Demand Models for Differentiated Products with Complementarity and Substitutability*

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May 1, 2018

Abstract

We develop a class of demand models for differentiated products based on the concept of entropy. The new models facilitate the BLP method (Berry et al., 1995) by avoiding numerical inversion of the demand system. They accommodate rich patterns of substitution and complementarity while being easily estimated with standard regression techniques and allowing very large choice sets. We use the new models to describe markets for differentiated products that exhibit segmentation along several dimensions and illustrate their application by estimating the demand for cereals in Chicago.

Keywords. Demand estimation; Differentiated products; Discrete choice; Generalized entropy; Representative consumer.
JEL codes. C26, D11, D12, L.

*First Draft: March, 2016. We are grateful for comments from Moshe Ben-Akiva, Xavier D'Haultfoeuille, Robin Lindsey, Laurent Linnemer, Yurii Nesterov, Bernard Salanié, Thibaud Vergé, Sophie Dantan and Hugo Molina as well as participants at the conference on Advances in Discrete Choice Models in honor of Daniel McFadden at the University of Cergy-Pontoise, and seminars participants at the Tinbergen Institute, the University of Copenhagen, Northwestern University, Université de Montréal, CREST-LEI, KU Leuven and ENS Paris-Saclay. We have received financial support from the Danish Strategic Research Council, iCODE, University Paris-Saclay, and the ARN project (Elitisme). Mogens Fosgerau is supported by the ERC Advanced Grant GEM 740369. This paper is a continuation of the paper "Demand system for market shares" that was first circulated in 2016.

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1 Introduction

This paper develops a new class of discrete choice demand models that allow for rich patterns of substitution and complementarity. These models can be employed to estimate the demand for differentiated products using the famous BLP method (Berry et al., 1995) for dealing with endogeneity concerns. The models also avoid the need to invert the demand system numerically, and therefore may be estimated using standard instrumental variables regression techniques on very large choice sets.

This approach builds on new insights regarding the relationship between the additive random utility model (ARUM) and the representative consumer model (RCM). Since the seminal paper by McFadden (1974), the ARUM has been widely used in many fields of economics including industrial organization, game theory, experimental economics, and transportation. Crucially, the ARUM explicitly models the preferences of a single consumer and relies on the assumption that each consumer buys one unit of the product that provides her the highest utility. As a consequence, products can only be substitutes in an ARUM, not complements.[1]

Among other applications, the RCM has been a workhorse of the international trade literature since Dixit and Stiglitz (1977) and Krugman (1979). In contrast to the ARUM, the RCM assumes the existence of a variety-seeking representative consumer who aggregates a population of consumers. The representative consumer chooses some quantity of every product, trading off variety against quality. In the standard representative consumer approach, consumers are not restricted to buying just one unit of one product (see e.g., Hausman et al., 1994; Pinkse et al., 2002; Pinkse and Slade, 2004) and products can be substitutes or complements.

Despite their fundamental differences, the two approaches are closely linked. Anderson et al. (1992) show the existence of a RCM corresponding to any given ARUM. Anderson et al. (1988) and Verboven (1996b) explicitly derive the representative consumer's direct utility for the logit model and the nested logit model, respectively. These direct utilities are specified as the sum of a term that captures the utility derived from consumption in the absence of interaction among products, and a second term that expresses the representative consumer's taste for variety. The concept of entropy plays an important role in these

[1] In this paper, complementarity (resp., substitutability) is defined by a negative (resp., positive) cross derivative of demand with respect to product-specific characteristics or price. This is the standard definition of complementarity or substitutability (Samuelson, 1974). The definition we use is similar to the definition of Gentzkow (2007) based on random utility.

relationships: in the logit model the taste for variety is captured by the Shannon (1948) entropy, while in the nested logit model it is captured by a sum of Shannon entropies.

In this paper, we define the class of generalized entropy models (GEM) as RCM in which the taste for variety is captured by a generalization of the Shannon entropy. With the Shannon entropy, all products are treated symmetrically. Hence the identity of products does not matter and it is not possible to account for the degree of similarity between products. In contrast, our class of generalized entropies allows for general relationships in the taste for variety, enabling us to capture rich patterns of substitution and complementarity.

The class of GEM is large. We show that we can always find a GEM that leads to the same choice probabilities as any given ARUM. This improves on the existence result of Anderson et al. (1992) by providing a construction of the representative consumer's direct utility for any ARUM. Our class of GEM is actually strictly larger than the class of ARUM. Whereas ARUM rule out complementarity, we find that there are GEM which allow for complements. This is a very attractive feature as there are many market settings in which complementarity is likely to occur (see Berry et al., 2014, 2017).

In their seminal paper, Berry et al. (1995) provide a method for estimating the demand for differentiated products while accounting for price endogeneity due to the presence of an unobserved characteristics term, which is the structural error of the model. They propose a generalized method-of-moments (GMM) estimator, together with an estimation algorithm to compute it. To evaluate the BLP's GMM objective function, the demand system must be inverted to obtain the structural error as a function of the data and parameters. When this cannot be performed analytically, Berry et al. (1995) propose inverting numerically using a contraction mapping nested into the GMM minimization procedure, which must be performed each time the GMM objective function is evaluated.[2]

In contrast, with the GEM, we obtain the structural error term directly as a known function of the data and parameters, enabling us to implement the BLP method with standard regression techniques. This is because GEM are formulated in the space of consumption, and not in the dual space of indirect utilities, making the inverse demand system directly available.[3] However, some GEM lead to demands which do not have analytic formula,

[2] To the best of our knowledge, only the logit and nested logit models allow for an analytical inversion (see Berry, 1994; Verboven, 1996b). For example, the inverse demands in the random coefficient logit model exist, but have no analytic form (see Berry et al., 1995).

[3] In this respect, GEM are alternatives to the algorithms in Dubé et al. (2012) and Lee and Seo (2015). Dubé et al. (2012) transform the BLP's GMM minimization into a mathematical program with equilibrium constraints (MPEC), which minimizes the GMM objective function subject to the constraint that observed market shares be equal to predicted market shares. Lee and Seo (2015) approximate by linearization the non-

suggesting that an inversion step must be done once to obtain predicted demands after estimates have been obtained. We establish existence and uniqueness of the inverse system. Our results supplement earlier results on demand invertibility in different settings (see e.g., Berry, 1994; Beckert and Blundell, 2008; Berry et al., 2013), but is not implied by them.

GEM lead to demands with a tractable and familiar form that generalizes the logit demand in a nontrivial way. We employ our general theory to build a generalized nested entropy (GNE) with a corresponding GEM that generalizes the nested logit model by allowing the nests to overlap in any way. This allows us to build GEM that are similar in spirit to existing generalized extreme value (GEV) models that have already proved useful for demand estimation purposes. We also show how to build a GNE model describing markets having a natural ordering of products (as in Small, 1987; Grigolon, 2017) and a cross-nested GNE model that is similar in spirit to the product differentiation logit model of Bresnahan et al. (1997), describing product segmentation along several dimensions.

We apply the cross-nested GNE model to estimate the demand for cereals in Chicago in 1991–1992 using aggregate data. The cross-nested GNE model provides rich patterns of substitution and complementarity, while being parsimonious, computationally fast and very easy to estimate. We show how it can be estimated by a linear regression model of market shares on product characteristics and terms related to segmentation.

The cross-nested GNE model is related to Hausman et al. (1994) who also classify products into groups, allow consumers to buy several products, and estimate their model by a sequence of linear regressions. However, they treat consumers' choice as a sequence of separate but related choices. They also require a large number of instruments, and thus cannot handle large choice sets.

The cross-nested GNE model is also related to Pinkse and Slade (2004) who construct a continuous-choice demand model. Their model is simple to estimate, allows rich substitution patterns, handles large choice sets, makes cross-price elasticities functions of the distance between products in characteristics space and allows consumers to buy multiple products. However, their model is not as parsimonious as the cross-nested GNE model and also requires a large number of instruments.

The remainder of the paper is organized as follows. Section 2 introduces the class of GE demand models and provides general methods for construction. Section 3 studies the linkages between ARUM, GEM and the class of perturbed utility models (introduced in that section). Section 4 shows how to estimate GEM with aggregate data and discusses

linear system of market shares for the random coefficient logit model, and, in turn, do inversion analytically.

identification. Section 5 applies the cross-nested GNE model to estimate the demand for cereals in Chicago. Section 6 concludes.

2 The Class of Generalized Entropy Models

2.1 Notation

We use italics for scalar variables and real-valued functions, boldface for vectors, matrices and vector-valued functions, and script for sets. By default, vectors are column vectors.

Let $\mathscr{J} = \{0, 1, \ldots, J\}$. Let $\mathbf{q} = (q_0, \ldots, q_J)^\intercal \in \mathbb{R}^{J+1}$ and $\boldsymbol{\delta} = (\delta_0, \ldots, \delta_J)^\intercal \in \mathbb{R}^{J+1}$ be two vectors. $|\mathbf{q}| = \sum_{j \in \mathscr{J}} |q_j|$ denotes the 1-norm of vector $\mathbf{q}$ and $\boldsymbol{\delta} \cdot \mathbf{q} = \sum_{j \in \mathscr{J}} \delta_j q_j$ denotes the vector scalar product.

Let $\Omega : \mathbb{R}^{J+1} \to \mathbb{R}$. Then, $\Omega_j(\mathbf{q}) = \frac{\partial \Omega(\mathbf{q})}{\partial q_j}$ denotes its partial derivative with respect to its jth entry and $\nabla_{\mathbf{q}} \Omega(\mathbf{q})$ denotes its gradient with respect to the vector $\mathbf{q}$. A univariate function $\mathbb{R} \to \mathbb{R}$ applied to a vector is a coordinate-wise application of the function, e.g., $\ln(\mathbf{q}) = (\ln(q_0), \ldots, \ln(q_J))$.

Let $\mathbf{S} : \mathbb{R}^{J+1} \to \mathbb{R}^{J+1}$ be a vector function composed of functions $S^{(j)} : \mathbb{R}^{J+1} \to \mathbb{R}$: $\mathbf{S}(\mathbf{q}) = \left(S^{(0)}(\mathbf{q}), \ldots, S^{(J)}(\mathbf{q})\right)$. Then, its Jacobian matrix $\mathbf{J_S}(\mathbf{q})$ has elements ij given by $\frac{\partial S^{(i)}(\mathbf{q})}{\partial q_j}$.

$\mathbf{A}^\intercal \in \mathbb{R}^{J \times J}$ denotes the transpose matrix of $\mathbf{A} \in \mathbb{R}^{J \times J}$. $\mathbf{0}_J = (0, \ldots, 0)^\intercal \in \mathbb{R}^J$ and $\mathbf{1}_J = (1, \ldots, 1)^\intercal \in \mathbb{R}^J$ denote the J-dimensional zero and unit vectors, respectively. $\mathbf{I}_J \in \mathbb{R}^{J \times J}$ and $\mathbf{1}_{JJ} \in \mathbb{R}^{J \times J}$ denote the $J \times J$ identity matrix and unit matrix (where every element equals one), respectively.

Let $\mathbb{R}^J_+ = [0, \infty)^J$ and $\mathbb{R}^J_{++} = (0, \infty)^J$. Let $\Delta = \left\{\mathbf{q} \in \mathbb{R}^{J+1}_+ : \sum_{j \in \mathscr{J}} q_j = 1\right\}$ denote the unit simplex, with $\operatorname{int}(\Delta) = \Delta \cap \mathbb{R}^{J+1}_{++}$ its interior and $\operatorname{bd}(\Delta) = \Delta \setminus \operatorname{int}(\Delta)$ its boundary.

2.2 Definitions

Consider a representative consumer facing a choice set of $J + 1$ differentiated products, $\mathscr{J} = \{0, 1, \ldots J\}$, and a homogeneous numéraire good, with demands for the differentiated products summing to one. Let p_j and v_j be the price and the quality of product $j \in \mathscr{J}$, respectively. We normalize the price of the numéraire good to 1 and assume that the representative consumer's income y is sufficiently high, $y > \max_{j \in \mathscr{J}} p_j$, that consumption of the numéraire good is necessarily strictly positive.

Let $\mathbf{q} = (q_0, \ldots, q_J)^\intercal$ be the vector of quantities consumed of the differentiated products and z be the quantity consumed of the numéraire good. The representative consumer's direct utility function u, which is quasi-linear in the numéraire, is given by

$$u(\mathbf{q}, z) = \alpha z + \sum_{j \in \mathscr{J}} v_j q_j + \Omega(\mathbf{q}), \tag{1}$$

where $\alpha > 0$ is the marginal utility of income, and Ω is a nonlinear function of $\mathbf{q}$.

The utility in (1) has two components: the first describes the utility derived from the consumption of $(\mathbf{q}, z)$ in the absence of interaction among products while the second expresses the consumer's taste for variety.

The representative consumer chooses $\mathbf{q} \in \Delta$ and $z \in \mathbb{R}_+$ so as to maximize her utility (1) subject to her budget constraint. She solves

$$\max_{(\mathbf{q},z) \in \Delta \times \mathbb{R}_+} u(\mathbf{q}, z), \quad \text{subject to} \quad \sum_{j \in \mathscr{J}} p_j q_j + z \leq y. \tag{2}$$

The budget constraint is always binding,[4] so that (2) can be rewritten as

$$\max_{\mathbf{q} \in \Delta} \left\{ \alpha y + \sum_{j \in \mathscr{J}} \delta_j q_j + \Omega(\mathbf{q}) \right\}, \tag{3}$$

where $\delta_j = v_j - \alpha p_j$ is the net utility that the consumer derives from consuming one unit of product j.[5]

We build the class of generalized entropy models (hereafter, GEM) by first defining a class of vector functions $\mathbf{S}$ that we call generators. A generalized entropy (hereafter, GE) is then defined in terms of a generator and a GEM is a RCM in which the taste for variety is modeled by a GE.

Definition 1 (Generator)**.** The vector function $\mathbf{S} = \left(S^{(0)}, \ldots, S^{(J)}\right) : \mathbb{R}_+^{J+1} \to \mathbb{R}_+^{J+1}$ is a generator if it is twice continuously differentiable, linearly homogeneous, and the Jacobian of $\ln \mathbf{S}$, $\mathbf{J}_{\ln \mathbf{S}}$, is positive definite and symmetric on $\operatorname{int}(\Delta)$.

[4] This is because $\alpha > 0$ and $y > \max_{j \in \mathscr{J}} p_j$.

[5] In the empirical industrial organization literature, δ_j is referred to as the mean utility of product j.

Definition 2 (GE)**.** A GE is a function $\Omega : \mathbb{R}_{+}^{J+1} \to \mathbb{R} \cup \{-\infty\}$ given by

$$\Omega(\mathbf{q}) = -\sum_{j \in \mathscr{J}} q_j \ln S^{(j)}(\mathbf{q}), \; \mathbf{q} \in \Delta, \tag{4}$$

with $\Omega(\mathbf{q}) = -\infty$ when $\mathbf{q} \notin \Delta$, where $\mathbf{S}$ is a generator.

Definition 3 (GEM)**.** A GEM is a demand system that solves (2), where $\Omega(\mathbf{q})$ is a GE.

For the sake of the exposition, our definition of GEM specifies how the net utilities $\boldsymbol{\delta}$ depend on prices. We use this specification in our empirical application. Alternatively, the definition of GEM can be based directly on (3) without specifying the dependence on prices or how the utility maximization problem in (3) relates to a budget constraint. This is useful, for example, in settings where prices are not relevant (Allen and Rehbeck, 2016).

We require the following additional condition on the generator $\mathbf{S}$ in order to rule out zero demands.[6] We retain Assumption 1 in the remainder of the paper, except when otherwise stated.

Assumption 1 (Positivity)**.** The 1-norm $|\ln \mathbf{S}(\mathbf{q})|$ approaches infinity as $\mathbf{q}$ approaches $\mathrm{bd}(\Delta)$.

The conditions imposed on the generator $\mathbf{S}$ imply the following lemma, which is essential in the sequel.

Lemma 1. Assume that $\mathbf{S}$ is a generator and let Ω be its corresponding GE. Then $\mathbf{S}$ is invertible on $\mathrm{int}(\Delta)$, satisfies the identity

$$\sum_{j \in \mathscr{J}} q_j \frac{\partial \ln S^{(j)}(\mathbf{q})}{\partial q_k} = 1, \quad k \in \mathscr{J}, \; \mathbf{q} \in \mathrm{int}(\Delta), \tag{5}$$

and Ω is strictly concave on $\mathrm{int}(\Delta)$.

Equation (5) follows from the Euler equation for homothetic functions (McElroy, 1969) and is crucial for deriving the simple expression for demand in Theorem 1 below.

The quasi-linearity of the direct utility function (1) has two implications. First, GEM demands for the differentiated products are independent of income, so that all income effects are captured by the numéraire.

[6] Hofbauer and Sandholm (2002) and Fudenberg et al. (2015) require similar conditions.

Second, in the GEM, as in any model with quasi-linear direct utility (see e.g., Vives, 2001), the assumption of a representative consumer is not restrictive. Indeed, consider a population of utility-maximizing consumers all with quasi-linear direct utility of the form (1), and assume that they all have the same constant marginal utility of income $\alpha > 0$. Then individual indirect utilities have the Gorman form and can thus be aggregated across consumers, meaning that consumers can be treated as if they were a single consumer, regardless of the distribution of unobserved consumer heterogeneity or of income.[7]

Lastly, GEM allow consumers to buy several products in varying quantities. This assumption is appropriate in situations in which consumers are observed to buy multiple products from the same category (see evidence in Dubé (2004) for soft drinks and in Kim et al. (2002) for yogurts) as well as products from multiple categories (see evidence in Thomassen et al., 2017). This variety-seeking behavior is one possible mechanism that may lead to complementarity. We will see below that GEM allow complementarity to occur.

2.3 Demand and Consumer Surplus

The utility maximizing demand in the GEM exists, since the utility function is continuous on the compact set Δ. The strict concavity of Ω established in Lemma 1 ensures that demand is unique, while Assumption 1 ensures that it is interior. The Euler-type equation (5), together with the invertibility of $\mathbf{S}$, allow us to derive a tractable and familiar form for demand in Theorem 1. We denote the inverse of $\mathbf{S}$ by $\mathbf{H} = \mathbf{S}^{-1}$.

Theorem 1. Let $\mathbf{S}$ be a generator. Under Assumption 1, GEM lead to non-zero GE demands

$$q_i(\boldsymbol{\delta}) = \frac{H^{(i)}\left(e^{\boldsymbol{\delta}}\right)}{\sum_{j \in \mathscr{J}} H^{(j)}\left(e^{\boldsymbol{\delta}}\right)}, \quad i \in \mathscr{J}. \tag{6}$$

where $H^{(i)}\left(e^{\boldsymbol{\delta}}\right) = S^{-1(i)}\left(e^{\boldsymbol{\delta}}\right)$. Net utility $\boldsymbol{\delta}$ and demand $\mathbf{q}$ are related through the generator

[7] Consider a population of N consumers. Suppose that each consumer n's maximizes her direct utility $u_n(\mathbf{q}_n, z_n) = \alpha z_n + \sum_{j \in \mathscr{J}} v_j q_{jn} + \Omega_n(\mathbf{q}_n)$ subject to her individual budget constraint. Then consumer n's indirect utility is given by $v_n(\boldsymbol{\delta}, y_n) = \alpha y_n + \ln\left(\sum_{j \in \mathscr{J}} H_n^{(j)}\left(e^{\boldsymbol{\delta}}\right)\right)$ and has the Gorman form $v_n(\mathbf{p}, y_n) = b(\mathbf{p}) y_n + a_n(\mathbf{p})$ with $b(\mathbf{p}) = \alpha$ that is identical for all consumers and $a_n(\mathbf{p}) = \ln\left(\sum_{j \in \mathscr{J}} H_n^{(j)}\left(e^{\boldsymbol{\delta}}\right)\right)$ that differs from consumer to consumer.

$\mathbf{S}$ and its inverse $\mathbf{H}$ by

$$\delta_i = \ln S^{(i)}(\mathbf{q}) + \ln\left(\sum_{j\in\mathscr{J}} H^{(j)}\left(e^{\boldsymbol{\delta}}\right)\right), \quad i \in \mathscr{J}, \ \mathbf{q} \in \operatorname{int}(\Delta). \tag{7}$$

The GE demand in (6) generalizes the logit demand in a nontrivial way through the presence of the function $\mathbf{H}$.

The mapping from demand to net utility in (7) is unique up to a constant. This shows that GEM generate demands with an explicit inverse which can be used as basis for demand estimation after specifying the functional form of the generator $\mathbf{S}$. Note that the log-sum term in the RHS of (7) is common across products.

For example, in the simplest possible case, the generator is the identity $\mathbf{S}(\mathbf{q}) = \mathbf{q}$ which implies that the inverse generator is also the identity $\mathbf{H}\left(e^{\boldsymbol{\delta}}\right) = e^{\boldsymbol{\delta}}$. In this case, the GE reduces to the Shannon entropy $\Omega(\mathbf{q}) = -\sum_{j\in\mathscr{J}} q_j \ln(q_j)$ and we obtain the logit demand (see Anderson et al., 1988)

$$q_i(\boldsymbol{\delta}) = \frac{e^{\delta_i}}{\sum_{j\in\mathscr{J}} e^{\delta_j}}. \tag{8}$$

In accordance with (7), utility $\boldsymbol{\delta}$ and logit demand $\mathbf{q}$ satisfy the relations

$$\delta_i = \ln(q_i) + \ln\left(\sum_{j\in\mathscr{J}} e^{\delta_j}\right), \quad i \in \mathscr{J}.$$

Let $G(\boldsymbol{\delta}) = \sum_{j\in\mathscr{J}} \delta_j q_j(\boldsymbol{\delta}) + \Omega(\mathbf{q}(\boldsymbol{\delta}))$ be the consumer surplus and $w(\boldsymbol{\delta}, y) = \alpha y + G(\boldsymbol{\delta})$ be the indirect utility, associated with the direct utility (1). The following proposition provides an expression for the consumer surplus.

Proposition 1. The consumer's surplus is given by

$$G(\boldsymbol{\delta}) = \ln\left(\sum_{j\in\mathscr{J}} H^{(j)}\left(e^{\boldsymbol{\delta}}\right)\right). \tag{9}$$

GE demands (6) are consistent with Roy's identity, i.e., $q_i = -\frac{\partial w(\boldsymbol{\delta},y)}{\partial p_i} / \frac{\partial w(\boldsymbol{\delta},y)}{\partial y}$, $i \in \mathscr{J}$, or equivalently, $q_i = \frac{\partial G(\boldsymbol{\delta})}{\partial \delta_i}$ for all $i \in \mathscr{J}$.

For the logit model, it is well known that the consumer surplus is the logarithm of the denominator of the demands. Proposition 1 shows that this is also the case for the entire class of GEM.

GE demands q_j given by (6) are increasing in their own utility component δ_j.[8] Proposition 2 provides an expression for the whole matrix of demand derivatives.

Proposition 2. The matrix of demand derivatives $\partial q_j / \partial \delta_i$ is given by

$$\mathbf{J}_{\mathbf{q}} = \left[\mathbf{J}_{\ln \mathbf{S}}\left(\mathbf{q}\right)\right]^{-1}\left[\mathbf{I}_{J+1} - \mathbf{1}_{J+1,J+1}\mathbf{q}^{\intercal}\right], \tag{10}$$

where $\mathbf{q} = \mathbf{q}\left(\boldsymbol{\delta}\right)$ given by Equation (6).

Since GEM do not allow for income effects, complementarity (resp., substitutability) between products is just understood as a negative (resp., positive) cross derivative of GE demands. While Proposition 2 is useful for calculating demand elasticities in applications, it does not allow to directly determine whether products are complements or substitutes. Example 2 below exhibits a GEM in which products may be complements.

2.4 Construction of GEM

To construct a GEM, it suffices to construct a generator that satisfies Definition 1.[9] We here propose a family of generators that lead to models that extend the nested logit (NL) model in a very intuitive way as follows.[10]

We first observe that the nested logit (NL) model can be cast as a GEM. Suppose that the choice set is partitioned into non-overlapping sets, called nests. Let g_j be the nest that contains product j. Then the generator that leads to the NL demands is given by

$$S^{(j)}\left(\mathbf{q}\right) = q_j^{\mu}\left(\sum_{i \in g_j} q_i\right)^{1-\mu},$$

where $\mu \in (0, 1)$ is the nesting parameter.[11]

[8]This property is due to the concavity of the GE and is equivalent to stating that demands are decreasing in their own prices.

[9]Similarly, different GEV models (see McFadden, 1981) are obtained from different specifications of a choice probability function (Fosgerau et al., 2013).

[10]In Appendix E, we provide a range of general methods for building generators along with illustrative examples.

[11]The corresponding GE is given by the sum of two Shannon entropies since $\Omega\left(\mathbf{q}\right) = -\mu \sum_{j \in \mathscr{J}} q_j \ln\left(q_j\right) - (1-\mu) \sum_{g=1}^{G} q_g \ln\left(q_g\right)$.

Multi-level NL models generalize the NL model. They are obtained by partitioning the choice set into nests and then further partitioning each nest into subnests, and so on (see e.g., Goldberg, 1995; Verboven, 1996b). This hierarchical structure implies that each product belongs to only one (sub)nest at each level, meaning that nests are not allowed to overlap by construction. The following proposition generalizes the NL model by giving a construction of generators through a nesting operation that allows the nests to overlap in any way.

Proposition 3 (General nesting)**.** Let $\mathscr{G} \subseteq 2^{\mathscr{J}}$ be a finite set of nests with associated nesting parameters μ_g , where $\mu_0 + \sum_{\{g\in\mathscr{G}|j\in g\}} \mu_g = 1$ for all $j \in \mathscr{J}$ with $\mu_g \geq 0$ for all $g \in \mathscr{G}$ and $\mu_0 > 0$. Let $\mathbf{S}$ be given by

$$S^{(j)}(\mathbf{q}) = q_j^{\mu_0} \prod_{\{g\in\mathscr{G}|j\in g\}} q_g^{\mu_g}, \tag{11}$$

where $q_g = \sum_{i\in g} q_i$. Then $\mathbf{S}$ is a generator.[12]

We label a GE with a generator of the form (11) as a generalized nested entropy (GNE). Using that allows us to build a wide range of models that are similar in spirit to the well-known GEV models based on nesting (see e.g., Train, 2009, Chapter 4 for details).

As a first example, we construct a GNE model describing a market having a natural ordering of products. Examples could be, e.g., hotels that can be ordered according to their number of stars or ready-to-eat cereals ordered according to sugar content. In the GNE model that we construct, products that are nearer each other in the ordering will be closer substitutes. This is similar to the GEV ordered models of Small (1987) and Grigolon (2017).

Example 1 (Ordered model)**.** Let product 0 be the outside option, and products $1, \ldots, J$ be ordered in ascending sequence. We make the ordering circular, letting product 1 follow product J. Let $\mu_0 > 0$ and $\mu_1, \mu_2, \mu_3 \geq 0$ with $\mu_0 + \mu_1 + \mu_2 + \mu_3 = 1$. The function $\mathbf{S}$ given by

$$S^{(j)}(\mathbf{q}) = \begin{cases} q_0, & j = 0 \\ q_j^{\mu_0} q_{\sigma_1(j)}^{\mu_1} q_{\sigma_2(j)}^{\mu_2} q_{\sigma_3(j)}^{\mu_3}, & j > 0, \end{cases}$$

[12]Without the term $q_j^{\mu_0}$, $\mathbf{S}$ is twice continuously differentiable and linearly homogeneous, and $\mathbf{J}_{\ln \mathbf{S}}$ is symmetric, but not necessarily positive definite. The general nesting operation leads to the following GE $\Omega(\mathbf{q}) = -\mu_0 \sum_{j\in\mathscr{J}} q_j \ln(q_j) - \sum_{\{g\in\mathscr{G}|j\in g\}} \left[\mu_g \sum_{j\in\mathscr{J}} q_j \ln(q_g)\right]$, where the first term is the Shannon entropy that expresses consumer's taste for variety over all products and the second term expresses consumer's taste for variety over products belonging to group g (see Verboven, 1996b).

with $q_{\sigma_1(j)} = q_{j-2} + q_{j-1} + q_j$, $q_{\sigma_2(j)} = q_{j-1} + q_j + q_{j+1}$, $q_{\sigma_3(j)} = q_j + q_{j+1} + q_{j+2}$, is a generator.

In Subsection 2.5, similarly to the Product-Differentiation Logit (PDL) model of Bresnahan et al. (1997), we build and study a cross-nested GNE model describing markets that exhibit product segmentation along several dimensions. We take this model to real-world data in Section 5.

The next example is a GEM in which products can be complements.

Example 2. Let $\mathbf{S}$ be defined by

$$\mathbf{S}(\mathbf{q}) = \begin{cases} q_0^{\mu}\left(q_0 + \frac{1}{2}q_1\right)^{1-\mu}, \\ q_1^{\mu}\left(q_0 + \frac{1}{2}q_1\right)^{\frac{1-\mu}{2}}\left(\frac{1}{2}q_1 + q_2\right)^{\frac{1-\mu}{2}}, \\ q_2^{\mu}\left(\frac{1}{2}q_1 + q_2\right)^{1-\mu}, \end{cases}$$

with $\mu \in (0,1)$. Then $\mathbf{S}$ is a generator.

We show in Appendix B that the first-order conditions for utility maximization imply that $\partial q_2/\partial \delta_0 > 0$ if and only if μ is small enough, i.e.,

$$\mu < \frac{q_1}{4q_0q_2 + 3q_1q_2 + 2q_1^2 + 3q_0q_1}.$$

At $\boldsymbol{\delta}$ such that $q_0 = q_1 = q_2 = 1/3$, the condition becomes $\mu < 1/4$, thereby showing that there exists combinations of parameters μ and utilities $\boldsymbol{\delta}$ at which some products are complements.

We define ARUM and discuss the link to GEM in the next section. It is, however, natural to mention one relationship already at this stage. We have just seen complementarity may arise in GEM. Combining this with the fact that all products are necessarily substitutes in an ARUM leads to the following result.

Proposition 4. Some GEM lead to demand systems that cannot be rationalized by any ARUM.

We show in Subsection 3.1 that any ARUM has a GEM counterpart that leads to the same choice probabilities. Combining this with Proposition 4 shows that the class of GEM is strictly larger than the class of ARUM.

2.5 The Cross-Nested GNE Model

In this section, we build the cross-nested GNE model describing markets that exhibit product segmentation along several dimensions. This model generalizes the multi-level NL models by breaking their hierarchical structure. We take this model to real-world data in Section 5.

Consider a market for differentiated products that exhibits product segmentation according to C *dimensions*, indexed by c. Each dimension c taken separately potentially provides a source of segmentation and defines a finite number of *nests*. Each product belongs to exactly C nests, one for each dimension, and the nesting structure is exogenous. The dimensions taken together define product *types*. Products of the same type are those that are grouped together according to all dimensions. Each dimension defines a concept of product closeness (or distance), so that products of the same type will be closer substitutes than products of different types.

Let nest $\sigma_c(j)$ be the set of products that are grouped with product j in dimension c, and $q_{\sigma_c(j),t} = \sum_{i \in \sigma_c(j)} q_{it}$ be the market share of nest $\sigma_c(j)$ in market t. Let $\boldsymbol{\Theta}_c$ be the nesting structure matrix for dimension c, having elements

$$(\boldsymbol{\Theta}_c)_{ij} = \begin{cases} 1, & \text{if } i \in \sigma_c(j), \\ 0, & \text{otherwise}, \end{cases} \tag{12}$$

and let $\boldsymbol{\Theta} = (\boldsymbol{\Theta}_1, \ldots, \boldsymbol{\Theta}_C)$ denote the array of nesting structure matrices.

We define the cross-nested GNE model as follows.

Definition 4. The cross-nested GNE model is a GEM with generator given by

$$S^{(j)}(\mathbf{q}) = \begin{cases} q_0, & j = 0, \\ q_j^{\mu_0} \prod_{c=1}^{C} q_{\sigma_c(j)}^{\mu_c}, & j > 0, \end{cases} \tag{13}$$

with $\mu_0 + \sum_{c=1}^{C} \mu_c = 1$, $\mu_0 > 0$, and $\mu_c \geq 0$ for all $c \in \{1, \ldots, C\}$.

This model satisfies Assumption 1, so that zero demands never arise. Product $j = 0$ is the outside option, which defines itself a product type and is the only product of its type. Let $\boldsymbol{\mu} = (\mu_0, \ldots, \mu_C)$ be the vector of nesting parameters. The parameter μ_0 measures the consumers' taste for variety over all products and each μ_c, $c \geq 1$, measures the consumers' taste for variety across nests in dimension c (see Verboven, 1996b). The following proposition is useful for understanding the behavior of the model.

Proposition 5. In the cross-nested GNE model, the Independence from Irrelevant Alternatives (IIA) property holds for products of the same type; but does not hold in general for products of different types.

It is well-known that the logit model exhibits the IIA property and that, in the NL model, IIA holds for products within each nest but not for products in different nests in general. IIA constitutes a restriction on the models since it implies that an improvement in one product draws demand proportionately from the other products, meaning that the cross-price elasticities does not depend on how close products are in the characteristics space. The cross-nested GNE model thus extends the logit and NL models by relaxing these restrictions.

Appendix C provides some simulation results investigating the patterns of substitution and complementarity as the nesting structure and market shares change. In summary, we find that (i) products of the same type are never complements, while products of different types may or may not be complements; and (ii) the size of the cross-elasticities depends on the degree of closeness between products as measured by the value of the nesting parameters and by the proximity of the products in the characteristics space used to form product types.

3 Linkages between Choice Models

In this section, we study first the relation between GEM and ARUM and show that the choice probabilities of any ARUM can be obtained as the demand of some GEM. Then we discuss how a GEM is a special case of a perturbed utility model, (PUM, defined below). Altogether we find that the GEM class of models is intermediate between ARUM and PUM.

In this section, we consider a consumer who faces a choice set $\mathscr{J} = \{0, 1, \ldots, J\}$ of $J + 1$ products with utility components $\boldsymbol{\delta} = (\delta_0, \ldots, \delta_J)^\intercal$. Recall that in the RCM, the utility components δ_j are linear in their own price due to the budget constraint. In the ARUM and the PUM, we do not require prices to enter utilities δ_j in any specific way.

3.1 ARUM as GEM

We first set up the ARUM. The consumer buys one unit of the product that provides her the highest (indirect) utility $u_j = \delta_j + \varepsilon_j$, $j \in \mathscr{J}$, where δ_j is a deterministic utility component and ε_j is a random utility component. The following assumption on $\boldsymbol{\varepsilon} = (\varepsilon_0, \ldots, \varepsilon_J)^\intercal$ is standard in the discrete choice literature.

Assumption 2. The random vector $\boldsymbol{\varepsilon}$ follows a joint distribution with finite means that is absolutely continuous, independent of $\boldsymbol{\delta}$, and has full support on $\mathbb{R}^{J+1}$.

Assumption 2 implies that utility ties $u_i = u_j$, $i \neq j$, occur with probability 0 (because the joint distribution of $\boldsymbol{\varepsilon}$ is absolutely continuous), meaning that the argmax set of the ARUM is almost surely a singleton. Furthermore, the choice probabilities are all everywhere positive (because $\boldsymbol{\varepsilon}$ has full support). Assumption 2 also rules out random coefficients, since the joint distribution of $\boldsymbol{\varepsilon}$ is required to be independent of $\boldsymbol{\delta}$.

Let $\overline{G} : \mathbb{R}^{J+1} \to \mathbb{R}$ given by

$$\overline{G}(\boldsymbol{\delta}) = \mathbb{E}\left(\max_{j \in \mathscr{J}} u_j\right) \tag{14}$$

be the expected maximum utility. Let $\mathbf{P} = (P_0(\boldsymbol{\delta}), \ldots, P_J(\boldsymbol{\delta})) : \mathbb{R}^{J+1} \to \Delta$ be the vector of choice probabilities with $P_j(\boldsymbol{\delta})$ being the probability of choosing product j.

From the Williams-Daly-Zachary theorem (McFadden, 1981), the choice probabilities and the derivatives of $\overline{G}(\boldsymbol{\delta})$ coincide, i.e.,

$$P_j(\boldsymbol{\delta}) = \frac{\partial \overline{G}(\boldsymbol{\delta})}{\partial \delta_j}, \quad j \in \mathscr{J}. \tag{15}$$

Let $\overline{\mathbf{H}} = \left(\overline{H}^{(0)}, \ldots, \overline{H}^{(J)}\right)$, with $\overline{H}^{(i)} : \mathbb{R}^{J+1}_{++} \to \mathbb{R}_{++}$ defined as the derivative of the exponentiated surplus with respect to its ith component, i.e.,

$$\overline{H}^{(i)}\left(e^{\boldsymbol{\delta}}\right) = \frac{\partial e^{\overline{G}(\boldsymbol{\delta})}}{\partial \delta_i}. \tag{16}$$

Note that $\sum_{j \in \mathscr{J}} \overline{H}^{(i)}\left(e^{\boldsymbol{\delta}}\right) = e^{\overline{G}(\boldsymbol{\delta})}$.[13] Then the ARUM choice probabilities may be written as

$$P_i(\boldsymbol{\delta}) = \frac{\overline{H}^{(i)}\left(e^{\boldsymbol{\delta}}\right)}{\sum_{j \in \mathscr{J}} \overline{H}^{(j)}\left(e^{\boldsymbol{\delta}}\right)}, \quad i \in \mathscr{J}, \tag{17}$$

which is exactly the same form as the GEM demand (6) if $\mathbf{H} = \overline{\mathbf{H}}$. To establish that the

[13]This follows since (16) may be written as $\overline{H}^{(i)}\left(e^{\boldsymbol{\delta}}\right) = \frac{\partial \overline{G}(\boldsymbol{\delta})}{\partial \delta_i} e^{\overline{G}(\boldsymbol{\delta})}$, $i \in \mathscr{J}$. Then by (15), $\overline{H}^{(i)}\left(e^{\boldsymbol{\delta}}\right) = P_i(\boldsymbol{\delta})\, e^{\overline{G}(\boldsymbol{\delta})}$ for all $i \in \mathscr{J}$. Finally, sum over $i \in \mathscr{J}$ and use that choice probabilities sum to one.

ARUM choice probabilities (17) can be generated by a GEM, it then only remains to show that $\overline{\mathbf{H}}$ has an inverse $\overline{\mathbf{S}} = \overline{\mathbf{H}}^{-1}$ and that this inverse is a generator. This is established in the following lemma.

Lemma 2. The function $\overline{\mathbf{H}}$ is invertible, and its inverse $\overline{\mathbf{S}} = \overline{\mathbf{H}}^{-1}$ is a generator.

Then the function $-\overline{G}^*$ given by

$$-\overline{G}^*(\mathbf{q}) = -\sum_{j\in\mathscr{J}} q_j \ln \overline{S}^{(j)}(\mathbf{q}), \; \mathbf{q} \in \Delta, \tag{18}$$

and $-\overline{G}^*(\mathbf{q}) = +\infty$ when $\mathbf{q} \notin \Delta$ is a GE. Fosgerau et al. (2017) show that $-\overline{G}^*$ is the convex conjugate of $\overline{G}$.[14]

We summarize these results as follows.

Theorem 2. The ARUM choice probabilities (17) with surplus function $\overline{G}$ given by (14) coincide with the GE demand system (6) with GE function $-\overline{G}^*$, where $\overline{G}^*$ is the convex conjugate of $\overline{G}$ given by (18).

According to Theorem 2, all ARUM have a GEM as counterpart that leads to the same demand. However, as shown in Example 2, the converse is not true: the class of GEM is strictly larger than the class of ARUM. When a GEM corresponds to an ARUM, the surplus function (9) and the maximum expected utility (14) coincide, i.e., $G = \overline{G}$; and similarly for their generators, i.e., $\mathbf{S} = \overline{\mathbf{S}}$. Figure 1 illustrates how ARUM and GEM are linked and shows how, beginning with some ARUM, we can determine a GEM with demand that is equal to the ARUM choice probabilities.

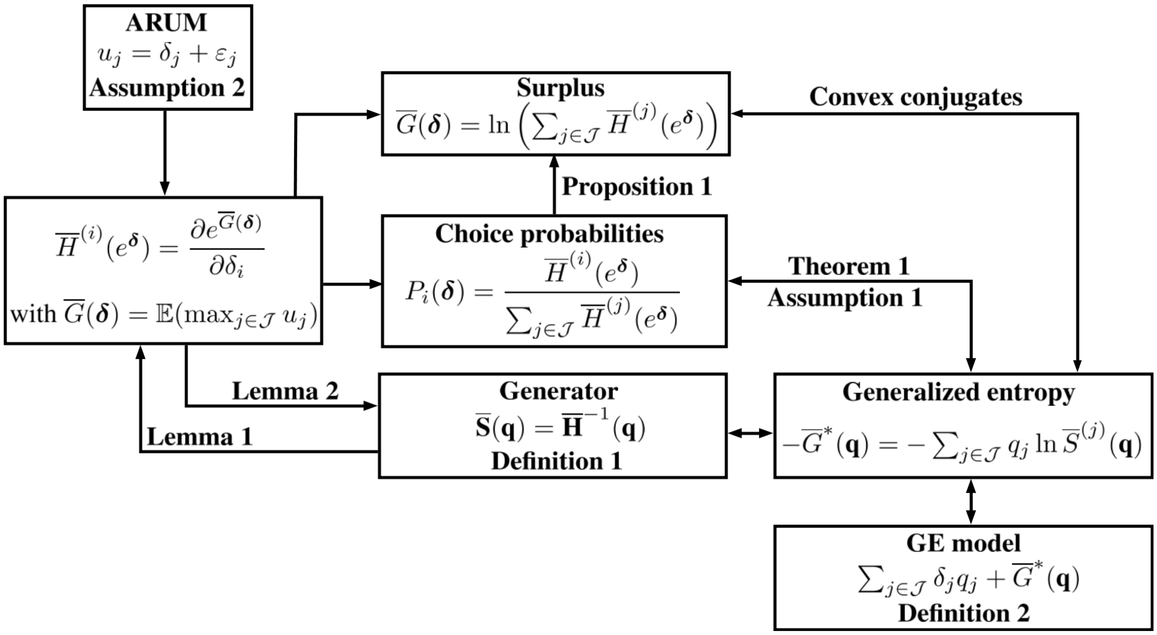
3.2 Link to Perturbed Utility Models

We now discuss briefly the relationship between GEM and perturbed utility model (PUM). In a PUM, the consumer chooses a vector of choice probabilities $\mathbf{q} \in \Delta$ to maximize her utility function

$$\sum_{j\in\mathscr{J}} \delta_j q_j + \Omega(\mathbf{q}), \tag{19}$$

[14] The latter result is well-known in the special case of the logit model, i.e. that the convex conjugate of the negative entropy $f(\mathbf{q}) = \sum_j q_j \ln(q_j)$ is the log-sum $f^*(\boldsymbol{\delta}) = \ln\left(\sum_j e^{\delta_j}\right)$ (see e.g., Boyd and Vandenberghe, 2004).

Figure 1: LINKAGES BETWEEN GEM AND ARUM



defined as the sum of an expected utility component, which is linear in $\mathbf{q}$, and a perturbation Ω, which is a concave and deterministic function of $\mathbf{q}$. See Hofbauer and Sandholm (2002), McFadden and Fosgerau (2012) and Fudenberg et al. (2015) for more details on PUM. [15]

We have shown in Lemma 1 above that a GE (4) is concave, which implies that any GEM is also a PUM. The converse, however, does not hold, as there are many concave functions that do not have the form of a GE. For example, Hofbauer and Sandholm (2002) mention the concave perturbation function $\sum_{j\in\mathscr{J}} \ln q_j$. The corresponding candidate generator $S^{(j)}(\mathbf{q}) = q_j^{1/q_j}$ is not linearly homogeneous and is hence not a generator according to Definition 1.

Hofbauer and Sandholm (2002) show that the choice probabilities generated by any ARUM can be derived from a PUM with a deterministic perturbation. In Theorem 2, we strengthen this result by showing that the GEM structure is sufficient to recover any ARUM.

[15]PUM have been used to model optimization with effort (Mattsson and Weibull, 2002), stochastic choices (Fudenberg et al., 2015), stochastic choice as the result of balancing multiple goals (?), and rational inattention (Matejka and McKay, 2015; Fosgerau et al., 2017).

4 Estimation of GEM

We are now able to estimate GEM. In Subsection 2.4, we proposed some general methods for constructing generators. In applications, the generators can be written as functions of the data and some parameters to be estimated with individual-level or aggregate data. In this section, we show how to estimate GEM with aggregate data.

4.1 Econometric Model

The aggregate data required to estimate GEM consists of the market shares, prices and characteristics for each product in each market (see e.g., Nevo, 2001).

Consider markets $t \in \{1, \dots, T\}$ with inside products $j \in \{1, \dots, J\}$ and an outside option $j = 0$. Let ξ_{jt} be the unobserved characteristics term of product j in market t.

The invertibility of $\mathbf{S}$ shown in Lemma 1, which is required for the GE demands (6) to be well defined, is not sufficient to ensure that there is a unique $\boldsymbol{\delta}_t$ that rationalizes $\mathbf{q}_t$, for all market t. To ensure uniqueness, we normalize $\delta_{0t} = 0$ for all markets t.

Parametrize net utility for the inside products as

$$\delta_{jt}\left(\mathbf{X}_{jt}, p_{jt}, \xi_{jt}; \boldsymbol{\theta}_1\right) = \beta_0 + \mathbf{X}_{jt}\boldsymbol{\beta} - \alpha p_{jt} + \xi_{jt},$$

where $\boldsymbol{\theta}_1 = (\alpha, \beta_0, \boldsymbol{\beta})$ is a vector of parameters that enter the linear part of the utility, and p_{jt} and $\mathbf{X}_{jt}$ are the price and (any function of) the characteristics of product j in market t, respectively. The intercept β_0 captures the value of consuming an inside product instead of the outside option; the parameter vector $\boldsymbol{\beta}$ represents the consumers' taste for the $\mathbf{X}_{jt}$'s; and the parameter $\alpha > 0$ is consumers' price sensitivity (i.e. the marginal utility of income).

Let $\boldsymbol{\theta}_2$ be a parameter vector that enters the nonlinear part of the utility and parametrize the generator as $S^{(j)}\left(\mathbf{q}_t; \boldsymbol{\theta}_2\right)$. Using (7) we have

$$\ln S^{(j)}\left(\mathbf{q}_t; \boldsymbol{\theta}_2\right) = \delta_{jt}\left(\mathbf{X}_{jt}, p_{jt}, \xi_{jt}; \boldsymbol{\theta}_1\right) + c_t,\ j \in \mathscr{J},\ t \in \{1, \dots, T\},$$

where $c_t \in \mathbb{R}$ denotes the log-sum term that is common across products on the same market, and $\mathbf{q}_t = (q_{0t}, \dots, q_{Jt})^\intercal$.

Subtracting the equations for the outside good, we end up with the $J \times T$ demand equations $\xi_{jt} = \xi_{jt}\left(\boldsymbol{\theta}\right)$, where the market-specific constant terms c_t have dropped out, and with

$$\xi_{jt}(\boldsymbol{\theta}) = \ln S^{(j)}(\mathbf{q}_t; \boldsymbol{\theta}_2) - \ln S^{(0)}(\mathbf{q}_t; \boldsymbol{\theta}_2) - (\beta_0 + \mathbf{X}_{jt}\boldsymbol{\beta} - \alpha p_{jt}), \tag{20}$$

where $\boldsymbol{\theta} = (\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)$ are the parameters to be estimated.

After transformation, GEM are nonlinear regression models, where the error is non-additive. GEM can thus be estimated using standard regression techniques and can handle very large choice sets, while, as seen before, being able of accommodating rich patterns of substitution and complementarity. These main features make GEM appealing for merger evaluation and for studying vertically related markets, as highlighted by Pinkse and Slade (2004).

4.2 Identification

Prices and market shares form two different sets of endogenous variables and require different sources of exogenous variation for the model to be identified. Prices are endogenous due to the presence of the unobserved product characteristics ξ_{jt}. Indeed, price competition models with differentiated products typically assume that firms consider both observed and unobserved product characteristics when setting prices, and make prices a function of marginal costs and a markup term. Since the markup term is a function of the (entire vector of) unobserved product characteristics, which constitute the error terms in Equations (20), prices are likely to be correlated with the error terms. Market shares are endogenous because demands are defined by a system of equations, where each demand depends on the entire vectors of endogenous prices and of unobserved product characteristics.

GEM provide a system of demand equations (20), where each equation has one unobservable ξ_{jt} and, under the standard assumption that products characteristics are exogenous, depends on $(J+1)$ endogenous variables, namely the market shares $\mathbf{q}_t$ and one price p_{jt}. The main identification assumption is the existence of as many excluded (from the demand equations) instruments z_t as there are endogenous variables. Recall that instruments are variables that are correlated with the endogenous variables (relevance) but are not correlated with the error term ξ_{jt} (exogeneity). Following Berry (1994) and Berry et al. (1995), we propose a GMM estimator based on the conditional moment restrictions $\mathbb{E}[\xi_{jt}(\boldsymbol{\theta})|z_t] = 0$, which lead to the unconditional moment restrictions $\mathbb{E}[z_t\xi_{jt}(\boldsymbol{\theta})] = 0$.

We require instruments for prices and for some functions of market shares, where the need for instruments for market shares depends on the structure of the generator. For example, in the case of the NL model,

$$\xi_{jt} = \ln\left(\frac{q_{jt}}{q_{0t}}\right) - \mu \ln\left(q_{jt|g_j}\right) + \alpha p_{jt} - \left(\mathbf{X}_{jt}\boldsymbol{\beta} + \beta_0\right),$$

where $q_{jt|g_j}$ is the share of product j within its corresponding nest g_j. This requires only two instruments, one for price p_{jt} and one for the share $q_{jt|g_j}$.

Following the prevailing literature (Berry and Haile, 2014; Reynaert and Verboven, 2014; Armstrong, 2016), both cost shifters and BLP instruments are required. Cost shifters separate exogenous variation in prices due to exogenous cost changes from endogenous variation in prices from unobserved product characteristics changes. They are valid under the assumption that variation in cost shifters is correlated with price variation, but not with changes in unobservable product characteristics. However, they are not sufficient on their own, because costs affect the endogenous market shares only through prices.

BLP instruments are functions of the characteristics of competing products and are valid instruments under the assumption that $\mathbf{X}_{jt}$ is exogenous (i.e., ξ_{jt} is independent of $\mathbf{X}_{jt}$). They separate exogenous variation in prices due to changes in $\mathbf{X}_{jt}$ from endogenous variation in prices from unobserved product characteristics changes. They are commonly used to instrument prices with the idea that characteristics of competing products are correlated with prices since the (equilibrium) markup of each product depends on how close products are in characteristics space (products with close substitutes will tend to have low markups and thus low prices relative to costs). They are also appropriate instruments for market shares on the RHS of Equation (20).[16] BLP instruments can suffice for identification but cost shifters are useful in practice (see e.g., Reynaert and Verboven, 2014).

4.3 Link to Berry Inversion

Berry inversion consists in inverting the system that equates observed market shares to predicted market shares, in which the terms ξ_{jt} enter non-linearly in general, to get a system of equations in which the terms ξ_{jt} enter linearly. Inversion can be done analytically or numerically, depending on whether the inverse system has a closed form or not. The inverse system thus obtained serves as a basis for demand estimation. The inverse system thus obtained serves as a basis for demand estimation.

[16]This is because identifying the effects of markets shares in the inverse demand system amounts to identifying the effects of $\mathbf{v}$ on market shares and that BLP instruments directly shifts $\mathbf{v}$.

Berry et al. (2013) generalize Berry (1994)'s invertibility result and show that their "connected substitutes" structure is sufficient for invertibility. They require that (i) products be weak gross substitutes (i.e., everything else equal, an increase in δ_j weakly decreases demand q_i for all other products) and (ii) the "connected strict substitution" condition holds (i.e., there is sufficient strict substitution between products to treat them in one demand system). Their structure can accommodate models with complementary products, but the first requirement is not always satisfied in GEM, meaning that Berry et al. (2013)'s results are not applicable.

GEM provide the system (7) which is just the inverse system obtained by Berry inversion. This is because GEM are formulated in the space of market shares and not in the space of indirect utilities. In the GEM, the inverse system is thus directly available and has a known and analytic formula. When GEM lead to demands that do not have analytic formula, we obtain predicted demands by inverting numerically the system (7) just once after estimation. In contrast, in the BLP method, the inversion step is performed each time the GMM objective function is evaluated and is nested into the GMM minimization procedure. Note also that the GEM inversion and the BLP inversion go in opposite directions.

We establish the existence and uniqueness of the inverse system as follows. Existence follows by homogeneity and invertibility of the generator $\mathbf{S}$. Uniqueness is obtained by the normalization of the outside option's net utility, $\delta_{0t} = 0$ for all t. Our result supplements earlier results on demand invertibility in different settings (Berry, 1994; Beckert and Blundell, 2008; Berry et al., 2013), but is not implied by these results.

5 Empirical Application: Demand for Cereals

In this section, we apply the GEM structure to estimate the demand for ready-to-eat (RTE) cereals in Chicago in 1991 – 1992. The assumption of the GEM that consumers buy multiple products in varying quantities seems more appropriate for RTE cereals than the single-unit purchase assumption of the ARUM, thereby justifying the use of a GEM. In addition, to take into account the feature that the RTE cereals market exhibits product segmentation along several dimensions, we use the cross-nested GNE model proposed in Subsection 2.5.

5.1 Product Segmentation on the Cereals Market

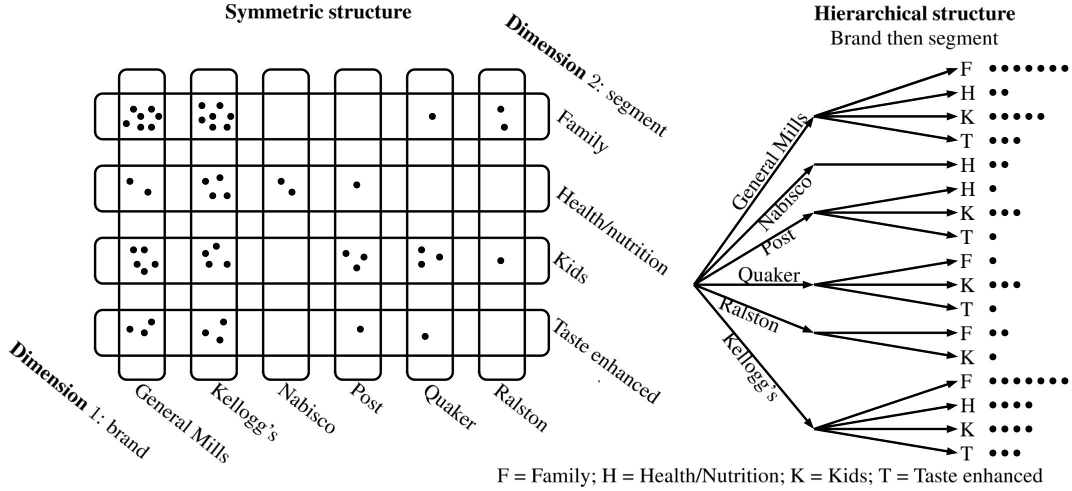
Data. We use data from the Dominick's Database made available by the James M. Kilts Center, University of Chicago Booth School of Business. This is weekly store-level scanner data, comprising information on 30 categories of packaged products at the UPC level for all Dominick's Finer Foods chain stores in the Chicago metropolitan area over the period 1989-1997. We consider the RTE cereal category during the period 1991–1992; and we supplement the data with the nutrient content of the RTE cereals using the USDA Nutrient Database for Standard Reference (fiber, sugar, lipid, protein, energy, and sodium), and with monthly sugar prices from the website www.indexmundi.com. Following the prevailing literature, we aggregate UPCs into brands (e.g., Kellogg's Special K), so that different size boxes are considered one brand, where a brand is a cereal (e.g., Special K) associated to its brand name (e.g., Kellogg's). We focus attention on the top 50 brands, which account for 73 percent of sales of the category in the sample we use. We define a product as a brand, and a market as a store-month pair. Market shares and prices are computed following Nevo (2001) (see Appendix F.1 for more details).

Product segmentation. For the application, we focus on two segmentation dimensions that form 17 product types: one measures the substitutability between products within the same market segment, where segments are family, kids, health/nutrition, and taste enhanced (see e.g., Nevo, 2001); and the other measures the advantages the brand-name reputation provides to the products, where brand names are General Mills, Kellogg's, Quaker, Post, Nabisco, and Ralston.

The structure of the cross-nested GNE model in the present application to cereals is illustrated in the left panel of Figure 2. Each dot illustrates the location of a product in the nesting structure and there are 17 non-empty types. The two segmentation dimensions are treated symmetrically in this model.

The right panel of Figure 2 illustrates one of the two NL models that are possible with the same two segmentations. The NL models have a hierarchical nesting structure, in which the second layer of nesting is a partitioning of the first. Both NL models can be represented as cross-nested GNE models and we estimate both for comparison. This is easily done using the same regression setup while changing only the nesting structure.

Figure 2: PRODUCT SEGMENTATION ON THE CEREALS MARKET



5.2 Estimation

We now turn to the estimation of the cross-nested GNE model. The demand equations (20) can be written as

$$\xi_{jt} = \mu_0 \ln\left(q_{jt}\right) + \sum_{c=1}^{C} \mu_c \ln\left(q_{\sigma_c(j),t}\right) - \ln\left(q_{0t}\right) - \left(\beta_0 + \mathbf{X}_{jt}\boldsymbol{\beta} - \alpha p_{jt}\right).$$

Using the parameter constraint $\mu_0 + \sum_{c=1}^{C} \mu_c = 1$, we obtain

$$\ln\left(\frac{q_{jt}}{q_{0t}}\right) = \beta_0 + \mathbf{X}_{jt}\boldsymbol{\beta} - \alpha p_{jt} + \sum_{c=1}^{C} \mu_c \ln\left(\frac{q_{jt}}{q_{\sigma_c(j),t}}\right) + \xi_{jt}, \tag{21}$$

for $j \in \mathscr{J} \setminus \{0\}$ and $t \in \{1, \ldots, T\}$, where $\boldsymbol{\theta} = (\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)$, with $\boldsymbol{\theta}_1 = (\beta_0, \boldsymbol{\beta}, \alpha)$ and $\boldsymbol{\theta}_2 = (\mu_1, \ldots, \mu_C)$, are the parameters to be estimated.

Equation (21) has the same form as the logit and NL equations (see Berry, 1994; Verboven, 1996a), except for the terms $\mu_c \ln\left(q_{jt}/q_{\sigma_c(j),t}\right)$; this suggests estimating the GNE model by a linear instrumental variables regression of market shares on product characteristics and terms related to market segmentation.

Price p_{jt} is endogenous due to the presence of the unobserved product characteristics ξ_{jt}. In addition, the C nesting terms $\ln\left(q_{jt}/q_{\sigma_c(j),t}\right)$ are endogenous by construction: any shock to ξ_{jt} that increases the dependent variable $\ln\left(q_{jt}/q_{0t}\right)$ also increases the nesting terms $\ln\left(q_{jt}/q_{\sigma_c(j),t}\right)$. Assuming that product characteristics are exogenous, identification requires finding at least one instrument for price and each of the C nesting terms.

We define markets as month-store pairs and products as brands. Following Bresnahan et al. (1997), we include brand name and segment fixed effects, ξ_s and ξ_b, and market-invariant continuous product characteristics $\mathbf{x}_j$ (i.e., fiber, sugar, lipid, protein, energy, and sodium). The fixed effects, ξ_b and ξ_s, capture market-invariant observed and unobserved brand name (i.e. company) and segment-specific characteristics. We also include month and store fixed effects, ξ_m and ξ_s, that capture monthly unobserved determinants of demand and time-invariant store characteristics, respectively. The structural error that remains in ξ_{jt} therefore captures the unobserved product characteristics varying across products and markets (e.g., changes in shelf-space, positioning of the products among others) that affect consumers utility and that consumers and firms (but not the modeller) observe so that they are likely to be correlated with prices.

We use two sets of instruments. First, as cost shifters, we use the market-level price of sugar times the sugar content of the cereals, interacted with brand name fixed effects. Multiplying the price of sugar by the sugar content allows the instrument to vary by product; and interacting this with fixed effects allows the price of sugar to enter the production function of each firm differently.

Second, we form BLP instruments by using other products' promotional activity in a given month, which varies both across stores for a given month and across months for a given store: for a given product, other products' promotional activity affects consumers' choices, and is thus correlated with the price of that product, but uncorrelated with the error term.[17] We use the number of other promoted products of rival firms and the number of other promoted products of the same firm, which we interact with brand names fixed effects. We also use these numbers over products belonging to the same segment, which we interact with segment fixed effects. We distinguish between products of the same firm and of rival firms, and interact instruments with brand name fixed effects with the idea that (equilibrium) markup is a function of the ownership structure since multi-product firms set prices so as to maximize their total profits. Interaction with segment fixed effects accounts for within-segment competitive conditions.

A potential problem is weak identification, which happens when instruments are only weakly correlated with the endogenous variables. With multiple endogenous variables, the

[17] The promotion is treated as an exogenous variable since, at Dominick's Finer Foods, the promotional calendar is known several weeks in advance of the weekly price decisions. In addition, we do not use functions of the continuous product characteristics as instruments since by construction of the data they are invariant across markets (see Nevo, 2001).

standard first-stage F-statistic is no longer appropriate to test for weak instruments. We therefore use Sanderson and Windmeijer (2016)'s F-statistic to test whether each endogenous variable is weakly identified. F-statistics are larger than 10, suggesting that we can be quite confident that instruments are not weak.

5.3 Empirical Results

Demand parameters. Table 1 presents the two-stage least squares (2SLS) estimates of demand parameters from the GNE model and the three-level NL models with nests for segment on top and with nests for brand on top, in columns (1), (2), and (3), respectively.

Table 1: PARAMETER ESTIMATES OF DEMAND

	(1) GNE		(2) 3NL1		(3) 3NL2	
Price ($-\alpha$)	-1.831	(0.116)	-2.908	(0.118)	-4.101	(0.156)
Promotion (β)	0.0882	(0.00278)	0.102	(0.00305)	0.144	(0.00365)
Constant (β_0)	-0.697	(0.0593)	-0.379	(0.0645)	-0.195	(0.0755)
Nesting Parameters (μ)						
Segment/nest (μ_1)	0.626	(0.00931)	0.771	(0.00818)	0.668	(0.0109)
Brand/subnest (μ_2)	0.232	(0.00944)	0.792	(0.00725)	0.709	(0.00961)
FE Segments (γ)						
Health/nutrition (γ_H)	-0.672	(0.00990)	-0.855	(0.00751)	-0.0693	(0.00538)
Kids (γ_K)	-0.433	(0.00875)	-0.529	(0.00869)	0.0705	(0.00522)
Taste enhanced (γ_T)	-0.710	(0.0102)	-0.903	(0.00747)	-0.0877	(0.00558)
FE Brand Names (θ)						
Kellogg's (θ_K)	0.0243	(0.00460)	-0.0563	(0.00344)	0.104	(0.00635)
Nabisco (θ_N)	-0.754	(0.0242)	-0.218	(0.0109)	-2.105	(0.0201)
Post (θ_P)	-0.485	(0.0144)	-0.187	(0.00830)	-1.364	(0.00931)
Quaker (θ_Q)	-0.553	(0.0150)	-0.329	(0.0137)	-1.508	(0.00653)
Ralston (θ_R)	-0.732	(0.0249)	-0.200	(0.0111)	-2.131	(0.0211)
Observations	99281		99281		99281	
RMSE	0.210		0.242		0.270	

Notes: The dependent variable is $\ln(q_{jt}/q_{0t})$. Regressions include fixed effects (FE) for brand names and segments, months, and stores, as well as the market-invariant continuous product characteristics (fiber, sugar, lipid, protein, energy, and sodium). Robust standard errors are reported in parentheses. The values of the F-statistics in the first stages suggest that weak instruments are not a problem.

Consider first the results from the cross-nested GNE model. The estimated parameters on the negative of price (α) and on promotion (β) are significantly positive. The estimated nesting parameters ($0 < \mu_2 < \mu_1 < 1$) are consistent with the GEM ($\mu_1 + \mu_2 < 1$); this provides an empirical check on the appropriateness of the cross-nested GNE model as no constraint was imposed on the estimates. The parameter estimates imply that there is product segmentation along both dimensions: products with the same brand name are closer substitutes than products with different brand names; and products within the same segment are closer substitutes than products from different segments. Overall, products of the same type are closer substitutes.

The advantages provided by the two dimensions are parametrized by the segment and brand name fixed effects (the γ's and θ's) and the nesting parameters (μ_1 and μ_2). The fixed effects measure the extent to which belonging to a nest shifts the demand for the product, and the nesting parameters measure the extent to which products within a nest are protected from competition from products from different nests along each dimension.

We find that the brand-name reputation of the cereals confers a significant advantage to products from General Mills and Kellogg's ($\theta_K > \theta_G = 0 > \theta_P > \theta_Q > \theta_R > \theta_N$); moreover cereals for family also benefit from a significant advantage ($\gamma_F = 0 > \gamma_K > \gamma_H > \gamma_T$). In addition, we find that $\mu_1 > \mu_2$, i.e., the segments confer more protection from competition than brand-name reputation does (products within the same segment are more protected from products from different segments than products with the same brand name are from products with different brand names).

Turn now to the results from the three-level NL models. They are both consistent with random utility maximization ($\mu_2 > \mu_1$), which means that it is not possible to decide between them based on this criterion. However, the Rivers and Vuong (2002) test strongly rejects both NL models in favor of the GNE model.[18]

Alternative specification with very large choice set. In many situations, consumers face choices involving a very large number of products (e.g., choice of a car or of a breakfast cereal). We have estimated an alternative model in which all brand-store combinations are considered as choice alternatives, as it is common in the vertical relationships literature (see e.g., Villas-Boas, 2007), while markets are taken to be months. The resulting model has more than 4,000 products, but was estimated very quickly without any issues. This shows

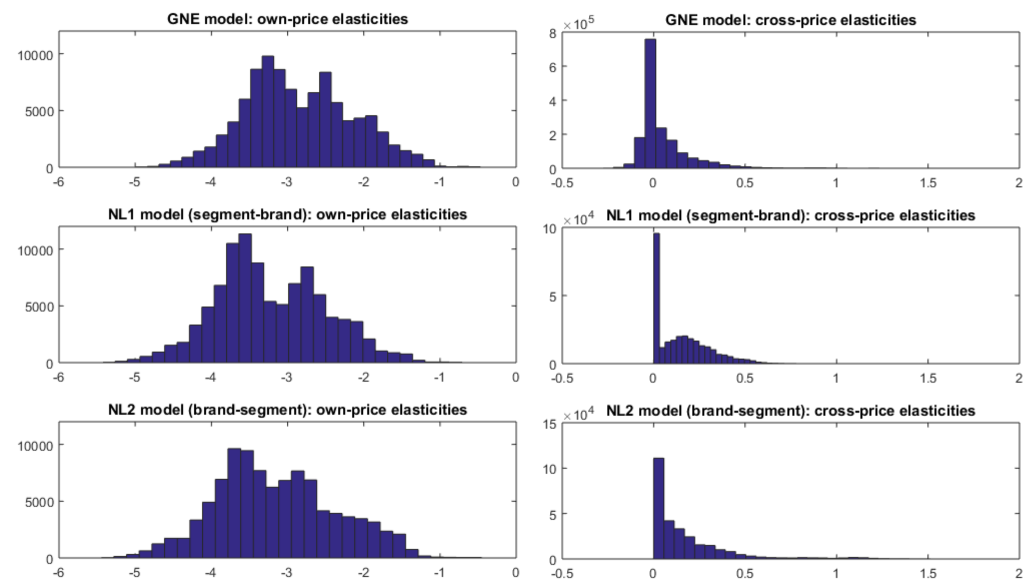
[18] The test statistic is given by $T_N = \frac{\sqrt{N}}{\hat{\sigma}} \left(\hat{Q}_1 - \hat{Q}_2 \right)$, where N is the number of observations, $\hat{Q}_i$ is the value of the estimated RMSE of model i, and $\hat{\sigma}^2$ is the estimated value of the variance of the difference between $\hat{Q}_i$'s. This statistic must be evaluated against the standard normal distribution and we estimate $\hat{\sigma}^2$ using 500 bootstrap replications. The test statistics of the two NL models (model 1 in the statistic) against the GNE model (model 2) are 2891.97 and 4879.82, respectively.

the ability of the GNE model to deal with very large choice sets.

The parameter estimates were not significantly affected by this change in specification, which indicates that the results are fairly robust.

Substitution patterns. Figure 3 presents the estimated density of the own- and cross-price elasticities of demands of the cross-nested GNE and NL models (see Tables 7 and 8 in Appendix F.2 for the estimated own- and cross-price elasticities of demands, averaged across markets and within product types). We compute elasticities after estimation in two steps. First, we compute the predicted market shares by solving the system of nonlinear equations (7) derived in Theorem 1, which amounts to solve the utility maximization program (2). As seen before, from Lemma 1, the solution exists and is unique. Second, we compute the matrix of derivatives using the formula obtained in Proposition 2 where the generator $\mathbf{S}$ is given by Equation (13), which we pre-multiply by the prices and post-multiply by the predicted market shares to obtain the elasticities (see Appendix F.1 for more details).

Figure 3: ESTIMATED ELASTICITIES



The estimated own-price elasticities are in line with the literature (see e.g., Nevo, 2001). On average, the estimated own-price elasticity of demands is -2.815 for the cross-nested GNE model. However, there is an important variation in price responsiveness across product types: demands for cereals for kids produced by General Mills exhibit a much higher own-price elasticity than cereals for health/nutrition produced by Post (-3.427 vs. -1.524).

Consider the cross-price elasticities. Among the 17×50 different cross-price elasticities in the cross-nested GNE model, 48.5 percent (resp., 51.5 percent) are negative (resp., positive), meaning that some cereals are substitutes while others are complements. For example, cereals for families produced by General Mills are complementary to those with taste enhanced produced by Kellogg's; but are substitutable with those for kids produced by General Mills.

6 Conclusion

In this paper, we have developed the class of generalized entropy models (GEM). They are based on the representative consumer approach of demand for differentiated products. In the GEM, the representative consumer's taste for variety is captured by a generalization of the Shannon entropy, which allows for general relationships in taste for variety.

We employ these models for two purposes. First, we explore the linkages between the ARUM and the RCM. We show that the class of GEM is strictly larger than the class of ARUM: the GEM structure allows to recover all ARUM, but the reverse is not true. In particular, in contrast to any ARUM, GEM allow for complementarity. This is a very attractive feature since complementarity is likely to occur in many markets. In our empirical application, we find that complementarity is a salient feature of the ready-to-eat cereals market, since about one half of the products are estimated to be complements. The presence of complementarity has important implications for many economic questions, such as the effect of a merger and the incentive to introduce a new product on the market.

Second, we use GEM for estimating the demand for differentiated products with aggregate data. GEM facilitate the BLP method. With the ARUM, the method requires inverting the demand system, which cannot generally be performed analytically. Berry et al. (1995) thus propose inverting numerically using a contraction mapping nested into a GMM minimization procedure, which must be performed each time the GMM objective function is evaluated. With the GEM, the inverse demand system is directly available, so that GEM can be estimated using the BLP method while avoiding this inversion step. However, some GEM lead to demands which do not have analytic formula, suggesting that must be done once to obtain predicted demands after estimation is complete.

With the GEM, we have opened the door to a new family of models. The further development of GEM provides many opportunities for research. Specifically, the models and methods developed in this paper can be extended in two ways. First, we have developed methods to estimate demands using aggregate data. It remains to study how the methods can be adapted to individual-level data. Second, we have considered static choice models. Further work to develop dynamic GEM that parallel the dynamic discrete choice model of Rust (1987) can help us to better understand the behavior of forward-looking consumers.

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A Preliminaries

For easy reference, this section states some mathematical results that are used in the proofs below.

Lemma 3. Let $\phi : \mathbb{R}^n_+ \to \mathbb{R}$ and $F : \mathbb{R} \to \mathbb{R}$ be two continuous and differentiable functions. Define $f : \mathbb{R}^n_+ \to \mathbb{R}$ by $f(\mathbf{x}) = F(\phi(\mathbf{x}))$, with $\mathbf{x} = (x_1, \ldots, x_n)^\intercal$, and $h : \mathbb{R} \to \mathbb{R}$ by $h = F^{-1}$. Assume that ϕ is linearly homogeneous. Then,

a. (Euler equation for homogeneous functions)

$$\phi(\mathbf{x}) = \sum_{i=1}^{n} \frac{\partial \phi(\mathbf{x})}{\partial x_i} x_i.$$

b. (Generalized Euler equation for homothetic functions (McElroy, 1969)) If F is non-decreasing, then f is homothetic, and

$$\sum_{i=1}^{n} \frac{\partial f(\mathbf{x})}{\partial x_i} x_i = \frac{h(y)}{h'(y)}.$$

Proof. a. See e.g., proof of Theorem M.B.2. in Mas-Colell et al. (1995).
b. Consider $h(y) = \phi(\mathbf{x})$. Differentiate with respect to x_i and rearrange terms to get

$$\frac{\partial y}{\partial x_i} = \frac{1}{h'(y)} \frac{\partial \phi(\mathbf{x})}{\partial x_i}.$$

Then

$$\begin{aligned}\sum_{i=1}^{n} \frac{\partial f(\mathbf{x})}{\partial x_i} \frac{x_i}{y} &= \sum_{i=1}^{n} \frac{\partial y}{\partial x_i} \frac{x_i}{y} = \sum_{i=1}^{n} \frac{1}{h'(y)} \frac{\partial \phi(\mathbf{x})}{\partial x_i} \frac{x_i}{y}, \\ &= \frac{1}{h'(y)\, y} \sum_{i=1}^{n} \frac{\partial \phi(\mathbf{x})}{\partial x_i} x_i = \frac{h(y)}{h'(y)\, y},\end{aligned}$$

where the last equality uses a. applied to the homogeneous function ϕ. Multiplying both side by y yields the required equality. $\square$

A matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ is said to be positive quasi-definite if its symmetric part $\frac{1}{2}\left(\mathbf{A}+\mathbf{A}^{\intercal}\right)$ is positive definite.

Lemma 4 (Gale and Nikaido 1965, Theorem 6)**.** If a differentiable mapping $\mathbf{F} : \Theta \to \mathbb{R}^n$, where Θ is a convex region (either closed or non-closed) of $\mathbb{R}^n$, has a Jacobian matrix that is everywhere quasi-definite in Θ, then $\mathbf{F}$ is injective on Θ.

Lemma 5 (Simon and Blume, 1994, Theorem 14.4)**.** Let $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ and $\mathbf{G} : \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable functions. Let $\mathbf{y} \in \mathbb{R}^n$ and $\mathbf{x} = \mathbf{G}\left(\mathbf{y}\right) \in \mathbb{R}^n$. Consider the composite function

$$\mathbf{C} = \mathbf{F} \circ \mathbf{G} : \mathbb{R}^n \to \mathbb{R}^n.$$

Let $\mathbf{J_F}\left(\mathbf{x}\right) \in \mathbb{R}^{n \times n}$ be the Jacobian matrix of the partial derivatives of $\mathbf{F}$ at $\mathbf{x}$, and let $\mathbf{J_G}\left(\mathbf{y}\right) \in \mathbb{R}^{n \times n}$ be the Jacobian matrix of the partial derivatives of $\mathbf{G}$ at $\mathbf{y}$. Then the Jacobian matrix $\mathbf{J_C}\left(\mathbf{y}\right)$ is given by the matrix product of the Jacobians as

$$\mathbf{J_C}\left(\mathbf{y}\right) = \mathbf{J_{F \circ G}}\left(\mathbf{y}\right) = \mathbf{J_F}\left(\mathbf{x}\right)\mathbf{J_G}\left(\mathbf{y}\right).$$

B Proofs for Section 2

Proof of Lemma 1. Lemma 1 is implied by Lemma 6. □

Lemma 6. Assume that $\mathbf{S}$ is twice continuously differentiable and linearly homogeneous. Then,

a. $\mathbf{J}_{\ln \mathbf{S}}$ is symmetric on $\operatorname{int}\left(\Delta\right)$ if and only if

$$\sum_{j \in \mathscr{J}} q_j \frac{\partial \ln S^{(j)}\left(\mathbf{q}\right)}{\partial q_k} = 1,\ k \in \mathscr{J},\ \forall \mathbf{q} \in \operatorname{int}\left(\Delta\right). \tag{22}$$

b. If $\mathbf{J}_{\ln \mathbf{S}}$ is symmetric and positive definite on $\operatorname{int}\left(\Delta\right)$, then Ω is strictly concave on $\operatorname{int}\left(\Delta\right)$.

c. If $\mathbf{J}_{\ln \mathbf{S}}$ is positive definite, then $\mathbf{S}$ is invertible on $\operatorname{int}\left(\Delta\right)$.

Proof of Lemma 6. a. Assume that $\mathbf{J}_{\ln \mathbf{S}}$ is symmetric. $S^{(k)}$ is linearly homogeneous, then $\ln S^{(k)}$ is homothetic.

Let $\phi(\mathbf{q}) = S^{(k)}(\mathbf{q})$ and $F(\mathbf{q}) = \ln(\mathbf{q})$, then $h(\delta) = \exp(\delta)$. Define $\delta = f(\mathbf{q}) = F(\phi(\mathbf{q})) = \ln\left(S^{(k)}(\mathbf{q})\right)$ and $h(\delta) = \phi(\mathbf{q}) = \exp(\delta)$. Then, by Lemma 3, $S^{(k)}$ satisfies

$$\sum_{j\in\mathscr{J}} q_j \frac{\partial \ln S^{(k)}(\mathbf{q})}{\partial q_j} = \frac{\exp(\delta)}{\exp(\delta)\,\delta}\delta = 1.$$

By symmetry of $\mathbf{J}_{\ln \mathbf{S}}$, we end up with

$$\sum_{j\in\mathscr{J}} q_j \frac{\partial \ln S^{(j)}(\mathbf{q})}{\partial q_k} = 1.$$

Assume now that $\sum_{j\in\mathscr{J}} q_j \frac{\partial \ln S^{(j)}(q)}{\partial q_k} = 1$. Then, for each $j, k \in \mathscr{J}$,

$$\frac{\partial \Omega(\mathbf{q})}{\partial q_j} = -\ln S^{(j)}(\mathbf{q}) - 1; \quad \frac{\partial \Omega(\mathbf{q})}{\partial q_k} = -\ln S^{(k)}(\mathbf{q}) - 1,$$

so that

$$\frac{\partial^2 \Omega(\mathbf{q})}{\partial q_j \partial q_k} = -\frac{\partial \ln S^{(j)}(\mathbf{q})}{\partial q_k}; \quad \frac{\partial^2 \Omega(\mathbf{q})}{\partial q_k \partial q_j} = -\frac{\partial \ln S^{(k)}(\mathbf{q})}{\partial q_j}.$$

Since Ω is twice continuously differentiable, then by Schwarz's theorem,

$$\frac{\partial^2 \Omega(\mathbf{q})}{\partial q_j \partial q_k} = \frac{\partial^2 \Omega(\mathbf{q})}{\partial q_k \partial q_j},$$

i.e.,

$$\frac{\partial \ln S^{(j)}(\mathbf{q})}{\partial q_k} = \frac{\partial \ln S^{(k)}(\mathbf{q})}{\partial q_j},$$

Then $\mathbf{J}_{\ln \mathbf{S}}$ is symmetric as required.

b. From Part 1, we find that $\mathbf{J}_{\ln \mathbf{S}}(\mathbf{q}) = \nabla^2_{\mathbf{q}}(-\Omega(\mathbf{q}))$, for all $\mathbf{q} \in \operatorname{int}(\Delta)$. Then Ω is strictly concave by positive definiteness of $\mathbf{J}_{\ln \mathbf{S}}$.

c. The function $\ln \mathbf{S}$ is differentiable on the convex region $\operatorname{int}(\Delta)$ of $\mathbb{R}^{J+1}$. In addition, $\mathbf{J}_{\ln \mathbf{S}}$ is positive quasi-definite on $\operatorname{int}(\Delta)$, since its symmetric part $\frac{1}{2}\left(\mathbf{J}_{\ln \mathbf{S}} + (\mathbf{J}_{\ln \mathbf{S}})^{\intercal}\right) = \mathbf{J}_{\ln \mathbf{S}}$ is positive definite on $\operatorname{int}(\Delta)$. Then, by Lemma 4, $\ln \mathbf{S}$ is injective, implying that $\mathbf{S}$ is injective. □

Proof of Theorem 1. The Lagrangian of the GEM is

$$\mathcal{L}\left(\mathbf{q},\lambda,\lambda_0,\ldots,\lambda_J\right)=\alpha y+\sum_{j\in\mathscr{J}}\delta_j q_j-\sum_{j\in\mathscr{J}}q_j\ln S^{(j)}\left(\mathbf{q}\right)+\lambda\left(1-q_j\right)+\sum_{j\in\mathscr{J}}\lambda_j q_j,$$

where $\lambda\geq 0$ and $\lambda_j\geq 0$ for all $j\in\mathscr{J}$.

The first-order conditions are

$$\delta_i-\ln S^{(i)}\left(\mathbf{q}\right)-\sum_{j\in\mathscr{J}}q_j\frac{\partial\ln S^{(j)}\left(\mathbf{q}\right)}{\partial q_k}-\lambda+\lambda_i=0,\quad i\in\mathscr{J},$$
$$\sum_{j\in\mathscr{J}}q_j=1.$$

Using Lemma 1, we get

$$\delta_i-\ln S^{(i)}\left(\mathbf{q}\right)-1-\lambda+\lambda_i=0,\quad i\in\mathscr{J},$$
$$\sum_{j\in\mathscr{J}}q_j=1.$$

Observe that if $\mathbf{q}\in\operatorname{bd}\left(\Delta\right)$, then $|\ln\mathbf{S}(\mathbf{q})|=+\infty$ by Assumption 1. Hence, $\mathbf{q}$ cannot solve the first-order conditions, since the λ_i's must be finite. Therefore the solution must be interior with $\lambda_i=0$ for all $i\in\mathscr{J}$. Then the first-order conditions reduce to

$$\mathbf{S}\left(\mathbf{q}\right)=e^{\boldsymbol{\delta}-1-\lambda}>0, \tag{23}$$
$$\sum_{j\in\mathscr{J}}q_j=1. \tag{24}$$

The linear homogeneity of $\mathbf{S}$ implies that also $\mathbf{H}=\mathbf{S}^{-1}$ is linearly homogeneous. Then (23) yields

$$\mathbf{q}=\mathbf{S}^{-1}\left(e^{\boldsymbol{\delta}-1-\lambda}\right)=\mathbf{H}\left(e^{\boldsymbol{\delta}-1-\lambda}\right)=e^{-(1+\lambda)}\mathbf{H}\left(e^{\boldsymbol{\delta}}\right).$$

Lastly, (24) implies that $e^{1+\lambda}=\sum_{j\in\mathscr{J}}H^{(j)}\left(e^{\boldsymbol{\delta}}\right)$ such that any solution to the first-order conditions satisfies

$$q_i=\frac{H^{(i)}\left(e^{\boldsymbol{\delta}}\right)}{\sum_{j\in\mathscr{J}}H^{(j)}\left(e^{\boldsymbol{\delta}}\right)},\quad i\in\mathscr{J}. \tag{25}$$

The strict concavity of the utility u on $\operatorname{int}\left(\Delta\right)$ implies that this solution is unique and is the argmax to the utility maximization problem.

***Relation (7) between δ and* q.** Note that if $\mathbf{q}$ is an interior solution to the utility maximization problem then it satisfies Equation (6), which, by invertibility and linear homogeneity of $\mathbf{S}$ implies that

$$\ln S^{(i)}(\mathbf{q}) + \ln\left(\sum_{j\in\mathscr{J}} H^{(j)}\left(e^{\boldsymbol{\delta}}\right)\right) = \delta_i, \quad i \in \mathscr{J}.$$

Conversely, if $\forall i \in \mathscr{J}$, we have $\delta_i = \ln S^{(i)}(\mathbf{q}) + \ln\left(\sum_{j\in\mathscr{J}} H^{(j)}\left(e^{\boldsymbol{\delta}}\right)\right)$, then $\mathbf{q}$ solves (6). □

Proof of Proposition 1. The surplus function G is defined by

$$G(\boldsymbol{\delta}) = \sum_{j\in\mathscr{J}} \delta_j q_j(\boldsymbol{\delta}) + \Omega(\mathbf{q}(\boldsymbol{\delta})),$$

with $q_j(\boldsymbol{\delta})$ given by (6). The log-sum (9) results substituting $q_j(\boldsymbol{\delta})$ by (6).

We now show that demands (6) satisfy Roy's identity, i.e.,

$$q_j(\boldsymbol{\delta}) = \frac{\partial G(\boldsymbol{\delta})}{\partial \delta_j}.$$

Let $\boldsymbol{\delta} = \ln \mathbf{S}(\mathbf{q})$, so that $(\ln \mathbf{S})^{-1}(\boldsymbol{\delta}) = \mathbf{H}\circ\exp(\boldsymbol{\delta}) = \mathbf{q}$. Then by Lemma 5,

$$\mathbf{J}_{\ln \mathbf{S}}(\mathbf{q}) = \left[\mathbf{J}_{(\ln \mathbf{S})^{-1}}(\ln \mathbf{S}(\mathbf{q}))\right]^{-1} = \left[\mathbf{J}_{\mathbf{H}\circ\exp}(\boldsymbol{\delta})\right]^{-1}. \tag{26}$$

Since $\mathbf{J}_{\ln \mathbf{S}}(\mathbf{q})$ is symmetric, $\mathbf{J}_{\mathbf{H}\circ\exp}$ is also symmetric, i.e.,

$$\frac{\partial H^{(i)}\left(e^{\boldsymbol{\delta}}\right)}{\partial \delta_j} = \frac{\partial H^{(j)}\left(e^{\boldsymbol{\delta}}\right)}{\partial \delta_i}. \tag{27}$$

This is because every positive definite matrix is invertible, and the inverse of a symmetric matrix is also a symmetric matrix. Then,

$$\begin{aligned}\frac{\partial G\left(e^{\boldsymbol{\delta}}\right)}{\partial \delta_i} &= \frac{\sum_{k\in\mathscr{J}} \frac{\partial H^{(k)}\left(e^{\boldsymbol{\delta}}\right)}{\partial \delta_i}}{\sum_{j\in\mathscr{J}} H^{(j)}\left(e^{\boldsymbol{\delta}}\right)} = \frac{\sum_{k\in\mathscr{J}} \frac{\partial H^{(i)}\left(e^{\boldsymbol{\delta}}\right)}{\partial \delta_k}}{\sum_{j\in\mathscr{J}} H^{(j)}\left(e^{\boldsymbol{\delta}}\right)}, \\ &= \frac{\sum_{k\in\mathscr{J}} \frac{\partial H^{(i)}\left(e^{\boldsymbol{\delta}}\right)}{\partial e^{\delta_k}} e^{\delta_k}}{\sum_{j\in\mathscr{J}} H^{(j)}\left(e^{\boldsymbol{\delta}}\right)} = \frac{H^{(i)}\left(e^{\boldsymbol{\delta}}\right)}{\sum_{j\in\mathscr{J}} H^{(j)}\left(e^{\boldsymbol{\delta}}\right)},\end{aligned}$$

where the second equality comes from the symmetry of $\mathbf{J}_{\mathbf{H}\circ\exp}$, and the last equality comes from the Euler equation (in Lemma 3) applied to the linearly homogeneous function $H^{(i)}$. $\square$

Proof of Proposition 2. From Theorem 1, (7) we obtain

$$\mathbf{I} = \mathbf{J}_{\ln \mathbf{S}}(\mathbf{q})\mathbf{J}_{\mathbf{q}} + \mathbf{1}\mathbf{q}^{\intercal}.$$

This can be solved to obtain the desired result since $\mathbf{J}_{\ln \mathbf{S}}(\mathbf{q})$ is invertible. $\square$

Proof for Example 2. Consider the generator $\mathbf{S}$ in Example 2 and write the corresponding first-order conditions (23) and (24). Differentiating them with respect to δ_0, we obtain the following system of equations:

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{\mu}{q_0} + \frac{1-\mu}{q_0+q_1/2} & \frac{(1-\mu)/2}{q_0+q_1/2} & 0 & 1 \\ \frac{(1-\mu)/2}{q_0+q_1/2} & \frac{\mu}{q_1} + \frac{(1-\mu)/4}{q_0+q_1/2} + \frac{(1-\mu)/4}{q_1/2+q_2} & \frac{(1-\mu)/2}{q_1/2+q_2} & 1 \\ 0 & \frac{(1-\mu)/2}{q_1/2+q_2} & \frac{\mu}{q_2} + \frac{(1-\mu)}{q_1/2+q_2} & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial q_0}{\partial \delta_0} \\ \frac{\partial q_1}{\partial \delta_0} \\ \frac{\partial q_2}{\partial \delta_0} \\ \frac{\partial \lambda}{\partial \delta_0} \end{pmatrix}$$

This can be solved to find that $\partial q_2/\partial \delta_0 > 0$ if and only if

$$\mu < \frac{q_1^2 + q_0 q_1 + q_1 q_2}{4q_0 q_2 + 3q_1 q_2 + 2q_1^2 + 3q_0 q_1},$$

and noting that $q_0 + q_1 + q_2 = 1$, if and only if

$$\mu < \frac{q_1}{4q_0 q_2 + 3q_1 q_2 + 2q_1^2 + 3q_0 q_1}.$$

$\square$

Proof of Proposition 5. Using the relation (7) between $\boldsymbol{\delta}$ and $\mathbf{q}$, we get, for any pair of products j and k,

$$\frac{q_j(\boldsymbol{\delta})}{q_k(\boldsymbol{\delta})} = \exp\left(\frac{\delta_j - \delta_k}{\mu_0} + \sum_{c=1}^{C} \frac{\mu_c}{\mu_0} \ln\left(\frac{q_{\sigma_c(k)}(\boldsymbol{\delta})}{q_{\sigma_c(j)}(\boldsymbol{\delta})}\right)\right). \tag{28}$$

Then, for products j and k of the same type (i.e., with $\sigma_c(k) = \sigma_c(j)$ for all c), (28) reduces to $\frac{q_j}{q_k} = \exp\left(\frac{\delta_j - \delta_k}{\mu_0}\right)$, and, in turn, the ratio q_j/q_k is independent of the characteristics or existence of all other products, i.e., IIA holds for products of the same type. However, for any two products of different types, this ratio can depend on the characteristics of other products, so that IIA does not hold in general for products of different types. □

C Numerical properties of the Cross-Nested GNE Model

Using Proposition 2, the matrix of own- and cross-price derivatives for the cross-nested GNE model is given by

$$\mathbf{J_q} = -\alpha \boldsymbol{\Psi}\left(\boldsymbol{\Theta}; \boldsymbol{\mu}\right) \operatorname{diag}\left(\mathbf{q}\right) \left[\mathbf{I}_{J+1} - \mathbf{1}_{J+1}\mathbf{q}^{\intercal}\right], \tag{29}$$

where

$$\boldsymbol{\Psi}\left(\boldsymbol{\Theta}; \boldsymbol{\mu}\right) = \left[\mu_0 \mathbf{I}_{J+1} + \sum_{c=1}^{C} \mu_c \boldsymbol{\Theta}_c \mathbf{Q}_{\sigma_c}\right]^{-1},$$

with $\boldsymbol{\Theta}_c$ given by (12) and $\mathbf{Q}_{\sigma_c}$ being the diagonal matrix of the market shares of products within their nest $\sigma_c(j)$, i.e., $\left(\mathbf{Q}_{\sigma_c}\right)_{jj} = \frac{q_j}{q_{\sigma_c(j)}}$. This means that we cannot obtain an analytic formula for each entry of the matrix of own- and cross-price derivatives independently. We therefore perform simulations to better understand substitution and complementarity patterns the cross-nested GNE model can accommodate.

To do so, we simulate NS different nesting structures (i.e. allocations of products in nests) along C dimensions (with M nests per dimension), NS different vectors of nesting parameters $\boldsymbol{\mu} = (\mu_0, \ldots, \mu_C)$, and NS different vectors of market shares $\mathbf{q} = (q_0, \ldots, q_J)$. Setting $NS = 20$, $C = 3$, $M = 3$, and $J = 30$, we end up with $8,000$ market structures by combining these dimensions. We obtain (i) a nesting structure by simulating a $NS \times C$ matrix of binomial random numbers; (ii) a vector of nesting parameters by simulating a $(C+1)$ vector of uniformly distributed random numbers where the first element is μ_0, then by normalizing the vector of the other nesting parameters to get a unit vector $\boldsymbol{\mu}$; (iii) a vector of market shares by simulating a $(J+1)$ vector of uniformly distributed random numbers where the first element is q_0, then by normalizing the vector of market shares of inside products to get a unit vector $\mathbf{q}$. The normalizations are to simulate markets with very low and very high values for μ_0 and q_0.

The following table gives summary statistics on the simulated data:

Table 2: Summary Statistics on the Simulated Data

Variable	Mean	Min	Max
q_0	0.5253	0.0064	0.9906
q_j	0.0158	3e-06	0.0697
μ_0	0.4662	0.0697	0.9532
μ_1	0.2014	0.0135	0.8480
μ_2	0.1420	0.0175	0.4036
μ_3	0.1904	0.0059	0.5212

Nesting structure. Table 3 shows the distribution of the own- and cross-price derivatives for the simulated data according to the proximity of the products in the characteristics space used to form product types.

Own-price elasticities are always negative, while cross-price elasticities can be either negative (complementarity) or positive (substitutability). Products of the same type are always substitutable. As products become different, products are less likely to be substitutable. Products that are very similar (i.e., that are grouped together according to all the dimensions, but one) are always substitutable too. However, products that are completely different can be either substitutable or complementary. To summarize, complementarity may or may not arise for products that are of different types, while products of the same type are always substitutable.

Table 3: Distribution of Price Derivatives according to the Number of Common Nests

Same nests	$\mathbf{J_q} > 0$	Median	Min	Max	Freq.
Own-price derivatives					
–	0.00%	-0.0222	-0.7781	-3e-06	100.00%
Cross-price derivatives					
0 (None)	45.33%	-7e-07	-0.1539	0.0251	25.09%
1	90.38%	0.0002	-0.1114	0.2082	43.59%
2	100.00%	0.0006	-1e-09	0.2641	26.47%
3 (All)	100.00%	0.0009	-1e-09	0.3100	4.85%
Total	82.09%	0.0002	-0.1539	0.3100	100.00%

Notes: Column "$\mathbf{J_q} > 0$" gives the percentage of positive cross-price elasticities (i.e., the percentage of substitutable products). Column "Freq." gives the frequencies (in percentage) of the cross-price elasticities (e.g., 4.85 percent of the cross-price elasticities involve products of the same type).

Nesting parameters. Table 4 shows the distribution of cross-price derivative according to the level of the closeness of products, as measured by the sum of nesting parameters $\mu_{jk} = \sum_{c=1}^{3} \mu_i \mathbf{1} \{j \in \sigma_c(k)\}$ for two products j and k.

As the parameter μ_{jk} increases, we observe first that the size of the derivatives decreases in their negatives values, and increases in their positive values; then that the share of substitutable products increases. This comes from the fact that a higher value of μ_{ik} indicates that products j and k are perceived as more similar.

Table 4: PERCENTAGE OF SUBSTITUTES ACCORDING TO THE VALUE OF μ_{jk}

μ_{jk}	$\mathbf{J_q} > 0$	Median	Min	Max
$[0, 0.1[$	65.60%	0.0000	-0.1539	0.0286
$[0.1, 0.2[$	96.37%	0.0002	-0.0538	0.1462
$[0.2, 0.3[$	93.52%	0.0003	-0.1114	0.1670
$[0.3, 0.4[$	94.16%	0.0007	-0.0673	0.2082
$[0.4, 0.5[$	93.89%	0.0009	-0.0432	0.2049
$[0.5, 0.6[$	100.00%	0.0020	1e-08	0.2295
$[0.6, 0.7[$	100.00%	0.0026	3e-08	0.2339
$[0.7, 0.8[$	100.00%	0.0032	3e-08	0.2641
$[0.8, 0.9[$	100.00%	0.0041	6e-08	0.1615
$[0.9, 1[$	100.00%	0.0130	2e-07	0.3100

D Supplemental Material and Proofs for Section 3

Define $\Lambda = \left\{ \boldsymbol{\delta} : \sum_j \delta_j = 0 \right\}$ as the tangent space of Δ. The following lemma collects some properties of the expected maximum utility $\overline{G}$.

Lemma 7. The surplus $\overline{G}$ has the following properties.

a. $\overline{G}$ is twice continuously differentiable, convex and finite everywhere.

b. $\overline{G}(\boldsymbol{\delta} + c\mathbf{1}) = \overline{G}(\boldsymbol{\delta}) + c$ for any $c \in \mathbb{R}$.

c. The Hessian of $\overline{G}$ is positive definite on Λ.

d. $\overline{G}$ is given in terms of the expected residual of the maximum utility product by

$$\overline{G}(\boldsymbol{\delta}) = \sum_{j \in \mathscr{J}} P_j(\boldsymbol{\delta}) \delta_j + \mathbb{E}(\varepsilon_{j^*} | \boldsymbol{\delta}).$$

Proof of Lemma 7. McFadden (1981) establishes convexity and finiteness of $\overline{G}$ as well as the homogeneity property (b.) and the existence of all mixed partial derivatives up to order J. This also implies that all second order mixed partial derivatives are continuous, since $J \geq 2$. Hofbauer and Sandholm (2002) show that the Hessian of $\overline{G}$ is positive definite on Λ (see the proof of their Theorem 2.1).

Let j^* be the index of the chosen product. The last statement of the lemma follows using the law of iterated expectations:

$$\begin{aligned}\overline{G}(\boldsymbol{\delta}) &= \sum_{j \in \mathscr{J}} \mathbb{E}\left(\max_{j \in \mathscr{J}} \{\delta_j + \varepsilon_j\} \,|j^* = j, \boldsymbol{\delta}\right) P_j(\boldsymbol{\delta}), \\ &= \sum_{j \in \mathscr{J}} \left(\delta_j + \mathbb{E}\left(\varepsilon_{j^*}|j^* = j, \boldsymbol{\delta}\right) P_j(\boldsymbol{\delta})\right), \\ &= \sum_{j \in \mathscr{J}} P_j(\boldsymbol{\delta})\,\delta_j + \mathbb{E}\left(\varepsilon_{j^*}|\boldsymbol{\delta}\right).\end{aligned}$$

□

Proof of Lemma 2. *Invertibility of* $\overline{\mathbf{H}}$. Note first that $\overline{\mathbf{H}}$ is differentiable.

In addition, the Jacobian of $\boldsymbol{\delta} \to \overline{\mathbf{H}}\left(e^{\boldsymbol{\delta}}\right)$, labeled $\mathbf{J}_{\overline{\mathbf{H}}}$, is positive quasi-definite on Λ. The Jacobian $\mathbf{J}_{\overline{\mathbf{H}}}$ has elements ij given by

$$\left\{e^{\overline{G}(\boldsymbol{\delta})}\overline{G}_i(\boldsymbol{\delta})\,\overline{G}_j(\boldsymbol{\delta})\right\} + \left\{e^{\overline{G}(\boldsymbol{\delta})}\overline{G}_{ij}(\boldsymbol{\delta})\right\}.$$

The first matrix is positive semi-definite. By part d. of Lemma 7, the second matrix is positive definite on Λ. The Jacobian is therefore positive definite on Λ. Lastly, since $\mathbf{J}_{\overline{\mathbf{H}}}$ is symmetric, its symmetric part is itself, and thus positive quasi-definiteness of $\mathbf{J}_{\overline{\mathbf{H}}}$ is equivalent to its positive definiteness. Then, by Lemma 4, $\overline{\mathbf{H}}$ is invertible on the range $\overline{\mathbf{H}}(e^{\Lambda})$. Global invertibility follows, since by the homogeneity property we have for $\boldsymbol{\delta} \in \mathbb{R}^{J+1}$ that

$$\overline{\mathbf{H}}(e^{\boldsymbol{\delta}}) = e^{\mathbf{1}^\intercal \boldsymbol{\delta}} e^{\overline{G}(\boldsymbol{\delta} - \mathbf{1}_{JJ}\boldsymbol{\delta})}\mathbf{P}(\boldsymbol{\delta} - \mathbf{1}_{JJ}\boldsymbol{\delta}).$$

The range of $\overline{\mathbf{H}}$ is $\mathbb{R}^{J+1}_{++}$ since the range of $\mathbf{P}$ is the interior of Δ. To conform to the definition of a generator, we need to extend $\overline{\mathbf{H}}$ continuously to have domain and range $\mathbb{R}^{J+1}_{+}$. Fosgerau et al. (2017, Proposition 2) show that $\overline{\mathbf{H}}$ does in fact have such a continuous and invertible extension $\overline{\overline{\mathbf{H}}}$.[19] We may therefore define a candidate generator $\overline{\mathbf{S}} : \mathbb{R}^{J+1}_{+} \to$

[19] The argument is fairly long, so we do not repeat it here.

$\mathbb{R}_+^{J+1}$ as the inverse of $\overline{\overline{\mathbf{H}}}$.

Generator $\overline{\mathbf{S}}$. The function $\overline{\mathbf{S}}$ is twice continuously differentiable and linearly homogeneous. As shown above, the Jacobian of $\overline{\mathbf{H}}$ is symmetric and positive definite. Then the same is true of the Jacobian of $\ln \overline{\mathbf{S}}$ (see Lemma 5). □

E Construction of Generators

In this section, we provide a range of general methods for building generators along with illustrative examples. According to Definition 1, candidate generators must be shown to be twice continuously differentiable, linearly homogeneous, and with a Jacobian of their logarithm that is symmetric and positive definite.

Constructing generators, we will encounter many instances where it is possible to construct a candidate generator that satisfies all the requirements for being a generator except the Jacobian of the log generator may be only positive semi-definite. We call such a candidate an *almost generator*. The first result in this section shows that averaging such an almost generator with a generator produces a new generator.

Proposition 6 (Averaging)**.** Let $\mathbf{T}_k : \mathbb{R}_+^{J+1} \to \mathbb{R}_+^{J+1}, k \in \{1, \ldots, K\}$, be almost generators with at least one being a generator. Let $(\alpha_1, \ldots, \alpha_K) \in \operatorname{int}(\Delta)$. Then $\mathbf{S} : \mathbb{R}_+^{J+1} \to \mathbb{R}_+^{J+1}$ given by

$$\mathbf{S}(\mathbf{q}) = \prod_{k=1}^{K} \mathbf{T}_k(\mathbf{q})^{\alpha_k} \tag{30}$$

is a generator.

Proof of Proposition 6. S given by (30) is twice continuously differentiable. It is also linearly homogeneous since for $\lambda > 0$

$$\begin{aligned}
\mathbf{S}(\lambda\mathbf{q}) &= \prod_{k=1}^{K} \mathbf{T}_k(\lambda\mathbf{q})^{\alpha_k} = \prod_{k=1}^{K} \lambda^{\alpha_k}\mathbf{T}_k(\mathbf{q})^{\alpha_k}, \\
&= \left(\prod_{k=1}^{K} \lambda^{\alpha_k}\right)\left(\prod_{k=1}^{K} \mathbf{T}_k(\mathbf{q})^{\alpha_k}\right), \\
&= \left(\lambda^{\sum_{k=1}^{K}\alpha_k}\right)\left(\prod_{k=1}^{K} \mathbf{T}_k(\mathbf{q})^{\alpha_k}\right) = \lambda\mathbf{S}(\mathbf{q}),
\end{aligned}$$

where the second equality comes from the linear homogeneity of the functions $\mathbf{T}_k$ and the fourth equality comes from the restrictions on parameters $\sum_{k=1}^{K} \alpha_k = 1$.

The Jacobian of $\ln \mathbf{S}$, given by $\mathbf{J}_{\ln \mathbf{S}} = \sum_{k=1}^{K} \alpha_k \mathbf{J}_{\ln \mathbf{T}_k}$, is symmetric as the linear combination of symmetric matrices; and positive definite as the linear combination of at most $K-1$ positive semi-definite matrices and at least one positive definite matrix. □

Proposition 6 has two corollaries: Proposition 3 stated in the main text and Corollary 1 given below.

Proof of Proposition 3. For each $g \in \mathscr{G}$, let $\mathbf{T}_g = \left(T_g^{(1)}, \ldots, T_g^{(J)}\right)$ with $T_g^{(j)}(\mathbf{q}) = q_g^{\mathbf{1}\{j \in g\}}$, and let $T_0^{(j)}(\mathbf{q}) = q_j$. Then the Jacobian of $\ln \mathbf{T}_g$ has elements jk given by $\frac{\mathbf{1}\{j \in g\} \mathbf{1}\{k \in g\}}{q_g}$, and thus $\mathbf{J}_{\ln \mathbf{T}_g} = \frac{\mathbf{1}_g \mathbf{1}_g^{\intercal}}{q_g}$ where $\mathbf{1}_g = (\mathbf{1}\{1 \in g\}, \ldots, \mathbf{1}\{J \in g\})^{\intercal}$. Each $\mathbf{T}_g, g \in \mathscr{G}$ is an almost generator while $\mathbf{T}_0$ is the logit generator. Lastly, $\sum_{\{g \in \mathcal{G} | j \in g\}} \mu_g + \mu_0 = 1$. Then the conditions for application of Proposition 6 are fulfilled. □

The following corollary provides another application of Proposition 6, which allows to build models with analytic formulae for both the demand functions and their inverse, as it is the case for the logit and NL models. Let un-normalized demands $\tilde{\mathbf{q}}$ be demands obtained before normalizing their sum to 1, i.e., $\mathbf{q} = \tilde{\mathbf{q}} / |\tilde{\mathbf{q}}|$.

Corollary 1 (Invertible nesting)**.** Let $\mathscr{G} = \{g_0, \ldots, g_J\}$ be a finite set of $J+1$ nests (i.e., the number of nests is equal to the number of products). Let $\mu_g > 0$, for all $g \in \mathscr{G}$, be the associated nesting parameters where $\sum_{\{g \in \mathscr{G} | j \in g\}} \mu_g = 1$ for all $j \in \mathscr{J}$, and $q_g = \sum_{i \in g} q_i$. Let $\mathbf{S}$ be given by

$$S^{(j)}(\mathbf{q}) = \prod_{\{g \in \mathscr{G} | j \in g\}} q_g^{\mu_g}. \tag{31}$$

Let $\mathbf{W} = \operatorname{diag}(\mu_{g_0}, \ldots, \mu_{g_J})$ and let $\mathbf{M} \in \mathbb{R}^{(J+1) \times (J+1)}$ with entries $M_{jk} = \mathbf{1}_{\{j \in g_k\}}$ (where rows correspond to products and columns to nests). If $\mathbf{M}$ is invertible, then $\mathbf{S}$ is a generator, and the un-normalized demands satisfy

$$\boldsymbol{\delta} = \ln \mathbf{S}(\tilde{\mathbf{q}}) \Leftrightarrow \tilde{\mathbf{q}} = \left(\mathbf{M}^T\right)^{-1} \exp\left(\mathbf{W}^{-1} \mathbf{M}^{-1} \boldsymbol{\delta}\right).$$

The generator (31) satisfies Assumption 1 only when there is at least one degenerate nest (i.e., a nest with a single product). This means that Corollary 1 allows for zero demands when there is no degenerate nest. Note that zero demands may also arise in an ARUM where the error terms have bounded support.

Proof of Corollary 1. Following the proof of Proposition 3, the (candidate) generator $\mathbf{S}$ given by (31) is clearly an almost generator. Thus, it remains to show that, if $\mathbf{M}$ is invertible, then the Jacobian of $\ln \mathbf{S}$ is positive definite.

Note that

$$\begin{aligned}\ln S^{(j)}(\mathbf{q}) &= \sum_{k\in\mathscr{J}} \mu_{g_k} \mathbf{1}\{j\in g_k\} \ln (q_{g_k}), \\ &= \sum_{k\in\mathscr{J}} \mu_{g_k} \mathbf{1}\{j\in g_k\} \ln \left(\sum_{i\in\mathscr{J}} \mathbf{1}\{i\in g_k\} q_i \right),\end{aligned}$$

and, in turn,

$$\frac{\partial \ln S^{(j)}(\mathbf{q})}{\partial q_l} = \sum_{k\in\mathscr{J}} \mu_{g_k} \frac{\mathbf{1}\{j\in g_k\}\mathbf{1}\{l\in g_k\}}{q_{g_k}},$$

which can be expressed in matrix form as

$$\mathbf{J}_{\ln \mathbf{S}}(\mathbf{q}) = \mathbf{M}\mathbf{V}\mathbf{M}^{\intercal},$$

where $\mathbf{V} = \text{diag}\left(\frac{\mu_{g_0}}{q_{g_0}}, \ldots, \frac{\mu_{g_J}}{q_{g_J}}\right)$. This is positive definite since all μ_g are strictly positive and $\mathbf{M}$ is invertible.

Lastly, with $\mathbf{M}$ invertible, un-normalized demands solve $\ln \mathbf{S}(\mathbf{q}) = \mathbf{M}\mathbf{W} \ln (\mathbf{M}^{\intercal}\mathbf{q}) = \boldsymbol{\delta}$ and are given by

$$\mathbf{q} = (\mathbf{M}^{\intercal})^{-1} \exp\left(\mathbf{W}^{-1}\mathbf{M}^{-1}\boldsymbol{\delta}\right).$$

□

Example 3. Define nests from the symmetric incidence matrix $\mathbf{M}$ with entries $M_{ij} = \mathbf{1}_{\{i\neq j\}}$, so that each product belongs to J nests. The inverse of the incidence matrix has entries ij equal to $\frac{1}{J} - 1_{\{i=j\}}$.

Let $\mu_g = 1/J$ for each nest $g = 1, \ldots, J$. Then the un-normalized demands are given by $\tilde{\mathbf{q}} = (\mathbf{M})^{-1} \exp\left[J\mathbf{M}^{-1}\boldsymbol{\delta}\right]$ which leads to the following demands

$$q_i = \frac{\tilde{q}_i}{\sum_{j\in\mathscr{J}} \tilde{q}_j} = \frac{\sum_{j\in\mathscr{J}} e^{-J\delta_j} - Je^{-J\delta_i}}{\sum_{j\in\mathscr{J}} e^{-J\delta_j}}. \tag{32}$$

These demands are non-negative only for values of $\boldsymbol{\delta}$ within some set. To ensure positive demands, it is possible to average with the simple logit generator, since then Assumption 1 is satisfied and Theorem 1 applies.

Demands (32) are not consistent with any ARUM since they do not exhibit the (restrictive) feature of the ARUM that the mixed partial derivatives of q_j alternate in sign (McFadden, 1981). Indeed, products are substitutes

$$\frac{\partial q_1}{\partial \delta_2} = -J^2 e^{-J(\delta_1+\delta_2)} / \left(\sum_{j \in \mathscr{J}} e^{-J\delta_j} \right)^2 < 0,$$

but

$$\frac{\partial^2 q_1}{\partial \delta_2 \partial \delta_3} = -2J^3 e^{-J(\delta_1+\delta_2+\delta_3)} / \left(\sum_{j \in \mathscr{J}} e^{-J\delta_j} \right)^3 < 0.$$

The following proposition shows how a generator can be transformed into a new generator by application of a location shift and a bistochastic matrix (i.e., a matrix with non-negative elements that sum to 1 across rows and columns).

Proposition 7 (Transformation)**.** Let $\mathbf{T}$ be a generator and $\mathbf{m} \in \mathbb{R}^{J+1}$ be a location shift vector. Let $\mathbf{A} \in \mathbb{R}^{(J+1)\times(J+1)}$ be an invertible bistochastic matrix, so that $a_{ij} \geq 0$ and $\sum_{i \in \mathscr{J}} a_{ij} = \sum_{j \in \mathscr{J}} a_{ij} = 1$. Then $\mathbf{S}$ given by

$$\mathbf{S}(\mathbf{q}) = \exp\left(\mathbf{A}^T \left[\ln\left(\mathbf{T}\left(\mathbf{A}\mathbf{q}\right)\right)\right] + \mathbf{m}\right) \tag{33}$$

is a generator, and the un-normalized demands are given by

$$\tilde{\mathbf{q}} = \mathbf{A}^{-1}\mathbf{T}^{-1}\left(\exp\left[\left(\mathbf{A}^{\intercal}\right)^{-1}\left(\boldsymbol{\delta} - \mathbf{m}\right)\right]\right).$$

Proof of Proposition 7. $\mathbf{S}$ defined by (33) is twice continuously differentiable. It is also linearly homogeneous since for $\lambda > 0$,

$$\begin{aligned} \mathbf{S}(\lambda\mathbf{q}) &= \exp\left(\mathbf{A}^{\intercal} \ln \mathbf{T}\left(\mathbf{A}\left(\lambda\mathbf{q}\right)\right) + \mathbf{m}\right), \\ &= \exp\left(\mathbf{A}^{\intercal} \ln \lambda + \mathbf{A}^{\intercal} \ln T\left(\mathbf{A}\mathbf{q}\right) + \mathbf{m}\right), \\ &= \exp\left(\ln \lambda + \mathbf{A}^{\intercal} \ln T\left(\mathbf{A}\mathbf{q}\right) + \mathbf{m}\right) = \lambda\mathbf{S}\left(\mathbf{q}\right), \end{aligned}$$

where the second equality comes from the linear homogeneity of $\mathbf{T}$, and the third equality comes from the fact that columns of $\mathbf{A}$ sum to 1.

By Lemma 5, the Jacobian of $\ln \mathbf{S}$ is $\mathbf{J}_{\ln S} = \mathbf{A}^{\intercal}\mathbf{J}_{\ln \mathbf{T}}\mathbf{A}$, which is symmetric and positive definite.

The final conclusion follows from solving $\ln \mathbf{S}(\tilde{\mathbf{q}}) = \boldsymbol{\delta}$. $\square$

By Proposition 7, if $\mathbf{A}$ is an invertible bistochastic matrix and Ω a GE, then $\mathbf{q} \to \Omega(\mathbf{Aq})$ is also a GE, since $\Omega(\mathbf{Aq}) = -\mathbf{q}^\intercal \mathbf{A}^\intercal \ln \mathbf{S}(\mathbf{Aq})$. This construction may be useful if choice products can be viewed as mixtures of another level of choice products. In addition, similarly to Corollary 1, Proposition 7 allows the construction of models with analytic formulae for both their demand functions and their inverse. Lastly, it allows for zero demands: this may arise when the generator $\mathbf{T}$ does not satisfy Assumption 1. We illustrate Proposition 7 with a generator that leads to demands where products may be complements.

Example 4. Let $J+1=3$, $\mathbf{m}=\mathbf{0}$, and $\mathbf{T}(\mathbf{q})=\mathbf{q}$, and

$$\mathbf{A} = \begin{pmatrix} p & 1-p & 0 \\ 1-p & p & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

with $p<0.5$. Then we obtain

$$\tilde{\mathbf{q}} = \mathbf{A}^{-1}\left(\exp\left[\left(\mathbf{A}^T\right)^{-1}\boldsymbol{\delta}\right]\right) = \begin{pmatrix} \frac{p}{2p-1} e^{\frac{p}{2p-1}\delta_1 - \frac{1-p}{2p-1}\delta_2} - \frac{1-p}{2p-1} e^{\frac{p}{2p-1}\delta_2 - \frac{1-p}{2p-1}\delta_1} \\ \frac{p}{2p-1} e^{\frac{p}{2p-1}\delta_2 - \frac{1-p}{2p-1}\delta_1} - \frac{1-p}{2p-1} e^{\frac{p}{2p-1}\delta_1 - \frac{1-p}{2p-1}\delta_2} \\ e^{\delta_3} \end{pmatrix},$$

so that

$$q_3 = \frac{e^{\delta_3}}{e^{\frac{p}{2p-1}\delta_1 - \frac{1-p}{2p-1}\delta_2} + e^{\frac{p}{2p-1}\delta_2 - \frac{1-p}{2p-1}\delta_1} + e^{\delta_3}},$$

and $\frac{\partial q_3}{\partial \delta_1} > 0$ if and only if $\delta_2 - \delta_1 > (2p-1)\ln\left(\frac{1-p}{p}\right)$.

F Supplemental Material for Section 5

F.1 Data

Data. We use data from the Dominick's Database made available by the James M. Kilts Center, University of Chicago Booth School of Business. They comprise all Dominick's Finer Foods chain stores in the Chicago metropolitan area over the period 1989-1997, and concern 30 categories of packaged products. They are weekly store-level scanner data at the UPC level, and include unit sales, retail price, and weekly stores traffic.

We supplement the data with the nutrient content of the cereals using the USDA Nutrient Database for Standard Reference. This dataset is made available by the United States Department of Agriculture and provides the nutrient content of more than 8,500 different foods including ready-to-eat cereals (in particular, we use releases SR11 and SR16 for sugar). We use six characteristics: fiber, sugar, lipid, protein in grams/100g of cereals, energy in Kcal/100g of cereals, and sodium in mg/100 g of cereals. We convert each characteristics into g/serve, Kcal/serve, and mg/serve, respectively.

We supplement the data with the sugar monthly price in dollars/kg. We use this variable to form a cost-based instrument: the price of the cereal's sugar content (i.e., sugar content in grams times the sugar monthly price in dollars/g).

Market shares and prices. Following Nevo (2001), we define market shares of the (inside) products by converting volume sales into number of servings sold, and then by dividing it by the total potential number of servings at a store in a given month.

To compute the potential market size, we assume that (i) an individual in a household consumes around 15 servings per month, and (ii) consumers visit stores twice a week.[20] Indeed, according to USDA's Economic Research Service, per capita consumption of RTE cereals was equal to around 14 pounds (that is, about 6350 grammes) in 1992, which is equivalent to serving 15 servings per month (without loss of generality, we assume that a serving weight is equal to 35 grammes). Then, the potential month-store market size (in servings) is computed as the weekly average number of households which visited that store in that given month, times the average household size for that store, divided by two, times the number of servings an individual consumes in a month. Using the weekly average number of households itself allows to take into account the fact that consumers visit stores once a week. The market share of the outside option is then the difference between one and the sum of the inside products market shares.

Following Nevo (2001), we compute the price of a serving weight by dividing the dollar sales by the number of servings sold, where the dollar sales reflect the price consumers paid.

[20] As a robustness check, we have also estimated the models with the alternative assumption that consumers visit stores once a week. Results do not change significantly.

Descriptive statistics. The sample we use consists of the six biggest companies mentioned above. Brand names seem to play a non-negligible role: Kellogg's is the biggest company with large market shares in all segments; and General Mills, the second biggest one, is especially present in the family and kids segments. Taken together they account for around 80 percent of the market. As regards market segments, the family and kids segments dominate and account for almost 70 percent of the market.

Table 5 shows the nutrient content of the cereals according to their market segment and brand name. We observe that cereals for health/nutrition contain less sugar, more fiber, less lipid, and less sodium, and are less caloric. Cereals for kids contain more sugar and more calories. Nabisco offers cereals with less sugar and less calories, and Quaker and Ralston offer cereals with more calories.

Implementing the GNE model. We first select the dimensions along which the market is segmented.

Then, we estimate the GNE model by 2SLS (or GMM) using cost shifters and BLP instruments as instruments for prices and nesting terms. Practically, we use `ivregress` or `ivreg2` commands of the software package STATA.

Lastly, to compute the own- and cross-price elasticities, we proceed as follows. First, we get the estimated net utility $\hat{\boldsymbol{\delta}}$, the estimated marginal utility of income $\hat{\alpha}$, and the estimated nesting parameters $\hat{\boldsymbol{\mu}} = (\hat{\mu}_1, \ldots, \hat{\mu}_C)$. Second, we compute the predicted market shares $\hat{\mathbf{q}}$ by solving for $\mathbf{q}$ the system of nonlinear equations $\ln \mathbf{S}(\mathbf{q}, \hat{\boldsymbol{\mu}}) = \hat{\boldsymbol{\delta}} + \mathbf{c}$ (see Equations (7)), with $\mathbf{S}$ defined by (13) and with $\mathbf{c} = -\ln(q_0) = -\ln\left(1 - \sum_{j=1}^{J} q_j\right)$ by normalization $\delta_0 = 0$. Practically, we use the Stata command `solvenl` or the Matlab command `fsolve`.[21] Third, we compute the matrix of elasticities $\boldsymbol{\eta} = \mathrm{diag}(\mathbf{p}) \mathbf{J}_{\mathbf{q}} \mathrm{diag}(\hat{\mathbf{q}})^{-1}$, where $\mathbf{J}_{\mathbf{q}}$ is given by (29) with $\alpha = \hat{\alpha}$, $\mathbf{q} = \hat{\mathbf{q}}$, and $\boldsymbol{\mu} = \hat{\boldsymbol{\mu}}$.

F.2 Results: Elasticities for the Main Specifications

Tables 7 and 8 give the estimated average own- and cross-price elasticities of demands for the main specifications, averaged over markets and product types.

[21] Equivalently, we can solving for $\mathbf{q}$ the utility maximization program (3) using Matlab command `fmincon`.

Table 5: SAMPLE STATISTICS BY SEGMENT AND BY BRAND NAME

Dimensions	Sugar g/serve	Energy Kcal/serve	Fiber g/serve	Lipid g/serve	Sodium mg/serve	Protein g/serve	N
Segment							
Family	7.54	130.41	2.22	0.99	269.66	2.88	17
	(5.27)	(9.83)	(2.61)	(0.71)	(88.64)	(1.03)	
Health/nutrition	5.03	122.54	3.16	0.54	168.54	3.84	9
	(3.69)	(5.78)	(1.31)	(0.21)	(133.62)	(1.35)	
Kids	13.40	137.75	1.00	1.35	211.38	2.01	16
	(4.17)	(3.80)	(0.69)	(0.79)	(44.77)	(0.87)	
Taste enhanced	9.70	129.28	3.32	2.22	166.43	3.16	8
	(2.05)	(15.50)	(1.12)	(1.93)	(76.38)	(0.34)	
Brand Name							
General Mills	9.92	132.09	1.99	1.51	230.69	2.65	17
	(4.67)	(7.69)	(0.98)	(0.82)	(60.83)	(0.83)	
Kellogg's	9.58	127.50	2.47	0.85	228.49	2.88	18
	(5.52)	(11.16)	(2.81)	(0.96)	(103.93)	(1.43)	
Nabisco	0.25	125.48	3.43	0.58	2.10	3.83	2
	(0.09)	(0.74)	(0)	(0)	(1.98)	(0.02)	
Post	12.02	130.76	2.09	1.03	212.03	2.49	5
	(4.64)	(14.83)	(2.02)	(0.78)	(22.31)	(1.15)	
Quaker	8.50	139.44	2.26	2.43	159.88	3.59	5
	(4.04)	(9.20)	(0.66)	(1.86)	(94.60)	(1.15)	
Ralston	7.09	138.48	0.58	0.51	305.43	2.04	3
	(6.61)	(1.41)	(0.08)	(0.65)	(71.57)	(0.39)	
Total	9.31	131.16	2.17	1.22	216.29	2.82	50
	(5.21)	(10.21)	(1.92)	(1.08)	(93.53)	(1.15)	

Notes: Standard deviations are reported in parentheses. Column "N" gives the number of brands by segment and by brand name.

Table 6: TOP 50 BRANDS

Nb.	Brand	Product Type	Brand name	Segment	Shares (%)	
					Dollars	Volume
1	Apple Cinnamon Cheerios	1	General Mills	Family	2.23	2.02
2	Cheerios	1	General Mills	Family	7.67	6.76
3	Clusters	1	General Mills	Family	1.03	0.89
4	Golden Grahams	1	General Mills	Family	2.28	2.12
5	Honey Nut Cheerios	1	General Mills	Family	4.82	4.47
6	Total Corn Flakes	1	General Mills	Family	0.87	0.59
7	Wheaties	1	General Mills	Family	2.59	2.75
8	Total	2	General Mills	Health/nutrition	1.29	1.00
9	Total Raisin Bran	2	General Mills	Health/nutrition	1.61	1.49
10	Cinnamon Toast Crunch	3	General Mills	Kids	2.16	1.94
11	Cocoa Puffs	3	General Mills	Kids	1.22	0.98
12	Kix	3	General Mills	Kids	1.68	1.29
13	Lucky Charms	3	General Mills	Kids	2.35	1.94
14	Trix	3	General Mills	Kids	2.43	1.75
15	Oatmeal (Raisin) Crisp	4	General Mills	Taste enhanced	2.05	2.09
16	Raisin Nut	4	General Mills	Taste enhanced	1.60	1.60
17	Whole Grain Total	4	General Mills	Taste enhanced	1.77	1.29
18	All Bran	5	Kellogg’s	Family	0.97	1.11
19	Common Sense Oat Bran	5	Kellogg’s	Family	0.49	0.46
20	Corn Flakes	5	Kellogg’s	Family	4.12	6.96
21	Crispix	5	Kellogg’s	Family	1.88	1.70
22	Frosted Flakes	5	Kellogg’s	Family	6.01	6.77
23	Honey Smacks	5	Kellogg’s	Family	0.85	0.84
24	Rice Krispies	5	Kellogg’s	Family	5.58	6.06
25	Bran Flakes	6	Kellogg’s	Health/nutrition	0.90	1.16
26	Frosted Mini-Wheats	6	Kellogg’s	Health/nutrition	3.35	3.69
27	Product 19	6	Kellogg’s	Health/nutrition	1.06	0.86
28	Special K	6	Kellogg’s	Health/nutrition	3.07	2.53
29	Apple Jacks	7	Kellogg’s	Kids	1.67	1.32
30	Cocoa Krispies	7	Kellogg’s	Kids	0.99	0.85
31	Corn Pops	7	Kellogg’s	Kids	1.80	1.52
32	Froot Loops	7	Kellogg’s	Kids	2.66	2.22
33	Cracklin’ Oat Bran	8	Kellogg’s	Taste enhanced	1.91	1.66
34	Just Right	8	Kellogg’s	Taste enhanced	1.07	1.12
35	Raisin Bran	8	Kellogg’s	Taste enhanced	3.96	4.83
36	Shredded Wheat	9	Nabisco	Health/nutrition	0.77	0.88
37	Spoon Size Shredded Wheat	9	Nabisco	Health/nutrition	1.59	1.63
38	Grape Nuts	10	Post	Health/nutrition	2.27	3.06
39	Cocoa Pebbles	11	Post	Kids	1.11	0.92
40	Fruity Pebbles	11	Post	Kids	1.14	0.94
41	Honey-Comb	11	Post	Kids	1.05	0.90
42	Raisin Bran	12	Post	Taste enhanced	0.93	1.10
43	Oat Squares	13	Quaker	Family	0.91	1.02
44	CapNCrunch	14	Quaker	Kids	1.00	1.10
45	Jumbo Crunch (Cap’n Crunch)	14	Quaker	Kids	1.27	1.35
46	Life	14	Quaker	Kids	1.73	2.24
47	100% Cereal-H	15	Quaker	Taste enhanced	1.42	1.84
48	Corn Chex	16	Ralston	Family	0.81	0.72
49	Rice Chex	16	Ralston	Family	1.15	1.03
50	Cookie-Crisp	17	Ralston	Kids	0.89	0.68

Table 7: AVERAGE PRICE ELASTICITIES FOR THE CROSS-NESTED GNE MODEL

	Own	Cross																
Type		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
1	-3.165	0.241	0.111	0.085	0.092	0.092	-0.038	-0.064	-0.056	0.003	0.013	-0.013	-0.005	0.133	-0.023	-0.016	0.077	-0.079
2	-3.146	0.072	0.379	0.069	0.074	-0.039	0.269	-0.041	-0.037	0.113	0.238	-0.072	-0.064	-0.003	-0.006	-0.002	-0.004	-0.006
3	-3.427	0.066	0.083	0.316	0.076	-0.026	-0.009	0.225	-0.016	0.002	-0.059	0.174	-0.064	-0.100	0.150	-0.090	-0.044	0.206
4	-2.950	0.067	0.083	0.071	0.384	-0.034	-0.018	-0.030	0.282	0.003	-0.035	-0.047	0.255	-0.051	-0.047	0.266	-0.004	-0.000
5	-2.560	0.097	-0.063	-0.035	-0.049	0.211	0.050	0.078	0.065	-0.007	-0.017	0.011	-0.003	0.125	-0.008	-0.021	0.074	-0.058
6	-2.700	-0.028	0.308	-0.008	-0.018	0.035	0.372	0.056	0.046	0.122	0.251	-0.066	-0.072	-0.015	0.005	-0.005	-0.009	0.011
7	-3.366	-0.037	-0.037	0.169	-0.024	0.044	0.044	0.250	0.057	-0.005	-0.062	0.144	-0.046	-0.084	0.122	-0.071	-0.036	0.170
8	-2.531	-0.042	-0.042	-0.016	0.288	0.046	0.046	0.072	0.376	-0.005	-0.056	-0.029	0.265	-0.063	-0.036	0.268	-0.009	0.017
9	-1.949	0.001	0.074	0.001	0.002	-0.003	0.069	-0.003	-0.003	0.909	0.062	-0.011	-0.009	0.005	0.005	0.006	0.005	0.005
10	-1.524	0.008	0.225	-0.047	-0.028	-0.010	0.208	-0.065	-0.045	0.090	—	0.399	0.396	0.033	-0.022	-0.002	0.016	-0.038
11	-3.175	-0.006	-0.046	0.092	-0.026	0.004	-0.036	0.102	-0.016	-0.011	0.266	0.404	0.272	-0.029	0.069	-0.050	-0.010	0.087
12	-1.949	-0.003	-0.050	-0.042	0.184	-0.001	-0.048	-0.040	0.186	-0.011	0.337	0.344	—	-0.005	-0.044	0.182	0.013	-0.026
13	-2.337	0.050	-0.002	-0.048	-0.027	0.045	-0.008	-0.053	-0.032	0.005	0.020	-0.025	-0.004	—	0.233	0.254	0.040	-0.058
14	-2.273	-0.011	-0.004	0.090	-0.030	-0.003	0.003	0.098	-0.022	0.005	-0.016	0.079	-0.040	0.285	0.387	0.266	-0.013	0.089
15	-1.778	-0.008	-0.001	-0.056	0.177	-0.010	-0.003	-0.059	0.174	0.006	-0.003	-0.058	0.170	0.326	0.277	—	0.012	-0.037
16	-2.607	0.028	-0.002	-0.019	-0.002	0.026	-0.005	-0.021	-0.004	0.005	0.009	-0.008	0.009	0.038	-0.009	0.008	0.808	0.759
17	-3.382	-0.021	-0.003	0.072	-0.000	-0.014	0.004	0.078	0.006	0.003	-0.017	0.057	-0.014	-0.041	0.051	-0.021	0.581	—

Notes: Elasticities are averaged over product types and over markets.

Table 8: AVERAGE PRICE ELASTICITIES FOR THE THREE-LEVEL NL MODELS

Type	3NL1				3NL2			
	Own	Cross			Own	Cross		
		Same subgroup	Same group	Different group		Same subgroup	Same group	Different group
1	-3.528	0.182	0.137	0.005	-3.524	0.208	0.147	0.007
2	-3.414	0.426	0.257	0.004	-3.530	0.332	0.104	0.005
3	-3.778	0.300	0.228	0.004	-3.876	0.226	0.129	0.006
4	-3.228	0.403	0.302	0.004	-3.398	0.255	0.117	0.006
5	-2.840	0.178	0.145	0.006	-2.868	0.169	0.124	0.008
6	-2.994	0.353	0.282	0.004	-3.181	0.186	0.089	0.006
7	-3.678	0.261	0.172	0.003	-3.763	0.199	0.080	0.005
8	-2.781	0.386	0.296	0.004	-2.971	0.215	0.092	0.006
9	-2.804	0.309	0.169	—	-2.030	1.102	—	0.003
10	-1.930	—	0.244	0.003	-1.605	—	0.507	0.005
11	-3.671	0.229	0.111	0.002	-3.435	0.488	0.329	0.003
12	-2.295	—	0.201	0.003	-2.028	—	0.396	0.004
13	-2.584	—	0.059	0.002	-2.270	—	0.298	0.003
14	-2.685	0.213	0.129	0.002	-2.480	0.434	0.321	0.004
15	-2.064	—	0.203	0.003	-1.845	—	0.357	0.004
16	-3.494	0.226	0.058	0.002	-2.713	1.029	0.800	0.003
17	-3.929	—	0.089	0.002	-3.236	—	0.667	0.002

Notes: Elasticities are averaged over product types and over markets.