Quantile-based Risk Sharing

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Abstract

We address the problem of risk sharing among agents using a two-parameter class of quantile-based risk measures, the so-called Range-Value-at-Risk (RVaR), as their preferences. The family of RVaR includes the Value-at-Risk (VaR) and the Expected Shortfall (ES), the two popular and competing regulatory risk measures, as special cases. We first establish an inequality for RVaR-based risk aggregation, showing that RVaR satisfies a special form of subadditivity. Then, the Pareto-optimal risk sharing problem is solved through explicit construction. To study risk sharing in a competitive market, an Arrow-Debreu equilibrium is established for some simple, yet natural settings. Further, we investigate the problem of model uncertainty in risk sharing, and show that, generally, a robust optimal allocation exists if and only if none of the underlying risk measures is a VaR. Practical implications of our main results for risk management and policy makers are discussed, and several novel advantages of ES over VaR from the perspective of a regulator are thereby revealed.

Keywords: Value-at-Risk, Expected Shortfall, risk sharing, regulatory capital, robustness, Arrow-Debreu equilibrium.

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1 Introduction

1.1 Risk sharing problems and quantile-based risk measures

A risk sharing problem concerns the redistribution of a total risk among multiple participants. In this paper, we address collaborative as well as competitive risk sharing problems in which participants are equipped with *monetary risk measures* (Artzner et al. (1999)). These generic risk sharing problems can be formulated in various contexts. For instance, it may represent regulatory capital reduction within affiliates of a single firm, equilibrium among a group of firms with costs associated with regulatory capital, insurance-reinsurance contracts and risk-transfer, or wealth redistribution among investors. Throughout this paper, we generally refer to a participant in the risk sharing problem as an *agent*, which may represent an affiliate, a firm, an insured, an insurer, or an investor in different contexts.

The most commonly used families of risk measures in practice are the Value-at-Risk (VaR) and the Expected Shortfall (ES); both are implemented in modern financial and insurance regulation (see Section 2 for definitions). During the past few years, there has been an extensive debate on the comparative advantages of VaR and ES; see the academic papers Embrechts et al. (2014) and Emmer et al. (2015) for comprehensive discussions, and BCBS (2014) and IAIS (2014) for contributions from regulators in banking and insurance, respectively.

The one-parameter families of risk measures, VaR and ES, are unified in a more general two-parameter family of risk measures, called the Range-Value-at-Risk (RVaR). The family of RVaR was introduced in Cont et al. (2010) in the context of *robustness* properties of risk measures (see Section 2). More importantly, RVaR can be seen as a bridge connecting VaR and ES, the two most popular but methodologically very different regulatory risk measures. This embedding of VaR and ES into RVaR helps us to understand many properties and comparative advantages of the former risk measures, and hence motivates our concentration on RVaR as the underlying risk measures in the problem of risk sharing discussed in this paper.

Since each of VaR, ES and RVaR can be represented as average quantiles of a random variable, we refer to the problems considered in this paper as *quantile-based* risk sharing. We hope that the methodological results obtained in this paper will be helpful to risk management and policy makers in designing risk allocations and appropriate regulatory risk measures.

1.2 Contribution and structure of the paper

First, some basic definitions and preliminaries on the risk measures used in this paper are given in Section 2.

Our theoretical contributions start with establishing a powerful inequality for the RVaR family in Section 3. This inequality later serves as a building block for the main results on quantile-based risk sharing;

it implies that the risk measures RVaR, including VaR and ES as special cases, satisfy a special form of *subadditivity*.

Section 4 contains results on (Pareto-)optimal allocations for agents whose preferences are characterized by the RVaR family. We first solve the optimal risk sharing problem by characterizing the *inf-convolution* of several RVaR measures with different parameters. An optimal allocation is given through an explicit construction.

In Section 5, we study competitive risk sharing in which each agent optimizes their own preferences, regardless of other participants. We show that, under suitable assumptions, the optimal allocation obtained in Section 4 is an equilibrium allocation in the sense of Arrow-Debreu. Moreover, the equilibrium pricing rule can be obtained explicitly; it has the form of a mixture of a constant and the reciprocal of the total risk.

We then proceed to discuss some relevant issues on optimal allocation in Section 6. In particular, we show that in general, a *robust optimal allocation* exists if and only if none of the underlying risk measures is a VaR, and a *comonotonic optimal allocation* exists only if there is at most one underlying risk measure which is not an ES.

Finally, in Section 7 we summarize our main results, and discuss some practical implications of our results for risk management and policy makers. As a consequence, we reveal several novel advantages of ES-based risk management. The proofs of our main results are put in Section 8, and some related technical details are included in the Appendices.

1.3 Related literature

In a seminal paper, Borch (1962) showed that within the context of concave utilities, Pareto-optimal allocations between agents are *comonotonic*. Since the introduction of *coherent* and *convex risk measures* by Artzner et al. (1999), Föllmer and Schied (2002) and Frittelli and Rosazza Gianin (2002, 2005), the problem of Pareto-optimal risk sharing has been extensively studied when the underlying risk measures are chosen as convex or coherent. As a relevant mathematical tool, the inf-convolution of convex risk measures was obtained in Barrieu and El Karoui (2005). For law-determined *monetary utility functions*, or equivalently, convex risk measures, Jouini et al. (2008) showed the existence of an optimal risk sharing for bounded random variables, which is always comonotonic. This result was generalized to non-monotone risk measures by Acciaio (2007) and Filipović and Svindland (2008), to multivariate risks by Carlier et al. (2012) and to cash-subadditive and quasi-convex risk measures by Mastrogiacomo and Rosazza Gianin (2015). Pareto-optimal risk sharing for Choquet expected utilities is studied by Chateauneuf et al. (2000). See Heath and Ku (2004), Tsanakas (2009) and Dana and Le Van (2010) for more on risk sharing with monetary and convex risk measures. A recent reprint Weber (2017) generalizes the results in Sections 3-4 to a class of distortion risk measures dominated by a VaR. On the design of insurance and reinsurance contracts using risk measures, see Cai et al. (2008), Cui et al. (2013) and Bernard et al. (2015). A summary on problems related to inf-convolution of monetary utility functions can be found in Delbaen (2012). For some recent developments on efficient risk sharing and equilibria of the Arrow and Debreu (1954) type with risk measures and rank-dependent utilities (RDU), see Cherny (2006), Carlier and Dana (2008, 2012), Madan and Schoutens (2012), Xia and Zhou (2016) and Jin et al. (2016). In particular, Xia and Zhou (2016) studied the existence of Arrow-Debreu equilibria for RDU agents and obtained solutions for the state-price density. As far as we are aware of, there is little existing research on non-convex monetary risk measures in risk sharing, and there are no explicit results on equilibrium allocations under such settings.

The extensive debate on desirable properties of regulatory risk measures, in particular VaR and ES, is summarized in Embrechts et al. (2014) and Emmer et al. (2015); see also BCBS (2016) for a recent discussion concerning market risk under Basel III and Sandström (2010, Chapter 14) for an overview in the context of Solvency II. For a critical voice on risk measures and capital requirements in the case of Solvency II, see Floreani (2013). Whereas there is a tendency to move from VaR to ES, for a while to come both risk measures will coexist for regulatory purposes. Our paper adds some guidance potentially useful in reaching more widely acceptable solutions. Many quantitative concepts may enter into this discussion; below we highlight some issues relevant for our discussion. An overriding concept no doubt is *model uncertainty* in its various guises. Robustness of risk measures is addressed in Cont et al. (2010), Kou et al. (2013), Krätschmer et al. (2012, 2014) and Embrechts et al. (2015). The concept of elicitability is closely related to risk measure forecasts. Osband (1985) and Weber (2006) contain key results that are used by Bellini and Bignozzi (2015) and Delbaen et al. (2016) to characterize one-dimensional elicitable risk measures. For recent progress on elicitability, forecasting and backtesting of risk measures, see Gneiting (2011), Ziegel (2016), Fissler and Ziegel (2016), Acerbi and Székely (2014), Kou and Peng (2016) and Davis (2016). Some papers addressing model uncertainty in risk aggregation are Embrechts et al. (2013), Bernard and Vanduffel (2015) and Wang et al. (2015), amongst others. The problems of currency exchange and regulatory arbitrage are discussed in Koch-Medina and Munari (2016) and Wang (2016), and model uncertainty in the context of stress-testing is studied in for instance Cambou and Filipović (2015).

An important feature of our contribution is the introduction of a concept of robustness into the problem of risk sharing. It is well-known that various concepts and applications of robustness exist in different fields. In the realm of statistics, Huber and Ronchetti (2009) is an excellent place to start. For a recent generalization of the classic notion of robustness in the context of tail functionals, see Krätschmer et al. (2014) and Zähle (2016). For discussions on robustness in economics, see for instance, Gilboa and Schmeidler (1989) and Maccheroni et al. (2006) in the theory of preferences, and the classic book Hansen and Sargent (2008). Within the theory of optimization, a standard reference is Ben-Tal et al. (2009). The concept of robustness in this paper relates to the practical consideration of model misspecification, and hence it is different from the problem of risk sharing under *robust utility functionals* as in for instance Knispel et al. (2016).

The risk sharing problem in this paper involves multiple firms in an economy, and as such *systemic risk* becomes relevant. We refer to Acharya (2009), Chen et al. (2013), Rogers and Veraart (2013), Adrian and Brunnermeier (2016), Feinstein et al. (2017) and the references therein for recent developments on systemic risk. In particular, from the results in this paper, some degree of regulation against risk sharing is important for the whole economy when VaR is used as a regulatory risk measure; see also Ibragimov et al. (2011) in a different setting.

2 Risk measures, the RVaR family, and basic terminology

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be an atomless probability space, and $\mathcal{X}$ be the set of real, integrable random variables (i.e. random variables with finite means) defined on $(\Omega, \mathscr{F}, \mathbb{P})$. We treat almost surely equal random variables as identical in this paper and we assume that for any $X \in \mathcal{X}$, there exists a $Y \in \mathcal{X}$ independent of X. A *risk measure* is a functional $\rho : \mathcal{X} \to [-\infty, \infty]$.

Below we list some properties for risk measures: for $X, Y \in \mathcal{X}$,

(a) Monotonicity: $\rho(X) \leqslant \rho(Y)$ if $X \leqslant Y$;

(b) Cash-invariance: $\rho(X + c) = \rho(X) + c$ for any $c \in \mathbb{R}$;

(c) Positive homogeneity: $\rho(\lambda X) = \lambda\rho(X)$ for any $\lambda > 0$;

(d) Subadditivity: $\rho(X + Y) \leqslant \rho(X) + \rho(Y)$;

(e) Law-determination: $\rho(X) = \rho(Y)$ if X and Y have the same distribution.

We refer to Föllmer and Schied (2016, Chapter 4) and Delbaen (2012) for interpretations of and discussions on these, by now standard properties of risk measures.

Definition 1. A *monetary risk measure* is a risk measure satisfying (a) and (b), and a *coherent risk measure* is a risk measure satisfying (a)-(d).

The Value-at-Risk (VaR) of $X \in \mathcal{X}$ at level $\alpha \in \mathbb{R}_+ := [0, \infty)$ is defined as the $100(1-\alpha)\%$ (generalized) quantile of X,

$$\mathrm{VaR}_\alpha(X) = \inf\{x \in [-\infty, \infty] : \mathbb{P}(X \leqslant x) \geqslant 1 - \alpha\}. \tag{1}$$

Note that in (1), for $\alpha \geqslant 1$, $\mathrm{VaR}_\alpha(X) = -\infty$ for all $X \in \mathcal{X}$. Certainly, only the case $\alpha \in [0, 1)$ is relevant in risk management; we do however allow α to take values greater than 1 in order to unify the main results in this paper. The risk measures VaR_α, $\alpha \geqslant 0$, are monotone, cash-invariant, positive homogeneous, and law-determined, but in general not subadditive; see McNeil et al. (2015) for an in-depth discussion on the various uses and misuses of VaR in Quantitative Risk Management.

The key family of risk measures we study in this paper is the family of the Range-Value-at-Risk (RVaR), truncated average quantiles of a random variable. For $X \in \mathcal{X}$, the RVaR at level $(\alpha,\beta) \in \mathbb{R}_+^2$ is defined as

$$\mathrm{RVaR}_{\alpha,\beta}(X) = \begin{cases} \frac{1}{\beta}\int_{\alpha}^{\alpha+\beta} \mathrm{VaR}_{\gamma}(X)\mathrm{d}\gamma & \text{if } \beta > 0, \\ \mathrm{VaR}_{\alpha}(X) & \text{if } \beta = 0. \end{cases} \tag{2}$$

For $X \in \mathcal{X}$ and $\alpha + \beta > 1$, since $\mathrm{VaR}_{\alpha+\beta-\varepsilon}(X) = -\infty$ for all $\varepsilon \in [0, \alpha+\beta-1]$, we have $\mathrm{RVaR}_{\alpha,\beta}(X) = -\infty$.

The family of RVaR is introduced by Cont et al. (2010) as *robust* risk measures, in the sense that for $\alpha > 0$ and $\alpha + \beta < 1$, $\mathrm{RVaR}_{\alpha,\beta}$ is continuous with respect to convergence in distribution (weak convergence). Similar to the case of VaR_α, $\mathrm{RVaR}_{\alpha,\beta}$ is also only relevant in practice for $\alpha+\beta < 1$. RVaR belongs to the large family of distortion risk measures (see Appendices A; for more on distortion risk measures, see e.g. Kusuoka (2001), Song and Yan (2009), Dhaene et al. (2012), Grigorova (2014), Wang et al. (2015) and the references therein). Though some of our results hold for the broader class of distortion risk measures, both for reasons of practical relevance as well as space constraints we restrict our attention to RVaR. This also allows for the explicit derivation of risk sharing formulas.

For all $X \in \mathcal{X}$, $\mathrm{VaR}_\alpha(X)$ is non-increasing and right-continuous in $\alpha \geqslant 0$, and hence we have

$$\mathrm{RVaR}_{\alpha,0}(X) = \mathrm{VaR}_\alpha(X) = \lim_{\beta\to 0^+} \mathrm{RVaR}_{\alpha,\beta}(X), \quad \alpha \geqslant 0.$$

Another special case of RVaR is the Expected Shortfall (ES, also known as CVaR and TVaR), defined as

$$\mathrm{ES}_\beta(X) = \mathrm{RVaR}_{0,\beta}(X), \quad \beta \geqslant 0.$$

Different from RVaR and VaR, an ES is subadditive. Therefore, ES_β, $\beta \in [0,1]$ are law-determined and coherent risk measures on $\mathcal{X}$. Note that by definition, for all $X \in \mathcal{X}$, $\mathrm{RVaR}_{\alpha,\beta}(X)$ is non-increasing in both $\alpha \in \mathbb{R}_+$ and $\beta \in \mathbb{R}_+$, and $\mathrm{RVaR}_{\alpha,\beta-\alpha}(X)$ is non-increasing in $\alpha \in [0,\beta]$.

Throughout this paper, we divide the set of risk measures $\{\mathrm{RVaR}_{\alpha,\beta} : \alpha,\beta \in \mathbb{R}_+\}$ into three subcategories. A risk measure VaR_α, $\alpha > 0$ is called a *true VaR*, a risk measure $\mathrm{RVaR}_{\alpha,\beta}$, $\alpha,\beta > 0$ is called a *true RVaR*, and ES_β, $\beta \geqslant 0$ is simply called an *ES*.

Remark 1. We adhere to the following convention: for $X \in \mathcal{X}$, positive values of X corresponds to losses. Mainly for notational convenience we write $\mathrm{VaR}_\alpha(X)$ for the $100(1-\alpha)\%$ quantile of the random variable X; the same notation is applied to ES_β. Whereas this convention (small $\alpha,\beta > 0$) can be widely found in the academic literature (see for instance Föllmer and Schied (2016) and Delbaen (2012)), we are well aware that in practice the notation $\mathrm{VaR}_\alpha(X)$ typically refers to the $100\alpha\%$ quantile of X (thus α is close to 1). With this notational convention, our main results like Theorems 1 and 2 below admit a much more elegant formulation. Moreover, the generic results of this paper on risk sharing are independent of this notational issue. As a consequence, the applicability for practice remains fully accessible to the (regulatory or industry) end-user.

Before we proceed, we introduce some common terminology and notation. Throughout this paper, for $p \in (0,1)$ and any non-decreasing function F, let

$$F^{-1}(p) = \inf\{x \in \mathbb{R} : F(x) \geqslant p\}.$$

Define U_X as a uniform random variable on $[0,1]$ such that $F^{-1}(U_X) = X$ almost surely where F is the distribution function of the random variable X. If X is continuously distributed, $U_X = F(X)$ almost surely. For a general random variable X, the existence of U_X is guaranteed; see for instance Lemma A.32 of Föllmer and Schied (2016). We say that a random variable with distribution F is *doubly continuous* if both F and F^{-1} are continuous; see also Proposition 1 (7) of Embrechts and Hofert (2013). For any $\beta_1, \dots, \beta_n \in \mathbb{R}$, write $\bigvee_{i=1}^n \beta_i = \max\{\beta_1, \dots, \beta_n\}$ and $\bigwedge_{i=1}^n \beta_i = \min\{\beta_1, \dots, \beta_n\}$.

3 Quantile inequalities

The following theorem establishes the relationship between the individual RVaR and the aggregate RVaR. To unify our results for all possible choices of $\alpha_1, \dots, \alpha_n$ and $\beta_1, \dots, \beta_n$, from now on the indefinite form $\infty - \infty$ is interpreted as $-\infty$. Note that $\mathrm{RVaR}_{\alpha,\beta}(X) = \infty$ may only happen in the very special case where $X \in \mathcal{X}$ is unbounded above and $\alpha = \beta = 0$.

Theorem 1. *For any $X_1, \dots, X_n \in \mathcal{X}$ and any $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n \geqslant 0$, we have*

$$\mathrm{RVaR}_{\sum_{i=1}^n \alpha_i, \bigvee_{i=1}^n \beta_i}\left(\sum_{i=1}^n X_i\right) \leqslant \sum_{i=1}^n \mathrm{RVaR}_{\alpha_i,\beta_i}(X_i). \tag{3}$$

By setting $\alpha_1 = \cdots = \alpha_n = 0$ and $\beta_1 = \cdots = \beta_n$, Theorem 1 reduces to the classic subadditivity of ES. Embrechts and Wang (2015) contains several proofs of the latter result, with each proof set into a different technical as well as pedagogical environment. By setting $\beta_1 = \cdots = \beta_n = 0$, we obtain the following inequality for VaR.

Corollary 1. *For any $X_1, \dots, X_n \in \mathcal{X}$ and any $\alpha_1, \dots, \alpha_n \geqslant 0$, we have*

$$\mathrm{VaR}_{\sum_{i=1}^n \alpha_i}\left(\sum_{i=1}^n X_i\right) \leqslant \sum_{i=1}^n \mathrm{VaR}_{\alpha_i}(X_i). \tag{4}$$

Theorem 1 and Corollary 1 imply that RVaR and VaR enjoy special forms of *subadditivity* as in (3) and (4). For $n = 2$, (3) reads as

$$\mathrm{RVaR}_{\alpha_1+\alpha_2, \beta_1 \vee \beta_2}(X_1 + X_2) \leqslant \mathrm{RVaR}_{\alpha_1,\beta_1}(X_1) + \mathrm{RVaR}_{\alpha_2,\beta_2}(X_2),$$

for all $X_1, X_2 \in \mathcal{X}$, $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}_+$. This subadditivity involves a combination of the summation of the random variables $X_1, \dots, X_n \in \mathcal{X}$, and the summation of the parameters $(\alpha_1, \beta_1), \dots, (\alpha_n, \beta_n) \in \mathbb{R}_+^2$ with respect to the two-dimensional additive operation $(+, \vee)$. Note that $\vee$-operation is known as the *tropical addition* in the max-plus algebra; see Richter-Gebert et al. (2005) and also Remark 4.

Remark 2. Recall that $\mathcal{X}$ is the set of integrable random variables in Theorem 1 and Corollary 1. For non-integrable random variables, the definition of VaR in (1) is still valid, and it is straightforward to see that (4) in Corollary 1 holds for all random variables $X_1, \dots, X_n$. For the case of RVaR, the definition (2) may involve ill-posed cases such as $\infty - \infty$. For instance, the integral $\int_0^1 \mathrm{VaR}_\gamma(X)\mathrm{d}\gamma = \mathbb{E}[X]$ is only properly defined on $\mathcal{X}$. Therefore, to make all results consistent throughout this paper, we focus on integrable random variables.

4 Optimal allocations in quantile-based risk sharing

In this section we study (Pareto-)optimal allocations in a risk sharing problem where the objectives of agents are described by the RVaR family, and the target is to minimize the aggregate risk value defined below. This setting is the most suitable if one assumes that the agents collectively work with each other to reach optimality. This may be interpreted as, for instance, the case where a single firm (e.g. a holding) redistributes an aggregate risk among its subsidiaries, which are assessed under separate regulatory regimes (e.g. these subsidiaries may belong to different countries). Competitive optimality, in which each agent optimizes their own objective without cooperation, will be discussed in Section 5.

4.1 Inf-convolution and Pareto-optimal allocations

Given $X \in \mathcal{X}$, we define the set of *allocations* of X as

$$\mathbb{A}_n(X) = \left\{ (X_1, \dots, X_n) \in \mathcal{X}^n : \sum_{i=1}^n X_i = X \right\}. \tag{5}$$

In a risk sharing problem, there are n agents equipped with respective risk measures $\rho_1, \dots, \rho_n$ and they will share a risk X by splitting it into an allocation $(X_1, \dots, X_n) \in \mathbb{A}_n(X)$. Throughout, we refer to $\rho_1, \dots, \rho_n$ in a risk sharing problem as the *underlying risk measures*, X as the *total risk*, and for an allocation $(X_1, \dots, X_n)$, we refer to $\sum_{i=1}^n \rho_i(X_i)$ as the *aggregate risk value*. The problem we consider here is an unconstrained allocation problem, that is, $X_1, \dots, X_n$ in (5) can be chosen over all integrable random variables.

The *inf-convolution* of n risk measures $\rho_1, \dots, \rho_n$ is a risk measure defined as

$$\mathop{\square}_{i=1}^{n} \rho_i(X) := \inf \left\{ \sum_{i=1}^n \rho_i(X_i) : (X_1, \dots, X_n) \in \mathbb{A}_n(X) \right\}, \quad X \in \mathcal{X}.$$

That is, the inf-convolution of n risk measures is the infimum over aggregate risk values for all possible allocations.

Definition 2. For risk measures $\rho_1, \ldots, \rho_n$ and $X \in \mathcal{X}$,

(i) an n-tuple $(X_1, \ldots, X_n) \in \mathbb{A}_n(X)$ is called an *optimal allocation* of X if $\sum_{i=1}^n \rho_i(X_i) = \square_{i=1}^n \rho_i(X)$;

(ii) an n-tuple $(X_1, \ldots, X_n) \in \mathbb{A}_n(X)$ is called a *Pareto-optimal allocation* of X if for any $(Y_1, \ldots, Y_n) \in \mathbb{A}_n(X)$ satisfying $\rho_i(Y_i) \leqslant \rho_i(X_i)$ for all $i = 1, \ldots, n$, we have $\rho_i(Y_i) = \rho_i(X_i)$ for all $i = 1, \ldots, n$.

In this paper, whenever an optimal allocation is mentioned, it is with respect to some underlying risk measures which should be clear from the context. The following statement, unifying optimal allocations and Pareto-optimal ones, can be found in Barrieu and El Karoui (2005) and Jouini et al. (2008) in the case of convex risk measures.

Proposition 1. *For any monetary risk measures $\rho_1, \ldots, \rho_n$, an allocation is Pareto-optimal if and only if it is optimal.*

In the sequel, we do not distinguish between optimal allocations and Pareto-optimal ones. In order to find an optimal allocation, we simply need to minimize the aggregate risk value over all allocations. In some situations, the n agents in a sharing problem have initial risks $\xi_1, \ldots, \xi_n$, respectively, and the total risk is $X = \xi_1 + \cdots + \xi_n$. With a given total risk X, the initial risks $\xi_1, \ldots, \xi_n$ do not affect Pareto-optimality and we do not take them into account in this section. They do play a role in the formulation of a competitive equilibrium; see Section 5.

4.2 Optimal allocations

In this section we find the optimal allocations and the corresponding aggregate risk value for the RVaR family of risk measures. The main result is the following theorem.

Theorem 2. *For $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n \geqslant 0$, we have*

$$\mathop{\square}_{i=1}^{n} \mathrm{RVaR}_{\alpha_i,\beta_i}(X) = \mathrm{RVaR}_{\sum_{i=1}^n \alpha_i, \bigvee_{i=1}^n \beta_i}(X), \quad X \in \mathcal{X}. \tag{6}$$

Moreover, if $p := \sum_{i=1}^n \alpha_i + \bigvee_{i=1}^n \beta_i < 1$, then, assuming $\beta_n = \bigvee_{i=1}^n \beta_i$, an optimal allocation $(X_1, \ldots, X_n)$ of $X \in \mathcal{X}$ is given by

$$X_i = (X - m)\,\mathrm{I}_{\{1-\sum_{k=1}^i \alpha_k < U_X \leqslant 1-\sum_{k=1}^{i-1} \alpha_k\}}, \quad i = 1, \ldots, n-1, \tag{7}$$

$$X_n = (X - m)\,\mathrm{I}_{\{U_X \leqslant 1-\sum_{k=1}^{n-1} \alpha_k\}} + m, \tag{8}$$

where $m \in (-\infty, \mathrm{VaR}_p(X)]$ is a constant and U_X is defined as in Section 2.

If $X \geqslant 0$, then by setting $m = 0$ in (7)-(8), the optimal allocation is

$$X_i = X\mathrm{I}_{\{1-\sum_{k=1}^i \alpha_k < U_X \leqslant 1-\sum_{k=1}^{i-1} \alpha_k\}}, \quad i = 1, \ldots, n-1, \tag{9}$$

$$X_n = X\mathrm{I}_{\{U_X \leqslant 1-\sum_{k=1}^{n-1} \alpha_k\}}. \tag{10}$$

The interpretation of the above allocation is clear: for each $i = 1, \ldots, n-1$, agent i takes a risk X_i with probability of loss $\mathbb{P}(X_i > 0) = \alpha_i$. This implies $\mathrm{RVaR}_{\alpha_i,\beta_i}(X_i) = 0$. The last agent (agent n) takes the rest of the risk, and $\mathrm{RVaR}_{\alpha_n,\beta_n}(X_n) = \mathrm{RVaR}_{\sum_{i=1}^n \alpha_i,\beta_n}(X)$ which is positive if $X > 0$. For each agent i, the parameter β_i can be seen as the sensitivity with respect to a loss exceeding the α_i-probability level. In view of the above discussion, we will refer to β_i as the *tolerance parameter* of agent i, and agent n as the *remaining-risk bearer*, who has the largest tolerance parameter among all agents.

Remark 3. Some observations on the optimal allocation in Theorem 2:

(i) Assuming $p < 1$ in Theorem 2, each $X_1, \ldots, X_n$ is a function of U_X in the optimal allocation (7)-(8). If X is continuously distributed, then $X_1, \ldots, X_n$ are also functions of X, since U_X can be taken as $F(X)$ where F is the distribution of X. In this case, the optimal allocation in (7)-(8) can be written as

$$X_i = (X - m)\,\mathrm{I}_{\{F^{-1}(1-\sum_{k=1}^{i}\alpha_k) < X \leqslant F^{-1}(1-\sum_{k=1}^{i-1}\alpha_k)\}}, \quad i = 1, \ldots, n-1, \text{ and} \tag{11}$$

$$X_n = (X - m)\,\mathrm{I}_{\{X \leqslant F^{-1}(1-\sum_{k=1}^{n-1}\alpha_k)\}} + m, \tag{12}$$

where $m \in (-\infty, \mathrm{VaR}_p(X)]$.

(ii) If $\alpha_i = \beta_i = 0$ for some $i = 1, \ldots, n$, assuming $n \geqslant 2$, one can always choose $X_i = 0$ in an optimal risk sharing $(X_1, \ldots, X_n) \in \mathbb{A}_n(X)$. This is because for any $\alpha, \beta \in \mathbb{R}_+$ and $X_1, X_2 \in \mathcal{X}$,

$$\mathrm{RVaR}_{\alpha,\beta}(X_1 + X_2) + \mathrm{VaR}_0(0) \leqslant \mathrm{RVaR}_{\alpha,\beta}(X_1 + \mathrm{VaR}_0(X_2)) = \mathrm{RVaR}_{\alpha,\beta}(X_1) + \mathrm{VaR}_0(X_2).$$

That is, it is not beneficial to allocate any risk to agent i, since she is extremely averse to taking any risk. This is already reflected in the construction in (7).

(iii) If $\sum_{i=1}^n \alpha_i + \bigvee_{i=1}^n \beta_i > 1$, as $\mathrm{RVaR}_{\sum_{i=1}^n \alpha_i, \bigvee_{i=1}^n \beta_i}(X) = -\infty$, no optimal allocation exists. There exists an allocation $(X_1, \ldots, X_n) \in \mathbb{A}_n(X)$ such that $\sum_{i=1}^n \mathrm{RVaR}_{\alpha_i,\,\beta_i}(X_i) < -m$ for any $m \in \mathbb{R}$. If $\sum_{i=1}^n \alpha_i + \bigvee_{i=1}^n \beta_i = 1$, from the proof of Theorem 2 parts (iii) and (iv), it follows that, depending on the choice of (α_i, β_i), $i = 1, \ldots, n$, an optimal allocation may or may not exist.

The following corollary for VaR now follows directly from Theorem 2.

Corollary 2. *For $\alpha_1, \ldots, \alpha_n \geqslant 0$, we have*

$$\mathop{\square}_{i=1}^{n} \mathrm{VaR}_{\alpha_i}(X) = \mathrm{VaR}_{\sum_{i=1}^n \alpha_i}(X), \quad X \in \mathcal{X}.$$

Moreover, if $p := \sum_{i=1}^n \alpha_i < 1$, an optimal allocation of $X \in \mathcal{X}$ is given by (7)-(8) where $m \in (-\infty, \mathrm{VaR}_p(X)]$.

Similarly to Corollary 1, Corollary 2 also holds for non-integrable random variables; see Remark 2.

Remark 4. From Theorem 2 and Corollary 2, the subset $\mathcal{G}$ of risk measures on $\mathcal{X}$,

$$\mathcal{G} = \left\{\mathrm{RVaR}_{\alpha,\beta} : (\alpha,\beta) \in \mathbb{R}_+^2\right\},$$

forms a commutative monoid (semi-group) equipped with the addition $\square$. Moreover, this monoid is isomorphic to the monoid $\mathbb{R}_+^2$ equipped with the addition $(+,\vee)$. The identity element in the monoid $(\mathcal{G},\square)$ is $\mathrm{RVaR}_{0,0} = \mathrm{ES}_0 = \mathrm{VaR}_0$, and the identity element in the monoid $(\mathbb{R}_+^2,(+,\vee))$ is simply $(0,0)$. The submonoid $\mathcal{G}_V = \{\mathrm{VaR}_\alpha : \alpha \in \mathbb{R}_+\}$ of $(\mathcal{G},\square)$ is isomorphic to the monoid $(\mathbb{R}_+,+)$, and the submonoid $\mathcal{G}_E = \{\mathrm{ES}_\beta : \beta \in \mathbb{R}_+\}$ of $(\mathcal{G},\square)$ is isomorphic to the monoid $(\mathbb{R}_+,\vee)$.

5 Competitive equilibria

In Section 4, (Pareto-)optimal allocations are obtained for the quantile-based risk sharing problem; these are more suitable for the study of cooperative games. If the agents represent a group of individual firms, there might not be a central coordination for these self-interested firms to reach Pareto-optimality. In this section, we investigate settings of non-cooperative equilibria. We shall see that the optimal allocation obtained in Section 4 is indeed part of an Arrow-Debrew equilibrium under a simple condition on the distribution function of X.

We consider a classic Arrow-Debreu economic equilibrium model (Arrow and Debreu (1954)) for agents whose objectives are characterized by the RVaR family. All discussions are based on the underlying risk measures $\mathrm{RVaR}_{\alpha_1,\beta_1},\dots,\mathrm{RVaR}_{\alpha_n,\beta_n}$, $\alpha_i,\beta_i \in [0,1)$ satisfying

$$\sum_{i=1}^n \alpha_i + \bigvee_{i=1}^n \beta_i < 1, \quad \beta_n = \bigvee_{i=1}^n \beta_i. \tag{13}$$

Note that we are assuming without loss of generality that the n-th agent has the largest tolerance parameter among all agents.

For $i = 1,\dots,n$, assume that agent i has an initial risk $\xi_i \in \mathcal{X}$. Let $X = \sum_{i=1}^n \xi_i$ be the total risk, and assume $X \geqslant 0$. Let Ψ be the set of bounded non-negative random variables. A random variable $\psi \in \Psi$ presents the pricing rule for the microeconomic market among the agents, so that the traded price of a risk $Y \in \mathcal{X}$ is given by $\mathbb{E}[\psi Y]$. Since a positive value of Y means loss, the value $\mathbb{E}[\psi Y]$ should be interpreted as the amount of money one needs to *pay* to transfer the loss Y to another agent. Up to a sign change from loss to profit, ψ is the same as a pricing density in asset pricing theory (see for instance Föllmer and Schied (2016)), except that we do not require it to be strictly positive here (see the discussion after Theorem 3 about the case $\psi = 0$).

For each $i = 1,\dots,n$, agent i may trade the initial risk ξ_i for a new position $X_i \in \mathcal{X}$. We assume that an agent is not allowed to take more than the total risk, or take less than zero, and she is allowed to make side-payments to other agents (represented by a cash amount s_i). More precisely, for a given pricing rule ψ, and each $i = 1, \dots, n$, the individual optimization problem is

$$\begin{aligned} &\text{to minimize} \quad \mathrm{RVaR}_{\alpha_i,\beta_i}(X_i) + s_i \;\text{ over } X_i \in \mathcal{X},\; s_i \in \mathbb{R} \\ &\text{subject to} \quad s_i + \mathbb{E}[\psi X_i] \geqslant \mathbb{E}[\psi \xi_i],\; 0 \leqslant X_i \leqslant X. \end{aligned} \tag{14}$$

In the optimization (14), s_i is the (negative) cash position of agent i, $s_i + \mathbb{E}[\psi X_i] \geqslant \mathbb{E}[\psi \xi_i]$ is the budget constraint, and $0 \leqslant X_i \leqslant X$ reflects that one's risk position is neither beyond the total risk nor less than zero. In the classic context of a one-period-two-date exchange economy, the cash position s_i in (14) is interpreted as the time-0 consumption of agent i; see e.g. Xia and Zhou (2016).

Obviously, the budget constraint in (14) is binding, and hence the objective in (14) can be rewritten as $\mathrm{RVaR}_{\alpha_i,\beta_i}(X_i) + \mathbb{E}[\psi(\xi_i - X_i)]$. Moreover, ξ_i is irrelevant in optimizing this objective. Therefore, the optimization problem (14) is equivalent to

$$\begin{aligned} &\text{to minimize} \quad \mathcal{V}_i(X_i) = \mathrm{RVaR}_{\alpha_i,\beta_i}(X_i) - \mathbb{E}[\psi X_i] \;\text{ over } X_i \in \mathcal{X} \\ &\text{subject to} \quad 0 \leqslant X_i \leqslant X, \end{aligned} \qquad i = 1, \dots, n. \tag{15}$$

To reach an equilibrium, the market clearing equation

$$\sum_{i=1}^n X_i^* = X = \sum_{i=1}^n \xi_i \tag{16}$$

needs to be satisfied, where X_i^* solves (15), $i = 1, \dots, n$. The corresponding side-payments are automatically cleared as well if (16) holds.

The constraint $0 \leqslant X_i \leqslant X$ is essential to the optimization (15). Note that the functional $X_i \mapsto \mathrm{RVaR}_{\alpha_i,\beta_i}(X_i) - \mathbb{E}[\psi X_i]$ is positively homogeneous. If we allow X_i to be taken over the full set $\mathcal{X}$, then the infimum value of (15) will always be either 0 or $-\infty$ (one cannot expect a non-trivial equilibrium to exist). In view of this, we consider non-negative random variables and write $\mathcal{X}_+ = \{X \in \mathcal{X} : X \geqslant 0\}$. Below we formally introduce an Arrow-Debreu equilibrium. For an introduction of Arrow-Debreu equilibria in finance, see Föllmer and Schied (2016, Section 3.6).

Definition 3 (Arrow-Debreu equilibrium)**.** Let $X \in \mathcal{X}_+$. A pair $(\psi, (X_1^*, \dots, X_n^*)) \in \Psi \times \mathbb{A}_n(X)$ is *an Arrow-Debreu equilibrium* for (15) if

$$X_i^* \in \arg\min \{\mathcal{V}_i(X_i) : X_i \in \mathcal{X},\ 0 \leqslant X_i \leqslant X\},\ i = 1, \dots, n. \tag{17}$$

The pricing rule ψ in an Arrow-Debreu equilibrium is called an *equilibrium pricing rule*, and the allocation $(X_1^*, \dots, X_n^*)$ in an Arrow-Debreu equilibrium is called an *equilibrium allocation*.

Certainly, the equilibrium pricing rule ψ, assuming it exists, is arbitrary on the set $\{X = 0\}$. Explicit solutions of Arrow-Debreu equilibria for non-convex objectives (or non-concave objectives in the framework of utility maximization), including the RVaR family, are very limited in the literature. We are not aware of any explicit solutions. For some recent development on Arrow-Debreu equilibria for rank-dependent utilities, see Xia and Zhou (2016) and Jin et al. (2016).

We first establish the Pareto efficiency of an Arrow-Debreu equilibrium by showing that an equilibrium allocation is necessarily an optimal one.

Proposition 2. *Let $X \in \mathcal{X}_+$ and assume (13) holds. Suppose that $(\psi, (X_1^*, \dots, X_n^*)) \in \Psi \times \mathbb{A}_n(X)$ is an Arrow-Debreu equilibrium for (15). Then $(X_1^*, \dots, X_n^*)$ is necessarily an optimal allocation for* $\mathrm{RVaR}_{\alpha_1,\beta_1}, \dots,$ $\mathrm{RVaR}_{\alpha_n,\beta_n}$.

Proposition 2 is a special version of the *First Welfare Economics Theorem* for the optimization (15), stating that an equilibrium allocation achieves Pareto efficiency under suitable assumptions (see e.g. Arrow (1951) and Arrow and Debreu (1954)).

Next we shall see that, with an extra condition on the value of $\mathbb{P}(X > 0)$, the optimal allocation in Theorem 2 is indeed an equilibrium allocation, and the corresponding equilibrium pricing rule is explicit. Recall that for $X \geqslant 0$ and assuming (13), an optimal allocation in Theorem 2 is given by

$$X_i^* = X\mathrm{I}_{\{1-\sum_{k=1}^{i}\alpha_k < U_X \leqslant 1-\sum_{k=1}^{i-1}\alpha_k\}}, \quad i = 1, \dots, n-1, \tag{18}$$

$$X_n^* = X\mathrm{I}_{\{U_X \leqslant 1-\sum_{k=1}^{n-1}\alpha_k\}}. \tag{19}$$

The following theorem establishes an explicit Arrow-Debreu equilibrium for (15).

Theorem 3. *Write $\alpha = \sum_{i=1}^n \alpha_i$, $\underline{\alpha} = \bigwedge_{i=1}^n \alpha_i$ and $\beta = \bigvee_{i=1}^n \beta_i = \beta_n$. Assume $\alpha + \beta < 1$, and $X \in \mathcal{X}_+$ satisfies $\mathbb{P}(X > 0) \leqslant \max\{\underline{\alpha} + \beta, \alpha\}$. Let $(X_1^*, \dots, X_n^*)$ be given by (18)-(19), and*

$$\psi = \min\left\{\frac{x}{X\beta}, \frac{1}{\beta}\right\}\mathrm{I}_{\{X\beta > 0\}} \quad \textit{where } x = \mathrm{VaR}_\alpha(X). \tag{20}$$

Then $(\psi, (X_1^*, \dots, X_n^*))$ is an Arrow-Debreu equilibrium for (15).

From Theorem 3, there are two cases for the equilibrium pricing rule ψ on $\{X > 0\}$:

(i) if $\mathbb{P}(X > 0) \leqslant \alpha$, then $\psi = 0$;

(ii) if $\alpha < \mathbb{P}(X > 0) \leqslant \underline{\alpha} + \beta$, then

$$\psi = \min\left\{\frac{x}{X\beta}, \frac{1}{\beta}\right\} = \frac{x}{X\beta}\mathrm{I}_{\{U_X \geqslant 1-\alpha\}} + \frac{1}{\beta}\mathrm{I}_{\{U_X < 1-\alpha\}} \quad \text{where } x = \mathrm{VaR}_\alpha(X). \tag{21}$$

In the above case (i), each agent takes a "free-lunch" risk X_i^* in (18) which does not contribute to their measure of risk. Note that in this case, $\square_{i=1}^n \mathrm{RVaR}_{\alpha_i,\beta_i}(X) \leqslant \mathrm{VaR}_\alpha(X) = 0$. This means that the total risk X

somehow "vanishes" from the the agents' point of view. This explains intuitively why the price becomes zero: no agent is willing to pay anything for a hedge of his risk. A special case of (i) is when all agents use true VaR.

The above case (ii) is somewhat remarkable. The equilibrium pricing rule ψ in (21) consists of two parts. If $X \geqslant x = \mathrm{VaR}_\alpha(X)$, the pricing rule is given by $\psi = \frac{x}{X\beta}$, a constant times the reciprocal of X. This form of equilibrium pricing rule is found in the Arrow-Debreu equilibrium for log utility maximizers (see e.g. Example 3.63 of Föllmer and Schied (2016)). If $0 < X < x$, ψ is equal to the constant $1/\beta$. If $X = 0$, as mentioned before, ψ is arbitrary and its value does not affect the optimization problem. For simplicity one can take $\psi = 1/\beta$ to unify with the previous case, so that ψ is a non-increasing function of X. The distribution of the equilibrium pricing rule ψ is a mixture of a scaled reciprocal of X given $X \geqslant x$ and a constant $1/\beta$ given $X < x$. We are not aware of any existing literature containing this particular form of equilibrium pricing rules.

Remark 5. The condition $\mathbb{P}(X > 0) \leqslant \max\{\underline{\alpha} + \beta, \alpha\}$ is crucial for the above Arrow-Debreu equilibrium. One can verify that if $\mathbb{P}(X > 0) > \max\{\underline{\alpha} + \beta, \alpha\}$, then $(\psi, (X_1^*, \dots, X_n^*))$ in (18)-(20) may no longer be an Arrow-Debreu equilibrium. It is not clear yet whether an Arrow-Debreu equilibrium exists in this case. We conjecture that the existence depends on other distributional properties of X.

So far we considered an Arrow-Debreu equilibrium in which each agent's objective is to minimize his or her risk measure $\mathrm{RVaR}_{\alpha_i,\beta_i}$. This can be interpreted as a setting of minimizing each firm's regulatory capital. Admittedly, it is simplistic to suggest that regulatory capital is the only concern of a firm in managing its risk. Next we consider a slightly more comprehensive model where each firm minimizes its expected loss plus the cost of capital. For $i = 1, \dots, n$, let the individual optimization problem be given by

$$\begin{array}{ll} \text{to minimize} & \mathbb{E}[X_i] + c_i \mathrm{RVaR}_{\alpha_i,\beta_i}(X_i) + s_i \ \text{ over } X_i \in \mathcal{X},\ s_i \in \mathbb{R} \\ \text{subject to} & s_i + \mathbb{E}[\psi X_i] \geqslant \mathbb{E}[\psi \xi_i],\ \ 0 \leqslant X_i \leqslant X, \end{array} \tag{22}$$

where $c_i > 0$ is a constant which represents the cost of raising one unit of capital for this firm. Similarly to (15), (22) is equivalent to the problem

$$\begin{array}{ll} \text{to minimize} & \mathcal{V}_i(X_i) = \mathbb{E}[X_i] + c_i \mathrm{RVaR}_{\alpha_i,\beta_i}(X_i) - \mathbb{E}[\psi X_i] \ \text{ over } X_i \in \mathcal{X} \\ \text{subject to} & 0 \leqslant X_i \leqslant X, \end{array} \quad i = 1, \dots, n. \tag{23}$$

It is not surprising that the cost-of-capital coefficients $c_1, \dots, c_n$ play a non-negligible role in an equilibrium for (23). In the following, let $d_i = \beta_i / c_i$ represent the *tolerance-to-cost ratio* of agent i, $i = 1, \dots, n$. Without loss of generality, we assume $d_n = \bigvee_{i=1}^n d_i$. That is, an agent with the largest tolerance-to-cost ratio is rearranged to be the n-th agent.

Theorem 4. *Write* $\alpha = \sum_{i=1}^n \alpha_i$, $\eta = \bigwedge_{i=1}^n(\alpha_i + \beta_i)$ *and* $d = \bigvee_{i=1}^n d_i = d_n$*. Assume* $\alpha + \bigvee_{i=1}^n \beta_i < 1$*, and* $X \in \mathcal{X}_+$ *satisfies* $\mathbb{P}(X > 0) \leqslant \max\{\eta, \alpha\}$*. Let* $(X_1^*, \ldots, X_n^*)$ *be given by* (18)-(19)*, and*

$$\psi = 1 + \min\left\{\frac{x}{Xd}, \frac{1}{d}\right\} \mathrm{I}_{\{Xd>0\}} \quad \text{where } x = \mathrm{VaR}_\alpha(X). \tag{24}$$

Then $(\psi, (X_1^*, \ldots, X_n^*))$ *is an Arrow-Debreu equilibrium for* (23).

Theorem 4 suggests that for the objectives in (23), there exists an Arrow-Debreu equilibrium in which the allocation is again (18)-(19), albeit the remaining-risk bearer (see Remark 3) in this problem is the agent with the largest tolerance-to-cost ratio, instead of the one with the largest tolerance parameter as in Theorem 3.

Remark 6. Noting $\eta = \bigwedge_{i=1}^n(\alpha_i + \beta_i) \leqslant \bigwedge_{i=1}^n \alpha_i + \bigvee_{i=1}^n \beta_i$, the constraint $\mathbb{P}(X > 0) \leqslant \max\{\eta, \alpha\}$ is slightly stronger than the one in Theorem 3, where $\mathbb{P}(X > 0) \leqslant \max\{\bigwedge_{i=1}^n \alpha_i + \bigvee_{i=1}^n \beta_i, \alpha\}$ is required. This technical condition was caused by the introduction of the possibly different coefficients $c_1, \ldots, c_n$, and does not seem to be dispensable.

6 Model misspecification, robustness and comonotonicity in risk sharing

As shown in Sections 4 and 5, the optimal allocations in (7)-(8) are prominent to various settings of risk sharing and equilibria when using the RVaR family of risk measures. In this section we discuss a few issues related to the above optimal allocations. If an allocation $(X_1, \ldots, X_n)$ is determined by X, it can be written as $(X_1, \ldots, X_n) = (f_1(X), \ldots, f_n(X)) \in \mathbb{A}_n(X)$ for some functions $f_1, \ldots, f_n$. We denote by $\mathbb{F}_n$ the set of *sharing principles* $(f_1, \ldots, f_n)$ where each $f_i : \mathbb{R} \to \mathbb{R}$, $i = 1, \ldots, n$, has at most finitely many points of discontinuity, $f_1(x) + \cdots + f_n(x) = x$ for all $x \in \mathbb{R}$, and $f_i(X) \in \mathcal{X}$ for $X \in \mathcal{X}$, $i = 1, \ldots, n$. As discussed in Remark 3, the cases in which $\sum_{i=1}^n \alpha_i + \bigvee_{i=1}^n \beta_i < 1$ and $\alpha_i + \beta_i > 0$ for each $i = 1, \ldots, n$ are most relevant for the existence of an optimal allocation, and we shall make this assumption in the following discussions.

6.1 Robust allocations

In this section we discuss risk sharing in the presence of model uncertainty by studying the resulting aggregate risk value when the distribution of the total risk $X \in \mathcal{X}$ is misspecified. We will see that this in general implies serious problems for VaR but not for RVaR or ES. This relates to the issue of the robustness of VaR and RVaR; for a relevant discussion on robustness properties for risk measures, see Cont et al. (2010), Kou et al. (2013), Krätschmer et al. (2014) and Embrechts et al. (2015); see also Remark 8 below. In contrast to the above literature, we are interested in the *robustness of the optimal allocation* instead of the robustness of the risk measures themselves.

Definition 4. For given risk measures $\rho_1, \ldots, \rho_n$ on $\mathcal{X}$, $X \in \mathcal{X}$ and a pseudo-metric π defined on $\mathcal{X}$, an allocation $(f_1(X), \ldots, f_n(X)) \in \mathbb{A}_n(X)$ with $(f_1, \ldots, f_n) \in \mathbb{F}_n$ is *π-robust* if the functional $Z \mapsto \sum_{i=1}^n \rho_i(f_i(Z))$ is continuous at $Z = X$ with respect to π.

A pseudo-metric is similar to a metric except that the distance between two distinct points can be zero. For instance, a metric on the set of distributions, such as the Lévy metric, induces a pseudo-metric π_W on $\mathcal{X}$. Commonly used pseudo-metrics π in risk management include the L^q metric for $q \geqslant 1$, the L^∞ metric (assuming X is bounded), or the (induced) Lévy metric π_W, which metrizes weak convergence (convergence in distribution). As we take the common domain $\mathcal{X}$ as the set of integrable random variables, we shall analyze the cases $\pi = L^1, L^\infty$ and π_W in the following.

In Definition 4, X represents an agreed-upon underlying risk. The n agents design a sharing principle $(f_1, \ldots, f_n)$ based on the knowledge of a model X. The true risk Z is unknown to the agents, and can be slightly different from the model X. If an optimal allocation is robust in the sense of Definition 4, then under a small model misspecification, the true aggregate risk value $\sum_{i=1}^n \rho_i(f_i(Z))$ would not be too far away from the optimized value for X. On the other hand, for a non-robust optimal allocation, a small model misspecification would destroy the optimality of the allocation.

Proposition 3. *Let $X \in \mathcal{X}$ be a continuously distributed random variable. Suppose that $Z_j \to X$ weakly as $j \to \infty$, then for $\alpha_i, \beta_i \in [0, 1)$, $\alpha_i + \beta_i < 1$, $i = 1, \ldots, n$, and $(f_1, \ldots, f_n) \in \mathbb{F}_n$, we have*

$$\liminf_{j\to\infty} \sum_{i=1}^n \mathrm{RVaR}_{\alpha_i,\beta_i}(f_i(Z_j)) \geqslant \sum_{i=1}^n \mathrm{RVaR}_{\alpha_i,\beta_i}(f_i(X)).$$

Proposition 3 suggests that if the actual risk Z is misspecified as X, then the aggregate risk value for an allocation of Z is asymptotically larger than that for an allocation of X. Proposition 3 remains valid if weak convergence is strengthened to L^1-convergence or L^∞-convergence.

The next proposition discusses the connection between the robustness property of the inf-convolution risk measure and that of the optimal allocation.

Proposition 4. *For given risk measures $\rho_1, \ldots, \rho_n$ on $\mathcal{X}$, $X \in \mathcal{X}$ and a pseudo-metric π defined on $\mathcal{X}$, if there exists a π-robust optimal allocation of X, then $\square_{i=1}^n \rho_i$ is π-upper-semicontinuous at X.*

In Section 6.2 below we shall see that π-continuity (which is stronger than π-upper-semicontinuity) of $\square_{i=1}^n \rho_i$ is not sufficient for the existence of a π-robust optimal allocation. More discussions on the relationship in Proposition 4 for the RVaR family and convex risk measures are presented in Remark 8.

Remark 7. Recently, Krätschmer et al. (2012, 2014) and Zähle (2016) developed robustness properties for statistical functionals (including law-invariant risk measures) on Orlicz hearts with respect to ψ-weak topologies. These concepts are well suited for studying convex risk measures; see Cheridito and Li (2009) for more on risk measures on Orlicz hearts. For $\mathrm{RVaR}_{\alpha,\beta}$ with $\alpha > 0$, the tail distribution of a risk beyond its $(1-\alpha)$-quantile level does not play a role, and hence the notions of Orlicz hearts and ψ-weak convergence are hardly relevant. In the case of $\mathrm{ES}_\beta = \mathrm{RVaR}_{0,\beta}$, the corresponding Orlicz heart is L^1 and the corresponding gauge function ψ is linear; see Krätschmer et al. (2014).

6.2 Robust allocations for quantile-based risk measures

In the following we characterize robust optimal allocations in the RVaR family. For technical reasons, we assume that the total risk X under study is doubly continuous; this includes practically all models used in risk management and robust statistics. Note that this does not imply that the random variables in an optimal allocation are continuously distributed.

Theorem 5. *For risk measures* $\mathrm{RVaR}_{\alpha_1,\beta_1},\dots,\mathrm{RVaR}_{\alpha_n,\beta_n}$, $\alpha_i,\beta_i \in [0,1)$, $\alpha_i+\beta_i > 0$, $i=1,\dots,n$, $\sum_{i=1}^n \alpha_i + \bigvee_{i=1}^n \beta_i < 1$ *and a doubly continuous random variable* $X \in \mathcal{X}$, *the following hold.*

(i) There exists an L^1*-robust optimal allocation of* X *if and only if* $\beta_1,\dots,\beta_n > 0$.

(ii) If X *is bounded, then there exists an* L^∞*-robust optimal allocation of* X *if and only if* $\beta_1,\dots,\beta_n > 0$.

(iii) There exists a π_W*-robust optimal allocation of* X *if and only if* $\beta_1,\dots,\beta_n > 0$ *and* $\alpha_i > 0$ *for some* $i = 1,\dots,n$.

From Theorem 5, if all of the underlying risk measures are true RVaR or ES, then an L^1-robust optimal allocation can be obtained. More interestingly, as soon as one of the underlying risk measures is a true VaR, not only the allocation in (11)-(12) is non-robust, but any optimal allocation is non-robust with respect to any commonly used metric.

A true RVaR is known to have a strong form of robustness (π_W-continuity), and hence it is not surprising that the strongest robustness in the optimal allocation is found for true RVaR. On the contrary, if one of $\beta_1,\dots,\beta_n$ is zero, even if $\square_{i=1}^n \mathrm{RVaR}_{\alpha_i,\beta_i}$ is π_W-continuous, and each of $\mathrm{RVaR}_{\alpha_i,\beta_i}$ is π_W-continuous at X (a VaR is π_W-continuous at any doubly continuous random variable), an L^∞-optimal allocation does not exist, not to say L^1- or π_W-robust ones. Thus, individual robustness of the underlying risk measures does not imply the existence of robust optimal allocations.

Remark 8. In the literature of risk measures, there is a well-known conflict between convexity and robustness. This is due to the fact that no convex risk measure is π_W-upper-semicontinuous on the set of bounded random variables (see Bäuerle and Müller (2006) and Cont et al. (2010)). If the underlying risk measures $\rho_1,\dots,\rho_n$ are convex risk measures, then $\square_{i=1}^n \rho_i$ is also a convex risk measure (Barrieu and El Karoui (2005)).

In this case, there does not exist a π_W-robust optimal allocation by Proposition 4. On the other hand, from Theorem 5 (iii), for a π_W-robust optimal allocation to exist, some of the underlying risk measures can be convex (ES), as long as at least one of them is a true RVaR, which is not convex. To summarize, the conflict between convexity and robustness still exists, and this only applies to weak convergence, not to L^∞ and L^1 metrics; to allow for a robust optimal allocation, some (but not all) of the underlying risk measures may be convex.

6.3 Comonotonicity in optimal allocations

Another important concept in the literature of risk sharing is comonotonicity, which relates to a type of moral hazard among collaborative agents sharing a risk. As we have seen from (7)-(8) in Theorem 2, the optimal allocation we construct may not be comonotonic. If the allocations are constrained to be comonotonic, general results on risk sharing for a general class of risk measures including RVaR are already known in the literature; see Jouini et al. (2008) and Cui et al. (2013). In this section we discuss whether an optimal allocation in a quantile-based risk sharing problem can be chosen as comonotonic.

Definition 5. Random variables $X_1, \dots, X_n$ are *comonotonic* if there exists a random variable Z and non-decreasing functions $f_1, \dots, f_n : \mathbb{R} \to \mathbb{R}$ such that $X_i = f_i(Z)$ almost surely for $i = 1, \dots, n$.

See Dhaene et al. (2002) for an overview on comonotonicity. In the following theorem, we show that, in a quantile-based risk sharing problem, a comonotonic optimal allocation exists if and only if all underlying risk measures are ES except for the one with the largest tolerance parameter.

Theorem 6. *For risk measures* $\mathrm{RVaR}_{\alpha_1,\beta_1}, \dots, \mathrm{RVaR}_{\alpha_n,\beta_n}$, $\alpha_i, \beta_i \in [0, 1)$, $\alpha_i + \beta_i \leqslant 1$, $i = 1, \dots, n$, *and any continuously distributed random variable* $X \in \mathcal{X}$, *there exists a comonotonic optimal allocation of* X *if and only if there exists* $i = 1, \dots, n$, *such that for all* $j = 1, \dots, n$, $j \neq i$, $\alpha_j = 0$ *and* $\beta_i \geqslant \beta_j$.

Remark 9. Comonotonicity is closely related to convex-order consistency and convexity (see Rüschendorf (2013) and Föllmer and Schied (2016)). Within the RVaR family, the latter two properties are only satisfied by ES. In view of this, it is not surprising that the existence of comonotonic optimal allocations relies on the presence of ES as the underlying risk measures.

7 Summary and discussions

7.1 Summary of main results

For underlying risk measures $\mathrm{RVaR}_{\alpha_1,\beta_1}, \dots, \mathrm{RVaR}_{\alpha_n,\beta_n}$, $\alpha_i, \beta_i \geqslant 0$, $i = 1, \dots, n$, we solve the optimal risk sharing problem of a total risk $X \in \mathcal{X}$ and construct corresponding Arrow-Debreu equilibria. The mathematical results are summarized below.

We first establish an inequality in Theorem 1,

$$\mathrm{RVaR}_{\sum_{i=1}^n \alpha_i, \bigvee_{i=1}^n \beta_i}\left(\sum_{i=1}^n X_i\right) \leqslant \sum_{i=1}^n \mathrm{RVaR}_{\alpha_i,\beta_i}(X_i),$$

which applies to all $X_1, \ldots, X_n \in \mathcal{X}$ and all $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n \in \mathbb{R}_+$.

Assuming $\sum_{i=1}^n \alpha_i + \bigvee_{i=1}^n \beta_i < 1$, a Pareto-optimal allocation $(X_1, \ldots, X_n) \in \mathbb{A}_n(X)$ can be constructed explicitly as in Theorem 2, with the aggregate risk value

$$\sum_{i=1}^n \mathrm{RVaR}_{\alpha_i,\beta_i}(X_i) = \mathrm{RVaR}_{\sum_{i=1}^n \alpha_i, \bigvee_{i=1}^n \beta_i}(X).$$

This optimal allocation turns out to be an Arrow-Debreu equilibrium allocation in the settings of Theorems 3 and 4, and the equilibrium pricing rule is obtained explicitly.

Some properties of the above optimal allocation are further characterized. In particular, in Theorems 5 and 6 we show that, to allow for an L^1-robust optimal allocation of X, the underlying risk measures should all be ES or true RVaR, and to allow for a comonotonic optimal allocation of X, all but one of the underlying risk measures should be ES.

7.2 Implications for the choice of a suitable regulatory risk measure

As mentioned in the introduction, there has recently been an extensive debate on the desirability of regulatory risk measures, and in particular, VaR or ES, in banking and insurance. It is a fact that currently VaR and ES coexist as regulatory risk measures throughout the broader financial industry. For example, within banking, where VaR used to rule as "the benchmark" (see Jorion (2006)), ES as an alternative is strongly gaining ground. This is for instance the case for internal models within the new regulatory guidelines for the trading book; see BCBS (2014). The "coexistence" becomes clear from the fact that Credit Risk is still falling under the VaR-regime. For Operational Risk we are at the moment in a transitionary phase where VaR-based internal models within the Advanced Measurement Approach (AMA) may be scaled down fully; see BCBS (2016). This less quantitative modeling approach towards Operational Risk is already standard in insurance regulation like the Swiss Solvency Test (SST) and Solvency II. Within the latter regulatory landscapes, we also witness a coexistence of VaR (Solvency II) and ES (SST) making the results of our paper more relevant.

Below we discuss some implications of our results to the above regulatory debates on risk measures. In particular, we discover some new advantages of ES, supporting the transition initiated by the Basel Committee on Banking Supervision. We like to stress however that, through various explicit formulas, our results are relevant for the ongoing discussion on the use of risk measures within Quantitative Risk Management more generally.

7.2.1 Capturing tail risk

"Tail risk" is currently of crucial concern for banking regulation. Below we quote the Basel Committee on Banking Supervision, Page 1 of BCBS (2016), Executive Summary:

"*... A shift from Value-at-Risk (VaR) to an Expected Shortfall (ES) measure of risk under stress. Use of ES will help to ensure a more prudent capture of "tail risk" and capital adequacy during periods of significant financial market stress.*"

From our results in Section 4, for any risk $X \geqslant 0$ with $\mathbb{P}(X > 0) < n\alpha$, one has $\square_{i=1}^n \mathrm{VaR}_\alpha(X) = \mathrm{VaR}_{n\alpha}(X) = 0$. Therefore, in the optimization of risk under true VaR (or true RVaR), there is a part of the loss undertaken by the firms, but its riskiness is completely ignored; this is also clear from the optimal allocation presented in Theorem 2. Note that although α is typically very small in practice, $n\alpha$ may be large for an economy of many participants, making $\mathbb{P}(X > 0) < n\alpha$ highly relevant.

Although the fact that VaR cannot capture tail risk is often argued from various perspectives, our results explain this fact mathematically for the first time within the framework of risk sharing and optimization. Within the RVaR family, to completely avoid such a phenomenon, one requires $\alpha_i = 0$, $i = 1, \ldots, n$, which offers further support to ES as a regulatory risk measure.

7.2.2 Model misspecification

Due to model uncertainty, a non-robust allocation may lead to a significantly higher aggregate risk value for the agents, that is, far away from the optimal one. Any model for the total risk X suffers from model uncertainty, be it at the level of statistical (parameter) uncertainty or at the level of the analytic structure of the model (e.g. which economic factors to include). The 2007 - 2009 financial crisis (unfortunately) gave ample proof of this, especially in the context of the rating of mortgage based derivatives; see, for instance, Donnelly and Embrechts (2010).

From our results in Section 6, as soon as one of underlying risk measures is a true VaR, an optimal allocation cannot be robust. Therefore, a true RVaR or an ES is a better choice than a VaR in the presence of model uncertainty. Our conclusion is consistent with the observations in Cont et al. (2010) that RVaR has advantages in robustness properties over VaR and ES, albeit our results come from a different mathematical setting. Remarkably, ES is more robust than VaR in our settings of risk sharing.

7.2.3 Understanding the least possible total capital

Let ρ be a regulatory risk measure in use for a given jurisdiction. Note that, via sharing, be it cooperative (e.g. fragmentation of a single firm; see Section 4) or competitive (see Section 5), the total risk in the economy remains the same while the total regulatory capital is reduced.

The mathematical results obtained in the paper give a guideline for calculating the least possible aggregate capital $\sum_{i=1}^{n} \rho(X_i)$ within an economy, when the regulatory risk measure is chosen within the RVaR family. In practice, a regulator may not know how risks are (will be) distributed among firms before she designs a regulatory risk measure; there are many possibilities. Our results can be seen as a worst-case scenario (least amount) of total regulatory capital within that economy. Another implication of our results is that, within a VaR-based regulatory system, constraints on the within-firm fragmentation have to be imposed; otherwise the total regulatory capital may be artificially reduced. Of course these statements are fairly stylised, but we do hope that they contain sufficiently interesting information for practitioners and regulators.

8 Proofs of main results

In this section we present the proofs of the most important results, Theorems 1, 2, and 3. The proofs of Theorems 4, 5, and 6 and Propositions 1, 2, 3, and 4 are put in Appendices B-H. Further background and some useful results on optimal comonotonic allocations are given in Appendices A.

Proof of Theorem 1. We only show the case of $n = 2$; for $n > 2$, an induction argument is sufficent. For any $X_1, X_2 \in \mathcal{X}$, we consider the following three cases respectively.

(i) $\alpha_1 + \alpha_2 + \beta_1 \vee \beta_2 < 1$.

Let $A_1 = \{U_{X_1} \geqslant 1 - \alpha_1\}$ and $A_2 = \{U_{X_2} \geqslant 1 - \alpha_2\}$. Then $\mathbb{P}(A_1 \cup A_2) \leqslant \mathbb{P}(A_1) + \mathbb{P}(A_2) = \alpha_1 + \alpha_2$. Take

$$Y_1 = \mathrm{I}_{A_1^c} X_1 - m\mathrm{I}_{A_1}, \quad Y_2 = \mathrm{I}_{A_2^c} X_2 - m\mathrm{I}_{A_2}, \tag{25}$$

where m is a real number satisfying $m > -\min\{\mathrm{VaR}_{\alpha_1+\beta_1}(X_1), \mathrm{VaR}_{\alpha_2+\beta_2}(X_2)\}$. It is straightforward to verify $\mathrm{RVaR}_{\alpha_1,\beta_1}(X_1) = \mathrm{ES}_{\beta_1}(Y_1)$ and $\mathrm{RVaR}_{\alpha_2,\beta_2}(X_2) = \mathrm{ES}_{\beta_2}(Y_2)$. It follows that

$$\mathrm{RVaR}_{\alpha_1,\beta_1}(X_1) + \mathrm{RVaR}_{\alpha_2,\beta_2}(X_2) = \mathrm{ES}_{\beta_1}(Y_1) + \mathrm{ES}_{\beta_2}(Y_2) \geqslant \mathrm{ES}_{\beta_1 \vee \beta_2}(Y_1 + Y_2), \tag{26}$$

where the last inequality holds since $\mathrm{ES}_\beta(X)$ is subadditive and non-increasing in the parameter $\beta \geqslant 0$. Moreover, for $\gamma \in [0, 1]$, we will show

$$\mathrm{VaR}_\gamma(Y_1 + Y_2) \geqslant \mathrm{VaR}_{\gamma+(\alpha_1+\alpha_2)}(X_1 + X_2). \tag{27}$$

Inequality (27) holds by the definition of VaR if $\gamma + \alpha_1 + \alpha_2 \geqslant 1$. If $\gamma + \alpha_1 + \alpha_2 < 1$, we have $(Y_1 + Y_2)\mathrm{I}_{A_1^c \cap A_2^c} = (X_1 + X_2)\mathrm{I}_{A_1^c \cap A_2^c}$ and hence for any $x \in \mathbb{R}$,

$$\mathbb{P}(Y_1 + Y_2 \geqslant x) \geqslant \mathbb{P}(X_1 + X_2 \geqslant x, A_1^c, A_2^c) \geqslant \mathbb{P}(X_1 + X_2 \geqslant x) - \mathbb{P}(A_1 \cup A_2).$$

Therefore,

$$\mathrm{VaR}_\gamma(Y_1 + Y_2) \geqslant \mathrm{VaR}_{\gamma+\mathbb{P}(A_1 \cup A_2)}(X_1 + X_2) \geqslant \mathrm{VaR}_{\gamma+(\alpha_1+\alpha_2)}(X_1 + X_2). \tag{28}$$

Hence (27) holds. If $\beta_1 \vee \beta_2 > 0$, by (26) and (27), we have

$$\begin{aligned}\mathrm{RVaR}_{\alpha_1,\beta_1}(X_1) + \mathrm{RVaR}_{\alpha_2,\beta_2}(X_2) &\geqslant \mathrm{ES}_{\beta_1\vee\beta_2}(Y_1+Y_2)\\ &= \frac{1}{\beta_1\vee\beta_2}\int_0^{\beta_1\vee\beta_2}\mathrm{VaR}_\alpha(Y_1+Y_2)\mathrm{d}\alpha\\ &\geqslant \frac{1}{\beta_1\vee\beta_2}\int_0^{\beta_1\vee\beta_2}\mathrm{VaR}_{\alpha+(\alpha_1+\alpha_2)}(X_1+X_2)\mathrm{d}\alpha\\ &= \mathrm{RVaR}_{\alpha_1+\alpha_2,\beta_1\vee\beta_2}(X_1+X_2). \end{aligned} \tag{29}$$

If $\beta_1 \vee \beta_2 = 0$, then by using (29), we have

$$\begin{aligned}\mathrm{RVaR}_{\alpha_1,0}(X_1) + \mathrm{RVaR}_{\alpha_2,0}(X_2) &= \lim_{\varepsilon\to0^+}\left(\mathrm{RVaR}_{\alpha_1,\varepsilon}(X_1) + \mathrm{RVaR}_{\alpha_2,0}(X_2)\right)\\ &\geqslant \lim_{\varepsilon\to0^+}\mathrm{RVaR}_{\alpha_1+\alpha_2,\varepsilon}(X_1+X_2)\\ &= \mathrm{RVaR}_{\alpha_1+\alpha_2,0}(X_1+X_2).\end{aligned}$$

In either case,

$$\mathrm{RVaR}_{\alpha_1,\beta_1}(X_1) + \mathrm{RVaR}_{\alpha_2,\beta_2}(X_2) \geqslant \mathrm{RVaR}_{\alpha_1+\alpha_2,\beta_1\vee\beta_2}(X_1+X_2). \tag{30}$$

(ii) $\alpha_1+\alpha_2 < 1$ and $\alpha_1+\alpha_2+\beta_1\vee\beta_2 = 1$.

In this case, (30) follows from the proof in (i) by using the left-continuity of $\mathrm{RVaR}_{\alpha,\beta}(X)$ in β for $0 < \beta \leqslant 1-\alpha$.

(iii) $\alpha_1+\alpha_2 \geqslant 1$ or $\alpha_1+\alpha_2+\beta_1\vee\beta_2 > 1$.

In this case, (30) holds trivially since $\mathrm{RVaR}_{\alpha_1+\alpha_2,\beta_1\vee\beta_2}(X_1+X_2) = -\infty$.

In summary, (3) holds for $n=2$; the case of $n \geqslant 3$ is obtained by induction. □

Proof of Theorem 2. Write $\rho_i = \mathrm{RVaR}_{\alpha_i,\beta_i}$, $i=1,\dots,n$. Since the order of $(\alpha_i,\ \beta_i)$, $i=1,\dots,n$, is irrelevant in (6), we may assume without loss of generality $\beta_n = \bigvee_{i=1}^n\beta_i$. To show (6), it suffices to show

$$\mathop{\square}_{i=1}^{n}\rho_i(X) \leqslant \mathrm{RVaR}_{\sum_{i=1}^n\alpha_i,\beta_n}(X); \tag{31}$$

indeed, Theorem 1 guarantees the reversed inequality. In all of the following cases, take $(X_1,\dots,X_n)$ in (7)-(8) with some $m\in\mathbb{R}$. It is easy to see $X_1+\cdots+X_n = X$, and for $i=1,\dots,n-1$, we have $\rho_i(X_i)\leqslant 0$ since $\mathbb{P}(X_i>0)\leqslant\alpha_i$. We discuss the following four possible cases.

(i) $p<1$.

Take $m \leqslant \mathrm{VaR}_p(X)$. It is easy to verify $\rho_n(X_n) = \mathrm{RVaR}_{\sum_{i=1}^n\alpha_i,\beta_n}(X)$, thus,

$$\mathop{\square}_{i=1}^{n}\rho_i(X) \leqslant \sum_{i=1}^n\rho_i(X_i) \leqslant \mathrm{RVaR}_{\sum_{i=1}^n\alpha_i,\beta_n}(X).$$

Therefore (31) holds, and $(X_1,\dots,X_n)$ is an optimal allocation.

(ii) $p > 1$.

Take $m < 0$. If $\alpha_n + \beta_n > 1$ then (31) holds trivially since $\sum_{i=1}^n \rho_i(X_i) = -\infty$. If $\alpha_n + \beta_n \leqslant 1$, using the subadditivity of ES, we have

$$
\begin{aligned}
\rho_n(X_n) &= \mathrm{RVaR}_{\alpha_n,\beta_n}\left(X\mathrm{I}_{\{U_X \leqslant 1-\sum_{k=1}^{n-1}\alpha_k\}} + m\mathrm{I}_{\{U_X > 1-\sum_{k=1}^{n-1}\alpha_k\}}\right) \\
&\leqslant \mathrm{ES}_{\alpha_n+\beta_n}\left(X\mathrm{I}_{\{U_X \leqslant 1-\sum_{k=1}^{n-1}\alpha_k\}} + m\mathrm{I}_{\{U_X > 1-\sum_{k=1}^{n-1}\alpha_k\}}\right) \\
&\leqslant \mathrm{ES}_{\alpha_n+\beta_n}\left(X\mathrm{I}_{\{U_X \leqslant 1-\sum_{k=1}^{n-1}\alpha_k\}}\right) + \mathrm{ES}_{\alpha_n+\beta_n}\left(m\mathrm{I}_{\{U_X > 1-\sum_{k=1}^{n-1}\alpha_k\}}\right) \\
&= \begin{cases} \mathrm{ES}_{\alpha_n+\beta_n}\left(X\mathrm{I}_{\{U_X \leqslant 1-\sum_{k=1}^{n-1}\alpha_k\}}\right) + m\frac{p-1}{\alpha_n+\beta_n} & \text{if } \sum_{k=1}^{n-1}\alpha_k < 1, \\ m & \text{if } \sum_{k=1}^{n-1}\alpha_k \geqslant 1, \end{cases} \\
&\to -\infty \qquad \text{as } m \to -\infty.
\end{aligned}
$$

This shows $\square_{i=1}^n \rho_i(X_i) = -\infty$ and hence (31) holds.

(iii) $p = 1, \beta_n = 0$.

Since $\mathbb{P}(X_n > m) \leqslant \alpha_n$, one has $\mathrm{VaR}_{\alpha_n}(X_n) \leqslant m \to -\infty$ as $m \to -\infty$. This shows $\square_{i=1}^n \rho_i(X_i) = -\infty$ and hence (31) holds.

(iv) $p = 1, \beta_n > 0$.

If $\alpha_n + \beta_n = 1$ then $\rho_n(X_n) = \rho_n(X) = \mathrm{RVaR}_{\alpha_n,\beta_n}(X)$, and therefore (31) holds.

If $\alpha_n + \beta_n < 1$, take $m = \mathrm{VaR}_q(X)$ for some $q \in (\alpha_n + \beta_n, 1) \cap (1-\beta_n, 1)$. We have

$$
\begin{aligned}
\rho_n(X_n) &= \mathrm{RVaR}_{\alpha_n,\beta_n}\left(X\mathrm{I}_{\{U_X \leqslant \alpha_n+\beta_n\}} + \mathrm{VaR}_q(X)\mathrm{I}_{\{U_X > \alpha_n+\beta_n\}}\right) \\
&= \frac{1}{\beta_n}\left(\int_{1-\beta_n}^{q} \mathrm{VaR}_\gamma(X)\mathrm{d}\gamma + (1-q)\mathrm{VaR}_q(X)\right) \\
&\to \frac{1}{\beta_n}\int_{1-\beta_n}^{1} \mathrm{VaR}_\gamma(X)\mathrm{d}\gamma \qquad \text{as } q \to 1.
\end{aligned}
$$

This shows $\square_{i=1}^n \rho_i(X_i) \leqslant \frac{1}{\beta_n}\int_{1-\beta_n}^1 \mathrm{VaR}_\gamma(X)\mathrm{d}\gamma = \mathrm{RVaR}_{1-\beta_n,\beta_n}(X)$ and thus (31) holds.

Combining the cases (i)-(iv), the proof is complete. □

Proof of Theorem 3. Recall that $\mathrm{RVaR}_{\alpha_n,\beta_n}(X_n^*) = \mathrm{RVaR}_{\alpha,\beta}(X)$ and $\mathrm{RVaR}_{\alpha_i,\beta_i}(X_i^*) = 0$ for $i = 1,\dots,n-1$. We consider two cases separately.

(i) Suppose $\mathbb{P}(X > 0) \leqslant \alpha$. This implies $\mathrm{RVaR}_{\alpha_n,\beta_n}(X_n^*) = \mathrm{RVaR}_{\alpha,\beta}(X) = 0$, $x = 0$ and $\psi = 0$. On the other hand, for any $0 \leqslant X_i \leqslant X$, we have $\mathrm{RVaR}_{\alpha_i,\beta_i}(X_i) - \mathbb{E}[\psi X_i] = \mathrm{RVaR}_{\alpha_i,\beta_i}(X_i) \geqslant 0$. Thus X_i^* satisfies (17), and hence $(\psi, (X_1^*, \dots, X_n^*))$ is an Arrow-Debreu equilibrium.

(ii) Suppose $\alpha < \mathbb{P}(X > 0) \leqslant \underline{\alpha} + \beta$. This implies $x, \beta > 0$. For $i = 1, \ldots, n$, take any $X_i \in \mathcal{X}$ such that $0 \leqslant X_i \leqslant X$. Note that by definition, $\psi X \leqslant x/\beta$. We have

$$\mathbb{E}[\psi \mathrm{I}_{\{U_{X_i} \geqslant 1-\alpha_i\}} X_i] \leqslant \mathbb{E}[\psi \mathrm{I}_{\{U_{X_i} \geqslant 1-\alpha_i\}} X] \leqslant \mathbb{E}\left[\frac{x}{\beta} \mathrm{I}_{\{U_{X_i} \geqslant 1-\alpha_i\}}\right] = \frac{x\alpha_i}{\beta}. \tag{32}$$

On the other hand, using $\psi \leqslant 1/\beta$ and $\mathbb{P}(X_i > 0) \leqslant \mathbb{P}(X > 0) \leqslant \alpha_i + \beta$,

$$\begin{aligned} \mathbb{E}[\psi \mathrm{I}_{\{U_{X_i} < 1-\alpha_i\}} X_i] &\leqslant \frac{1}{\beta} \mathbb{E}[\mathrm{I}_{\{U_{X_i} < 1-\alpha_i\}} X_i] \\ &= \frac{1}{\beta} \int_{\alpha_i}^{1} \mathrm{VaR}_\gamma(X_i) \mathrm{d}\gamma \\ &= \frac{1}{\beta} \int_{\alpha_i}^{\alpha_i+\beta} \mathrm{VaR}_\gamma(X_i) \mathrm{d}\gamma = \mathrm{RVaR}_{\alpha_i,\beta}(X_i) \leqslant \mathrm{RVaR}_{\alpha_i,\beta_i}(X_i). \end{aligned} \tag{33}$$

Combining (32) and (33), we have

$$\mathbb{E}[\psi X_i] \leqslant \frac{x\alpha_i}{\beta} + \mathrm{RVaR}_{\alpha_i,\beta_i}(X_i).$$

Equivalently,

$$\mathrm{RVaR}_{\alpha_i\beta_i}(X_i) - \mathbb{E}[\psi X_i] \geqslant -\frac{x\alpha_i}{\beta}.$$

Next we verify that $\mathrm{RVaR}_{\alpha_i,\beta_i}(X_i^*) - \mathbb{E}[\psi X^*]$ is equal to $-x\alpha_i/\beta$. Write

$$A_i = \left\{1 - \sum_{k=1}^{i} \alpha_k < U_X \leqslant 1 - \sum_{k=1}^{i-1} \alpha_k\right\} \subset \{U_X \geqslant 1 - \alpha\}.$$

Note that $\psi = \frac{x}{X\beta} \mathrm{I}_{\{U_X \geqslant 1-\alpha\}} + \frac{1}{\beta} \mathrm{I}_{\{U_X < 1-\alpha\}}$. We have $X_i^* = X\mathrm{I}_{A_i}$ for $i = 1, \ldots, n-1$, and $X_n^* = X\mathrm{I}_{A_n} + X\mathrm{I}_{\{U_X < 1-\alpha\}}$. For $i = 1, \ldots, n-1$,

$$\begin{aligned} \mathrm{RVaR}_{\alpha_i,\beta_i}(X_i^*) - \mathbb{E}[\psi X_i^*] = -\mathbb{E}[\psi X_i^*] &= -\mathbb{E}\left[\frac{x}{X\beta} \mathrm{I}_{\{U_X \geqslant 1-\alpha\}} X \mathrm{I}_{A_i}\right] \\ &= -\mathbb{E}\left[\frac{x}{\beta} \mathrm{I}_{A_i}\right] = -\frac{x\alpha_i}{\beta}. \end{aligned}$$

For the last agent, we have

$$\begin{aligned} \mathbb{E}[\psi X_n^*] &= \mathbb{E}\left[\frac{x}{X\beta} \mathrm{I}_{\{U_X \geqslant 1-\alpha\}} X \mathrm{I}_{A_n}\right] + \mathbb{E}\left[\frac{1}{\beta} \mathrm{I}_{\{U_X < 1-\alpha\}} X\right] \\ &= \frac{x\alpha_n}{\beta} + \frac{1}{\beta} \int_{\alpha}^{1} \mathrm{VaR}_\gamma(X) \mathrm{d}\gamma \\ &= \frac{x\alpha_n}{\beta} + \frac{1}{\beta} \int_{\alpha}^{\alpha+\beta} \mathrm{VaR}_\gamma(X) \mathrm{d}\gamma = \frac{x\alpha_n}{\beta} + \mathrm{RVaR}_{\alpha,\beta}(X). \end{aligned}$$

Rearranging the above equation, and using $\mathrm{RVaR}_{\alpha_n,\beta_n}(X_n^*) = \mathrm{RVaR}_{\alpha,\beta}(X)$, we obtain

$$\mathrm{RVaR}_{\alpha_n,\beta_n}(X_n^*) - \mathbb{E}[\psi X_n^*] = \mathrm{RVaR}_{\alpha,\beta}(X) - \mathbb{E}[\psi X_n^*] = -\frac{x\alpha_n}{\beta}.$$

In summary, for $i = 1, \ldots, n$,

$$\mathrm{RVaR}_{\alpha_i,\beta_i}(X_i) - \mathbb{E}[\psi X_i] \geqslant -\frac{x\alpha_i}{\beta} = \mathrm{RVaR}_{\alpha_i,\beta_i}(X_i^*) - \mathbb{E}[\psi X_i^*].$$

Therefore, $(\psi, (X_1^*, \ldots, X_n^*))$ is an Arrow-Debreu equilibrium. □

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Appendices

A Comonotonic risk sharing for distortion risk measures

For $\alpha,\beta \in [0,1)$ and $\alpha+\beta \leqslant 1$, $\mathrm{RVaR}_{\alpha,\beta}$ belongs to the class of *distortion risk measures*, that is, risk measures ρ_h of the Stieltjes integral form

$$\rho_h(X) = \int_0^1 \mathrm{VaR}_\alpha(X)\mathrm{d}h(\alpha), \quad X \in \mathcal{X}, \tag{34}$$

for some non-decreasing and left-continuous function $h : [0,1] \to [0,1]$ satisfying $h(0) = 0$ and $h(1) = 1$, such that the above integral is properly defined. Here h is called a *distortion function*. For $\alpha,\beta \in [0,1)$ and $\alpha+\beta \leqslant 1$, the distortion function of $\mathrm{RVaR}_{\alpha,\beta}(X)$ is given by

$$h^{(\alpha,\beta)}(t) := \begin{cases} \min\{\mathrm{I}_{\{t>\alpha\}}\frac{t-\alpha}{\beta}, 1\} & \text{if } \beta > 0, \\ \mathrm{I}_{\{t>\alpha\}} & \text{if } \beta = 0, \end{cases} \quad t \in [0,1]. \tag{35}$$

The set of comonotonic allocations is defined as

$$\mathbb{A}_n^+(X) = \{(X_1,\dots,X_n) \in \mathbb{A}_n(X) : \ X_i \uparrow X, \ i = 1,\dots,n\},$$

where $X_i \uparrow X$ means that X_i and X are comonotonic.

The *constrained inf-convolution* of risk measures $\rho_1,\dots,\rho_n$ is defined as

$$\mathop{\boxplus}_{i=1}^{n} \rho_i(X) := \inf\left\{\sum_{i=1}^n \rho_i(X_i) : (X_1,\dots,X_n) \in \mathbb{A}_n^+(X)\right\}.$$

Definition 6. Let $\rho_1, \ldots, \rho_n$ be risk measures and $X \in \mathcal{X}$. An n-tuple $(X_1, X_2, \ldots, X_n) \in \mathbb{A}_n^+(X)$ is called an *optimal constrained allocation* of X if $\sum_{i=1}^n \rho_i(X_i) = \boxplus_{i=1}^n \rho_i(X)$.

It is obvious that $\square_{i=1}^n \rho_i(X) \leqslant \boxplus_{i=1}^n \rho_i(X)$. Hence, if an optimal allocation of X is comonotonic, then it is also an optimal constrained allocation, and $\square_{i=1}^n \rho_i(X) = \boxplus_{i=1}^n \rho_i(X)$. In Jouini et al. (2008) it is shown that for law-determined convex risk measures on L^∞, optimal constrained allocations are also optimal allocations. This statement remains true if the underlying risk measures preserve convex order; this is based on the *comonotone improvement* in Landsberger and Meilijson (1994) and Ludkovski and Rüschendorf (2008).

A solution to the optimal constrained allocation can be found in Jouini et al. (2008) for convex risk measures and in Cui et al. (2013) for general distortion risk measures in the context of the design of optimal reinsurance contracts. We give a self-contained proof here which we believe is simpler than the existing ones in the literature.

Proposition 5. *For n distortion functions $h_1, \ldots, h_n$ such that ρ_{h_i} is finite on $\mathcal{X}$ for $i = 1, \ldots, n$, we have*

$$\boxplus_{i=1}^n \rho_{h_i}(X) = \int_0^1 \mathrm{VaR}_\alpha(X) \mathrm{d}h(\alpha), \quad X \in \mathcal{X}, \tag{36}$$

where $h(t) = \min\{h_1(t), \ldots, h_n(t)\}$. Moreover, an optimal constrained allocation $(X_1, \ldots, X_n)$ of $X \in \mathcal{X}$ is given by $X_i = f_i(X)$, $i = 1, \ldots, n$, where

$$f_i(x) = \int_0^x g_i(t) \mathrm{d}t, \quad x \in \mathbb{R},$$

and

$$g_i(t) = \begin{cases} 0 & \text{if } h_i(1 - F(t)) > h(1 - F(t)), \\ 1/k(t) & \text{otherwise}, \end{cases}$$

for $t \in \mathbb{R}$ and $k(t) = \#\{j = 1, \ldots, n : h_j(1 - F(t)) = h(1 - F(t))\}$.

Proof. We first show

$$\boxplus_{i=1}^n \rho_{h_i}(X) \geqslant \int_0^1 \mathrm{VaR}_\alpha(X) \mathrm{d}h(\alpha). \tag{37}$$

For two left-continuous distortion functions f and g, we have $\rho_f(X) \leqslant \rho_g(X)$ if $f \leqslant g$ (see Lemma A.1 of Wang et al. (2015)). Therefore, for any $(X_1, X_2, \ldots, X_n) \in \mathbb{A}_n^+(X)$, by the comonotonic additivity of VaR, we have

$$\int_0^1 \mathrm{VaR}_\alpha(X) \mathrm{d}h(\alpha) = \int_0^1 (\mathrm{VaR}_\alpha(X_1) + \cdots + \mathrm{VaR}_\alpha(X_n)) \, \mathrm{d}h(\alpha) \leqslant \sum_{i=1}^n \rho_{h_i}(X_i).$$

Thus, (37) holds. Conversely, let F be the distribution of X. Since $f_1(t), \ldots, f_n(t)$ are Lipschitz continuous and non-decreasing, we have

$$\begin{aligned}
\sum_{i=1}^n \rho_{h_i}(f_i(X)) &= \sum_{i=1}^n \int_0^1 \mathrm{VaR}_t(f_i(X))\mathrm{d}h_i(t) \\
&= \sum_{i=1}^n \int_0^1 f_i(\mathrm{VaR}_t(X))\mathrm{d}h_i(t) \\
&= \sum_{i=1}^n \int_0^1 \int_0^{\mathrm{VaR}_t(X)} g_i(s)\mathrm{d}s\mathrm{d}h_i(t) \\
&= \sum_{i=1}^n \left(\int_0^\infty h_i(1-F(s))g_i(s)\mathrm{d}s - \int_{-\infty}^0 (1-h_i(1-F(s)))g_i(s)\mathrm{d}s \right) \\
&= \int_0^\infty h(1-F(s))\mathrm{d}s - \int_{-\infty}^0 (1-h(1-F(s)))\mathrm{d}s = \rho_h(X),
\end{aligned}$$

where the fourth equality follows from Fubini's Theorem and the last equality

$$\rho_h(X) = \int_0^\infty h(1-F(x))\mathrm{d}x - \int_{-\infty}^0 (1-h(1-F(x)))\mathrm{d}x \tag{38}$$

is given in, for instance, Theorem 6 of Dhaene et al. (2012). Thus,

$$\mathop{\boxplus}_{i=1}^n \rho_{h_i}(X) \leqslant \int_0^1 \mathrm{VaR}_\alpha(X)\mathrm{d}h(\alpha).$$

The desired result follows. □

Since RVaRs belong to the family of distortion risk measures, their optimal constrained allocations can be constructed analogously, as summarized in the following corollary.

Corollary 3. *For $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n \in [0, 1)$ such that $\alpha_i + \beta_i \leqslant 1$, $i = 1, \ldots, n$, we have*

$$\mathop{\boxplus}_{i=1}^n \mathrm{RVaR}_{\alpha_i,\beta_i}(X) = \int_0^1 \mathrm{VaR}_\alpha(X)\mathrm{d}h(\alpha), \quad X \in \mathcal{X}, \tag{39}$$

where $h(t) = \min\{h^{(\alpha_1,\beta_1)}(t), \ldots, h^{(\alpha_n,\beta_n)}(t)\}$, $t \in [0, 1]$.

B Proof of Proposition 1

Proof. It is trivial to check that an optimal allocation is always Pareto-optimal. To show the other direction, suppose that $(X_1, \ldots, X_n) \in \mathbb{A}_n(X)$ is not optimal. Then there exists an allocation $(Y_1, \ldots, Y_n) \in \mathbb{A}_n(X)$ such that $\sum_{i=1}^n \rho_i(Y_i) < \sum_{i=1}^n \rho_i(X_i)$. Take $c_i = \rho_i(X_i) - \rho_i(Y_i)$, $i = 1, \ldots, n$ and $c = \sum_{i=1}^n c_i > 0$. Then we have

$$(Y_1 + c_1 - c/n, \ldots, Y_n + c_n - c/n) \in \mathbb{A}_n(X),$$

and

$$\rho_i(Y_i + c_i - c/n) < \rho_i(Y_i + c_i) = \rho_i(X_i).$$

Therefore, $(X_1, \ldots, X_n)$ is not Pareto-optimal. □

C Proof of Proposition 2

Proof. By the construction in (11)-(12), there exists $(Y_1, \dots, Y_n) \in \mathbb{A}_n(X)$, $0 \leqslant Y_i \leqslant X$, $i = 1, \dots, n$, such that

$$\sum_{i=1}^{n} \mathrm{RVaR}_{\alpha_i,\beta_i}(Y_i) = \mathrm{RVaR}_{\alpha,\beta}(X)$$

where $\alpha = \sum_{i=1}^{n} \alpha_i$ and $\beta = \bigvee_{i=1}^{n} \beta_i$. Since $(\psi, (X_1^*, \dots, X_n^*))$ is an Arrow-Debreu equilibrium, we have for $i = 1, \dots, n$,

$$\mathrm{RVaR}_{\alpha_i,\beta_i}(X_i^*) - \mathbb{E}[\psi X_i^*] \leqslant \mathrm{RVaR}_{\alpha_i,\beta_i}(Y_i) - \mathbb{E}[\psi Y_i].$$

It follows from $\sum_{i=1}^{n} X_i^* = X = \sum_{i=1}^{n} Y_i$ that

$$\begin{aligned}\sum_{i=1}^{n} \mathrm{RVaR}_{\alpha_i,\beta_i}(X_i^*) - \mathbb{E}[\psi X] &= \sum_{i=1}^{n} (\mathrm{RVaR}_{\alpha_i,\beta_i}(X_i^*) - \mathbb{E}[\psi X_i^*]) \\ &\leqslant \sum_{i=1}^{n} (\mathrm{RVaR}_{\alpha_i,\beta_i}(Y_i) - \mathbb{E}[\psi Y_i]) = \mathrm{RVaR}_{\alpha,\beta}(X) - \mathbb{E}[\psi X].\end{aligned}$$

Therefore $\sum_{i=1}^{n} \mathrm{RVaR}_{\alpha_i,\beta_i}(X_i^*) \leqslant \mathrm{RVaR}_{\alpha,\beta}(X)$. By Theorem 2, $(X_1^*, \dots, X_n^*)$ is an optimal allocation. □

D Proof of Theorem 4

Proof. Similarly to the proof of Theorem 3, we consider two cases separately.

(i) Suppose $\mathbb{P}(X > 0) \leqslant \alpha$. This implies $\mathrm{RVaR}_{\alpha_i,\beta_i}(X_i^*) = 0$ for $i = 1, \dots, n$, and $\psi = 1$. On the other hand, for any $0 \leqslant X_i \leqslant X$, we have $\mathbb{E}[X_i] + c_i \mathrm{RVaR}_{\alpha_i,\beta_i}(X_i) - \mathbb{E}[\psi X_i] = c_i \mathrm{RVaR}_{\alpha_i,\beta_i}(X_i) \geqslant 0$. Thus X_i^* satisfies (17), and hence $(\psi, (X_1^*, \dots, X_n^*))$ is an Arrow-Debreu equilibrium.

(ii) Suppose $\alpha < \mathbb{P}(X > 0) \leqslant \eta$. This implies one of $\beta_1, \dots, \beta_n$ is positive, and therefore $d > 0$. For $i = 1, \dots, n$, take any $X_i \in \mathcal{X}$ such that $0 \leqslant X_i \leqslant X$. Note that by definition, $(\psi - 1)X \leqslant x/d$. We have

$$\mathbb{E}[(\psi - 1)\mathrm{I}_{\{U_{X_i} \geqslant 1-\alpha_i\}} X_i] \leqslant \mathbb{E}[(\psi - 1)\mathrm{I}_{\{U_{X_i} \geqslant 1-\alpha_i\}} X] \leqslant \mathbb{E}\left[\frac{x}{d}\mathrm{I}_{\{U_{X_i} \geqslant 1-\alpha_i\}}\right] = \frac{x\alpha_i}{d}. \tag{40}$$

On the other hand, using $(\psi - 1) \leqslant 1/d$ and $\mathbb{P}(X_i > 0) \leqslant \mathbb{P}(X > 0) \leqslant \alpha_i + \beta_i$,

$$\begin{aligned}\mathbb{E}[(\psi - 1)\mathrm{I}_{\{U_{X_i} < 1-\alpha_i\}} X_i] \leqslant \frac{1}{d}\mathbb{E}[\mathrm{I}_{\{U_{X_i} < 1-\alpha_i\}} X_i] &\leqslant \frac{c_i}{\beta_i}\int_{\alpha_i}^{1} \mathrm{VaR}_\gamma(X_i)\mathrm{d}\gamma \\ &= \frac{c_i}{\beta_i}\int_{\alpha_i}^{\alpha_i+\beta_i} \mathrm{VaR}_\gamma(X_i)\mathrm{d}\gamma \\ &= c_i \mathrm{RVaR}_{\alpha_i,\beta_i}(X_i). \end{aligned} \tag{41}$$

Combining (40) and (41), we have

$$\mathbb{E}[(\psi - 1)X_i] \leqslant \frac{x\alpha_i}{d} + c_i \mathrm{RVaR}_{\alpha_i\beta_i}(X_i).$$

Equivalently,

$$\mathbb{E}[X_i] + c_i \mathrm{RVaR}_{\alpha_i\beta_i}(X_i) - \mathbb{E}[\psi X_i] \geqslant -\frac{x\alpha_i}{d}.$$

Next we verify that $\mathcal{V}_i(X_i^*)$ is equal to $-x\alpha_i/d$. Write $A_i = \{1 - \sum_{k=1}^{i} \alpha_k < U_X \leqslant 1 - \sum_{k=1}^{i-1} \alpha_k\} \subset \{U_X \geqslant 1 - \alpha\}$. We have $X_i^* = X\mathrm{I}_{A_i}$ for $i = 1, \dots, n-1$, and $X_n^* = X\mathrm{I}_{A_n} + X\mathrm{I}_{\{U_X < 1-\alpha\}}$. For $i = 1, \dots, n-1$,

$$\begin{aligned}\mathbb{E}[X_i] + c_i \mathrm{RVaR}_{\alpha_i,\beta_i}(X_i^*) - \mathbb{E}[\psi X_i^*] = \mathbb{E}[(1-\psi)X_i^*] &= -\mathbb{E}\left[\frac{x}{Xd}\mathrm{I}_{\{U_X \geqslant 1-\alpha\}} X\mathrm{I}_{A_i}\right] \\ &= -\mathbb{E}\left[\frac{x}{d}\mathrm{I}_{A_i}\right] = -\frac{x\alpha_i}{d}.\end{aligned}$$

For the last agent, we have

$$\begin{aligned}\mathbb{E}[(\psi - 1)X_n^*] &= \mathbb{E}\left[\frac{x}{Xd}\mathrm{I}_{\{U_X \geqslant 1-\alpha\}} X\mathrm{I}_{A_n}\right] + \mathbb{E}\left[\frac{1}{d}\mathrm{I}_{\{U_X < 1-\alpha\}} X\right] \\ &= \frac{x\alpha_n}{d} + \frac{1}{d}\int_{\alpha}^{1} \mathrm{VaR}_\gamma(X)\mathrm{d}\gamma \\ &= \frac{x\alpha_n}{d} + \frac{c_n}{\beta_n}\int_{\alpha}^{\alpha+\beta} \mathrm{VaR}_\gamma(X)\mathrm{d}\gamma \\ &= \frac{x\alpha_n}{d} + c_n \mathrm{RVaR}_{\alpha,\beta}(X).\end{aligned}$$

Therefore,

$$\mathbb{E}[X_n^*] + c_n \mathrm{RVaR}_{\alpha_n,\beta_n}(X_n^*) - \mathbb{E}[\psi X_n^*] = c_n \mathrm{RVaR}_{\alpha,\beta}(X) - \mathbb{E}[(\psi-1)X_n^*] = -\frac{x\alpha_n}{d}.$$

In summary, for $i = 1, \dots, n$,

$$\mathcal{V}_i(X_i) \geqslant -\frac{x\alpha_i}{d} = \mathcal{V}_i(X_i^*). \tag{42}$$

By definition, $(\psi, (X_1^*, \dots, X_n^*))$ is an Arrow-Debreu equilibrium for (23). □

E Proof of Proposition 3

Proof. For fixed $i = 1, \dots, n$, we will show that for any $\alpha, \beta \in [0, 1)$, $\alpha + \beta < 1$, the inequality

$$\liminf_{j\to\infty} \mathrm{RVaR}_{\alpha,\beta}(f_i(Z_j)) \geqslant \mathrm{RVaR}_{\alpha,\beta}(f_i(X)). \tag{43}$$

holds. Then the proposition follows from taking $(\alpha, \beta) = (\alpha_i, \beta_i)$ in (43) and summing up over $i = 1, \dots, n$.

Since X is continuously distributed, by the Continuous Mapping Theorem, we have $f_i(Z_j) \to f_i(X)$ weakly. Then, $\mathrm{VaR}_\gamma(f_i(Z_j)) \to \mathrm{VaR}_\gamma(f_i(X))$ for almost every $\gamma \in (0, 1)$. By noting that $\mathrm{VaR}_{\alpha+\beta}(X) > -\infty$, we have that $\mathrm{VaR}_{\alpha+\beta}(Z_j)$ is bounded below for $j \in \mathbb{N}$, and hence Fatou's Lemma gives us

$$\liminf_{j\to\infty} \mathrm{RVaR}_{\alpha,\beta}(f_i(Z_j)) \geqslant \frac{1}{\beta}\int_{\alpha}^{\alpha+\beta} \liminf_{j\to\infty} \mathrm{VaR}_\gamma(f_i(Z_j))\mathrm{d}\gamma = \mathrm{RVaR}_{\alpha,\beta}(f_i(X)), \quad \beta > 0. \tag{44}$$

For any $\gamma > 0$, since $\mathrm{VaR}_\gamma(X)$ is non-increasing in $\gamma \in [0,1)$, using (44), we have

$$\liminf_{j\to\infty} \mathrm{VaR}_\alpha(f_i(Z_j)) \geqslant \liminf_{j\to\infty} \mathrm{RVaR}_{\alpha,\gamma}(f_i(Z_j)) \geqslant \mathrm{RVaR}_{\alpha,\gamma}(f_i(X)).$$

By letting $\gamma \downarrow 0$, we obtain

$$\liminf_{j\to\infty} \mathrm{VaR}_\alpha(f_i(Z_j)) \geqslant \mathrm{VaR}_\alpha(f_i(X)). \tag{45}$$

Therefore, (43) follows from (44)-(45). □

F Proof of Proposition 4

Proof. Let $(f_1(X),\ldots,f_n(X)) \in \mathbb{A}_n(X)$ be a π-robust optimal allocation of X. For any $Z_j \to X$ in π as $n\to\infty$, we have

$$\square_{i=1}^n \rho_i(Z_j) \leqslant \sum_{i=1}^n \rho_i(f_i(Z_j)) \to \sum_{i=1}^n \rho_i(f_i(X)) = \square_{i=1}^n \rho_i(X).$$

Therefore, $\sum_{i=1}^n \rho_i$ is π-upper-semicontinuous at X. □

G Proof of Theorem 5

Proof. Since the risk sharing problem is invariant under a constant shift in X, without loss of generality we may assume $\mathrm{VaR}_p(X) = 0$, where $p = \sum_{i=1}^n \alpha_i + \bigvee_{i=1}^n \beta_i < 1$. Similar to the proof of Theorem 2, we may also assume $\beta_n = \bigvee_{i=1}^n \beta_i$. Let F be the distribution of X.

Part 1. We first show that, in all cases (i)-(iii), the optimal allocation in (11)-(12) is robust. The optimal allocation in (11)-(12) can be written as $(f_1(X),\ldots,f_n(X))$, where

$$f_i(x) = x\mathrm{I}_{\{F^{-1}(1-\sum_{k=1}^{i}\alpha_k) < x \leqslant F^{-1}(1-\sum_{k=1}^{i-1}\alpha_k)\}}, \quad i = 1,\ldots,n-1, \ x\in\mathbb{R}, \text{ and} \tag{46}$$

$$f_n(x) = x\mathrm{I}_{\{x\leqslant F^{-1}(1-\sum_{k=1}^{n-1}\alpha_k)\}}, \quad x\in\mathbb{R}. \tag{47}$$

To show the cases (i) and (ii), suppose that $\beta_1,\ldots,\beta_n > 0$. Let $Z_j \in \mathcal{X}$, $j\in\mathbb{N}$, be a sequence of random variables such that $Z_j \to X$ in L^1, $j\to\infty$. Note that this implies that $\{Z_j : j\in\mathbb{N}\}$ is uniformly integrable. By the Continuous Mapping Theorem, we have $f_i(Z_j) \to f_i(X)$ in probability. For each $i = 1,\ldots,n$, since $f_i(x) \leqslant x\mathrm{I}_{\{x\geqslant 0\}}$ and $\{Z_j : j\in\mathbb{N}\}$ is uniformly integrable, $\{f_i(Z_j) : j\in\mathbb{N}\}$ is also uniformly integrable. Hence, we have $f_i(Z_j) \to f_i(X)$ in L^1. Note that $\mathrm{RVaR}_{\alpha,\beta}$, $\alpha,\beta > 0$, is continuous with respect to weak convergence (see Cont et al. (2010)) and ES_β, $\beta > 0$ is continuous with respect to L^1-convergence (see Emmer et al. (2015)). Therefore, as $j\to\infty$, for $i = 1,\ldots,n$,

$$\mathrm{RVaR}_{\alpha_i,\beta_i}(f_i(Z_j)) \to \mathrm{RVaR}_{\alpha_i,\beta_i}(f_i(X)). \tag{48}$$

Thus, $(f_1(X),\ldots,f_n(X))$ is an L^1-robust optimal allocation of X. Note that if X is bounded, then L^∞-robustness is weaker than L^1 robustness, and hence $(f_1(X),\ldots,f_n(X))$ is an L^∞-robust optimal allocation of X.

To show the case (iii), suppose that $\beta_1, \ldots, \beta_n > 0$ and $\alpha_1 > 0$ without loss of generality (in fact, if $\alpha_1 = 0$, then $f_1(X) = 0$ and we can proceed to consider the next agent). Let $Z_j \in \mathcal{X},\ j \in \mathbb{N}$, be a sequence of random variables such that $Z_j \to X$ in $\pi_W,\ j \to \infty$. By the Continuous Mapping Theorem, we have $f_i(Z_j) \to f_i(X)$ weakly. Since $\mathrm{RVaR}_{\alpha,\beta},\ \alpha, \beta > 0$, is continuous with respect to weak convergence, we have

$$\mathrm{RVaR}_{\alpha_1,\beta_1}(f_1(Z_j)) \to \mathrm{RVaR}_{\alpha_1,\beta_1}(f_1(X)). \tag{49}$$

Note that all $f_i(X)$, $i = 2, \ldots, n$ are bounded above by $\mathrm{VaR}_{\alpha_1}(X)$. By a simple argument of the Dominated Convergence Theorem, we have, for $i = 2, \ldots, n$, regardless of whether $\alpha_i = 0$,

$$\mathrm{RVaR}_{\alpha_i,\beta_i}(f_i(Z_j)) \to \mathrm{RVaR}_{\alpha_i,\beta_i}(f_i(X)). \tag{50}$$

Thus, $(f_1(X), \ldots, f_n(X))$ is an π_W-robust optimal allocation of X.

Part 2. Next we show the other direction of the statements in (i)-(iii).

(1) (i) and (ii), $n = 2$: Suppose that $\beta_k = 0$ and $\alpha_k > 0$ for some $k = 1, \ldots, n$. We first look at the case $n = 2$, and we may assume that the first agent uses a true VaR. That is, $\alpha_1 > 0$ and $\beta_1 = 0$. Recall that we have assumed $\mathrm{VaR}_{\alpha_1+\alpha_2+\beta_2}(X) = 0$.

Suppose that (X_1, X_2) is an optimal allocation of X where $X_1 = f_1(X)$ and $X_2 = f_2(X)$ for some $(f_1, f_2) \in \mathbb{F}_2$. Since $(X_1 + c, X_2 - c)$ is also optimal for any $c \in \mathbb{R}$ and the robustness property of $(X_1 + c, X_2 - c)$ is the same as (X_1, X_2), we may assume without loss of generality $\mathrm{VaR}_{\alpha_1}(X_1) = 0$. As (X_1, X_2) is optimal, we have, from Theorem 2,

$$\mathrm{VaR}_{\alpha_1}(X_1) + \mathrm{RVaR}_{\alpha_2,\beta_2}(X_2) = \mathrm{RVaR}_{\alpha_1+\alpha_2,\beta_2}(X). \tag{51}$$

Writing (51) in an integral form, we have

$$\beta_2 \mathrm{VaR}_{\alpha_1}(X_1) + \int_0^{\beta_2} \mathrm{VaR}_{\alpha_2+\beta}(X_2)\mathrm{d}\beta = \int_0^{\beta_2} \mathrm{VaR}_{\alpha_1+\alpha_2+\beta}(X)\mathrm{d}\beta. \tag{52}$$

Note that from Corollary 1, we have, for any $\beta \geqslant 0$,

$$\mathrm{VaR}_{\alpha_1}(X_1) + \mathrm{VaR}_{\alpha_2+\beta}(X_2) \geqslant \mathrm{VaR}_{\alpha_1+\alpha_2+\beta}(X). \tag{53}$$

Therefore, the inequalities in (53) are equalities for almost every $\beta \geqslant 0$. By noting that both sides of (53) are right-continuous, the inequalities in (53) are indeed equalities for all $\beta \geqslant 0$. In particular, we have

$$\mathrm{VaR}_{\alpha_1}(X_1) + \mathrm{VaR}_{\alpha_2+\beta_2}(X_2) = \mathrm{VaR}_{\alpha_1+\alpha_2+\beta_2}(X). \tag{54}$$

Thus,

$$\mathrm{VaR}_{\alpha_1}(X_1) = \mathrm{VaR}_{\alpha_2+\beta_2}(X_2) = \mathrm{VaR}_{\alpha_1+\alpha_2+\beta_2}(X) = 0, \tag{55}$$

which implies $\mathbb{P}(X_2 > 0) \leqslant \alpha_2 + \beta_2$.

Let $A_1 = \{U_{X_1} > 1 - \alpha_1\}$, $A_2 = \{U_{X_2} > 1 - \alpha_2\}$ and $A = \{U_{X_2} > 1 - \alpha_2 - \beta_2\}$. Note that by (55), $\{X_1 > 0\} \subseteq A_1$ and $\{X_2 > 0\} \subseteq A$. However, since $\mathbb{P}(X > 0) = \alpha_1 + \alpha_2 + \beta_2$, and

$$\{X > 0\} \subseteq (\{X_1 > 0\} \cup \{X_2 > 0\}),$$

we have

$$\begin{aligned}\alpha_1 + \alpha_2 + \beta_2 \leqslant \mathbb{P}(\{X_1 > 0\} \cup \{X_2 > 0\}) &\leqslant \mathbb{P}(X_1 > 0) + \mathbb{P}(X_2 > 0) \\ &\leqslant \mathbb{P}(A_1) + \mathbb{P}(A) = \alpha_1 + \alpha_2 + \beta_2. \end{aligned} \tag{56}$$

Therefore, all the inequalities in (56) are equalities, and in particular, $\mathbb{P}(\{X_1 > 0\} \cup \{X_2 > 0\}) = \mathbb{P}(X_1 > 0) + \mathbb{P}(X_2 > 0)$ implies

$$\mathbb{P}(X_1 > 0, X_2 > 0) = 0. \tag{57}$$

From (29) in the proof of Theorem 1, we can see that (51) implies that the inequalities in (28) are equalities for almost every $\gamma \in [0, \beta_2]$, where Y_1, Y_2 are defined in (25) and m is some constant. In particular, by taking $\gamma \downarrow 0$ in

$$\mathrm{VaR}_\gamma(Y_1 + Y_2) = \mathrm{VaR}_{\gamma+\alpha_1+\alpha_2}(X) \ \text{ for almost every } \gamma \in [0, \beta_2],$$

and since both sides are right-continuous in γ, we have

$$\mathrm{VaR}_0(Y_1 + Y_2) = \mathrm{VaR}_{\alpha_1+\alpha_2}(X).$$

That is, $X \leqslant \mathrm{VaR}_{\alpha_1+\alpha_2}(X)$ almost surely on $A_1^c \cap A_2^c$, and equivalently,

$$\{X > \mathrm{VaR}_{\alpha_1+\alpha_2}(X)\} \subset (A_1 \cup A_2) \ \text{ a.s.}$$

It follows that

$$\alpha_1 + \alpha_2 = \mathbb{P}(X > \mathrm{VaR}_{\alpha_1+\alpha_2}(X)) \leqslant \mathbb{P}(A_1 \cup A_2) \leqslant \mathbb{P}(A_1) + \mathbb{P}(A_2) = \alpha_1 + \alpha_2,$$

and therefore all the inequalities above are equalities. In particular, we have $\mathbb{P}(X > \mathrm{VaR}_{\alpha_1+\alpha_2}(X)) = \mathbb{P}(A_1 \cup A_2)$ and hence

$$\{X > \mathrm{VaR}_{\alpha_1+\alpha_2}(X)\} = (A_1 \cup A_2) \ \text{ a.s.}$$

From $\mathbb{P}(X_1 > 0, X_2 > 0) = 0$ in (57), $X_2 \leqslant 0$ almost surely on A_1. Finally, since $X_1 = X - X_2$, we have $X_1 > \mathrm{VaR}_{\alpha_1+\alpha_2}(X)$ almost surely on A_1, and this further implies

$$\{X_1 > \mathrm{VaR}_{\alpha_1+\alpha_2}(X)\} = A_1 \ \text{ a.s.} \tag{58}$$

We consider the cases $\beta_2 > 0$ and $\beta_2 = 0$ separately:

(a) If $\beta_2 > 0$, then since $\mathrm{VaR}_\gamma(X)$ is strictly decreasing in $\gamma \in [0,1]$ (implied by the continuity of F; see Proposition 1 of Embrechts and Hofert (2013)), we have $\mathrm{VaR}_{\alpha_1+\alpha_2}(X) > \mathrm{VaR}_{\alpha_1+\alpha_2+\beta_2}(X) = 0$.

(b) If $\beta_2 = 0$, since f_1 and f_2 have at most finitely many discontinuity points, there is a constant $c \in (0, \mathrm{VaR}_0(X))$ such that f_1 and f_2 are continuous on the interval $(0,c)$. Since $\mathrm{VaR}_\gamma(X) = F^{-1}(1-\gamma)$ is continuous and strictly decreasing in γ, we have that for any subinterval $(a,b) \subset (0,c)$, one has $\mathbb{P}(X \in (a,b)) > 0$. From (57), we have $\mathbb{P}(f_1(X) > 0, f_2(X) > 0) = 0$, and hence for almost every $x \in (0, \mathrm{VaR}_0(X))$, $f_1(x) > 0$ implies $f_2(x) \leqslant 0$. Moreover, since $f_1(x)+f_2(x) = x$, $x \in (0,c)$, we know that $f_1(x)$ and $f_2(x)$ cannot be in the interval $(0,x)$. By the continuity of f_1 and f_2, we know that either $f_1(x) \leqslant 0$ for all $x \in (0,c)$ or $f_2(x) \leqslant 0$ for all $x \in (0,c)$. Without loss of generality, assume $f_1(x) \leqslant 0$ for all $x \in (0,c)$. Then, together with $\mathbb{P}(f_1(X) > 0, f_2(X) > 0) = 0$, we have $\{X_1 > c\} = A_1$ almost surely.

In both (a) and (b), there is a constant $c_0 > 0$ such that $\{X_1 > c_0\} = A_1$ almost surely. Define

$$B = \{x \in \mathbb{R} : f_1(x) > c_0\},$$

and thus $\{X \in B\} = \{X_1 > c_0\}$. From (58), $\mathbb{P}(X \in B) = \mathbb{P}(X_1 > c_0) = \mathbb{P}(A_1) = \alpha_1$. For $\varepsilon > 0$, let Y_ε be a Uniform$[-\varepsilon, \varepsilon]$ random variable independent of X and

$$Z_\varepsilon = X + Y_\varepsilon \mathrm{I}_{\{X \notin B\}}.$$

We can easily see that $Z_\varepsilon \to X$ in L^1 (in L^∞ if X is bounded) as $\varepsilon \downarrow 0$, and $\mathbb{P}(Z_\varepsilon \in B) > \alpha_1$ which means $\mathrm{VaR}_{\alpha_1}(f_1(Z_\varepsilon)) \geqslant c_0$. On the other hand, from (43), we have

$$\liminf_{\varepsilon \downarrow 0} \mathrm{RVaR}_{\alpha_2,\beta_2}(f_2(Z_\varepsilon)) \geqslant \mathrm{RVaR}_{\alpha_2,\beta_2}(f_2(X)),$$

and hence

$$\liminf_{\varepsilon \downarrow 0} \Big(\mathrm{VaR}_{\alpha_1}(f_1(Z_\varepsilon)) + \mathrm{RVaR}_{\alpha_2,\beta_2}(f_2(Z_\varepsilon))\Big) - \Big(\mathrm{VaR}_{\alpha_1}(f_1(X)) + \mathrm{RVaR}_{\alpha_2,\beta_2}(f_2(X))\Big)$$
$$\geqslant c_0 > 0.$$

Thus, $(f_1(X), f_2(X))$ is not L^1-robust (and not L^∞-robust if X is bounded).

(2) (i) and (ii), $n > 2$: We may assume $\alpha_1 > 0, \beta_1 = 0$, that is, the first agent uses a true VaR. Suppose that $(f_1(X), \dots, f(X_n))$ is an optimal allocation of X where $(f_1, f_2, \dots, f_n) \in \mathbb{F}_n$. Write $\alpha = \sum_{i=2}^n \alpha_i$, $\beta = \bigvee_{i=2}^n \beta_i$ and $g(x) = f_2(x) + \cdots + f_n(x)$, $x \in \mathbb{R}$; it is easy to see that $(f_1, g) \in \mathbb{F}_2$. From Theorems 1 and 2,

$$\mathrm{RVaR}_{\sum_{i=1}^n \alpha_i, \bigvee_{i=1}^n \beta_i}(X) = \sum_{i=1}^n \mathrm{RVaR}_{\alpha_i,\beta_i}(f_i(X)) \geqslant \mathrm{VaR}_{\alpha_1}(f_1(X)) + \mathrm{RVaR}_{\alpha,\beta}(g(X))$$
$$\geqslant \mathrm{RVaR}_{\sum_{i=1}^n \alpha_i, \bigvee_{i=1}^n \beta_i}(X).$$

Hence, the above inequalities are all equalities, and in particular,

$$\mathrm{VaR}_{\alpha_1}(f_1(X)) + \mathrm{RVaR}_{\alpha,\beta}(g(X)) = \mathrm{RVaR}_{\alpha+\alpha_1,\beta}(f_1(X) + g(X)).$$

Thus, $(f_1(X), g(X))$ is an optimal allocation of X for the underlying risk measures VaR_{α_1} and $\mathrm{RVaR}_{\alpha,\beta}$. From part (ii), we know that there exists Z_ε, such that $Z_\varepsilon \to X$ in L^1 as $\varepsilon \downarrow 0$ and

$$\liminf_{\varepsilon\downarrow 0} \left(\mathrm{VaR}_{\alpha_1}(f_1(Z_\varepsilon)) + \mathrm{RVaR}_{\alpha,\beta}(g(Z_\varepsilon))\right) - \left(\mathrm{VaR}_{\alpha_1}(f_1(X)) + \mathrm{RVaR}_{\alpha,\beta}(g(X))\right) > 0.$$

Using Theorem 1 again, we have, for $\varepsilon > 0$,

$$\mathrm{VaR}_{\alpha_1}(f_1(Z_\varepsilon)) + \sum_{i=2}^{n} \mathrm{RVaR}_{\alpha_i,\beta_i}(f_i(Z_\varepsilon)) \geqslant \mathrm{VaR}_{\alpha_1}(f_1(Z_\varepsilon)) + \mathrm{RVaR}_{\alpha,\beta}(g(Z_\varepsilon)).$$

Therefore,

$$\liminf_{\varepsilon\downarrow 0} \left(\mathrm{VaR}_{\alpha_1}(f_1(Z_\varepsilon)) + \sum_{i=2}^{n} \mathrm{RVaR}_{\alpha_i,\beta_i}(f_i(Z_\varepsilon))\right) - \mathrm{RVaR}_{\sum_{i=1}^n \alpha_i, \bigvee_{i=1}^n \beta_i}(X) > 0.$$

Thus, $(f_1(X), \ldots, f_n(X))$ is not robust (and not L^∞-robust if X is bounded).

(3) (iii): Suppose that there exists a π_W-robust optimal allocation. Since π_W-robustness is stronger than L^1-robustness, we know that $\beta_1, \ldots, \beta_n > 0$. If $\alpha_1 = \cdots = \alpha_n = 0$, then $\square_{i=1}^n \mathrm{RVaR}_{\alpha_i,\beta_i} = \mathrm{ES}_{\beta_n}(X)$. As ES_{β_n} is not upper-semicontinuous at any X with respect to weak convergence (see Cont et al. (2010)), by Proposition 4 there cannot exist any π_W-robust optimal allocation. Hence, in order to allow for a π_W-robust optimal allocation, all of $\beta_1, \ldots, \beta_n$ have to be positive, and at least one of $\alpha_1, \ldots, \alpha_n$ has to be positive. □

H Proof of Theorem 6

Proof. For the "if" part, take $X_i = X$ and $X_j = 0$ for $j \neq i$. We can see that

$$\sum_{j=1}^{n} \mathrm{RVaR}_{\alpha_j,\beta_j}(X_j) = \mathrm{RVaR}_{\alpha_i,\beta_i}(X) = \mathop{\square}_{i=1}^{n} \mathrm{RVaR}_{\alpha_i,\beta_i}(X)$$

and thus the "if" part holds.

In the following we show the "only-if" part. Suppose that there exists a comonotonic optimal allocation. This implies

$$\mathop{\square}_{i=1}^{n} \mathrm{RVaR}_{\alpha_i,\beta_i}(X) = \mathop{\boxplus}_{i=1}^{n} \mathrm{RVaR}_{\alpha_i,\beta_i}(X).$$

By Theorem 2 and Corollary 3, we have

$$\mathop{\square}_{i=1}^{n} \mathrm{RVaR}_{\alpha_i,\beta_i}(X) = \mathrm{RVaR}_{\sum_{i=1}^n \alpha_i, \bigvee_{i=1}^n \beta_i}(X),$$

and

$$\boxplus_{i=1}^{n} \mathrm{RVaR}_{\alpha_i,\beta_i}(X) = \int_0^1 \mathrm{VaR}_\alpha(X)\mathrm{d}h(\alpha),$$

where h is given in Corollary 3.

Let $\alpha = \sum_{i=1}^n \alpha_i$, $\beta = \max\{\beta_i : i = 1,\ldots,n\}$, and $g(t) = h^{(\alpha,\beta)}(t)$, $t \in [0,1]$. It is easy to see $h(t) \geqslant g(t)$. By (38), we have

$$0 = \int_0^1 \mathrm{VaR}_\gamma(X)\mathrm{d}h(\gamma) - \int_0^1 \mathrm{VaR}_\gamma(X)\mathrm{d}g(\gamma) = \int_{-\infty}^{+\infty} \left(h(1-F(x)) - g(1-F(x))\right) \mathrm{d}x,$$

where F is the distribution of X. Since $h(t) \geqslant g(t)$, we have $h(1-F(x)) = g(1-F(x))$ for almost every $x \in \mathbb{R}$, and as X is continuously distributed, this leads to $h(t) = g(t)$ for almost every $t \in [0,1]$. Thus, $h^{(\alpha,\beta)}(t) = \min\{h^{(\alpha_1,\beta_1)}(t),\ldots,h^{(\alpha_n,\beta_n)}(t)\}$. Simple algebra shows that there exists $i \in \{1,\ldots,n\}$ such that for all $j \neq i$, $\alpha_j = 0$ and $\beta_i \geqslant \beta_j$. □