Budget-Constrained Procurement*

Alexandre Belloni,† Giuseppe Lopomo,‡
Leslie M. Marx,§ Roberto Steri¶

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Abstract

Procurement auctions have been successfully applied in a variety of settings. We study optimal procurement mechanisms for a buyer with a fixed budget that wishes to purchase units of a homogeneous good, up to a maximum demand amount, from symmetric suppliers with privately known constant marginal costs. We show that the nature of the optimal mechanism depends crucially on the normalized budget (the ratio between the buyer's budget and its demand) relative to the support of suppliers' costs. In particular, there is an intermediate range of the normalized budget for which there is a gap between formulations with interim incentive constraints and those with ex post constraints. We characterize the optimal mechanism subject to interim incentive compatibility and individually rationality and, for the case of two suppliers, we characterize the optimal ex post mechanism and provide a dynamic implementation.

Keywords: mechanism design, budget constraints, auctions, ex post implementation
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†The Fuqua School of Business, Duke University, 100 Fuqua Drive, Durham, NC 27708, USA: Email: abn5@duke.edu.

‡The Fuqua School of Business, Duke University, 100 Fuqua Drive, Durham, NC 27708, USA: Email: glopomo@duke.edu.

§The Fuqua School of Business, Duke University, 100 Fuqua Drive, Durham, NC 27708, USA: Email: marx@duke.edu.

¶HEC Lausanne, Quartier Unil Dorigny, CH-1015 Lausanne, Switzerland: Email: robert.steri@unil.ch.

1 Introduction

Procurement accounts for a substantial amount of economic activity, and procurement officials regularly face the problem of procuring multiple units of a good subject to a budget constraint. Examples include smaller-scale applications, such as a new business furnishing its offices or an art museum expanding its collection, and larger-scale applications, such as governmental purchases of pharmaceuticals for distribution to health care providers or of broadband coverage for underserved areas.[1]

Surprisingly, the optimal procurement mechanism for a budget-constrained buyer has remained an open question in the economics literature. Authors such as Che and Gale (1996, 1998) have analyzed budget constraints on the bidders' side, but not on the side of the mechanism designer. In this paper, we identify the optimal mechanism for a budget-constrained buyer in an independent private values framework satisfying regularity conditions. As we show, the optimal budget-constrained mechanism differs from its non-budget-constrained counterpart in that its implementation requires payments to suppliers other than the lowest-cost supplier for some type realizations. Further, and also in contrast to the case without budget constraints, the optimal budget-constrained mechanism subject to interim incentive compatibility and individual rationality constraints does not satisfy the ex post versions of those constraints.

In a procurement setting, it is natural to think that each seller's cost of providing the good is its own private information. Thus, the buyer has an incentive to employ a mechanism that elicits information from the sellers in a way that maximizes its expected surplus subject to its budget constraint. The theory of optimal procurement is well understood when the buyer is not financially constrained, but challenges remain when dealing with a budget-constrained buyer. Indeed, in such cases, how the budget constraint limits the gains from trade depends on how the mechanism elicits information from agents. Moreover, as noted by Bergemann and Morris (2005), budget constraints yield a payoff environment that is "nonseparable" in

[1]Since 2007, the state of São Paulo, Brazil, has routinely purchased a vast array of goods and services using procurements that are subject to budget constraints. The Brazilian Constitution states that all public works, services, purchases and transfers of ownership must be contracted through a process of public tender. The volume of federal government purchases rose from R$40.6 billion in 2007 to R$72.6 billion in 2012 (oecd.org). In September 2012, the U.S. Federal Communications Service (FCC) held the Mobility Fund Phase I reverse auction to award one-time support to carriers that committed to provide 3G or better mobile voice and broadband services in areas where such services were unavailable, without exceeding the budget of $300 million. Bids were ranked from lowest to highest in terms of dollars per road-mile covered and bidders were paid until the budget was exhausted. The use of reverse auctions for this purpose has been proposed in the past, notably by Milgrom (1996) and Weller (1999). As discussed in Wallsten (2009), a number of countries have used reverse auctions to allocate universal service support, to varying degrees of success. For details of the FCC's auction, see "Report and Order and Further Notice of Proposed Rulemaking In the Matter of Connect America Fund, A National Broadband Plan for Our Future, et al.," FCC 11-161, Released November 18, 2011, available at https://apps.fcc.gov/edocs_public/attachmatch/FCC-11-161A1.pdf, especially Section XVII.I–K.

their language, and so the interim implementation no longer implies ex post implementation.

As we show, in a setup with budget constraints, pooling can arise not because of ironing, which addresses nonlocal incentive compatibility constraints, but rather to provide further screening by restricting trade for some types and thereby allowing more trade with better types. Further, pooling due to budget constraints yields dual sourcing. Although many companies regard dual sourcing as important and would not agree to long-term contracts with a single supplier out of concern for hold-up or delivery risk (Anton and Yao, 1992), in our model dual sourcing arises for different reasons. Because the budget constraint potentially restricts trade, dual sourcing allows for more effective screening because a supplier with a better type has an incentive to be the only producer in order to obtain a larger payment, instead of splitting the available budget with another supplier.

In this paper, we develop a model that reflects the procurement environment described above. We consider a setup in which a buyer with a fixed budget B wishes to purchase units of a homogeneous product up to a maximum demand amount D. Suppliers' marginal costs are their private information and are modeled as independent draws from a common distribution. We show that the nature of the optimal mechanism depends crucially on the "normalized budget," $b \equiv B/D$, relative to the support of costs $[c_L, c_H]$. Broadly speaking there are three cases:

(i) large normalized budget: $b \geq c_H$

(ii) intermediate normalized budget: $b \in (c_L, c_H)$

(iii) small normalized budget: $b \leq c_L$

The large normalized budget case is well understood in the mechanism design literature (Myerson, 1981). We use this known case to introduce notation and illustrate the techniques and ideas. In the small normalized budget case, we show that the buyer's problem can also be solved using standard mechanism design techniques after we leverage the analogous roles played by payments and allocations. In both cases, the optimal mechanism does not change depending on whether we consider interim or ex post incentive constraints—there is no revenue gap between the interim and ex post case for a large or small normalized budget.

However, in the intermediate range for the normalized budget, there is a revenue gap between formulations with interim incentive constraints and ex post incentives constraints. We characterize the mechanism that maximizes the buyer's expected surplus, subject to interim incentive compatibility and individually rationality, assuming that each buyer has a constant marginal cost and that the type distribution satisfies regularity conditions. We also characterize the optimal mechanism subject to ex post incentive compatibility and individually rationality for the case of two suppliers. We summarize our results in Table 1, which refers to the second-price auction (SPA), in which the buyer purchases its full demand from the lowest bidder at a unit price equal to the second-lowest bid, and the second-unit-price auction (SUPA), in which the lowest bidder is paid the full budget and supplies as many units as can be purchased at a unit price equal to the second-lowest bid. Indeed, the SPA has a long history in auction theory and is widely used in practice in its dynamic implementation of a clock auction, and the SUPA corresponds to the "dual Dutch auction" described by Crawford and Kuo (2003).[2] Other mechanisms mentioned in the table are defined in the body of the paper.

Table 1: The table displays a brief description of the optimal mechanisms, where SPA is second-price auction and SUPA is second-unit-price auction, for each combination of normalized budget b and incentive constraints under the regularity conditions discussed in the body of the paper. The support of the distribution of costs is $[c_L, c_H]$, and the thresholds b_L and b_H are defined in (17) and (20).

	normalized budget $b = B/D$				
Incentive constraints	small $b \leq c_L$	intermediate $c_L < b < b_L$	$b_L \leq b \leq b_H$	$b_H < b < c_H$	large $c_H \leq b$
interim	SUPA	SUPA reduced-form	Clipped reduced-form	SPA reduced-form	SPA
ex post	SUPA	$S = 2$, partial dual sourcing auction $S \geq 3$, open question			SPA

An underlying motivation for this paper is the desire for mechanisms that are practical and possibly optimal for a large class of cost distributions. It is arguably the case that mechanisms that are ex post incentive compatible lead to easier and more practical implementations (e.g., SPA, SUPA). Additional motivation for ex post mechanisms is provided by the literature on robust mechanism design, e.g., Bergemann and Morris (2005). A key finding of our paper is that in a symmetric environment with two suppliers, the optimal ex post incentive compatible, ex post individual rationality mechanism has a simple form that can be implemented through a descending clock auction. The mechanism is straightforward to communicate and implement for an arbitrary number of suppliers, although optimality is no longer guaranteed for more than two suppliers. Roughly speaking, bidding starts at the level of the budget and continues downward until all remaining active bidders can be accommodated within the budget. The mechanism we identify has the appealing features that it is robust in the sense that bidders have dominant strategies (the strategic problem of each participant is just a sequence of in-out decisions, where conjectures about the behavior of others do not matter), and it is simple in the sense of having an equilibrium that involves truthful bidding, payments only to winners, and no random awards. This auction, as well as an auction that we define that combines the second-price auction and the second-unit-price

[2]Implementations that lead to dual sourcing have also been considered in Alcalde and Dahm (2016).

auction into a single mechanism, belong to a more general family of dynamic implementations that we call sequential auctions with promises.

Our mechanism-design approach to the problem of budget-constrained procurement relies on previous methodological contributions including: Mussa and Rosen (1978), Myerson (1981), Rochet (1985), Matthews and Moore (1987), McAfee and McMillan (1988), Armstrong (1996), Rochet and Choné (1998), and Manelli and Vincent (2006, 2007). A result of this literature is that many mechanism design problems can be formulated as a linear programming problem, which a more recent literature has relied upon. See, e.g., Bikhchandani et al. (2001), Belloni et al. (2010), Vohra (2011), Du (2017), and Carroll and Segal (2017). There is a related literature on auctions with budget-constrained bidders, e.g., Pitchik and Schotter (1989), Shleifer and Vishny (1992), Maskin (1992), Che and Gale (1996, 1998), Benoît and Krishna (2001), and Pitchik (2009); however, this literature typically focuses on a comparison of standard auction formats and the optimal order in which to sell objects rather than taking a general mechanism-design approach. For example, Che and Gale (1996, 1998) show that with budget-constrained bidders, revenue equivalence between various auction formats no longer holds. Similarly, for the case of a budget-constrained procurement, Dastidar (2008) compares results for first and second-price procurement auctions. Using a mechanism design approach, Pai and Vohra (2014) show that the optimal mechanism for selling to budget-constrained buyers requires pooling both at the top and in the middle even with monotone hazard rates. Che and Gale (2000) show that the optimal mechanism for selling a divisible good to a single budget-constrained buyer generally involves nontrivial price discrimination, with different types trading different quantities.

Our approach to establishing the optimality of certain mechanisms relies on studying the linear programming problem associated with the budget-constrained procurement problem. In the intermediate normalized budget case, it is not a priori clear which constraints are relevant in determining the optimal mechanism (in contrast to the small and large normalized budget cases). A key step is to find a suitable way to combine the constraints to facilitate the optimality proof (that is, suitable Lagrange multipliers to produce an optimality certificate). Related duality approaches have been used by Du (2017) in the context of the sale of a common-value good and by Carroll and Segal (2017) for an auction with possible resale opportunities among bidders. Other papers that rely on related approaches include Bergemann and Morris (2005) and Carrasco et al. (2015).

The paper proceeds as follows. In Section 2, we describe our setup. In Section 3 we provide results for the cases of large and small normalized budgets. In Section 4, we extend the analysis to include the intermediate normalized budget case, where both demand and budget constraints are binding with positive probability. We characterize the optimal reduced-form mechanism satisfying interim incentive compatibility and interim individual rationality. Section 5 considers the intermediate normalized budget case with ex post incentive constraints.

We provide a mechanism, and its dynamic implementation, that is optimal for the case of two suppliers. Section 6 contains additional discussion, including the single supplier case and numerical examples illustrating the performance of various mechanisms calibrated based on auctions used in the Brazilian exchange system. Section 7 concludes.

2 Setup

A buyer faces a set $\mathcal{S} = \{1, ..., S\}$ of suppliers, where $S \geq 2$. Each supplier $s \in \mathcal{S}$ can produce any nonnegative amount q_s of a homogeneous good at a constant and privately known unit cost c_s. From the buyer's perspective, the cost profile $\mathbf{c} = (c_1, ..., c_S)$ is a vector of independent and identically distributed random variables drawn from the twice continuously differentiable cumulative distribution function F with density function f that is strictly positive on the support $[c_L, c_H]$, where $0 < c_L < c_H$.[3]

If supplier s produces q_s units of the good and is paid a nonnegative amount m_s, its profit is given by

$$m_s - c_s q_s.$$

The buyer cannot spend more than an exogenously given budget B and has a constant unit willingness to pay $v > 0$ for the good, up to D units.[4] Its surplus is given by

$$v \min\{D, \sum_{s\in\mathcal{S}} q_s\} - \sum_{s\in\mathcal{S}} m_s.$$

We focus on the a setup in which the buyer's value v is sufficiently large that the buyer's value does not define a binding reserve. This allows us to focus on the effects of budget constraints on the optimal mechanism without the additional complexity of binding reserves. In what follows, we make precise the required lower bound on v.

The buyer can commit to any feasible trading mechanism and aims at maximizing its expected surplus. By the revelation principle, any equilibrium of any feasible trading mechanism can be implemented by a direct revelation mechanism, which in this case comprises for $s \in \mathcal{S}$, $q_s : [c_L, c_H]^S \to \mathbb{R}$ and $m_s : [c_L, c_H]^S \to \mathbb{R}$ that satisfy individual rationality and incentive compatibility.

Because the suppliers are ex ante identical, the search for optimal mechanisms can be restricted without loss of generality to symmetric (anonymous) mechanisms.[5] Thus, we

[3]The assumption that c_L is strictly positive allows us to consider the small normalized budget case in which the buyer's budget constraint always binds.

[4]The amount D for which the buyer has positive willingness to pay can be normalized to 1 with no loss of generality. However, we retain the variable to facilitate relating our model to data in which the buyer's demand varies.

[5]The justification for this claim, due to Maskin and Riley (1984), is as follows: If an asymmetric mechanism $(\mathbf{q}, \mathbf{m})$ is optimal, due to the ex ante symmetry of the suppliers, the mechanism $(\mathbf{q}', \mathbf{m}')$ obtained from

search for a symmetric mechanism,[6] which is determined by just two functions,

$$q : [c_L, c_H]^S \to \mathbb{R} \ \text{ and } \ m : [c_L, c_H]^S \to \mathbb{R},$$

which are invariant to permutations of the last $S-1$ arguments.

To be feasible, a mechanism (q, m) must satisfy the nonnegativity constraints,

$$\forall (c_s, \mathbf{c}_{-s}) \in [c_L, c_H]^S, \quad q(c_s, \mathbf{c}_{-s}) \geq 0 \ \text{ and } \ m(c_s, \mathbf{c}_{-s}) \geq 0, \tag{1}$$

the budget constraints,

$$\forall (c_s, \mathbf{c}_{-s}) \in [c_L, c_H]^S, \quad \sum_{s \in \mathcal{S}} m(c_s, \mathbf{c}_{-s}) \leq B, \tag{2}$$

and the demand constraints,

$$\forall (c_s, \mathbf{c}_{-s}) \in [c_L, c_H]^S, \quad \sum_{s \in \mathcal{S}} q(c_s, \mathbf{c}_{-s}) \leq D. \tag{3}$$

In addition, to be outcome equivalent to an equilibrium of a trading game, (q, m) must satisfy incentive compatibility and individual rationality. We first consider the set of all Bayesian equilibria of all trading games and thus impose incentive compatibility and individual rationality at the interim level:

$$\forall c_s, c_s' \in [c_L, c_H], \quad M(c_s) - c_s Q(c_s) \geq \ M(c_s') - c_s Q(c_s') \tag{4}$$

and

$$\forall c_s \in [c_L, c_H], \ \ M(c_s) - c_s Q(c_s) \geq 0, \tag{5}$$

where

$$Q(c_s) \equiv \int_{[c_L, c_H]^{S-1}} q(c_s, \mathbf{c}_{-s}) \, dF_{-s}(\mathbf{c}_{-s}) \ \text{ and } \ M(c_s) \equiv \int_{[c_L, c_H]^{S-1}} m(c_s, \mathbf{c}_{-s}) \, dF_{-s}(\mathbf{c}_{-s}). \tag{6}$$

Following Border (1991), we refer to the mechanism (Q, M) as the reduced-form mechanism associated with (q, m). Given a reduced-form mechanism $(Q, M) : [c_L, c_H] \to \mathbb{R}^2$ and a

$(\mathbf{q}, \mathbf{m})$ by reversing the roles of the buyers must also be optimal. But then, because the objective function is linear, the symmetric mechanism $(\frac{1}{2}\mathbf{q} + \frac{1}{2}\mathbf{q}', \frac{1}{2}\mathbf{m} + \frac{1}{2}\mathbf{m}')$ is also optimal. The argument generalizes to more than two suppliers.

[6] A mechanism (q, m) is symmetric if the suppliers' identities $1, ..., S$ do not matter, i.e., for every permutation π of $\mathcal{S}$ we have for all $\mathbf{c} \in [c_L, c_H]^S$ and $s \in \mathcal{S}$,

$$q_s(c_{\pi(1)}, ..., c_{\pi(S)}) = q_{\pi(s)}(\mathbf{c}) \ \text{ and } m_s(c_{\pi(1)}, ..., c_{\pi(S)}) = m_{\pi(s)}(\mathbf{c}).$$

symmetric mechanism $(q,m) : [c_L, c_H]^S \to \mathbb{R}^2$, we say that (q,m) implements (Q,M) if (q,m) is feasible (i.e., satisfies (1)–(3)) and satisfies (6).

We also consider the effects of restricting attention to mechanisms with equilibria in dominant strategies, and thus in places we impose both incentive compatibility and individual rationality ex post:

$$\forall c_s, c_s' \in [c_L, c_H],\ \forall \mathbf{c}_{-s} \in [c_L, c_H]^{S-1},\ m(c_s, \mathbf{c}_{-s}) - c_s q(c_s, \mathbf{c}_{-s}) \geq m(c_s', \mathbf{c}_{-s}) - c_s q(c_s', \mathbf{c}_{-s}) \quad (7)$$

and

$$\forall c_s \in [c_L, c_H], \quad \forall \mathbf{c}_{-s} \in [c_L, c_H]^{S-1}, \quad m(c_s, \mathbf{c}_{-s}) - c_s q(c_s, \mathbf{c}_{-s}) \geq 0. \quad (8)$$

The *buyer's problem* is to select a mechanism (q,m) that maximizes the buyer's expected surplus,

$$\int_{[c_L, c_H]^S} \sum_{s \in \mathcal{S}} (vq(c_s, \mathbf{c}_{-s}) - m_s(c_s, \mathbf{c}_{-s})) \prod_{s \in \mathcal{S}} dF(c_s), \quad (9)$$

subject to (1), (2), (3), and either the interim incentive constraints (4) and (5), or the ex post incentive constraints (7) and (8).

In what follows, we show that the optimal mechanism depends in a fundamental way on the normalized budget $b = B/D$. Indeed, this quantity provides information regarding whether the budget constraint or the demand constraint is more relevant. In particular, there are three main regimes that lead to different optimal mechanisms: small normalized budget ($b \leq c_L$), intermediate normalized budget ($c_L < b < c_H$), and large normalized budget ($b \geq c_H$). In the next sections, we address each of these regimes in detail and their interplay with the type of incentive constraints (interim or ex post).

3 Large and small normalized budgets

In this section, we consider the two (extreme) cases of a large normalized budget and a small normalized budget. In both cases, the analysis is closely related to the literature. Indeed, the large normalized budget case is well understood in the mechanism design literature. In what follows, we discuss it briefly for completeness and to introduce notation for later use. We then study the small normalized budget case and show how the buyer's problem in that case can also be solved using standard mechanism design techniques.

3.1 Large normalized budget

Consider the case of a large normalized budget, $b \geq c_H$. In this case, the buyer can purchase its entire demand D with its budget B, even when all suppliers have cost c_H. Thus, the budget constraints, (2), can be ignored and the buyer's problem reduces to a well-known application of optimal auction theory (see, e.g., Myerson, 1981). We define the virtual marginal value

$$w(x) \equiv v - x - F(x)/f(x), \tag{10}$$

which is v minus the virtual cost for a supplier with type x, $x + F(x)/f(x)$. Standard mechanism design arguments (see, e.g., Myerson, 1981, or Krishna, 2002, Chapter 5) imply that the optimization problem amounts to:

$$\max_q \int_{[c_L, c_H]^S} \sum_{s \in \mathcal{S}} w(c_s) q(c_s, \mathbf{c}_{-s}) \prod_{s \in \mathcal{S}} dF(c_s),$$

subject to the nonnegativity and demand constraints, (1) and (3), and individual rationality, (5). It follows that if w is nonincreasing and nonnegative on $[c_L, c_H]$, then it is optimal for the buyer to purchase the entire amount D from the lowest-cost supplier at a unit price equal to the second-lowest cost. This mechanism is a second-price auction (SPA) with no reserve,[7] which can be formally defined as

$$q^{SPA}(c_s, \mathbf{c}_{-s}) \equiv D \cdot \mathbf{1}_{[c_s = \mathbf{c}_{(1)}]} \text{ and } m^{SPA}(c_s, \mathbf{c}_{-s}) \equiv \mathbf{c}_{(2)} D \cdot \mathbf{1}_{[c_s = \mathbf{c}_{(1)}]}, \tag{11}$$

where $\mathbf{c}_{(2)}$ denotes the second-lowest cost among $c_1, ..., c_S$. In the large normalized budget case, the SPA satisfies the budget constraint because the buyer's total payment is no more than B, even when $\mathbf{c}_{(2)} = c_H$.

For future reference, we next define the regularity assumption required to guarantee that w is nonincreasing, which is the same as that of Myerson (1981).[8] We refer to this standard Myersonian regularity condition as "regularity-0" to distinguish it from two other variants, which we introduce shortly.

Definition 1 Regularity-0 holds *if for all* $x \in [c_L, c_H]$, $\frac{d}{dx}[x + F(x)/f(x)] \geq 0$.

A sufficient condition for regularity-0 is that the reversed hazard rate $f(x)/F(x)$ be nonincreasing, which is satisfied for a range of distributions (Burkschat and Torrado, 2014). Given the definition of w in (10), regularity-0 implies that $w'(c) \leq 0$. Thus, under regularity-0, we need only require that v be sufficiently large that $w(c_H) \geq 0$ to ensure that w is nonnegative on $[c_L, c_H]$. For completeness and easy reference, we record the optimality result for the large normalized budget case as Theorem 1.

Theorem 1 (Optimal mechanism for a large normalized budget) *In the case with* $b \geq c_H$, *if regularity-*0 *holds and* $v \geq c_H + 1/f(c_H)$, *then the mechanism* (q^{SPA}, m^{SPA}) *maximizes the buyer's expected surplus* (9), *subject to* (1)–(5). *Moreover, because the mechanism*

[7] In this case, the entire amount D is treated as a single object.

[8] Myerson (1981) requires strict monotonicity, but because we do not insist on the uniqueness of our mechanisms, weak monotonicity is sufficient for our results.

(q^{SPA}, m^{SPA}) *satisfies the ex post incentive constraints, it also maximizes the buyer's expected surplus* (9), *subject to* (1)–(3) *and* (7)–(8).

Theorem 1 also states the well-known result that in the SPA, truth telling is in dominant strategies, and so the SPA also solves the buyer's problem when incentive compatibility and individual rationality are imposed ex post.

3.2 Small normalized budget

We now consider the case in which the normalized budget is small, $b \leq c_L$. In this setting, the budget is so small that the buyer can never purchase its full demand D, even when all suppliers have the lowest cost. Therefore, the demand constraints, (3), can be ignored, and the buyer's problem can again be solved using familiar mechanism design techniques, but with a slight twist. Specifically, we can use the envelope theorem to eliminate q (instead of m, as is normally done) from the problem. This allows us to rewrite the buyer's objective as

$$\max_{m} \int_{[c_L, c_H]^S} \left(\sum_{s \in \mathcal{S}} \psi(c_s) m(c_s, \mathbf{c}_{-s}) \right) \prod_{s \in \mathcal{S}} dF(c_s), \tag{12}$$

where ψ is the virtual marginal value of payments,[9]

$$\psi(x) \equiv \frac{v}{x^2}(x - F(x)/f(x)) - 1, \tag{13}$$

subject to the nonnegativity constraints on the payments, (1), the budget constraint, (2), and individual rationality, (5). Analogously to the large normalized budget case, if the function ψ is nondecreasing and nonnegative, then it is optimal for the buyer to pay the entire budget to the lowest-cost supplier and to purchase quantity $B/\mathbf{c}_{(2)}$ from that supplier. We refer to this mechanism as the second-unit-price auction (SUPA):

$$q^{SUPA}(c_s, \mathbf{c}_{-s}) \equiv \frac{B}{\mathbf{c}_{(2)}} \cdot 1_{[c_s = \mathbf{c}_{(1)}]} \quad \text{and} \quad m^{SUPA}(c_s, \mathbf{c}_{-s}) \equiv B \cdot 1_{[c_s = \mathbf{c}_{(1)}]}. \tag{14}$$

In the small normalized budget case, the SUPA satisfies the demand constraints because the quantity purchased is less than D, even when $\mathbf{c}_{(2)} = c_L$. Figure 1 contrasts the SUPA for the small normalized budget case with the SPA for the large normalized budget case.

We formally define the required regularity assumption, which we refer to as regularity-1. Similar to regularity-0 defined above, regularity-1 depends on the reverse hazard rate.

Definition 2 Regularity-1 holds *if for all* $x \in [c_L, c_H]$, $\frac{d}{dx}\left[\frac{x - F(x)/f(x)}{x^2}\right] \leq 0$.

[9] In this formulation we eliminate q so that the coefficients ψ carry the value of the additional allocation induced by a payment m.

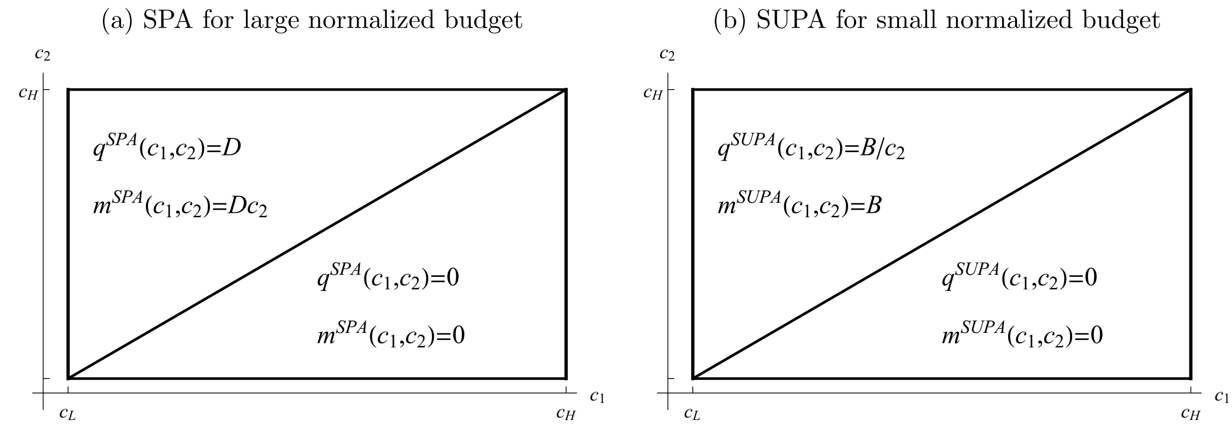

Figure 1: Illustration of the SPA and SUPA mechanisms when there are two suppliers with types c_1 and c_2. Panel (a): SPA mechanism with a large normalized budget $b \geq c_H$. Panel (b): SUPA mechanism with a small normalized budget $b \leq c_L$.

As with regularity-0, regularity-1 holds for a broad range of distributions. Regularity-1 implies that $\psi'(x) \leq 0$, so under regularity-1, having v sufficiently large that $\psi(c_H) \geq 0$ is sufficient to ensure that ψ is nonnegative on $[c_L, c_H]$. We record the optimality of the SUPA in the next theorem.

Theorem 2 (Optimal mechanism for a small normalized budget) *In the case with $b \leq c_L$, if regularity-1 holds and $v \geq c_H + \frac{1}{f(c_H)c_H - 1}$, then the mechanism (q^{SUPA}, m^{SUPA}) maximizes the buyer's expected surplus* (9)*, subject to* (1)–(5)*. Moreover, because the mechanism (q^{SUPA}, m^{SUPA}) satisfies the ex post incentive constraints, it also maximizes the buyer's expected surplus* (9)*, subject to* (1)–(3) *and* (7)–(8)*.*

For the small normalized budget case, Theorem 2 shows that the SUPA solves the buyer's problem under interim incentive compatibility and individual rationality. Moreover, as for the SPA in the large normalized budget case, for the SUPA in the small normalized budget case, truth telling is in dominant strategies. Thus, in the small normalized budget case, the SUPA also solves the buyer's problem when incentive compatibility and individual rationality are imposed ex post.

3.3 Implementation via the SUPA-SPA mechanism

In light of the optimality of the SUPA and the SPA for the small and large normalized budget cases, it is natural to consider a direct mechanism that combines both of them, so that we have a mechanism that is feasible for all values of the normalized budget and is optimal for small and large normalized budgets. We call it the SUPA-SPA. In its dynamic implementation, loosely speaking, one starts a descending clock auction based on the SUPA, which delivers a descending marginal price. If two of more suppliers remain in the auction when the price reaches $b = B/D$, one starts a SPA with those bidders. The mechanism stops when a single supplier remains. The allocation and payment functions are defined as

$$(q^{SUPA\text{-}SPA}(c_s, \mathbf{c}_{-s}), m^{SUPA\text{-}SPA}(c_s, \mathbf{c}_{-s})) \equiv \begin{cases} (B/\mathbf{c}_{(2)}, B), & \text{if } c_s = \mathbf{c}_{(1)} \text{ and } b < \mathbf{c}_{(2)} \\ (D, D\mathbf{c}_{(2)}), & \text{if } c_s = \mathbf{c}_{(1)} \text{ and } \mathbf{c}_{(2)} \leq b \\ 0, & \text{otherwise.} \end{cases} \quad (15)$$

Figure 2 illustrates the SUPA-SPA for the case of two suppliers. In the large normalized budget case it reduces to the SPA (which is optimal by Theorem 1), and in the small normalized budget case it reduces to the SUPA (which is optimal by Theorem 2).

This mechanism has a number of appealing features. First, its direct implementation is simple and handles an arbitrary number of suppliers. Second, it does not require knowledge of the support $[c_L, c_H]$. Third, only the supplier with the lowest cost produces a positive quantity.

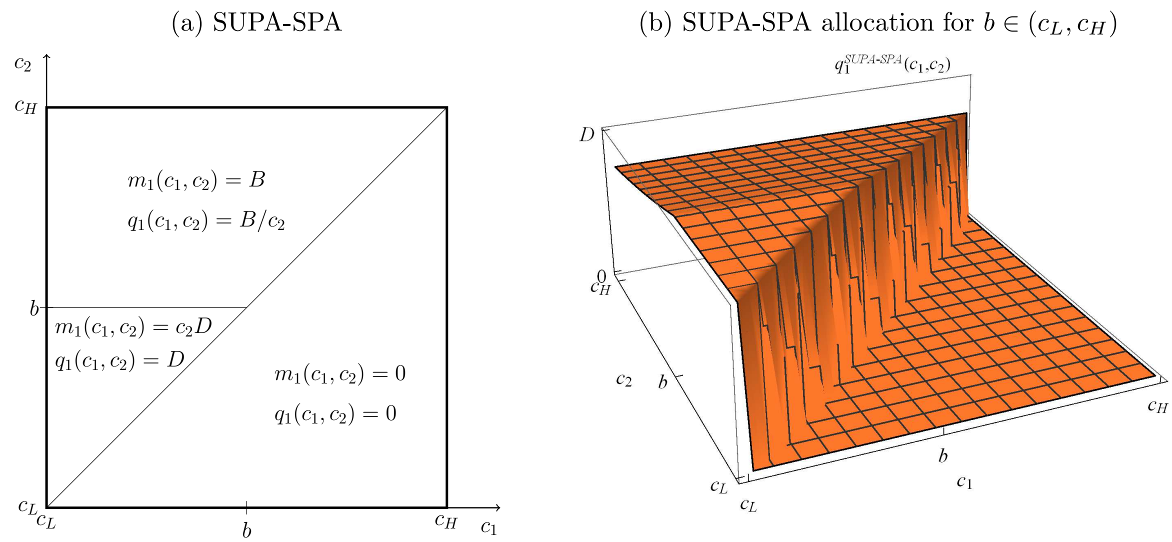

Figure 2: SUPA-SPA with two suppliers. In the large normalized budget case, $b \geq c_H$, and so the mechanism is the SPA. In the small normalized budget case, $b \leq c_L$, and so the mechanism is the SUPA. Panel (b) shows the allocation for the intermediate normalized budget case, $b \in (c_L, c_H)$.

Another interesting feature of the SUPA-SPA mechanism is that it is feasible for any normalized budget, including the intermediate case. Because the SUPA-SPA mechanism is optimal for both the small and large normalized budget ranges, it is natural to conjecture that it would also be optimal for the intermediate normalized budget range. However, it is not, as established by the following result.

Proposition 1 (Nonoptimality of the SUPA-SPA for an intermediate normalized budget) *Assume that regularity-*1 *holds and* $v \geq c_H + \frac{1}{f(c_H)c_H - 1}$. *For any* $b \in (c_L, c_H)$, *the SUPA-SPA does not maximize the buyer's expected surplus subject to subject to* (1)–(3) *and* (7)–(8).

Proof. See the Appendix.

The negative result in Proposition 1 holds quite generally, including for a uniform type distribution. The source of the nonoptimality is that the mechanism does not elicit information (i.e., does not screen agents) in a way that mitigates the trade restriction imposed by the budget constraints. Intuitively, if both suppliers are known to have costs at most $b + \varepsilon$ (which is elicited by the decreasing clock of the SUPA), it is likely that at least one of the supplier's types is less than b, so that the buyer can procure D units. It follows that to wait for the clock to get to b is not optimal. As we will see, a more aggressive strategy is optimal. Despite its nonoptimality, the SUPA-SPA mechanism is still of interest because of its simplicity and properties that are attractive for implementation.

Because Proposition 1 establishes the nonoptimality of the SUPA-SPA among ex post incentive compatible mechanisms, it immediately implies its nonoptimality among interim incentive compatible mechanisms. As we will see, the intermediate normalized budget case differs substantially from the large and small normalized budget cases in that the optimal interim mechanism is not ex post implementable. Because the interim and ex post formulations require different techniques, we address the interim formulation in Section 4 and the ex post formulation in Section 5.

4 Intermediate normalized budget with interim incentive constraints

In this section, we consider the intermediate normalized budget case, $b \in (c_L, c_H)$, when incentive compatibility and individual rationality are imposed at the interim level: that is, the buyer maximizes (9) subject to nonnegativity, budget, and demand constraints, (1)–(3), and subject to interim incentive compatibility and individual rationality, (4)–(5). In this case, the available budget is neither so large relative to demand that the budget constraint never binds nor so small relative to the demand that the budget constraint always binds. For a given mechanism, we can have the budget and/or the demand constraints binding for different type realizations.

The characterization of the optimal mechanism in the intermediate normalized budget case poses additional challenges relative to the small and large normalized budget cases. For the case of an intermediate normalized budget, the buyer's problem cannot be solved with the classic Myersonian approach, which relies on the convenient fact that, after using the envelope theorem to eliminate all payment variables (or all quantity variables), the residual problem can be solved pointwise. The special structure that arises when one can eliminate either all of the payment variables or all of the quantity variables is critical for allowing the problem to be solved pointwise. In the intermediate case, when budget constraints bind for some types and demand constraints for others, the structure required for pointwise optimization is not present and additional tools are needed.

Our analysis proceeds in two steps. First, we invoke a result by Border (1991) that has been used to express demand constraints in terms of the reduced-form mechanism. However, in our setting we apply Border's representation to both the budget constraints and the demand constraints to write them in terms of the reduced-form mechanism (Q, M). Second, we provide a "dual" variable for each constraint in order to construct a certificate of optimality for a given mechanism. This allows us to argue that the conjectured solution defined below maximizes the buyer's expected surplus, subject to a single "aggregate constraint" that is satisfied by all feasible points, and thus that it also solves the buyer's problem.

4.1 Reduced-form formulation

Similarly to Border (1991), we consider a formulation of the problem based on the reduced-form mechanism (Q, M). We focus on the following relaxed optimization problem, where $P_k(x) \equiv 1 - (1 - F(x))^k$ is the cumulative distribution function for the minimum of k independent draws from the distribution F:

$$
\begin{aligned}
&\max_{Q,M} \quad S \int_{c_L}^{c_H} (vQ(t) - M(t))\, dF(t) \\
&\text{s.t. } \forall x, x' \in [c_L, c_H]: \\
&\qquad\qquad \int_{c_L}^{x} Q(t)dF(t) \leq \tfrac{1}{S} D \; P_S(x) \qquad (\mathcal{D}_x) \\
&\qquad\qquad \int_{c_L}^{x} M(t)dF(t) \leq \tfrac{1}{S} B \; P_S(x) \qquad (\mathcal{B}_x) \\
&\qquad -M(x) + x\,Q(x) + M(x') - x\,Q(x') \leq 0 \qquad (IIC_{x,x'}) \\
&\qquad -M(c_H) + c_H\,Q(c_H) \leq 0 \qquad (IIR_{c_H}).
\end{aligned}
\tag{16}
$$

As shown in (16), the objective is to maximize the buyer's expected surplus, and the constraints are demand constraints $\mathcal{D}_x$, budget constraints $\mathcal{B}_x$, local incentive compatibility constraints $IIC_{x,x'}$, and individual rationality for the worst type of agent IIR_{c_H}.

Border (1991, Proposition 3.2) shows that the demand constraints in (3) imply the demand constraints in (16): The left side of $\mathcal{D}_x$ is the expected quantity purchased from a supplier with cost less than x, and the right side of $\mathcal{D}_x$ is per-capita demand D/S times the probability that the lowest-cost supplier has cost below x.[10] Similarly, the budget constraints in (2) imply the budget constraints in (16). As in Border (1991), individual rationality is only imposed for the worst type and only the local incentive compatibility constraints are imposed. The monotonicity constraints (i.e., the nonlocal incentive compatibility constraints) are ignored, and all nonnegativity constraints on Q are also ignored, although, of course, we confirm that our solution satisfies all constraints.

4.2 Reduced-forms of the SPA and SUPA are optimal in the upper and lower ends of the intermediate range

To introduce the techniques that we use to solve the buyer's problem for the intermediate normalized budget case and to connect with the results for the large and small normalized budget cases, we begin by showing that the reduced forms of the SPA and SUPA mechanisms continue to be optimal in regions at the upper and lower ends, respectively, of the intermediate normalized budget range.

The SPA in its ex post implementation cannot be used in the intermediate normalized budget case because, for some type realizations, the budget constraint would be violated. Specifically, if the second-lowest cost is greater than b, then a mechanism that pays D times the second-lowest cost violates the budget constraint. However, the reduced-form of the SPA is still feasible in the intermediate normalized budget case if b is sufficiently large.[11]

The reduced-form of the SPA has allocation and payment rules defined as, for each $x \in [c_L, c_H]$,

$$Q^{SPA}(x) = D(1 - P_{S-1}(x)) \quad \text{and} \quad M^{SPA}(x) = \int_x^{c_H} Dt \; dP_{S-1}(t),$$

which satisfies $\mathcal{B}_x$ for all $x \in [c_L, c_H]$ when $b \geq b_H$, where:[12]

$$b_H \equiv S \int_{c_L}^{c_H} tF(t) \; dP_{S-1}(t) \in (c_L, c_H). \tag{17}$$

This means that when $b \in [b_H, c_H)$, the SPA in its ex post implementation is not feasible, but

[10] More formally, Border (1991) shows that $Q : [c_L, c_H] \to [0, D]$ is implementable (meaning there exists q that implements it) if and only if $\forall a \in [0, D]$, $S \int_{E_a} Q(z) \, dF(z) \leq (1 - (\int_{E_a^C} dF(z))^S)D$, where $E_a = \{x \mid Q(x) \geq a\}$ and E_a^C is the complement of E_a. In our case, because Q must be nonincreasing by incentive compatibility, the sets E_a are intervals of the form $[c_L, a]$, and thus the inquality above becomes $\forall a \in [0, D]$, $S \int_{c_L}^a Q(z) \, dF(z) \leq D(1 - (1 - F(x))^S) = DP_S(x)$.

[11] Similar remarks apply to the SUPA mechanism but its reduced form is still feasible in the intermediate budget case if b is sufficiently small.

[12] To see that $b_H \in (c_L, c_H)$, note that $b_H = E[X]$, where $X \sim G$, where G is the distribution of the second-lowest of S independent draws from F, with associated density $SF(x)dP_{S-1}(x)$ for $x \in [c_L, c_H]$.

its reduced from is feasible.[13] Thus, for $b \geq b_H$, analogous to the large normalized budget case, we can focus on the relaxed problem of

$$\max_Q \int_{c_L}^{c_H} w(t)Q(t)dF(t),$$

subject to the demand constraints that for all $x \in [c_L, c_H]$,

$$\int_{c_L}^{x} Q(t)dF(t) \leq \frac{1}{S}D\ P_S(x). \tag{18}$$

Under the assumptions of Theorem 1, $w(c_H) \geq 0$ and $-w'(x) \geq 0$, which means that any Q that satisfies (18) also satisfies

$$w(c_H)\int_{c_L}^{c_H} Q(t)dF(t) \leq w(c_H)\frac{1}{S}D$$

and

$$\int_{c_L}^{c_H} (-w'(x))\int_{c_L}^{x} Q(t)dF(t)dx \leq \int_{c_L}^{c_H} (-w'(x))\frac{1}{S}DP_S(x)dx.$$

Summing these inequalities, we get the following aggregate constraint:

$$w(c_H)\int_{c_L}^{c_H} Q(t)dF(t) + \int_{c_L}^{c_H} (-w'(x))\int_{c_L}^{x} Q(t)dF(t)dx \leq \frac{w(c_H)D}{S} + \int_{c_L}^{c_H} \frac{-w'(x)DP_S(x)}{S}dx,$$

which we can rearrange using integration by parts to get

$$\int_{c_L}^{c_H} w(x)Q(x)dF(x) \leq \int_{c_L}^{c_H} w(x)\frac{D}{S}dP_S(x). \tag{19}$$

Because the left side of (19) is the buyer's objective, the right side of (19) provides an upper bound for the buyer's objective for all Q in a superset of the feasible set. Because $Q^{SPA}(x) = D(1-P_{S-1}(x))$ is feasible in the original problem and satisfies (19) with equality, it must be optimal.

This illustrates the method of proof that we use in the remainder of this section. The coefficients $w(c_H)$ and $\{-w'(x)\}_{x\in[c_L,c_H]}$ are Lagrange multipliers that yield a dual certificate of optimality.

Although we omit the details, an analogous approach can be used in the intermediate normalized budget case when b is sufficiently small. The SUPA in its ex post implementation is no longer feasible, but the reduced-form of the SUPA is feasible. Under the assumptions of Theorem 2, the reduced-form of the SUPA, which has allocation and payment rules defined

[13] Proposition 2(c) below shows that the reduced form of the SPA is feasible when $b \geq b_H$.

as, for each $x \in [c_L, c_H]$,

$$Q^{SUPA}(x) = \int_x^{c_H} \frac{B}{t} dP_{S-1}(t) \text{ and } M^{SUPA}(x) = B(1 - P_{S-1}(x)),$$

is optimal when $b \leq b_L$, where:[14]

$$b_L \equiv \left(\int_{c_L}^{c_H} \frac{1}{t} \, dP_{S-1}(t) \right)^{-1} \in (c_L, b_H). \tag{21}$$

The following theorem summarizes the discussion above.

Theorem 3 (Optimal mechanism for low and high normalized budget within the intermediate range with interim incentive constraints) *If regularity-*0 *holds,* $v \geq c_H + 1/f(c_H)$*, and* $b \geq b_H$*, then the reduced-form mechanism* (Q^{SPA}, M^{SPA}) *maximizes the buyer's expected surplus* (9)*, subject to* (1)–(5)*. Moreover, if regularity-*1 *holds,* $v \geq c_H + 1/(c_H f(c_H) - 1)$*, and* $b \leq b_L$*, then the reduced-form mechanism* (Q^{SUPA}, M^{SUPA}) *maximizes the buyer's expected surplus* (9)*, subject to* (1)–(5)*.*

We postpone discussing the implementation of the reduced-form SPA for $b \in [b_H, c_H)$ and of the reduced-form SUPA for $b \in (c_L, b_L]$ because the reduced-form mechanism that we define below for $b \in (b_L, b_H)$ also applies to the low and high ends of the intermediate range and so encompasses the reduced-forms of the SPA and SUPA.

We now apply this approach to the general case, including budget levels $b \in (b_L, b_H)$, in which case neither the reduced-form of the SPA nor the reduced-form of the SUPA is optimal.

4.3 Clipped reduced-form mechanism is optimal for the center of the intermediate range

Next we turn to the middle range (b_L, b_H) of the intermediate normalized budget range. To study this case, we first introduce a family of reduced-form mechanisms (parameterized by $b \in (b_L, b_H)$) for which we establish optimality. These mechanisms interpolate the reduced forms of the SPA and SUPA in an incentive compatible way by suitably clipping the reduced-form functions of the SPA and SUPA mechanisms.

[14] To see that $b_L \in (c_L, c_H)$, note that $b_L = E\,[1/X]^{-1}$, where $X \sim P_{S-1}$. To see that $b_L < b_H$, recall G as defined in footnote 12 and note that $G(t) < P_{S-1}(t)$ for $t \in (c_L, c_H)$, so G first-order stochastically dominates P_{S-1}. It then follows using Jensen's inequality that

$$b_L = E_{X \sim P_{S-1}}\,[1/X]^{-1} \leq E_{X \sim P_{S-1}}\,[X] \leq E_{X \sim G}\,[X] = b_H. \tag{20}$$

Define the clipped reduced-form (CRF) mechanism as:[15]

$$Q^{CRF}(x;c_b) \equiv \begin{cases} D(1-P_{S-1}(x)), & \text{if } x \in [c_L, c_b) \\ \int_x^{c_H} \frac{B}{t}\, dP_{S-1}(t), & \text{if } x \in [c_b, c_H] \end{cases} \tag{22}$$

and

$$M^{CRF}(x;c_b) \equiv \begin{cases} Dx(1-P_{S-1}(x)) + \int_x^{c_b} D(1-P_{S-1}(t))dt \\ \qquad +c_b \int_{c_b}^{c_H} B/t^2(1-P_{S-1}(t))dt, & \text{if } x \in [c_L, c_b) \\ B(1-P_{S-1}(x)), & \text{if } x \in [c_b, c_H], \end{cases} \tag{23}$$

where c_b is defined as the unique value in (c_L, c_H) such that

$$\int_{c_L}^{c_H} M^{CRF}(t;c_b)dF(t) = B/S. \tag{24}$$

Proposition 2 below establishes that given $b \in (b_L, b_H)$, c_b is well (and uniquely) defined. In what follows, we drop the argument c_b in Q^{CRF} and M^{CRF}.

We illustrate the relation between c_b and b in Figure 3(a). In Figure 3(b), we illustrate Q^{CRF} assuming two suppliers with uniformly distributed types and a normalized budget $b = (c_L + c_H)/2$, which for the chosen parameters satisfies $b \in (b_L, b_H)$. For comparison, Figure 3(b) also shows the reduced-form allocation rule for the SPA. Proposition 2 establishes basic features of the CRF mechanism, including its feasibility for the buyer's (interim) problem.

Proposition 2 (Features of the CRF mechanism) *In the case with* $b \in (b_L, b_H)$*:*
(a) $c_b \in (c_L, c_H)$ *is uniquely defined;*
(b) $\lim_{x\uparrow c_b} Q^{CRF}(x) > \lim_{x\downarrow c_b} Q^{CRF}(x)$ *and* $\lim_{x\uparrow c_b} M^{CRF}(x) > \lim_{x\downarrow c_b} M^{CRF}(x)$*;*
(c) (Q^{CRF}, M^{CRF}) *satisfies the constraints in* (16)*, including satisfying* IIR_{c_H} *with equality,*

[15] To see how M^{CRF} connects with Q^{CRF}, note that for $x \in [c_L, c_b)$, we can write $M^{CRF}(x)$ as

$$M^{CRF}(c_b) + \int_x^{c_b} tDP_{S-1}(t)dt + c_b \left(\lim_{y\uparrow c_b} Q^{CRF}(y) - Q^{CRF}(c_b) \right),$$

which can be rewritten as

$$B(1-P_{S-1}(c_b)) + \int_x^{c_b} tD\, dP_{S-1}(t) + c_b(1-P_{S-1}(c_b))D - c_b \int_{c_b}^{c_H} \frac{B}{t} dP_{S-1}(t).$$

Integrating by parts and rearranging gives the expression in the body of the paper.

$IIC_{x,x'}$ with equality for all $x, x' \in [c_L, c_H]$, D_x with equality for all $x \in [c_L, c_b]$, and B_x with equality for all $x \in [c_b, c_H]$.

Proof. See the Appendix.

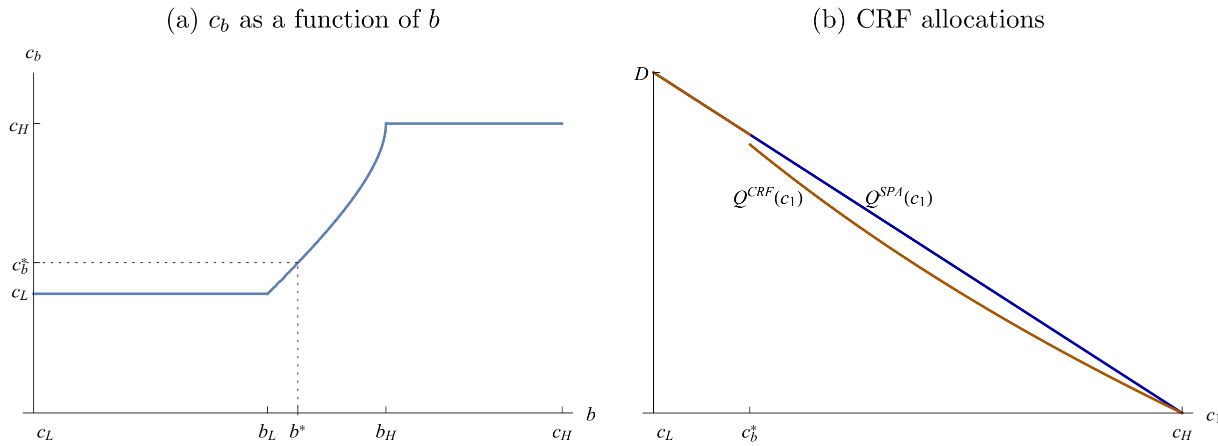

Figure 3: Panel (a): Type c_b as a function of b. Also shown is c_b^* corresponding to $b^* \equiv (c_L + c_H)/2$. Panel (b): Allocation for supplier 1 for the CRF mechanism when $b = b^*$ and, as a point of comparison, for the reduced-form of the SPA. Both panels assume two suppliers drawing costs from the uniform distribution on $[c_L, c_H]$.

A number of observations are in order regarding the CRF mechanism. Proposition 2(a) establishes that c_b is well defined. Proposition 2(b) establishes that Q^{CRF} and M^{CRF} jump down at c_b when $b \in (b_L, b_H)$. Figure 3(b) illustrates this feature of Q^{CRF}. As shown there, the allocation rule in the CRF mechanism is discontinuous at c_b when $b \in (b_L, b_H)$. (The payment rule in the CRF mechanism is correspondingly discontinuous at c_b.) This reflects the transition from having an expected quantity that is constrained by the buyer's demand to one that is constrained by a combination of incentive compatibility and the buyer's budget. The change in the marginal value of $\int_{c_L}^{x} Q^{CRF}(t)dF(t)$ across this transition translates into a discontinuity in Q^{CRF}. This is precisely the point at which we "clip" the reduced-form mechanism. Finally, Proposition 2(c) shows that the CRF mechanism is feasible in that it satisfies the interim incentive constraints, budget constraints, and demand constraints for $b \in (b_L, b_H)$.

In the remainder of this section, we discuss and state the optimality of the CRF mechanism. The proof is based on an approach similar to that used in Section 4.2. Specifically, we show that the CRF mechanism satisfies with equality an aggregate constraint that holds for a superset of the feasible set, thereby establishing the optimality of the mechanism. The key step in establishing the optimality of the CRF mechanism is to construct dual variables associated with the constraints in the relaxed problem in (16).

In order to state our regularity condition we define the following constant:

$$\Phi_b \equiv \frac{F(c_b)}{F(c_b) + c_b f(c_b)} \in [0, 1). \tag{25}$$

The factor Φ_b appears in the construction of the multipliers, which are defined in the proof of Theorem 4 in equations (31)–(33). The following regularity condition ensures that these multipliers are nonnegative.[16]

Definition 3 *Regularity-*2 *holds if regularity-*0 *and regularity-*1 *hold and for all* $x \in [c_L, c_H]$, $\frac{d}{dx}\left[\left(F(c_b)\Phi_b + f(x)x - F(x)\right)/\left(x^2 f(x)\right)\right] \leq 0.$

It is straightforward to show that regularity-2 is satisfied for a range of distributions and parameter values, including power distributions $F(x) = (x^a - c_L^a)/(c_H^a - c_L^a)$ for all $a \geq 1$, and Beta distributions with parameters $\alpha, \beta \geq 1$. Because regularity-2 encompasses both regularity-0 and regularity-1, it implies that $w'(x) \leq 0$ and $\psi'(x) \leq 0$.

Analogously to our analysis of the large and small normalized budget cases, we assume that v is sufficiently large that reserves based on v can be ignored. Also as before, such reserves could be incorporated, but at the cost of obscuring the effects of budget constraints, which are our primary interest. For the intermediate normalized budget case, we need v to be sufficiently large that the conditions of both Theorem 1 and 2 are satisfied. To reduce the repetition of notation in what follows, we define the following lower bound for v:

$$\underline{v} \equiv c_H + \max\left\{\frac{1}{c_H f(c_H) - 1},\ 1/f(c_H)\right\}. \tag{26}$$

As noted above, when $v \geq \underline{v}$, it follows that $w(c_H) \geq 0$ and $\psi(c_H) \geq 0$.

As in the argument described in Section 4.2, we use the dual variables to construct a certificate of optimality for the proposed mechanism. The following result formally states the optimality of the CRF mechanism. Because the CRF mechanism reduces to the reduced form of the SUPA for $b \leq b_L$ and to the reduced form of the SPA for $b \geq b_H$, the CRF is optimal for all values of the normalized budget.

Theorem 4 (Optimal mechanism for the intermediate normalized budget with interim incentive constraints) *If regularity-*2 *holds,* $v \geq \underline{v}$*, and* $b > 0$*, then the reduced-form mechanism* (Q^{CRF}, M^{CRF}) *maximizes the buyer's expected surplus* (9)*, subject to* (1)–(5)*.*

Proof. See the Appendix.

[16] More specifically, the multipliers associated with the demand constraints are positive on $[c_L, c_b)$ and zero on $(c_b, c_H]$, and the multipliers associated with the budget constraints are zero in $[c_L, c_b)$ and positive in $(c_b, c_H]$.

Theorem 4 establishes the optimality of the mechanism (Q^{CRF}, M^{CRF}) for the relaxed program. It follows from Border (1991) that an implementation of this mechanism exists that satisfies the interim incentive constraints of the original problem. However, unlike the results of Theorem 1 and 2, as we show in what follows, there is no implementation of the CRF that satisfies the ex post versions of the incentive constraints for the intermediate normalized budget case.

The SPA and SUPA cannot be used in the intermediate normalized budget case because they violate the budget constraints and demand constraints, respectively, for some type realizations. For $b \in (c_L, b_L]$, the CRF (which is equivalent in this case to the reduced form of the SUPA) can be implemented by a mechanism that trades only with the lowest-cost supplier, paying B for quantity $E_{y \sim P_{S-1}}[B/y \mid \mathbf{c}_{(1)} < y]$. However, this mechanism is not ex post incentive compatible. For the remaining portion of the intermediate normalized budget range $b \in (b_L, c_H)$, any implementation of the CRF mechanism is distinctly different from the SPA and SUPA in that it must pay a supplier other than the lowest-cost supplier for a positive measure set of type realizations.[17]

Proposition 3 (Optimality of paying non-lowest-cost suppliers) *If $b \in (b_L, c_H)$, any implementation of the CRF mechanism makes a positive payment to at least one supplier other than the lowest-cost supplier for a positive measure set of type realizations.*

Proof. See the Appendix.

We relegate additional discussion of the implementation of the CRF mechanism to the online appendix. As shown in the next section, Theorem 4 is sufficient to establish the gap between interim and ex post implementability in the intermediate budget case.

5 Intermediate normalized budget with ex post incentive constraints

In this section, we analyze the intermediate normalized budget case, $b \in (c_L, c_H)$, subject to ex post incentive compatibility and individual rationality. As shown in Section 3, for the large normalized budget case with $b \geq c_H$ and the small normalized budget case with $b \leq c_L$, the SPA and SUPA satisfy both interim and ex post versions of the constraints and are optimal mechanisms, respectively. As anticipated in Table 1, here we introduce the partial

[17] For $b \in (b_L, b_H)$ the proof follows from Proposition 2(b), which shows that $M^{CRF}(x)$ is strictly greater than $B(1 - P_{S-1}(c_b))$ for x slightly less than c_b, with the resulting implication that a payment rule that never pays suppliers other than the lowest-cost supplier cannot both replicate $M^{CRF}(x)$ for x slightly less than c_b and respect the budget constraint. The proof for $b \in [b_H, c_H)$ shows that such a mechanism must violate the budget constraint for types close to c_H.

dual sourcing auction (PDSA), a class of ex post incentive compatible mechanisms that is optimal when the buyer faces two suppliers, but might be suboptimal if there are three or more suppliers. Nonetheless, it strictly improves upon the SUPA-SPA mechanism described earlier. (The proof of the nonoptimality of the SUPA-SPA in Theorem 1 is completed via the family of PDSA.)

Define a mechanism family called a partial dual sourcing auction (PDSA), parametrized by $H \in [b, c_H]$ and defined in terms of $H_{\mathbf{c}} \equiv \max\{b, \min\{H, \mathbf{c}_{(3)}\}\} \in [b, c_H]$ and $h_{\mathbf{c}} \equiv bH_{\mathbf{c}}/(2H_{\mathbf{c}} - b) \in [c_L, b]$, as follows:

$$(q^{PDSA}(c_s, c_{-s}), m^{PDSA}(c_s, c_{-s})) \equiv \begin{cases} (B/\mathbf{c}_{(2)}, B), & \text{if } c_s = \mathbf{c}_{(1)} \text{ and } H < \mathbf{c}_{(2)} \\ (B/(2H_{\mathbf{c}}), B/2), & \text{if } h_{\mathbf{c}} \leq \mathbf{c}_{(1)} \leq c_s \leq \mathbf{c}_{(2)} < H_{\mathbf{c}} \\ (D, B), & \text{if } c_s = \mathbf{c}_{(1)} < h_{\mathbf{c}} \text{ and } h_{\mathbf{c}} \leq \mathbf{c}_{(2)} \leq H_{\mathbf{c}} \\ (D, \mathbf{c}_{(2)}D), & \text{if } c_s = \mathbf{c}_{(1)} \text{ and } \mathbf{c}_{(2)} \leq h_{\mathbf{c}} \\ (0, 0), & \text{otherwise.} \end{cases}$$

This family includes interesting mechanisms. For example, if we set $H = b$, then $h_{\mathbf{c}} = b$ for every realization of costs, and we recover the SUPA-SPA mechanism. Under our regularity conditions, we can optimize over $H \in [b, c_H]$ for the case of two suppliers by setting $H = \min\{c_H, \hat{H}\}$, where $\hat{H}$ is implicitly defined by

$$\psi(b\hat{H}/(2\hat{H} - b)) - \psi(\hat{H}) = \frac{(v - b\hat{H}/(2\hat{H} - b))F(b\hat{H}/(2\hat{H} - b))}{\hat{H}^2 f(\hat{H})}.$$

Moreover, in the case of two suppliers, we set $\mathbf{c}_{(3)} = \infty$ so that $H_{\mathbf{c}} = \max\{b, H\} = H$ is independent of the type realizations.

Figure 4 illustrates the mechanism and also makes clear why we refer to it as a partial dual sourcing auction. As shown in Figure 4, for some regions of the type space, namely for $(c_1, c_2) \in [h, H]^2$, the buyer dual sources, splitting the budget between the two suppliers. In other regions, the buyer single sources, purchasing only from the lowest-cost supplier.

We establish the feasibility of the PDSA in the following proposition:

Proposition 4 (Feasibility of the PDSA with ex post incentive constraints) *For any $b > 0$, the PDSA is feasible for the buyer's problem and satisfies ex post incentive compatibility and ex post individual rationality,* (7) *and* (8).

Proof. See the Appendix.

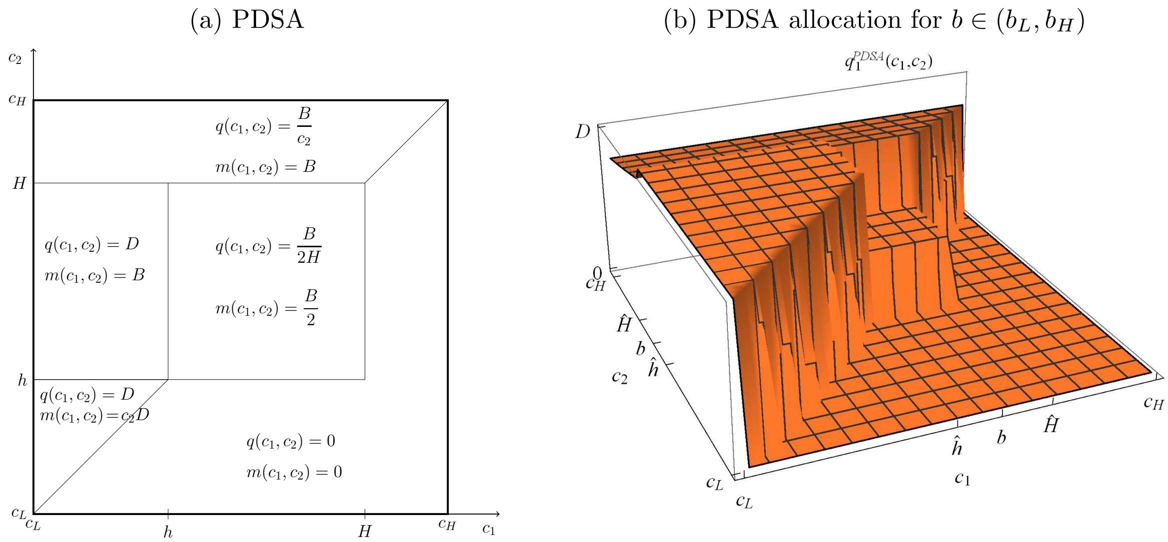

Figure 4: PDSA for the case of two suppliers. Panel (b) assumes costs drawn from the uniform distribution on $[c_L, c_H]$.

Applying the same proof strategy as before, we construct the required multipliers. However, in the ex post formulation, the multipliers are functions with domain $[c_L, c_H]^S$. This is in contrast to the reduced-form formulations, which allowed us to work with multipliers that were functions with domain $[c_L, c_H]$.

In the following theorem, we establish revenue performance guarantees for the PDSA for general numbers of suppliers and optimality for the case of two suppliers under ex post incentive constraints.

Theorem 5 (PDSA performance guarantees for the intermediate normalized budget with ex post incentive constraints) *Assume that regularity-*1 *holds and that* $v \geq c_H + \frac{1}{f(c_H)c_H - 1}$. *In the case with* $b \in (c_L, c_H)$, *there is a choice of* H *such that the PDSA with parameter* H *achieves strictly greater expected buyer surplus than the SUPA-SPA. Moreover, in the case of two suppliers, if regularity-*2 *holds and* $v \geq \underline{v}$, *then the mechanism* (q^{PDSA}, m^{PDSA}) *with* $H = \min\{c_H, \hat{H}\}$ *maximizes the buyer's expected surplus* (9), *subject to* (1)–(3) *and* (7)–(8).

Proof. See the Appendix.

Theorem 5, when contrasted with Theorem 4, shows that the optimal ex post and interim mechanisms differ for the intermediate normalized budget case. Further, it is straightforward to show numerically that the buyer's expected payoff under the CRF mechanism is greater than its expected payoff under the PDSA with $H = \min\{c_H, \hat{H}\}$ for a range of distributions and parameters, giving us the following result.

Corollary 1 *If regularity-*2 *holds,* $v \geq \underline{v}$, *and* $b \in (c_L, c_H)$, *in the case of two suppliers, the buyer's maximized expected surplus subject to the interim incentive constraints is weakly* *greater (and strictly greater for a range of distributions and parameters) than the buyer's maximized expected surplus subject to the ex post incentive constraints.*

5.1 Dynamic implementation of the ex post mechanisms

It is well known that the SPA has a dynamic implementation in the form of a descending clock auction. The clock price decreases from a level sufficiently high to ensure that all bidders are active, and the bidders choose at what price to exit, where exit is irreversible. The clock stops when only one active bidder remains. That bidder wins and supplies the buyer's entire demand D and receives a per-unit payment equal to the final clock price, which in equilibrium is the second-lowest unit cost.

It is straightforward to show that the SUPA, which is optimal for the case of a sufficiently small normalized budget, has a similar dynamic implementation. A descending clock can also be used, but the final remaining active bidder is paid the entire budget B and supplies a quantity equal to B divided by the final clock price. In equilibrium, the lowest-cost supplier sells all that the buyer can afford with a budget B at a unit price equal to the second-lowest cost.

It is perhaps less obvious that the PDSA mechanism has a straightforward dynamic implementation. We focus on the case of two suppliers. In that case, once again a descending clock price can be used.

To define the dynamic implementation of the PDSA, we first define a more general class of dynamic mechanisms that includes the SPA and SUPA. We focus on the case of two bidders. Define a sequential auction with promises (SAP) with parameters p_0 and p_1 as follows: The clock price declines until there is an exit. If the first bidder to exit exits at a price greater than or equal to p_0, then the remaining bidder supplies quantity B divided by the final clock price and is paid B. If a bidder exits at a clock price less than p_0 but greater than p_1, then the clock continues until either the other bidder exits or the clock price reaches p_1. If the other bidder exits prior to the clock price reaching p_1, then each bidder supplies quantity $B/(2p_0)$ and is paid $B/2$, but if the clock price reaches p_1, then the remaining active bidder supplies quantity D and is paid B. Finally, if no bidder exits until the clock price is less than or equal to p_1, then the final active bidder supplies quantity D and is paid D multiplied by the final clock price.

The SAP with parameters $p_0 = p_1 = b$ implements the SUPA-SPA. The SAP with parameters $p_0 = H$ and $p_1 = h$ implements the PDSA, giving us the following result.

Proposition 5 (Dynamic implementation of the PDSA) *In the intermediate normalized budget case with $b \in (c_L, c_H)$, the PDSA has a dynamic implementation that preserves ex post incentive compatibility and ex post individually rationality.*

6 Discussion

6.1 Numerical illustration

In this section, we illustrate the mechanisms developed here based on simulations with parameters calibrated based on data from the Brazilian Exchange System. We consider data from auctions realized in 2015 to procure water for injection ("Água para injećão"). We collected bids from two suppliers that participated in procurement auctions (Cirúrgica São José Ltda. and Farmace Industria Quimico-Farmaceutica Cearense Lt.). During that year, these two bidders were regularly the only two suppliers of this specific good. Quantities purchased in an individual procurement ranged from 1000 to 8000 units, with an average price of 1.77 BRL per unit.

In the simulations below, we use $c_L = 1.5$ BRL and $c_H = 2.3$ BRL. Based on a kernel parameter, we estimate a probability density function f around which we build our simulations. For simplicity, we set the buyer's value of one unit to $v = 10$ BRL which can be viewed as the price to obtain such water for injection on short notice. Figure 5 displays the empirically calibrated density function.

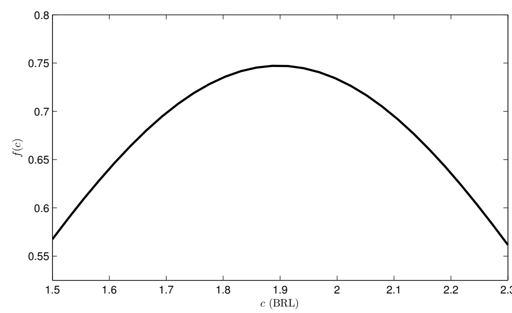

Figure 5: Probability density function calibrated based on the data from procurement auctions for water for injection in the 2015 Brazilian Exchange System.

Next, we provide results to compare various mechanisms as we vary the budget across different regimes. We set the demand at $D = 5000$ units and consider the budget values of $B = Dc_L$ (small), $B = D(c_L + b_L)/2$ (intermediate, $b \leq b_L$), $B = D(b_L + b_H)/2$ (intermediate, $b_L \leq b \leq b_H$), $B = D(b_H + c_H)/2$ (intermediate, $b \geq b_H$), and $B = Dc_H$ (large).[18] The Brazilian government's policy (BGP) can be interpreted as an SPA with a reservation price that handles the budget constraint by restricting trade to types for which it can afford the full demand.[19] We compare this mechanism with the mechanisms proposed here: SUPA-SPA, CRF, and PDSA.

[18] We note that the values of b_L and b_H vary with the number of suppliers S.

[19] In practice, the reservation price arises from a renegotiation phase involving the Brazilian government and the winner of a descending auction phase of the procurement.

Our results are depicted in Table 2 (to be completed).

	S	Incentive constraints	Normalized budget b $b \leq c_L$	$c_L < b < b_L$	$b_L \leq b \leq b_H$	$b_H < b < c_H$	$c_H \leq b$
BGP	2	ex post					
SUPA-SPA	2	ex post					
PDSA	2	ex post					
CRF	2	interim					
BGP	3	ex post					
SUPA-SPA	3	ex post					
PDSA	3	ex post					
CRF	3	interim					
BGP	5	ex post					
SUPA-SPA	5	ex post					
PDSA	5	ex post					
CRF	5	interim					

Table 2: Comparison of mechanisms: BGP (Brazilian government procurement), SUPA-SPA, PDSA, and CRF.

6.2 Single supplier case

Up to now, we have considered the case of $S \geq 2$ suppliers. In that case. suppliers place competitive pressure on each other, which is critical for all the mechanisms considered so far. Next we consider the case of a single supplier.

Formally, with one supplier, the buyer's problem is

$$\begin{aligned} &\max_{q,m} \int_{c_L}^{c_H} \left(v\ q\left(x\right) - m\left(x\right) \right) dF\left(x\right) \\ &\text{s.t.} \quad 0 \leq q\left(x\right) \leq 1 && (\mathcal{D}_x) \quad \forall x \in [c_L, c_H]; \\ &\qquad m\left(x\right) \leq B && (\mathcal{B}_x) \quad \forall x \in [c_L, c_H]; \\ &\qquad m\left(x\right) - xq\left(x\right) \geq 0 && (IR_x) \quad \forall x \in [c_L, c_H]; \\ &\qquad m\left(x\right) - x\ q\left(x\right) \geq m\left(x'\right) - x\ q\left(x'\right) && (IC_{x,x'}) \quad \forall x, x' \in [c_L, c_H]. \end{aligned} \tag{27}$$

The next theorem provides the optimal mechanism for the single supplier case. As expected, it depends on the normalized budget value b relative to c_H.

Theorem 6 (Optimal mechanism for the single supplier case) *Assume $S = 1$. If $b > c_H$*

and for all $x \in [c_L, c_H]$, $w(x) \geq 0$, then the optimal mechanism is, for $x \in [c_L, c_H]$,

$$q_H^1(x) \equiv D \quad \text{and} \quad m_H^1(x) \equiv c_H D. \tag{28}$$

If $b < c_H$ and $\psi(x) \geq 0$, then the optimal mechanism is, for all $x \in [c_L, c_H]$,

$$q_L^1(x) \equiv \frac{B}{c_H} \quad \text{and} \quad m_L^1(x) \equiv B. \tag{29}$$

Theorem 6 establishes that for a large normalized budget, we recover the classical solution when a budget constraint is not present. When the normalized budget satisfies $b < c_H$, it is optimal to restrict the quantity rather than restrict trade with high-cost types. The optimal mechanism does not attempt to further screen types by offering a menu with two of more quantity levels.

6.3 Dual multipliers and implementation via regularization

In this section, we discuss an approach via regularization to obtain dual multipliers as well as implementations of the reduced-form mechanisms.
(to be completed)

7 Conclusion

We study the optimal procurement mechanisms for a buyer that has a fixed budget and wishes to purchase units of a homogenous product up to a maximum demand amount from suppliers whose constant marginal costs are their private information.

As we discuss, the case of a large normalized budget, where the budget constraint never binds, is well understood. A well-known application of mechanism design theory shows that a second-price auction is optimal—the buyer satisfies its demand from the supplier with the lowest cost at a unit price equal to the second-lowest cost. As we show, the case of a small normalized budget, where the budget constraint always binds, can be solved with analogous techniques. The buyer purchases as much as it can afford from the supplier with the lowest cost at a unit price equal to the second-lowest cost. We refer to this mechanism as a second-unit-price auction. In contrast to the large and small normalized budget cases, the case of an intermediate normalized budget presents new challenges. Because both demand and budget constraints bind in a positive measure region of the type space, standard Myersonian techniques that allow the mechanism design problem to be solved pointwise do not apply.

Using techniques based on duality theory, we identify the optimal reduced-form mechanism and an associated implementation. As we show, and in contrast to the large and small normalized budget cases, in the intermediate normalized budget case, the optimal reduced-form mechanism does not satisfy ex post incentive compatibility and individual rationality. However, we are able to identify the optimal ex post mechanism for the case of two suppliers and show that it has a straightforward dynamic implementation.

A Appendix: Proofs

A.1 Proofs for Section 3.3

Proof of Proposition 1. Let $\mathbf{c}_{(3)}$ denote the third-lowest cost, define $\hat{b} \equiv (c_H + b)/2$, and consider the following class of mechanisms parametrized by $H \in [b, \hat{b})$: if $\mathbf{c}_{(3)} < \hat{b}$, then q and m are as in the SUPA-SPA mechanism defined in (15); if $\mathbf{c}_{(3)} > \hat{b}$, then the two lowest-cost suppliers are treated as in the PDSA with parameter H. Note that because $\mathbf{c}_{(3)} > \hat{b} > H$, this mechanism has a dual sourcing piece that is independent of $\mathbf{c}_{(3)}$.

For $\mathbf{c}_{(3)} < \hat{b}$, the mechanism agrees with the SUPA-SPA. Therefore, it suffices to consider the type profiles in the set $\{c \in [c_L, c_H]^S \mid \mathbf{c}_{(3)} > \hat{b}\}$. For any $x \leq y \leq c_H$, define

$$\Psi(x, y) \equiv \psi(x) f(x) f(y) \int_{[\max\{y, \hat{b}\}, c_H]^{S-2}} \prod_{s=3}^{S} dF_s(c_s),$$

where the integral represents the probability that $\mathbf{c}_{(3)}$ is above $\max\{y, \hat{b}\}$. The buyer's expected surplus generated by this mechanism with parameter H, conditional on $\hat{b} < \mathbf{c}_{(3)}$, denoted $V(H)$, can be written as

$$\begin{aligned} V(H) = \; & B \int_H^{c_H} \int_{c_L}^{y} \Psi(x, y)\, dx\, dy + B \int_h^H \int_{c_L}^{h} \Psi(x, y)\ dx\ dy \\ & + \frac{B}{2} \int_h^H \int_h^H \Psi(x, y)\, dx\, dy + D \int_{c_L}^{h} y \int_{c_L}^{y} \Psi(x, y)\ dx\ dy. \end{aligned}$$

The first derivative is

$$\begin{aligned} V'(H) \;\; = \;\; & -\frac{B}{2} \int_h^H \Psi(h, y) dy \cdot \left| \frac{dh}{dH} \right| + \int_h^H \Psi(H, y) dy \\ & -\frac{B}{2} \int_h^H \Psi(x, h) dx + \frac{B}{2} \int_h^H \Psi(x, h) dx \cdot \left| \frac{dh}{dH} \right| \\ & + (B - hD) \int_{c_L}^{h} \Psi(x, h) dx \cdot \left| \frac{dh}{dH} \right|. \end{aligned}$$

At $h = H = b$, we have $V'(b) = 0$ because the first four integrals are equal to zero and the last term is also zero ($B - bD = 0$). The second derivative (ignoring all integral terms, which vanish at $h = H = b$) is

$$\begin{aligned} V''(H) &= -\frac{B}{2}\Psi(h,H) - \frac{B}{2}\Psi(h,h)\left|\frac{dh}{dH}\right| + \frac{B}{2}\Psi(H,H) + \frac{B}{2}\Psi(H,h) \\ &\quad -\frac{B}{2}\Psi(H,H) - \frac{B}{2}\Psi(h,H) + \frac{B}{2}\Psi(H,h) + \frac{B}{2}\Psi(h,h) \\ &\quad +\left|\frac{dh}{dH}\right| D\int_{c_L}^{h}\Psi(x,h)dx. \end{aligned}$$

Because $h = bH/(2H-b)$, $H = b$ implies $h = b$, and we have $\frac{dh}{dH} = -b^2/(2H-b)^2$ so that, at $H = b$, we have $\left|\frac{dh}{dH}\right| = 1$. Thus,

$$V''(b) = D\int_{c_L}^{b}\Psi(x,b)dx > 0.$$

Because $b \in (c_L, c_H)$ and $\psi(x) > 0$, it follows that the buyer's expected surplus increases as H increases above b, establishing that the hybrid mechanism $(\hat{q}, \hat{m})$ cannot be optimal for the buyer's problem. ■

A.2 Proofs for Section 4.3

Proof of Proposition 2. We begin the proof by stating and proving a lemma that provides properties of $\int_{c_L}^{c_H} M^{CRF}(t;c)dF(t)$.

Lemma A.1 *Letting* $\mu(c) \equiv \int_{c_L}^{c_H} M^{CRF}(t;c)dF(t)$, *it follows that*
(a) $\mu'(c) = X(c)\frac{d}{dc}[cF(c)]$, *where* $X(c) \equiv D(1 - P_{S-1}(c)) - \int_c^{c_H}\frac{B}{t}dP_{S-1}(t)$;
(b) $\mu''(c) = (-D + B/c)P'_{S-1}(c) + X(c)\frac{d^2}{dc^2}[cF(c)]$;
(c) $\mu'(c) = 0$ *implies* $\mu''(c) > 0$, *so that* μ *is quasiconvex;*
(d) *if* $b \leq b_L$, *then* $\mu(c) = B/S$ *only if* $c = c_L$;
(e) *if* $b \geq b_H$, *then* $\mu(c_H) \leq B/S$.

Proof of Lemma A.1. To show part (a), we write out the expression for $\mu(c)$. By definition (23), M^{CRF} is piecewise differentiable. Using integration by parts on each differentiable piece (to account for possible discontinuities),

$$\int_{c_L}^{c_H} M^{CRF}(x)\,dF(x) = \int_{c_L}^{c} M^{CRF}(x)\,dF(x) + \int_{c}^{c_H} M^{CRF}(x)\,dF(x)$$

$$= \lim_{x\uparrow c} M^{CRF}(x)F(c) - \int_{c_L}^{c} M^{CRF\prime}(x)\,F(x)\,dx - \lim_{x\downarrow c} M^{CRF}(x)F(c) - \int_{c}^{c_H} M^{CRF\prime}(x)\,F(x)\,dx$$

$$= \left(Dc(1-P_{S-1}(c)) - c\int_{c}^{c_H}\frac{B}{y}dP_{S-1}(y)\right)F(c) - \int_{c_L}^{c_H} M^{CRF\prime}(x)\,F(x)\,dx$$

$$= cF(c)\left(D(1-P_{S-1}(c)) - \int_{c}^{c_H}\frac{B}{y}dP_{S-1}(y)\right) + \int_{c_L}^{c} DxF(x)dP_{S-1}(x) + \int_{c}^{c_H} BF(x)dP_{S-1}(x).$$

Using the definition of $X(c)$ and explicitly indicating the dependence of M^{CRF} on c, we can write this as

$$\int_{c_L}^{c_H} M^{CRF}(x;c)\,dF(x) = cF(c)X(c) + \int_{c_L}^{c} DxF(x)\;dP_{S-1}(x) + \int_{c}^{c_H} BF(x)\,dP_{S-1}(x).$$

Taking the derivative with respect to c, we have

$$\frac{d}{dc}\int_{c_L}^{c_H} M^{CRF}(t;c)dF(t)$$

$$= X(c)\frac{d}{dc}[cF(c)] + cF(c)\left(-DP'_{S-1}(c) + \frac{B}{c}P'_{S-1}(c)\right) + DcP'_{S-1}(c)F(c) - BP'_{S-1}(c)F(c)$$

$$= X(c)\frac{d}{dc}[cF(c)].$$

Part (b) follows simply from differentiating $X(c)\frac{d}{dc}[cF(c)]$.

To show part (c), note that by part (a), $\mu'(c) = 0$ only if $X(c) = 0$. Therefore, for any c such that $\mu'(c) = 0$, it follows from part (b) that $\mu''(c) = (-D + B/c)P'_{S-1}(c)$. Moreover, because $c \in (c_L, c_H)$ and $X(c) = 0$, we have

$$D(1-P_{S-1}(c)) = \int_{c}^{c_H}\frac{B}{t}dP_{S-1}(t) < \frac{B}{c}(1-P_{S-1}(c)),$$

and so $D < B/c$, which implies that $\mu''(c) = (-D + B/c)P'_{S-1}(c) > 0$.

To show part (d), suppose that $b \le b_L$. By part (a), $\mu'(c) = X(c)\frac{d}{dc}[cF(c)]$. Because $\frac{d}{dc}[cF(c)] = F(c) + cf(c) > 0$, under $c > 0$ and $f(c) > 0$ for any $c \in [c_L, c_H]$, it follows that the sign of $\mu'(c)$ is equal to the sign of $X(c)$. Using that $b \le b_L$, we show that $\mu'(c) < 0$ for every $c \in (c_L, c_H)$ and $\mu'(c) = 0$ for $c = c_L$. Because $\mu(c)$ is continuous and $\mu(c_L) = B/S$, this implies that $c_b = c_L$. Because the sign of $\mu'(c)$ is equal to the sign of $X(c)$ and $b \leq b_L$,

$$X(c_L) = D - \int_{c_L}^{c_H} \frac{B}{y} dP_{S-1}(y) = D - B/b_L \geq D - B/b = 0.$$

Then, using that $b \leq b_L$, we can write

$$\begin{aligned} X(c) &= D(1 - P_{S-1}(c)) - \textstyle\int_c^{c_H} \frac{B}{t} dP_{S-1}(t) \\ &\geq \tfrac{B}{b_L}(1 - P_{S-1}(c)) - B \textstyle\int_c^{c_H} \frac{1}{t} dP_{S-1}(t) \\ &= B(1 - P_{S-1}(c)) \left(\textstyle\int_{c_L}^{c_H} \frac{1}{t} dP_{S-1}(t) - \int_c^{c_H} \frac{1}{t(1-P_{S-1}(c))} dP_{S-1}(t) \right) \\ &= B(1 - P_{S-1}(c))(\mathrm{E}_{t\sim P_{S-1}}[1/t] - \mathrm{E}_{t\sim P_{S-1}}[1/t \mid t \geq c]) \\ &> 0, \end{aligned}$$

where the final inequality uses $\mathrm{E}_{t\sim P_{S-1}}[1/t] > \mathrm{E}_{t\sim P_{S-1}}[1/t \mid t \geq c]$. (A weak inequality follows trivially because $1/t$ is larger in the domain that is being excluded. A strict inequality follows from the density $dG(t)$ being strictly positive in the support $[c_L, c_H]$. Therefore, $\mu'(c) > 0$ for any $c \in (c_L, c_H)$, which implies that $\mu(c) > B/S$ for any $c \in (c_L, c_H]$. Therefore, $c_b = c_L$.)

To show part (e), suppose that $b \geq b_H$ and note that

$$\mu(c_H) = D \int_{c_L}^{c_H} tF(t) dP_{S-1}(t) = D\frac{b_H}{S} \leq D\frac{b}{S} = \frac{B}{S},$$

which completes the proof. □

Continuation of the Proof of Proposition 2. In what follows, as in Lemma A.1, we use the definitions

$$\mu(c) \equiv \int_{c_L}^{c_H} M^{CRF}(t; c_H) dF(t)$$

and

$$X(c) \equiv D(1 - P_{S-1}(c)) - \int_c^{c_H} \frac{B}{t} dP_{S-1}(t).$$

Proof of Proposition 2(a). For $b \in (b_L, b_H)$,

$$\mu(c_H) = D \int_{c_L}^{c_H} tF(t) dP_{S-1}(t) = D\frac{b_H}{S} > D\frac{b}{S} = \frac{B}{S}.$$

By construction, $\mu(c_L) = B/S$. Further, by Lemma A.1(a), $\mu'(c) = X(c)\frac{d}{dc}[cF(c)]$. Because

$\frac{d}{dc}[cF(c)] = F(c) + cf(c) > 0$ and because $c > 0$ and $f(c) > 0$ for any $c \in [c_L, c_H]$, it follows that the sign of $\mu'(c)$ is equal to the sign of $X(c)$. Thus, $X(c_L) = D - B/b_L < 0$, so that $\mu'(c_L) < 0$. It then follows from $\mu(c_L) = B/S$ and the continuity of $\mu'(c)$, that $\mu(c) < B/S$ for all $c \in (c_L, c')$ for some $c' > c_L$. By the Intermediate Value Theorem, there exists $c_b \in (c_L, c_H)$ such that $\mu(c_b) = B/S$. Further, by Lemma A.1(c), μ is quasiconvex, and so c_b is uniquely defined and $\mu'(c_b) > 0$.

Proof of Proposition 2(b). By Proposition 2(a), $b \in (b_L, b_H)$ implies $c_b \in (c_L, c_H)$. For $x \in (c_b, c_H]$, $Q^{CRF}(x) = \int_x^{c_H} B/t \, dP_{S-1}(t)$, so it follows that

$$X(c_b) = \lim_{x \uparrow c_b} Q^{CRF}(x) - \lim_{x \downarrow c_b} Q^{CRF}(x).$$

As established in part (a), the sign of $X(c)$ is the sign of $\mu'(c)$ and $\mu'(c_b) > 0$. Thus, $X(c_b) > 0$, which implies that

$$\lim_{x \uparrow c_b} Q^{CRF}(x) = D(1 - P_{S-1}(c_b)) > Q^{CRF}(c_b) = \lim_{x \downarrow c_b} Q^{CRF}(x). \tag{30}$$

Inequality (30) implies a corresponding result for M^{CRF}:

$$\begin{aligned}
\lim_{x \uparrow c_b} M^{CRF}(x) &= B(1 - P_{S-1}(c_b)) + Dc_b(1 - P_{S-1}(c_b)) - c_b \lim_{x \downarrow c_b} Q^{CRF}(x) \\
&> B(1 - P_{S-1}(c_b)) + Dc_b(1 - P_{S-1}(c_b)) - c_b D(1 - P_{S-1}(c_b)) \\
&= B(1 - P_{S-1}(c_b)) \\
&= \lim_{x \downarrow c_b} M^{CRF}(x),
\end{aligned}$$

where the first equality uses the definitions of M^{CRF} and Q^{CRF}, the inequality uses (30), the second equality simplifies, and the third equality uses the definition of M^{CRF}.

Proof of Proposition 2(c). We begin with the demand constraints D_x. We show that for $x \in (c_b, c_H]$

$$\int_{c_L}^{x} Q^{CRF}(t) dF(t) \leq \frac{1}{S} D P_S(x).$$

Because the definition of Q^{CRF} implies that this is satisfied at $x \in [c_L, c_b]$, and because by

Proposition 2(b), Q^{CRF} jumps down relative to $D(1-P_{S-1}(x))$ at $x = c_b$,

$$D(1-P_{S-1}(c_b)) = \lim_{x\uparrow c_b} Q^{CRF}(x) > \lim_{x\downarrow c_b} Q^{CRF}(x) = \frac{B}{c_b}(1-P_{S-1}(c_b)) - \int_{c_b}^{c_H} \frac{B}{t^2}(1-P_{S-1}(t))dt,$$

it is sufficient to show that $Q^{CRF}(x)$ does not cross $D(1-P_{S-1}(x))$ for $x \in (c_b, c_H)$.

We proceed by way of contradiction. Suppose the existence of an $\hat{x} \in (c_b, c_H)$ such that $Q^{CRF}(\hat{x}) = D(1-P_{S-1}(\hat{x}))$. By the definition of $\hat{x} \in (c_b, c_H)$,

$$\frac{B}{\hat{x}}(1-P_{S-1}(\hat{x})) - \int_{\hat{x}}^{c_H} \frac{B}{t^2}(1-P_{S-1}(t))dt = D(1-P_{S-1}(\hat{x})) > 0,$$

which, because $\hat{x} < c_H$, implies that

$$(B/\hat{x} - D)\,(1-P_{S-1}(\hat{x})) = \int_{\hat{x}}^{c_H} \frac{B}{t^2}(1-P_{S-1}(t))dt > 0.$$

Therefore, we have $B/\hat{x} > D$ which implies $\hat{x} < b$. It follows then that

$$Q^{CRF\prime}(\hat{x}) = -(B/\hat{x})\ P'_{S-1}(\hat{x}) < -(B/b)\ P'_{S-1}(\hat{x}) = -D\ P'_{S-1}(\hat{x}) = \frac{d}{dx}[D(1-P_{S-1}(x))],$$

where the first equality uses the definition of Q^{CRF}, the inequality uses $\hat{x} < b$, and second equality uses $b = B/D$, and the final equality rearranges. This implies that $Q^{CRF}(x)$ crosses $D(1-P_{S-1}(x))$ from above at $x = \hat{x}$, which is a contradiction because $\lim_{x\downarrow c_b} Q^{CRF}(x) \leq D(1-P_{S-1}(c_b))$. Thus, we conclude that no such $\hat{x}$ exists, and so $Q^{CRF}(x)$ does not cross $D(1-P_{S-1}(x))$ for $x \in (c_b, c_H)$, completing the proof that the demand constraints are satisfied for $x \in (c_b, c_H]$.

It remains to show that the budget constraints are satisfied. The definitions of c_b and M^{CRF} guarantee that the budget constraints B_x are satisfied with equality for $x \in [c_b, c_H]$. The budget constraint is trivially satisfied with equality at $x = c_L$. It remains to show that for $x \in (c_L, c_b)$,

$$\int_{c_L}^{x} M^{CRF}(t; c_b)dF(t) \leq \frac{1}{S}B \cdot P_S(x).$$

Because this is satisfied at $x = c_L$ and at $x = c_b$, and because $M^{CRF}(x)$ jumps down relative to $B(1-P_{S-1}(x))$ at $x = c_b$ by Proposition 2(b) (i.e., $\lim_{x\uparrow c_b} M^{CRF}(x) > B(1-P_{S-1}(c_b))$), it is sufficient to show that $M^{CRF}(x)$ crosses $B(1-P_{S-1}(x))$ only once for $x \in (c_L, c_b)$. For this, it is sufficient to show that whenever they cross at $x \in (c_L, c_b)$, $M^{CRF}(x)$ crosses $B(1-P_{S-1}(x))$ from below. Suppose there exists $\hat{x} \in (c_L, c_b)$ such that $M^{CRF}(\hat{x}) = B(1-P_{S-1}(\hat{x}))$. Then, by the definition of M^{CRF},

$$D\hat{x}(1-P_{S-1}(\hat{x})) + \int_{\hat{x}}^{c_b} D(1-P_{S-1}(t))dt + c_b\int_{c_b}^{c_H} \frac{B}{t^2}(1-P_{S-1}(t))dt = B(1-P_{S-1}(\hat{x})),$$

which implies that

$$(B - D\hat{x})(1 - P_{S-1}(\hat{x})) = \int_{\hat{x}}^{c_b} D(1 - P_{S-1}(t))dt + c_b \int_{c_b}^{c_H} \frac{B}{t^2}(1 - P_{S-1}(t))dt,$$

which, using $\hat{x} < c_b$, implies that $B - D\hat{x} > 0$, i.e., $\hat{x} < b$. Then note that

$$M^{CRF\prime}(\hat{x}) = -D\hat{x}P'_{S-1}(\hat{x}) > -DbP'_{S-1}(\hat{x}) = -BP'_{S-1}(\hat{x}) = \frac{d}{dx}[B(1 - P_{S-1}(x))],$$

where the first equality uses the definition of M^{CRF}, the inequality uses $\hat{x} < b$, the second equality uses $b = B/D$, and the third equality rearranges. Thus, whenever $M^{CRF}(x)$ crosses $B(1 - P_{S-1}(x))$, it crosses from below, completing proof. ■

Proof of Theorem 4. We define three functions that will serve as our dual variables. First, we have λ as the dual variable for the incentive compatibility and individual rationality constraints:

$$\lambda(x) \equiv \begin{cases} \frac{v}{c_b} F(x)(1 - \Phi_b), & \text{if } c_L \leq x < c_b \\ \frac{v}{x} F(x)\left(1 - \frac{F(c_b)}{F(x)}\Phi_b\right), & \text{if } c_b \leq x \leq c_H. \end{cases} \tag{31}$$

Second, we have δ for the demand constraints:

$$\delta(x) \equiv \begin{cases} -\frac{v}{c_b}(1 - \Phi_b)\, w'(x), & \text{if } c_L \leq x < c_b \\ 0, & \text{if } c_b \leq x \leq c_H. \end{cases} \tag{32}$$

Third, we have β for the budget constraints:

$$\beta(x) \equiv \begin{cases} 0, & \text{if } c_L \leq x < c_b \\ -v\, F(c_b)\Phi_b \frac{d}{dx}\left(\frac{1}{x^2 f(x)}\right) - \psi'(x), & \text{if } c_b \leq x < c_H \\ \frac{v\, F(c_b)\Phi_b}{c_H^2\, f(c_H)} + \psi(c_H), & \text{if } x = c_H. \end{cases} \tag{33}$$

Next we state a technical result that under regularity-2 and for v sufficiently large, the functions w and ψ as well as the functions defined in (32)–(33) are nonnegative.

Lemma A.2 *If regularity-2 holds and $v \geq \underline{v}$, then for all $x \in [c_L, c_H]$, $w(x) \geq 0$, $\psi(x) \geq 0$, $\delta(x) \geq 0$, $\lambda(x) \geq 0$, and $\beta(x) \geq 0$.*

Proof of Lemma A.2. The results that w and ψ are nonnegative follow from the regularity assumption (specifically regularity-0 and regularity-1) and the lower bound on v. The result that δ is nonnegative holds because $Sv\phi_b > 0$ and $w'(x) \leq 0$ (by regularity-0). Now consider λ. Because $\phi_b > 0$, we have $\lambda(x) \geq 0$ for all $x \in [c_L, c_b]$. Also, λ is continuous at c_b, i.e.

$$\lim_{x \uparrow c_b} \lambda(x) - \lim_{x \downarrow c_b} \lambda(x) = S\frac{v}{c_b}(1-\Phi_b)\ F(c_b) - Sv\,\frac{F(c_b)(1-\Phi_b)}{c_b} = 0.$$

Thus, $\lim_{x \downarrow c_b} \lambda(x) > 0$, which implies that for all $c > c_b$, $F(x)(1-\Phi_b) > 0$, and so $\lambda(x) > 0$ for all $x \in (c_b, c_H]$. The result that $\beta(x) \geq 0$ follows from regularity-2 for $x \in [c_b, c_H)$ and from regularity-1 and the lower bound on v for $x = c_H$. $\square$

Continuation of the Proof of Theorem 4: Using the dual variables to construct an aggregate constraint. We now make use of the dual variables λ, δ, and β just defined by multiplying the constraints in (16) by these functions, using λ for the incentive compatibility and individual rationality constraints, δ for the demand constraints, and β for the budget constraints. We then sum these to form a single aggregate constraint. The details are as follows:

The incentive compatibility constraint $IIC_{x,x'}$, when applied to adjacent types, implies that for all $x \in [c_L, c_H]$, $dM(x) - x\,dQ(x) \leq 0$. Thus, multiplying by $\lambda(x)$ and integrating over $x \in [c_L, c_H]$, we have:[20]

$$\int_{c_L}^{c_H} \lambda(x)\left(dM(x) - x\,dQ(x)\right) \leq 0. \tag{34}$$

Using the result from Lemma A.2 that $\lambda(c_H) \geq 0$, the individual rationality constraint IIR_{c_H} implies

$$\lambda\left(c_H\right)\left(-M\left(c_H\right) + c_H\,Q\left(c_H\right)\right) \leq 0. \tag{35}$$

Taking the demand constraint D_x and multiplying by $\delta(x)$, which is nonnegative by Lemma A.2, and integrating over $x \in [c_L, c_b)$, we have

$$\int_{c_L}^{c_b} \delta(x)\left(\int_{c_L}^{x} q(t)dF(t)\right)dx \leq \frac{1}{S}D\int_{c_L}^{c_b} \delta(x)P(x)dx. \tag{36}$$

Similarly, taking the budget constraint B_x and multiplying by $\beta(x)$, which is nonnegative by Lemma A.2, and integrating over $x \in [c_b, c_H)$, we have

$$\int_{c_b}^{c_H} \beta(x)\int_{c_L}^{x} M\left(t\right)dF\left(t\right)dx \leq \frac{1}{S}B\int_{c_b}^{c_H} \beta(x)P(x)dx. \tag{37}$$

[20] We note that the set of dual variables associated with nonincreasing functions consists of $\{\mu : \int_{c_L}^{t} \mu(t)dt \geq 0, \forall t \in [c_L, c_H]\}$. The nonnegative multiplier λ trivially belongs to this set.

Focusing on the worst type of supplier, the budget constraint B_{c_H} implies that

$$\beta\left(c_H\right)\int_{c_L}^{c_H} M\left(t\right)dF\left(t\right) \leq \frac{1}{S}B\beta\left(c_H\right)P(c_H). \tag{38}$$

Summing all inequalities in (34)–(38) yields the following "aggregate constraint":

$$\begin{aligned}
&\int_{c_L}^{c_b}\delta(x)\left(\int_{c_L}^{x} q\left(t\right)dF\left(t\right)\right)dx + \int_{c_b}^{c_H}\beta(x)\int_{c_L}^{x} M\left(t\right)dF\left(t\right)dx + \beta\left(c_H\right)\int_{c_L}^{c_H} M\left(t\right)dF\left(t\right) \\
&-\int_{c_L}^{c_H}\lambda(x)x\,dQ(x) + \int_{c_L}^{c_H}\lambda(x)dM(x) + \lambda\left(c_H\right)\left[-M\left(c_H\right)+c_H\,Q\left(c_H\right)\right] \\
&\leq \tfrac{1}{S}D\int_{c_L}^{c_b}\delta(x)P(x)dx + \tfrac{1}{S}B\int_{c_b}^{c_H}\beta(x)P(x)dx + \tfrac{1}{S}B\beta(c_H)P(c_H).
\end{aligned} \tag{39}$$

Relating the aggregate constraint to the buyer's objective. We now relate the aggregate constraint (39) to the buyer's objective in (16), showing that the left side of (39) is equal to the buyer's objective. As shown in Lemma A.3, one can rearrange the left side of (39) to collect the terms that involve each of the variables $Q(x)$ for $x \in [c_L, c_b)$, $Q(x)$ for $x \in [c_b, c_H]$, $M(x)$ for $x \in [c_L, c_b)$, and $M(x)$ for $x \in [c_b, c_H]$. Then one can show that the coefficients on these variables have a simple form and that the left side of (39) is equal to the buyer's objective.

We begin by rearranging the left side of the aggregate constraint as shown in the following lemma:

Lemma A.3 *The left side of* (39) *can be rewritten as*

$$\int_{c_L}^{c_H}\left(vQ(x)-M(x)\right)dF\left(x\right).$$

Proof of Lemma A.3. We begin by gathering terms that involve $q(x)$ for $x \leq c_b$:

$$\begin{aligned}
&\int_{c_L}^{c_b}\delta(x)\left(\int_{c_L}^{x} q\left(t\right)dF\left(t\right)\right)dx - \int_{c_L}^{c_b}\lambda(x)x\,dq(x) \\
=\;&\int_{c_L}^{c_b}\left[f(x)\int_{x}^{c_b}\delta\left(z\right)dz\right]\,q(x)dx - \overbrace{\lambda\left(c_b\right)\;c_b\;q\left(c_b\right)}^{\text{will cancel}} + \int_{c_L}^{c_b}\frac{d}{dx}\left[x\lambda(x)\right]\,q(x)dx \\
=\;&\int_{c_L}^{c_b}\left[f(x)\int_{x}^{c_b}\delta\left(z\right)dz + \frac{d}{dx}\left[x\lambda(x)\right]\right]q(x)dx - \overbrace{\lambda\left(c_b\right)\;c_b\;q\left(c_b\right)}^{\text{will cancel}},
\end{aligned}$$

where the first equality uses $\lambda(c_L) = 0$.

Turning to terms involving $Q(x)$ for $x \geq c_b$, we have

$$-\int_{c_b}^{c_H} \lambda(x)x\, dQ(x) + \lambda(c_H)\, c_H\, Q(c_H) = \overbrace{\lambda(c_b)\, c_b Q(c_b)}^{\text{will cancel}} + \int_{c_b}^{c_H} \frac{d}{dx}\left[x\lambda(x)\right] Q(x)dx.$$

For $m(x)$ for $x \leq c_b$, we have

$$\begin{aligned} &\int_{c_b}^{c_H} \beta(x) \int_{c_L}^{c_b} m(t)\, dF(t)\, dx + \beta(c_H) \int_{c_L}^{c_b} m(t)\, dF(t) + \int_{c_L}^{c_b} \lambda(x)\, dm(x) \\ = \quad &\int_{c_L}^{c_b} \left[f(x) \left(\int_{c_b}^{c_H} \beta(t)\, dt + \beta(c_H)\right) - \lambda'(x)\right] m(x)dx + \overbrace{\lambda(c_b)\, m(c_b)}^{\text{will cancel}}, \end{aligned}$$

which again uses $\lambda(c_L) = 0$. Finally, for $M(x)$ with $x \geq c_b$, we have

$$\begin{aligned} &\int_{c_b}^{c_H} \beta(x) \int_{c_b}^{x} M(t)\, dF(t)\, dx + \beta(c_H) \int_{c_b}^{c_H} M(t)\, dF(t) \\ &+ \int_{c_b}^{c_H} \lambda(x) dM(x) + \lambda(c_H)\left[-M(c_H) + c_H\, Q(c_H)\right] \\ = \quad &\int_{c_b}^{c_H} f(x) \left(\int_{x}^{c_H} \beta(t)\, dt + \beta(c_H)\right) M(x)dx - \overbrace{\lambda(c_b)\, M(c_b)}^{\text{will cancel}} - \int_{c_b}^{c_H} \lambda'(x)M(x)dx. \end{aligned}$$

Summing these expressions, we obtain

$$\begin{aligned} &\int_{c_L}^{c_b} \left(f(x) \int_{x}^{c_b} \delta(z)\, dz + \tfrac{d}{dx}\left[x\lambda(x)\right]\right) Q(x)dx + \int_{c_b}^{c_H} \tfrac{d}{dx}\left[x\lambda(x)\right] Q(x)dx \\ &+ \int_{c_L}^{c_b} \left(f(x) \left(\int_{c_b}^{c_H} \beta(t)\, dt + \beta(c_H)\right) - \lambda'(x)\right) M(x)dx \\ &+ \int_{c_b}^{c_H} \left(f(x) \left(\int_{x}^{c_H} \beta(t)\, dt + \beta(c_H)\right) - \lambda'(x)\right) M(x)dx. \end{aligned} \tag{40}$$

Now we further simplify the expression of Lemma A.3. Starting with the first term in

(40), the integrand in that term is, for $x \in [c_L, c_b)$,

$$\begin{aligned}
f(x)\int_x^{c_b}\delta(z)\,dz + \frac{d}{dx}[x\lambda(x)] &= f(x)\int_x^{c_b}\left(-v\,\frac{(1-\Phi_b)}{c_b}w'(z)\right)dz + v\,\frac{(1-\Phi_b)}{c_b}\frac{d}{dx}[xF(x)] \\
&= -f(x)v\,\frac{(1-\Phi_b)}{c_b}(w(c_b)-w(x)) + v\,\frac{(1-\Phi_b)}{c_b}\frac{d}{dx}[xF(x)] \\
&= vf(x)\,\frac{(1-\Phi_b)}{c_b}\left(c_b + \frac{F(c_b)}{f(c_b)}\right) \\
&= vf(x),
\end{aligned}$$

where the first equality uses the definitions of δ and λ, the second equality integrates, the third equality uses the definition of w and simplifies, and the fourth equality uses the definition of $\frac{(1-\Phi_b)}{c_b}$. Similarly, referring to the integrand in the second term in (40), one can show using the definition of λ that for $x \in [c_b, c_H]$,

$$\frac{d}{dx}[x\lambda(x)] = \frac{d}{dx}[v\,(F(x) - F(c_b)\Phi_b)] = vf(x).$$

Turning to the remaining terms in (40), we show that the integrands in the third and fourth term are equal to $-f(x)$. Considering the third term of (40), the integrand is, for $x \in [c_L, c_b)$,

$$\begin{aligned}
& f(x)\left(\int_{c_b}^{c_H}\beta(t)\,dt + \beta(c_H)\right) - \lambda'(x) \\
=\ & -f(x)\int_{c_b}^{c_H}\left(v\,F(c_b)\Phi_b\,\frac{d}{dt}\left(\frac{1}{t^2 f(t)}\right) + \psi'(t)\right)dt + f(x)\beta(c_H) - \lambda'(x) \\
=\ & -f(x)\left(\frac{v\,F(c_b)\Phi_b}{c_H^2 f(c_H)} - \frac{v\,F(c_b)\Phi_b}{c_b^2 f(c_b)} + \psi(c_H) - \psi(c_b)\right) + f(x)\left(\frac{v\,F(c_b)\Phi_b}{c_H^2\ f(c_H)} + \psi(c_H)\right) \\
& -v\,\frac{(1-\Phi_b)}{c_b}f(x) \\
=\ & f(x)\left(\frac{v\,F(c_b)\Phi_b}{c_b^2 f(c_b)} + \psi(c_b) - v\,\frac{(1-\Phi_b)}{c_b}\right) \\
=\ & -f(x),
\end{aligned}$$

where the first equality uses the definition of β, the second equality integrates and uses the definitions of $\beta(c_H)$ and λ, the third equality simplifies, and the fourth equality uses the definitions of Φ_b and ψ. Similarly, the integrand in the fourth term in (40) is, for $x \in [c_b, c_H]$,

$$f(x)\left(\int_x^{c_H}\beta(t)\,dt + \beta(c_H)\right) - \lambda'(x) = -f(x),$$

where, relative to the previous case, the different definition of $\lambda(x)$ for $x \in [c_b, c_H]$ versus $x \in [c_L, c_b)$ combines with the different lower bound of integration (x versus c_b) to give the same result.

Substituting these expressions in for the integrands in (40), we can rewrite (40) as

$$\begin{aligned}&\int_{c_L}^{c_b} vQ(x)dF(x) + \int_{c_b}^{c_H} vQ(x)dF(x) - \int_{c_L}^{c_b} M(x)dF(x) - \int_{c_b}^{c_H} M(x)dF(x)\\ = &\int_{c_L}^{c_H} (vQ(x) - M(x))\, dF(x),\end{aligned}$$

which completes the proof. □

Continuation of the Proof of Theorem 4. Using Lemma A.3, the buyer's objective the aggregate constraint (39) can be written as

$$\begin{aligned}&S\int_{c_L}^{c_H} (vQ(x) - M(x))\, dF(x)\\ \leq\ &D\int_{c_L}^{c_b} \delta(x)P(x)dx + B\int_{c_b}^{c_H} \beta(x)P(x)dx + B\beta(c_H)P(c_H).\end{aligned} \tag{41}$$

Because the left side of (41) is the buyer's objective, if a candidate solution satisfies all of the constraints in the original problem and satisfies (41) with equality, then the candidate solution is optimal. We establish feasibility of the CRF mechanism in Proposition 2(c). To show that the CRF mechanism satisfies (41) with equality, recall from Proposition 2(c) that the CRF mechanism satisfies all constraints with equality except D_x for $x \in [c_b, c_H]$ and B_x for $x \in [c_L, c_b)$. But the multiplier on the demand constraints, $\delta(x)$, is zero for $x \in [c_b, c_H]$, and the multiplier on the budget constraints, $\beta(x)$, is zero for $x \in [c_L, c_b)$. Thus, the CRF mechanism satisfies (41) with equality. ■

Proof of Proposition 3. Assume $b \in (b_L, b_H)$, in which case $c_b \in (c_L, c_H)$. Let m be the payment rule in a feasible mechanism that implements the CRF mechanism and that for any type profile specifies a zero payment for any supplier that does not have the lowest cost. By Proposition 2(2), $\lim_{x\uparrow c_b} M^{CRF}(x) > B(1 - P_{S-1}(c_b))$. Thus, by the continuity of M^{CRF} below c_b and of F, there exists $\varepsilon > 0$ sufficiently small such that

$$M^{CRF}(c_b - \varepsilon) > B(1 - P_{S-1}(c_b - \varepsilon)). \tag{42}$$

Then we have the following contradiction:

$$\begin{aligned}\int_{[c_L,c_H]^{S-1}} m(c_b-\varepsilon,\mathbf{c}_{-s})dF_{-s}(\mathbf{c}_{-s}) &= M^{CRF}(c_b-\varepsilon)\\ &> B(1-P_{S-1}(c_b-\varepsilon))\\ &= \int_{[c_b-\varepsilon,c_H]^{S-1}} BdF_{-s}(\mathbf{c}_{-s})\\ &\geq \int_{[c_b-\varepsilon,c_H]^{S-1}} m(c_b-\varepsilon,\mathbf{c}_{-s})dF_{-s}(\mathbf{c}_{-s})\\ &= \int_{[c_L,c_H]^{S-1}} m(c_b-\varepsilon,\mathbf{c}_{-s})dF_{-s}(\mathbf{c}_{-s}),\end{aligned}$$

where the first equality uses the assumption that m implements M^{CRF}, the strict inequality uses (42), the second equality uses the Fundamental Theorem of Calculus, the weak inequality uses the assumption that m is feasible and so for all c_{-s}, $m(c_b-\varepsilon,c_{-s})\leq B$, and the final equality uses the assumption that payments are zero for a supplier without the lowest cost, so that for all c_{-s} with at least one component less than $c_b-\varepsilon$, $m(c_b-\varepsilon,c_{-s})=0$. Given this contradiction, it follows that any feasible mechanism that implements M^{CRF} specifies a positive payment for more that one supplier for at least some type realizations.

Now assume $b\in[b_H,c_H)$, in which case $c_b=c_H$ and there exists $\varepsilon>0$ such that $c_H-\varepsilon>b$, which implies that

$$(c_H-\varepsilon)D>B. \tag{43}$$

Then we have the following contradiction:

$$
\begin{aligned}
B(1-P_{S-1}(c_H-\varepsilon)) &\geq \int_{[c_H-\varepsilon,c_H]^{S-1}} m(c_H-\varepsilon,\mathbf{c}_{-s})dF_{-s}(\mathbf{c}_{-s}) \\
&= \int_{[c_L,c_H]^{S-1}} m(c_H-\varepsilon,\mathbf{c}_{-s})dF_{-s}(\mathbf{c}_{-s}) \\
&= M^{CRF}(c_H-\varepsilon) \\
&= \int_{c_H-\varepsilon}^{c_H} yDdP_{S-1}(y) \\
&> (c_H-\varepsilon)D(1-P_{S-1}(c_H-\varepsilon)) \\
&> B(1-P_{S-1}(c_H-\varepsilon)),
\end{aligned}
$$

where the first inequality uses the feasibility of m, the first equality uses the assumption that a supplier without the lowest cost receives zero payment, the second equality uses the assumption that m implements M^{CRF}, the third equality uses the definition of M^{CRF} when $c_b = c_H$, the second inequality replaces y with the lower bound of integration in one place and integrates, and the last inequality uses (43). So once again, any feasible mechanism that implements M^{CRF} specifies a positive payment for more that one supplier for at least some type realizations. ■

A.3 Proofs for Section 5

Proof of Proposition 4. It suffices to show that the PDSA satisfies ex post incentive compatibility (7), which is equivalent to $q^{PDSA}(\cdot, y)$ nonincreasing for each $y \in [c_L, c_H]$, and

$$\forall (x,y) \in [c_L,c_H]^2, \quad \int_x^{c_H} q^{PDSA}(t,y)dt \;=\; m^{PDSA}(x,y) - x \cdot q^{PDSA}(x,y).$$

In what follows, to conserve on notation, we drop the superscript "PDSA" on the mechanism. We have: for each $y \in (H, c_H]$,

$$
\begin{aligned}
x &\in [y,c_H], &\quad \int_x^{c_H} q(t,y)dt &= 0 = m(x,y) - x \cdot q(x,y) \\
x &\in [c_L,y], &\quad \int_x^{c_H} q(t,y)dt &= \int_x^{y} \frac{B}{y}dt = B - x\frac{B}{y} = m(x,y) - x \cdot q(x,y),
\end{aligned}
$$

and for each $y \in [h, H]$,

$$
\begin{aligned}
x \in [y, c_H], &\qquad \int_x^{c_H} q(t,y)dt = 0 = m(x,y) - x \cdot q(x,y) \\
x \in [h, H], &\qquad \int_x^{c_H} q(t,y)dt = \int_x^H \frac{B}{2H}dt = \frac{B}{2} - x\frac{B}{2H} = m(x,y) - x \cdot q(x,y) \\
x \in [c_L, h], &\qquad \int_x^{c_H} q(t,y)dt = \int_x^h Ddt + \int_h^H \frac{B}{2H}dt = B - xD = m(x,y) - x \cdot q(x,y),
\end{aligned}
$$

where the last equality is obtained using the relation between H and h, and for each $y \in [c_L, h]$,

$$
\begin{aligned}
x \in [y, c_H], &\qquad \int_x^{c_H} q(t,y)dt = 0 = m(x,y) - x \cdot q(x,y) \\
x \in [c_L, y], &\qquad \int_x^{c_H} q(t,y)dt = \int_x^y Ddt = yD - xD = m(x,y) - x \cdot q(x,y).
\end{aligned}
$$

Proof of Theorem 5. See the online appendix.

A.4 Proofs for Section 6.2

Proof of Theorem 6. Consider first the case with $c_H D \leq B$. Using the envelope theorem to eliminate all m variables, we can rewrite the buyer's problem as

$$
\begin{cases}
\underset{q}{\text{Max}} \quad \int_{c_L}^{c_H} w(x)\, q(x) \;\, dF(x) & \\
\text{s.t.} \quad 0 \leq q(x) \leq 1 & (\mathcal{D}_x) \quad \forall x \in [c_L, c_H] \\
\qquad\;\; xq(x) + \int_x^{c_H} q(t)\, dt \leq B & (\mathcal{B}_x) \quad \forall x \in [c_L, c_H]\,.
\end{cases}
\tag{44}
$$

Multiplying each demand constraint by $w(x)\, f(x)$ and integrating yields

$$
\int_{c_L}^{c_H} w(x)\, q(x) \;\, dF(x) \leq D \int_{c_L}^{c_H} w(x) \;\, dF(x)\,.
$$

The function q_H^1 defined in (28) maximizes the objective in (44), subject to the last inequality. Substituting q_H^1 into the left side of the budget constraints of (44) yields m_H^1. Thus q_H^1 also satisfies the budget constraints of (44). We can then conclude that q_H^1 solves the linear program in (44). Because (q_H^1, m_H^1) is feasible in the original buyer problem (27), it also solves it.

The proof for the case with $B \leq c_H D$ is similar. Using the envelope theorem to eliminate all m variables, we can rewrite the buyer's problem as

$$\begin{cases} \underset{m}{\text{Max}} & \int_{c_L}^{c_H} \psi(x)\, m(x)\ dF(x) \\ s.t.: & m(x) \leq B \qquad\qquad (\mathcal{B}_x) \quad \forall x \in [c_L, c_H] \\ & \frac{m(x)}{x} + \int_x^{c_H} \frac{1}{t^2} m(t)\, dt \leq D \qquad (\mathcal{D}_x) \quad \forall x \in [c_L, c_H]. \end{cases} \tag{45}$$

Multiplying each demand constraint by $\psi(x) f(x)$ and integrating yields

$$\int_{c_L}^{c_H} \psi(x)\, m(x)\ dF(x) \leq B \int_{c_L}^{c_H} \psi(x)\, dF(x).$$

The function m_L^1 defined in (29) maximizes the objective in (45), subject to the last inequality. Substituting m_L^1 into the left side of the demand constraints of (45) yields q_L^1. Thus m_L^1 also satisfies the demand constraints of (45). We can then conclude that m_L^1 solves the linear program in (45). Because (m_L^1, q_L^1) is feasible in the original buyer problem (27), it also solves it. ■

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