017).

Finally, we present the result for a more general class of risk measures, the Range-Value-at-Risk (RVaR), as studied in Embrechts et al. (2017). Recall that for $X \in \mathcal{X}$ and

$Q \in \mathcal{P}$, the RVaR at level $(\alpha, \beta) \in [0,1]^2$, $\alpha + \beta \leqslant 1$ is defined as

$$\mathrm{RVaR}^Q_{\alpha,\beta}(X) = \begin{cases} \frac{1}{\beta}\int_\alpha^{\alpha+\beta} \mathrm{VaR}^Q_\gamma(X)\mathrm{d}\gamma & \text{if } \beta > 0, \\ \mathrm{VaR}^Q_\alpha(X) & \text{if } \beta = 0. \end{cases}$$

Clearly, the RVaR family includes both VaR and ES as special cases. For more details on RVaR, see Embrechts et al. (2017). To study risk sharing problems for RVaR agents, the key observation is that RVaR is the inf-convolution of VaR and ES, namely,

$$\mathrm{RVaR}^Q_{\alpha,\beta} = \mathrm{VaR}^Q_\alpha \,\Box\, \mathrm{ES}^Q_\beta;$$

see Theorem 2 of Embrechts et al. (2017). With this result, we can use Theorem 6.1 to calculate the inf-convolution of RVaR and identify its corresponding Pareto-optimal allocations, by decomposing each RVaR agent into two "imaginary" VaR and ES agents. To guarantee the existence of a Pareto-optimal allocation, or equivalently $\Box_{i=1}^n \mathrm{RVaR}^{Q_i}_{\alpha_i,\beta_i}(X) > -\infty$, we require

$$\bigvee_{i=1}^n \frac{Q_i(A_i)}{\alpha_i} > 1 \quad \text{for all } (A_1, \dots, A_n) \in \pi_n(\Omega), \tag{6.7}$$

and

$$\sum_{i=1}^n \frac{Q_i(B_{2i})}{\beta_i} \geqslant 1 \qquad \begin{array}{c} \text{for all } (B_{11}, B_{21}, \dots, B_{1n}, B_{2n}) \in \pi_{2n}(\Omega) \\ \text{with } Q_i(B_{1i}) \leqslant \alpha_i,\ i = 1, \dots, n. \end{array} \tag{6.8}$$

Then the following corollary follows directly from Theorem 6.1.

Corollary 6.2. *Let $X \in \mathcal{X}$ and $\alpha_i, \beta_i \in (0,1)$, $i = 1, \dots, n$. Assume that (6.7) and (6.8) hold. Then*

$$\begin{aligned} &\mathop{\Box}_{i=1}^n \mathrm{RVaR}^{Q_i}_{\alpha_i,\beta_i}(X) \\ &= \min\left\{ \sum_{i=1}^n \frac{1}{\beta_i}\mathbb{E}^{Q_i}\left[(X-x)_+ \mathrm{I}_{B_{2i}}\right] + x \;\middle|\; \begin{array}{l} (B_{11}, B_{21}, \dots, B_{1n}, B_{2n}) \in \pi_{2n}(\Omega) \\ Q_i(B_{1i}) \leqslant \alpha_i,\ i = 1, \dots, n,\ x \in \mathbb{R} \end{array} \right\}. \end{aligned}$$

We conclude this section by observing that, similarly to the case of VaR agents, a competitive equilibrium does not exist for mixed VaR and ES agents or RVaR agents, unless all agents are ES agents.

7 Conclusion

By solving various optimization problems, we obtain in explicit forms Pareto-optimal allocations and competitive equilibria for quantile-based risk measures with belief heterogeneity. For ES agents, we show that Pareto-optimal allocations and equilibrium allocations are equivalent, and the equilibrium price is uniquely determined. In the case of VaR agents, Pareto-optimal allocations are obtained, but competitive equilibria do not exist. Our results and economic interpretations differ significantly from those of Embrechts et al. (2017) where belief homogeneity is assumed. In view of the prominent usage of internal models for various financial institutions, belief heterogeneity seems to be a more reasonable assumption for studying risk sharing games in the context of regulatory capital calculation and its practical implications.

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