Asset Pricing with Heterogeneous Agents and Long-Run Risk*

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Abstract

This paper examines the effects of the heterogeneity of agents' beliefs about the persistence of long-run risks in consumption-based asset-pricing models. Agents who believe in a lower persistence level dominate the economy rather quickly, even if their belief is wrong. In a standard calibration of the long-run risk model, this dominance drives the equity premium down below the level observed in the data. Simultaneously, belief heterogeneity can generate significant excess volatility and priced consumption risk due to changes in the wealth distribution. This effect in turn helps to explain several asset-pricing puzzles such as the large countercyclical variation of expected risk premia, the volatility of the price–dividend ratio, and the predictability of cash flows and returns. A new calibration of the heterogeneous-agents long-run risk model can simultaneously explain the large equity premium and the aforementioned puzzles.

Keywords: asset pricing, heterogeneous agents, long-run risk, recursive preferences.

JEL codes: G11, G12.

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1 Introduction

This paper examines the effect of agent belief heterogeneity about the persistence of long-run risks in consumption-based asset-pricing models. Agents who believe in a lower persistence level dominate the economy rather quickly, even if their belief is wrong. In a standard calibration of the long-run risks model, this dominance drives the equity premium down below the level observed in the data. Simultaneously, belief heterogeneity can generate excess volatility and priced consumption risk due to significant changes in the wealth distribution. We show that a new calibration of the heterogeneous agents long-run risk model can simultaneously explain the large equity premium and, in particular, generate a large counter-cyclical variation in expected risk premia consistent with the empirical findings in Martin (2017) and Martin and Wagner (2018). Moreover, the model helps to explain several asset-pricing puzzles such as the large volatility of the price-dividend ratio, and the predictability of cash flows and returns. In sum, the paper demonstrates that belief heterogeneity can significantly improve the explanatory power of long-run risk asset-pricing models.

The Bansal–Yaron long-run risk model (Bansal and Yaron (2004)) has emerged as one of the premier consumption-based asset-pricing models. It can generate many of the features of aggregate stock prices that have long been considered puzzles. The model generates a high equity premium by combining two mechanisms—investors with a taste for the early resolution of uncertainty, and very persistent shocks to the growth rate of consumption. For long-run risk to generate a high equity premium, the level of persistence must be very close to a unit root. The amount of persistence in the data is very difficult to measure, and arguments for a range of estimates have appeared in the literature (Bansal, Kiku, and Yaron (2016), Schorfheide, Song, and Yaron (2018), and Grammig and Schaub (2014)). This literature suggests that there is considerable scope for disagreement over the true value.

In this paper, we consider the consequences if the agents themselves disagree about the persistence of exogenous shocks. Naturally, the analysis of belief differences requires us to move from the standard representative-agent model of the consumption-based asset-pricing framework to a model with multiple agents. For this purpose, we begin the paper with a description of a heterogeneous-agent economy with complete markets. Solving this model reveals a critical difference from the representative-agent model. Even for Markovian shocks, equilibrium allocations are no longer a function of the exogenous state alone. As a result, the standard solution methods from consumption-based asset pricing are not applicable. We employ a reformulation of the first-order conditions for the equilibrium that is recursive, through the device of introducing new endogenous state variables. These state variables have a clear interpretation in terms of time-varying weights in a social planner's problem. The weights capture the relative trend in an agent's consumption—an agent who has a declining share of consumption will have a declining weight.[1]

Using the theoretical framework for the heterogenous-agent asset-pricing model, we can provide an in-depth numerical analysis of the effects of belief differences. Before we do so, however, we first provide some helpful economic intuition on the different effects of belief heterogeneity in models with CRRA (constant relative risk aversion) preferences and in models with Epstein–Zin preferences. Most importantly, we show why the general result (Sandroni (2000), Blume and Easley (2006), Yan (2008)) for CRRA preferences—namely, that agents with wrong beliefs do not survive in the long run—does not always hold for Epstein–Zin agents. (Borovička (2018) provides a theoretical analysis of this phenomenon in a continuous-time framework with i.i.d. consumption growth.) For CRRA preferences, agents gamble on the realization of the state next period. How much they gamble depends on their risk aversion and their subjective beliefs—we call this the speculation motive—but in the long run, only the investors with the correct beliefs survive. For Epstein–Zin preferences, agents are also willing to pay a premium to hedge future risks. Agents believing in high persistence levels demand large risk premia in the long-run risk model. Agents who believe in a lower persistence are therefore willing to provide insurance against these long-run risks—we call this mechanism the risk-sharing motive. As long as risk premia in the economy are sufficiently large, this motive will, on average, transfer wealth to the agents who believe in a lower persistence irrespective of the true data-generating process.

The interaction of the economic effects resulting from the two motives has interesting economic implications. To analyze these implications, we perform a comprehensive numerical analysis of the complete-markets heterogeneous-agent economy. As our baseline calibration of the model, we employ the calibration (without stochastic volatility) of Bansal and Yaron (2004, Case I) with the only exception being that we use two different persistence levels close to their original persistence for the long-run risk.

If the disagreement is sufficiently large, the speculation motive is stronger than the risk-sharing motive and agents whose beliefs are more correct dominate the economy, in accordance with the market selection hypothesis of Alchian (1950) and Friedman (1953). For small

[1] Similar approaches have been used to solve models with multiple goods (Colacito and Croce (2013)), discrete state models without growth, and risk-sensitive preferences (Anderson (2005)), overlapping generation models with different preference parameters (Gârleanu and Panageas (2015)) and different beliefs (Collin-Dufresne, Johannes, and Lochstoer (2016a)), and models with i.i.d. consumption growth and belief differences (Borovička (2018)).

differences, the situation changes dramatically and the risk-sharing motive dominates the speculation motive. Investors who believe in a lower persistence level not only survive in the long run, they, in fact, enjoy a larger share of consumption, even if their belief is wrong, because they collect a premium for providing the insurance against long-run risks. If they initially hold only a very small consumption share, their share increases rapidly over time. As small differences in beliefs about the long-run risk process have large effects on asset prices, we report a drop in the equity premium of 2 percent within a century.

When we increase the belief difference, the speculation motive becomes stronger and it becomes more important which investor holds the correct beliefs. If the investor who believes in the larger persistence has the correct beliefs, both agents survive and due to the interaction of the two motives we observe a large variation in the consumption shares. Hence, belief heterogeneity itself can serve as a source of endogenous asset-price volatility. This result gives a model-based explanation for the empirical findings of Carlin, Longstaff, and Matoba (2014), who use data from the mortgage-backed security market and show that higher disagreement leads to higher volatility. They also show that, as in our model, disagreement is time-varying and correlated with macroeconomic variables.

In the final step of our analysis, we show that the heterogenous-agent model with long-run risk can jointly generate a large equity premium and explain several asset-pricing puzzles. In particular, it can explain the large countercyclical time variation in risk premia recently reported in Martin (2017) and Martin and Wagner (2018). We modify the calibration of Bansal and Yaron (2004, Case I) by replacing the original persistence value of the long-run risk process with two different values; compared to the original value, the first agent believes in a slightly larger and the second agent in a slightly smaller persistence level. We show that belief heterogeneity endogenously adds priced consumption risks to the model due to persistent changes in the wealth distribution. These risk premia are countercyclical and time varying. The risk-sharing motive implies that when there are negative shocks to the long-run risk component the insurance provided by the investors with lower beliefs about the persistence will pay off. Hence, when the economy enters a recession (multiple negative shocks to the long-run risk component), there is a wealth transfer to the investor who beliefs in a larger value for the persistence. As these are the investors that demand the larger risk premia, belief heterogeneity adds significant countercyclical variation in risk premia to the long-run risk model.

We report a standard deviation of expected risk premia of 5.73 percent in line with the values reported in Martin (2017). Representative-agent long-run risk models are not able to replicate this finding even when including exogenous stochastic volatility, a feature that is deliberately included to obtain time variation in risk premia (see Bansal and Yaron (2004)). Furthermore, the heterogeneous belief model generates a large and significant equity premium and also addresses other empirical deficiencies of the representative-agent model, which have been emphasized by Beeler and Campbell (2012). Beeler and Campbell (2012) show that the long-run risk model cannot explain the large volatility of the price–dividend ratio observed in the data (a value of 0.45 compared to 0.28 in the model). In the heterogeneous-agent setup, the shifts in the wealth distribution increase the volatility of the price–dividend ratio to levels close to the data (0.38) as the impact of the different agents on asset prices varies over time. Furthermore, the variation in the wealth distribution helps to address the predictability puzzle pointed out by Beeler and Campbell (2012). The endogenous variation in asset prices increases the predictability of returns while simultaneously decreasing the predictability of consumption and dividend growth.

Our results are complementary to the findings of Collin-Dufresne, Johannes, and Lochstoer (2016b) and Bidder and Dew-Becker (2016), which show that the asset-pricing implications of long-run risk can emerge endogenously from parameter uncertainty, even without long-run risk being present. Collin-Dufresne, Johannes, and Lochstoer (2016b) show that if investors learn the growth rate from the data, then innovations to expectations of growth rates are permanent. Agents then price in the risk from this permanent shock to their expected growth rates. Bidder and Dew-Becker (2016) show that ambiguity-averse investors will price in long-run risk if they cannot rule it out a priori. In our setup, neither investor suffers from model uncertainty, but despite this difference a clear picture of the effect of long-run risk emerges.

While in the present paper the agents agree to disagree about the long-run risks in the economy, Andrei, Carlin, and Hasler (2016) provide an explanation of how this disagreement can arise from model uncertainty as market participants calibrate their models differently. They find that uncertainty about long-run risks can explain many stylized facts of stock return volatilities, such as large volatilities during recessions and booms and persistent volatility clustering. Andrei, Hasler, and Jeanneret (2017) show how model uncertainty can lead to long-run-risk-like behavior in the presence of a noisy signal of the growth rate.

The remainder of the paper is organized as follows. In Section 2 we describe the general asset-pricing model with heterogeneous investors and recursive preferences. Section 3 explains the main economic mechanism of the two-agent economy in a stylized version of the model. In Section 4 we present results for the model based on the calibration by Bansal and Yaron (2004) and explain the underlying economic mechanism in the full model. Section 5 presents a recalibration of the model showing that the heterogeneous-agent framework not only generates the high equity premium but also significantly improves upon the representative-agent long-run-risk models in multiple dimensions. Section 6 concludes. Online appendices with a discussion of additional literature, the proofs of all theoretical results, a description of the numerical solution method, and additional results complete the paper.

2 Theoretical Framework

We consider a standard infinite-horizon discrete-time endowment economy with a finite number of heterogeneous agents. Agents can differ with respect to both their utility functions and their subjective beliefs. We restrict our attention to the complete-markets setting, which allows us to reformulate the problem as a social planner's problem. Here we run into a critical difference from the representative-agent problem—even for a Markov economy, equilibrium allocations are no longer required to be functions of the exogenous state alone. This failure of recursiveness occurs for essentially economic reasons—even if aggregate consumption does not contain a trend, the individual consumption allocations can do so. This feature defeats most of the approaches to solving for equilibrium in an infinite-horizon asset-pricing model.

We present a reformulation of the first-order conditions for equilibrium that is recursive. This reformulation involves introducing new endogenous state variables. These state variables have a clear interpretation in terms of time-varying weights in the social planner's problem. The weights capture the relative trend in an agent's consumption—an agent who has a declining share of consumption will have a declining weight.

2.1 The Heterogeneous-Agents Economy

Time is discrete and indexed by $t = 0, 1, 2, \ldots$. Let y_t denote the exogenous state of the economy in period t. The state has continuous support and may be multidimensional. The economy is populated by a finite number of infinitely lived agents, $h \in \mathbb{H} = \{1 \ldots H\}$. Agents choose individual consumption at time t as a function of the entire history of the exogenous state, y^t, where $y^t = (y_0, \ldots, y_t)$. Let $C^h(y^t)$ be the individual consumption for agent h. Similarly, $C(y^t) \in \mathbb{R}_{++}$ denotes the aggregate consumption of all agents as a function of the history, y^t. The individual consumption levels satisfy the usual market-clearing condition,

$$\sum_{h=1}^{H} C^h(y^t) = C(y^t). \tag{1}$$

Agents have subjective beliefs about the stochastic process of the exogenous state. We denote the expectation operator for agent h at time t by E_t^h. Each agent has recursive utility. Let $\{C^h\}_t = \{C^h(y^t), C^h(y^{t+1}), \ldots\}$ denote the consumption stream of agent h from time t forward. The utility of agent h at time t, $U^h(\{C^h\}_t)$, is specified by an aggregator, $F^h(c, x)$, and a certainty equivalence, $G^h(x)$,

$$U^h(\{C^h\}_t) = F^h\left(C^h(y^t), R_t^h\left[U^h(\{C^h\}_{t+1})\right]\right), \tag{2}$$

with

$$R_t^h[x] = G_h^{-1}\left(E_t^h[G_h(x)]\right). \tag{3}$$

We assume that the functions F^h and G^h are both continuously differentiable. This preference framework includes both Epstein–Zin utility and discounted expected utility, for the appropriate choices of F^h and G^h. To simplify the analysis, we ensure that agents never choose zero consumption, in any state of the world, by imposing an Inada condition on the aggregator F^h; so, $F_1^h(c, x) \to \infty$ as $c \to 0$, where F_1^h denotes the derivative of F^h with respect to the first argument.

We also impose a condition on the agents' beliefs. Let $P_{t,t+1}^h$ be the subjective conditional distribution of y_{t+1} given y^t, and $P_{t,t+1}$ be the true conditional distribution. We assume that each agent's expectation can be written in terms of the true distribution as

$$E_t^h[x] = E_t\left[x\frac{\mathrm{dP}_{t,t+1}^h}{\mathrm{dP}_{t,t+1}}\right],$$

for some measurable function $\mathrm{dP}_{t,t+1}^h/\mathrm{dP}_{t,t+1}$. In mathematical terms, every agent's conditional distribution is absolutely continuous with respect to the true distribution. Then, by the Radon–Nikodym theorem (see Billingsley (1999, Chapter 32)) such a $\mathrm{dP}_{t,t+1}^h/\mathrm{dP}_{t,t+1}$ must exist. Accordingly, $\mathrm{dP}_{t,t+1}^h/\mathrm{dP}_{t,t+1}$ is known as the Radon–Nikodym derivative of $\mathrm{P}_{t,t+1}^h$ with respect to $\mathrm{P}_{t,t+1}$. We also assume that, vice versa, the true distribution is absolutely continuous with respect to every agent's subjective distribution.

To solve for equilibrium, we assume that markets are complete so that we can reformulate equilibrium as a social welfare problem (Mas-Colell and Zame (1991)). The social planner maximizes a weighted sum of the individual agents' utilities at $t = 0$. Let $\boldsymbol{\lambda} = \left(\bar{\lambda}^1, \ldots, \bar{\lambda}^H\right) \in \mathbb{R}_{++}^H$ be a vector of positive Negishi weights and let $\{\boldsymbol{C}\}_0 = \left(\{C^1\}_0, \ldots, \{C^H\}_0\right)$ be an H-

vector of the agents' consumption processes. Then, the social planner maximizes

$$SP(\{\boldsymbol{C}\}_0;\boldsymbol{\lambda}) = \sum_{h=1}^{H} \bar{\lambda}^h U^h \left(\{C^h\}_0\right) \tag{4}$$

subject to the market-clearing equation (1). We denote an optimal solution to the social planner's problem for given Negishi weights $\boldsymbol{\lambda}$ by $\{\boldsymbol{C}\}_0^*$. For each agent $h \in \mathbb{H}$, let $U_t^h = U^h(\{C^h\}_t^*)$ be the utility in period t at the optimal solution. Also, for ease of notation, we suppress the state dependence of consumption and simply write C_t^h for $C^h(y^t)$.

Theorem 1. *The vector of consumption processes $\{\boldsymbol{C}\}_0^*$ solves the social planner's problem (4,1) for given Negishi weights $\boldsymbol{\lambda} = \left(\bar{\lambda}^1, \ldots, \bar{\lambda}^H\right)$ if and only if the consumption processes satisfy the following first-order conditions in each period $t \geq 0$:*

$$\lambda_t^h F_1^h(C_t^h, R_t^h\left[U_{t+1}^h\right]) = \lambda_t^1 F_1^1(C_t^1, R_t^1\left[U_{t+1}^1\right]), \tag{5}$$

where the weights λ_t^h satisfy

$$\lambda_0^h = \bar{\lambda}^h, \tag{6}$$

$$\frac{\lambda_{t+1}^h}{\lambda_{t+1}^1} = \frac{\Pi_{t+1}^h}{\Pi_{t+1}^1}\frac{\lambda_t^h}{\lambda_t^1}, \qquad t \geq 0, h \in \{2, \ldots H\}, \tag{7}$$

with Π_{t+1}^h given by

$$\Pi_{t+1}^h = F_2^h\left(C_t^h, R_t^h[U_{t+1}^h]\right) \cdot \frac{G_h'(U_{t+1}^h)}{G_h'(R_t^h[U_{t+1}^h])}\frac{\mathrm{dP}_{t,t+1}^h}{\mathrm{dP}_{t,t+1}}. \tag{8}$$

Appendix B contains the proof of this theorem as well as those of the theoretical results presented later in this section.

In each period t, the weights λ_t^h are only determined up to a scalar factor, so we are free to choose a normalization. For numerical purposes, the normalization requiring the weights λ_t^h to lie in the unit simplex in every period is convenient. From a conceptual point of view, an attractive choice is to let $\lambda_{t+1}^1 = \Pi_{t+1}^1 \lambda_t^1$, because then for all h, $\lambda_{t+1}^h = \Pi_{t+1}^h \lambda_t^h$.

If the aggregator F^h is additively separable, then the allocation of consumption in (5) depends only on the current value for the weights λ_t^h. Additive separability is the most common case in applications. Discounted expected utility is additively separable, while Epstein–Zin can be transformed to be so. In this particular case, the Negishi weights and individual agents' consumption allocations are closely linked. The following theorem provides an asymptotic result relating the limits of weights λ_t^h to the limits of consumption.

Theorem 2. *Suppose that F^h is additively separable for all $h \in \mathbb{H}$ and that the aggregate endowment is bounded, $C_t \in [\underline{C}, \overline{C}]$ for finite constants $\overline{C} \geq \underline{C} > 0$. If $\lambda_t^j/\lambda_t^i \to \infty$, then $C_t^i \to 0$. If $C_t^i \to 0$, then for at least one other agent j, $\limsup_t \lambda_t^j/\lambda_t^i = \infty$.*

Note that $\limsup_t \lambda_j/\lambda_i$ is a random variable—the limit can depend on the history. Theorem 2 generalizes a similar result by Blume and Easley (2006).

2.2 The Growth Economy with Epstein–Zin Preferences

We now consider the special case of our heterogeneous-agent economy in which aggregate consumption is expressed exogenously in terms of growth rates and agents have Epstein–Zin preferences (see Epstein and Zin (1989) and Weil (1989)). For this popular parametrization of asset-pricing models, we can sharpen the general results of Theorems 1 and 2. Here we state the equilibrium conditions for this model parametrization and refer any interested reader to Appendix B.2 for a proper derivation of those conditions.

If agent h has Epstein–Zin preferences, then

$$F^h(c,x) = \left[(1-\delta^h)c^{\rho^h} + \delta^h x^{\rho^h}\right]^{1/\rho^h} \tag{9}$$

$$G_h(x) = x^{\alpha^h} \tag{10}$$

with parameters $\rho^h \neq 0, \alpha^h < 1$. In this case, the equations are all homogeneous, so we can divide through by aggregate consumption and express the equilibrium allocations in terms of individual consumption shares, $s_t^h = C_t^h/C_t$. Market clearing (1) implies that

$$\sum_{h=1}^{H} s_t^h = 1. \tag{11}$$

Let V_t^h be agent h's value function. We also normalize this function by aggregate consumption, $v_t^h = V_t^h/C_t$. Let $c_t = \log C_t$ and $\Delta c_{t+1} = c_{t+1} - c_t$. The normalized value function of agent h satisfies the following fixed-point equation,

$$v_t^h = \left[(1-\delta^h)(s_t^h)^{\rho^h} + \delta^h R_t^h\left(v_{t+1}^h e^{\Delta c_{t+1}}\right)^{\rho^h}\right]^{\frac{1}{\rho^h}}, \qquad h \in \mathbb{H}, \tag{12}$$

with $R_t^h(x) = \left(E_t^h\left[x^{\alpha^h}\right]\right)^{\frac{1}{\alpha^h}}$. The parameter δ^h is the discount factor, $\rho^h = 1 - \frac{1}{\psi^h}$ determines the elasticity of intertemporal substitution (EIS), ψ^h, and $\alpha^h = 1 - \gamma^h$ determines the relative risk aversion, γ^h, of agent h.

To accompany the normalized value function we introduce a normalized Negishi weight, $\underline{\lambda}_t^h = \frac{\lambda_t^h}{(v_t^h)^{\rho^h-1}}$. In Appendix B.2 we show that the consumption share s_t^h of agent h is given by

$$\underline{\lambda}_t^h(1-\delta^h)(s_t^h)^{\rho^h-1} = \underline{\lambda}_t^1(1-\delta^1)(s_t^1)^{\rho^1-1}. \tag{13}$$

Finally, the equations for $\underline{\lambda}_t^h$ simplify to

$$\begin{aligned}
\frac{\underline{\lambda}_{t+1}^h}{\underline{\lambda}_{t+1}^1} &= \frac{\underline{\Pi}_{t+1}^h}{\underline{\Pi}_{t+1}^1}\frac{\underline{\lambda}_t^h}{\underline{\lambda}_t^1} \\
\underline{\Pi}_{t+1}^h &= \delta^h e^{\rho^h \Delta c_{t+1}} \frac{\mathrm{dP}_{t,t+1}^h}{\mathrm{dP}_{t,t+1}} \frac{\left(v_{t+1}^h e^{\Delta c_{t+1}}\right)^{\alpha^h-\rho^h}}{R_t^h\left(v_{t+1}^h e^{\Delta c_{t+1}}\right)^{\alpha^h-\rho^h}}, \qquad h \in \mathbb{H}^-.
\end{aligned}$$

This simplification gives us $H-1$ nonlinear equations for the equilibrium. In our numerical calculation, we complete the system by requiring that $\sum \underline{\lambda}_t^h = 1$ when we solve for the weights, $\underline{\lambda}_t^h$, given by

$$\begin{aligned}
\underline{\lambda}_{t+1}^h &= \frac{\underline{\lambda}_t^h \underline{\Pi}_{t+1}^h}{\sum_{h=1}^H \underline{\lambda}_t^h \underline{\Pi}_{t+1}^h} \\
\underline{\Pi}_{t+1}^h &= \underbrace{\delta^h e^{\rho^h \Delta c_{t+1}} \frac{\mathrm{dP}_{t,t+1}^h}{\mathrm{dP}_{t,t+1}}}_{\text{CRRA-Term}} \underbrace{\frac{\left(v_{t+1}^h e^{\Delta c_{t+1}}\right)^{\alpha^h-\rho^h}}{R_t^h\left(v_{t+1}^h e^{\Delta c_{t+1}}\right)^{\alpha^h-\rho^h}}}_{\text{Additional EZ-Term}}, \qquad h \in \mathbb{H}^-.
\end{aligned} \tag{14}$$

Unlike in the discounted expected utility case, the dynamics of the weights $\underline{\lambda}_t^h$ depend on the value functions (12), which in turn depend on the consumption decisions (13). Hence, to compute the equilibrium we need to jointly solve equations (11)–(14). As there are—to the best of our knowledge—no closed-form solutions for the general model, we present in Appendix C.1 a numerical solution approach, which is based on projection methods as proposed in Pohl, Schmedders, and Wilms (2018) to approximate for the equilibrium functions.

In this setting, we can derive an improvement over Theorem 2—the limiting behavior for λ_t^h drives the limiting behavior for an agent's share of aggregate consumption. This result requires no assumptions on aggregate consumption, only that agents have utility in the Epstein–Zin family.

Theorem 3. *Suppose all agents in the economy have Epstein–Zin preferences. If $\underline{\lambda}_t^j/\underline{\lambda}_t^i \to \infty$, then $s_t^i \to 0$. If $s_t^i \to 0$, then for at least one agent j, $\limsup_t \underline{\lambda}_t^j/\underline{\lambda}_t^i = \infty$.*

This completes our discussion of the theoretical framework for our analysis. Appendix B provides proofs for the three theorems in this section. Along the way, we derive a system of first-order conditions for Epstein–Zin preferences. This system constitutes the foundation for our numerical solution method (see Appendix C).

3 Belief Differences and Long-Run Risks: Model Setup and Economic Intuition

We consider a standard long-run risk model as in Bansal and Yaron (2004), in which log aggregate consumption growth Δc_{t+1} and log aggregate dividend growth Δd_{t+1} are given by

$$\begin{aligned} \Delta c_{t+1} &= \mu_c + x_t + \eta_{c,t+1} \\ x_{t+1} &= \rho_x x_t + \eta_{x,t+1} \\ \Delta d_{t+1} &= \mu_d + \Phi x_t + \eta_{d,t+1}. \end{aligned} \tag{15}$$

The process x_t captures the long-run variation in the mean of consumption and of dividend growth and $\eta_{c,t+1}$, $\eta_{x,t+1}$, and $\eta_{d,t+1}$ are independent normally distributed shocks with mean 0 and standard deviations σ, $\phi_x\sigma$, and $\phi_d\sigma$ respectively. A key feature of long-run risk models is highly persistent shifts in the growth rate of consumption. With a preference for the early resolution of risks ($\gamma > \frac{1}{\psi}$), investors will dislike shocks in x_t and require a large premium for bearing those risks. Hence, the results in the long-run risk literature rely on a highly persistent state process x_t, or, put differently, the parameter ρ_x needs to be very close to 1 (0.979 in the original calibration of Bansal and Yaron (2004)).

In this paper, we analyze the equilibrium implications of differences in beliefs with regard to the long-run risk process. As x_t is not directly observable from the data, it is reasonable to assume that investors disagree—at least slightly—about the data-generating process of x_t. In Appendix D, we provide evidence that even for 500 years of data, point estimates of ρ_x show significant variation. Hence, as there are less than 100 years of data available, it is reasonable to assume that there are differences in the beliefs about ρ_x.

Our baseline setup is an economy with $H = 2$ agents in which the first agent believes that ρ_x is close to 1 while the second agent believes that ρ_x is slightly smaller. We denote by ρ_x^h the belief of agent h about ρ_x. As x_{t+1} conditional on time t information is normally distributed with mean $\rho_x x_t$ and variance $\sigma_x^2 = \phi_x^2\sigma^2$, agents' beliefs $\mathrm{dP}^h_{t,t+1}$ are given by

$$\mathrm{dP}^h_{t,t+1} = \frac{1}{\sqrt{2\pi}\sigma_x} \exp\left(-\frac{1}{2}\left(\frac{x_{t+1} - \rho_x^h x_t}{\sigma_x}\right)^2\right).$$

We observe that for $x_t = 0$ both agents have the same beliefs and that the belief difference increases the further away x_t is from its long-run mean. Hence, the state of the economy plays an important role for the equilibrium consumption shares. Before we analyze the full long-run risk model, we provide some intuition to better understand the influence of the belief differences. To obtain such an intuition, it is helpful to understand the difference between the standard case of CRRA preferences and the new mechanism that emerges due to the importance of continuation values for Epstein–Zin utility (see Equation (14)). For CRRA preferences, it is always the investor with the more correct beliefs who dominates the economy in the long run (see, for example, Blume and Easley (2006) and Yan (2008)). This result does not hold true for Epstein–Zin preferences. Borovička (2018) shows that agents with more optimistic beliefs (believing in a larger mean growth rate) can dominate the economy in the long run even if their beliefs are wrong. In the following, we first show how the belief differences affect the equilibrium consumption shares in the long-run risk economy with CRRA preferences and then show the difference to the Epstein–Zin setup.

3.1 The CRRA Case

We begin our analysis by considering the CRRA case as a simple benchmark. In this case, only the speculation motive matters, in contrast to the general Epstein–Zin case. We denote the two investors by A and B. In Appendix E we show that the log consumption share of agent A is a linear function of $\eta_{x,t+1}$,

$$\log\left(s^A_{t+1}\right) = a^{CRRA} + b^{CRRA}\eta_{x,t+1}. \tag{16}$$

The coefficients are given by

$$\begin{aligned} b^{CRRA} &= \frac{(1-s^A_t)x_t(\rho^A_x - \rho^B_x)}{\sigma^2_x\gamma} \\ a^{CRRA} &= \log\left(s^A_t\right) + \frac{(1-s^A_t)x^2_t}{2\sigma^2_x\gamma}\left[(\rho_x - \rho^B_x)^2 - (\rho_x - \rho^A_x)^2\right]. \end{aligned}$$

The slope b^{CRRA} determines how the consumption share of investor A changes—in response to shocks to x_{t+1}. Assume that $\rho^A_x > \rho^B_x$. The sign of b^{CRRA} depends on the sign of x_t. If x_t is positive (negative), b^{CRRA} is positive (respectively, negative); this sign implies that the larger the shock to x_{t+1}, the larger (smaller) will be $\log s^A_1$ and, hence, the larger (smaller) the consumption share of agent A. The intuition is that investor B believes in faster mean reversion and hence puts more probability weight on states where x_{t+1} moves toward its long-run mean of 0. So investors bet on states depending on the subjective probabilities they assign to those states. This speculation motive increases with $|\rho_x^A - \rho_x^B|$ and $|x_t|$ and decreases with the risk aversion γ. The larger the difference in the beliefs, $|\rho_x^A - \rho_x^B|$, the larger is the difference in probabilities that the investors assign to different states. For $x_t = 0$ investors share the same beliefs but the larger $|x_t|$, the more important becomes the difference in the beliefs about the speed of mean reversion. Finally, the more risk averse investors are, the less they are willing to speculate on future outcomes.

We observe that this speculation motive is independent of the true persistence ρ_x. However, the true persistence does influence the average change in the consumption share. Assume that investor A has the correct beliefs, $\rho_x = \rho_x^A$. The average change in the log consumption share is then given by

$$\begin{aligned} E_t\left(\log\left(s_{t+1}^A\right)\right) - \log\left(s_t^A\right) &= a^{CRRA} - \log\left(s_t^A\right) \\ &= \frac{(1-s_t^A)x_t^2}{2\sigma_x^2\gamma}(\rho_x^A - \rho_x^B)^2 \geq 0. \end{aligned} \tag{17}$$

We observe that—independent of the states s_0^A and x_0 and whether ρ_x^A is larger or smaller than ρ_x^B—the consumption share of investor A, who has the correct beliefs, will always increase on average. So for CRRA utility, the only thing that matters for the average change in the consumption shares is which investor has the correct beliefs. The speed at which he or she accumulates wealth depends on the risk aversion of the investor. So the more risk averse the investor, the less he or she will be willing to speculate on future outcomes and, hence, the slower will be the wealth accumulation.

3.2 The Epstein–Zin Case

We next consider a stylized version of the Epstein–Zin model. Instead of the fully stochastic setup (15), we consider an infinite-horizon economy, but there is only a single shock in the model, to long-run risk in the first period. Subsequently, x_t slowly converges back to the long-run mean of zero. In period zero, agents disagree about the persistence of x_t, but they learn about the true persistence once all uncertainty is resolved in period 1.[2] We abstract from shocks to Δc_{t+1} as they do not affect the equilibrium consumption shares.[3]

[2] If belief differences persist into period 1, the model has no equilibria.

[3] As agents agree on Δc_{t+1}, $\eta_{c,t+1}$ only affects the equilibrium consumption share (14) if agents have different preference parameters, which is not the case in this setup.

In the initial period, the two agents have the following beliefs about the economy:

$$\begin{aligned}\Delta c_1 &= \mu + x_0 \\ x_1 &= \rho_x^h x_0 + \eta_{x,1}, \qquad \eta_{x,1} \sim N(0, \sigma_x^2),\end{aligned}$$

with the true persistence being ρ_x. In period 1 all uncertainty is resolved and both investors know the true persistence ρ_x:

$$\begin{aligned}\Delta c_{t+1} &= \mu + x_t \\ x_{t+1} &= \rho_x x_t, \qquad \forall t \geq 1.\end{aligned}$$

To solve the model, we follow the approach in Colacito, Croce, and Liu (2018b).[4] As in Colacito, Croce, and Liu (2018b), we set the EIS to 1 ($\psi^1 = \psi^2 = 1$). The value functions of the investors are then given by

$$v_t^h = (1-\delta)\log s_t^h + \frac{\delta}{1-\gamma}\log E_t^h\left(e^{(1-\gamma)(v_{t+1}^h + \Delta c_{t+1})}\right), \quad h \in A, B. \tag{18}$$

In Appendix F we show how to derive the solutions for the model. As there is no disagreement and hence no trading after period 0, the solution is characterized by the equilibrium consumption share in period 1. We show that, as in the CRRA case, the log consumption share of investor A in period 1, $\log s_1^A$, is a linear function of the shock $\eta_{x,1}$

$$\log\left(s_1^A\right) = a + b\eta_{x,1} \tag{19}$$

and the coefficients are given by

$$b = \frac{(1-s_0^A)x_0(\rho_x^A - \rho_x^B)}{\gamma\sigma_x^2} \tag{20}$$

$$\begin{aligned}a = {} & \log\left(s_0^A\right) + \frac{(1-s_0^A)x_0^2}{2\sigma_x^2}\left[(\rho_x - \rho_x^B)^2 - (\rho_x - \rho_x^A)^2\right] \\ & +(1-s_0^A)\frac{\delta}{1-\delta\rho_x}(\rho_x^B - \rho_x^A)x_0\frac{(1-\gamma)}{\gamma} \\ & +\frac{x_0^2(1-\gamma)^2(\rho_x^A - \rho_x^B)^2(1-s_0^A)(2s_0^A - 1)}{2\sigma_x^2\gamma^2}.\end{aligned} \tag{21}$$

We observe that the slope coefficient is the same for Epstein–Zin and CRRA utility with the

[4] We consider the infinite-horizon economy instead of the simple 2-period setup in Colacito, Croce, and Liu (2018b) to obtain effects on continuation values due to the slow convergence of x_t.

same coefficient of risk aversion ($b = b^{CRRA}$). So investors bet again on the occurrence of future states, depending on their subjective beliefs. We call this mechanism the speculation motive. However, the constant terms a differ significantly. Assume again that investor A has the correct beliefs, $\rho_x = \rho_x^A$. The average change in the consumption share is then given by

$$
\begin{aligned}
E_0\left(\log\left(s_1^A\right)\right) - \log\left(s_0^A\right) &= a - \log\left(s_0^A\right) \\
&= \underbrace{\frac{(1-s_0^A)x_0^2}{2\sigma_x^2}(\rho_x^A - \rho_x^B)^2}_{>0}\left[\underbrace{1}_{\text{CRRA-Term}} + \underbrace{\frac{(1-\gamma)^2(2s_0^A-1)}{\gamma^2}}_{\text{EZ-Risk Adjustment}}\right] \\
&+ \underbrace{(1-s_0^A)\frac{\delta}{1-\delta\rho_x}(\rho_x^B - \rho_x^A)x_0\frac{(1-\gamma)}{\gamma}}_{\text{New EZ-Channel}}.
\end{aligned}
$$

The first term in the second line is the same as the CRRA term with risk aversion 1. So there is a wealth transfer to the investor with the correct beliefs irrespective of the state. The second term in the first line is a risk adjustment, which is negative for $s_0^A < 0.5$ and positive for $s_0^A > 0.5$. So for small levels of s_0^A, investor A is more conservative relative to the CRRA case with unit risk aversion and is not willing to bet as much money. Therefore, the average increase of investor A's consumption share is not as strong as for CRRA utility. Note that $1 + \frac{(1-\gamma)^2(2s_0^A-1)}{\gamma^2} > 0$ for $\gamma > 1$. Thus, as in the CRRA case, the term in the second line always increases the wealth of investor A, who holds the correct beliefs.[5]

The third line shows the key difference between the Epstein–Zin and the CRRA case. The sign of the third line depends on the sign of x_0 as well as the sign of the difference $\rho_x^A - \rho_x^B$. So the average change in the consumption share depends on whether an investor believes in a larger or smaller ρ_x compared to the other agent and not only on which investor has the correct beliefs. Note that $\frac{(1-\gamma)}{\gamma}$ is negative for $\gamma > 1$ and we assume $\rho_x^A - \rho_x^B > 0$. For $x_0 < 0$ the EZ channel is therefore negative and there is a wealth transfer to investor B who beliefs in a smaller ρ_x. Hence, even though investor A has the correct beliefs, there is a channel that decreases his consumption share if $x_0 < 0$. Put differently, investors who believe in a larger persistence ρ_x, are, on average, willing to give away parts of their consumption share to hedge against risks in x_1 if $x_0 < 0$. This channel increases with the degree of risk aversion of the investor since $\left|\frac{(1-\gamma)}{\gamma}\right|$ increases for $\gamma > 1$. The larger the degree of risk aversion the more are those investors, who believe in a larger ρ_x, willing to pay to hedge against future risks. This

[5] As $1 + \frac{(1-\gamma)^2(2s_0^A-1)}{\gamma^2}$ is increasing in s_0^A, its minimum is obtained for $s_0^A = 0$. This yields $1 - \frac{(1-\gamma)^2}{\gamma^2} = \frac{2\gamma-1}{\gamma^2}$, which is > 0 for $\gamma > 1$.

risk-sharing motive is the key channel that drives our results in the following section.

Finally, note that in the stylized economy with no uncertainty after period 1, the average change in the consumption share for $x_0 = 0$ is always zero. For $x_0 = 0$ investors agree on the outcomes in the first period and, hence, there is neither speculation nor risk-sharing demand. In the fully stochastic economy in the next section this is not true. Even for $x_0 = 0$ the investors face the risk that x_t becomes negative after period 1 and, hence, are willing to pay a premium to insure against these risks. We show that this risk-sharing motive is present and strong; even for $x_0 > 0$, investors who are (more) highly afraid of bad shocks to x_t will be willing to pay a premium to insure against these risks.

4 Belief Differences in the Standard Calibration

In the following, we present the results for the fully stochastic infinite-horizon economy with the exogenous processes given in system (15). As before, we assume that there are two investors who have different beliefs about ρ_x. In light of the results from the representative-agent literature on long-run risks, the majority of investors need to believe in a highly persistent long-run risk process. Otherwise, asset prices would be determined by those investors who don't believe, or who believe less, in long-run risks; and, hence, the model outcomes would certainly not be consistent with the data. Therefore, we assume that a majority of investors believe in a highly persistent long-run risk process. Then we address the question of what happens if there is a small fraction of investors who believe in slightly less persistent shocks—that is, who are somewhat skeptical of the presence of long-run risks. We do not make a specific assumption about which agent has the correct beliefs. In fact, we show below that for small belief differences the true distribution has a negligible influence on equilibrium outcomes.

Most long-run risk models calibrate the underlying cash-flow parameters in order to match asset-pricing data. For example, Bansal and Yaron (2004) use a value of $\rho_x = 0.979$. Bansal, Kiku, and Yaron (2012) use $\rho_x = 0.975$, and Drechsler and Yaron (2011) assume $\rho_x = 0.976$. They obtain high values of ρ_x by construction, as otherwise the models would not be consistent with the high equity premium and other pricing moments observed in the data. The study by Bansal, Kiku, and Yaron (2016) relies on cash-flow and asset-pricing data to estimate the long-run risk model parameters and reports a value of $\rho_x \approx 0.98$ with a standard error of 0.01. For our baseline calibration, we assume that the first agent believes that $\rho_x^1 = 0.985$. This value implies an equity premium of 6.53 percent for the representative-agent economy, which is consistent with the value observed in the data. The second agent believes in a slightly smaller persistence level, $\rho_x^2 = 0.975$. Both values lie well within the confidence interval provided by Bansal, Kiku, and Yaron (2016). A small change in ρ_x has large effects on asset prices. For $\rho_x = 0.975$, the equity premium decreases to 2.76 percent in the representative-agent economy. For $\rho_x = 0.95$, it collapses to 0.26 percent and the influence of x_t on asset prices is negligible. For completion, we also analyze the model for the values $\rho_x^1 = 0.985$ and $\rho_x^2 = 0.95$ of the persistence parameter.

While the two agents have different beliefs, they have identical Epstein–Zin utility parameters in the benchmark economy. They share the properties of the representative agent of Bansal and Yaron (2004) with $\psi^1 = \psi^2 = 1.5$, $\gamma^1 = \gamma^2 = 10$, and $\delta^1 = \delta^2 = 0.998$. For the remaining parameters of the state processes (15) we also use the calibration from Bansal and Yaron (2004, Case I) (model without stochastic volatility), with $\mu_c = \mu_d = 0.0015, \sigma = 0.0078, \Phi = 3, \phi_d = 4.5$, and $\phi_x = 0.044$. (This calibration is used for all the results in the present paper.)

4.1 Equilibrium Consumption Shares in the Infinite-Horizon Economy

Before we look at the consumption dynamics and asset-pricing implications of the model, we investigate how the belief differences affect the equilibrium consumption shares and relate the results to the intuition obtained in the previous section. Colacito and Croce (2013) show that the equilibrium consumption shares in the infinite-horizon economy can be characterized by a mean-variance trade-off of the continuation utilities of the investors. Let $\bar{v}_t^h = \frac{(v_t^h)^{\rho^h}}{\rho^h}$. The value functions of the investors (12) are then given by

$$\bar{v}_t^h = (1-\delta^h)\frac{(s_t^h)^{\rho^h}}{\rho^h} + \delta^h E_t^h \left((\bar{v}_{t+1}^h)^{\frac{\alpha^h}{\rho^h}} e^{\alpha^h \Delta c_{t+1}} \right)^{\frac{\rho^h}{\alpha^h}}, \qquad h \in \mathbb{H}. \tag{22}$$

If all investors share the same preference parameters $\gamma^h = \gamma$ and $\psi^h = \psi$, then $\bar{v}_{t+1}^h$ is independent of Δc_{t+1} (see Equation (14)). Colacito and Croce (2013) then show that by assuming log-normality for $\bar{v}_{t+1}^h$, the equation can be rewritten as

$$\bar{v}_t^h \approx (1-\delta^h)\frac{(s_t^h)^{\rho}}{\rho} + \delta k_t E_t^h(\bar{v}_{t+1}^h) - \delta k_t \frac{\rho-\alpha}{2\rho} \frac{Var_t^h(\bar{v}_{t+1}^h)}{E_t^h(\bar{v}_{t+1}^h)}, \qquad h \in \mathbb{H}, \tag{23}$$

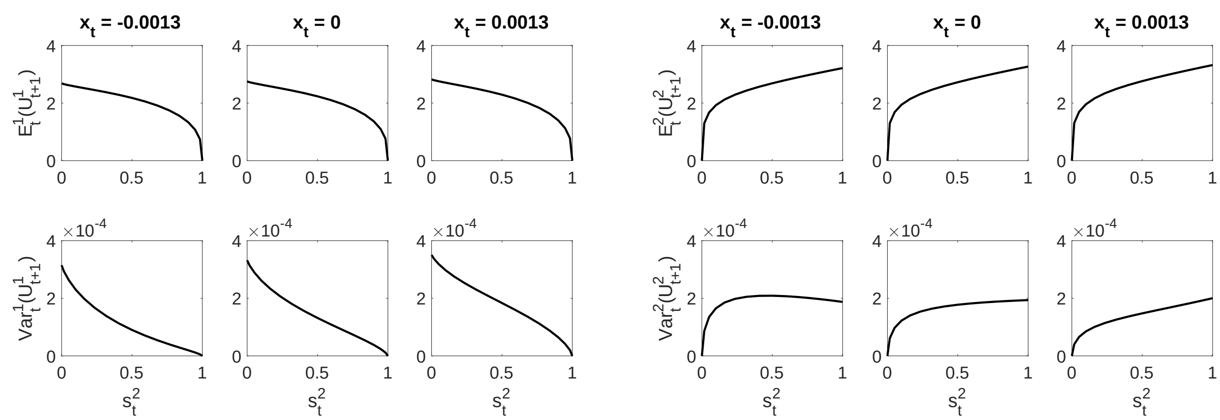
where $k_t = E_t^h(e^{\alpha \Delta c_{t+1}})^{\frac{\rho}{\alpha}}$. Note that $\rho E_t^h(\bar{v}_{t+1}^h)$ is positive and, hence, for $\rho > \alpha$—that is, for preferences for the early resolution of risks—a higher variance of the continuation utility,

$Var_t^h(\bar{v}_{t+1}^h)$, reduces welfare. So the investor is willing to trade off expected future utility $E_t^h(\bar{v}_{t+1}^h)$ for uncertainty about future utility $Var_t^h(\bar{v}_{t+1}^h)$. As there is a one-to-one mapping between the continuation values v_t^h and the lifetime wealth of the agents (see Epstein and Zin (1989)), the mean-variance trade-off of continuation utilities can also be interpreted as the effects on the wealth shares of the investors.

Figure 1 shows this trade-off for our benchmark economy. As expected, the conditional mean of the continuation utilities of investor 1 (2) decreases (respectively, increases) with the consumption share of investor 2. The conditional variance of investor 1 gradually increases when investor 1's consumption share increases. In contrast, for investor 2, who is less afraid of long-run risks, the variance sharply increases when moving from a share of 0 to a small share of consumption. Hence, the presence of a small fraction of investors who are less afraid

Figure 1: Conditional Mean and Variance of Continuation Utilities

(a) Investor 1 (b) Investor 2



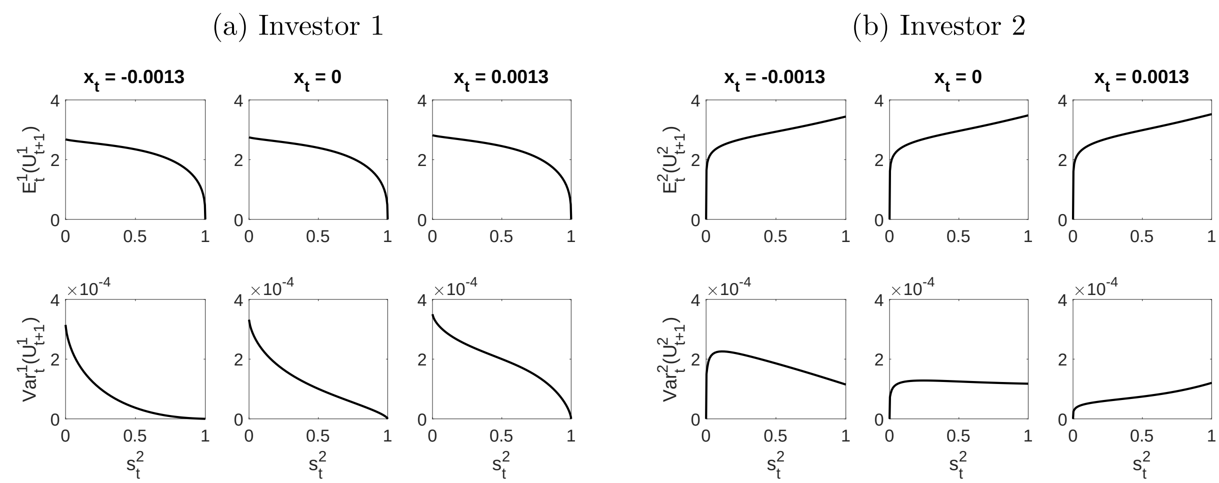
The figure plots the conditional mean and variance of continuation utilities as a function of the consumption share of the second investor. The left panel shows the results for investor 1, who believes that $\rho_x^1 = 0.985$, and the right panel for investor 2, who believes that $\rho_x^2 = 0.975$. Results are shown for three different values of x_t. For the data-generating process, we use $\mu_x = 0, \sigma_x = 0.0003432, \mu = 0.0015$, and $\sigma = 0.0078$.

of long-run risks is sufficient to induce strong risk sharing. In line with the results obtained in the previous section, this mechanism is especially strong for small ("more negative") x_t—that is, when the influence of long-run risks is large. Hence, in low x-states, where the marginal utility of investor 1 is low due to the bad state of the economy, investor 1 is willing to trade future wealth in order to reduce his risks. Investor 2, who is less afraid of the risks, is willing to take on the additional risks but she must be compensated by higher future wealth shares. So the risk sharing will transfer wealth from the first to the second investor.

Figure 2 shows the corresponding results for an increase in the belief difference ($\rho_x^2 =$

0.965 instead of 0.975). We observe that the willingness of investor 2 to provide risk sharing increases. For $x_t = -0.0013$, $Var_t^2(\bar{v}_{t+1}^2)$ increases significantly compared to the case where investor 2 holds all the wealth (moving from the right corner of the graphs to the left). Furthermore, the variance of investor 1 decreases sharply compared to the case where investor 1 holds all the wealth (moving from the left corner of the graph to the right). As investor 2 must be compensated by investor 1 for bearing these additional risks, there is a wealth transfer from investor 1 to investor 2.

Figure 2: Conditional Mean and Variance of Continuation Utilities—Large Disagreement



The figure plots the conditional mean and variance of continuation utilities as a function of the consumption share of the second investor. The left panel shows the results for investor 1, who believes that $\rho_x^1 = 0.985$, and the right panel for investor 2, who believes that $\rho_x^2 = 0.965$. Results are shown for three different values of x_t. For the data-generating process, we use $\mu_x = 0, \sigma_x = 0.0003432, \mu = 0.0015$, and $\sigma = 0.0078$.

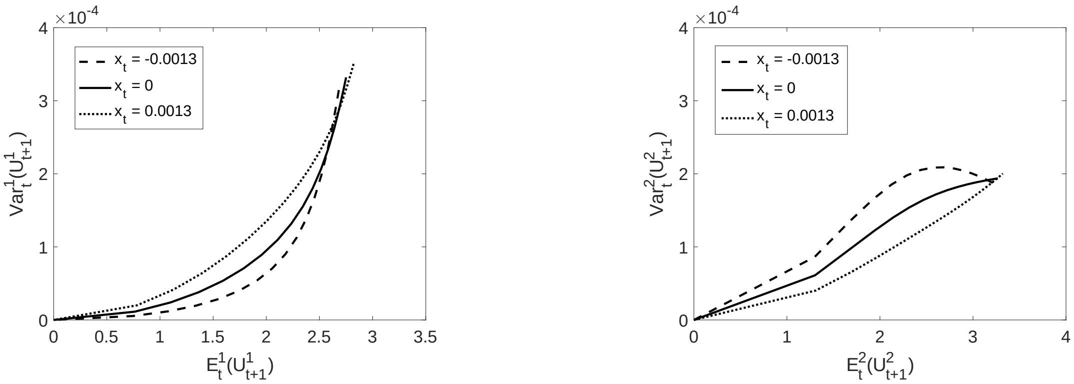
For a different view of the described trade-off between the conditional variances and conditional means of the continuation utilities, we plot them directly against each other. Figures 3 and 4 are the corresponding plots to Figures 1 and 2. In accordance with our observations above, we observe a strict convexity in the graph of the first investor's conditional variance versus the conditional mean of that investor's continuation utility. Moreover, the plots nicely show that the trade-off is more extreme the worse the level of the long-run-risk state, x_t.

4.2 Simulation Results

The previous section has shown that investors who are more afraid of long-run risks (large ρ_x) are willing to trade future wealth to hedge against risks in x_t. Investors with smaller ρ_x are willing to provide this "insurance". This channel transfers wealth from investors with high

Figure 3: Mean-Variance Frontier of Continuation Utilities

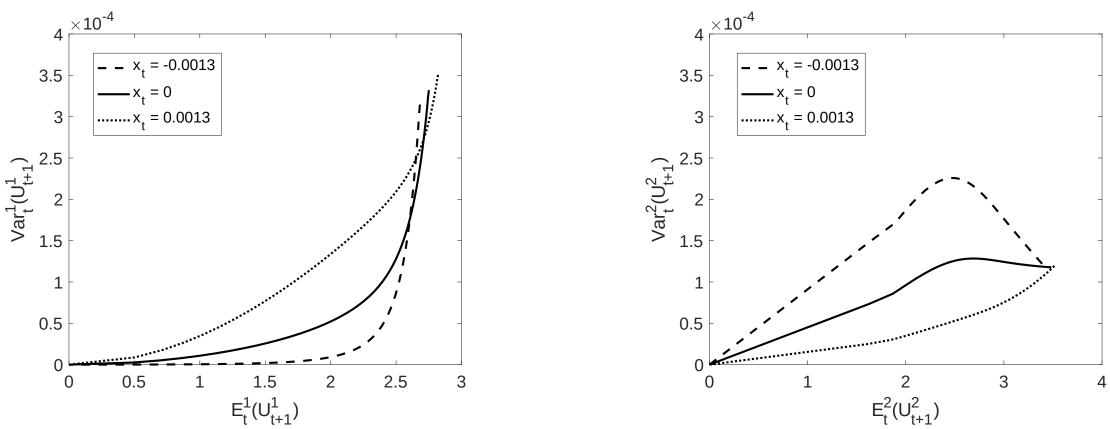
(a) Investor 1 (b) Investor 2



The figure plots the conditional variance against the conditional mean of continuation utilities. The left panel shows the results for investor 1, who believes that $\rho_x^1 = 0.985$, and the right panel for investor 2, who believes that $\rho_x^2 = 0.975$. For the data-generating process, we use $\mu_x = 0, \sigma_x = 0.0003432, \mu = 0.0015$, and $\sigma = 0.0078$.

Figure 4: Mean-Variance Frontier of Continuation Utilities—Large Disagreement

(a) Investor 1 (b) Investor 2



The figure plots the conditional variance against the conditional mean of continuation utilities. The left panel shows the results for investor 1, who believes that $\rho_x^1 = 0.985$, and the right panel for investor 2, who believes that $\rho_x^2 = 0.965$. For the data-generating process, we use $\mu_x = 0, \sigma_x = 0.0003432, \mu = 0.0015$, and $\sigma = 0.0078$.

ρ_x to investors with low ρ_x irrespectively of the true persistence value. In the following we analyze how this channel affects consumption shares and asset prices.

We begin with an analysis of the equilibrium dynamics of the consumption shares of the individual agents. Figure 5 shows the consumption share of the second, skeptical agent ($\rho_x^2 = 0.975$) over time for different initial shares $s_0^2 = \{0.01, 0.05, 0.5\}$. We report the median, 5%, and 95% quantile paths using 1,000 samples each consisting of 500 years of simulated data. To minimize the influence of the initial value of x_t, we initialize each simulated path by running a "burn-in" period of 1,000 years before using the output. The left panel shows the results for $\rho_x = \rho_x^1 = 0.985$ (the first agent has the correct beliefs) and the right panel for $\rho_x = \rho_x^2 = 0.975$ (the second agent has the correct beliefs).

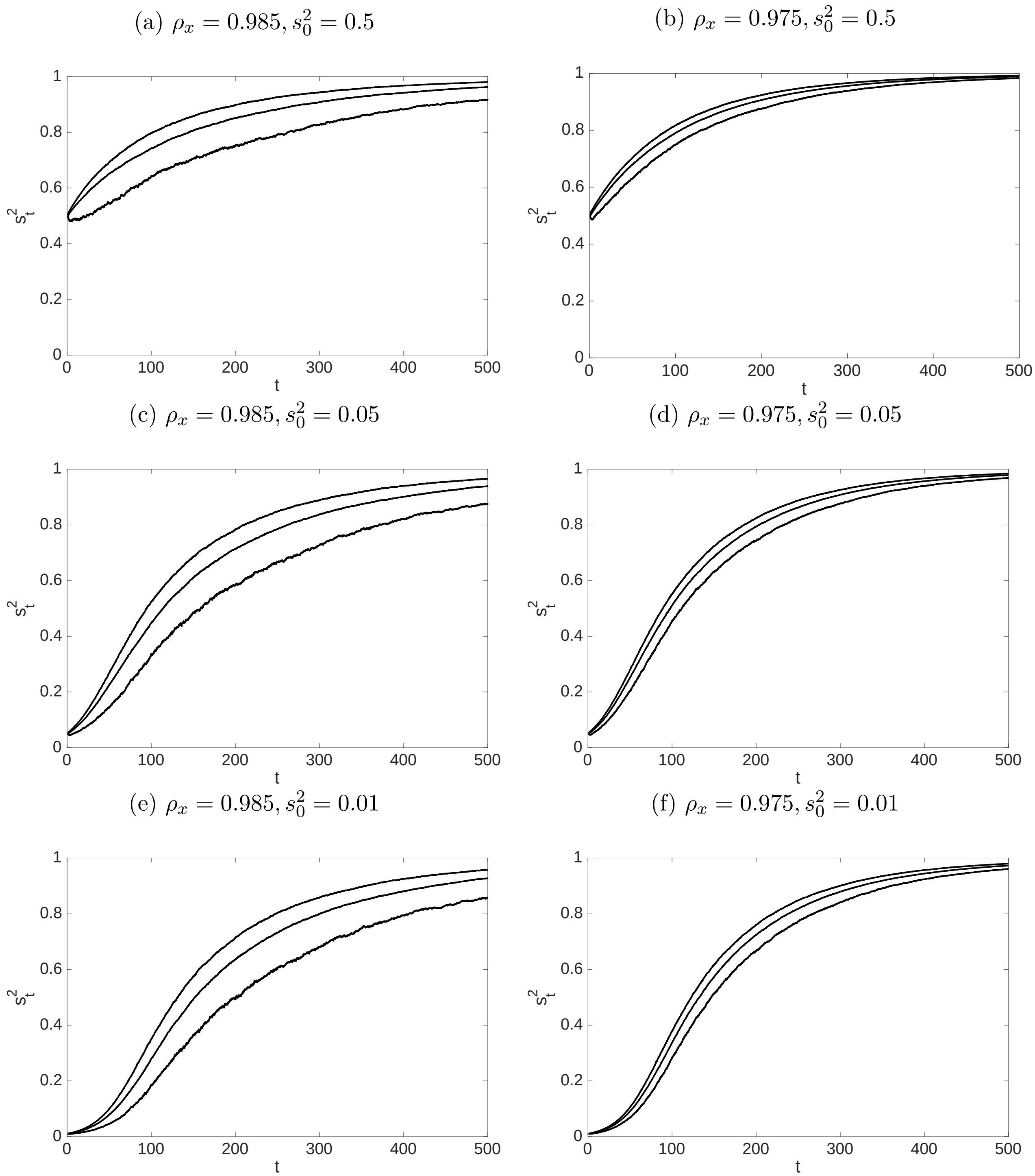
We observe that in all cases the consumption share of the skeptical agent, agent 2, strongly increases over time. While this increase occurs faster if agent 2 has the correct beliefs (right panel), the increase is almost as strong if agent 1 has the correct beliefs (left panel). Hence, given a small difference in the beliefs, independent of whether agent 1 or agent 2 has the correct beliefs, in the long run the agent who believes in a smaller ρ_x will dominate the economy. So, using the terminology from the stylized model in Section 3.2, the influence of the risk-sharing motive is stronger than the influence of the speculation motive. Most importantly, even if the economy is initially almost entirely populated by agent 1, $s_0^2 = 0.01$, his consumption share decreases sharply, leading the agent to lose significant share in a short amount of time. This observation is in line with the results from Section 4.1, which show that the risk-sharing motive is especially strong if there is a small fraction of skeptical investors.

Table 1 reports the corresponding median consumption shares for different time horizons and for values $s_0^2 \in \{0.01, 0.05, 0.5\}$. We observe that for $s_0^2 = 0.01$ the consumption share of agent 1 has decreased by more than 27% after 100 years, by 62% after 200 years, and by almost 92% after 500 years.

Figure 6 shows the corresponding results for $\rho_x^2 = 0.95$ and an initial allocation of $s_0^2 = 0.01$. The left panel shows the results for $\rho_x = 0.985$ (agent 1 has the correct beliefs). We observe that the initial increase in the consumption share is stronger compared to the case with $\rho_x^2 = 0.975$ but that the median share does not become as large in the long run (the median shares of the second agent after 100, 200, and 500 years are given by 32.59%, 37.82%, and 40.19%, respectively). As shown in the previous section, the larger the belief difference, the stronger becomes the influence of the risk-sharing channel for small s_t^2 and hence the larger is the wealth transfer to the second agent.

The 5% and 95% quantile paths and the grey sample path show that there is significantly

Figure 5: Consumption Shares—Simulations



(a) $\rho_x = 0.985, s_0^2 = 0.5$

(b) $\rho_x = 0.975, s_0^2 = 0.5$

(c) $\rho_x = 0.985, s_0^2 = 0.05$

(d) $\rho_x = 0.975, s_0^2 = 0.05$

(e) $\rho_x = 0.985, s_0^2 = 0.01$

(f) $\rho_x = 0.975, s_0^2 = 0.01$

The figure shows the median, 5%, and 95% quantile paths of the consumption share of agent 2 for 1,000 samples each consisting of 500 years of simulated data. Agent 2 believes that $\rho_x^2 = 0.975$ and agent 1 believes that $\rho_x^1 = 0.985$. Results are shown for different initial consumption shares ($s_0^2 = \{0.01, 0.05, 0.5\}$). The left panel depicts the case where the skeptical agent, agent 2, has the wrong beliefs about the long-run risk process ($\rho_x = 0.985 = \rho_x^1$) and in the right panel the skeptical agent has the right beliefs ($\rho_x = 0.95 = \rho_x^2$).

Table 1: Consumption Shares—Summary Statistics

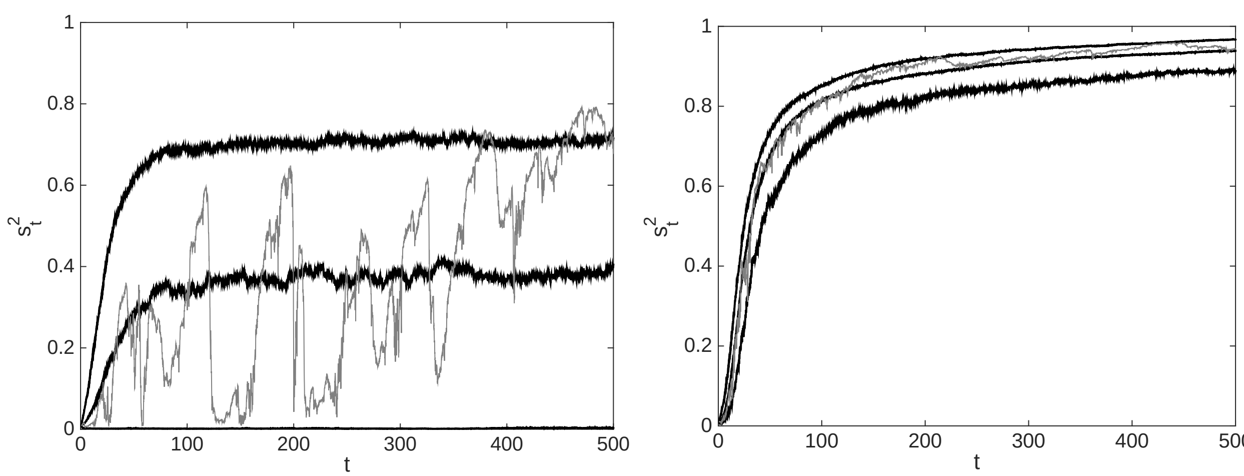
	$\rho_x = 0.985$			$\rho_x = 0.975$		
Years	100	200	500	100	200	500
$s_0^2 = 0.5$	0.7429	0.8515	0.9628	0.8904	0.9683	0.9995
	(0.0500)	(0.0481)	(0.0212)	(0.0118)	(0.0072)	(4.2e-5)
$s_0^2 = 0.05$	0.4507	0.7143	0.9393	0.5140	0.7947	0.9787
	(0.0589)	(0.0636)	(0.0293)	(0.0301)	(0.0252)	(0.0049)
$s_0^2 = 0.01$	0.2824	0.6376	0.9278	0.3404	0.7249	0.9732
	(0.0509)	(0.0681)	(0.0326)	(0.0294)	(0.0293)	(0.0060)

The table shows the median and the standard deviation (in parenthesis) of the consumption share of agent 2 using 1,000 samples each consisting of 500 years of simulated data. Agent 2 believes that $\rho_x^2 = 0.975$ and agent 1 believes that $\rho_x^1 = 0.985$. Summary statistics are shown for different initial consumption shares ($s_0^2 = \{0.01, 0.05, 0.5\}$) and different time periods $T = \{100, 200, 500\}$ years. The left panel depicts the case where the skeptical agent, agent 2, has the wrong beliefs about the long-run risk process ($\rho_x = 0.985 = \rho_x^1$) and in the right panel the skeptical agent has the right beliefs ($\rho_x = 0.95 = \rho_x^2$).

Figure 6: Consumption Shares for $\rho_x^2 = 0.95$—Simulations

(a) $\rho_x = 0.985$

(b) $\rho_x = 0.95$



The figure shows the median, 5%, and 95% quantile paths of the consumption share of agent 2 for 1,000 samples each consisting of 500 years of simulated data as well as a sample path (grey line). Agent 2 believes that $\rho_x^2 = 0.95$ and agent 1 believes that $\rho_x^1 = 0.985$. Results are shown for an initial consumption share of $s_0^2 = 0.01$. The left panel depicts the case where the skeptical agent, agent 2, has the wrong beliefs about the long-run risk process ($\rho_x = 0.985 = \rho_x^1$) and in the right panel the skeptical agent has the right beliefs ($\rho_x = 0.95 = \rho_x^2$).

more variation in the shares. We observe that there are large drops and recoveries in the consumption share. The large drops occur because the second agent assigns "wrong" probabilities to extreme states and hence "bets" on states that turn out to occur less often in the long run. As belief differences are large, this speculation motive has a large influence. This channel works in favor of agent 2, once she has the correct beliefs and is therefore more likely to bet on the correct states. This case is shown in the right panel ($\rho_x = \rho_x^2 = 0.95$), where we indeed observe that the increase in the consumption share is much stronger and that the large drops in consumption are no longer present. The increase in the left panel occurs much more slowly as the speculation motive works against agent 2 on average. However, the risk-sharing channel transfers wealth to investor 2. As belief differences are large and the first investor strongly dislikes shocks in x_t, that investor is willing to pay a high premium to insure against these risks, while the second investor is willing to provide this insurance. This risk-sharing channel will transfer significant amounts of wealth to the second investor. We provide a more detailed analysis of the speculation channel and the risk-sharing channel in the economy in Section 4.3 below.

What does the change in the consumption shares imply for asset prices and aggregate financial market statistics? We assume that the economy is initially almost entirely populated by agent 1 in order to generate a high equity premium consistent with the data. But the consumption share of the first agent decreases rapidly, and so will that agent's influence on asset prices. In Table 2 we show the annualized equity premium in the years 0, 100, 200, and 500, assuming an initial share of $s_0^2 = 0.01$.[6] The left panel shows the results for $\rho_x^2 = 0.975$ where agent 1 has the correct beliefs. For the initial allocation $s_t^2 = 0.01$, when agent 1 dominates the economy, the aggregate risk premium is 6.42%. This value is very close to that of the representative-agent economy populated only by the first agent, which generates a premium of 6.53%. After 100 years, when the share of agent 1 has decreased from 99% to 72%, the premium decreases to 4.59%. Hence, even if agent 1 holds almost all the wealth initially, which implies a high risk premium, the premium will drop by almost 2% within a century. After 200 years, the premium decreases by almost 3% and after 500 years it is almost at the level of the representative-agent economy populated only by agent 2, with a premium of 2.89%. The right panel shows the corresponding results for $\rho_x^2 = 0.95$. We observe that the sharp increase in the consumption share decreases the premium from 5.42% initially to

[6] Note that Table 2 does not report the premium starting with a given value for s_0^2 and simulating a long time series, but that we report the average premium for a given consumption share $s_t^2 = \bar{s}$. Hence, we take the expectation over all x_t while keeping the consumption share constant at $\bar{s}$. The population moment for 500 years of simulated data is given in Table 3.

Table 2: Equity Premium for Different Consumption Shares

	$\rho_x^2 = 0.975$		$\rho_x^2 = 0.95$	
	s_t^2	Equity Premium	s_t^2	Equity Premium
Rep. Agent 1	0	6.53	0	6.53
0 Years	0.01	6.42	0.01	5.42
100 Years	0.2824	4.59	0.3259	1.84
200 Years	0.6376	3.49	0.3782	1.64
500 Years	0.9278	2.89	0.4019	1.56
Rep. Agent 2	1	2.76	1	0.26

The table shows the annualized equity premium for a specific consumption share $s_t^2 = \bar{s}$. The premium is reported for the equilibrium allocations after 0, 100, 200, and 500 years of simulated data assuming an initial share of $s_0^2 = 0.01$ (see Table 1). Agent 1 has the correct beliefs with $\rho_x^1 = \rho_x = 0.985$. The left panel depicts the case for $\rho_x^2 = 0.975$ and the right panel for $\rho_x^2 = 0.95$.

1.84% after 100 years—a decrease of more than 3.5% in a century. Hence, the difference in beliefs brings down the equity premium to well below the levels observed in the data even if the agent who is skeptical about the presence of long-run risks does not have the correct beliefs. (Table 8 in Appendix G reports the corresponding results for the case where agent 2 rather than agent 1 has the correct beliefs. As expected, we observe that the drop in the equity premium is even more severe.)

While the consumption dynamics have a negative impact on first moments, they might positively influence second moments due to the large variation in consumption shares. Table 3 shows selected moments from the 1,000 sample paths. We report the mean and the standard deviation of the annualized log price–dividend ratio, the annualized equity premium, and the risk-free return. Results are shown for the case in which agent 1 has the correct beliefs. In addition to the two-agent economy, the table also shows the two representative-agent cases, where the economy is populated only by agent 1 ($s_t^2 = 0$) or agent 2 ($s_t^2 = 1$). Note that the results only serve to obtain intuition about the effects on second moments and we conduct a full recalibration exercise in Section 5.

We observe that the volatility of the log price–dividend ratio is significantly larger for the two-agent economy compared to both representative-agent economies, respectively. This effect is especially strong for $\rho_x^2 = 0.95$, where the volatility is 0.48 compared to 0.25 and 0.14 for the two representative-agent economies. These results are driven by shifts in the wealth distribution. As the wealth distribution shifts weights back and forth between the agents,

Table 3: Annualized Asset-Pricing Moments

	$E(p_t - d_t)$	$\sigma(p_t - d_t)$	$E\left(r_t^m - r_t^f\right)$	$E\left(r_t^f\right)$	$\sigma(r_t^m)$	$\sigma\left(r_t^f\right)$
			$\boldsymbol{\rho_x^2 = 0.975}$			
$s_t^2 = 0$	2.68	0.25	6.53	2.32	17.84	1.50
Two-Agent Economy	3.10	0.29	3.98	2.58	17.19	1.51
$s_t^2 = 1$	3.29	0.20	2.83	2.71	16.55	1.53
			$\boldsymbol{\rho_x^2 = 0.95}$			
$s_t^2 = 0$	2.68	0.25	6.53	2.32	17.84	1.50
Two-Agent Economy	3.60	0.48	2.63	2.47	20.37	1.58
$s_t^2 = 1$	6.27	0.14	0.26	2.93	14.80	1.52

The table shows selected moments from 1,000 samples each containing 500 years of simulated data starting with an initial share of $s_0^2 = 0.01$. It shows the mean and the standard deviation of the annualized log price–dividend ratio, the annualized market over the risk-free return, and the risk-free return. Agent 1 has the correct beliefs with $\rho_x^1 = \rho_x = 0.985$. All returns are shown in percentages, so a value of 1.5 is a 1.5% annualized figure.

second moments increase in response to the time variation in the wealth shares.[7]

Beeler and Campbell (2012) argue that one of the deficiencies of the long-run risk models of Bansal and Yaron (2004) and Bansal, Kiku, and Yaron (2012) is that they significantly underestimate the volatility of the price–dividend ratio (they report values of 0.28 compared to 0.45 observed in the financial market data). Our results show that differences in beliefs can potentially resolve this puzzle, since they lead to a significant increase in the volatility figures. In Section 5 we conduct a re-calibration exercise to analyze whether the heterogeneous-agent model can match both first and second moments as well as other stylized financial markets features.

In sum, if there are different investors who all believe in long-run risks but use slightly different estimates for the long-run risk process, the investor who is more skeptical about ρ_x eventually dominates the economy. The investor who believes in a larger value of ρ_x rapidly loses wealth, no matter whether those beliefs are correct or not. Recall that a large ρ_x is needed to obtain a high risk premium in the long-run risk model. Even if this investor with the belief in a large ρ_x almost entirely populates the economy initially, that investor's consumption share decreases so fast that the equity premium in the economy declines considerably in a short amount of time. On the positive side, different beliefs about ρ_x introduce variations

[7] While for this calibration the first moments lie well within the values observed for the representative-agent economies, in Section 5 we show that belief heterogeneity can endogenously add persistent consumption risks to the model, which in turn generate significant risk premia.

in the consumption shares, which in turn induces significant excess volatility of the price–dividend ratio. We have also seen that when the more skeptical agent's estimate of ρ_x is a bit further away from the true value then both agents may survive in the long run. (And, obviously, the skeptical agent would not survive when his or her estimate is sufficiently small and thus sufficiently far away from the correct value of ρ_x.)

4.3 The Speculation and the Risk-Sharing Motives

In this section, we analyze in more detail how the speculation motive and the risk-sharing motive influence the equilibrium consumption shares.

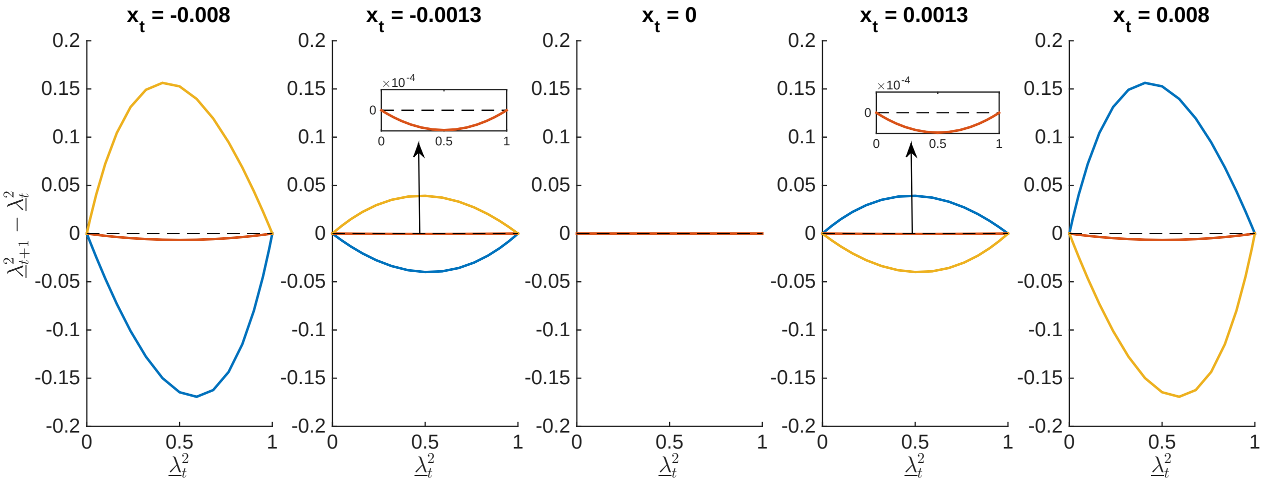
4.3.1 The Speculation Motive

In Section 3.1 we have seen that the speculation motive alone determines equilibrium outcomes in the special case of CRRA preferences. The investors assign different subjective probabilities to future states and buy assets that pay off in states they believe are more likely. Hence, for CRRA utility the agent with the more correct beliefs will accumulate wealth in the long run, as the investor with the more distorted beliefs bets on states that have a small probability under the true probability measure.

To demonstrate how the speculation motive affects equilibrium outcomes in the long-run risk model with different beliefs, we first consider the special case of CRRA preferences (see Section 3.1 for the log-linear solutions for the CRRA case). In Figure 7 we show the change in the Pareto weights $\underline{\lambda}^2_{t+1} - \underline{\lambda}^2_t$ as a function of $\underline{\lambda}^2_t$. Note that a positive (negative) change in the Pareto weight also implies a positive (negative) change in the consumption share (see Equation (13)). The blue and yellow lines depict the cases of a negative shock ($x_{t+1} - \rho_x x_t = -0.001$) and a positive shock ($x_{t+1} - \rho_x x_t = 0.001$) in x_{t+1}, respectively. The red line shows the average over all shocks. From left to right, the results are shown for $x_t = -0.008, x_t = -0.0013, x_t = 0, x_t = 0.0013$, and $x_t = 0.008$. Agent 1 has the correct belief, $\rho^1_x = \rho_x = 0.985$, while agent 2 believes that $\rho^2_x = 0.975$.

The second agent believes that x_t converges faster to its long-run mean than does agent 1. Hence, if $x_t < 0$, agent 2 assigns larger probabilities to large x_{t+1} and bets on those states as $\rho^2_x x_t > \rho^1_x x_t$ (left panels). The opposite holds true for $x_t > 0$. So agent 2 loses wealth if x_t is low and the shock in x_t is negative (blue line in the left-hand figures) or if x_t is high and the shock in x_t is also high (yellow line in the right-hand figures). Taking the average over all future realizations of x_{t+1} (red line), agent 2 loses wealth on average (red line). For $x_t = 0$ both agents share the same beliefs ($\rho^2_x x_t = \rho^1_x x_t$) and hence they assign the same probabilities

Figure 7: Changes in the Wealth Distribution—The CRRA Case



The figure shows the change in the optimal weights $\underline{\lambda}^2_{t+1} - \underline{\lambda}^2_t$ as a function of $\underline{\lambda}^2_t$. From left to right, the change is shown for $x_t = \{-0.008, -0.0013, 0, 0.0013, 0.008\}$ ($\pm$ 4 standard deviations). The blue line depicts the case of a negative shock in x_{t+1} ($x_{t+1} - \rho_x x_t = -0.001$) and the yellow line of a positive shock in x_{t+1} ($x_{t+1} - \rho_x x_t = 0.001$). The red line shows the average over all shocks. Baseline calibration with $\rho_x = \rho^1_x$ and CRRA preferences.

to x_{t+1} (the blue and yellow lines coincide with the red line and are not visible). As agent 2 loses wealth on average for all x_t except for $x_t = 0$, this agent will eventually vanish in the long run. Note that the influence of the speculation motive becomes stronger the larger $|x_t|$ is, as the belief dispersion grows the more x_t deviates from its unconditional mean, $E(x_t) = 0$.

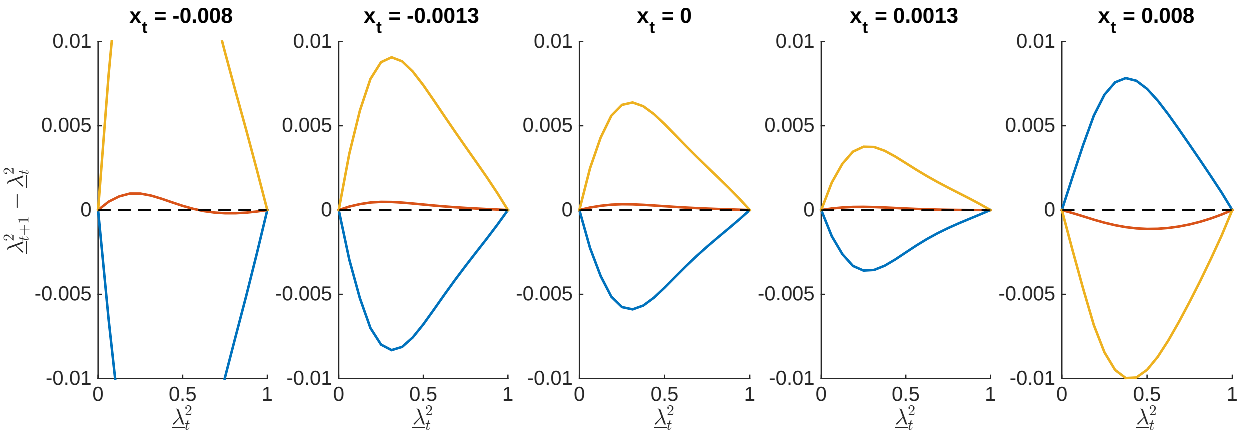
The speculation motive entirely determines the equilibrium in the standard case of CRRA preferences. For general Epstein–Zin preferences equilibrium dynamics become more complex. In the following we first describe the general effects of the risk-sharing motive and then analyze how the two effects interact and influence equilibrium outcomes.

4.3.2 The Risk-Sharing Motive

Section 4.1 has shown that the risk-return trade-offs are not the same among agents with Epstein–Zin preferences; instead, agents with smaller ρ_x are willing to take more risks. Recall from Section 3.2 that an investor who believes in a large ρ_x is afraid of large negative realizations of x_t and would therefore like to buy insurance against these risks. The skeptical investor—believing in a relatively smaller value for ρ_x—will be willing to provide this insurance as she is less afraid of the long-run risks. As there is a premium for bearing these risks, the risk-sharing motive will transfer wealth from the investors who are (more) afraid of long-run risks to the more skeptical investors.

In Figure 8 we demonstrate how this channel affects model outcomes. The figure shows the corresponding results to Figure 7 but for the general case of Epstein–Zin preferences. First,

Figure 8: Changes in the Wealth Distribution—The Epstein–Zin Case



The figure shows the change in the optimal weights $\underline{\lambda}^2_{t+1} - \underline{\lambda}^2_t$ as a function of $\underline{\lambda}^2_t$. From left to right, the change is shown for $x_t = \{-0.008, -0.0013, 0, 0.0013, 0.008\}$ ($\pm$ 4 standard deviations). The blue line depicts the case of a negative shock in x_{t+1} ($x_{t+1} - \rho_x x_t = -0.001$) and the yellow line of a positive shock in x_{t+1} ($x_{t+1} - \rho_x x_t = 0.001$). The red line shows the average over all shocks. Baseline calibration with $\rho_x = \rho_x^1$.

consider the center panel, where $x_t = 0$ and hence the speculation motive has no effect on equilibrium outcomes (see Figure 7). Agent 1 is more afraid of negative shocks to x_{t+1} than is agent 2. Therefore, agent 1 buys insurance against the long-run risks, which pays off in bad times when there is a negative shock to x_{t+1} (the blue line is negative, which implies an increase in the weights of the first agent for all $\underline{\lambda}^2_t$). For this insurance, agent 1 has to pay a premium in good times. So, for a positive shock to x_{t+1} the results reverse (yellow line). The average over all shocks (red line) is positive, so agent 1 pays a positive premium to insure against long-run risk, which is why this agent loses wealth on average. The effect is stronger for small $\underline{\lambda}^2_t$ and decreases for large $\underline{\lambda}^2_t$. A small value of $\underline{\lambda}^2_t$ implies that there is a large share of agents who want to buy insurance against long-run risks. Hence, they are willing to pay a higher price. The larger the share of the skeptical investors becomes, the lower becomes the demand for the insurance and, hence, the increase in the Pareto weights also becomes less pronounced.

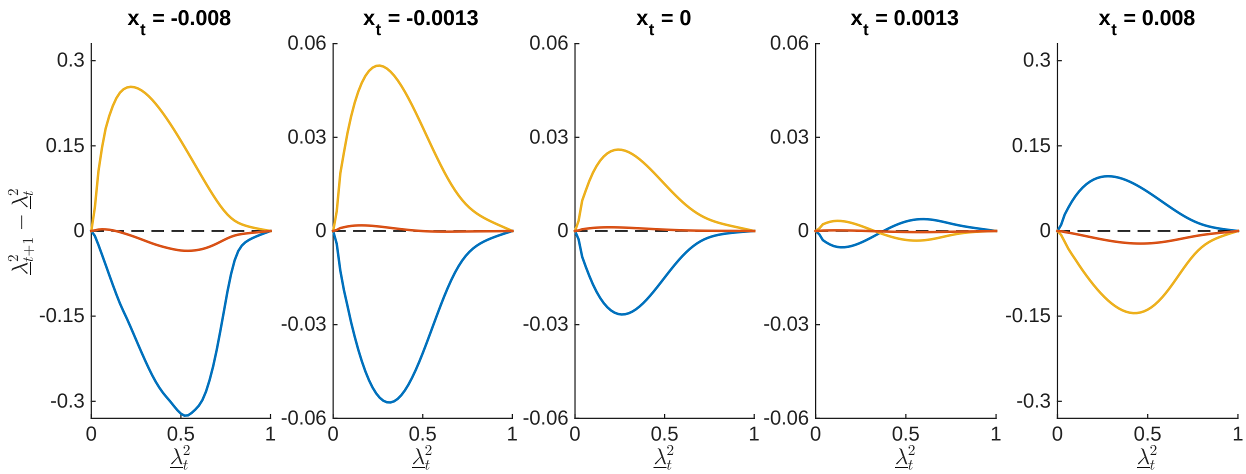
The case of $x_t = 0$ shows a key difference between the fully stochastic model and the simplified setup where all uncertainty is resolved in the first period (see Section 3.2). In the simplified setup, for $x_0 = 0$ investors will not trade and consumption shares remain constant as (i) there is no belief difference, and (ii) investors know that uncertainty is resolved in period 1, so there is no incentive to hedge risks that occur after period 1. As a result, the risk-sharing motive is much stronger in the fully stochastic setup.

Decreasing x_t has two effects. First of all, agent 1 becomes more afraid of long-run risks (given a negative value of x_t, a large negative realization of x_{t+1} becomes more likely due to the belief in a high persistence of ρ_x), which is why agent 1 wants to buy more insurance against long-run risks and is willing to pay a higher premium. We observe this effect in the second panel from the left ($x_t = -0.0013$) in Figure 8, where the average increase in the Pareto weight of the second agent (red line) increases compared to the results for $x_t = 0$. This result is in line with the risk-sharing motive described in Section 3.2. Additionally, the belief difference, and hence the difference between the subjective probabilities, becomes more pronounced for large $|x_t|$. So the influence of the speculation motive becomes stronger the further x_t is away from its unconditional mean. This potentially shifts wealth to the first agent, who has the correct beliefs about ρ_x. We observe this pattern in the left panel ($x_t = -0.008$), where for large $\underline{\lambda}_t^2$ the average change in the weights $\underline{\lambda}_{t+1}^2 - \underline{\lambda}_t^2$ becomes negative. For small $\underline{\lambda}_t^2$, the risk-sharing motive is larger (see Section 4.1) and, hence, the risk-sharing dominates the speculation motive.

For positive x_t agent 1 becomes less afraid of long-run risks and hence is less willing to pay to insure against them. Therefore, the average increase in the weights of agent 2 decreases for $x_t = 0.0013$ compared to $x_t = 0$. For very large x_t (right panel) the influence of the speculation motive dominates and hence the results reverse. The second agent wins if there is a negative shock (blue line), but loses if there is a positive shock (yellow line). The risk-sharing motive becomes negligible and the second agent loses on average as this agent bets on states that have a vanishing probability under the true measure (see Figure 7). So, the risk-sharing motive dominates the speculation motive for x_t close to its unconditional mean; only for very large x_t does the speculation motive dominate and then agent 2 potentially loses wealth (on average). However, values of $x_t = 0.008$ (+4 standard deviations of x_t) occur only very rarely; most of the time the process stays within the range where the risk-sharing motive clearly dominates the speculation motive and so, on average, agent 2's consumption share increases.

In Figure 9 we show the corresponding results for $\rho_x^2 = 0.95$ instead of $\rho_x^2 = 0.975$. The decrease in ρ_x^2 increases the influence of the speculation motive as the beliefs of the second agent are "more wrong" on average and hence will shift wealth to the first investor. Furthermore, the second agent is (even) less afraid of long-run risks and therefore will be willing to sell more insurance. So the influence of the risk-sharing motive also increases, which—on the other hand—shifts wealth to the second investor. Looking at the aggregate effects, we observe that for $x_t = 0$ the change in the weights $\underline{\lambda}_{t+1}^2 - \underline{\lambda}_t^2$ becomes larger on average. (Note the different scale. For a better visualization we show the average change separately in Figure 12 in Appendix G.) This increase reflects the increasing influence of the risk-sharing motive

Figure 9: Changes in the Wealth Distribution—The Epstein–Zin Case ($\rho_x^2 = 0.95$)



The figure shows the change in the optimal weights $\underline{\lambda}_{t+1}^2 - \underline{\lambda}_t^2$ as a function of $\underline{\lambda}_t^2$. From left to right, the change is shown for $x_t = \{-0.008, -0.0013, 0, 0.0013, 0.008\}$ ($\pm$ 4 standard deviations). The blue line depicts the case of a negative shock in x_{t+1} ($x_{t+1} - \rho_x x_t = -0.001$) and the yellow line of a positive shock in x_{t+1} ($x_{t+1} - \rho_x x_t = 0.001$). The red line shows the average over all shocks. Calibration with $\rho_x = \rho_x^1 = 0.985$ and $\rho_x^2 = 0.95$.

compared to the case with $\rho_x^2 = 0.975$. However, for larger $|x_t|$ the influence of the speculation motive quickly increases and only for small $\underline{\lambda}_t^2$—where there is a large share of investors who want to buy insurance against long-run risks—does the risk-sharing motive dominate. This observation explains why the median consumption share in Figure 6 only increases to a certain level and does not converge further toward 1. The magnitude of the change in the weights explains the large drops and recoveries that we observe in Figure 6. For example, for the extreme case with $x_t = -0.008$ a large negative shock implies a drop in the weights of more than 0.3 for $\underline{\lambda}_t^2 = 0.5$. This implies a decrease in the consumption share of the second agent of more than 0.3. But as the influence of the risk-sharing motive increases for small $\underline{\lambda}_t^2$ (see Section 4.1), the second agent recovers rather quickly, as can be observed from Figure 6.

4.4 Robustness of the Results

In this section we run several robustness checks that also provide more intuition about the results. In Figure 10a we show the median consumption share of agent 2 (as in Figure 5) for different degrees of risk aversion $\gamma^h = \{2, 5, 10\}$. In Section 3.2 we have shown that the risk-sharing motive increases with the degree of risk aversion, while the speculation motive decreases. Hence, the smaller is γ, the less wealth should be transferred to the skeptical investors who hold the wrong beliefs. And indeed, this is exactly what we observe in Figure 10a. For $\gamma^h = 10$ (yellow line) the influence of the risk-sharing motive is strong. Hence, agent 2 profits from selling the insurance against long-run risks and rapidly accumulates wealth. For $\gamma^h = 5$ (red line) this effect becomes less severe and agent 2's consumption share increases less quickly. For $\gamma^h = 2$ (blue line) the risk-sharing motive has little influence as risk premia in the economy are small and the speculation motive dominates equilibrium outcomes. As $\rho_x = \rho_x^1$, the speculation motive works in favor of agent 1 (agent 2 bets on states that have a vanishing probability under the true probability measure) and agent 1 dominates the economy in the long run. If agent 2 has the correct beliefs $\rho_x = \rho_x^2$, the speculation motive works in favor of agent 2. We show this case in Figure 10b. The blue line shows the consumption shares for $\rho_x = \rho_x^1$ and the red line for $\rho_x = \rho_x^2$. So in the absence of the risk-sharing motive, the speculation motive determines equilibrium outcomes.

In Figure 10c we depict the robustness of our findings with regard to the level of the persistences of x_t. We show the consumption paths for $\rho_x^2 = 0.6$ and $\rho_x^1 = 0.5$ instead of for 0.975 and 0.985, respectively. Lowering the persistence will—similarly to the decrease in risk aversion—decrease the risk-sharing motive. Risk premia in the economy are only large for ρ_x close to one but collapse for smaller ρ_x (see Bansal and Yaron (2004)). Hence, even those investors who believe in a larger (but significantly smaller than 1) value for ρ_x have only small hedging demands. Consequently, we observe that in this setup the dynamics of the consumption shares strongly depend on the true value of ρ_x as the speculation motive dominates—that is, the agent with the correct beliefs will dominate the economy.

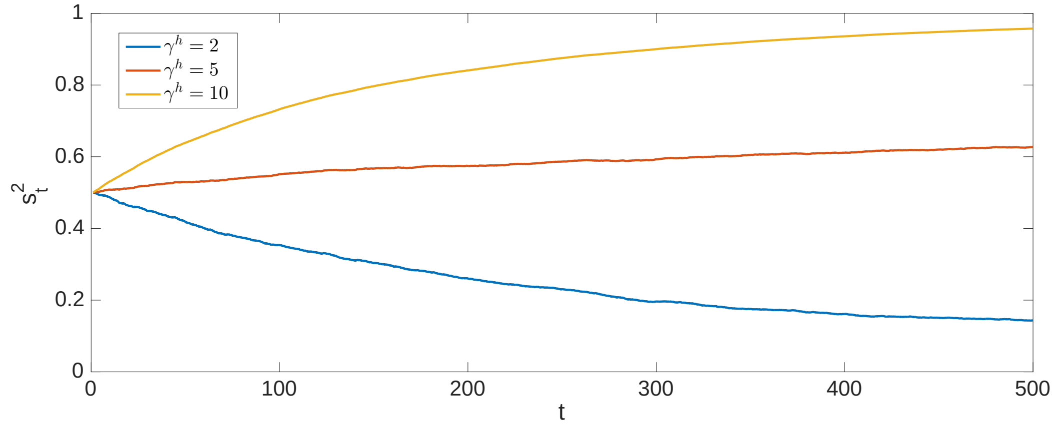
4.5 Correcting for the Difference in Mean Consumption Growth

Different beliefs about the persistence of the long-run risk process imply that—everything else being equal—the agent also has different beliefs about the mean of the gross growth rate of consumption $E\left(\frac{C_{t+1}}{C_t}\right)$ due to Jensen's inequality. In this section we show that our results are not driven by this simple mean effect, but rather by the time-varying risk-sharing motive as demonstrated in the previous section. In fact, when we correct for the belief difference in the mean growth rate of consumption, the consumption share of the skeptical investor increases even faster. For the long-run risk model (15), the mean growth rate of consumption is given by

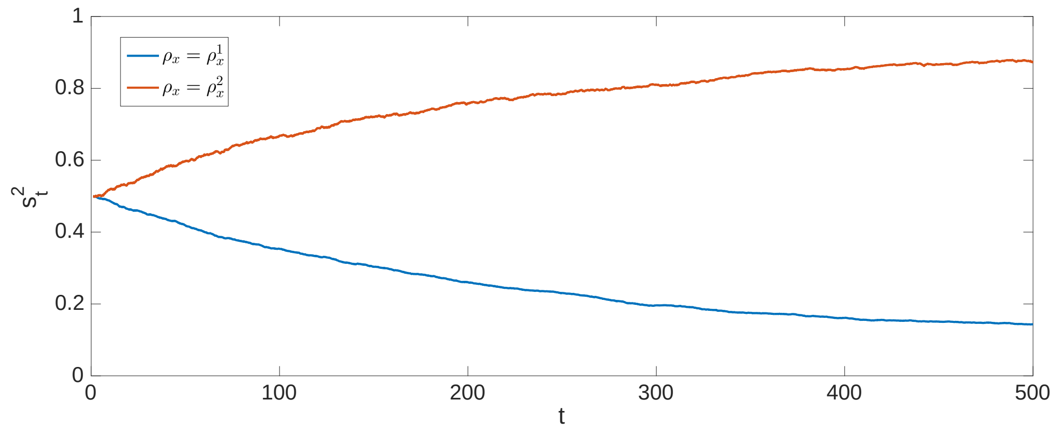
$$E\left(\frac{C_{t+1}}{C_t}\right) = E\left(e^{\Delta c_{t+1}}\right) = e^{\mu_c + 0.5\sigma^2 + 0.5\frac{\phi_x^2\sigma^2}{1-(\rho_x)^2}}. \tag{24}$$

Figure 10: The Risk Sharing and Speculation Motives

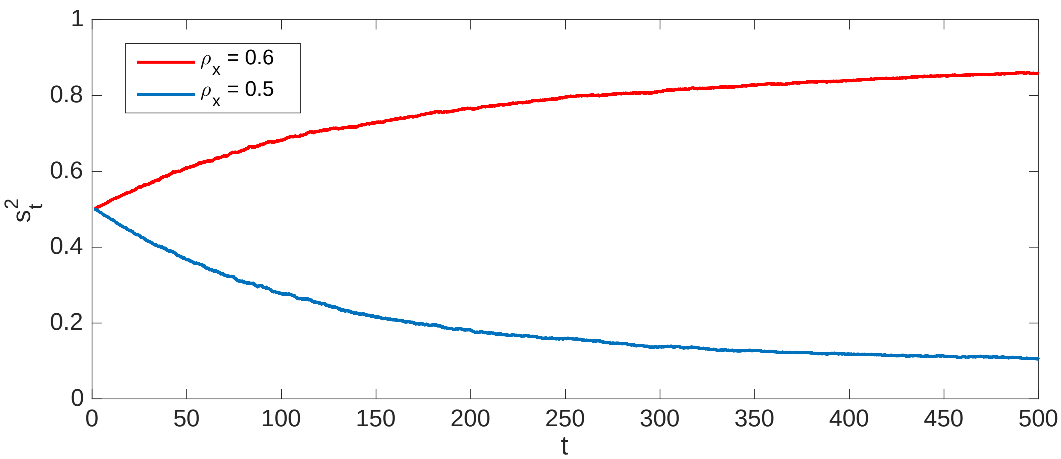
(a) $\rho_x^1 = \rho_x = 0.985$, $\rho_x^2 = 0.975$



(b) $\gamma^h = 2$, $\rho_x^1 = 0.985$, $\rho_x^2 = 0.975$



(c) $\gamma^h = 10$, $\rho_x^2 = 0.6$, $\rho_x^1 = 0.5$



The figure shows the median consumption share of agent 2 for 1,000 samples each consisting of 500 years of simulated data. Panel (a) shows the time series for different degrees of risk aversion $\gamma^h \in \{2, 5, 10\}$. Agent 2 believes that $\rho_x^2 = 0.975$ and agent 1 has the correct beliefs with $\rho_x^1 = \rho_x = 0.985$. Panel (b) shows the time series for $\gamma^h = 2$, $\rho_x^1 = 0.985$, and $\rho_x^2 = 0.975$ for the two cases in which either agent 1 (blue line) or agent 2 (red line) has the correct beliefs. Panel (c) shows the time series for $\gamma^h = 10$, $\rho_x^1 = 0.6$, and $\rho_x^2 = 0.5$ for the two cases where either agent 1 (blue line) or agent 2 (red line) has the correct beliefs.

For $\rho_x^2 < \rho_x^1 = \rho_x$ we have that

$$E^2\left(\frac{C_{t+1}}{C_t}\right) = e^{\mu_c + 0.5\sigma^2 + 0.5\frac{\phi_x^2\sigma^2}{1-(\rho_x^2)^2}} < E\left(\frac{C_{t+1}}{C_t}\right). \tag{25}$$

So the second agent believes in a lower mean growth rate of consumption as she believes in a lower persistence and hence a lower unconditional volatility of the long-run risk process. We correct for this belief difference by setting the subjective belief of the second agent with regard to mean log consumption growth to $\mu_c^2 = \mu_c + 0.5\frac{\phi_x^2\sigma^2}{1-(\rho_x)^2} - 0.5\frac{\phi_x^2\sigma^2}{1-(\rho_x^2)^2}$. Once we correct for this difference, the consumption shares of the skeptical investor increase even faster. For the original specification with an initial allocation of $s_0^2 = 0.01$, the consumption shares of the skeptical investor increased to 0.2824, 0.6376, and 0.9278 after 100, 200, and 500 years, respectively (see Table 1). With the corrected mean we obtain values of 0.2827, 0.6379, and 0.9281. Hence, our results are not driven by the effect of different mean beliefs about consumption growth. This result is also in accordance with Borovička (2018), who shows that underestimation of the mean growth rate lowers the chances of survival while overestimation has the opposite effect due to the positive risk-sharing motive. Consequently, in our model specification, the effect of the mean growth rate should lead the skeptical investor to have lower consumption shares. And indeed, once we correct the mean growth rate estimate of the skeptical investor we obtain a faster increase in the consumption shares of that skeptical investor.

5 Belief Differences in a Modified Calibration

In this final section of the paper, we demonstrate that a two-agent economy in which both agents survive can explain several asset-pricing puzzles. In particular, it can explain the large time variation in expected risk premia recently reported by Martin (2017). Furthermore, it can jointly generate a large equity premium and excess volatility of the price–dividend ratio as well as help explain the predictability of consumption, dividends, and returns. Importantly, our model does not require stochastic volatility, which is essential in the standard log-run risk model to generate time variation in risk premia and a large volatility of the price dividend ratio (see Bansal, Kiku, and Yaron (2012)). But even with stochastic volatility, the implied moments in the model of Bansal, Kiku, and Yaron (2012) are significantly smaller than observed in the data and we show that belief heterogeneity can resolve this discrepancy.

In order to demonstrate the usefulness of the two-agent model for asset-pricing analysis, we deliberately restrict ourselves to the calibration of Bansal and Yaron (2004), with the exception of the value for the long-run risk parameter. Instead of their value of $\rho_x = 0.979$, we choose two different values, one that is slightly larger and one that in slightly smaller than their value. The values of all other parameters in our two-agent economy are identical to those in Bansal and Yaron (2004, Case I) (the calibration without stochastic volatility). We emphasize that we do not attempt to find a perfect calibration of the heterogeneous-agent model but instead only attempt to impose minimal changes to the standard calibration.

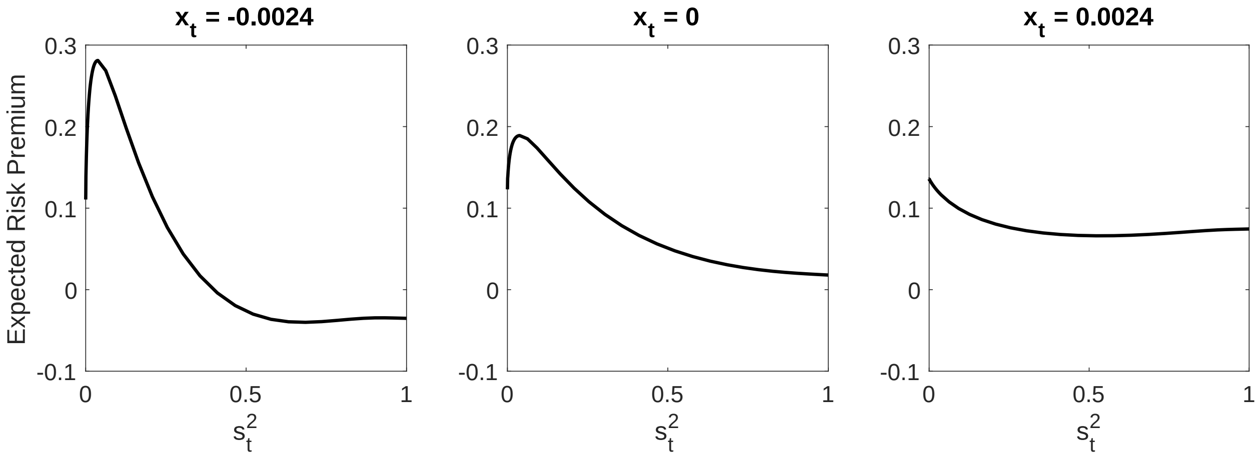
We consider an economy where the first investor strongly believes in long-run risks ($\rho_x^1 = 0.99$) while the second investor is more skeptical about them ($\rho_x^2 = 0.96$). The first investor has the correct beliefs, $\rho_x^1 = \rho_x$. In this setup, the long-term average consumption share is about 0.5, so both investors hold the same average share in the long run. We first show that the model implies significant countercyclical time variation in expected risk premia. Figure 11 shows the annualized expected risk premium as a function of s_t^2. The results are shown for the correct beliefs, $\rho_x = 0.99$. We find that the expected risk premium changes significantly with the consumption share. The expected risk premium is large when investor 1 holds a larger consumption share, while it is significantly smaller when agent 2 dominates.

Recall from Section 4.3 that for Epstein–Zin preferences negative shocks to x_t will increase the consumption share of investor 1 with the larger ρ_x, as these investors buy insurance against such bad shocks (see Figure 8). So when the economy enters a recession—that is, for a series of negative shocks to x_t—the consumption share of investor 1 will increase. This increase, in turn, increases the expected risk premium, as Figure 11 shows. Hence, the changes in the wealth distribution lead to countercyclical variations in the expected equity risk premium. Is this variation economically significant?

In Table 4 we show the mean and standard deviation of the annualized expected risk premium for 1,000 samples each containing 77 years of simulated data initialized at the long-run mean of $s_0^2 = 0.5$.[8] Martin (2017) constructs a lower bound for the expected risk premium in terms of its risk-neutral variance and shows that there is significantly more time variation in the premium than previous studies have shown. We test the ability of the heterogeneous-beliefs model to explain this finding and compare it to the standard long-run risk model. Martin (2017) reports a mean expected risk premium of 5% per year with a standard deviation of 4.6%. The heterogeneous-agent setup implies similar numbers with a mean of 5.42% and a standard deviation of 5.73% (see Table 4). In the standard long-run risk model without stochastic

[8] We do not encounter any survival issues in this setup. Even after 1,000 years of simulated data, all agents survive and the mean consumption share remains almost constant at about $s_t^2 = 0.5$.

Figure 11: Time Variation in Risk Premia



The figure shows the annualized expected risk premium as a function of s_t^2. The results are shown for $\rho_x^2 = 0.96$, $\rho_x^1 = 0.99$, and $\rho_x = 0.99$. Results are shown for ± 1 unconditional standard deviations of x_t around its unconditional mean of $x_t = 0$, as well as for $x_t = 0$.

volatility, expected risk premia are constant as shown by Bansal and Yaron (2004). They propose to add stochastic volatility to the model to generate countercyclical time variation in risk premia as observed in the data. Table 4 shows that while the model of Bansal and Yaron (2004) is able to generate a large mean premium, the standard deviation is considerably smaller compared to the data (1.13%). In the new calibration of Bansal, Kiku, and Yaron (2012), the influence of the stochastic volatility channel is increased in order to match second-order moments. This should also generate more volatility in expected risk premia. However, while the value of Bansal and Yaron (2004) is slightly improved, it is still considerably lower compared to the data (1.61%). Hence, belief heterogeneity in the long-run risk model provides a solution that accounts for the large variation in expected risk premia reported in the data.

Furthermore, Figure 11 reveals that belief heterogeneity can increase the model-implied expected risk premium compared to the representative-agent cases. For small s_t^2, the premium is larger compared to the representative-agent case of $s_t^2 = 0$. Hence, belief heterogeneity endogenously generates priced consumption risk—that is, investors require a premium for the expected changes in the wealth distribution. How does this mechanism work? Consider first the case of $x_t = 0$. In Section 4.3, we have shown that for small s_t^2 the influence of the risk-sharing motive is large (see, for example, Figure 8). So if s_t^2 is small, the consumption share of investor 2 is expected to increase strongly. As an increase in s_t^2 leads to larger price–dividend ratios (investor 2 believes in a smaller ρ_x and therefore requires less risk compensation), the expected increase in s_t^2 implies an increase in the expected risk premium. Consistent with the

Table 4: Expected Risk Premium

	Data	Het. Agents	BY (2004)	BKY (2012)
Mean	5.00	5.42	5.59	6.38
Std. dev.	4.60	5.73	1.13	1.61

The table shows the annualized mean and volatility of the expected risk premium. The first column shows the empirical values reported in Table 1 of Martin (2017). The second column shows the results for the heterogeneous-agent setup with $\rho_x^1 = \rho_x = 0.99$ and $\rho_x^2 = 0.96$ for the correct beliefs. The data is obtained from 1,000 samples—each containing 77 years of simulated data—starting with an initial share of $s_0^2 = 0.5$. Columns three and four show the results for the models of Bansal and Yaron (2004, Case II) and Bansal, Kiku, and Yaron (2012), respectively, using the same size for the simulated data set. All returns are shown in percentages, so a value of 1.5 is a 1.5% annualized figure.

results from Section 4.3, this effect is even stronger for $x_t = -0.0024$, where risk prices are larger and, hence, the consumption share of investor 2 is expected to increase even faster. For $x_t = 0.0024$ the speculation motive is stronger and the effect vanishes. Looking at the time series properties of s_t^2, we observe a standard deviation of 0.18 and a persistence of 0.9926 in the finite data set. Hence, belief heterogeneity endogenously adds persistent consumption risk to the model, which increases the model-implied risk premia.

Can the heterogeneous-agent setup also improve the long-run risk in other dimensions? In Table 5 we show the annualized asset-pricing moments for the two-agent economy as well as the representative-agent economies populated by either of the two investors.

Table 5: Annualized Asset-Pricing Moments

	$E(p_t - d_t)$	$\sigma(p_t - d_t)$	$AC1(p_t - d_t)$	$E\left(R_t^m - R_t^f\right)$	$E\left(R_t^f\right)$	$\sigma(R_t^m)$	$\sigma\left(r_t^f\right)$
$s_t^2 = 0$	2.36	0.28	0.79	12.52	1.72	18.66	1.65
Two-Ag.	3.52	0.38	0.80	5.42	2.29	21.79	1.92
$s_t^2 = 1$	3.97	0.17	0.70	2.07	2.94	15.59	1.66
Data	3.36	0.45	0.87	7.09	0.57	20.28	2.86

The table shows selected moments from 1,000 samples—each containing 77 years of simulated data—starting with an initial share of $s_0^2 = 0.5$. It reports the mean, the standard deviation, and the first-order autocorrelation of the annualized log price–dividend ratio as well as the mean and the standard deviation of both the annualized equity premium and the risk-free return. Agent 1 has the correct beliefs with $\rho_x^1 = \rho_x = 0.99$ and $\rho_x^2 = 0.96$. All returns are shown in percentages, so a value of 1.5 is a 1.5% annualized figure. The estimates from the data are taken from Bansal, Kiku, and Yaron (2012).

We observe that even for the representative-agent case with the high ρ_x of 0.99, the volatility of the price–dividend ratio implied by the long-run risk model is significantly smaller compared to the data. As documented in Section 4.2, the two-agent economy with large belief differences generates significant excess volatility (0.38 compared to 0.28 and 0.17 for the representative-agent economies populated by investor 1 and 2, respectively). This value is close to the volatility of the price dividend ratio in the data (0.45). Also, it significantly improves upon the value for the calibration of Bansal, Kiku, and Yaron (2012) (0.28), that deliberately includes stochastic volatility to increase the volatility of the price–dividend ratio.

Simultaneously, as both agents maintain a significant consumption share in the long-run, the equity premium is still large and significant with a value of 5.42 percent. Furthermore, the two-agent economy generates excess volatility in the market return as well as the volatility of the risk-free rate. The level of the risk-free rate is too high in the two-agent economy. Recall that we deliberately have taken all parameters from Bansal and Yaron (2004) and only varied the belief in the persistence parameter. A lower risk-free rate could be achieved by slightly increasing the subjective discount factor δ.

In Table 6 we report the implications of the two-investor economy for the predictability of returns and cash flows. Beeler and Campbell (2012) argue that in the long-run risk model, the price–dividend ratio has too much predictive power for consumption and dividend growth while the predictability of returns is too low compared to the values observed in the data. Bansal, Kiku, and Yaron (2012) propose a solution by increasing the importance of the stochastic volatility channel and decreasing the importance of the growth-rate channel (x_t). Beeler and Campbell (2012) acknowledge that the statistical rejections of the model are less extreme for this calibration, but it requires extremely persistent movements in the volatility process. We show that the heterogeneous-agent economy helps to explain the predictability puzzle.

Table 6 reports R^2 statistics and regression coefficients from regressing cumulative log excess returns, consumption growth, and dividend growth on the lagged log price–dividend ratio. Statistics are shown for the annualized time series with one-, three-, and five-year horizons. We observe that the predictability of returns is low in the representative-agent economies and that regression coefficients are even positive for the case with a small ρ_x. In the two-agent setup, prices become more volatile and revert to the mean of the stationary distribution. Therefore, we observe more return predictability, which also increases with the horizon. For consumption and dividends, we observe the opposite pattern. First, predictability is too large in the representative-agent economy. In the two-agent economy, the predictability decreases significantly compared to both representative-agent economies due to the endogenous vari-

Table 6: Predictability of Excess Returns, Consumption, and Dividends

	R^2			β		
	1Y	3Y	5Y	1Y	3Y	5Y
$\sum_h^H (r_{m,t+h} - r_{f,t+h}) = \alpha + \beta(p_t - d_t) + \epsilon_{t+H}$						
$s_t^2 = 0$	0.0070	0.0190	0.0277	-0.0134	-0.0496	-0.1116
Two-Ag.	0.0129	0.0350	0.0540	-0.0563	-0.1676	-0.2639
$s_t^2 = 1$	0.0438	0.0757	0.0700	0.1899	0.4651	0.6031
Data	0.04	0.19	0.31	-0.09	-0.27	-0.43
$\sum_h^H (\Delta c_{t+h}) = \alpha + \beta(p_t - d_t) + \epsilon_{t+H}$						
$s_t^2 = 0$	0.5121	0.5738	0.5075	0.0829	0.2187	0.3124
Two-Ag.	0.3043	0.3346	0.2825	0.0495	0.1283	0.1805
$s_t^2 = 1$	0.4535	0.5074	0.4494	0.1341	0.3525	0.5012
Data	0.060	0.01	0.000	0.01	0.01	0.00
$\sum_h^H (\Delta d_{t+h}) = \alpha + \beta(p_t - d_t) + \epsilon_{t+H}$						
$s_t^2 = 0$	0.4881	0.4903	0.4393	0.3044	0.7074	0.9705
Two-Ag.	0.3088	0.2868	0.2486	0.1882	0.4233	0.5910
$s_t^2 = 1$	0.5658	0.4909	0.4161	0.5615	1.2088	1.6225
Data	0.09	0.06	0.04	0.07	0.11	0.09

The table reports R^2 statistics and regression coefficients from regressing cumulative log excess returns, consumption growth, and dividend growth on the lagged log price–dividend ratio from 1,000 samples each containing 77 years of simulated data starting with an initial share of $s_0^2 = 0.5$. Statistics are shown for the annualized time series with one-, three-, and five-year horizons. The estimates from the data are taken from Bansal, Kiku, and Yaron (2012).

ation in the consumption shares. Overall the predictability of cash flows is still too large compared to the data but the results show that investor heterogeneity can help resolve the predictability deficiencies of the long-run risk model.

6 Conclusion

This paper presents a discrete-time consumption-based asset-pricing model with heterogeneous agents. In contrast to the standard representative-agent model, equilibrium allocations are no longer a function of the exogenous state alone. As a result, the standard solution methods from consumption-based asset pricing are not applicable. For the purpose of solving the model, we use a recursive reformulation of the first-order conditions, which involves new endogenous state variables. These state variables have a clear interpretation in terms of time-varying weights in a social planner's problem.

Using the recursive formulation, we have performed a detailed study of heterogeneity in agents' beliefs for the long-run risk model of Bansal and Yaron (2004). In particular, we consider agents with different beliefs about the level of persistence of long-run risk. For the standard calibration of the long-run risk model, we find that for modest levels of disagreement agents who believe in a lower level of persistence come to dominate the economy rather quickly relative to agents who believe in a higher level of persistence. This result holds even if the agent who believes in the higher level of persistence holds the correct belief. As a consequence, the model's equity premium falls much below the level observed in the data. Simultaneously, belief heterogeneity can generate significant excess volatility due to endogenous movements in the wealth distribution.

These observations motivate a modestly altered calibration of the heterogeneous-agent model, where both agents survive. In this calibration, one agent believes in a slightly smaller amount of persistence relative to the original paper, while one agent believes in a slightly higher amount (and is correct). This model not only generates a large and significant equity premium, it also addresses many of the empirical deficiencies of the representative-agent model. Notably, it adds significant countercyclical time variation in expected risk premia to the model, consistent with the data reported in Martin (2017). Furthermore, shifts in the wealth distribution increase the volatility of the price–dividend ratio to levels close to the data as the impact of the different agents on asset prices varies over time. The variation in the wealth distribution also helps to address the predictability puzzle pointed out by Beeler and Campbell (2012). The endogenous variation in asset prices increases the predictability of returns while simultaneously decreasing the predictability of consumption and dividend growth.

These findings suggest that belief heterogeneity can contribute to a solution for several asset-pricing puzzles.

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Appendix

A Additional Literature

The study of agent belief heterogeneity begins with the market selection hypothesis of Alchian (1950) and Friedman (1953). By analogy with natural selection, the market selection hypothesis states that agents with systematically wrong beliefs will eventually be driven out of the market. The influence of agent heterogeneity on market outcomes under the standard assumption of discounted expected utility is well understood, and consistent with market selection. Sandroni (2000) and Blume and Easley (2006) find strong support for this hypothesis under the assumption of time-separable preferences in an economy without growth. Yan (2008) and Cvitanić, Jouini, Malamud, and Napp (2012) analyze the survival of investors in a continuous-time framework where there are not only differences in beliefs but also potentially differences in the utility parameters of the investors. They show that it is always the investor with the lowest survival index[9] who survives in the long run. However, the "long run" can indeed be very long and, therefore, irrational investors can have significant effects on asset prices even under the assumption of discounted expected utility. David (2008) considers a similar model setup, in which both agents have distorted estimates of the mean growth rate of the economy, showing that—as agents with lower risk aversion undertake more aggressive trading strategies—the equity premium increases the lower the risk aversion is. Chen, Joslin, and Tran (2012) analyze how differences in beliefs about the probability of disasters affect asset prices. They show that even if there is only a small fraction of investors who are optimistic about disasters, this fraction sells insurance for the disaster states and so eliminates most of the risk premium associated with disaster risk. Bhamra and Uppal (2014) consider the case of habit utility.

For recursive utility, this qualitative behavior changes fundamentally. However, there has been less research in this area as solving such models is anything but trivial. Lucas and Stokey (1984) observe in the deterministic case that the problem of finding all Pareto-optimal allocations can be made recursive if we allow the weights in the social welfare function to be time-varying. This approach is extended by Kan (1995) to the stochastic case with finite state

[9] Yan (2008) shows that the survival index increases with belief distortion, risk aversion, and the subjective time discount rate of the investor.

spaces. Anderson (2005) develops an extensive theory for the special case of risk-sensitive preferences, no growth, and finite state spaces, and finds first-order conditions similar to those we use below. In particular, he shows how to characterize the equilibrium by a single value function instead of one value function for each agent. Collin-Dufresne, Johannes, and Lochstoer (2015) derive similar first-order conditions to ours for recursive utility by equating marginal utilities, but use a different procedure to solve for their allocations. Duffie, Geoffard, and Skiadas (1994) formulate the problem in continuous time, while Dumas, Uppal, and Wang (2000) reformulate it in terms of variational utility. Borovička (2018) uses this formulation to explore the question of the survival of agents with recursive utility in continuous time, and shows that agents with fundamentally wrong beliefs can survive or even dominate. So, inferences about market selection and equilibrium outcomes fundamentally differ under the assumption of general recursive utility compared to the special case of standard time-separable preferences. While Borovička (2018) concentrates on the special case of i.i.d. consumption growth, Branger, Dumitrescu, Ivanova, and Schlag (2011) generalize the results to a model with long-run risks as a state variable.

However, most papers with heterogeneous investors and recursive preferences consider only an i.i.d. process for consumption growth. For example, Gârleanu and Panageas (2015) analyze the influence of heterogeneity in the preference parameters on asset prices in a two-agent OLG economy. Roche (2011) considers a model in which the heterogeneous investors can only invest in a stock but there is no risk-free bond. Hence, as there is no savings trade-off, the impact of recursive preferences on equilibrium outcomes will be quite different.

Exceptions that relax the i.i.d. assumptions include the papers by Branger, Konermann, and Schlag (2015) and Collin-Dufresne, Johannes, and Lochstoer (2016a). Both papers reexamine the influence of belief differences regarding disaster risk with Epstein–Zin instead of CRRA preferences as in Chen, Joslin, and Tran (2012). Branger, Konermann, and Schlag (2015) provide evidence that the influence of investors with more optimistic beliefs about disasters is less profound when the disaster occurs to the growth rate of consumption and show that the risk sharing mechanism persists even when markets are incomplete. Collin-Dufresne, Johannes, and Lochstoer (2016a) make a similar claim but for a different reason. They show that if the investors can learn about the probability of disasters and if they have recursive preferences, the impact of the optimistic investor on asset prices decreases. Optimists are uncertain about the probability of disasters and hence will provide less insurance to pessimistic investors. Collin-Dufresne, Johannes, and Lochstoer (2016a) use an OLG model with two generations to model optimists and pessimists. Hence—in contrast to the results in the present study—the consumption shares of the investors are fixed and the increasing influence of optimistic agents over time is not captured.

A different strand of literature, which does not rely on the i.i.d. assumption, is comprised of papers on international asset pricing. This area (advanced by Riccardo Colacito and Mariano Croce in particular), considers models with Epstein–Zin preferences, two investors, and also two goods for which investors have different preferences (home bias). For example, Colacito and Croce (2013) argue that a model with highly correlated international long-run components in output can explain both the low correlation between consumption differentials and the tendency of high interest rate currencies to appreciate. The authors furthermore show that an increase in capital mobility can explain the structural break in the data for the pre- and post-1970 period. Colacito, Croce, and Liu (2018b) provide the theoretical foundation for the multiple good economy by providing results on equilibrium existence and agents' survival; they also compare computational methods to solve the model. Furthermore, Colacito, Croce, Ho, and Howard (2018a) use a model with Epstein–Zin preference and short- and long-run productivity shocks to study the effects of these shocks on international investment flows.

In a different direction, Epstein, Farhi, and Strzalecki (2014) argue that an Epstein–Zin investor dislikes long-run risk to the extent that he or she would pay a substantial premium to get rid of it. In a model with two agents, the agent who believes that risk is longer term than the other is willing to pay an insurance premium to the other agent to hedge against long-run risk.

B Proofs and Details

In this appendix, we provide proofs for the theoretical results presented in Section 2. Along the way, we derive a system of first-order conditions for Epstein–Zin preferences. This system constitutes the foundation for our numerical solution method (see Appendix C).

B.1 Proofs for Section 2.1

Proof of Theorem 1. Let $\boldsymbol{\lambda} = \{\bar{\lambda}^1, \ldots, \bar{\lambda}^H\}$ be a set of Negishi weights and let $\{\boldsymbol{C}\}_0 = \{\{C^1\}_0, \ldots, \{C^H\}_0\}$ denote a vector of agents' consumption processes. The optimal decision $\{\boldsymbol{C}\}_0^*$ of the social planner in the initial period assigns consumption streams to all individual agents for all periods and possible states. Obviously, the optimal decisions must satisfy the market-clearing condition (1) in all periods and states. For ease of notation we again abbreviate the state dependence; we use C_t^h for $C^h(y^t)$ and $U_{\{t\}}^h$ for $U^h\left(\{C^h\}_t\right)$.

To derive the first-order conditions, we borrow a technique from the calculus of variations. For any function f_t, we can vary the consumption of two agents by

$$\begin{aligned} C_t^h \to & \quad C_t^h + \epsilon f_t \\ C_t^l \to & \quad C_t^l - \epsilon f_t. \end{aligned} \tag{26}$$

It is sufficient to consider the variation with $l = 1$ and $h \in \mathbb{H}^-$. For an optimal allocation it must be true that

$$\left.\frac{\mathrm{d}SP(\{\boldsymbol{C}\}_0; \boldsymbol{\lambda})}{\mathrm{d}\epsilon}\right|_{\epsilon=0} = 0. \tag{27}$$

This gives us

$$\bar{\lambda}^h \hat{U}_{0,t}^h = \bar{\lambda}^1 \hat{U}_{0,t}^1, \qquad h \in \mathbb{H}^-, \tag{28}$$

where $\hat{U}_{t,t+k}^h$ is defined as

$$\hat{U}_{t,t+k}^h = \left.\frac{\mathrm{d}U^h(C_t^h, \ldots, C_{t+k}^h + \epsilon f_{t+k}, \ldots)}{\mathrm{d}\epsilon}\right|_{\epsilon=0}. \tag{29}$$

Using the expression given in Equation (2), the derivative $\hat{U}_{t,t+k}^h$ satisfies a recursive equation with the initial condition

$$\hat{U}_{t,t}^h = \left.\frac{\mathrm{d}U^h(C_t^h + \epsilon f_t, \ldots)}{\mathrm{d}\epsilon}\right|_{\epsilon=0} = F_1^h\left(C_t^h, R_t[U_{\{t+1\}}^h]\right) \cdot f_t, \tag{30}$$

where $F_k^h\left(C_t^h, R_t^h[U_{\{t+1\}}^h]\right)$ denotes the derivative of $F^h\left(C_t^h, R_t^h[U_{\{t+1\}}^h]\right)$ with respect to its kth argument. The recursive step is given by

$$\begin{aligned} \hat{U}_{t,t+k}^h &= \left.\frac{\mathrm{d}F^h\left(C_t^h, R_t^h\left[U^h(C_{t+1}^h, \ldots C_{t+k}^h + \epsilon f_{t+k}, \ldots)\right]\right)}{\mathrm{d}\epsilon}\right|_{\epsilon=0} \\ &= F_2^h\left(C_t^h, R_t^h[U_{\{t+1\}}^h]\right) \cdot \left.\frac{\mathrm{d}R_t^h\left[U^h(\cdot)\right]}{\mathrm{d}\epsilon}\right|_{\epsilon=0} \\ &= F_2^h\left(C_t^h, R_t^h[U_{\{t+1\}}^h]\right) \cdot \frac{\mathrm{d}G_h^{-1}\left(E_t^h G_h\left[U^h(\cdot)\right]\right)}{\mathrm{d}E_t^h G_h[U^h(\cdot)]} \cdot \left.\frac{\mathrm{d}E_t^h G_h[U^h(\cdot)]}{\mathrm{d}\epsilon}\right|_{\epsilon=0} \\ &= F_2^h\left(C_t^h, R_t^h[U_{\{t+1\}}^h]\right) \cdot \frac{1}{G_h'(G_h^{-1}(E_t^h G_h[U_{\{t+1\}}^h]))} \cdot E_t^h\left(G_h'(U_{\{t+1\}}^h) \cdot \hat{U}_{t+1,t+k}^h\right) \\ &= F_2^h\left(C_t^h, R_t^h[U_{\{t+1\}}^h]\right) \cdot \frac{E_t^h\left(G_h'(U_{\{t+1\}}^h) \cdot \hat{U}_{t+1,t+k}^h\right)}{G_h'(R_t^h[U_{\{t+1\}}^h])}, \end{aligned} \tag{31}$$

where we use $\frac{\partial G^{-1}(x)}{\partial x} = \frac{1}{G'(G^{-1}(x))}$ and abbreviate $U^h(C_{t+1}^h, \ldots C_{t+k}^h + \epsilon f_{t+k}, \ldots)$ by $U^h(\cdot)$. We can recast this recursion into a useful form. For this purpose, we define a second recursion $U^h_{t,t+k}$ by

$$U^h_{t,t} = F^h_1\left(C^h_t, R^h_t[U^h_{\{t+1\}}]\right) \tag{32}$$

and

$$U^h_{t,t+k} = \Pi^h_{t+1} \cdot U^h_{t+1,t+k}, \tag{33}$$

where

$$\Pi^h_{t+1} = F^h_2\left(C^h_t, R^h_t[U^h_{\{t+1\}}]\right) \cdot \frac{G'_h(U^h_{\{t+1\}})}{G'_h(R^h_t[U^h_{\{t+1\}}])} \frac{\mathrm{dP}^h_{t,t+1}}{\mathrm{dP}_{t,t+1}}. \tag{34}$$

A simple induction shows that

$$\hat{U}^h_{t,t+k} = E_t(U^h_{t,t+k} f_t). \tag{35}$$

Plugging (35) into the optimality condition (28) we obtain

$$E_0\left((\bar{\lambda}^h U^h_{0,t} - \bar{\lambda}^1 U^1_{0,t}) f_t\right) = 0, \qquad h \in \mathbb{H}^-. \tag{36}$$

Under a broad range of regularity conditions, this condition implies that

$$\bar{\lambda}^h U^h_{0,t} = \bar{\lambda}^1 U^1_{0,t}, \qquad h \in \mathbb{H}^-. \tag{37}$$

For example, if $\bar{\lambda}^h U^h_{0,t} - \bar{\lambda}^1 U^1_{0,t}$ has finite variance, then this holds for the Riesz Representation Theorem for L^2 random variables. We can then split Expression (37) into two parts. First define $\lambda^h_0 \equiv \bar{\lambda}^h$ to obtain

$$\frac{\lambda^h_0}{\lambda^1_0} = \frac{U^1_{0,t}}{U^h_{0,t}} = \frac{\Pi^1_0}{\Pi^h_0}\frac{U^1_{1,t}}{U^h_{1,t}} = \frac{\Pi^1_0}{\Pi^h_0}\frac{\lambda^h_1}{\lambda^1_1}, \qquad h \in \mathbb{H}^-,$$

where λ^h_1 denotes the Negishi weight in the social planner's optimal solution in $t = 1$. Generalizing this equation for any period t, we obtain the following dynamics for the optimal weight[10] λ^h_{t+1}:

$$\frac{\lambda^h_{t+1}}{\lambda^1_{t+1}} = \frac{\Pi^h_{t+1}}{\Pi^1_{t+1}}\frac{\lambda^h_t}{\lambda^1_t}, \qquad h \in \mathbb{H}^-. \tag{38}$$

Inserting the initial condition (32) into (37) for $t = 0$ and generalizing it for any social planner's

[10] Note that we can either solve the model in terms of the ratio $\frac{\lambda^h_t}{\lambda^1_t}$ (this is equal to setting $\lambda^1_t = 1$ for all t as done in Judd, Kubler, and Schmedders (2003)) or we can normalize the weights so that they remain bounded in $(0, 1)$. Our solution method uses the latter approach as it obtains better numerical properties.

optimal solution at time t yields

$$\lambda_t^h F_1^h \left(C_t^h, R_t^h[U_{\{t+1\}}^h]\right) = \lambda_t^1 F_1^1 \left(C_t^1, R_t^1[U_{\{t+1\}}^1]\right), \qquad h \in \mathbb{H}^-. \tag{39}$$

Equation (39) states the optimality conditions for the individual consumption choices at any time t. This completes the proof of Theorem 1. □

Note that for time-separable utility, $F_1^h \left(C_t^h, R_t^h[U_{\{t+1\}}^h]\right)$ is simply the marginal utility of agent h at time t, and so we obtain the same optimality condition as, for example, Judd, Kubler, and Schmedders (2003) (see Equation (7) on page 2209). In this special case the Negishi weights can be pinned down in the initial period and thereafter remain constant. For general recursive preferences this is not true. The optimal weights vary over time following the law of motion described by Equation (38).

We can use Equations (38) and (39) together with the market-clearing condition (1) to compute the social planner's optimal solution. We therefore define $\boldsymbol{\lambda}_t^- = \{\lambda_t^2, \lambda_t^3, \ldots, \lambda_t^H\}$ and let V^h denote the value function of agent $h \in \mathbb{H}$. We are looking for model solutions of the form $V^h(\boldsymbol{\lambda}_t^-, y^t)$. So, the model solution depends on both the exogenous state y^t and the time-varying Negishi weights $\boldsymbol{\lambda}_t^-$. An optimal allocation is then characterized by the following four equations:

- the market-clearing condition (1)

$$\sum_{h=1}^{H} C^h(\boldsymbol{\lambda}_t^-, y^t) = C(y^t); \tag{40}$$

- the value functions (2) of the individual agents

$$V^h(\boldsymbol{\lambda}_t^-, y^t) = F^h \left(C^h(\boldsymbol{\lambda}_t^-, y^t), R_t^h[V^h(\boldsymbol{\lambda}_{t+1}^-, y^{t+1})]\right), \qquad h \in \mathbb{H}; \tag{41}$$

- the optimality conditions (39) for the individual consumption decisions for $h \in \mathbb{H}^-$

$$\lambda_t^h F_1^h \left(C^h(\boldsymbol{\lambda}_t^-, y^t), R_t^h[V^h(\boldsymbol{\lambda}_{t+1}^-, y^{t+1})]\right) = \lambda_t^1 F_1^1 \left(C^1(\boldsymbol{\lambda}_t^-, y^t), R_t^1[V^1(\boldsymbol{\lambda}_{t+1}^-, y^{t+1})]\right); \tag{42}$$

- the equations (38) for the dynamics of $\boldsymbol{\lambda}_t^-$

$$\frac{\lambda_{t+1}^h}{\lambda_{t+1}^1} = \frac{\Pi_{t+1}^h}{\Pi_{t+1}^1} \frac{\lambda_t^h}{\lambda_t^1}, \qquad h \in \mathbb{H}^-, \tag{43}$$

with

$$\Pi^h_{t+1} = F^h_2\left(C^h(\boldsymbol{\lambda}^-_t, y^t), R^h_t[V^h(\boldsymbol{\lambda}^-_{t+1}, y^{t+1})]\right) \cdot \frac{G'_h(V^h(\boldsymbol{\lambda}^-_{t+1}, y^{t+1}))}{G'_h(R^h_t[V^h(\boldsymbol{\lambda}^-_{t+1}, y^{t+1})])} \frac{\mathrm{dP}^h_{t,t+1}}{\mathrm{dP}_{t,t+1}}. \tag{44}$$

This concludes the general description of the equilibrium obtained from the social planner's optimization problem.

To prove Theorem 2, we first derive a variant of Lemma 1 in Blume and Easley (2006).

Lemma 1. *Let X^i_t, $i = 1, 2, \ldots, H$, be a family of positive random variables for each $t = 0, 1, 2, \ldots$, such that $A \leq \sum_i X^i_t \leq B$ with $B \in \mathbb{R}_{++}$. Let $f^i : \mathbb{R}_{++} \to \mathbb{R}_{++}$, $i = 1, 2, \ldots, H$, be a family of decreasing functions such that $f^i(x) \to \infty$ as $x \to 0$. If $f^i(X^i_t)/f^j(X^j_t) \to \infty$, then $X^i_t \to 0$ for $t \to \infty$. If $X^i_t \to 0$, then for at least one j, $\limsup_t f^i(X^i_t)/f^j(X^j_t) = \infty$.*

Proof. Since X^i_t is positive, $X^i_t \leq B$ for all i, t. By assumption, $0 < f^j(B) \leq f^j(X^j_t)$. Thus, $f^i(X^i_t)/f^j(X^j_t) \to \infty$ if and only if $f^i(X^i_t) \to \infty$, which happens when $X^i_t \to 0$ as $t \to \infty$.

Conversely, assume $X^i_t \to 0$. Every period, for at least one j, $X^j_t \geq A/H$ (otherwise $\sum_{i=1}^H X^i_t < A$). Since there are only finitely many random variables, for at least one j we have $X^j_t \geq A/H$ infinitely often. Then, by assumption, $f^j(X^j_t) \leq f^j(A/H)$ infinitely often, and so $\limsup f^i(X^i_t)/f^j(X^j_t) = \infty$. ☐

Proof of Theorem 2. By the first-order condition (5), $\lambda^j_t/\lambda^i_t = F^i_1(C^i_t, R^i_t)/F^j_1(C^j_t, R^j_t)$. Since F^h is additively separable, F^h_1 is a function of consumption alone. Let $f^i = F^i_1, f^j = F^j_1$, $A = \underline{C}$, and $B = \overline{C}$, and apply Lemma 1. ☐

B.2 Proofs for Section 2.2

In this section we provide the specific expressions for V^h, F^h_1, F^h_2, and Π^h when the heterogeneous investors have recursive preferences as in Epstein and Zin (1989) and Weil (1989). The value function for Epstein–Zin (EZ) preferences is given by[11]

$$V^h_t = \left[(1-\delta^h)(C^h_t)^{\rho^h} + \delta^h R^h_t\left(V^h_{t+1}\right)^{\rho^h}\right]^{\frac{1}{\rho^h}} \tag{45}$$

with

$$\begin{aligned} R^h_t\left(V^h_{t+1}\right) &= G^{-1}_h\left(E^h_t\left[G_h(V^h_{t+1})\right]\right) \\ G_h(V^h_{t+1}) &= \left(V^h_{t+1}\right)^{\alpha^h}. \end{aligned}$$

[11] For ease of notation, we again abbreviate the dependence on the exogenous state y_t and the endogenous state $\underline{\boldsymbol{\lambda}}^-_t$. Hence we write V^h_t for $V^h(\boldsymbol{\lambda}^-_t, y_t)$ or C^h_t for $C^h(\boldsymbol{\lambda}^-_t, y_t)$.

Recall that the parameter δ^h is the discount factor, $\rho^h = 1 - \frac{1}{\psi^h}$ determines the EIS, ψ^h, and $\alpha^h = 1 - \gamma^h$ determines the relative risk aversion γ^h of agent h. The derivatives of $F^h\left(C_t^h, R_t^h[V_{t+1}^h]\right) = V_t^h$ with respect to its first and second argument are then given by

$$F_{1,t}^h = (1-\delta^h)(C_t^h)^{\rho^h - 1}(V_t^h)^{1-\rho^h} \tag{46}$$

and

$$F_{2,t}^h = \delta^h R_t^h \left(V_{t+1}^h\right)^{\rho^h - 1}(V_t^h)^{1-\rho^h}. \tag{47}$$

In this paper we focus on growth economies. Therefore, we introduce the following normalization to obtain a stationary formulation of the model. We define the consumption share of agent h by $s_t^h = \frac{C_t^h}{C_t}$ and the normalized value functions, $v_t^h = \frac{V_t^h}{C_t}$. Recall that $\Delta c_{t+1} = c_{t+1} - c_t$ with $c_t = \log(C_t)$. The value function (45) is then given by

$$v_t^h = \left[(1-\delta^h)(s_t^h)^{\rho^h} + \delta^h R_t^h\left(v_{t+1}^h e^{\Delta c_{t+1}}\right)^{\rho^h}\right]^{\frac{1}{\rho^h}}. \tag{48}$$

By inserting (46) into (42) we obtain the optimality condition for the individual consumption decisions

$$\lambda_t^h F_1^h\left(C^h(\boldsymbol{\lambda}_t^-, y^t), R_t^h[V^h(\boldsymbol{\lambda}_{t+1}^-, y^{t+1})]\right) = \lambda_t^1 F_1^1\left(C^1(\boldsymbol{\lambda}_t^-, y^t), R_t^1[V^1(\boldsymbol{\lambda}_{t+1}^-, y^{t+1})]\right),$$

which simplifies to

$$\lambda_t^h(1-\delta^h)(C_t^h)^{\rho^h-1}(V_t^h)^{1-\rho^h} = \lambda_t^1(1-\delta^1)(C_t^1)^{\rho^1-1}(V_t^1)^{1-\rho^1}. \tag{49}$$

Recall the definition of the normalized Negishi weights, $\underline{\lambda}_t^h = \frac{\lambda_t^h}{(v_t^h)^{\rho^h-1}}$. From Equation (49) we obtain

$$\underline{\lambda}_t^h(1-\delta^h)(s_t^h)^{\rho^h-1} = \underline{\lambda}_t^1(1-\delta^1)(s_t^1)^{\rho^1-1}. \tag{50}$$

This equation is the optimality condition for the individual consumption decisions we employ for solving for the model with Epstein–Zin preferences. Inserting the de-trended weight $\underline{\lambda}_t^h$ into the dynamics for the weights (43), we obtain

$$\frac{\lambda_{t+1}^h}{\lambda_{t+1}^1} = \frac{\underline{\lambda}_{t+1}^h(v_{t+1}^h)^{\rho^h-1}}{\underline{\lambda}_{t+1}^1(v_{t+1}^1)^{\rho^1-1}} = \frac{\underline{\lambda}_t^h(v_t^h)^{\rho^h-1}}{\underline{\lambda}_t^1(v_t^1)^{\rho^1-1}}\frac{\Pi_{t+1}^h}{\Pi_{t+1}^1}, \qquad h \in \mathbb{H}^-. \tag{51}$$

Plugging the expressions for Epstein–Zin preferences (45)–(47) into Equation (44), we obtain the following expression for Π^h_{t+1}:

$$\begin{aligned}\Pi^h_{t+1} &= \delta^h R^h_t \left(V^h_{t+1}\right)^{\rho^h-1} (V^h_t)^{1-\rho^h} \frac{\left(V^h_{t+1}\right)^{\alpha^h-1}}{R^h_t \left(V^h_{t+1}\right)^{\alpha^h-1}} \frac{\mathrm{dP}^h_{t,t+1}}{\mathrm{dP}_{t,t+1}} \\ &= \delta^h (V^h_t)^{1-\rho^h} \frac{\left(V^h_{t+1}\right)^{\alpha^h-1}}{R^h_t \left(V^h_{t+1}\right)^{\alpha^h-\rho^h}} \frac{\mathrm{dP}^h_{t,t+1}}{\mathrm{dP}_{t,t+1}}. \end{aligned} \tag{52}$$

Using the normalized value function $v^h_t = \frac{V^h_t}{C_t}$, we have

$$\Pi^h_{t+1} = \delta^h (v^h_t)^{1-\rho^h} \frac{\left(v^h_{t+1} e^{\Delta c_{t+1}}\right)^{\alpha^h-1}}{R^h_t \left(v^h_{t+1} e^{\Delta c_{t+1}}\right)^{\alpha^h-\rho^h}} \frac{\mathrm{dP}^h_{t,t+1}}{\mathrm{dP}_{t,t+1}}. \tag{53}$$

Equation (51) can then be written as

$$\frac{\underline{\lambda}^h_{t+1}}{\underline{\lambda}^1_{t+1}} = \frac{\underline{\lambda}^h_t}{\underline{\lambda}^1_t} \frac{\underline{\Pi}^h_{t+1}}{\underline{\Pi}^1_{t+1}}, \qquad h \in \mathbb{H}^-, \tag{54}$$

where

$$\underline{\Pi}^h_{t+1} = \underbrace{\delta^h e^{\rho^h \Delta c_{t+1}} \frac{\mathrm{dP}^h_{t,t+1}}{\mathrm{dP}_{t,t+1}}}_{\text{CRRA-Term}} \underbrace{\frac{\left(v^h_{t+1} e^{\Delta c_{t+1}}\right)^{\alpha^h-\rho^h}}{R^h_t \left(v^h_{t+1} e^{\Delta c_{t+1}}\right)^{\alpha^h-\rho^h}}}_{\text{New EZ-Term}}. \tag{55}$$

For $\alpha^h = \rho^h$, we obtain the standard term for CRRA preferences; the dynamics of $\underline{\lambda}^h_{t+1}$ only depend on the subjective discount factor, the EIS, and the subjective beliefs of the investors. For Epstein–Zin preferences, we obtain an extra term that reflects the time trade-off. Using the normalization $\sum_{h=1}^H \underline{\lambda}^h_t = 1$, the dynamics for $\underline{\lambda}^h_{t+1}$ are then given by

$$\underline{\lambda}^h_{t+1} = \frac{\underline{\lambda}^h_t \underline{\Pi}^h_{t+1}}{\sum_{h=1}^H \underline{\lambda}^h_t \underline{\Pi}^h_{t+1}}. \tag{56}$$

Hence, for Epstein–Zin preferences we obtain the following system for the first-order conditions (40)-(44):

The market-clearing condition:

$$\sum_{h=1}^H s^h_t = 1. \tag{MC}$$

The optimality condition for the individual consumption decisions:

$$\underline{\lambda}_t^h(1-\delta^h)(s_t^h)^{\rho^h-1} = \underline{\lambda}_t^1(1-\delta^1)(s_t^1)^{\rho^1-1}, \quad h \in \mathbb{H}^-, \tag{CD}$$

with $\sum_{h=1}^H \underline{\lambda}_t^h = 1$.

The value functions of the individual agents:

$$v_t^h = \left[(1-\delta^h)(s_t^h)^{\rho^h} + \delta^h R_t^h \left(v_{t+1}^h e^{\Delta c_{t+1}}\right)^{\rho^h}\right]^{\frac{1}{\rho^h}}, \quad h \in \mathbb{H}. \tag{VF}$$

The equation for the dynamics of $\underline{\lambda}_t^h$:

$$\begin{aligned} \underline{\lambda}_{t+1}^h &= \frac{\underline{\lambda}_t^h \underline{\Pi}_{t+1}^h}{\sum_{h=1}^H \underline{\lambda}_t^h \underline{\Pi}_{t+1}^h} \\ \underline{\Pi}_{t+1}^h &= \delta^h e^{\rho^h \Delta c_{t+1}} \frac{\mathrm{dP}_{t,t+1}^h}{\mathrm{dP}_{t,t+1}} \frac{\left(v_{t+1}^h e^{\Delta c_{t+1}}\right)^{\alpha^h-\rho^h}}{R_t^h \left(v_{t+1}^h e^{\Delta c_{t+1}}\right)^{\alpha^h-\rho^h}}, \quad h \in \mathbb{H}^-. \end{aligned} \tag{Dλ}$$

Note that the conditions (MC, CD, VF, Dλ) are just the equilibrium conditions (11)–(14) stated in Section 2.2. We observe that Equation (CD) and hence the individual consumption decisions s_t^h only depend on time t information and that there is no intertemporal dependence. This feature allows us to first solve for s_t^h given the current state of the economy, and in a second step to solve for the dynamics of the Negishi weights. Hence, we can separate solving the optimality conditions (11)–(14) into two steps in order to reduce the computational complexity. In Appendix C we describe this approach in detail.

Using condition (CD) we can prove Theorem 3. Recall that $\rho^h = 1 - \frac{1}{\psi^h} < 1$ for all possible values of an agent's EIS, $\psi^h > 0$.

Proof of Theorem 3. Condition (CD) implies

$$\frac{\underline{\lambda}_t^j}{\underline{\lambda}_t^i} = \frac{(1-\delta^i)(s_t^i)^{\rho^i-1}}{(1-\delta^j)(s_t^j)^{\rho^j-1}}.$$

Now let $f^i(s) = s^{\rho^i-1}$, $f^j(s) = s^{\rho^j-1}$, and $A = B = 1$, and apply Lemma 1. □

C Solution Method

We describe our solution method for asset-pricing models with heterogeneous agents and recursive preferences.

C.1 Computational Procedure—A Two-Step Approach

For ease of notation the following procedures are described for $H = 2$ agents and a single state variable $y_t \in \mathbb{R}^1$. However, the approach can analogously be extended to the general case of $H > 2$ agents and multiple states. We solve the social planner's problem using a collocation projection. For this we perform the usual transformation from an equilibrium described by the infinite sequences (with a time index t) to the equilibrium being described by functions of some state variable(s) x on a state space X. We denote the current exogenous state of the economy by y and the subsequent state in the next period by y' with the state space $Y \in \mathbb{R}$. The term $\underline{\lambda}_2$ denotes the current endogenous state of the Negishi weight and $\underline{\lambda}_2'$ denotes the corresponding state in the subsequent period with $\underline{\Lambda}_2 \in (0,1)$.

We approximate the value functions of the two agents, $v^h(\underline{\lambda}_2, y), h = \{1,2\}$, by two-dimensional cubic splines and we denote the approximated value functions by $\hat{v}^h(\underline{\lambda}_2, y)$. For the collocation projection we have to choose a set of collocation nodes $\{\underline{\lambda}_{2_k}\}_{k=0}^n$ and $\{y_l\}_{l=0}^m$ at which we evaluate $\hat{v}^h(\underline{\lambda}_2, y)$. The individual consumption shares only depend on the endogeneous state $\underline{\lambda}_{2_k}$. So in the following we show how to first solve for the individual consumption shares at the collocation nodes $s_k^h = s^h(\underline{\lambda}_{2_k})$, which are then used to solve for the value functions v^h and the dynamics of the endogenous state $\underline{\lambda}_2$.

Step 1: *Computing Optimal Consumption Allocations*

Equation (13) has to hold at each collocation node $\{\underline{\lambda}_{2_k}\}_{k=0}^n$:

$$\underline{\lambda}_{2_k}(1-\delta^2)\left(s_k^2\right)^{\rho^2-1} = (1-\underline{\lambda}_{2_k})(1-\delta^1)\left(s_k^1\right)^{\rho^1-1}.$$

Together with the market-clearing condition (11) we get

$$\underline{\lambda}_{2_k}(1-\delta^2)\left(s_k^2\right)^{\rho^2-1} = (1-\underline{\lambda}_{2_k})(1-\delta^1)\left(1-s_k^2\right)^{\rho^1-1}. \tag{57}$$

So for each node $\{\underline{\lambda}_{2_k}\}_{k=0}^n$ the optimal consumption choice s_k^2 can be computed by solving Equation (57) and s_k^1 is obtained by the market-clearing condition (11).[12] For the special case of $\rho^2 = \rho^1$ we can solve for s^2 as a function of $\underline{\lambda}_2$ analytically, and hence we do not have to solve the system of equations for each node.

[12] Note that in the case of H agents we have to solve a system of $H-1$ equations that pin down the $H-1$ individual consumption choices $s^h \in \mathbb{H}^-$.

Step 2: *Solving for the Value Function and the Dynamics of the Negishi Weights*

Solving for the value function is not as straight-forward, as it depends on the dynamics of the endogenous state $\underline{\lambda}_2$, which are unknown and follow Equation (14). We compute the expectation over the exogenous state by a Gaussian quadrature with Q quadrature nodes. This implies that the values for y' at which we evaluate v^h are given by the quadrature rule. We denote the corresponding quadrature nodes by $\{y^{'}_{l,g}\}_{l=0,g=1}^{m,Q}$ and the weights by $\{\omega_g\}_{g=1}^{Q}$.[13] We can then solve Equation (14) for a given pair of collocation nodes $\{\underline{\lambda}_{2_k}, y_l\}_{k=0,l=0}^{n,m}$ and the corresponding quadrature nodes $\{y^{'}_{l,g}\}_{l=0,g=1}^{m,Q}$ to compute a vector $\vec{\underline{\lambda}}_2^{\prime}$ of size $(n{+}1)\times(m{+}1)\times Q$ that consists of the corresponding values $\underline{\lambda}'_{2_{k,l,g}}$ for each node. For each $\underline{\lambda}'_{2_{k,l,g}}$, Equation (14) then reads

$$\begin{aligned} \underline{\lambda}'_{2_{k,l,g}} &= \frac{\underline{\lambda}_{2_k}\underline{\Pi}^2}{(1-\underline{\lambda}_{2_k})\underline{\Pi}^1 + \underline{\lambda}_{2_k}\underline{\Pi}^2} \\ \underline{\Pi}^h &= \delta^h e^{\rho^h \Delta c(y'_{l,g})} \left(\frac{v^h(\underline{\lambda}'_{2_{k,l,g}}, y'_{l,g}) e^{\Delta c(y'_{l,g})}}{R^h\left[v^h(\underline{\lambda}'_2, y')e^{\Delta c(y')} | \underline{\lambda}_{2_k}, y_l\right]} \right)^{\alpha^h - \rho^h} \frac{\mathrm{dP}^h(y'_{l,g}|y_l)}{\mathrm{dP}(y'_{l,g}|y_l)}, \end{aligned} \tag{58}$$

where

$$R^h\left[v^h(\underline{\lambda}'_2, y')e^{\Delta c(y')} | \underline{\lambda}_{2_k}, y_l\right] = G_h^{-1}\left(E\left[G_h\left(v^h(\underline{\lambda}'_2, y')e^{\Delta c(y')}\right) \frac{\mathrm{dP}^h(y')}{\mathrm{dP}(y')} \middle| \underline{\lambda}_{2_k}, y_l \right]\right).$$

Note that $\underline{\lambda}'_{2_{k,l,g}}$ depends on the full distribution of $\underline{\lambda}'_2$ through the expectation operator. By applying the Gaussian quadrature to compute the expectation we get

$$E\left[G_h\left(v^h(\underline{\lambda}'_2, y')e^{\Delta c(y')}\right) \frac{\mathrm{dP}^h(y')}{\mathrm{dP}(y')} \middle| \underline{\lambda}_{2_k}, y_l \right] \approx \sum_{g=1}^{Q} G_h\left(v^h(\underline{\lambda}'_{2_{k,l,g}}, y_{l,g})e^{\Delta c(y_{l,g})}\right)\cdot \omega_g.$$

By computing the expectation with the quadrature rule, we do not need the full distribution of $\underline{\lambda}'_2$; instead, we only have to evaluate v^h at those values $\underline{\lambda}'_{2_{k,l,g}}$ that can be obtained by solving (58) for each pair of collocation nodes $\{\underline{\lambda}_{2_k}, y_l\}_{k=0,l=0}^{n,m}$ and the corresponding quadrature nodes $\{y^{'}_{l,g}\}_{l=0,g=1}^{m,G}$. So at the end we have a square system of equations with $(n+1)\times(m+1)\times G$ unknowns, $\underline{\lambda}'_{2_{k,l,g}}$, and as many equations (58) for each $\{k, l, g\}$.

The value function is in general not known so we have to compute it simultaneously when

[13] Note that the quadrature nodes $\{\{y^{'}_{l,g}\}_{g=0}^{G}\}_{l=0}^{m}$ depend on the state today, $\{y_l\}_{l=0}^{m}$.

solving for $\underline{\lambda}'_{2_{k,l,g}}$. Plugging the approximation $\hat{v}^h(\underline{\lambda}_2, y)$ into the value function (12) yields

$$\hat{v}^h(\underline{\lambda}_{2_k}, y_l) = \left[(1-\delta^h) \left(s_k^h \right)^{\rho^h} + \delta^h R^h \left(\hat{v}^h(\underline{\lambda}'_2, y') e^{\Delta c(y')} \middle| \underline{\lambda}_{2_k}, y_l \right) \right]^{\frac{1}{\rho^h}}. \tag{59}$$

The collocation projection conditions require that the equation has to hold at each collocation node $\{\underline{\lambda}_{2_k}, y_l\}_{k=0,l=0}^{n,m}$. So we obtain a square system of equations with $(n+1)\times(m+1)\times 2$ equations (59) and as many unknowns for the spline interpolation at each collocation node, which we solve simultaneously with the system for $\underline{\lambda}'_{2_{k,l,g}}$ described above.

C.2 Computational Details

For the projection method outlined above we need to choose certain collocation nodes. In this paper we use 17 uniform nodes for the $\underline{\lambda}^2$ dimension and 13 uniform nodes for the x_t dimension for the results with $\rho^2 = 0.975$ and $\rho^1 = 0.985$. For solving the models with $\rho^2 = 0.95, \rho^1 = 0.985$ and $\rho^2 = 0.96, \rho^1 = 0.99$, respectively, we use 51 uniform nodes for the $\underline{\lambda}^2$ dimension and 23 uniform nodes for the x_t dimension. For $\underline{\lambda}^2$ the minimum and maximum values are given by 0 and 1. For x_t we choose the approximation interval to cover ± 4 standard deviations around the unconditional mean of the process. We approximate the value functions using two-dimensional cubic splines with not-a-knot end conditions. We provide the solver with additional information that we can formally derive for the limiting cases. For example, we know that for $\underline{\lambda}_t^2 = 1$ ($\underline{\lambda}_t^2 = 0$) agent 2 (1) consumes everything, so it corresponds to the representative-agent economy populated only by agent 2 (1). Hence, we require that the value function for these cases equals the value function for the corresponding representative-agent economy. We also know that for $\underline{\lambda}_t^2 = 0$ ($\underline{\lambda}_t^2 = 1$) the consumption of agent 2 (1) is 0, and hence the value function is also 0. As the shocks in the model are normally distributed, we compute the expectations over the exogenous states by Gauss–Hermite quadrature using 5 nodes for the shock in x_{t+1} and 3 nodes for the shock in Δc_{t+1}. Euler errors for the value function approximations evaluated on a 200×200 uniform grid for both states are less than 1×10^{-6}, suggesting a high accuracy of our results. We double-checked the accuracy by increasing the approximation interval as well as the number of collocation nodes, with no significant change in the results.

D Plausibility of Persistent Belief Differences

The benchmark economy exhibits two agents with persistent belief differences. A critical reader may argue that this assumption is unrealistic, since we may expect the agents to learn the true exogenous growth processes over time. To address such potential criticism, we now examine the speed of learning in the long-run risk model. For this purpose, we suppose that investors need to estimate model parameters from the data. We show that it is difficult to obtain a precise estimate for the persistence parameter ρ_x of the x_t process in small, finite samples. Very long time series—much longer than those observed in our simulations (of up to 500 years)—are required for the belief differences to vanish. Therefore, learning the true persistence parameter is a very slow process.

Suppose the true persistence parameter of the long-run risk process is $\rho_x = 0.985$, which is just the value of ρ_x^1 in the benchmark economy. Now suppose an investor does not know this parameter but estimates it from a finite sample. To analyze this estimation, we simulate 1,000 time series consisting of 500 years of monthly data and calculate estimates after 100, 200, and 500 years. As a first estimation approach, we assume that the investor directly observes x_t and simply estimates the AR(1) process

$$x_{t+1} = \mu_x + \rho_x x_t + \sigma_x \eta_{x,t+1}. \tag{60}$$

We distinguish two cases of this estimation approach; first, the investor estimates the process with the constant μ_x; and second, the investor estimates the process without a constant and knows that $\mu_x = 0$. We use least-squares to obtain consistent estimates. In reality, the process x_t is not directly observable but must be inferred from the consumption growth time series. Therefore, as a second approach, we also estimate the full state-space model (15) using the Kalman filter:

$$\begin{aligned} \Delta c_{t+1} &= \mu_c + x_t + \sigma \eta_{c,t+1} \\ x_{t+1} &= \rho_x x_t + \sigma_x \eta_{x,t+1}. \end{aligned} \tag{61}$$

Table 7 reports the results of the two estimation approaches. We observe that for 100 years of data there is the usual significant finite-sample downward bias in the mean of the point estimates $\hat{\rho}_x$ (see, for example, James and Smith (1998)). Kendall (1954) shows that the bias is approximately $-(1+3\rho_x)/T = -0.0033$ for the model with a constant, which is in accordance with the value we observe. (The investor can approximate the bias using the point estimate $\hat{\rho}_x$ and the number of periods, $T = 1200$.) The table also reports the 5% and 1% quantiles of the point estimates from the 1,000 simulations. After 100 years, even after adjusting for

Table 7: Parameter Estimates from Simulated Data

		x_t observable		x_t unobservable
		with constant	w/o constant	all parameters
100 Years	$\hat{\rho}_x$	0.9817	0.9837	0.9715
	$\hat{\rho}_{x,0.05}$	0.9709	0.9742	0.9329
	$\hat{\rho}_{x,0.01}$	0.9660	0.9692	0.8252
200 Years	$\hat{\rho}_x$	0.9834	0.9844	0.9806
	$\hat{\rho}_{x,0.05}$	0.9760	0.9773	0.9627
	$\hat{\rho}_{x,0.01}$	0.9727	0.9742	0.9408
500 Years	$\hat{\rho}_x$	0.9844	0.9847	0.9833
	$\hat{\rho}_{x,0.05}$	0.9804	0.9810	0.9745
	$\hat{\rho}_{x,0.01}$	0.9786	0.9789	0.9691

The table shows the mean point estimates of ρ_x as well as the 5% and 1% quantiles after 100, 200, and 500 years obtained from simulating 1,000 monthly time series of data. In the first approach, Equation (60) is used for x_t, assuming the process is directly observable, and least-squares is used to estimate the model parameters; we distinguish the two cases of estimating the AR(1) model, with and without a constant. The second approach assumes that x_t is unobservable and the full state-space model (61) is estimated using the Kalman filter. For the data-generating process, we use $\mu_x = 0, \rho_x = 0.985, \sigma_x = 0.0003432, \mu = 0.0015$, and $\sigma = 0.0078$.

the bias, the 5% quantile is still smaller than $\rho_x^2 = 0.975$ in the benchmark economy. After 200 years, again after adjusting for the bias, the 1% quantile is still smaller than 0.975. If the investor knows that $\mu_x = 0$, then both the standard errors of the estimation and the bias become slightly smaller.

In reality, however, the investor does not observe x_t but must estimate the full model (61). In this case, the bias in $\hat{\rho}_x$ becomes significantly larger with a mean value of 0.9715 and a 5% quantile value of 0.9329. Hence, also the second value of $\rho_x^2 = 0.95$ used for the second agent is well above this quantile after 100 years. After 200 years, the standard error and the bias become smaller, but a value of 0.95 is still well within the 1% quantile. After 500 years the bias slowly vanishes but $\rho_x^2 = 0.975$ is still within the 1% range (even after correcting for a bias).

In light of the estimation results, we conclude that even if the investor might learn about the true data-generating process after 500 or more years, it is reasonable to assume that any nontrivial initial belief differences persist for at least 100 years, if not for much longer.

E Closed-Form Solutions for the CRRA Case

In this section we derive closed form solutions for the long-run risk model with two investors and CRRA utility. Assume that the two agents A and B have CRRA utility with the same degree of risk aversion. The equilibrium conditions (54) and (50) then simplify to

$$\left(\frac{s_{t+1}^A}{s_{t+1}^B}\right)^\gamma = \left(\frac{s_t^A}{s_t^B}\right)^\gamma \frac{\mathrm{dP}_{t,t+1}^A}{\mathrm{dP}_{t,t+1}^B}. \tag{62}$$

Taking logs and using the market-clearing condition yields

$$\log\left(\frac{s_{t+1}^A}{1-s_{t+1}^A}\right) = \log\left(\frac{s_t^A}{1-s_t^A}\right) + \frac{1}{\gamma}\log\left(\frac{\mathrm{dP}_{t,t+1}^A}{\mathrm{dP}_{t,t+1}^B}\right). \tag{63}$$

We log-linearize $\log(1-s_{t+1}^A)$ around $\log(s_{t+1}^A) = \log(s_t^A)$. This step gives us

$$\begin{aligned}
\log\left(1-e^{\log(s_{t+1}^A)}\right) &\approx \log\left(1-s_t^A\right) - \frac{s_t^A}{1-s_t^A}(\log\left(s_{t+1}^A\right) - \log\left(s_t^A\right)) \\
&= \log\left(1-s_t^A\right) + \frac{s_t^A}{1-s_t^A}\log\left(s_t^A\right) - \frac{s_t^A}{1-s_t^A}a - \frac{s_t^A}{1-s_t^A}b\eta_{x,t+1}.
\end{aligned} \tag{64}$$

Using the linearization (77) and the expression for $\left(\frac{\mathrm{dP}_{t,t+1}^A}{\mathrm{dP}_{t,t+1}^B}\right)$ given in Equation (78), we find that the consumption share in $t+1$ is a linear function of $\eta_{x,t+1}$,

$$\log\left(s_{t+1}^A\right) = a^{CRRA} + b^{CRRA}\eta_{x,t+1} \tag{65}$$

and the coefficients are given by

$$\begin{aligned}
b^{CRRA} &= \frac{(1-s_0^A)x_0(\rho_x^A-\rho_x^B)}{\sigma_x^2\gamma} \\
a^{CRRA} &= \log\left(s_0^A\right) + \frac{(1-s_0^A)x_0^2}{2\sigma_x^2\gamma}\left[(\rho_x-\rho_x^B)^2 - (\rho_x-\rho_x^A)^2\right].
\end{aligned}$$

F Closed-Form Solutions for the Unit EIS Case

In this section we show how to solve a special case of the long-run risk model with two investors in closed form. We denote the two investors by A and B to avoid confusion with the time indices we later introduce for the first two periods. We derive closed-form solutions under the assumption that both investors have an EIS of $\psi^A = \psi^B = 1$. We consider the following setup.

F.1 Model Setup

In period 0 agents are endowed with a given consumption share. The only uncertainty in the model arises in period 1. Investors have different beliefs about the state of the economy in period 1 and investors trade based on their beliefs. After period 1, there is no more uncertainty and investors know the true distribution of the state. So in periods $2, \ldots, \infty$ the investors keep the consumption share that results from the trading in period 1.

F.2 The Unit EIS Case

We assume that both investors have an EIS of 1. The value functions are then given by

$$v_t^h = (1-\delta^h)\log s_t^h + \frac{\delta^h}{1-\gamma^h}\log E_t^h\left(e^{(1-\gamma^h)(v_{t+1}^h+\Delta c_{t+1})}\right), \quad h \in \{A, B\}. \tag{66}$$

Following the methodology of Appendix B, we obtain the following equilibrium conditions; first, we have the market-clearing condition

$$s_t^A + s_t^B = 1. \tag{67}$$

The optimality condition for the individual consumption decisions simplifies to

$$\frac{\lambda_t^A}{\lambda_t^B} = \frac{s_t^A}{s_t^B} \tag{68}$$

and the equation for the dynamics of λ_t is given by

$$\begin{aligned} \frac{\lambda_{t+1}^A}{\lambda_{t+1}^B} &= \frac{\lambda_t^A}{\lambda_t^B}\frac{\underline{\Pi}_{t+1}^A}{\underline{\Pi}_{t+1}^B} \\ \underline{\Pi}_{t+1}^h &= \delta^h \frac{\mathrm{dP}_{t,t+1}^h}{\mathrm{dP}_{t,t+1}} \frac{e^{(1-\gamma^h)(v_{t+1}^h+\Delta c_{t+1})}}{E_t^h\left(e^{(1-\gamma^h)(v_{t+1}^h+\Delta c_{t+1})}\right)}, \quad h \in \{A, B\}. \end{aligned} \tag{69}$$

We assume that investors have the same preference parameters ($\gamma = \gamma^A = \gamma^B, \delta = \delta^A = \delta^B$). Plugging (68) into (69) and making use of the market-clearing condition, we obtain the following equilibrium condition:

$$\frac{s_{t+1}^A}{1-s_{t+1}^A} = \frac{s_t^A}{1-s_t^A}\frac{\mathrm{dP}_{0,1}^A}{\mathrm{dP}_{0,1}^B} e^{(1-\gamma)(v_{t+1}^A - v_{t+1}^B)} \frac{E_0^B\left(e^{(1-\gamma)(v_{t+1}^B+\Delta c_{t+1})}\right)}{E_0^A\left(e^{(1-\gamma)(v_{t+1}^A+\Delta c_{t+1})}\right)}. \tag{70}$$

We are interested in how shocks to the mean growth rate of consumption x_t affect the consumption shares and the risk sharing of investors. For this task, we consider a simplified version of the exogenous processes (15), in which we omit the short-run shocks to consumption $\eta_{c,t+1}$. As mentioned above, the only uncertainty arises in period 1, so we have

$$\begin{aligned}\Delta c_1 &= \mu + x_0 \\ x_1 &= \rho_x^h x_0 + \eta_{x,1}, \qquad \eta_{x,1} \sim N(0, \sigma_x^2)\end{aligned}$$

and

$$\begin{aligned}\Delta c_{t+1} &= \mu + x_t \\ x_{t+1} &= \rho_x x_t, \qquad \forall t \geq 1.\end{aligned}$$

Before we analyze the risk-sharing motives in period 0, we first need to derive the value function in period 1. The value function (66) in the deterministic case simplifies to

$$v_t^h = (1-\delta)\log s_t^h + \delta v_{t+1}^h + \delta \Delta c_{t+1}. \tag{71}$$

Iterating forward yields

$$v_t^h = \sum_{i=1}^{\infty} \delta^i \mu + \sum_{i=0}^{\infty} \delta^i (1-\delta) \log(s_t^h) + \sum_{i=1}^{\infty} \delta^i \rho_x^{i-1} x_t.$$

As there is no more disagreement, agents no longer trade and hence $s_t^h = s_{t+1}^h$ for $t \geq 1$. We also have

$$\sum_{i=1} \delta^i \rho_x^{i-1} x_t = \delta x_t \sum_{i=0} \delta^i \rho_x^i = \delta x_t / (1 - \delta \rho_x).$$

Therefore, we obtain

$$v_t^h = \frac{\delta}{1-\delta}\mu + \frac{\delta}{1-\delta\rho_x} x_t + \log s_t^h. \tag{72}$$

All uncertainty is resolved in period 1; the value functions of the two agents are then given by

$$v_1^A = \frac{\delta}{1-\delta}\mu + \frac{\delta}{1-\delta\rho_x} x_1 + \log s_1^A \tag{73}$$

$$v_1^B = \frac{\delta}{1-\delta}\mu + \frac{\delta}{1-\delta\rho_x} x_1 + \log(1 - s_1^A). \tag{74}$$

Given the value functions in the first period, we can determine the risk-sharing rule. The risk-sharing rule (70) for period 1 is given by

$$\frac{s_1^A}{1-s_1^A} = \frac{s_0^A}{1-s_0^A}\frac{\mathrm{dP}_{0,1}^A}{\mathrm{dP}_{0,1}^B}e^{(1-\gamma)(v_1^A-v_1^B)}\frac{E_0^B(e^{(1-\gamma)(v_1^B)})}{E_0^A(e^{(1-\gamma)(v_1^A)})}. \tag{75}$$

Plugging in the value functions, taking logs, and simplifying yields

$$\begin{aligned} \log\left(\frac{s_1^A}{1-s_1^A}\right) &= \log\left(\frac{s_0^A}{1-s_0^A}\right) + \log\left(\frac{\mathrm{dP}_{0,1}^A}{\mathrm{dP}_{0,1}^B}\right) + (1-\gamma)\log\left(\frac{s_1^A}{1-s_1^A}\right) \\ &+ \log\left(E_0^B(e^{(1-\gamma)(\frac{\delta}{1-\delta\rho_x}x_1+\log(1-s_1^A))})\right) - \log\left(E_0^A(e^{(1-\gamma)(\frac{\delta}{1-\delta\rho_x}x_1+\log s_1^A)})\right) \end{aligned} \tag{76}$$

We now guess and verify that $\log(s_1^A) = a + b\eta_{x,1}$ holds true (given a log-linearization). We log-linearize $\log(1-s_1^A)$ around $\log(s_1^A) = \log(s_0^A)$. This step gives us

$$\begin{aligned} \log\left(1-e^{\log(s_1^A)}\right) &\approx \log\left(1-s_0^A\right) - \frac{s_0^A}{1-s_0^A}(\log\left(s_1^A\right) - \log\left(s_0^A\right)) \\ &= \log\left(1-s_0^A\right) + \frac{s_0^A}{1-s_0^A}\log\left(s_0^A\right) - \frac{s_0^A}{1-s_0^A}a - \frac{s_0^A}{1-s_0^A}b\eta_{x,1}, \end{aligned} \tag{77}$$

which we need below. In the following we derive the different terms in (76). Since $x_1 \sim N(\rho_x x_0, \sigma_x^2)$, the probability ratio is given by

$$\begin{aligned} \log\left(\frac{\mathrm{dP}_{0,1}^A}{\mathrm{dP}_{0,1}^B}\right) &= \log\left(e^{-0.5\frac{(\rho_x x_0+\eta_{x,1}-\rho_x^A x_0)^2}{\sigma_x^2}+0.5\frac{(\rho_x x_0+\eta_{x,1}-\rho_x^B x_0)^2}{\sigma_x^2}}\right) \\ &= \frac{x_0^2}{2\sigma_x^2}\left((\rho_x-\rho_x^B)^2-(\rho_x-\rho_x^A)^2\right) + \frac{x_0}{\sigma_x^2}\left(\rho_x^A-\rho_x^B\right)\eta_{x,1}, \end{aligned} \tag{78}$$

which is linear in $\eta_{x,1}$. We know that $E_0^A(e^{c_1+c_2\eta_{x,1}}) = e^{c_1+0.5c_2^2\sigma_x^2}$ for constants c_1 and c_2 since $\eta_{x,1} \sim N(0,\sigma_x^2)$. So, for the last two terms in (76) we obtain

$$\begin{aligned} &\log\left(E_0^A e^{(1-\gamma)(\frac{\delta}{1-\delta\rho_x}x_1+\log s_1^A)}\right) \\ =\ &\log\left(E_0^A e^{(1-\gamma)(\frac{\delta}{1-\delta\rho_x}\rho_x^A x_0+\frac{\delta}{1-\delta\rho_x}\eta_{x,1}+a+b\eta_{x,1})}\right) \\ =\ &(1-\gamma)a + (1-\gamma)\frac{\delta}{1-\delta\rho_x}\rho_x^A x_0 + 0.5(1-\gamma)^2(b+\frac{\delta}{1-\delta\rho_x})^2\sigma_x^2 \end{aligned} \tag{79}$$

and, using (77),

$$
\begin{aligned}
& \log\left(E_0^B e^{(1-\gamma)(\frac{\delta}{1-\delta\rho_x}x_1+\log(1-s_1^A))}\right) \\
\approx\ & \log\left(E_0^B e^{(1-\gamma)(\frac{\delta}{1-\delta\rho_x}\rho_x^B x_0+\frac{\delta}{1-\delta\rho_x}\eta_{x,1}+\log\left(1-s_0^A\right)+\frac{s_0^A}{1-s_0^A}\log\left(s_0^A\right)-\frac{s_0^A}{1-s_0^A}a-\frac{s_0^A}{1-s_0^A}b\eta_{x,1})}\right) \\
=\ & (1-\gamma)\frac{\delta}{1-\delta\rho_x}\rho_x^B x_0+(1-\gamma)\log\left(1-s_0^A\right)+(1-\gamma)\frac{s_0^A}{1-s_0^A}\log\left(s_0^A\right) \\
-\ & (1-\gamma)\frac{s_0^A}{1-s_0^A}a+0.5(1-\gamma)^2(\frac{\delta}{1-\delta\rho_x}-\frac{s_0^A}{1-s_0^A}b)^2\sigma_x^2.
\end{aligned}
$$

Plugging (78), (79), and (80) into (76) yields

$$
\begin{aligned}
& a+b\eta_{x,1}-\log\left(1-s_0^A\right)-\frac{s_0^A}{1-s_0^A}\log\left(s_0^A\right)+\frac{s_0^A}{1-s_0^A}a+\frac{s_0^A}{1-s_0^A}b\eta_{x,1} \\
=\ & \log\left(s_0^A\right)-\log\left(1-s_0^A\right)+\frac{x_0^2}{2\sigma_x^2}((\rho_x-\rho_x^B)^2-(\rho_x-\rho_x^A)^2)+\frac{x_0}{\sigma_x^2}(\rho_x^A-\rho_x^B)\eta_{x,1} \\
+\ & (1-\gamma)a+(1-\gamma)b\eta_{x,1}-(1-\gamma)\log\left(1-s_0^A\right)-(1-\gamma)\frac{s_0^A}{1-s_0^A}\log\left(s_0^A\right) \\
+\ & (1-\gamma)\frac{s_0^A}{1-s_0^A}a+(1-\gamma)\frac{s_0^A}{1-s_0^A}b\eta_{x,1} \\
+\ & (1-\gamma)\frac{\delta}{1-\delta\rho_x}\rho_x^B x_0+(1-\gamma)\log\left(1-s_0^A\right)+(1-\gamma)\frac{s_0^A}{1-s_0^A}\log\left(s_0^A\right) \\
-\ & (1-\gamma)\frac{s_0^A}{1-s_0^A}a+0.5(1-\gamma)^2(\frac{\delta}{1-\delta\rho_x}-\frac{s_0^A}{1-s_0^A}b)^2\sigma_x^2 \\
-\ & (1-\gamma)a-(1-\gamma)\frac{\delta}{1-\delta\rho_x}\rho_x^A x_0-0.5(1-\gamma)^2(b+\frac{\delta}{1-\delta\rho_x})^2\sigma_x^2.
\end{aligned}
$$

Simplifying gives

$$
\begin{aligned}
\frac{a}{1-s_0^A}+\frac{b}{1-s_0^A}\eta_{x,1}\ =\ & \frac{\log\left(s_0^A\right)}{1-s_0^A}+\frac{x_0^2}{2\sigma_x^2}((\rho_x-\rho_x^B)^2-(\rho_x-\rho_x^A)^2)+\frac{x_0}{\sigma_x^2}(\rho_x^A-\rho_x^B)\eta_{x,1} \\
+\ & (1-\gamma)\frac{b}{1-s_0^A}\eta_{x,1}+(1-\gamma)\frac{\delta}{1-\delta\rho_x}(\rho_x^B-\rho_x^A)x_0 \\
+\ & 0.5(1-\gamma)^2(\frac{\delta}{1-\delta\rho_x}-\frac{s_0^A}{1-s_0^A}b)^2\sigma_x^2 \\
-\ & 0.5(1-\gamma)^2(b+\frac{\delta}{1-\delta\rho_x})^2\sigma_x^2.
\end{aligned}
\tag{80}
$$

Equation (80) has to hold for every $\eta_{x,1}$, so we can collect coefficients for $\eta_{x,1}$ and for the constant term. For the terms with $\eta_{x,1}$ we have

$$\frac{b}{1-s_0^A}\eta_{x,1} = \frac{x_0}{\sigma_x^2}(\rho_x^A - \rho_x^B)\eta_{x,1} + (1-\gamma)\frac{b}{1-s_0^A}\eta_{x,1}, \tag{81}$$

which gives

$$b = \frac{(1-s_0^A)x_0(\rho_x^A - \rho_x^B)}{\gamma\sigma_x^2}. \tag{82}$$

For the constant terms we have

$$\begin{aligned}
a &= \log\left(s_0^A\right) + \frac{(1-s_0^A)x_0^2}{2\sigma_x^2}\left[(\rho_x - \rho_x^B)^2 - (\rho_x - \rho_x^A)^2\right] \\
&+ (1-\gamma)(1-s_0^A)\frac{\delta}{1-\delta\rho_x}(\rho_x^B - \rho_x^A)x_0 \\
&+ 0.5(1-\gamma)^2(1-s_0^A)(\frac{\delta}{1-\delta\rho_x} - \frac{s_0^A}{1-s_0^A}b)^2\sigma_x^2 \\
&- 0.5(1-\gamma)^2(1-s_0^A)(b + \frac{\delta}{1-\delta\rho_x})^2\sigma_x^2,
\end{aligned} \tag{83}$$

which equals

$$\begin{aligned}
a &= \log\left(s_0^A\right) + \frac{(1-s_0^A)x_0^2}{2\sigma_x^2}\left[(\rho_x - \rho_x^B)^2 - (\rho_x - \rho_x^A)^2\right] \\
&+ (1-\gamma)(1-s_0^A)\frac{\delta}{1-\delta\rho_x}(\rho_x^B - \rho_x^A)x_0 \\
&+ 0.5(1-\gamma)^2\sigma_x^2\left(b^2\frac{2s_0^A - 1}{1-s_0^A} - 2b\frac{\delta}{1-\delta\rho_x}\right).
\end{aligned}$$

Inserting b yields

$$\begin{aligned}
a &= \log\left(s_0^A\right) + \frac{(1-s_0^A)x_0^2}{2\sigma_x^2}\left[(\rho_x - \rho_x^B)^2 - (\rho_x - \rho_x^A)^2\right] \\
&+ (1-s_0^A)\frac{\delta}{1-\delta\rho_x}(\rho_x^B - \rho_x^A)x_0\frac{(1-\gamma)}{\gamma} \\
&+ \frac{x_0^2(1-\gamma)^2(\rho_x^A - \rho_x^B)^2(1-s_0^A)(2s_0^A - 1)}{2\sigma_x^2\gamma^2}.
\end{aligned} \tag{84}$$

The first line is the CRRA term. The second line, which is the most interesting, shows that the change in the weights depends on whether ρ_x^A is smaller or larger than ρ_x^B. The third line is a risk adjustment term.

G Additional Results

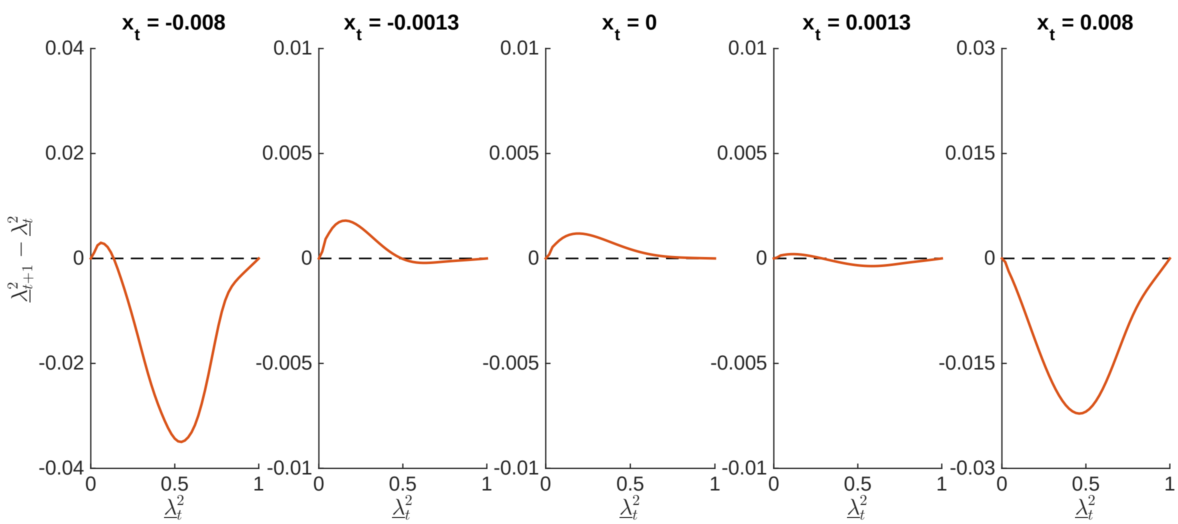
This section presents some additional results, which have been referenced in the main body of the paper. Table 8 reports equity premia for the economy in which agent 2 has the correct beliefs, $\rho_x = \rho_x^2$ (see Section 4.2). Figure 12 displays plots of the average changes in the wealth distribution for the economy with the larger belief differences, $\rho_x^2 = 0.95$. (see Section 4.3.2).

Table 8: Equity Premia for Different Consumption Shares ($\rho_x = \rho_x^2$)

	$\boldsymbol{\rho_x^2 = 0.975}$		$\boldsymbol{\rho_x^2 = 0.95}$	
	s_t^2	Equity Premium	s_t^2	Equity Premium
Rep. Agent 1	0	6.49	0	6.50
0 Years	0.01	6.38	0.01	5.36
100 Years	0.3404	4.39	0.8147	0.49
200 Years	0.7249	3.33	0.8810	0.41
500 Years	0.9732	2.83	0.9388	0.34
Rep. Agent 2	1	2.75	1	0.25

The table shows the annualized equity premium for a specific consumption share $s_t^2 = \bar{s}$. The premium is reported for the equilibrium allocations after 0, 100, 200, and 500 years of simulated data assuming an initial share of $s_0^2 = 0.01$ (see Table 1). Agent 1 believes that $\rho_x^1 = 0.985$ and agent 2 has the correct belief ($\rho_x = \rho_x^2$). The left panel depicts the case for $\rho_x^2 = 0.975$ and the right panel for $\rho_x^2 = 0.95$.

Figure 12: Changes in the Wealth Distribution—The Epstein–Zin Case 2 ($\rho_x^2 = 0.95$)



The figure shows the change in the optimal weights $\underline{\lambda}_{t+1}^2 - \underline{\lambda}_t^2$ as a function of $\underline{\lambda}_t^2$. From left to right, the change is shown for $x_t = \{-0.008, -0.0013, 0, 0.0013, 0.008\}$ ($\pm$ 4 standard deviations). The red line shows the average over all shocks in x_{t+1}. Calibration with $\rho_x = \rho_x^1 = 0.985$ and $\rho_x^2 = 0.95$.