Gradual Bargaining in Decentralized Asset Markets*

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Abstract

We introduce a new approach to bargaining, with both axiomatic and strategic foundations, into models of decentralized asset market. Gradual bargaining, which assumes that portfolios of assets are sold sequentially, one unit of asset at a time, has strong normative justifications: it increases the surplus of asset owners, it reduces asset misallocation, and it can implement first best. In the presence of multiple assets our theory generates a pecking order, a structure of asset returns based on asset negotiability, and differences in turnover. We apply our model to the study of open-market operations and the determination of the exchange rate in the presence of multiple (crypto-)currencies.

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1 Introduction

Both modern monetary theory and financial economics formalize asset trades in the context of decentralized markets where agents meet bilaterally (e.g., Duffie et al., 2005; Lagos and Wright, 2005). The extensive and intensive margins of trade are captured by two core components: a technology through which buyers and sellers meet one another and a mechanism through which prices and trade sizes are determined. This paper focuses on the latter: the negotiation of asset prices and trade sizes.

While there is a long tradition in the search-theoretic literature to place stark restrictions on individual inventories of assets and goods, going back to Diamond (1982), recent advances have allowed for unrestricted portfolios (e.g., Lagos and Wright; 2005; Lagos and Rocheteau, 2009; Uslu, 2016). In models with a single indivisible asset, the only item to negotiate is the price of the asset in terms of a divisible commodity.[1] In contrast, in models with multiple divisible assets, there are many ways to liquidate a portfolio, e.g., agents can sell their whole portfolio at once, as a large block, or they can negotiate the sale of assets gradually over time. This raises the following questions. What is the optimal way to sell an asset portfolio, e.g., should the portfolio be divided in smaller parts? Does the order according to which assets are sold matter for prices and allocations? Does the outcome depend on the side choosing the agenda of the negotiation, i.e., what to negotiate and when?

Our contribution is to introduce a new approach with both strategic and axiomatic foundations, to bargaining over portfolios of assets into a model of decentralized asset market. This approach assumes that agents sell their assets gradually, one unit at a time. It is a natural extension of the bargaining protocol in Shi (1995) and Trejos and Wright (1995). In those models, assets holdings are restricted to {0,1} and agents negotiate some amount of divisible output for one unit of asset. Similarly, we divide asset holdings into N equal parts and consider an extensive-form bargaining game composed of N rounds. In each round, agents negotiate some amount of output in exchange for at most a fraction $1/N$ of the overall assets.[2] For simplicity, there is one player in each round making an ultimatum offer and the identity of the proposer alternates across rounds. (We also consider alternating offers within each round.) While the Rubinstein's (1982) alternating-offer bargaining game has a (quasi-)stationary structure, our alternating-ultimatum-offer bargaining game is nonstationary since payoffs change over time as units of assets are sold. We show the

[1] In Osborne and Rubinstein (1990) agents trade an indivisible consumption good and pay with transferable utility. The interpretation is reversed in Shi (1995) and Trejos and Wright (1995) where the indivisible good is fiat money and agents negotiate over a divisible consumption good. In Duffie et al. (2005) the indivisible good is a consol and agents pay with transferable utility.

[2] The gradual aspect of asset trades is a key characteristic of many trading practices observed on financial markets. For example, broker-dealers are known to break large orders ("block orders") into smaller ones and execute them over the span of several days (see, e.g., Chan and Lakonishok, 1995).

existence and uniqueness of a subgame perfect equilibrium and characterize equilibrium payoffs through a system of differential equations in the limit as N goes to infinity.

We check the robustness of our solution by adopting an axiomatic approach that abstracts from the details of the extensive-form game in order to focus on some fundamental properties of the outcome. The relevant axiomatic approach comes from O'Neill et al. (2004) that extends Nash (1953) by adding the agenda of the negotiation, formalized as a collection of expanding bargaining sets. We choose the agenda to be consistent with our strategic game, i.e., agents add assets on the negotiating table gradually over time, and reach a definitive agreement over each unit added. The solution of O'Neill et al. (2004) is a path that shares three axioms with the Nash (1953) solution, Pareto optimality, scale invariance, and symmetry, and satisfies two new axioms, continuity and time consistency. The unique solution satisfying the five axioms of O'Neill et al. (2004) coincides with the subgame perfect equilibrium of the alternating ultimatum offer bargaining game.

A common thread throughout the paper is the need to specify an agenda for the negotiation of asset portfolios. In order to compare different agendas we extend our extensive form game so that agents play an aternating-offer game with exogenous risk of break-down, as in Rubinstein (1982), in each of the N rounds. Our game admits the Nash solution and the gradual solution as particular cases when $N = 1$ or $N = +\infty$. We show that asset owners maximize their surplus when $N = +\infty$. We also study an alternative agenda according to which agents bargain gradually over the decentralized market good, which can be interpreted as an illiquid asset sold over the counter. In that case the gradual solution coincides with the proportional solution of Kalai (1975). So our model provides, as a by-product, new strategic and axiomatic foundations to the use of the proportional solution in the context of decentralized asset markets.

Our next step consists in incorporating bargaining solutions with an agenda into a general equilibrium model of decentralized asset markets where portfolios are endogenous. We augment the analysis by introducing a new asset characteristic – negotiability – defined as the amount of time required for the sale of each unit of the asset to be concluded, e.g., each asset added to the negotiation table needs to be authenticated and ownership rights take time to transfer.[3] We make this negotiability relevant by assuming that the time agents have to complete their negotiation is stochastic and exponentially distributed – which can be interpreted as a risk of breakdown or discounting.[4] While we interpret the negotiability of an asset as an

[3]The concept of negotiability dates back to the 17th century and referred to institutional arrangements aiming at enhancing liquidity by "centralizing all rights to the underlying asset in a single physical document, [...] reducing the costs a prospective purchaser incurs in acquiring [...] information about the asset" (Mann, 1996). The concept of blockchains - immutable, decentralized ledgers that can record ownership and transfer of intangible assets - can be seen as a digital incarnation of the original idea of negotiability.

[4]According to Duffie (2012) search and matching frictions encompass not only "delays associated with reaching an awareness of trading opportunities" but also delays due to "arranging financing and meeting suitable legal restrictions, negotiating trades, executing trades, and so on." For evidence on these delays, see, e.g., Saunders et al. (2012) and Pagnotta and Phillipon (2017).

exogenous technological parameter in most of the paper, we develop an extension to endogenize it. The general equilibrium spread between the rate of return of the asset and the rate of time preference is the product of four components: the search friction, the bargaining power, the negotiability friction, and a measure of liquidity needs. An increase in the asset supply, or a reduction in search frictions, raises both the rate of return of the asset and its negotiability.

In terms of the normative properties of the equilibrium, if the asset is scarce, the decentralized choice of asset negotiability is too low relative to the planner's choice, even if asset owners have all the bargaining power, because of a pecuniary externality. The equilibrium under all-at-once bargaining ($N = 1$) features asset misallocation because a fraction of the asset supply end up being held by agents with no liquidity needs. In contrast, under gradual bargaining ($N = +\infty$), assets are held by agents with liquidity needs, and the first best is implemented as long as the asset supply is sufficiently abundant.

Finally, we extend our environment to allow for an arbitrary number of assets. All assets, except fiat money, generate the same stream of dividends but differ by their negotiability. For instance, more complex assets take more time to be negotiated than simpler ones. We let asset owners choose the agenda of the negotiation, i.e., the order according to which assets are negotiated. Our model generates an endogenous pecking order: assets that are more negotiable are put on the negotiating table before the less negotiable ones. This pecking order has implications for asset prices and velocities: the most negotiable assets have lower rates of return and higher velocities. Hence, our model explains rate-of-return difference differences of seemingly identical assets. Moreover, we show that interest spreads can be expressed as the sum of a liquidity and a negotiability premia. The liquidity premium measures the effect of an increases in wealth on the marginal utility of consumption assuming the negotiation lasts for long enough for the wealth to be spent. The negotiability premium measures the marginal utility gain from spending an asset that is relatively more negotiable than other assets, thereby allowing for larger trades in a given negotiation time.

We conclude the paper by considering two applications of our multiple-asset model. The first application has money and government bonds and studies the effects of open-market operations. Our model predicts that an open market sale of bonds raises the nominal interest rate and reduces output because less-negotiable bonds are replaced with fiat money, which is more negotiable. Our second application is a dual-currency economy where the two currencies have different money growth rates and negotiabilities. For example, the time it takes for crypto-currencies transactions to be confirmed differs greatly between the most popular coins.[5] We show that the exchange rate is determinate: the currency with higher negotiability appreciates vis-

[5] As of November 2017, it took on average less than 4 seconds for a Ripple transaction to be confirmed, against 5 minutes with Ethereum, 12 minutes with Litecoin, 15 minutes with Dash, and 45 minutes with Bitcoin.

a-vis the high-return currency if the frequency of trades increases, if consumers' bargaining power increases, or if the time horizon of the negotiation shortens.

Literature

The standard approach to price formation in decentralized asset markets consists in applying axiomatic bargaining solutions, such as Nash (1950), or equivalent extensive-form games, as described in Osborne and Rubinstein (1990). Early applications to monetary economies were provided by Shi (1995) and Trejos and Wright (1995). In models with unrestricted asset holdings, the Nash solution has properties that have been largely seen as undesirable, e.g., its lack of monotonicity (e.g., Aruoba et al., 2007). The proportional solution of Kalai (1977) avoids this issue by being strongly monotone, but it is not invariant to affine transformation of utilities.[6]

The gradual bargaining solution was developed by O'Neill et al. (2004). A key innovation consists in introducing as part of the primitives of the bargaining problem the agenda of the negotiation represented by a continuum of Pareto frontiers.[7] To the best of our knowlege we provide its first application.[8] One conceptual difficulty is to identify the proper agenda of the portfolio negotiation. We show how to address this question in the context of mainstream models of decentralized asset trades.

While O'Neill et al. (2004) are silent about the strategic foundations of the solution, an earlier working paper by Wiener and Winter (1998) conjectures that a bargaining game with alternating offers should generate the same outcome. We formalize this conjecture in details in the context of our model of asset markets by considering an alternating ultimatum-offer bargaining game. This game is not stationary because the amount left to negotiate varies as the negotiation progresses. Somewhat related, Coles and Wright (1998) describe the strategic negotiation of units of money in continuous time in the non-stationary monetary equilibria of the model of Shi (1995) and Trejos and Wright (1995). Tsoy (2016) proposes a model with bargaining and delays in equilibrium and applies it to OTC markets. Gerardi and Maestri (2017) formalize the bargaining of a divisible asset whose quality, unknown to buyers, can only be assessed by observing the seller's response to take-it-or-leave-it offers; they show that gradual trading emerges endogenously for high-quality assets.

[6] Other trading mechanisms studied in the context of these models include competitive search (Rocheteau and Wright, 2005; Lester et al., 2015), price taking (Rocheteau and Wright, 2005), auctions (Galenianos and Kircher, 2008), price posting (Jean et al., 2010), and monopolistic competition (Silva, 2017). Socially optimal mechanisms were characterized by Hu et al. (2009).

[7] Multi-issue bargaining with agendas was studied by Fershtman (1990), Bac and Raff (1996), Inderst (2000), and In and Serrano (2003, 2004) among others. In these studies, the agenda refers to the order of the multiple issues. In Fershtman (1990), the agenda is exogenously given, whereas in Bac and Raff (1996), Inderst (2000), and In and Serrano (2003, 2004), the agenda is endogenously determined within the bargaining games.

[8] An early application can be found in the working paper of Rocheteau and Waller (2005) in the context of a pure currency economy.

We incorporate the gradual bargaining game into two general equilibrium models of asset markets. The main framework is the decentralized asset market with divisible Lucas trees from Geromichalos et al. (2007) and Lagos (2010).[9] We also consider a variant where agents trade assets because of idiosyncratic valuations, as in Duffie et al. (2005) and Lagos and Rocheteau (2009) for a version with unrestricted portfolios.[10] This second version is closely connected to Geromichalos and Herrenbrueck (2016), Lagos and Zhang (2018), and Wright, Xiao, and Zhu (2018), who study the reallocation of assets in OTC trades financed with money.

Our extension with multiple assets contributes to the literature on asset price puzzles in markets with search frictions, e.g., Vayanos and Weill (2008) based on increasing-returns-to-scale matching technologies; Rocheteau (2011), Li et al. (2012) and Hu (2013) based on informational asymmetries; and Lagos (2013) based on self-fulfilling beliefs in the presence of assets' extrinsic characteristics. Our emphasis on the negotiation is tied to Zhu and Wallace (2007) and Nosal and Rocheteau (2013) but in contrast to those models we do not let the bargaining power depend on the portfolio.[11]

Our paper is also related to the literature on the optimal execution of large asset orders, e.g. Bertsimas and Lo (1998), Almgren and Chriss (1999), Almgren and Chriss (2001), Almgren (2003). These papers formalize the trade-off between trading large volumes quickly and breaking the order into small pieces sold gradually. Obizhaeva and Wang (2006) endogenize the price impact of trading aggressively by formalizing the dynamics of supply and demand through a limit book order market (see also Alfonsi et al., 2010).

2 Environment

Time is discrete, continues forever, and each period is divided into two stages. There is a continuum of agents with measure two evenly divided between two types, called consumers and producers. An agent's type corresponds to his role in the first stage, where only consumers wish to consume while only producers have the technology to produce. Throughout most of the paper we think of consumers as natural asset holders who receive liquidity shocks that make them want to sell their assets while producers are potential buyers of those assets. During that stage, labeled DM (for decentralized market), a fraction α of consumers and producers are matched bilaterally. The second stage, labeled CM (for centralized market), features a centralized Walrasian market. There is a one good in each stage and we take the CM good as numeraire.

Consumers' preferences are represented by the period utility function, $u(y) - h$, where y is DM consumption and h is the CM supply of labor. Producers' preferences are represented by $-v(y) + c$, where y

[9] In those models, the asset owner has all the bargaining power. Rocheteau and Wright (2013) adopt the proportional bargaining solution, endogenize participation, and consider non-stationary equilibria. Lester et al. (2012) introduce a costly acceptability problem. Rocheteau (2011) and Li et al. (2012) add informational asymmetries.

[10] See Trejos and Wright (2016) for a model that nests Shi (1995), Trejos, Wright (1995) and Duffie et al. (2005).

[11] Hu and Rocheteau (2013, 2015) show that having the bargaining power depend on portfolios is part of an optimal mechanism.

corresponds to the production of the DM good and c is the consumption of the CM good. The DM good can be given different interpretations, e.g., a perishable consumption good, a real asset, or services from financial assets. We assume $u'(y) > 0$, $u''(y) < 0$, $u(0) = v(0) = v'(0) = 0$, $v'(y) > 0$, $v''(y) > 0$, and $v(\bar{y}) = u(\bar{y})$ for some $\bar{y} > 0$. Let y^* denote the solution to $u'(y^*) = v'(y^*)$. All agents share the same discount factor across periods, $\beta \equiv (1+\rho)^{-1} \in (0, 1)$.

Agents are anonymous and hence cannot issue private IOUs to finance their DM consumption. There is an exogenous supply of Lucas trees, A_t, that are perfectly durable, storable at no cost, and non-counterfeitable. Each Lucas tree pays off $d \geq 0$ units of numeraire in the CM, where the case $d = 0$ corresponds to fiat money. The supply grows at rate π, $A_{t+1} = (1 + \pi)A_t$, where new trees are allocated to consumers in a lump-sum fashion. We set $\pi = 0$ when $d > 0$ but we allow $\pi \in (\beta - 1, +\infty)$ when $d = 0$. We denote ϕ_t the price of Lucas trees in terms of the numeraire.

3 Preliminary results

We first derive some preliminary results that will be useful to set up the bargaining problem in the DM. We restrict our attention to stationary equilibria where the price of Lucas trees is constant at ϕ and hence their gross rate of return is also constant and equal to $R = 1 + r = (\phi + d)/\phi$. We measure a consumer's asset holdings in the DM in terms of their value in the coming CM. More precisely, a units of asset in the DM are worth

$$z = (\phi + d)a.$$

The lifetime expected utility of a consumer (i.e., buyer of DM goods) with wealth z in the CM is

$$W^b(z) = \max_{z',h} \left\{-h + \beta V^b(z')\right\} \quad \text{s.t.} \quad z' = R\left(z + h + T\right), \tag{1}$$

where T denotes lump-sum transfers (expressed in terms of CM goods), z' are next-period asset holdings, and $V^b(z')$ is the value function at the start of the DM. From (1) the consumer chooses his supply of labor and future asset holdings in order to maximize his discounted continuation value net of the disutility of work. According to the budget constraint, next-period asset holdings are equal to current asset holdings, plus labor income and net transfer, everything multiplied by the gross rate of return of assets. Substituting h by its expression coming from the budget identity into the objective, we obtain

$$W^b(z) = z + T + \max_{z' \geq 0} \left\{ -\frac{z'}{R} + \beta V^b(z') \right\}. \tag{2}$$

As is standard, W^b is linear in wealth. By a similar reasoning, the value function of a producer is

$$W^s(z) = z + \max_{z' \geq 0} \left\{ -\frac{z'}{R} + \beta V^s(z') \right\}.$$

The lifetime expected utility of a consumer holding z assets in the DM solves

$$V^b(z) = \alpha \left\{ u\left[y(z)\right] + W^b\left[z - p\left(z\right)\right] \right\} + \left(1 - \alpha\right) W^b(z), \tag{3}$$

where $y(z)$ is the consumer's consumption and $p(z)$ is his sale of Lucas trees in the DM in terms of numeraire. Note that we conjecture (and verify later) that the terms of trade in a bilateral match, $[y(z), p(z)]$, only depend on the consumer's wealth. According to (3) a consumer meets a producer with probability α, in which case he enjoys $y(z)$ units of DM consumption in exchange for $p(z)$ units of real balances. With probability $1 - \alpha$ the consumer is unmatched and enters the CM with z units of asset. We now turn to the choice of asset holdings in the CM. Substituting $V^b(z)$ by its expression given by (3), the consumer's choice of asset holdings solves

$$\max_{z \geq 0} \left\{ -sz + \alpha \left\{ u\left[y(z)\right] - p\left(z\right) \right\} \right\}, \tag{4}$$

where s is the spread between the real interest rate on an illiquid asset that cannot be traded in the DM and the real rate on liquid Lucas trees,

$$s = \frac{\rho - r}{R} \geq 0. \tag{5}$$

According to (4), the consumer chooses his asset holdings in order to maximize his expected surplus from trading in the DM net of the cost of holding liquid assets measured by s. By a similar reasoning, the lifetime expected utility of a producer at the start of the DM solves

$$V^s(z) = \alpha \left\{ -v\left[y(z^b)\right] + p\left(z^b\right) \right\} + W^s(z),$$

where z^b are the consumer's assets in equilibrium. For all $s > 0$ it is weakly optimal for the producer to choose $z = 0$.

4 Gradual bargaining

We introduce gradual bargaining to determine the terms of trade in pairwise meetings. We first propose an extensive-form game and then adopt an axiomatic approach to show the robustness of the solution. Under both approaches we make the assumption that it takes time to negotiate the sale of assets, e.g., it takes time to authenticate assets to avoid fraud and counterfeiting and it also takes time to secure the transfer of ownership of the asset. We index time within the negotiation by τ. The technology to autheticate and transfer assets is such that δ units of assets can be negotiated per unit of time. Hence, the higher δ, the more negotiable the asset. In order to make the time dimension and negotiability relevant, we assume that there is a time limit, $\bar{\tau}$, to complete the negotiation. For now we assume that $\bar{\tau}$ is sufficiently large so that it is not a binding constraint.

4.1 The alternating-ultimatum-offers bargaining game

We start by considering an extensive-form game between a consumer (i.e., buyer of the DM good) holding $z > 0$ units of assets, expressed in terms of the numeraire, and a producer (i.e., seller of the DM good). The game has N rounds. In each round, the consumer can negotiate at most z/N units of assets for some output.[12] Agreements reached in each round are final. Each round corresponds to a two-stage ultimatum game: in the first stage an offer is made; in the second stage the offer is accepted or rejected. In order to maintain some symmetry between the two players (in particular when N is large) we assume that the identity of the proposer alternates across rounds. We assume N is odd and the consumer is the one making the first offer. These assumptions will be inconsequential when we consider the limit as N becomes large.

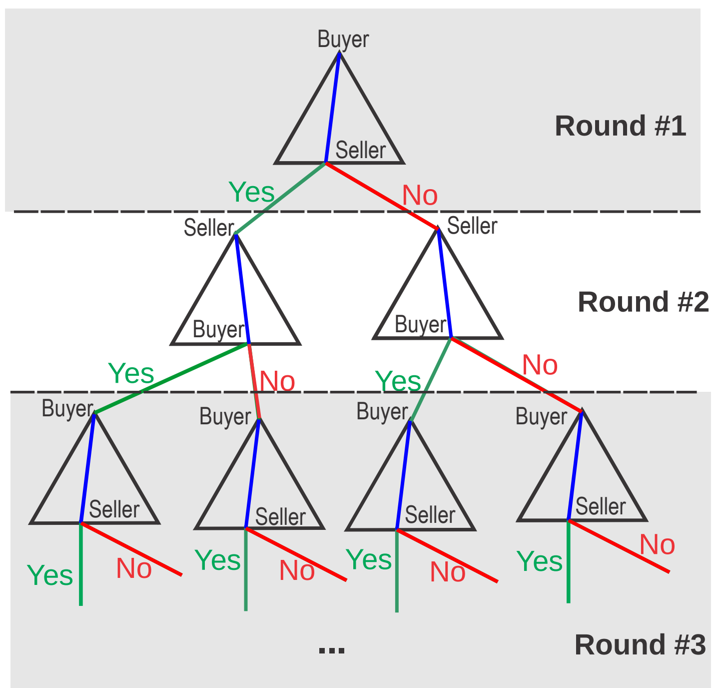

Figure 1: Game tree of the alternating ultimatum offer game

We define $\tau \equiv nz/(\delta N)$. Given that δ units of asset can be negotiated per unit of time, τ is the time until the end of the n^{th} round. The utility accumulated by the consumer up to τ is

$$u^b(\tau) = u\left[y(\tau)\right] + W^b\left[z - p(\tau)\right] = u\left[y(\tau)\right] - p(\tau) + u_0^b, \tag{6}$$

where $(y(\tau), p(\tau))$ is the intermediate agreement and $u_0^b = W^b(z)$. The utility accumulated by the producer

[12] Each round is similar to the negotiations described in earlier monetary search models by Shi (1995) and Trejos and Wright (1995) where agents would negotiate some output in exchange for an indivisible unit of money. A difference is that z/N is divisible in our analysis.

up to τ is

$$u^s(\tau) = -\upsilon\left[y(\tau)\right] + p(\tau) + u^s_0, \tag{7}$$

where $u^s_0 = W^s(0)$. Given the feasibility constraint $p(\tau) \leq \delta\tau$, we obtain a Pareto frontier for each τ. These Pareto frontiers will play a key role to characterize the subgame perfect equilibrium of our game.

Lemma 1 ***(Pareto frontiers)*** *The Pareto frontier at time τ satisfies*

$$H(u^b, u^s, \tau) = 0, \tag{8}$$

where

$$H(u^b, u^s, \tau) = \begin{cases} u(y^*) - \upsilon(y^*) - (u^b - u^b_0) - (u^s - u^s_0) & \text{if } u^s - u^s_0 \leq \delta\tau - \upsilon(y^*) \\ \delta\tau - \upsilon[u^{-1}(\delta\tau + u^b - u^b_0)] - (u^s - u^s_0) & \text{otherwise} \end{cases}. \tag{9}$$

The function H is continuously differentiable, increasing in τ (strictly so if $y < y^*$), decreasing in u^b and u^s. Consequently, each Pareto frontier has a negative slope:

$$\left.\frac{\partial u^s}{\partial u^b}\right|_{H(u^b,u^s,\tau)=0} = \begin{cases} -1 & \text{if } u^s - u^s_0 \leq \delta\tau - \upsilon(y^*) \\ -\frac{\upsilon'(y)}{u'(y)} & \text{otherwise} \end{cases}$$

The Pareto frontier is linear when $y = y^*$. When $y < y^*$, it is strictly concave. We are now in position to characterize subgame perfect equilibria of the bargaining game. We call a bargaining round an *active round* if there is equilibrium trade in that round. We say that a subgame perfect equilibrium is *simple* if in each active round the buyer offers z/N units of assets (except possibly for the last active round) and active rounds are followed by inactive rounds (if any).

Proposition 1 ***(Subgame-perfect equilibria of the alternating ultimatum offers game.)*** *There exists a subgame perfect equilibrium (SPE) in each alternating-ultimatum offer game, and all SPE share the same final payoffs. When the output level corresponding to the final payoffs is less than y^*, the SPE is unique and is simple; otherwise, there is a unique simple SPE. Moreover, in any simple SPE, the intermediate payoffs, $\{(u^b_n, u^s_n)\}_{n=1,2,\ldots,N}$, converge to the solution $(u^b(\tau), u^s(\tau))$ to the following differential equations as N approaches ∞ with $\tau = nz/N$:*

$$u^{b\prime}(\tau) = -\frac{1}{2}\frac{\partial H(u^b, u^s, \tau)/\partial\tau}{\partial H(u^b, u^s, \tau)/\partial u^b} \tag{10}$$

$$u^{s\prime}(\tau) = -\frac{1}{2}\frac{\partial H(u^b, u^s, \tau)/\partial\tau}{\partial H(u^b, u^s, \tau)/\partial u^s}. \tag{11}$$

An increase in τ by one unit expands the bargaining set by $\partial H/\partial\tau$. According to (10), the consumer enjoys half of the maximum utility gain generated by the expansion of the bargaining set. We combine (10) and (11) to obtain the slope of the gradual agreement path:

$$\frac{\partial u^s}{\partial u^b} = \frac{\partial H(u^b, u^s, \tau)/\partial u^b}{\partial H(u^b, u^s, \tau)/\partial u^s}. \tag{12}$$

According to (12), the slope of the gradual bargaining path is equal to the opposite of the slope of the Pareto frontier.[13] We represent the solution in Figure 2.

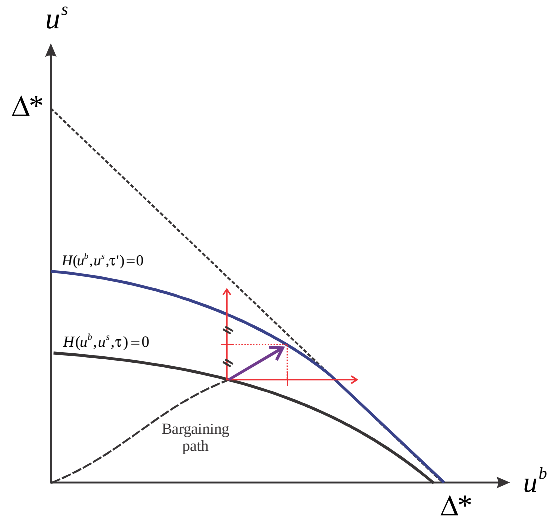

Figure 2: Solution to a gradual bargaining problem

The proof of Proposition 1 consists of two steps: first, we characterize the subgame perfect equilibrium (SPE) for any game (or subgame) with an arbitrary odd number of rounds, N. In the second part, we establish that the sequence of intermediate payoffs of the SPE converges to the solution to the system of differential equations, (10) and (11), as N approaches ∞. The logic goes as follows. Suppose the negotiation enters its last round, N, and the two agents have agreed upon some intermediate payoffs (u^b_{N-1}, u^s_{N-1}). The buyer makes the last take-it-or-leave offer, which maximizes his payoff by keeping the seller's payoff unchanged at u^s_{N-1}. Graphically, the final payoffs are constructed from the intermediate payoffs by moving horizontally from the lower Pareto frontier to which (u^b_{N-1}, u^s_{N-1}) belongs to the upper Pareto frontier corresponding to an increase in real balances of z/N, as shown in Figure 3. We now move backward in the game by one round. Suppose that the negotiation enters round $N-1$ with some intermediate payoffs (u^b_{N-2}, u^s_{N-2}), with the seller making the offer. The offer makes the buyer indifferent between accepting it and rejecting it. Now, if the buyer rejects the seller's offer, the negotiation enters its last round and the buyer's payoff is obtained as before, i.e., by moving horizontally from the lower frontier to the upper frontier. This determines the

[13] Another geometric interpretation of the solution is that the direction of the agreement path is orthogonal to the flipped gradient.

buyer's payoff. Given this payoff, the seller's payoff is obtained such that the pair of payoffs is located on the last Pareto frontier. Graphically, there is first a horizontal move from the initial payoff, (u^b_{N-2}, u^s_{N-2}), to the next Pareto frontier that determines the buyer's terminal payoff, (u^b_{N-1}, u^s_{N-2}), and then a vertical move to the following frontier that determines the seller's payoff, (u^b_{N-1}, u^s_N), as shown in Figure 3. We can iterate this procedure until we reach the start of the game. In order to pin down the terminal payoffs we need a starting point. We use the fact that the negotiation starts with initial payoffs (u^b_0, u^s_0). The sequence of payoffs is then obtained by alternating horizontal and vertical moves across consecutive frontiers.

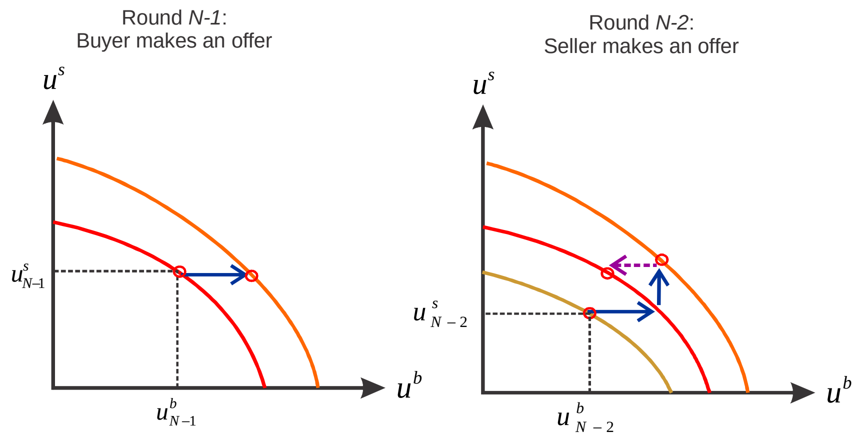

Figure 3: Left panel: Offer in last round; Right panel: offer in $(N-1)^{\text{th}}$ round

Once we have the terminal payoffs, we use another backward induction to determine the sequence of intermediate payoffs. The intermediate payoffs on the $(N-1)^{\text{th}}$ frontier are obtained by moving horizontally from the N^{th} frontier to the $(N-1)^{\text{th}}$ frontier since the buyer is making the last offer. The intermediate payoffs on $(N-2)^{\text{th}}$ frontier are obtained by moving vertically from the N^{th} frontier to the $(N-1)^{\text{th}}$ frontier and then horizontally from the $(N-1)^{\text{th}}$ frontier to the $(N-2)^{\text{th}}$ frontier by using the same reasoning as above. It turns out that the two sequences constructed above get closer to one another as N becomes large, and, both converge to the gradual bargaining path according to (12).

A feature of our game is that if an offer is rejected, the z/N of assets that are unsold cannot be renegotiated. The solution to our game, however, is robust to this feature. In the appendix we study a variant of the game where agents have $\bar{\tau}$ units of time to negotiate, where $\bar{\tau}$ can be larger than z/δ, the time required to sell the whole portfolio. As long as the whole portfolio has not be sold and the time limit has not been reached, agents can keep on negotiating. The SPE payoffs of this game solve (10)-(11).

4.2 An axiomatic approach

One might wonder how the solution to our extensive game depends on the details of the bargaining protocol, e.g., the ultimatum game in each round. An axiomatic approach, by abstracting from the details of the bargaining game, provides a sense of the robustness of our solution. O'Neill et al. (2004) developed an axiomatic approach that extends Nash (1953) to formalize negotiation that take place gradually over time. A gradual bargaining problem admits as a primitive a family of feasible sets indexed by the different items that are up for negotiation at a given point time in the negotiation.[14] Formally, in the context of our model:

Definition 1 *A gradual bargaining problem between a consumer holding* z *units of asset and a producer is a collection of Pareto frontiers,* $\langle H(u^b, u^s, \tau) = 0, \tau \in [0, z/\delta] \rangle$ *and a pair of disagreement points,* (u_0^b, u_0^s)*.*

A gradual agreement path is a function, $o : [0, z] \to \mathbb{R}_+ \times [0, z]$, that specifies an allocation (y, p) for all $\tau \in [0, z/\delta]$ and associated utility levels, $\langle u^b(\tau), u^s(\tau) \rangle$. The gradual Nash solution of O'Neill et al. (2004) is the unique solution to satisfy five axioms: Pareto optimality, covariance with respect to positive linear transformations of utility, symmetry, directional continuity, and time-consistency. The first three axioms are axioms imposed by Nash (1950).[15] The last two axioms are specific to the new definition of the bargaining problem. Directional continuity imposes a notion of continuity for the bargaining path with respect to changes in the agenda. More importantly, the key addition is the requirement of time-consistency according to which if the negotiation were to start at time τ with that agreement being the disagreement point, then the bargaining path going onward would be the same as the one obtained starting at $\tau = 0$. The key theorem of O'Neill et al. is the following:

Theorem 1 ***(Ordinal solution of O'Neill et al., 2004)*** *There is a unique solution to the gradual bargaining problem given by* $\langle H(u^b, u^s, \tau) = 0, \tau \in [0, z/\delta] \rangle$ *and it satisfies (10)-(11).*

It follows from this theorem that the solution to the alternating ultimatum offers bargaining game coincides with the axiomatic solution from O'Neill et al. (2004). Finally, it is worth noticing that while scale invariance was imposed as an axiom, the solution exhibits ordinality endogenously: the solution is covariant with respect to any order-preserving transformation.[16]

[14] In contrast Nash (1950) defines a bargaining problem as a single set of utility levels for the two parties and a pair of disagreement points.

[15] This axiomatization does not require Nash's fourth and more controversial axiom, independence of irrelevant alternatives.

[16] This result is noteworthy because Shapley (1969) shows that in the standard Nash framework, with two players, no single-valued solution can satisfy Pareto efficiency, symmetry, and ordinality.

4.3 Negotiated price and trade size

We now turn to the implications of the gradual bargaining solution for asset prices and trade sizes. From the definition of H in (9), the solution to the bargaining game, (10)-(11), can be reexpressed as

$$u^{b\prime}(\tau) = \delta \frac{u'(y) - \upsilon'(y)}{2\upsilon'(y)} \quad (13)$$

$$u^{s\prime}(\tau) = \delta \frac{u'(y) - \upsilon'(y)}{2u'(y)}, \quad (14)$$

if $\delta\tau < u^s - u_0^s + \upsilon(y^*)$ and $u^{b\prime}(\tau) = u^{s\prime}(\tau) = 0$ otherwise. From (13) and (14) the slope of this gradual bargaining path is $\partial u^s/\partial u^b = \upsilon'(y)/u'(y)$, which is increasing in y, i.e., it becomes steeper as the negotiation progresses.

Proposition 2 ***(Prices and trade sizes)*** *Along the gradual bargaining path, the price of the asset in terms of DM consumption is*

$$\frac{y'(\tau)}{\delta} = \frac{1}{2}\left(\overbrace{\frac{1}{\upsilon'(y)}}^{bid\ price} + \overbrace{\frac{1}{u'(y)}}^{ask\ price}\right) \quad \textit{for all } y < y^*. \quad (15)$$

The overall payment for y units of consumption is

$$p(y) = \int_0^y \frac{2\upsilon'(x)u'(x)}{u'(x) + \upsilon'(x)} dx. \quad (16)$$

If $z \geq p(y^*)$ then $y = y^*$ and $y = p^{-1}(z)$ otherwise.

Equation (15) has a simple interpretation. The bid price of one unit of asset at time τ, i.e., the maximum price in terms of DM goods that the producer is willing to pay to acquire it, is equal to $1/\upsilon'(y)$. The ask price at time τ, i.e., the minimum price in terms of DM goods that the consumer is willing to accept to give it up, is $1/u'(y)$. So, according to (15), the negotiated price is the arithmetic average of the bid and ask prices. Note that the bid price decreases with y because the producer incurs a convex cost to acquire an additional unit of asset. The ask price increases with y because the consumer enjoys a decreasing marginal utility in exchange of an additional unit of asset. So the negotiated price can be non-monotone with the size of the trade. A natural case is when the cost of the seller is linear, $\upsilon'(y) = 1$, e.g., think of the buyer of the asset as a large dealer. In this case the bid price is constant and equal to one, and the negotiated price is $y'(\tau)/\delta = \left(1 + [u'(y)]^{-1}\right)/2$. It increases with the quantities of assets sold. This result captures the idea that larger trades are more expensive. From (16) we can compute the buyer's surplus from a trade:

$$u(y) - p(y) = \int_0^y \frac{u'(x)\left[u'(x) - \upsilon'(x)\right]}{u'(x) + \upsilon'(x)} dx, \quad \text{for all } y \leq y^*.$$

The surplus increases with y, is strictly concave for all $y < y^*$, and is maximum when $y = y^*$.

4.4 Asymmetric gradual bargaining

The gradual bargaining solution presented so far treats the two players symmetrically. For several applications, however, it is useful to allow for asymmetric bargaining powers. In the following we modify the strategic game to provide a noncooperative foundation for asymmetric bargaining powers. In each round where the consumer is making the offer, the amount of assets that can be negotiated is now $2\theta z/N$ where $\theta \in [0, 1]$. In each round where the producer is making the offer, the amount of assets up for negotiation is $2(1-\theta)z/N$. Note that $\theta = 1/2$ corresponds to the bargaining game studied earlier. See the Appendix B for details. The solution to this bargaining game generalizes (10)-(11) as follows:

$$u^{b\prime}(\tau) = -\theta \frac{\partial H(u^b, u^s, \tau)/\partial \tau}{\partial H(u^b, u^s, \tau)/\partial u^b} \tag{17}$$

$$u^{s\prime}(\tau) = -(1-\theta) \frac{\partial H(u^b, u^s, \tau)/\partial \tau}{\partial H(u^b, u^s, \tau)/\partial u^s}, \tag{18}$$

where $\theta \in [0, 1]$ is interpreted as the buyer's bargaining power.[17] This solution coincides with the axiomatic solution of Wiener and Winter (1998).

By the same reasoning as above, the DM price of assets evolves according to

$$\frac{y'(\tau)}{\delta} = \left(\theta \overbrace{\frac{1}{v'(y)}}^{\text{bid price}} + (1-\theta) \overbrace{\frac{1}{u'(y)}}^{\text{ask price}} \right). \tag{19}$$

It is now a weighted average of the bid and ask prices where the weights are given by the relative bargaining powers of the consumer and the producer. From (19) the DM price of the asset is increasing in θ. The payment for y units of DM consumption is

$$p(y) = \int_0^y \frac{u'(x)\, v'(x)}{\theta u'(x) + (1-\theta) v'(x)} dx \quad \text{for all } y \leq y^*. \tag{20}$$

5 More on the agenda

The agenda of a negotiation specifies how much of each asset to put up for negotiation at different stages. The literature has implicitly assumed that portfolios were sold all at once, e.g., according to the Nash solution. In contrast, we described a negotiation where assets were sold gradually. In the following we compare the outcomes of the two agendas, including players' payoffs. In a second part, we set up an alternative agenda under which agents negotiate gradually over the DM good (instead of the liquid assets). We conclude by letting one player pick the agenda.

[17] One could make the bargaining power a function of time, τ, or output traded, y, without affecting the results significantly.

5.1 Bundled vs gradual asset sales

We generalize the extensive-form game to allow for alternating offers in each round in accordance with Rubinstein's (1982). We will show that our game admits the Nash solution in the limiting case where there is a single round and the gradual solution in the limiting case where the number of rounds becomes infinite. As before the extensive-form game has N rounds. Each round, $n \in \{1, ..., N\}$, is composed of an infinite number of stages during which the two players bargain over z/N units of assets following an alternating-offer protocol as in Rubinstein (1982). The buyer is the first proposer if n is odd, and the seller is the first proposer otherwise. The round-game is as follows. In the initial stage, the first proposer makes an offer and the other agent either accepts it or rejects it. If the offer is accepted, round n ends and agents move to round $n+1$. If the offer is rejected then there are two cases. With probability $(1-\xi)$ round n is terminated and the player moves to round $n+1$ without having reached an agreement. With probability ξ the negotiation continues and the responder becomes the proposer in the following stage. We will consider the limit where ξ approaches one in each round.

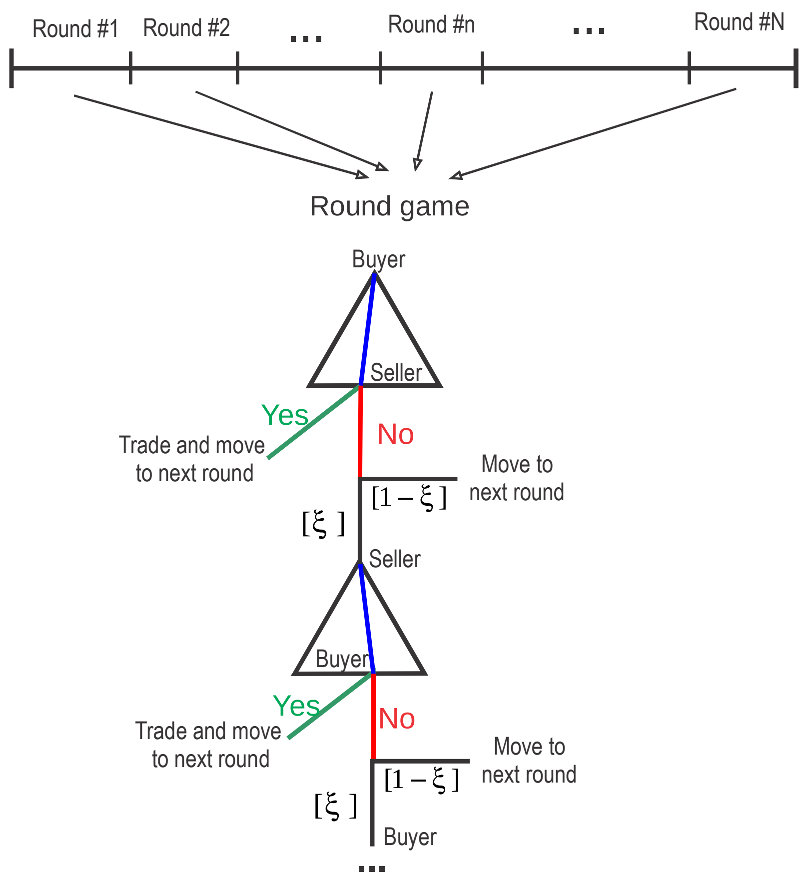

Figure 4: Game tree with alternating offers in each round

Proposition 3 ***(Repeated Rubinsten game.)*** *There exists a SPE of the repeated Rubinstein game characterized by a sequence of intermediate allocations, $\{(y_n, p_n)\}_{n=0}^N$, solution to:*

$$(y_n, p_n) \in \arg\max_{y,p} [u(y) - p - u(y_{n-1}) + p_{n-1}] [-\upsilon(y) + p + \upsilon(y_{n-1}) - p_{n-1}] \quad s.t. \quad p \leq \frac{nz}{N}, \tag{21}$$

for all $n \in \{1, ..., N\}$ with $(y_0, p_0) = (0, 0)$. As $N \to \infty$ the solution converges to the solution of the alternating ultimatum offer game.

In each round the intermediate payoffs coincide with the Nash solution where the endogenous disagreement points are given by the intermediate payoffs of the previous round. From (21) $\{(y_n, p_n)\}_{n=0}^N$ is the solution to

$$\int_{y_{n-1}}^{y_n} \frac{\upsilon'(y_n)u'(x) + u'(y_n)\upsilon'(x)}{u'(y_n) + \upsilon'(y_n)} dx \leq \frac{z}{N} \quad " = " \text{ if } y_n < y^*, \tag{22}$$
$$p_n - p_{n-1} = \min\left\{ \frac{[u(y^*) - u(y_{n-1})] + [\upsilon(y^*) - \upsilon(y_{n-1})]}{2}, \frac{z}{N} \right\},$$

with $y_0 = 0$. When the liquidity constraint, $p_n \leq nz/N$, binds, then the payment is equal to the weighted sum of the marginal utility of consumption and the marginal cost of production going from y_{n-1} to y_n.

Proposition 4 *Consumers obtain a higher surplus by negotiating the sale of their assets gradually over time, $N = +\infty$, instead of bundling assets across a finite number of rounds, $N < +\infty$.*

Proof. (Complement of proof) Summing (22) from $n = 1$ to N:

$$\sum_{n=1}^N \left[\int_{y_{n-1}}^{y_n} \frac{\upsilon'(y_n)}{u'(y_n) + \upsilon'(y_n)} u'(x)dx + \int_{y_{n-1}}^{y_n} \frac{u'(y_n)}{u'(y_n) + \upsilon'(y_n)} \upsilon'(x)dx \right] = z.$$

It can be expressed more compactly as

$$\int_0^{y_N} \left[1 - \Theta\left(x; \frac{z}{N}\right)\right] u'(x) + \Theta\left(x; \frac{z}{N}\right) \upsilon'(x)dx = z,$$

where

$$\Theta\left(x; \frac{z}{N}\right) = \sum_{n=1}^N \frac{u'(y_n)}{u'(y_n) + \upsilon'(y_n)} 1_{(y_{n-1}, y_n]}(x).$$

Note that for all $N < +\infty$ and for all $x \notin \{y_n\}$,

$$\Theta\left(x; \frac{z}{N}\right) < \frac{u'(x)}{u'(x) + \upsilon'(x)}.$$

Hence,

$$\int_0^{y_N} \left[1 - \Theta\left(x; \frac{z}{N}\right)\right] u'(x) + \Theta\left(x; \frac{z}{N}\right) \upsilon'(x)dx > \int_0^{y_N} \frac{2\upsilon'(x)u'(x)}{u'(x) + \upsilon'(x)} dx.$$

So for all $N < +\infty$, the payment to finance y_N units of consumption, the left side of the inequality, is larger than the one when $N = +\infty$, the right side of the inequality. Hence, the consumer extracts the largest surplus when $N = +\infty$. ■

In order to illustrate Proposition 4, consider the two limiting cases, $N = 1$ and $N = +\infty$. If $N = 1$ the SPE outcome corresponds to the symmetric Nash solution, in which case $z = p_1(y)$ where

$$p_1(y) \equiv \frac{\upsilon'(y)u(y) + u'(y)\upsilon(y)}{u'(y) + \upsilon'(y)}.$$

Relative to the gradual solution where $N = +\infty$ the consumer must pay an additional

$$p_1(y) - p_\infty(y) = \int_0^y \left[\frac{\upsilon'(y)}{u'(y) + \upsilon'(y)} - \frac{\upsilon'(x)}{u'(x) + \upsilon'(x)}\right] \left[u'(x) - \upsilon'(x)\right] dx.$$

The difference arises from the fact that under Nash bargaining the seller's share in each increment of the match surplus is $\upsilon'(y)/\left[u'(y) + \upsilon'(y)\right]$, which is larger than the share they get under gradual bargaining, $\upsilon'(x)/\left[u'(x) + \upsilon'(x)\right]$ for all $x < y$. Selling all the assets at once has a negative impact on the price that can be mitigated by selling them through small quantities.[18]

5.2 Gradual bargaining over DM goods

So far we described an agenda according to which agents add assets on the negotiation table gradually over time. Alternatively, suppose that agents add DM output on the negotiation table gradually over time and bargain over the price of each unit. This agenda is still consistent with gradual bargaining over assets if y is interpreted as an (illiquid) asset traded over-the-counter, as in Duffie et al. (2005) and Lagos and Rocheteau (2009). In that case each Pareto frontier in the definition of the gradual bargaining problem is indexed by the amount of DM good, $\bar{y}$, that is up for negotiation at a given point in time. With no loss in generality we normalize $u_0^b = u_0^s = 0$.

Lemma 2 *Assume agents are bargaining gradually over the DM good. For a given asset holding z, the bargaining problem is a collection of Pareto frontiers, $\left\langle H(u^b, u^s, \bar{y}) = 0,\ \bar{y} \in [0, y^*]\right\rangle$ where:*

$$H(u^b, u^s, \bar{y}) = \begin{cases} u(\bar{y}) - \upsilon(\bar{y}) - u^b - u^s & \text{if } u^s \leq z - \upsilon(\bar{y}) \\ z - \upsilon \circ u^{-1}\left(u^b + z\right) - u^s & \text{otherwise} \end{cases}, \tag{23}$$

for all $u^s \leq \min\left\{u(\bar{y}) - \upsilon(\bar{y}), z - \upsilon \circ u^{-1}(z)\right\}$.

As long as the DM output to be negotiated is sufficiently small relative to the consumer's real balances, $z \geq u(\bar{y})$, then the Pareto frontier is entirely linear. It is Pareto optimal to trade $\bar{y} \leq y^*$ and the real balances

[18] We compare the two solutions taking into account risk of termination when bargaining gradually. Once can generalize our result that showing that there exists $\bar{\delta} \in (0, +\infty)$ such that for all $\delta > \bar{\delta}$ buyers prefer gradual bargaining to all-at-once bargaining.

are used to split the surplus. In contrast, if $z < u(\bar{y})$ then the payment constraint binds if the seller receives a sufficiently large surplus. In that case the Pareto frontier is strictly concave.

The alternative ultimatum offer game associated with this agenda is analogous to the one described earlier. It is composed of N rounds with two stages each. In the first stage an offer is made; in the second stage the offer is accepted or rejected. The producer can now transfer at most y^*/N units of DM goods for some liquid assets in each round. The transfer of liquid asset is also subject to a feasibility constraint according to which the consumer cannot transfer more liquid asset than what he holds in a given round (taking into account the assets spent in earlier rounds). So the game ends when either the N^{th} round has been reached or the liquid assets of the consumer have been depleted. The identity of the proposer (the consumer or the producer) alternates across rounds.

We now apply the gradual solution to this bargaining problem.

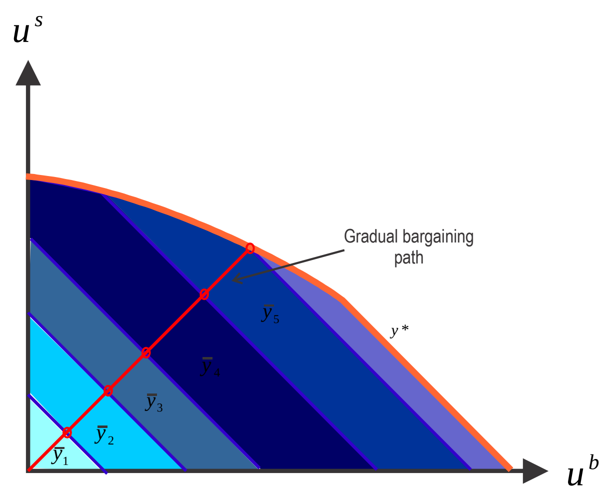

Figure 5: Bargaining gradually over output

Proposition 5 ***(Gradual bargaining over DM output)*** *Suppose agents bargain gradually over the DM output. The payment function is*

$$p(y) = \frac{1}{2}\left[u(y) + \upsilon(y)\right].$$

The outcome of the bargaining is given by y that solves $p(y) = \min\{z, p(y^*)\}$.

Proposition 5 shows that the payment made by the consumer is the arithmetic mean of the utility of the consumer and the cost of the producer. As a result, the surplus is shared equally between the consumer and the producer and the gradual bargaining path is linear. Equivalently, the gradual bargaining solution coincides with the egalitarian solution. The proportional solution has been used extensively in the monetary literature since Aruoba et al. (2007) because of its tractability and strong monotonicity property. However, two types of criticisms have been formulated against the proportional solution. First, it is not scale invariant. Second, it does not have strategic foundations in terms of an extensive form game. Proposition 5 shows that these two criticisms are unwarranted since our solution is ordinal and has strategic foundations in terms of an alternating offers game.[19]

We now endogenize the agenda by adding a stage prior to the negotiation where one of the players is picked at random to choose whether to bargain gradually over the DM good or the asset. For simplicity, we assume that there is no constraint on the horizon of the negotiation.

Proposition 6 ***(Endogenous agenda).*** *Suppose that either the consumer or the producer of the DM good has to choose the agenda of the negotiation. The consumer chooses to bargain gradually over the asset while the producer chooses to bargain gradually over the DM good.*

If we let the asset owner (the consumer) decide the agenda of the negotiation, then he will decide to bargain gradually over his asset holdings, one unit of asset at a time. In contrast, the producer would prefer to bargain gradually over the DM good. In both cases, the agent choosing the agenda prefers to negotiate gradually the asset or good he has to offer.

6 Asset prices and negotiability

We now move to the general equilibrium implications of the gradual bargaining protocol for asset prices, allocations, and welfare. We first study the pricing of Lucas trees ($d > 0$ and $\pi = 0$) in a New-Monetarist model with idiosyncratic spending opportunities (e.g., Geromichalos et al., 2007; Lagos, 2010) taking the negotiability of assets, δ, as exogenous. In the second part we endogenize negotiability by describing it as a costly investment decision.

In order for negotiability to matter, we assume that the total time for the negotiation is a random variable $\bar{\tau}$, which is exponentially distributed with mean $1/\lambda$. It is realized at the beginning of a match. It captures the idea that agents might have more or less time to negotiate the sale of their assets in order to take advantage of idiosyncratic expenditure opportunities. Our assumption is also reminiscent to the existence of a risk of breakdown in bargaining models with alternating offers (e.g., Osborne and Rubinstein, 1990). Finally, throughout the section we assume that the buyer's bargaining power is $\theta \in [0, 1]$.

[19] The strategic foundations we present in section 4.1 provide microfoundations for the egalitarian solution in that context. Dutta (2012) also proposes non-cooperative foundations for the Kalai solution, however not in the spirit of Rubinstein's alternating-offers game since players must simultaneously coordinate on an allocation.

6.1 Negotiability, asset prices, and welfare

We rewrite the portfolio problem, (4), as a choice of DM consumption, taking into account that the amount of assets a consumer can sell, $\delta\bar{\tau}$, is exponentially distributed, and the payment function, $p(y)$, is given by (20). It becomes:

$$\max_{y\geq 0}\left\{-sp(y)+\alpha\int_0^y e^{-\frac{\lambda}{\delta}p(x)}\frac{\theta u'(x)\left[u'(x)-\upsilon'(x)\right]}{\theta u'(x)+(1-\theta)\upsilon'(x)}dx\right\}. \tag{24}$$

From (24) the consumer chooses asset holdings, and hence DM output, to maximize his expected surplus from trade, net of the cost of holding liquid assets, $sp(y)$. The second term in the objective function corresponds to the consumer's expected surplus from a DM trade by holding $p(y)$ assets (its full derivation is given in the Appendix). For any $x \leq y$, with probability $e^{-\lambda p(x)/\delta}$ there is sufficient time to negotiate $p(x)$ units of assets, and the consumer can purchase x units of DM good by selling his first $p(x)$ units of asset; the terms of trade is then given by the gradual bargaining solution. The gradual bargaining solution keeps the choice of asset holdings tracatable. Indeed, the objective function is continuous and strictly concave for all $y \in (0, y^*)$.

By market clearing,

$$p(y) \leq \left(\frac{1+\rho}{\rho-s}\right)Ad, \quad " = " \ \text{ if } \ s>0, \tag{25}$$

where we have used that the cum-dividend price of the asset is $\phi+d=(1+\rho)d/(\rho-s)$. When $s>0$, buyers hold exactly $p(y)=(\phi+d)A$. If $s=0$, then from (27) $y=y^*$. The total supply of the asset, $(\phi+d)A$, is larger or equal than $p(y^*)$ since assets can also be held as a pure store of value. An equilibrium can be reduced to a pair (s, y) solution to (24) and (25). We measure social welfare as the sum of surpluses in pairwise meetings but do not take into account the output from Lucas trees, Ad:

$$\mathcal{W}=\alpha\int_0^y e^{-\frac{\lambda}{\delta}p(x)}\left[u'(x)-\upsilon'(x)\right]dx. \tag{26}$$

Proposition 7 ***(Asset prices and welfare.)*** *An equilibrium exists and is unique.*

1. *If $Ad \geq \rho p(y^*)/(1+\rho)$ then $s=0$ and y^* is implemented in a fraction $e^{-\frac{\lambda}{\delta}p(y^*)}$ of all matches. Social welfare is independent of Ad but it increases with δ and decreases with λ.*

2. *If $Ad < \rho p(y^*)/(1+\rho)$ then*

$$s=\alpha\theta e^{-\frac{\lambda}{\delta}p(y)}\ell(y)>0, \tag{27}$$

where $\ell(y)=u'(y)/\upsilon'(y)-1$, and y^* is never implemented. The asset spread, s, decreases with Ad and λ but increases with δ. Social welfare increases with Ad and δ but decreases with λ.

3. *Suppose $\lambda = 0$ and $\theta = 1/2$. If $Ad \geq \rho p(y^*)/(1+\rho)$ then equilibrium under gradual bargaining implements the first best. In contrast, the equilibrium under Nash bargaining never implements the first best, i.e., $y < y^*$ for all $A > 0$.*

Proposition 7 identifies two regimes. In the first regime consumers hold enough wealth to buy y^* provided that the negotiation lasts long enough, with probability $e^{-\frac{\lambda}{\delta}p(y^*)}$. As λ decreases or δ increases then the fraction of matches where y^* is implemented increases and welfare increases. The asset price, however, is not affected by λ or δ. In the second regime consumers hold less than $p(y^*)$ and hence trades are inefficient in all matches. From (27) the interest rate spread is the product of four components: the search friction, α, the bargaining power, θ, the negotiability friction, $e^{-\frac{\lambda}{\delta}p(y)}$, and the marginal value of wealth in the DM. It decreases with A, and it increases with both α and δ.

The last part of Proposition 7 compares equilibria under symmetric Nash bargaining and equilibria under symmetric gradual Nash bargaining when $\lambda = 0$, i.e., there is enough time to negotiate the whole asset portfolio. Under gradual bargaining, if A is sufficiently large, then $y = y^*$. In contrast, under Nash bargaining, $y \leq \tilde{y} < y^*$, the equilibrium never achieves first best. Indeed, the asset is misallocated since a fraction of the asset supply is held by producers even though they have no liquidity needs while consumers are liquidity constrained. This result shows that gradual bargaining is not only desirable for asset owners to increase their surplus (Proposition 4), it is also socially desirable to avoid the misallocation of assets.

6.2 Endogenous negotiability

We now endogenize the negotiability of assets, δ, by allowing consumers to choose the speed at which their assets are negotiated and transferred.[20] Buyers choose δ when a match is formed but before $\bar{\tau}$ is realized, where $\bar{\tau}$ is exponentially distributed with mean $1/\lambda$. There is a cost, $\psi(\delta)$, associated with the speed of the transaction, where $\psi(0) = \psi'(0) = 0$, $\psi'(\delta) > 0$ and $\psi''(\delta) > 0$. We can think of it as the cost of computer power to execute a trade and transfer assets safely.

The consumer's choice of asset holdings and speed of negotiation can be written compactly as:

$$\max_{z^b,\delta} \left\{-sz^b + \alpha\left[-\psi(\delta) + S^b(z^b,\delta)\right]\right\}, \quad \text{where } S^b(z^b,\delta) = \int_0^y e^{-\lambda\frac{p(x)}{\delta}} \frac{\theta u'(x)\left[u'(x) - v'(x)\right]}{\theta u'(x) + (1-\theta)v'(x)} dx, \tag{28}$$

i.e. $S^b(z^b,\delta)$ is the expected surplus of a consumer holding z^b assets when the speed of negotiation is δ and $y = p^{-1}(z^b)$. The novelty in (28) is the first term in squared brackets that represents the cost to invest in a technology to negotiate assets at speed δ. Despite the lack of concavity of the problem we can still

[20] Blockchain technologies provide us with a topical example, allowing market participants to choose the speed at which they want to finalize transactions. For instance, in the case of crypto-currencies, sellers of currencies can choose among a menu of fees to remunerate the miners who will check their transactions and add them to the blockchain.

fully characterize its solution and in the Appendix (see Lemma 5) we show that it is generically unique. Moreover, as the cost of holding asset, s, increases consumers reduce both their asset holdings and the speed of negotiation. A reduction in search frictions raises the demand for assets and the speed of negotiation.

The following proposition shows the existence of a unique general equilibrium with endogenous negotiability, and compares the equilibrium outcome to the constrained efficient δ. The speed of negotiation is constrained efficient if it maximizes the social welfare subject to the same cost as private agents, $\psi(\delta)$, and subject to the same trading protocols in the DM and CM. It means that the pricing in the DM is given by $p(y)$ and the asset spread in the CM is a market clearing price.

Proposition 8 ***(Equilibrium with endogenous negotiability.)***

1. *There exists a steady-state equilibrium and the equilibrium spread, s, is uniquely determined. If $Ad \geq \rho p(y^*)/(1+\rho)$ then $s = 0$ and δ is maximum. If $Ad < \rho p(y^*)/(1+\rho)$ then an increase in A reduces s, but raises δ.*

2. *Asset negotiability is constrained-efficient if and only if $Ad \geq \rho p(y^*)/(1+\rho)$ and $\theta = 1$.*

The first part of the above proposition shows that an increase in A reduces the spread s, which leads to a higher δ. Intuitively, if consumers have to sell more assets, they will find it worthwhile to increase the speed at which they can negotiate those assets. The second part shows that equilibrium negotiability is constrained efficient if and only if A is abundant, so that $s = 0$, and consumers have all the bargaining power. This result is intuitive since the costly investment in asset negotiability creates a holdup problem that can only be solved by having the ones making the investment receive the whole match surplus. However, if A is low so that $s > 0$, then the investment in δ is inefficiently low even when $\theta = 1$. This inefficiency occurs because of a pecuniary externality according to which the demand for the asset, and hence its price, increases with δ. As the asset becomes more valuable, consumers' wealth increases, which relaxes their liquidity constraint. The planner understands this externality and hence chooses a δ larger than the one that consumers would choose even if they had all the bargaining power.

7 Gradual bargaining and prices in OTC markets

In order to illustrate the versatility of our approach we now reinterpret our model as one where agents, called investors, have idiosyncratic valuations for an illiquid asset that can only be traded through pairwise meetings, similar to Duffie, Garleanu, and Pedersen (2005, 2007). At the end of each period, each agent receives an equal endowment of Lucas trees, Ω, that pay off at the end of the following period. The payoff from holding ω units of trees is $\varepsilon f(\omega)$ where $\varepsilon \in \{\varepsilon_h, \varepsilon_\ell\}$ is an idiosyncratic valuation with $\varepsilon_h > \varepsilon_\ell > 0$.[21] Upon entering the DM half of the agents draw ε_h while the other half draw ε_ℓ. These Lucas trees can only be traded in an over-the-counter market, through pairwise meetings, in the DM. The efficient trade size is such that $\varepsilon_h f'(\Omega + y^*) = \varepsilon_\ell f'(\Omega - y^*)$.

In accordance with the literature on OTC markets, investors can either meet directly or they can trade through dealers. Dealers are risk-neutral agents with linear preferences for the numeraire who have access to a competitive interdealer market in the DM.[22] Upon contact with a dealer, investors can buy and sell assets at the competitive interdealer price in terms of the numeraire, q, in exchange for the payment of an intermediation fee, φ, also expressed in the numeraire.

Investors, who cannot commit, must accumulate liquid assets to pay for illiquid Lucas trees. The liquid asset takes the form of fiat money with $d = 0$ and $\pi \in (\beta - 1, \infty)$. We denote $i \equiv (1 + \rho)(1 + \pi) - 1$ as the cost of holding money. We assume that dealers can commit to deliver the assets they purchase on behalf of investors in the interdealer market.

The matching technology in the OTC market is described as follows. We denote α^u the product of the probability of drawing a high (low) valuation times the probability of being matched with a low (high) valuation investor. We denote α^d the probability of drawing a high (low) valuation times the probability of meeting a dealer.

We need to make assumptions on how agents bargain in these different meetings. For simplicity we assume that $\lambda = 0$, which corresponds to the case where the time constraint never binds. In matches between investors, we follow our approach in Section 6 and assume that agents bargain gradually over the liquid asset, here fiat money. We later compare the equilibrium outcome to the one where agents bargain gradually over the illiquid asset. In matches between a dealer and an investor, we assume that the agents bargain gradually over the asset that the investor wants to sell, i.e., money in matches with h-investors and the illiquid asset in matches with ℓ-investors. As shown in Proposition 6, this choice corresponds to each investor's preferred agenda.

Consider a match between an h-investor and an ℓ-investor. The solutions from the previous sections apply, where we define $u(y) \equiv \varepsilon_h [f(\Omega + y) - f(\Omega)]$ and $\upsilon(y) \equiv \varepsilon_\ell [f(\Omega) - f(\Omega - y)]$. It follows that the payment function for the illiquid asset is

[21] One can interpretation $f(\omega)$ as a production function and ω as physical capital, as in Nosal and Rocheteau (2011, 2017) or Wright et al. (2017). In that case ε is an idiosyncratic productivity term. One can also think of $f(\omega)$ as some reduced form utility for different services provided by the asset, e.g., liquidity and hedging services, as in Duffie, Garleanu, and Pedersen (2005) and Lagos and Rocheteau (2009).

[22] In our environment, dealers are not endowed with Lucas trees. In a version with long-lived DM assets that can also be traded in the CM Lagos and Zhang (2018) show that dealers might have incentives to hold those assets in equilibrium since they are better at trading them.

$$p^u(y) = \int_0^y \frac{2\varepsilon_\ell f'(\Omega - x)\varepsilon_h f'(\Omega + x)}{\varepsilon_h f'(\Omega + x) + \varepsilon_\ell f'(\Omega - x)} dx. \tag{29}$$

Hence, at the margin, the price of an illiquid asset is

$$p^{u\prime}(y) = 2\left(\frac{1}{\varepsilon_\ell f'(\Omega - y)} + \frac{1}{\varepsilon_h f'(\Omega + y)}\right)^{-1}.$$

The price is the harmonic mean of the marginal productivities of the buyer and the seller. At the efficient quantity, the price is the marginal productivity of both agents, $p^{u\prime}(y^*) = \varepsilon_h f'(\Omega + y^*)$.

We now turn to a match between an h-investor holding z real balances and a dealer. An allocation, (y, φ^a), specifies a quantity of assets purchased by the dealer on behalf of the investor and a payment (in real balances) equal to $qy + \varphi^a$, where $q + \varphi^a/y$ is interpreted as an average ask price, and φ^a is the intermediation fee to the dealer associated with this ask price. The allocation is subject to the feasibility constraint, $qy + \varphi^a \leq z$.[23] The surplus of the investor is $u^b = \varepsilon_h f(\Omega + y) - qy - \varphi^a - \varepsilon_h f(\Omega)$ while the dealer's profits are $u^d = \varphi^a$. Applying the gradual bargaining solution where the agenda specifies that the h-investor sells his real balances gradually over time, the marginal surplus of the buyer is (see Appendix for detailed derivations)

$$u^{b\prime}(z) = \frac{\varepsilon_h f'(\Omega + y)/q - 1}{2}, \tag{30}$$

if $y \leq \tilde{y}_q^h$ where $\varepsilon_h f'\left(\Omega + \tilde{y}_q^h\right) = q$ and $u^{b\prime}(z) = 0$ otherwise. According to (30) the increase in the buyer's surplus from an additional unit of real balances is half of the gains that the buyer would enjoy by purchasing assets in the interdealer market directly. By the definition of the buyer's payoff, $u^{b\prime}(z) = \varepsilon_h f'(\Omega + y)\partial y/\partial z - 1$. Substituting this expression into (30) and integrating, the total payment for y units of assets is

$$p^a(y) = \varphi^a(y) + qy = q\int_0^y \frac{2\varepsilon_h f'(\Omega + x)}{\varepsilon_h f'(\Omega + x) + q} dx,$$

for all $y \leq \tilde{y}_q^h$. This payment function is increasing and concave in y. Hence, the average ask price decreases with trade size and increases with the investor's valuation, ε_h.

In a match between an ℓ-investor and a dealer, an allocation, (y, φ^b), specifies the quantity y of assets purchased by the dealer in exchange for a payment $qy - \varphi^b$, where $q - \varphi^b/y$ is the average bid price and φ^b is the intermediation fee to the dealer associated with this bid price. The investor's surplus is $u^s = \varepsilon_\ell f(\Omega - y) + qy - \varphi^b - \varepsilon_\ell f(\Omega)$ and the dealer's profits are φ^b. If the ℓ-investor sells his assets gradually over time, then the total payment function is given by the egalitarian solution:

$$p^b(y) = qy - \varphi^b(y) = \frac{qy + \varepsilon_\ell f(\Omega) - \varepsilon_\ell f(\Omega - y)}{2},$$

[23] This feasibility constraint differs from the one in Lagos and Zhang (2017) where it is assumed that $qy \leq z$ and φ^a is financed with credit repaid in the CM. This formulation makes their model with linear f and Nash bargaining more tractable.

for all $y \leq \tilde{y}_q^\ell$ where $\varepsilon_\ell f'\left(\Omega - \tilde{y}_q^\ell\right) = q$. This function is increasing and convex in y. Hence, the average bid price is increasing in y. The optimal y maximizes $\varepsilon_\ell f(\Omega - y) + p^b(y)$, i.e., assuming an interior solution,

$$q = \varepsilon_\ell f'(\Omega - y^d), \tag{31}$$

where we use y^d to denote the amount of assets traded between an ℓ-investor and a dealer. In equilibrium, this will also be the amount traded between an h-investor and a dealer.

The investor's optimal choice of real balances, assuming an interior solution, satisfies a generalized version of (27) that is derived in the Appendix, i.e.,

$$i = \frac{\alpha^u}{2}\left[\frac{\varepsilon_h f'(\Omega + y^u)}{\varepsilon_\ell f'(\Omega - y^u)} - 1\right] + \frac{\alpha^d}{2}\left[\frac{\varepsilon_h f'\left(\Omega + y^d\right)}{\varepsilon_\ell f'(\Omega - y^d)} - 1\right], \tag{32}$$

where $y^u = \min\left\{y^*, (p^u)^{-1}(z)\right\}$ is the amount of asset traded in direct trades, $y^d = \min\left\{\tilde{y}_q^h, (p^a)^{-1}(z; q)\right\}$ is the amount of asset traded in intermediated trades, and we have replaced q by its expression above. The first term on the right side of (32) is the marginal benefit of real balances to the investor in direct trades. The second term is the marginal benefit in intermediated trades. An equilibrium is a list (z, y^u, y^d, q) solution to (31), (32), and the bargaining outcomes.

Consider first an OTC market without dealers, $\alpha^d = 0$.[24] The trade size is uniquely determined by (32) and it is such that $\partial y^u/\partial i < 0$. Moreover, as i approaches 0, y^u approaches y^*. The same results hold if agents bargain gradually over the illiquid asset since in that case the bargaining solution coincides with the proportional solution. However, the trade size is larger if agents bargain gradually over the liquid asset instead of the illiquid one. This is another illustration of how the agenda of the negotiation matters for allocations and welfare. If agents bargain according to Nash, then $y^u < y^*$ even when i is driven to 0. So trade volume is inefficiently low. Gradual bargaining leads to larger trade sizes and larger trade volume by allowing agents to capture some of the gains from trade that each unit of real balances generates. We summarize these results in the following proposition.

Proposition 9 ***(Gradual bargaining in OTC markets)*** *Suppose $\alpha^d = 0$.*

1. ***(Gradual bargaining over real balances)*** *If $(\varepsilon_h - \varepsilon_\ell)/(2\varepsilon_\ell) > i/\alpha$, then there exists a unique steady-state monetary equilibrium. It is such that y approaches y^* as i approaches 0.*

2. ***(Gradual bargaining over illiquid assets)*** *If $(\varepsilon_h - \varepsilon_\ell)/(\varepsilon_h + \varepsilon_\ell) > i/\alpha$, then there exists a unique steady-state monetary equilibrium. It is such that y approaches y^* as i approaches 0. The trade size, y, is lower if agents bargain gradually over the DM asset instead of bargaining gradually over real balances.*

[24] Models of OTC markets without dealers include Afonso and Lagos (2016), Hugonnier, Lester, and Weill (2017), Uslu (2017), and Wright, Xiao, and Zhu (2018).

3. ***(Nash bargaining)*** *In any steady-state monetary equilibrium,* $y < y^*$.

Consider the other polar case of a pure dealer market where all trades are intermediated, $\alpha^u = 0$.[25] From (32), the equilibrium trade size is the solution to:

$$\frac{\varepsilon_h f'\left(\Omega + y^d\right)}{\varepsilon_\ell f'(\Omega - y^d)} \leq 1 + \frac{2i}{\alpha^d}, \quad " = " \text{ if } y^d > 0. \tag{33}$$

The trade size decreases with i and increases with α^d. As i goes to 0 then y^d tends to y^*. In accordance with Proposition 9, the Friedman rule implements the first best trade size under gradual bargaining while it fails to do so under Nash bargaining. From (31) the interdealer price decreases with i because as i goes up, investors reduce their real balances, which reduces the demand for illiquid assets.

Finally, consider an economy with both $\alpha^u > 0$ and $\alpha^d > 0$. First, replacing q by its expression given by (31) into $p^a(y)$, we obtain $p^u(y) < p^a(y)$ for all $y \leq y^d$. For the same trade size, buyers pay less in direct trades than in intermediated trades. It follows that for i close to 0, investors trade the first best in direct trades, $y^u = y^*$, while they are liquidity constrained in trades with dealers, i.e., y^d solves (33). So for low interest rates, an increase in i does not affect prices and trade sizes in direct trades but it reduces trade sizes in intermediated trades.

8 Bargaining with multiple assets and endogenous agenda

In Section 7 we described an economy with two assets: an illiquid assets traded in an OTC market and a liquid asset that can be traded in both a centralized market and an OTC market. We now extend our model to have multiple liquid assets in order to investigate cross-sectional differences in asset prices. We will show that our model can generate a pecking order of payments and rate-of-return differences across assets.

There are J types of one-period lived Lucas trees indexed by $j \in \{1, ..., J\}$, where each Lucas tree born in $t-1$ pays off one unit of numeraire in the CM of t. The supply of each Lucas tree is denoted A_j and the new Lucas trees are distributed to buyers in a lump-sum fashion at the beginning of each CM. We index fiat money by $j = 0$. Since assets are negotiated gradually over time, a natural dimension to distinguish different assets is the time it takes to negotiate their sale, which includes assessing and authenticating the asset and securing the transfer of the ownership. The negotiability of asset j is δ_j.[26] We rank assets according to their negotiability, $\delta_0 \geq \delta_1 \geq \delta_2 \geq ... \geq \delta_J$. So, by assumption, fiat money is the most negotiable asset. In each

[25] This corresponds to the version of the model by Lagos and Rocheteau (2007, 2009), and Lagos and Zhang (2018).

[26] The idea that it is costly to authenticate assets has been formalized recently by Lester, Postlewaite, and Wright (2012). In their model, the recognizability cost in terms of utils is incurred before matches are formed and does not interfere with the negotiation. In our model, authentication is a time cost while agents bargain that delays an agreement.

pairwise meeting, the negotiation ends at time $\bar{\tau}$ where $\bar{\tau}$ is exponentially distributed with mean $1/\lambda$. The consumer's bargaining power is constant over time and equal to θ.[27]

We let consumers choose the agenda of the negotiation defined as the order according to which assets are sold. The amount of asset of type j up for negotiation at time τ is denoted $\omega_j(\tau)$ and total wealth up for negotiation is $\omega(\tau)$. They obey the following law of motion:

$$\begin{aligned} \omega'(\tau) &= \sum_{j=0}^{J} \omega_j'(\tau) \\ \omega_j'(\tau) &= \delta_j \sigma_j(\tau) \text{ for all } j \in \{0,1,...,J\}, \end{aligned} \tag{34}$$

where $\sigma_j(\tau) \in [0,1]$ is the fraction of time devoted to the sale of asset j at time τ and $\sum_{j=0}^{J} \sigma_j(\tau) = 1$. Moreover, feasibility implies $\sigma_j(\tau) \in [0,1]$ if $\omega_j(\tau) < a_j$ and $\sigma_j(\tau) = 0$ otherwise. In words, an agent can add asset j on the negotiating table at time τ only if he has not sold all his holdings of asset j prior to τ. Replacing δ by ω_j' in (19), the change in the consumer's consumption and the change in the overall payment over time are

$$y'(\tau) = \frac{\theta u'(y) + (1-\theta) v'(y)}{u'(y)v'(y)} \omega' \tag{35}$$

$$p'(\tau) = \omega', \tag{36}$$

if $y(\tau) < y^*$ and $y'(\tau) = p'(\tau) = 0$ otherwise.

The expected surplus of a consumer in a DM match with portfolio $\mathbf{a} = [a_j]_{j=0}^{J}$ is:

$$S(\mathbf{a}) = \int_0^{+\infty} \lambda e^{-\lambda x} \int_0^x \{u'[y(\tau)]\, y'(\tau) - p'(\tau)\}\, d\tau dx = \theta \int_0^{+\infty} e^{-\lambda \tau} \ell[y(\tau)] \omega'(\tau) d\tau. \tag{37}$$

Over a small time interval of length $d\tau$ the consumer raises his consumption by $y'(\tau)d\tau$, where consumption is valued according to the marginal utility $u'(y)$, and increases his payment by $p'(\tau)d\tau \leq \omega'(\tau)d\tau$. The negotiation ends at some random time, x, that is exponentially distributed. From (35)-(36) $y(\tau)$ and $p(\tau)$ depend on the portfolio $\mathbf{a}$ through the feasibility constraints according to which if $\omega_j(\tau) < a_j$ then $\sigma_j(\tau) = 0$. The right side of (37) is obtained by changing the order of integration in the middle term and replacing $y'(\tau)$ and $p'(\tau)$ by their expressions given by (35) and (36). It states that the consumer's surplus is the discounted sum of the marginal surpluses along the bargaining path where the discount rate is the survival rate of the negotiation, $e^{-\lambda \tau}$.

In order to characterize the optimal strategy to sell assets we denote $T_0 = 0$ and

$$T_j(\mathbf{a}) = \sum_{k=0}^{j-1} \frac{a_k}{\delta_k} \text{ for all } j \in \{1,2,...,J+1\}. \tag{38}$$

[27] One could allow θ to be a function of τ, which would not affect our results qualitatively. One could also assume that θ varies with the type of asset that is currently under negotiation. Such extension would allow our theory to encompass the explanations for rate-of-return differences across assets by Zhu and Wallace (2007) and Nosal and Rocheteau (2011).

So T_j is the time that it takes to sell the first $j-1$ most negotiable assets.

Lemma 3 ***(Pecking order)*** *For any portfolio* $\mathbf{a}$, *the optimal choice* $\boldsymbol{\sigma}^* = [\sigma_j^*]$ *is given by*

$$\sigma_j^*(\tau) = \begin{cases} 1 & \text{if } T_j < \tau \leq T_{j+1} \\ 0 & \text{otherwise} \end{cases} .$$

Lemma 3 shows that it is optimal to adopt a pecking order to sale assets.[28] Consumers start paying with money. When their money holdings are exhausted, they start selling asset 1. And so on. Hence, our theory endogenizes and generalizes cash-in-advance constraints. In a fraction $1 - e^{-\lambda T_1}$ of matches only money is used to finance consumption, where T_1 is endogenous and depends on $\mathbf{a}$. In a fraction $e^{-\lambda T_1} - e^{-\lambda T_2}$ of matches both money and type-1 Lucas trees serve as means of payments. And so on. Given this pecking order the maximized surplus of the consumer is:

$$S(\mathbf{a}) = \theta \sum_{j=0}^{J} \delta_j \int_{T_j}^{T_{j+1}} e^{-\lambda\tau} \ell[y(\tau)] d\tau. \tag{39}$$

Over the time interval $[T_j, T_{j+1}]$ agents negotiate asset j where the speed of the negotiation is given by δ_j. The asset owner gets a fraction θ of the surplus of the negotiation.

We now turn to the asset pricing implications of this pecking order. The portfolio problem in the CM is given by

$$\max_{\mathbf{a} \geq \mathbf{0}} \left\{ -\mathbf{s}\mathbf{a} + \alpha S(\mathbf{a}) \right\}, \tag{40}$$

where $\mathbf{s} = [s_j]$ is the vector of asset spreads, i.e., $s_j = (i - i_j) / (1 + i_j)$ where the nominal interest rate of asset j is i_j. For fiat money, $i_0 = 0$ and $s_0 = i$. According to (40) the consumer maximizes his expected DM surplus net of the costs of holding assets as measured by the spreads $[s_j]$. The FOCs of the maximization problem (40) are:

$$s_j = \alpha \frac{\partial S(\mathbf{a})}{\partial a_j}. \tag{41}$$

The left side of (41) is the opportunity cost of holding asset j. The right side is the probability α that the consumer receives an opportunity to spend, α, times the marginal liquidity value from holding asset j. The expression of this last term is given in the following lemma.

Lemma 4 *The marginal value of asset* j *to a consumer with portfolio* $\mathbf{a}$ *is*

$$\frac{\partial S(\mathbf{a})}{\partial a_j} = \overbrace{\theta\lambda \sum_{k=j+1}^{J} \int_{T_k}^{T_{k+1}} \frac{(\delta_j - \delta_k)}{\delta_j} e^{-\lambda\tau} \ell[y(\tau)] d\tau}^{\text{negotiability value}} + \overbrace{\theta e^{-\lambda T_{J+1}} \ell[y(T_{J+1})]}^{\text{liquidity value}}. \tag{42}$$

[28] For a pecking-order theory of payments based on informational asymmetries between consumers and producers, see Rocheteau (2011).

From (42), holding an additional unit of a_j has two benefits to the consumer. First, the consumer has more wealth, which relaxes his liquidity constraint and allows him to consume more if the negotiation is not terminated before the whole portfolio has been sold. This effect, which corresponds to the liquidity value of the asset, is captured by the last term on the right side, which is analogous to the expression for the spread in the one asset case. This term is common to all assets, and hence it cannot explain rate-of-return differences. The novelty is given by the first term according to which asset j speeds up the negotiation relative to less negotiable assets of types $j+k$. We interpret this term as the negotiability value of the asset. This term is asset specific, as it depends on δ_j, and it can potentially help explain differences in rates of return across assets.

By market clearing $a_j = A_j$ for all $j \geq 1$. Hence, an equilibrium can be reduced to a list $\left\langle a_0, \{s_j\}_{j=1}^J \right\rangle$ solution to (41). In the following proposition we measure the liquidity of an asset by its velocity or turnover defined as

$$\mathcal{V}_j \equiv \frac{\alpha \int_0^{+\infty} \lambda e^{-\lambda x} \int_0^x \omega_j^{*\prime}(\tau) 1_{\{\omega^*(\tau)<p(y^*)\}} d\tau dx}{A_j}. \tag{43}$$

The numerator corresponds to the aggregate quantity of asset j sold in pairwise meetings while the denominator is the supply of the asset.

Proposition 10 ***(The negotiability structure of asset yields.)*** *For all $\{A_j\}_{j=1}^J$ if $\delta_0 > \delta_1$ then there is a $\bar{\imath} > 0$ such that for all $i < \bar{\imath}$ there exists a unique steady-state monetary equilibrium with aggregate real balances $A_0(i) > 0$. Let $\Omega_1 = A_0(i)$ and for each $j = 2, .., J$, let $\Omega_j = A_0(i) + \sum_{k=1}^{j-1} A_k$.*

1. *If $\Omega_{j+1} < p(y^*)$ and $\delta_j > \delta_{j+1}$, then $s_j > s_{j+1}$. If $\Omega_{j+1} \geq p(y^*)$, then $s_{j+k} = 0$ for all $k \geq 0$.*
2. *If $\delta_j > \delta_{j+1}$ and $p(y^*) > \Omega_j$, then $\mathcal{V}_j > \mathcal{V}_{j+1}$. If $p(y^*) \leq \Omega_j$ then $\mathcal{V}_j = 0$.*
3. *As λ approaches 0, $|s_j - s_{j'}|$ approaches 0 for all $j, j' \in \{0, ..., J\}$. Asset velocity, $\mathcal{V}_j$, approaches α for all j such that $\Omega_j \leq p(y^*)$, 0 for all j such that $\Omega_j \geq p(y^*)$, and $\alpha \left[p(y^*) - \Omega_j\right]/A_j$ for j such that $p(y^*) \in (\Omega_j, \Omega_{j+1})$.*

Proposition 10 has several implications. First, fiat money is valued for low i irrespective of the supply of Lucas trees. Even if the capitalization of all Lucas trees, $\sum_{k=1}^J A_k$, is larger than liquidity needs, $p(y^*)$, money is useful because it allows agents to secure some consumption when the time horizon of the negotiation is short.

Second, even though all Lucas trees yield identical dividends, our model generates rate-of-return differences across assets. Provided that asset supplies are not too large, assets with a high negotiability will command a lower interest rate than assets with a low negotiability, i.e., $i_j < i_{j+1}$ if $\delta_j > \delta_{j+1}$. We obtain this result even though there is no informational asymmetries regarding the intrinsic values of the assets. The key components of our theory is that negotiation takes time as assets are sold gradually, and not all assets can be sold at equal speed due to technological differences to authenticate and transfer assets. Part 2 of Proposition 10 shows that assets that are more negotiable have a higher velocity, which is a consequence of the endogenous pecking order. As a result, there is a positive correlation between velocity and asset prices.

Finally, Part 3 of Proposition 10 considers the limit when the expected time horizon of the negotiation becomes arbitrarily large. If the risk that the negotiation ends before the portfolio of assets has been sold goes to zero, then the rates of return of all assets converge to the same value, i.e., there is rate of return equality. In that case the negotiability of assets, and the order according to which they are negotiated, does not affect their rates of return. The order at which assets are sold, however, matters for velocities. Indeed, only a fraction of assets are used for transactions and those assets have a maximum velocity equal to α. There is a fraction of assets that are not used for transaction so that their velocity is 0.

9 Two applications

We now propose two applications of our model with multiple assets. In the first application, we study the effects of open-market operations in equilibria where money and interest-bearing government bonds coexist. The second application considers an economy with two currencies that differ by their inflation rate and their negotiability, allowing us to break exchange rate indeterminacy.

9.1 Money and bonds

We now illustrate some novel comparative statics of our model regarding the effects of open-market operations (OMOs) on aggregate output. We consider the case where $J = 1$ with asset 1 being interpreted as short-term government bonds. We start with the case where $\bar{\tau}$ is deterministic, which will allow us to build intuition for the results, and we will return to the case where $\bar{\tau}$ is exponentially distributed later.

The buyer's portfolio problem in the CM is given by

$$\max_{(a_0,a_1)} -ia_0 - s_1 a_1 + \alpha\{u[y(a_0,a_1)] - p[y(a_0,a_1)]\},$$

where DM output is

$$y(a_0,a_1) = \begin{cases} p^{-1}\left(\delta_0\bar{\tau}\right) & \leq a_0/\delta_0 \\ p^{-1}\left[a_0\left(1-\delta_1/\delta_0\right)+\delta_1\bar{\tau}\right] \quad \text{if} \ \ \bar{\tau} & \in (a_0/\delta_0, a_0/\delta_0 + a_1/\delta_1] \\ p^{-1}\left(a_0+a_1\right) & \geq a_0/\delta_0 + a_1/\delta_1 \end{cases}.$$

While $a_1 = A_1$ by market clearing, a_0 is endogenous and depends on policy through both i and A_1. We interpret an open-market operation as a change in A_1 associated with a change of opposite sign of the money supply. Because money is neutral, only the change in A_1 is relevant (e.g., Rocheteau, Xiao, and Wright, 2018). We distinguish four regimes represented in the parameter space $(\bar{\tau}, A_1)$ in Figure 6, where y_1 satisfies $i = \alpha\theta\left[(\delta_0 - \delta_1)/\delta_0\right]\left[u'(y)/v'(y) - 1\right]$ and y_2 satisfies $i = \alpha\theta\left[u'(y)/v'(y) - 1\right]$.

In regime I, $\bar{\tau} = T_1$, the buyer holds just enough real balances to spend them all by the time the negotiation ends. In such an endogenous "cash-in-advance" regime, $y = p^{-1}(a_0)$, $i = \alpha\theta\ell(y)$, and $s_1 = 0$. In regime II, $\bar{\tau} \in (T_1, T_2)$, only a fraction of bonds can be sold before the negotiation ends. Hence, $s_1 = 0$. Output and real balances solve $y = p^{-1}\left[a_0(1 - \delta_1/\delta_0) + \delta_1\bar{\tau}\right]$ and $i = (\delta_0 - \delta_1)\,\alpha\theta\ell(y)/\delta_0$. In both regimes I and II a change in A_1 has no effect on interest rates and output. In regime IV, $\bar{\tau} > T_2$, changes in A_1 are also ineffective. In such equilibria the negotiability constraint does not bind. Hence, $y = \min\left\{p^{-1}(a_0 + a_1), y^*\right\}$, $s_1 = i$, and $i_1 = 0$.

We will now focus on regime III, $T_2 = \bar{\tau}$, where the consumer's portfolio is sold in exactly $\bar{\tau}$ units of time. The following proposition describes the effects of an open-market operation and money growth on output and interest rates.

Proposition 11 ***(Coexistence of money and interest-bearing bonds and policy.)*** *A monetary equilibrium with $T_2 = \bar{\tau}$ exists if*

$$\frac{\delta_1\left[\delta_0\bar{\tau} - p(y_2)\right]}{\delta_0 - \delta_1} < A_1 < \min\left\{\frac{\delta_1\left[\delta_0\bar{\tau} - p(y_1)\right]}{\delta_0 - \delta_1}, \delta_1\bar{\tau}\right\} \quad \text{and} \quad \frac{p(y_1)}{\delta_0} < \bar{\tau} < \frac{p(y_2)}{\delta_1}.$$

Output and the interest-rate spread are determined recursively according to:

$$\begin{aligned} y &= p^{-1}\left[\delta_0\bar{\tau} - \left(\frac{\delta_0 - \delta_1}{\delta_1}\right)A_1\right] \\ s_1 &= \frac{\delta_0}{\delta_1}i - \alpha\theta\left(\frac{\delta_0 - \delta_1}{\delta_1}\right)\ell(y). \end{aligned}$$

An open-market sale of bonds raises i_1 and reduces y. An increase in the money growth rate has no effect on output. Assuming ρ and π are close to 0, money growth affects nominal interest rates according to

$$\frac{\partial i}{\partial \pi} \approx 1, \; \frac{\partial i_1}{\partial \pi} \approx \frac{\delta_1 - \delta_0}{\delta_1} < 0.$$

As A_1 increases buyers reduce a_0 so that they are still able to sell their whole portfolio in $\bar{\tau}$ units of time. But bonds take more time than money to be sold, and hence buyers' consumption decreases. Formally, $a_0/\delta_0 + A_1/\delta_1 = \bar{\tau}$ and hence $\partial a_0/\partial A_1 = -\delta_0/\delta_1 < -1$. So real balances fall by more than the increase in the bond supply. This type of equilibrium captures the view that an open market sale of bonds reduces the overall liquidity of the economy and hence it reduces aggregate output.

In this equilibrium where agents would hold more real balances if they were not constrained by $\bar{\tau}$, an increase in the money growth rate does not affect real balances. The interest rate on illiquid bonds increases

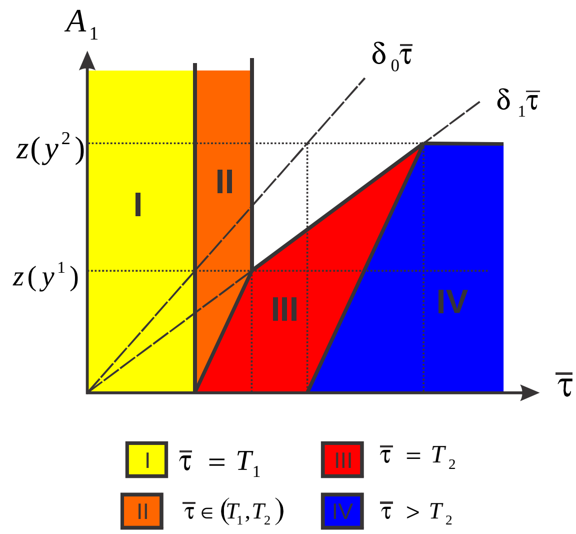

Figure 6: Typology of equilibria with money and bonds

one-to-one by the Fisher effect. Interestingly, the interest rate on liquid government bonds decreases with inflation according to the Mundell-Tobin effect.

By fixing the time horizon of the negotiation we have been able to isolate the negotiability effect of OMOs. We now present a numerical example where $\bar{\tau}$ is exponentially distributed to allow for both the negotiability and liquidity effects of OMOs.[29] The black line on Figure 7 plots y as a function of $\bar{\tau}$ when $A_1 = 0.15$. Following an increase in A_1 (from 0.15 to 0.35) consumers reduce their real balances but T_2 increases. For low values of $\bar{\tau}$, y is not affected by the change in A_1. If $\bar{\tau}$ falls into an intermediate range, then output is lower. This negotiability effect of OMOs corresponds to the gray region in Figure 7: the OMO decreases output by crowding out a highly negotiable asset, money, for a less negotiable asset, bonds. If $\bar{\tau}$ is big enough, output is higher through a liquidity effect, as visible in the blue region. The impact on aggregate output depends on the relative sizes of the negotiability and liquidity effects, which are eventually determined by the distribution of $\bar{\tau}$. In our example, the weight on the blue region is high enough for the liquidity effect to dominate, causing the expected aggregate output to increase overall.

9.2 Multiple (crypto-)currencies

Our model can be applied to economies with multiple fiat monies. This application is topical given the development of multiple cryptocurrencies, such as Bitcoins, Litecoin, Ethereum, and others. A transaction

[29] Preferences are defined by $u(y) = 2\sqrt{y}$ and $v(y) = (2/3)y^{3/2}$, thus $y^* = 1$. We pick $\alpha = 1$, $\theta = 0.5$, $\delta_0 = 2$, $\delta_1 = 1$, $i = 0.15$. $\lambda = 3.33$, so that the mean time horizon is 0.3.

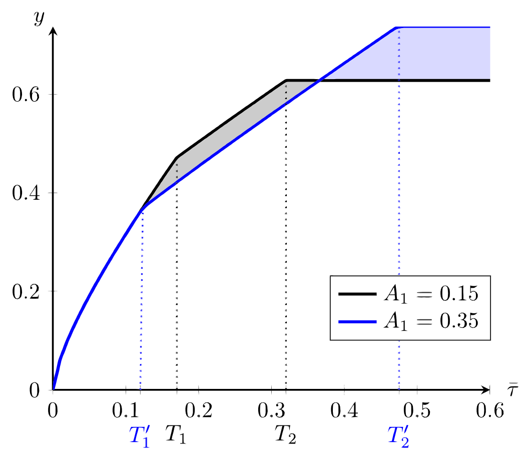

Figure 7: Stochastic case with money and bonds: Output distribution

with cryptocurrencies requires confirmation that takes time and confirmation times vary across currencies.[30] Our negotiability parameter, δ, is a proxy for the time it takes to transfer the ownership of coins.

We now consider an economy with two currencies, currency 0 with money growth rate π_0 and currency 1 with money growth rate π_1. Currency 0 has lower confirmation times and can be transferred faster than currency 1, i.e., $\delta_0 > \delta_1$. We focus on steady-state equilibria where the rate of return of each currency is constant. If $\pi_0 \leq \pi_1$ then agents will not want to hold the currency 1. Hence, we focus on the case where $\pi_0 > \pi_1$, i.e., currency 1 has a lower inflation rate than currency 0.

We start with the simple case where $\bar{\tau}$ is deterministic. For the two monies to coexist the equilibrium must feature $\bar{\tau} \geq T_2$ since otherwise one of the two currencies would have no utility as means of payment at the margin. The FOCs are

$$-i_0 - \frac{\mu}{\delta_0} + \alpha\theta\ell(y) \quad \leq \quad 0, \quad \text{“}=\text{”} \quad \text{if} \quad a_0 > 0 \tag{44}$$

$$-i_1 - \frac{\mu}{\delta_1} + \alpha\theta\ell(y) \quad \leq \quad 0, \quad \text{“}=\text{”} \quad \text{if} \quad a_1 > 0. \tag{45}$$

where $\mu \geq 0$ is the Lagrange multiplier associated with the negotiability constraint. If the negotiability constraint does not bind, $\mu = 0$, then (44)-(45) imply $i_0 = i_1$. The two currencies must have the same rate of return, which requires $\pi_0 = \pi_1$ in a steady-state equilibrium. If the negotiability constraint binds, $\mu > 0$, then the two currencies will be held only if $i_0 > i_1$. Moreover, by market clearing, the values of the two currencies solve $\phi_{0,t}A_{0,t} + \phi_{1,t}A_{1,t} = p(y)$ and the nominal exchange rate is $e_t = \phi_{0,t}/\phi_{1,t}$.

Proposition 12 ***(Dual currency economy.)***

[30] For instance, it takes on average 10 minutes with Bitcoins to receive a network confirmation. This transfer time is lowered to 2.5 minutes with Litecoins, 2 minutes for Monero, 14 seconds for Ethereum, and 3.5 second for Ripple.

1. *Suppose* $i_0 = i_1 = i$. *If* $\delta_0 \bar{\tau} > p(y)$, *where* y *solves* $u'(y)/v'(y) = 1 + i/\alpha\theta$, *then there exists a steady-state equilibrium where currencies 0 and 1 are valued. If* $\delta_1 \bar{\tau} \geq p(y)$ *then any* $e \in (0, +\infty)$ *is an equilibrium exchange rate. If* $\delta_1 \bar{\tau} < p(y)$ *then there is a positive lower bound for the exchange rate equal to*

$$\underline{e} = \frac{A_{1,t}}{A_{0,t}} \frac{\delta_0}{\delta_1} \frac{[p(y) - \delta_1 \bar{\tau}]}{[\delta_0 \bar{\tau} - p(y)]}. \tag{46}$$

2. *Suppose* $i_0 > i_1$. *There are thresholds* $0 < \bar{\tau}_0 < \bar{\tau}_1$ *such that for all* $\tau \in (\bar{\tau}_0, \bar{\tau}_1)$ *there exists a unique steady-state equilibrium where both currencies 0 and 1 are valued and output solves*

$$\frac{i_0 \delta_0 - i_1 \delta_1}{\delta_0 - \delta_1} = \alpha \theta \ell(y). \tag{47}$$

   Inflation rates affect output according to $\partial y / \partial \pi_0 < 0$ *and* $\partial y / \partial \pi_1 > 0$. *Moreover, currency 0 appreciates vis-a-vis currency 1 as* α *or* θ *increases or as* $\bar{\tau}$ *decreases.*

The first part of Proposition 12 shows that if currency 0 is sufficiently negotiable, then there exists an equilibrium where both currencies are valued. Moreover, if currency 1 is also sufficiently negotiable, then the nominal exchange rate between the two currencies, $e = \phi_0/\phi_1$, can be anything, in accordance with the indeterminacy result of Kareken and Wallace (1981). However, if the negotiability of currency 1 is limited, then the range of equilibrium values for e is reduced, i.e., there is a lower bond for the exchange rate.

The second part of Proposition 12 focuses on equilibria where $\mu > 0$ and $T_2 = \bar{\tau}$, i.e., $a_0/\delta_0 + a_1/\delta_1 = \bar{\tau}$ and $p(y) = a_0 + a_1$. The determination of a dual currency equilibrium is illustrated in Figure 8. The condition $a_0 + a_1 = p(y)$ is represented by the red line while the negotiability constraint, $a_0/\delta_0 + a_1/\delta_1 = \bar{\tau}$, is represented by the blue line. If an intersection exists, then it is unique. There exists an equilibrium where the two currencies with different inflation rates coexist provided that the time allocated to the negotiation, $\bar{\tau}$, is neither too small nor too large. If $\bar{\tau}$ is small, agents will choose to trade with the most negotiable currency only. If $\bar{\tau}$ is large, agents will choose to only hold the currency with the lowest inflation rate. For intermediate values for $\bar{\tau}$ agents choose a diversified portfolio of currencies. One can interpret such an equilibrium as one where different crypto-currencies with different technologies to record transactions coexist. One can also think of a dollarization equilibrium where the high-inflation domestic currency coexist with the low-inflation foreign currency.

If the inflation rate of the most negotiable currency increases, then output decreases. However, as i_1 increases, agents find it optimal to reduce their holdings of currency 1 and raise their holdings of currency 0. As a result, they can buy more output over the time horizon $\bar{\tau}$. In the context of a dollarization equilibrium this would mean that an increase of the inflation rate of the foreign currency raises output.

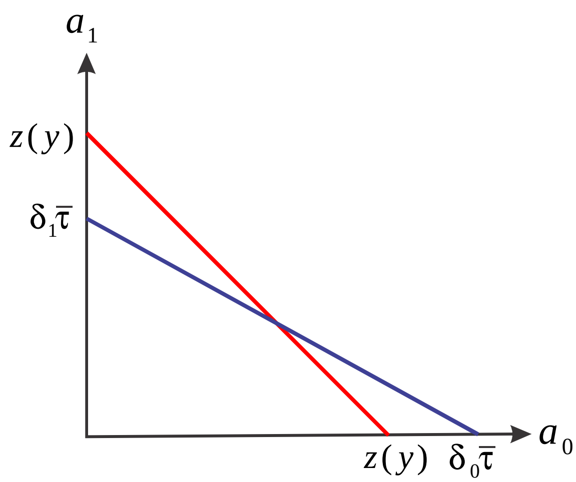

Figure 8: Equilibrium with 2 currencies

Our model provides a resolution to the Kareken-Wallace indeterminacy result.[31] In a two-currency equilibrium the nominal exchange rate is uniquely determined and given by

$$e_t = \frac{\delta_0}{\delta_1}\frac{p(y) - \delta_1\bar{\tau}}{\delta_0\bar{\tau} - p(y)}\frac{A_{1,t}}{A_{0,t}}. \tag{48}$$

It is the product of three terms: the ratio of the negotiability parameters; the ratio of the money supplies; and a middle term that captures the loss of purchasing power from using currency 1 only relative to the gain in purchasing power from using currency 0 only. As the frequency of trading opportunities or the buyer's bargaining power increase, buyers shift their portfolios toward the most negotiable currency, which leads to an appreciation of the exchange rate. Conversely, as the time to negotiate increases, agents reallocate their portfolios toward the currency with the highest rate of return, and hence the exchange rate depreciates.

10 Conclusion

The objective of this paper was to introduce a new approach to bargaining into a model of decentralized asset market with unrestricted portfolios. Following O'Neill et al. (2004) we define the bargaining problem between the owner of a portfolio of assets and a potential consumer as a collection of Pareto frontiers that expand with asset holdings. This definition captures the idea that the different items in a portfolio are sold sequentially, with each sale being final. In addition to standard axioms (Pareto efficiency, scale invariance, symmetry), the gradual bargaining solution is required to be continuous and time consistent. The solution that obeys these five axioms is characterized by a system of differential equations that can be solved in closed

[31] For information-based theories of the determinacy of the nominal exchange rate, see Zhang (2014) based on Lester et al. (2012) and Gomis-Porqueras et al. (2017) based on Li et al. (2012).

form. We show that the portfolio choice problem induced by this bargaining solution is concave, which makes the model tractable.

We showed that gradual bargaining has important positive and normative implications that distinguish it from other bargaining solutions. Relative to Nash, gradual bargaining is incentive-compatible when portfolios are private information. Moreover, gradual bargaining can implement socially-efficient outcomes while Nash cannot. Relative to Kalai bargaining, the gradual bargaining solution is not only scale invariant—as required by the axioms—but it is also ordinal. Thus, in contrast to all other solutions, the outcome is unaffected by any monotone transformation of utilities. On the positive side, gradual bargaining implements a wider range of liquidity premia, which matters for the existence of equilibria. For instance, under Kalai bargaining, monetary equilibria can break down for finite inflation rates even if the marginal utility of consumption approaches infinity when agents' real balances approach zero. This is not the case with gradual bargaining: under the Inada conditions, a monetary equilibrium exists for all inflation rates.

We extended the gradual bargaining problem to the case of portfolios composed of different assets, to allow for time-consuming technologies to negotiate the sale of assets as well as time-varying bargaining powers. We showed that the trading mechanism used by Zhu and Wallace (2007) to explain rate-of-return dominance is a special case of a gradual bargaining solution. We generalized this mechanism to arbitrary bargaining powers and investigated the implications for policy.

We proposed an alternative explanation for the rate-of-return dominance puzzle based on the idea that the time it takes to negotiate a portfolio of assets is finite and units of money are negotiated faster than units of bonds, e.g., because they are easier to authenticate. This version of the model can generate a regime where bonds pay interest and open-market operations are effective. A sale of bonds raises their interest rate and decreases aggregate output. Moreover, an increase in the money growth rate can at the same time raise the nominal rate on illiquid bonds (a Fisher effect) and lower the nominal rate on government bonds (a liquidity effect).

References

[1] Aruoba, S. Boragan, Guillaume Rocheteau, and Christopher Waller. "Bargaining and the Value of Money." Journal of Monetary Economics 54, no. 8 (2007): 2636-2655.

[2] Bac, Mehmet, and Horst Raff. "Issue-by-issue negotiations: the role of information and time preference." Games and Economic Behavior 13, no. 1 (1996): 125-134.

[3] Coles, Melvyn G., and Randall Wright. "A dynamic equilibrium model of search, bargaining, and money." Journal of Economic theory 78, no. 1 (1998): 32-54.

[4] Duffie, Darrell, Nicolae Gârleanu, and Lasse Heje Pedersen. "Over-the-counter Markets." Econometrica 73, no. 6 (2005): 1815-1847.

[5] Fershtman, Chaim. "The importance of the agenda in bargaining." Games and Economic Behavior 2, no. 3 (1990): 224-238.

[6] Galenianos, Manolis, and Philipp Kircher. "A model of money with multilateral matching." Journal of Monetary Economics55, no. 6 (2008): 1054-1066.

[7] Geromichalos, Athanasios, Juan Manuel Licari, and Jose Suirez-Llede. "Monetary policy and asset prices." Review of Economic Dynamics 10, no. 4 (2007): 761-779.

[8] Gu, Chao, and Randall Wright. "Monetary mechanisms." Journal of Economic Theory 163 (2016): 644-657.

[9] Hu, Tai-Wei. "Imperfect recognizability and coexistence of money and higher-return assets." Economic Theory 53, no. 1 (2013): 111-138.

[10] Hu, Tai-wei, John Kennan, and Neil Wallace. "Coalition-proof trade and the Friedman rule in the Lagos-Wright model." Journal of Political Economy 117, no. 1 (2009): 116-137.

[11] Hu, Tai-Wei, and Guillaume Rocheteau. "On the coexistence of money and higher-return assets and its social role." Journal of Economic Theory 148, no. 6 (2013): 2520-2560.

[12] Hu, Tai-Wei, and Guillaume Rocheteau. "Monetary policy and asset prices: A mechanism design approach." Journal of Money, Credit and Banking 47, no. S2 (2015): 39-76.

[13] Jean, Kasie, Stanislav Rabinovich, and Randall Wright. "On the multiplicity of monetary equilibria: Green-Zhou meets Lagos-Wright." Journal of Economic Theory 145, no. 1 (2010): 392-401.

[14] Kalai, Ehud. "Proportional solutions to bargaining situations: interpersonal utility comparisons." Econometrica: Journal of the Econometric Society (1977): 1623-1630.

[15] Kalai, Ehud, and Meir Smorodinsky. "Other solutions to Nash's bargaining problem." Econometrica: Journal of the Econometric Society (1975): 513-518.

[16] Lagos, Ricardo. "Asset prices and liquidity in an exchange economy." Journal of Monetary Economics 57, no. 8 (2010): 913-930.

[17] Lagos, Ricardo. "Moneyspots: extraneous attributes and the coexistence of money and interest-bearing nominal bonds." Journal of Political Economy 121, no. 1 (2013): 127-185.

[18] Lagos, Ricardo, and Guillaume Rocheteau. "Inflation, output, and welfare." International Economic Review 46, no. 2 (2005): 495-522.

[19] Lagos, Ricardo, and Guillaume Rocheteau. "Liquidity in asset markets with search frictions." Econometrica 77, no. 2 (2009): 403-426.

[20] Lagos, Ricardo, Guillaume Rocheteau, and Pierre-Olivier Weill. "Crises and liquidity in over-the-counter markets." Journal of Economic Theory 146, no. 6 (2011): 2169-2205.

[21] Lagos, Ricardo, and Randall Wright. "A unified framework for monetary theory and policy analysis." Journal of political Economy 113, no. 3 (2005): 463-484.

[22] Lester, Benjamin, Andrew Postlewaite, and Randall Wright. "Information, liquidity, asset prices, and monetary policy." The Review of Economic Studies 79, no. 3 (2012): 1209-1238.

[23] Lester, Benjamin, Ludo Visschers, and Ronald Wolthoff. "Meeting technologies and optimal trading mechanisms in competitive search markets." Journal of Economic Theory 155 (2015): 1-15.

[24] Li, Yiting, Guillaume Rocheteau, and Pierre-Olivier Weill. "Liquidity and the threat of fraudulent assets." Journal of Political Economy 120, no. 5 (2012): 815-846.

[25] Nash Jr, John F. "The bargaining problem." Econometrica: Journal of the Econometric Society (1950): 155-162.

[26] Nosal, Ed, and Guillaume Rocheteau. "Pairwise trade, asset prices, and monetary policy." Journal of Economic Dynamics and Control 37, no. 1 (2013): 1-17.

[27] Osborne, Martin J., and Ariel Rubinstein. Bargaining and markets. Academic press, 1990.

[28] O'Neill, Barry, Dov Samet, Zvi Wiener, and Eyal Winter. "Bargaining with an agenda." Games and Economic Behavior48, no. 1 (2004): 139-153

[29] Vayanos, Dimitri, and Pierre-Olivier Weill. "A Search-based Theory of the On-the-run Phenomenon." The Journal of Finance 63, no. 3 (2008): 1361-1398.

[30] Rocheteau, Guillaume. "Payments and liquidity under adverse selection." Journal of Monetary Economics 58, no. 3 (2011): 191-205.

[31] Rocheteau, Guillaume, and Ed Nosal. Money, Payments, and Liquidity. MIT Press, 2017.

[32] Rocheteau, Guillaume, and Randall Wright. "Money in search equilibrium, in competitive equilibrium, and in competitive search equilibrium." Econometrica 73, no. 1 (2005): 175-202.

[33] Rocheteau, Guillaume, and Randall Wright. "Inflation and welfare with trading frictions," in Monetary Policy in Low Inflation Economics. Eds. Ed Nosal and David Altig. (2009).

[34] Rocheteau, Guillaume, and Randall Wright. "Liquidity and asset-market dynamics." Journal of Monetary Economics 60, no. 2 (2013): 275-294.

[35] Shi, Shouyong. "Money and prices: a model of search and bargaining." Journal of Economic Theory 67, no. 2 (1995): 467-496.

[36] Silva, Mario. "New monetarism with endogenous product variety and monopolistic competition." Journal of Economic Dynamics and Control 75, (2017): 158-18.

[37] Trejos, Alberto, and Randall Wright. "Search, bargaining, money, and prices." Journal of political Economy 103, no. 1 (1995): 118-141.

[38] Tsoy, Anton. "Over-the-counter markets with bargaining delays: the role of public information in market liquidity." (2016).

[39] Wiener, Zvi, and Eyal Winter. "Gradual bargaining." (1998).

[40] Zhu, Tao, and Neil Wallace. "Pairwise trade and coexistence of money and higher-return assets." Journal of Economic Theory 133, no. 1 (2007): 524-535.

Appendix A: Proofs of Lemmas and Propositions

Proof of Lemma 1. The Pareto frontier is derived from the program

$$u^b = \max_{y,p\leq\delta\tau}\left\{u(y) - p + u_0^b\right\} \quad \text{s.t. } p - \upsilon(y) + u_0^s \geq u^s.$$

The consumer chooses the terms of trade, (y,p), to maximize his utility subject the constraint that he must guarantee some utility level u^s to the producer. If $\delta\tau \geq u^s - u_0^s + \upsilon(y^*)$, then $y = y^*$ and $p = u^s - u_0^s + \upsilon(y^*)$. Moreover, $u^b + u^s = u(y^*) - \upsilon(y^*) + u_0^b + u_0^s$. If $\delta\tau < u^s - u_0^s + \upsilon(y^*)$, then $p = \delta\tau = u^s - u_0^s + \upsilon(y)$, i.e., $y = \upsilon^{-1}(\delta\tau - u^s + u_0^s)$. ■

Proof of Proposition 2. By the definition of the consumer's utility, $u^b(\tau) = u_0^b + u\left[y(\tau)\right] - \delta\tau$, it follows that

$$u^{b\prime}(\tau) = u'\left(y\right)\frac{\partial y}{\partial\tau} - \delta. \tag{49}$$

The change in the consumer's utility along the gradual bargaining path is determined by the change in DM consumption as the consumer adds assets to the negotiating table. From (13) and (49), we obtain (15). The total transfer of assets is $p(y) = \int_0^y \delta\frac{\partial\tau}{\partial x}dx$ where from (15) $\partial\tau/\partial x$ coincides with $1/y'(\tau)$ evaluated at x. ■

Proof of Proposition 4. The payment under Nash can be reexpressed as:

$$\begin{aligned}
\tilde{p}(y) &= \frac{\upsilon'(y)}{u'(y)+\upsilon'(y)}\int_0^y u'(x)dx + \frac{u'(y)}{u'(y)+\upsilon'(y)}\int_0^y \upsilon'(x)dx \\
&= \int_0^y \frac{\upsilon'(y)u'(x) + u'(y)\upsilon'(x)}{u'(y)+\upsilon'(y)}dx \\
&= \int_0^y \frac{\upsilon'(y)}{u'(y)+\upsilon'(y)}\left[u'(x) - \upsilon'(x)\right] + \upsilon'(x)dx
\end{aligned}$$

Using that

$$\frac{\upsilon'(x)}{u'(x)+\upsilon'(x)} < \frac{\upsilon'(y)}{u'(y)+\upsilon'(y)} \quad \forall x < y,$$

and $u'(x) - \upsilon'(x) > 0$ for all $x < y \leq y^*$, we have:

$$\begin{aligned}
\tilde{p}(y) &> \int_0^y \frac{\upsilon'(x)}{u'(x)+\upsilon'(x)}\left[u'(x) - \upsilon'(x)\right] + \upsilon'(x)dx \\
&> \int_0^y \frac{2\upsilon'(x)u'(x)}{u'(x)+\upsilon'(x)}dx = p(y).
\end{aligned}$$

■

Derivation of buyer surplus (24). The expected surplus is given by

$$
\begin{aligned}
U^b(\delta; z^b) &= \int_0^\infty \lambda e^{-\lambda\overline{\tau}}\{u[y(\min\{\overline{\tau}, z^b/\delta\})] - p[y(\min\{\overline{\tau}, z^b/\delta\})]\}d\overline{\tau} & (50)\\
&= \int_0^\infty \lambda e^{-\lambda\overline{\tau}} \int_0^{p^{-1}(\min\{\delta\overline{\tau}, z^b\})} \frac{\theta u'(x)\left[u'(x) - \upsilon'(x)\right]}{\theta u'(x) + (1-\theta)\upsilon'(x)} dx d\overline{\tau} & (51)\\
&= \int_0^\infty \int_0^{p^{-1}(\min\{\delta\overline{\tau}, z^b\})} \lambda e^{-\lambda\overline{\tau}} \frac{\theta u'(x)\left[u'(x) - \upsilon'(x)\right]}{\theta u'(x) + (1-\theta)\upsilon'(x)} dx d\overline{\tau} & (52)\\
&= \int_0^{p^{-1}(z^b)} \int_{p(x)/\delta}^\infty \lambda e^{-\lambda\overline{\tau}} \frac{\theta u'(x)\left[u'(x) - \upsilon'(x)\right]}{\theta u'(x) + (1-\theta)\upsilon'(x)} d\overline{\tau} dx & (53)\\
&= \int_0^y e^{-\frac{\lambda}{\delta}p(x)} \frac{\theta u'(x)\left[u'(x) - \upsilon'(x)\right]}{\theta u'(x) + (1-\theta)\upsilon'(x)} dx, & (54)
\end{aligned}
$$

where $y = p^{-1}(z^b)$. The derivation for social welfare follows exactly the same steps except for replacing $\frac{\theta u'(x)\left[u'(x)-\upsilon'(x)\right]}{\theta u'(x)+(1-\theta)\upsilon'(x)}$ by $[u'(x) - \upsilon'(x)]$. ■

Proof of Lemma 2. The Pareto frontier is the solution to the following problem:

$$
u^b = \max_{y,p} \{u(y) - p\} \text{ s.t. } -\upsilon(y) + p = u^s,\ p \leq z,\ y \leq \bar{y},
$$

where u^b is the buyer's surplus and u^s is the seller's surplus. The payment cannot be greater than the buyer's asset holdings and the output is not greater than the upper bound $\bar{y}$. Substitute $p = u^s + \upsilon(y)$ into the constraints and rewrite the problem as:

$$
u^b = \max_{y,p} \{u(y) - \upsilon(y) - u^s\} \text{ s.t. } y \leq \min\left\{\upsilon^{-1}\left(z - u^s\right), \bar{y}\right\}.
$$

If $u^s \leq z - \upsilon(\bar{y})$ then $y = \bar{y}$ (note that we assume $\bar{y} \leq y^*$) and the equation of the Pareto frontier is simply

$$
u^b + u^s = u(\bar{y}) - \upsilon(\bar{y}).
$$

If the payment constraint binds then $y = \upsilon^{-1}\left(z - u^s\right)$ and

$$
u^b = u \circ \upsilon^{-1}\left(z - u^s\right) - z.
$$

This gives a negative relationship between u^b and u^s since $\partial u^b/\partial u^s = -u'(y)/\upsilon'(y)$. Moreover, $\partial^2 u^b/(\partial u^s)^2 < 0$, i.e., the Pareto frontier is strictly concave. ■

Proof of Proposition 5. The gradual bargaining solution is the solution to

$$
\begin{aligned}
u^{b\prime}(\bar{y}) &= -\frac{1}{2}\frac{\partial H(u^b, u^s, \bar{y})/\partial\bar{y}}{\partial H(u^b, u^s, \bar{y})/\partial u^b} & (55)\\
u^{s\prime}(\bar{y}) &= -\frac{1}{2}\frac{\partial H(u^b, u^s, \bar{y})/\partial\bar{y}}{\partial H(u^b, u^s, \bar{y})/\partial u^s}. & (56)
\end{aligned}
$$

Note that by (23), as long as the payment constraint does not bind, $p < z$, then $u^s < z - \upsilon(\bar{y})$ and the equation of the Pareto frontier is linear, at least locally. In that case the gradual solution gives:

$$
u^{b\prime}(\bar{y}) = u^{s\prime}(\bar{y}) = \frac{1}{2}\left[u'(\bar{y}) - \upsilon'(\bar{y})\right]. \qquad (57)
$$

The change in the buyer's surplus and the change in the seller's surplus are equal to half of the increase in the match surplus from raising the output that can be negotiated. Using the definition of the buyer's surplus as $u^b = u(y) - p$ it follows that the change in the payment over the gradual bargaining path is given by:

$$u^{b\prime}(\bar{y}) = u'(\bar{y}) - \frac{\partial p}{\partial \bar{y}} = \frac{1}{2}\left[u'(\bar{y}) - \upsilon'(\bar{y})\right].$$

Hence,

$$\frac{\partial p}{\partial \bar{y}} = \frac{1}{2}\left[u'(\bar{y}) + \upsilon'(\bar{y})\right].$$

The marginal payment (the price of the output) is equal to the average of the marginal utility of consumption and the marginal cost of production. Integrating from $\bar{y} = 0$ to $\bar{y} = y$, we obtain $p(y) = \frac{1}{2}\left[u(y) + \upsilon(y)\right]$, the payment function in the proposition.

Finally we verify that the payment constraint does not bind up to z. Let $\widetilde{y} = \min\{y^*, p^{-1}(z)\}$ with $p(y) = \frac{1}{2}\left[u(y) + \upsilon(y)\right]$. If $\widetilde{y} = y^*$, then it is easy to see that the constraint never binds. Otherwise, for all $\bar{y} < \widetilde{y}$, the constraint $p(\bar{y}) \leq z$ is not binding and hence the differential equations (55)-(56) apply. ■

Proof of Proposition 6. If agents bargain gradually over the asset then the payment function is:

$$p^1(y) = \int_0^y \frac{2u'(x)\,\upsilon'(x)}{u'(x) + \upsilon'(x)} dx.$$

Using that

$$\frac{2u'(x)\,\upsilon'(x)}{u'(x) + \upsilon'(x)} = \left[\frac{\upsilon'(x)}{u'(x) + \upsilon'(x)} u'(x) + \frac{u'(x)}{u'(x) + \upsilon'(x)} \upsilon'(x)\right] < \frac{u'(x) + \upsilon'(x)}{2},$$

for all $x < y^*$ since $u'(x) > \upsilon'(x)$, we obtain the following inequality:

$$p^1(y) < \int_0^y \frac{1}{2}\left[u'(x) + \upsilon'(x)\right] dx = \frac{u(y) + \upsilon(y)}{2} = p^2(y),$$

where $p^2(y)$ is the payment function if agents bargain gradually over the DM good. We denote $y^1(z)$ as the solution to $p^1(y^1) = \min\left\{z, p^1(y^*)\right\}$ and $y^2(z)$ as the solution to $p^2(y^2) = \min\left\{z, p^2(y^*)\right\}$. Using the inequality above it follows that $y^1(z) > y^2(z)$ for all z such that $y^2(z) < y^*$. We can now compare the consumer's surpluses under the two agendas: $u(y) - z^1(y) > u(y) - z^2(y)$ and $y^1(z) \geq y^2(z)$ for all z. Using that surpluses are monotone increasing in y it follows that consumers are better off with the first agenda than the second. We now compare the producer's surpluses under the two agendas. Let $z^{1*} = z^1(y^*)$ and $z^{2*} = z^2(y^*)$. For all $z \leq z^{1*}$, $z - \upsilon\left[y^2(z)\right] > z - \upsilon\left[y^1(z)\right]$ since $y^2(z) < y^1(z)$. For all $z \in \left(z^{1*}, z^{2*}\right)$, $z - \upsilon\left[y^2(z)\right] > z^{1*} - \upsilon\left(y^*\right)$ since the bargaining solution is monotone. It follows that producers always prefer to bargain gradually over the DM good. ■

Proof of Proposition 7. For each $y \in (0, y^*]$, equation (27) gives a negative relationship between s

and y, denoted by $s = s^r(y)$, with $\lim_{y \to 0} s^r(y) = +\infty$ and $s^r(y^*)$, and s^r is strictly decreasing. Given this function, equilibrium is given by y that satisfies (25). Since the left-side of (25) is strictly increasing in y and the right-side is strictly increasing in s and hence strictly decreasing in y with $s = s^r(y)$, and since the right-side of (25) is positive at $y = 0$, there is unique y that satisfies (25).

(1) Since $p(y^*) \le (1+\rho)Ad/\rho$ and $s^r(y^*) = 0$, $y = y^*$ is the unique equilibrium. In this equilibrium, the time it takes to sell $p(y^*)$ units of wealth is $\tau^* = p(y^*)/\delta$ and the probability that $\bar{\tau} \ge \tau^*$ is $e^{-\frac{\lambda}{\delta}p(y^*)}$. From (26), social welfare is

$$\mathcal{W} = \alpha \int_0^{y^*} e^{-\frac{\lambda}{\delta}p(x)} \left[u'(x) - v'(x)\right] dx,$$

which is independent of Ad but decreasing with λ/δ.

(2) Since $p(y^*) > (1+\rho)Ad/\rho$, the unique equilibrium features $y < y^*$ and $s > 0$. From (27) and (25) the spread is the unique $s \in (0, \rho)$ solution to

$$s = \alpha\theta e^{-\frac{\lambda}{\delta}\left(\frac{1+\rho}{\rho-s}\right)Ad} L\left[\left(\frac{1+\rho}{\rho-s}\right)Ad\right].$$

The right side is decreasing in Ad and λ/δ. Hence, s decreases with Ad and λ but increases with δ. From (27) y is a decreasing function of s, hence y increases with Ad and, from (26), social welfare increases with Ad. Similarly, y decreases with λ/δ and hence $\mathcal{W}$ decreases with λ/δ.

(3) From (1), when $p(y^*) \le (1+\rho)Ad/\rho$, $s^r(y^*) = 0$, $y = y^*$. Thus, as λ approaches zero, the probability that $\bar{\tau} \ge \tau^*$ approaches 1, and hence the social welfare approaches the first-best. ■

Before the proof of Proposition 8, we need the following lemma that characterize optimal speed of trade and the asset holding.

Lemma 5 *For each $s \ge 0$, there exists a solution, $[z^e(s), \delta^e(s)]$, to (28). It is unique for all but at most countably many s, upper-hemi continuous, non-increasing in s, and non-decreasing in α. If $s = 0$ then buyers hold at least $p(y^*)$. As s tends to infinity, $[z^e(s), \delta^e(s)]$ goes to $(0, 0)$.*

Proof. First we compute the partial derivative

$$\begin{aligned} \frac{\partial S^b(z^b,\delta)}{\partial z^b} &= e^{-\lambda \frac{z^b}{\delta}}\theta\ell(y) \ge 0, \\ \frac{\partial S^b(z^b,\delta)}{\partial \delta} &= \int_0^y \frac{\lambda}{\delta^2} e^{-\lambda\frac{p(x)}{\delta}} p(x) \frac{\theta u'(x)\left[u'(x) - v'(x)\right]}{\theta u'(x) + (1-\theta)v'(x)} dx > 0, \text{ with } y = p^{-1}(z^b). \end{aligned}$$

Moreover, $\partial^2 S^b(z^b,\delta)/\partial z^b \partial\delta > 0$ if $y < y^*$, a fact that we will use later. So there are complementarities between the choice of asset holdings and the speed of negotiation. The first-order condition with respect to δ is then

$$\psi'(\delta) = \int_0^y \frac{\lambda}{\delta^2} e^{-\lambda\frac{p(x)}{\delta}} p(x) \frac{\theta u'(x)\left[u'(x) - v'(x)\right]}{\theta u'(x) + (1-\theta)v'(x)} dx. \tag{58}$$

The buyer's surplus is bounded above by $u(y^*) - v(y^*)$. Hence, it is never optimal to choose a δ larger than $\bar{\delta} = \psi^{-1}[u(y^*) - v(y^*)]$. Similarly, for all $s > 0$ it is not optimal to accumulate more than $p(y^*)$ units of assets. Hence, with no loss in generality, we restrict the maximization problem to the compact set, $[0, \bar{\delta}] \times [0, p(y^*)]$. The objective in (28) is continuous. By the Theorem of the Maximum, a solution exists and it is upper hemi-continuous in s.

To show generic uniqueness and monotonic statics, consider the buyer problem in two steps. First, for any given $z^b \in [0, p(y^*)]$, consider

$$\bar{S}(z^b) = \max_{\delta \in [0,\bar{\delta}]} \left\{ -\psi(\delta) + S^b(z^b, \delta) \right\} \tag{59}$$

The objective function has strictly increasing differences in (z^b, δ) since $\partial S^b(z^b, \delta)/\partial z^b$ is strictly increasing in δ. By Theorem 2.8.2 and 2.8.4 in Topkis (1998) $\arg\max_{\delta \in [0,\bar{\delta}]} \left\{ -\psi(\delta) + S^b(z^b, \delta) \right\}$ is increasing in $z^b < p(y^*)$ and the set of maximizers is increasing in $z^b < p(y^*)$ as well. Now, for any z^b, the corresponding optimal δ solves

$$\psi'(\delta) = \int_0^y \frac{\lambda}{\delta^2} e^{-\lambda \frac{p(x)}{\delta}} p(x) \frac{\theta u'(x)\left[u'(x) - v'(x)\right]}{\theta u'(x) + (1-\theta)v'(x)} dx,$$

with $y = p^{-1}(z^b)$. Since the right-side of the above equation is strictly increasing in $z^b < p(y^*)$, the set of maximizers has to be strictly increasing as well; indeed, if $z_1^b < z_2^b$, and δ_1 and δ_2 are the corresponding maximizers, it must be the case that $\delta_1 \neq \delta_2$ as the same δ cannot satisfy the two FOC's at the same time. Now, if $\bar{S}(z_1^b) = \bar{S}(z_2^b)$, then

$$S^b(z_1^b, \delta_2) - S^b(z_1^b, \delta_1) < \psi(\delta_2) - \psi(\delta_1) = S^b(z_2^b, \delta_2) - S^b(z_1^b, \delta_1) < S^b(z_2^b, \delta_2) - S^b(z_2^b, \delta_1),$$

but the second inequality implies that $S^b(z_1^b, \delta_1) > S^b(z_2^b, \delta_1)$, a contradiction to the fact that $S^b(z^b, \delta_1)$ is increasing in z_b. Thus, $\bar{S}(z_1^b) < \bar{S}(z_2^b)$. Moreover, since $\bar{S}(z^b)$ is strictly increasing, it is also differentiable for all but at most a countably many points.

Let $\overline{z}(s)$ be the correspondence that solves (59). Consider now two spreads, $s^1 < s^2$, with associated choices of real balances z^1 and z^2. It follows that:

$$\begin{aligned} -s^1 z^1 + \alpha \bar{S}(z^1) &\geq -s^1 z^2 + \alpha \bar{S}(z^2), \\ -s^2 z^2 + \alpha \bar{S}(z^2) &\geq -s^2 z^1 + \alpha \bar{S}(z^1). \end{aligned}$$

Rearrange these inequalities to obtain:

$$s^1\left(z^1 - z^2\right) \leq \alpha\left[\bar{S}(z^1) - \bar{S}(z^2)\right] \leq s^2(z^1 - z^2).$$

Using that $s^2 > s^1$ it follows that $z^1 \geq z^2$. To show generic uniqueness, we first prove the following. Let s be given, and suppose that $z_1 < z_2 \in \overline{z}(s)$. We claim that if $s' > s$, then $z_2 \notin \overline{z}(s')$. Suppose, by contradiction, that $z_2 \in \overline{z}(s')$. Then,

$$\begin{aligned} -sz_1 + \alpha\bar{S}(z_1) &= -sz_2 + \alpha\bar{S}(z_2), \\ -s'z_2 + \alpha\bar{S}(z_2) &\geq -s'z_1 + \alpha\bar{S}(z_1). \end{aligned}$$

It then follows that

$$-s'z_2 + \alpha\bar{S}(z_2) \geq -(s'-s)z_1 - sz_2 + \alpha\bar{S}(z_2),$$

that is,

$$(s'-s)z_1 \geq (s'-s)z_2,$$

a contradiction to $z_1 < z_2$ and $s' > s$. Similarly, if $s'' < s$, then $z_1 \notin \overline{z}(s'')$. Suppose, by contradiction, that $z_1 \in \overline{z}(s'')$. Then,

$$\begin{aligned} -sz_1 + \alpha\bar{S}(z_1) &= -sz_2 + \alpha\bar{S}(z_2), \\ -s''z_1 + \alpha\bar{S}(z_1) &\geq -s''z_2 + \alpha\bar{S}(z_2). \end{aligned}$$

It then follows that

$$-s''z_1 + \alpha\bar{S}(z_1) \geq -s''z_2 + \alpha\bar{S}(z_2) = (s-s'')z_2 - sz_1 + \alpha\bar{S}(z_1),$$

that is,

$$(s-s'')z_1 \geq (s-s'')z_2,$$

a contradiction to $z_1 < z_2$ and $s > s''$. Now, for each s, let $\widetilde{z}(s) = \max \overline{z}(s)$. Then, $\widetilde{z}(s)$ is a decreasing function, and hence has at most countably many gaps. Note that only gaps in $\widetilde{z}(s)$ corresponds to nondegenerate values of $\overline{z}(s)$. This shows that the optimum is generically unique. ■

Proof of Proposition 8. (1) Note that, since $\overline{z}(s)$ is upper hemi-continous, for each s, $\widehat{z}(s) = \lim_{s' \downarrow s} \widetilde{z}(s) \in \overline{z}(s)$, and that $\widetilde{z}(s)$ is left-continuous. Thus, if we define $Z(s) = [\widehat{z}(s), \widetilde{z}(s)]$, then $Z(s)$ is weakly decreasing, and is the minimal convex set that contains $\overline{z}(s)$. Moreover, for each $z \in Z(s)$, it corresponds to η fraction of buyers holding $\widetilde{z}(s)$ units of assets and $(1-\eta)$ holding $\widehat{z}(s)$ with

$$\eta(z,s) = \frac{z - \widehat{z}(s)}{\widetilde{z}(s) - \widehat{z}(s)}.$$

Market clearing requires then

$$\frac{1+\rho}{\rho - s} Ad \in Z(s)$$

whenever $s > 0$, and when $s = 0$, we only need

$$\frac{1+\rho}{\rho} Ad \leq p(y^*).$$

Thus, a fixed point of the following correspondence

$$\rho - \frac{(1+\rho)Ad}{Z(s)}$$

is an equilibrium. Since $Z(s)$ is upper hemi-continuous and convex-valued, Kakutani's fixed point theorem ensures that a fixed point exists.

Finally, we show that equilibrium is in fact unique. Suppose that $\widetilde{s}$

$$\frac{1+\rho}{\rho-\widetilde{s}}Ad \in Z(\widetilde{s}) = [\widehat{z}(\widetilde{s}), \widetilde{z}(\widetilde{s})].$$

Then, for any $s > \widetilde{s}$, $z \in Z(s)$, $z \leq \widehat{z}(\widetilde{s})$ but $\frac{1+\rho}{\rho-s}Ad > \frac{1+\rho}{\rho-\widetilde{s}}Ad \geq \widehat{z}(\widetilde{s})$ and hence cannot be an equilibrium. The other case is symmetric.

(2) The planner's problem solves:

$$\max_{z,\delta,\,s} \left\{ -\psi(\delta) + \int_0^y e^{-\lambda \frac{p(x)}{\delta}} \left[u'(x) - v'(x)\right] dx \right\} \tag{60}$$

$$\text{s.t. } z \in \arg\max_z \left\{-sz + \alpha S^b(z,\delta)\right\} \tag{61}$$

$$p(y) \leq \left(\frac{1+\rho}{\rho-s}\right) Ad, \text{ ``}=\text{'' if } s > 0 \tag{62}$$

According to (60) the planner maximizes the expected surplus of each match net of the negotiability cost. It is subject to (61) according to which buyers choose their asset holdings optimally taking as given the negotiability of the asset and its cost (which is omitted from the buyer's objective). From (62) the spread, s, is consistent with market clearing.

From (1), if $Ad \geq \rho p(y^*)/(1+\rho)$ then equilibrium is such that $s = 0$ and $y = y^*$ irrespective of δ. Hence, the solution to (60) is

$$\psi'(\delta) = \int_0^{y^*} \frac{\lambda}{\delta^2} p(x) e^{-\lambda \frac{p(x)}{\delta}} \left[u'(x) - v'(x)\right] dx.$$

It coincides with (58) if and only if $\theta = 1$. If $\theta < 1$ then the decentralized choice of δ is smaller than the planner's choice. For the case $Ad < \rho p(y^*)/(1+\rho)$, we proved in Proposition 7 that s increases with δ. From market clearing $p(y) = \left(\frac{1+\rho}{\rho-s}\right) Ad$, and hence y is an increasing function of s. Hence, the solution to the planner's problem is:

$$\psi'(\delta) = \int_0^{y} \frac{\lambda}{\delta^2} p(x) e^{-\lambda \frac{p(x)}{\delta}} \left[u'(x) - v'(x)\right] dx + e^{-\lambda \frac{p(y)}{\delta}} \left[u'(y) - v'(y)\right] \frac{\partial y}{\partial \delta}.$$

The second term on the right side captures the effect of an increase of negotiability on the spread and hence y. Even if $\theta = 1$ this condition does not coincide with (58). ∎

Derivation of (32). First note that the expected surplus for an ℓ-investor does not depend on his money holdings, and hence, to solve the CM money holding problem, we only need to worry about h-investors. The expected surplus for a h-investor is given by (note that $\lambda = 0$)

$$\alpha^u[\varepsilon_h f(\Omega + y^u) - f(\Omega) - p^u(y^u)] + \alpha^d[\varepsilon_h f(\Omega + y^d) - f(\Omega) - p^a(y^d)]$$
$$= \alpha^u \left[\int_0^{y^u} \frac{\varepsilon_h f'(\Omega + x)[\varepsilon_h f'(\Omega + x) - \varepsilon_\ell f'(\Omega - x)]}{\varepsilon_h f'(\Omega + x) + \varepsilon_\ell f'(\Omega - x)} dx - f(\Omega) \right] + \alpha^d \left[\int_0^{y^d} \frac{\varepsilon_h f'(\Omega + x)\left[\varepsilon_h f'(\Omega + x) - q\right]}{\varepsilon_h f'(\Omega + x) + q} dx - f(\Omega) \right],$$

where $y^u = \min\left\{y^*, (p^u)^{-1}(z)\right\}$ and $y^d = \min\left\{\tilde{y}^h_q, (p^a)^{-1}(z; q)\right\}$. Thus, the CM problem then becomes

$$\max_{z \geq 0} -iz + \alpha^u \left[\int_0^{y^u} \frac{\varepsilon_h f'(\Omega + x)[\varepsilon_h f'(\Omega + x) - \varepsilon_\ell f'(\Omega - x)]}{\varepsilon_h f'(\Omega + x) + \varepsilon_\ell f'(\Omega - x)} dx - f(\Omega) \right] + \alpha^d \left[\int_0^{y^d} \frac{\varepsilon_h f'(\Omega + x)\left[\varepsilon_h f'(\Omega + x) - q\right]}{\varepsilon_h f'(\Omega + x) + q} dx - f(\Omega) \right].$$

The FOC is given by

$$-i + \frac{\alpha^u}{2} \frac{\varepsilon_h f'(\Omega + y^u) - \varepsilon_\ell f'(\Omega - y^u)}{\varepsilon_\ell f'(\Omega - y^u)} + \frac{\alpha^d}{2} \frac{\varepsilon_h f'\left(\Omega + y^d\right) - q}{q} = 0,$$

and, in equilibrium, $q = \varepsilon_\ell f'(\Omega - y^d)$. This then simplifies to (32). ■

Proof of Proposition 9. (1) The equilibrium condition is given by (32) with $\alpha^d = 0$, which can be rewritten as

$$\frac{f'(\Omega + y)}{f'(\Omega - y)} = \frac{\varepsilon_\ell}{\varepsilon_h} \left(1 + \frac{2i}{\alpha^u}\right), \tag{63}$$

To have a solution with $y > 0$, it is necessary and sufficient that $\frac{\varepsilon_\ell}{\varepsilon_h}\left(1 + \frac{2i}{\alpha^u}\right) < 1$, that is, $(\varepsilon_h - \varepsilon_\ell)/2\varepsilon_\ell > i/\alpha^u$.

(2) First we derive the equilibrium condition as in (32). When the agents bargain over DM asset, the payment is determined by Egalitarian solution and hence

$$p^{DM}(y) = \frac{\varepsilon_h\left[f(\Omega + y) - f(\Omega)\right] + \varepsilon_\ell\left[f(\Omega) - f(\Omega - y)\right]}{2}.$$

Thus, the FOC for the consumer is given by

$$-ip^{DM\prime}(y) + \alpha^u[u'(y) - p^{DM\prime}(y)] = 0,$$

which can be rewritten as

$$\frac{f'(\Omega + y)}{f'(\Omega - y)} = \frac{\varepsilon_\ell}{\varepsilon_h} \left(\frac{i + \alpha^u}{\alpha^u - i}\right), \tag{64}$$

To have a solution with $y > 0$, it is necessary and sufficient that $\frac{\varepsilon_\ell}{\varepsilon_h}\left(\frac{i+\alpha^u}{\alpha^u - i}\right) < 1$, that is, $(\varepsilon_h - \varepsilon_\ell)/(\varepsilon_h + \varepsilon_\ell) > i/\alpha^u$. Moroever, since

$$\left(\frac{i + \alpha^u}{\alpha^u - i}\right) > \left(1 + \frac{2i}{\alpha^u}\right),$$

the y that solves (63) is larger than that that solves (64).

(3) Following Proposition 4, the payment determined by Nash solution is given by

$$p^N(y) = \frac{\varepsilon_\ell f'(\Omega - y)}{\varepsilon_h f'(\Omega + y) + \varepsilon_\ell f'(\Omega - y)} \varepsilon_h\left[f(\Omega + y) - f(\Omega)\right] + \frac{\varepsilon_\ell f'(\Omega - y)}{\varepsilon_h f'(\Omega + y) + \varepsilon_\ell f'(\Omega - y)} \varepsilon_\ell\left[f(\Omega) - f(\Omega - y)\right],$$

Hence, the payoff of the h-buyer is

$$\frac{\varepsilon_h f'(\Omega+y)\{\varepsilon_h\left[f(\Omega+y)-f(\Omega)\right]-\varepsilon_\ell\left[f(\Omega)-f(\Omega-y)\right]\}}{\varepsilon_h f'(\Omega+y)+\varepsilon_\ell f'(\Omega-y)}.$$

It is easy to check that close to y^* this surplus is decreasing. Hence, under Nash bargaining the trade size is inefficiently low for all $i \geq 0$. ■

Proof of results with intermediated trades. Here we provide the derivation of Pareto frontiers for bargaining between the buyer and the dealer under intermediated trades. First, suppose that the buyer and the dealer bargain over the real banalces. The equation of the Pareto frontier is: let $\widetilde{y}$ solve $\max\{\varepsilon_h f(\Omega+\widetilde{y})-q\widetilde{y}\}$,

$$u^b = \begin{cases} \{\varepsilon_h f(\Omega+\widetilde{y})-q\widetilde{y}\}-u^d-\varepsilon_h f(\Omega) & \text{if } q\widetilde{y}+u^d \leq z, \\ \varepsilon_h f\left(\Omega+\frac{z-u^d}{q}\right)-z-\varepsilon_h f(\Omega) & \text{otherwise.} \end{cases}$$

As before, the Pareto frontier is linear if the investor is unconstrained and strictly concave otherwise. Now, we may represent the Pareto frontier as

$$H(u^b,u^d,z) = \begin{cases} -u^b+\{\varepsilon_h f(\Omega+\widetilde{y})-q\widetilde{y}\}-u^d-\varepsilon_h f(\Omega) & \text{if } q\widetilde{y}+u^d \leq z, \\ -u^b+\varepsilon_h f\left(\Omega+\frac{z-u^d}{q}\right)-z-\varepsilon_h f(\Omega) & \text{otherwise.} \end{cases}$$

Thus, for $y < \widetilde{y}$, gradual bargaining implies

$$(u^b)'(z) = -\frac{1}{2}\frac{\partial H(u^b,u^d,z)/\partial z}{\partial H(u^b,u^d,z)/\partial u^b} = \frac{1}{2}[\varepsilon_h f'\left(\Omega+y\right)/q-1].$$

As mentioned, this implies payment function

$$p^a(y;q) = q\int_0^y \frac{2\varepsilon_h f'\left(\Omega+x\right)}{\varepsilon_h f'\left(\Omega+x\right)+q}dx.$$

In contrast, the payment function under direct trade is given by

$$p^u(y) = \int_0^y \frac{2\varepsilon_h f'\left(\Omega+x\right)\varepsilon_\ell f'\left(\Omega-x\right)}{\varepsilon_h f'\left(\Omega+x\right)+\varepsilon_\ell f'\left(\Omega-x\right)}dx.$$

Equilibrium requires

$$z \leq p^u(y^u) \text{ and } z \leq p^d(y^d;q),$$

with equality whenever $y^u < y^*$ and/or $y^d < y^*$. Now, for all $y < y^d$, $q = \varepsilon_\ell f'\left(\Omega-y^d\right) > \varepsilon_\ell f'\left(\Omega-x\right)$ for all $x \leq y$, and hence $p^u(y) < p^d(y;q)$. Thus, in equilibrium $y^u > y^d$. We show that for a range of i near 0, in equilibrium $y^u = y^*$ and y^d solves

$$\frac{f'(\Omega+y^d)}{f'(\Omega-y^d)} = \frac{\varepsilon_\ell}{\varepsilon_h}\left(1+\frac{2i}{\alpha^d}\right). \tag{65}$$

To see this, let $y^d(i)$ solve (65), and let $\bar{i}$ be such that $q(i) = \varepsilon_\ell f'[\Omega-y^d(i)]$ and

$$p^u(y^*) = p^d[y^d(\bar{i});q(\bar{i})].$$

Since at y^*, $p^u(y^*) < p^d[y^*; q(0)]$ and since $p^d[y^d(i); q(i)]$ is strictly decreasing in i, such $\bar{i}$ exists and is unique. Then, for all $i \in (0, \bar{i}]$, we can construct an equilibrium with $z = p^d[y^d(i); q(i)]$.

Derivation of (37). First note that $y(\tau)$ depends on τ wholly through $\omega(\tau)$. Indeed, if we let $x(\omega)$ solves

$$\frac{\partial x}{\partial \omega} = \delta \frac{\theta u'(x) + (1-\theta) v'(x)}{u'(x) \, v'(x)}, \tag{66}$$

then $y(\tau) = x[\omega(\tau)]$. Here we derive (37).

$$\begin{aligned}
S(\mathbf{a}) &= \int_0^{+\infty} \lambda e^{-\lambda x} \int_0^x \{u'[y(\tau)] \, y'(\tau) - p'(\tau)\} \, d\tau dx & (67)\\
&= \theta \int_0^{+\infty} \int_0^x \lambda e^{-\lambda x} \ell[y(\tau)] \omega'(\tau) d\tau dx & (68)\\
&= \theta \int_0^{+\infty} \left[\int_\tau^{+\infty} \lambda e^{-\lambda x} \ell[y(\tau)] \omega'(\tau) dx \right] d\tau & (69)\\
&= \theta \int_0^{+\infty} \left[\int_\tau^{+\infty} \lambda e^{-\lambda x} dx \right] \ell[y(\tau)] \omega'(\tau) d\tau & (70)\\
&= \theta \int_0^{+\infty} e^{-\lambda \tau} \ell[y(\tau)] \omega'(\tau) d\tau. & (71)
\end{aligned}$$

■

Proof of Lemma 3. Using the notation (66), we can rewrite (37) as:

$$\begin{aligned}
S(\mathbf{a};\sigma_j) &= \theta \int_0^{+\infty} e^{-\lambda \tau} \left[u'\{x[\omega'(\tau)]\} x'[\omega(\tau)] - 1 \right] \omega'(\tau) d\tau \\
&= \theta \left\{ \lim_{\tau \to \infty} e^{-\lambda \tau} \left[u\{x[\omega(\tau)]\} - \omega(\tau) \right] + \int_0^{+\infty} \left[u\{x[\omega(\tau)]\} - \omega(\tau) \right] \lambda e^{-\lambda \tau} d\tau \right\} \\
&= \theta \int_0^{+\infty} \left[u\{x[\omega(\tau)]\} - \omega(\tau) \right] \lambda e^{-\lambda \tau} d\tau,
\end{aligned}$$

with $\omega'(\tau) = \sum_{j=0}^J \delta_j \sigma_j(\tau)$. Note that $\lambda e^{-\lambda \tau} > 0$ for all $\tau > 0$. Now, for any list of functions $[\sigma_j(\tau)]$ such that

$$\sum_{j=0}^J \sigma_j(\tau) = 1 \text{ and } \sigma_j(\tau) = 0 \text{ if } \omega_j(\tau) = a_j \quad \text{for all } \tau \geq 0,$$

we have $\omega(\tau) \leq \omega^*(\tau)$ for all $\tau \geq 0$. Thus, noting that $u[x(\omega)] - \omega$ is strictly increasing in ω for all $\omega < p^{-1}(y^*)$ and hence

$$\int_0^{+\infty} \left[u\{x[\omega(\tau)]\} - \omega(\tau) \right] \lambda e^{-\lambda \tau} d\tau \leq \int_0^{+\infty} \left[u\{x[\omega^*(\tau)]\} - \omega^*(\tau) \right] \lambda e^{-\lambda \tau} d\tau,$$

and hence $S(\mathbf{a};\sigma_j) \leq S(\mathbf{a};\sigma_j^*)$. To obtain (39) use that $\omega'(\tau) = \sum_{j=0}^J \delta_j \sigma_j^*(\tau)$ into (37). ■

Proof of Lemma 4. Define $\Omega_j(\mathbf{a}) = \sum_{k=0}^{j-1} a_k$ for all $j = 1, ..., J+1$ with $\Omega_0(\mathbf{a}) = 0$. Note that for all $\omega \in (\Omega_j, \Omega_{j+1})$,

$$\begin{aligned}
(\omega^*)^{-1}(\omega) &= \frac{(\omega - \Omega_j)}{\delta_j} + T_j, \\
\frac{d}{d\omega}(\omega^*)^{-1}(\omega) &= \frac{1}{\delta_j}.
\end{aligned}$$

Then, we use the change of variable $\tau = (\omega^*)^{-1}(\omega)$ to rewrite (39) as

$$S(\mathbf{a}) = \theta \sum_{j=0}^{J} \int_{\Omega_j}^{\Omega_{j+1}} e^{-\lambda\left[\frac{(\omega-\Omega_j)}{\delta_j}+T_j\right]} L(\omega) d\omega.$$

Now, let $k \geq 0$ be given. Then, for $j < k$,

$$\frac{\partial}{\partial a_k} \int_{\Omega_j}^{\Omega_{j+1}} e^{-\lambda\left[\frac{(\omega-\Omega_j)}{\delta_j}+T_j\right]} L(\omega) d\omega = 0,$$

and

$$\frac{\partial}{\partial a_k} \int_{\Omega_k}^{\Omega_{k+1}} e^{-\lambda\left[\frac{(\omega-\Omega_k)}{\delta_k}+T_k\right]} L(\omega) d\omega = -e^{-\lambda T_k} L(\Omega_k),$$

and for $j > k$,

$$\begin{aligned} & \frac{\partial}{\partial a_k} \int_{\Omega_j}^{\Omega_{j+1}} e^{-\lambda\left[\frac{(\omega-\Omega_j)}{\delta_j}+T_j\right]} L(\omega) d\omega \\ = \quad & -e^{-\lambda T_j} \ell\left[z^{-1}(\Omega_j)\right] + e^{-\lambda\left[\frac{(\Omega_{j+1}-\Omega_j)}{\delta_j}+T_j\right]} L(\Omega_{j+1}) + \int_{\Omega_j}^{\Omega_{j+1}} \lambda \left[\frac{1}{\delta_j} - \frac{1}{\delta_k}\right] e^{-\lambda\left[\frac{(\omega-\Omega_j)}{\delta_j}+T_j\right]} L(\omega) d\omega \\ = \quad & -e^{-\lambda T_j} \ell\left[z^{-1}(\Omega_j)\right] + e^{-\lambda T_{j+1}} L(\Omega_{j+1}) + \int_{\Omega_j}^{\Omega_{j+1}} \lambda \left[\frac{1}{\delta_j} - \frac{1}{\delta_k}\right] e^{-\lambda\left[\frac{(\omega-\Omega_j)}{\delta_j}+T_j\right]} L(\omega) d\omega. \end{aligned}$$

Thus,

$$\frac{\partial}{\partial a_k} S(\mathbf{a}) = \theta \sum_{j=k+1}^{J} \int_{\Omega_j}^{\Omega_{j+1}} \lambda \left[\frac{1}{\delta_j} - \frac{1}{\delta_k}\right] e^{-\lambda\left[\frac{(\omega-\Omega_j)}{\delta_j}+T_j\right]} L(\omega) d\omega + e^{-\lambda T_{J+1}} L(\Omega_{J+1}),$$

where note that the terms $e^{-\lambda T_j} L(\Omega_j)$ cancels one another except for the very last one. Equation (42) is obtained by another change of variable back to τ. ■

Proof of Proposition 10. (1) The equilibrium is solved recursively. The FOC (41) when $j = 0$ determines a_0, which is equivalent to the following (again, by a change of variable):

$$i = \theta \sum_{j=1}^{J} \int_{\Omega_j}^{\Omega_{j+1}} \lambda \left[\frac{1}{\delta_j} - \frac{1}{\delta_0}\right] e^{-\lambda\left[\frac{(\omega-\Omega_j)}{\delta_j}+T_j\right]} L(\omega) d\omega + e^{-\lambda T_{J+1}} L(\Omega_{J+1}), \tag{72}$$

where $\Omega_j = \sum_{k=0}^{j-1} A_k$ by equilibrium cond ition with $A_0 = a_0$. First we show that the RHS of (72) is strictly decreasing in a_0. Note that Ω_j is strictly increasing in a_0, the range for $c = \frac{(\omega-\Omega_j)}{\delta_j} + T_j$ for ω over Ω_j to Ω_{j+1} does not change when one changes a_0, but $L(\omega)$ is strictly decreasing in ω until it hits zero and stays there. Thus, the first term is strictly decreasing in a_0. For the second term, note that both T_{J+1} and Ω_{J+1} are strictly increasing in a_0 but the term is strictly decreasing in both T_{J+1} and Ω_{J+1}. Now, the RHS of (72) is also strictly positive at $a_0 = 0$ provided that $\delta_0 > \delta_1$ and equal to 0 as a_0 goes to ∞. The threshold for the nominal interest rate below which a monetary equilibrium exists is

$$\begin{aligned} \bar{\imath} \quad = \quad & \alpha\theta\lambda \sum_{k=1}^{J} \frac{(\delta_0 - \delta_k)}{\delta_0} \int_{T_k}^{T_{k+1}} e^{-\lambda\tau} \ell\left[y(\tau)\right] d\tau \\ & + \alpha\theta e^{-\lambda T_{J+1}} \ell\left[y(T_{J+1})\right], \end{aligned}$$

where $T_1 = 0$, and $T_j = \sum_{k=1}^{j-1} A_k/\delta_k$ for all $j \in \{2, ..., J+1\}$. Given a_0, the spreads $\{s_j\}_{j=1}^{J}$ are determined by (41), with $A_0 = a_0$ and $T_j = \sum_{k=0}^{j-1} A_k/\delta_k$ for all $j \in \{1, ..., J+1\}$. From (41) we can compute the difference between two consecutive spreads:

$$s_j - s_{j+1} = \alpha\theta\lambda \frac{(\delta_j - \delta_{j+1})}{\delta_j} \int_{T_{j+1}}^{T_{j+2}} e^{-\lambda\tau} \ell\left[y(\tau)\right] d\tau.$$

Hence, $s_j - s_{j+1} > 0$ requires $\delta_j - \delta_{j+1} > 0$ and $y(T_{j+1}) < y^*$, i.e., $\sum_{k=0}^{j} A_k < p(y^*)$.

(2) The velocity of asset j is

$$\mathcal{V}_j = \frac{\alpha\delta_j e^{-\lambda T_j}\left[1 - e^{-\frac{\lambda}{\delta_j}\left[\min\{p(y^*)-\Omega_j, A_j\}\right]}\right]}{A_j\lambda}, \tag{73}$$

By changing the order of integration we can simplify it to:

$$\mathcal{V}_j = \frac{\alpha\int_0^{+\infty} e^{-\lambda\tau}\omega_j^{*\prime}(\tau) 1_{\{\omega^*(\tau)<p(y^*)\}} d\tau}{A_j}.$$

Using Lemma 3 and the fact that $\omega_j^{*\prime}(\tau) = \delta_j 1_{\{T_j \le \tau < T_{j+1}\}}$ it can be rewritten as:

$$\mathcal{V}_j = \frac{\alpha\int_{T_j}^{T_{j+1}} e^{-\lambda\tau}\delta_j 1_{\{\omega^*(\tau)<p(y^*)\}} d\tau}{A_j}.$$

Using the expressions for T_j and T_{j+1} we distinguish three cases:

$$\mathcal{V}_j = \begin{array}{c} A_j^{-1}\lambda^{-1}\alpha\delta_j e^{-\lambda T_j}\left(1 - e^{-\frac{\lambda}{\delta_j}A_j}\right) \\ A_j^{-1}\lambda^{-1}\alpha\delta_j e^{-\lambda T_j}\left[1 - e^{-\frac{\lambda}{\delta_j}\left[p(y^*)-\Omega_j\right]}\right] \\ 0 \end{array} \quad \text{if} \quad z(y^*) \begin{array}{l} \ge \Omega_{j+1} \\ \in (\Omega_j, \Omega_{j+1}) \\ \le \Omega_j \end{array}$$

and the result follows immediately.

(3) It follows directly from (41) and the fact that:

$$\begin{aligned} |s_j - s_{j+1}| &= \alpha\theta\lambda\frac{(\delta_j - \delta_{j+1})}{\delta_j}\int_{T_{j+1}}^{T_{j+2}} e^{-\lambda\tau}\left[\frac{u'\left[y(\tau)\right] - v'\left[y(\tau)\right]}{v'\left[y(\tau)\right]}\right] d\tau \\ &\le \alpha\theta\lambda\frac{(\delta_j - \delta_{j+1})}{\delta_j} e^{-\lambda T_{j+1}}\left[\frac{u'\left[y(T_{j+1})\right] - v'\left[y(T_{j+1})\right]}{v'\left[y(T_{j+1})\right]}\right] d\tau, \end{aligned} \tag{74}$$

which converges to zero as $\lambda \to \infty$. ■

Proof of Proposition 11. Since we focus on equilibria with $\overline{\tau} = T_2$, the buyer portfolio problem can be written as

$$\max_{(a_m, a_b)} -ia_0 - sa_1 + \alpha\{u[y(\tau)] - p[y(\tau)]\}, \tag{75}$$

$$\text{s.t.} \quad \frac{a_0}{\delta_0} + \frac{a_1}{\delta_1} \le \overline{\tau}. \tag{76}$$

The FOC's are given as (here $\mu \le 0$ is the Lagrange multiplier for (76))

$$-i + \alpha\theta\ell(y) - \mu\frac{1}{\delta_0} \quad \le \quad 0 \text{ (with equality if } a_0 > 0), \tag{77}$$

$$-s + \alpha\theta\ell(y) - \mu\frac{1}{\delta_1} \quad \le \quad 0 \text{ (with equality if } a_1 > 0). \tag{78}$$

When $\mu > 0$, it implies that the constraint (76) is binding and hence adding more assets does not help increase liquidity the buyer possess. To characterize equilibrium outcomes, first we define the following three functions: let $\widetilde{y}^1(i,s)$ solve

$$i\frac{\delta_0}{\delta_1} - s = \alpha\theta\ell(y)\frac{\delta_0 - \delta_1}{\delta_1};$$

Note that $\widetilde{y}^1(i,s)$ is strictly increasing in s and strictly decreasing in i with $y^1(s)$ is strictly decreasing in s, $\widetilde{y}^1(i,0) = y_2$ and that $\widetilde{y}^1(i,i) = y_2$.

Given the conditions

$$\frac{\delta_1}{\delta_0 - \delta_1}\left[\delta_0\bar{\tau} - p(y_2)\right] < A_1 < \min\left\{\frac{\delta_1}{\delta_0 - \delta_1}\left[\delta_0\bar{\tau} - p(y_1)\right], \delta_1\bar{\tau}\right\} \tag{79}$$

$$\text{and } \frac{p(y_1)}{\delta_0} < \bar{\tau} < \frac{p(y_2)}{\delta_1}, \tag{80}$$

we show that

$$\begin{aligned} y &= \widetilde{y}(\bar{\tau}) = p^{-1}\left[\delta_0\bar{\tau} - \frac{\delta_0 - \delta_1}{\delta_1}A_1\right], \\ s &= \frac{\delta_0}{\delta_1}i - \alpha\theta\left(\frac{\delta_0 - \delta_1}{\delta_1}\right)\ell[\widetilde{y}(\bar{\tau})], \\ a_0 &= \left(\bar{\tau} - \frac{A_1}{\delta_1}\right)\delta_0, \end{aligned}$$

form a monetary equilibrium. First note that by (79)-(80),

$$p[\widetilde{y}(\bar{\tau})] = \delta_0\bar{\tau} - \frac{\delta_0 - \delta_1}{\delta_1}A_1 \in (0, p(y_2)),$$

and is strictly increasing in $\bar{\tau}$. Moreover, $A_1 < \delta_1\bar{\tau}$ implies that $a_0 > 0$. Now, let

$$\mu = -\delta_0 i + \delta_0\alpha\theta\ell[\widetilde{y}(\bar{\tau})] > 0,$$

where the last inequality follows form the fact that $\widetilde{y}(\bar{\tau}) < y_2$. Note also that (a_0, A_1) satisfies (76) by construction. Finally, in equilibrium we have

$$s = \frac{\delta_0}{\delta_1}i - \alpha\theta\left(\frac{\delta_0 - \delta_1}{\delta_1}\right)\ell[\widetilde{y}(\bar{\tau})].$$

Note that $s > 0$ if and only if

$$i - \alpha\theta\left(\frac{\delta_0 - \delta_1}{\delta_0}\right)\ell[\widetilde{y}(\bar{\tau})] > 0,$$

that is, if and only if $\widetilde{y}(\bar{\tau}) > y_1$, which is further equivalent to

$$\bar{\tau} > \frac{p(y_1) - A_1}{\delta_0} + \frac{A_1}{\delta_1},$$

which is guaranteed by (79)-(80). ■

Proof of Proposition 12. (1) Assume $i_0 = i_1 = i$. The two FOCs (44) and (45) hold at equality if and only if $\mu = 0$. Hence, y solves $u'(y)/v'(y) = 1 + i/\alpha\theta$. The negotiability constraint is slack if

$$\frac{e_t \phi_{1,t} A_{0,t}}{\delta_0} + \frac{\phi_{1,t} A_{1,t}}{\delta_1} \leq \bar{\tau},$$

where we used that $a_0 = \phi_0 A_0$ and $a_1 = \phi_1 A_1$ by market clearing. Moreover, the outcome of the negotiation is

$$e_t \phi_{1,t} A_{0,t} + \phi_{1,t} A_{1,t} = p(y).$$

Solving for $\phi_{1,t} = p(y)/\left(e_t A_{0,t} + A_{1,t}\right)$ and substituting into the negotiability constraint we obtain:

$$\frac{p(y)}{e_t A_{0,t} + A_{1,t}} \left(\frac{e_t A_{0,t}}{\delta_0} + \frac{A_{1,t}}{\delta_1} \right) \leq \bar{\tau}.$$

We rearrange the inequality to obtain:

$$\frac{A_{1,t}}{e_t A_{0,t} + A_{1,t}} \leq \frac{\delta_1}{\delta_0 - \delta_1} \frac{\delta_0 \bar{\tau} - p(y)}{p(y)}.$$

There exists a $e_t > 0$ such that this inequality holds iff $\delta_0 \bar{\tau} - p(y) > 0$, and this is the necessary and sufficient condition for both currency to be valued in equilibrium. Moreover, given that the left side is decreasing in e_t, if the inequality holds for $e_t = 0$, then it holds for all $e_t > 0$. This is the case if $p(y) \leq \delta_1 \bar{\tau}$. If $p(y) > \delta_1 \bar{\tau}$ then there is a lowest value for e_t consistent with the inequality. This value $\underline{e}$ is such that the inequality holds at equality.

(2) We have seen that for both currencies to be valued we need (47), which determines y. Equilibrium then requires $a_0/\delta_0 + a_1/\delta_1 = \bar{\tau}$ and $p(y) = a_0 + a_1$, which determine a_0 and a_1:

$$\begin{aligned} a_1 &= \frac{\delta_1}{\delta_0 - \delta_1} \left[\delta_0 \bar{\tau} - p(y) \right] \\ a_0 &= \frac{\delta_0}{\delta_0 - \delta_1} \left[p(y) - \delta_1 \bar{\tau} \right]. \end{aligned}$$

Thus, to have both $a_0 > 0$ and $a_1 > 0$, it is necessary and sufficient that

$$\delta_1 \bar{\tau} < p(y) < \delta_0 \bar{\tau}.$$

This condition can be rewritten as $\bar{\tau} \in (\bar{\tau}_0, \bar{\tau}_1)$ where $\bar{\tau}_0 = p(y)/\delta_0$, $\bar{\tau}_1 = p(y)/\delta_1$. It is immediate from (47) that $\partial y/\partial i_0 < 0$ and $\partial y/\partial i_1 > 0$. Similarly, $\partial y/\partial \alpha > 0$ and $\partial y/\partial \theta > 0$ which from (48) gives $\partial e_t/\partial \alpha > 0$ and $\partial e_t/\partial \theta > 0$. ■ ■

Appendix B: Proof of Proposition 1 and Extensions

As assumed in the main text, the number of bargaining rounds, N, is even, and the producer is the first to make an offer while the consumer the last. We obtain essentially the same results for the other cases (either N is odd or the producer is making the last offer), and we will comment in our proof how to handle those. Here we also normalize $u_0^b = u_0^s = 0$.

We define intermediate payoffs as the utilities that the players would enjoy based on the agreements reached up to some round $n \in \{1, ..., N\}$. Let (y_n, p_n) denote the cumulative offers that are agreed upon up to round n. Feasibility requires $0 \leq p_n - p_{n-1} \leq z/N$ and $0 \leq y_n - y_{n-1}$ for all $n = 1, ..., N$ and $p_0 = 0$ and $y_0 = 0$. Hence, equations (6) and (7) would correspond to the following intermediate payoffs for the consumer and the producer:

$$u_n^b = u(y_n) - p_n, \tag{81}$$

$$u_n^s = -v(y_n) + p_n. \tag{82}$$

The payoffs over terminal histories are simply u_N^b and u_N^s. If we restrict $y \in [0, y^*]$, then there is a one-to-one correspondence between the intermediate allocation (y, p) and the intermediate payoff (u^b, u^s) such that

$$H(u^b, u^s, p) = 0.$$

The rest of the section consists in proving Proposition 1 followed by two extensions: one with explicit negotiation time limit and the other with asymmetric bargaining powers. The proof contains four parts: the first gives a full characterization of the equilibrium payoffs of any subgame; the second gives equilibrium intermediate payoffs; the third proves the uniqueness claim; the fourth shows that those intermediate payoffs converge to the gradual bargaining solution as N goes to infinity.

Final equilibrium payoffs

To solve the game, we need to solve all possible subgames. A subgame is characterized by the intermediate payoff, denoted by (u_0^b, u_0^s) with the corresponding allocation denoted by (y_0, p_0), and the number of rounds remaining for bargaining, denoted by J. That is, the subgame begins at round $N-J+1$, with the intermediate payoff (u_0^b, u_0^s) that results from the bargaining in the first $N - J$ rounds. The entire game then has $(u_0^b, u_0^s) = (0, 0)$ and $J = N$. Feasibility requires $p_0 \leq (N - J)z/N$, and we only consider $y_0 < y^*$ so that there are still gains from trade to be exploited. Our first lemma describes the final payoffs of such a game. Let $S(y) = u(y) - v(y)$ and $S^* = S(y^*)$.

Lemma 6 *Consider a game* $[(u_0^b, u_0^s), J]$ *with* $0 \leq u_0^b + u_0^s < S^*$. *Equilibrium final payoffs are* $(\widetilde{u}_J^b, \widetilde{u}_J^s)$ *which correspond to the last term of the sequence defined as* $(\widetilde{u}_0^b, \widetilde{u}_0^s) = (u_0^b, u_0^s)$,

$$H(\widetilde{u}_j^b, \widetilde{u}_{j-1}^s, p_0 + jz/N) = 0 \text{ and } \widetilde{u}_j^s = \widetilde{u}_{j-1}^s, \text{ for } j \geq 1 \text{ odd}, \tag{83}$$

$$H(\widetilde{u}_{j-1}^b, \widetilde{u}_j^s, p_0 + jz/N) = 0 \text{ and } \widetilde{u}_j^b = \widetilde{u}_{j-1}^b, \text{ for } j \geq 2 \text{ even}, \tag{84}$$

where

$$p_0 = u[S^{-1}(u_0^b + u_0^s)] - u_0^b = u_0^s + v[S^{-1}(u_0^b + u_0^s)]. \tag{85}$$

Here we give an outline for the proof of Lemma 6, which uses backward induction. When $J = 1$, the game $[(u_0^b, u_0^s), 1]$ is a standard take-it-or-leave-it offer game (with the consumer making the offer). In equilibrium, the consumer makes an offer that leaves the producer indifferent between rejecting or accepting, with the final payoff to the producer $\widetilde{u}_1^s = u_0^s$. Taking this as given, the consumer spends up to the additional z/N units of assets so that his final payoff $\widetilde{u}_1^b$ satisfies $H(\widetilde{u}_1^b, u_0^s, p_0 + z/N) = 0$. (Note that the buyer will spend exactly z/N unless y^* is achieved with a slack liquidity constraint.) This proves (83) with $J = 1$.

Now consider $J = 2$, and the producer makes the first offer. If the consumer rejects the offer, the subgame becomes $[(u_0^b, u_0^s), 1]$, and the consumer can guarantee himself a final payoff of $\widetilde{u}_1^b$, which we call the consumer's *reservation payoff*. Take this as given, the producer's offer is acceptable as long as the offer would lead to a consumer final payoff no less than $\widetilde{u}_1^b$. Thus, the producer's offer would maximize his final payoff, u_2^s, subject to $u_2^b \geq \widetilde{u}_1^b$. Equivalently, the producer final payoff $\widetilde{u}_2^s$ solves $H(\widetilde{u}_1^b, \widetilde{u}_2^s, p_0 + 2z/N) = 0$. This proves (84) with $J = 2$. We illustrate this logic in Figure 9.

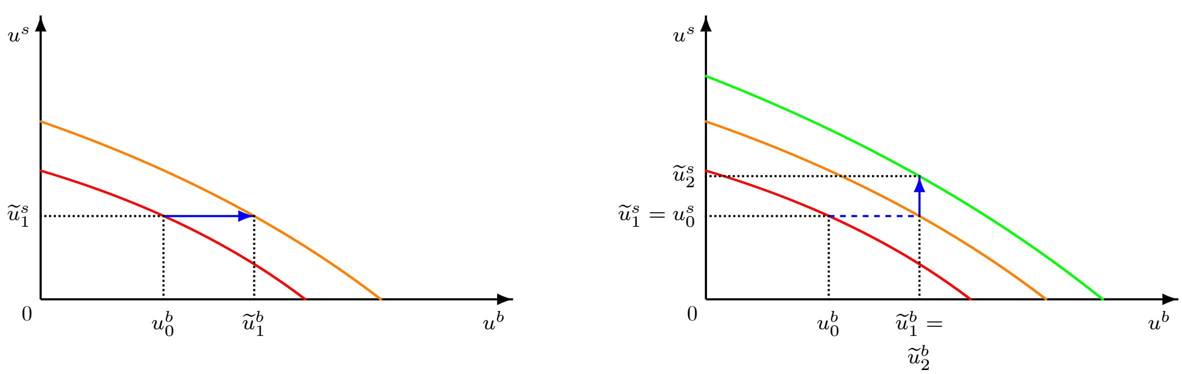

Figure 9: Construction of $\widetilde{u}_1^b$ and $\widetilde{u}_2^s$

We can continue this argument by induction. Suppose that the final payoffs are given by (83) and (84) for any game $[(u_0^b, u_0^s), J - 1]$ with $J \geq 2$ and consider a game $[(u_0^b, u_0^s), J]$ with J odd and the consumer is making the first offer. If the producer rejects the offer, his reservation payoff would be $\widetilde{u}^s_{J-1}$ by induction. Following the same logic, the consumer's offer maximize his final payoff u^J_b subject to the constraint that the producer's final payoff has to be no less than his reservation payoff, $\widetilde{u}^s_{J-1}$. Thus, the final payoffs in the game $[(u^b_0, u^s_0), J]$, denoted by $(\widetilde{u}^b_J, \widetilde{u}^s_J)$, would solve $H(\widetilde{u}^b_J, \widetilde{u}^s_J, p_0 + Jz/N) = 0$ with $\widetilde{u}^s_J = \widetilde{u}^s_{J-1}$. The case for J even is similar. This proves (83) and (84) for J.

Implicitly in the argument we assume that the agents can find the appropriate intermediate payoffs to achieve the final payoffs given by (83) and (84). In the following we explicitly construct the intermediate payoffs (and the corresponding allocations and offers) that will lead to those final payoffs at each round of the bargaining and we ensure that the offers are feasible at all rounds.

To construct the equilibrium intermediate payoffs, we expand the notation slightly to explicate the recursive nature of the sequence $\{(\widetilde{u}^b_j, \widetilde{u}^s_j)\}^J_{j=0}$. As mentioned, at each step according to (83)-(84), the next payoff is computed by either a rightward or upward shift to the next Pareto frontier. Formally, we can define two operations, $F_r(u^b, u^s)$ and $F_u(u^b, u^s)$ given by

$$F_r(u^b, u^s) = (u^{b\prime}, u^{s\prime}) \text{ such that } u^{s\prime s} \text{ and } H(u^{b\prime s}, p + z/N), \tag{86}$$

$$F_u(u^b, u^s) = (u^{b\prime}, u^{s\prime}) \text{ such that } u^{b\prime b} \text{ and } H(u^b, u^{s\prime}, p + z/N), \tag{87}$$

where p is given by (85). $F_r(u^b, u^s)$ moves from (u^b, u^s) to the next Pareto frontier by a rightward shift, and $F_u(u^b, u^s)$ moves from (u^b, u^s) to the next Pareto frontier by a upward shift. It then follows directly from (83) and (84) that, for all j even,

$$(\widetilde{u}^b_{j+1}, \widetilde{u}^s_{j+1}) = F_r(\widetilde{u}^b_j, \widetilde{u}^s_j), \tag{88}$$

$$(\widetilde{u}^b_{j+2}, \widetilde{u}^s_{j+2}) = F_u(\widetilde{u}^b_{j+1}, \widetilde{u}^s_{j+1}) = (F_u \circ F_r)(\widetilde{u}^b_j, \widetilde{u}^s_j). \tag{89}$$

To compute the whole sequence, we really only need the two functions F_r and F_u: for all j even,

$$(\widetilde{u}^b_j, \widetilde{u}^s_j) = (F_u \circ F_r)^{j/2}(u^b_0, u^s_0), \tag{90}$$

$$(\widetilde{u}^b_{j+1}, \widetilde{u}^s_{j+1}) = F_r[(F_u \circ F_r)^{j/2}(u^b_0, u^s_0)]. \tag{91}$$

Before we proceed, we give some comments on how to handle the case when the first-best is reached at some point of the game. Once we reach y^*, that is, once $\widetilde{u}^b_j + \widetilde{u}^s_j = u(y^*) - v(y^*)$, the sequence is constant afterwards and in equilibrium there is no trade in rounds after j. Note that this is consistent with our definition of simple SPE. Thus, we may only consider the case where

$$\widetilde{u}^b_{J-1} + \widetilde{u}^s_{J-1} < S^*. \tag{92}$$

Equilibrium Intermediate Payoffs

Our construction of equilibrium intermediate payoffs follows backward induction from the final payoffs constructed in Lemma 6. Consider a game $[(u_0^b, u_0^s), J]$ with J even. Lemma 6 shows that the final payoffs to the agents are given by $(\widetilde{u}_J^b, \widetilde{u}_J^s)$. Let $(\widehat{u}_{J-1}^b, \widehat{u}_{J-1}^s)$ denote the equilibrium intermediate payoff for the agents at the end of round-$(J-1)$ bargaining. Applying Lemma 6 to the game $[(\widehat{u}_{J-1}^b, \widehat{u}_{J-1}^s), 1]$, the equilibrium payoff to that game is given by $F_1(\widehat{u}_{J-1}^b, \widehat{u}_{J-1}^s)$. Thus, subgame perfection requires

$$F_r(\widehat{u}_{J-1}^b, \widehat{u}_{J-1}^s) = (\widetilde{u}_J^b, \widetilde{u}_J^s). \tag{93}$$

The solution to (93) is to move from $(\widetilde{u}_J^b, \widetilde{u}_J^s)$ leftward to the next Pareto frontier: formally, it is given by

$$H[\widehat{u}_{J-1}^b, \widetilde{u}_J^s, p_0 + (J-1)z/N] = 0, \ \widehat{u}_{J-1}^s = \widetilde{u}_J^s. \tag{94}$$

This process is shown in Figure 10.

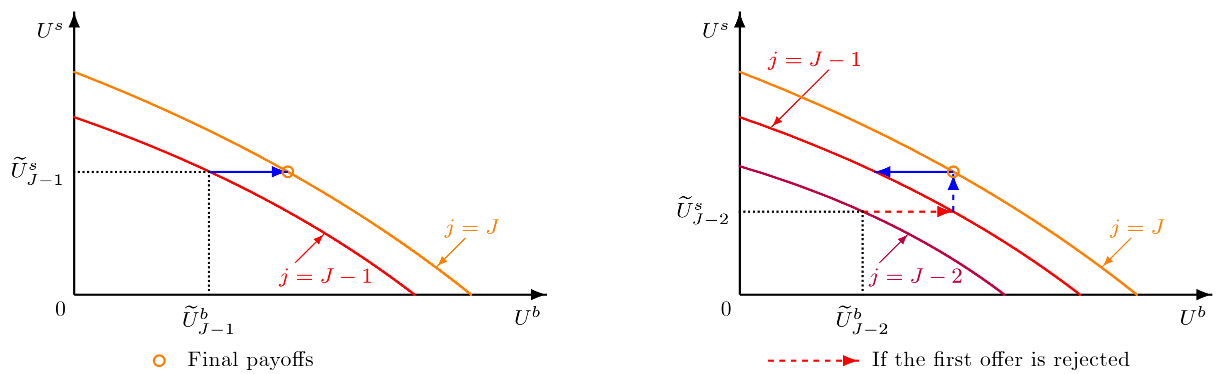

Figure 10: Backward induction

In general, the same argument shows that the equilibrium intermediate payoff at the end of round-$(J-j)$ bargaining, denoted by $(\widehat{u}_{J-j}^b, \widehat{u}_{J-j}^s)$, must satisfy

$$\begin{aligned} (F_u \circ F_r)^{j/2}(\widehat{u}_{J-j}^b, \widehat{u}_{J-j}^s) &= (\widetilde{u}_J^b, \widetilde{u}_J^s) \text{ for } j \text{ even}, \\ F_r[(F_u \circ F_r)^{(j-1)/2}(\widehat{u}_{J-j}^b, \widehat{u}_{J-j}^s)] &= (\widetilde{u}_J^b, \widetilde{u}_J^s) \text{ for } j \text{ odd}. \end{aligned} \tag{95}$$

The solution to (95) can in fact be computed recursively as follows by constructing a sequence of payoffs: $(\overline{u}_J^b, \overline{u}_J^s) = (\widetilde{u}_J^b, \widetilde{u}_J^s)$, and

$$H(\overline{u}_{J-j}^b, \overline{u}_{J-j+1}^s, p_0 + (J-j)z/N) = 0, \text{ and } \overline{u}_{J-j}^s = \overline{u}_{J-j+1}^s \text{ for } j \geq 1 \text{ odd}, \tag{96}$$

$$H(\overline{u}_{J-j+1}^b, \overline{u}_{J-j}^s, p_0 + (J-j)z/N) = 0, \text{ and } \overline{u}_{J-j}^b = \overline{u}_{J-j+1}^b \text{ for } j \geq 2 \text{ even}. \tag{97}$$

Graphically, for j odd, $(\overline{u}^b_{J-j}, \overline{u}^s_{J-j})$ is obtained from $(\overline{u}^b_{J-j+1}, \overline{u}^s_{J-j+1})$ by moving toward left to the next lower Pareto frontier; for j even, $(\overline{u}^b_{J-j}, \overline{u}^s_{J-j})$ is obtained from $(\overline{u}^b_{J-j+1}, \overline{u}^s_{J-j+1})$ by moving downward to the next lower Pareto frontier. Note that $(\overline{u}^b_{J-1}, \overline{u}^s_{J-1}) = (\widehat{u}^b_{J-1}, \widehat{u}^s_{J-1})$ given by (94). Note also that $(\widetilde{u}^b_{J-j}, \widetilde{u}^s_{J-j})$ is situated in the same Pareto frontier as $(\overline{u}^b_{J-j}, \overline{u}^s_{J-j})$ but they alternate in terms of moving downward and leftward. By construction, the sequence then has the following properties:

$$F_r(\overline{u}^b_{J-1}, \overline{u}^s_{J-1}) = (\overline{u}^b_J, \overline{u}^s_J) = (\widetilde{u}^b_J, \widetilde{u}^s_J), \tag{98}$$

$$(F_u \circ F_r)(\overline{u}^b_{J-j-2}, \overline{u}^s_{J-j-2}) = (\overline{u}^b_{J-j}, \overline{u}^s_{J-j}) \text{ for all } j \geq 1 \text{ odd.} \tag{99}$$

Thus, by repeated use of (89), (99) implies that for all $j \geq 2$ even

$$(F_u \circ F_r)^{j/2}(\overline{u}^b_{J-j-1}, \overline{u}^s_{J-j-1}) = (\overline{u}^b_{J-1}, \overline{u}^s_{J-1}),$$

and hence, by (88) and (98), $(\overline{u}^b_{J-j}, \overline{u}^s_{J-j})$ satisfies (95) for all j odd. For $j \geq 0$ and j even, by (89),

$$(F_u \circ F_r)(\widetilde{u}^b_{J-j-2}, \widetilde{u}^s_{J-j-2}) = (\widetilde{u}^b_{J-j}, \widetilde{u}^s_{J-j}),$$

which implies, by repeated use of (89), that $(\widetilde{u}^b_{J-j}, \widetilde{u}^s_{J-j})$ satisfies (95) for all j even. To summarize, the solution to (95) is given by

$$(\widehat{u}^b_{J-j}, \widehat{u}^s_{J-j}) = (\overline{u}^b_{J-j}, \overline{u}^s_{J-j}) \text{ for } j \text{ odd,} \tag{100}$$

$$(\widehat{u}^b_{J-j}, \widehat{u}^s_{J-j}) = (\widetilde{u}^b_{J-j}, \widetilde{u}^s_{J-j}) \text{ for } j \text{ even.} \tag{101}$$

When J is odd, we have the dual construction of the second sequence that begins by going downward first from $(\widetilde{u}^b_J, \widetilde{u}^s_J)$. We have the following lemma.

Lemma 7 *Consider a game $[(u^b_0, u^s_0), J]$ satisfying (92) with $J \geq 2$ even. The equilibrium intermediate payoffs at the end of round-j bargaining is then given by $(\widehat{u}^b_j, \widehat{u}^s_j)$ computed by (100)-(101).*

The proof of Lemma 7 follows directly from the above discussion, but we need to ensure that the intermediate payoffs can be supported by feasible offers, which is guaranteed by the following lemma.

Lemma 8 *Let a game $[(u^b_0, u^s_0), J]$ be given with J even. The sequence $\{(\widehat{u}^b_{J-j}, \widehat{u}^s_{J-j})\}_{j=0}^{J-1}$ given by (100)-(101) enjoys the following properties (where $\{\widehat{y}_{J-j}\}_{j=0}^{J-1}$ is the sequence of intermediate allocations corresponding to $\{(\widehat{u}^b_{J-j}, \widehat{u}^s_{J-j})\}_{j=0}^{J-1}$):*

$$\widehat{u}^b_{J-j} > \widehat{u}^b_{J-j-1} \text{ for all } j = 1, ..., J-2;\ \widehat{u}^b_1 > u^b_0; \tag{102}$$

$$\widehat{u}^s_{J-j} > \widehat{u}^s_{J-j-1} \text{ for all } j = 1, ..., J-2;\ \widehat{u}^s_1 > u^s_0; \tag{103}$$

$$\widehat{y}_j > \widehat{y}_{j-1} \text{ for all } j = 1, ..., J-2;\ \widehat{y}_1 > y_0. \tag{104}$$

The proof of Lemma 8 is based on induction on j and uses the fact that $u(y) - v(y)$ is strictly concave. The proof is rather straightforward but tedious and hence we refer the detailed proof to the Online Appendix XXX. However, Lemma 8 implies that the two sequences, $\{(\widetilde{u}^b_j, \widetilde{u}^s_j)\}_{j=1}^{J-1}$ and $\{(\overline{u}^b_j, \overline{u}^s_j)\}_{j=1}^{J-1}$ in fact nests one another, and hence, if one sequence converges, so would the other to the same limit.

Uniqueness of SPE

Here we prove our uniqueness claim. For this we need another lemma. Its proof is omitted but can be found in the Online Appendix XXX. Let $F_r = [F^b_r, F^s_r]$ and $F_u = [F^b_u, F^s_u]$.

Lemma 9 *Let (u^b, u^s) be given such that the output corresponding to $(F_u \circ F_r)(u^b, u^s)$ is strictly less than y^*. Then, $F^b_u[F_r(u^b, u^s)]$ is strictly increasing in u^b and decreasing in u^s and $F^s_u[F_r(u^b, u^s)]$ is strictly decreasing in u^b and strictly increasing in u^s.*

First we show that, for any subgame, $[(u^b_0, u^s_0), J]$, the equilibrium final payoffs in any SPE (not just simple SPE) is given by (83)-(84), denoted by $(\widetilde{u}^b_J, \widetilde{u}^s_J)$. For $J = 1$ this is the standard ultimatum game and the uniqueness is standard. Suppose that we have uniqueness for $J-1$, $J \geq 2$. Then, fix a SPE and consider the game at first bargaining round, and, without loss of generality, assume that consumer is making an offer and J is odd. Suppose that equilibrium intermediate payoff is given by $(\widehat{u}^b_1, \widehat{u}^s_1)$. Let (u^b_1, u^s_1) be such that

$$(F_u \circ F_r)^{(J-1)/2}(u^b_1, u^s_1) = (\widetilde{u}^b_J, \widetilde{u}^s_J) = F_r[(F_u \circ F_r)^{(J-1)/2}(u^b_0, u^s_0)] \equiv F_r(\widetilde{u}^b_{J-1}, \widetilde{u}^s_{J-1}).$$

Lemma 8 shows that such intermediate payoff is achievable with some offer (y_1, d_1). Moreover, by rejecting the consumer offer, by the induction hypothesis, the unique equilibrium payoff to the producer is $\widetilde{u}^s_{J-1} = \widetilde{u}^s_J$. Thus, any offer that leads to a final payoff lower than $\widetilde{u}^s_J$ will be rejected. Now, by offering $(y_1 - \varepsilon, d_1)$ for ε small the consumer can guarantee producer acceptance and hence, taking ε to zero, the consumer can guarantee a final payoff of $\widetilde{u}^b_J$. Since the payoff $(\widetilde{u}^b_J, \widetilde{u}^s_J)$ lies on the Pareto frontier for which the two agents can achieve by the end of the game, and each can guarantee the payment, this final payoff is unique.

Now we show that the intermediate payoffs we constructed are unique in simple SPE. Note first that in a simple SPE, the game effectively ends when active rounds end. Let J be the number of active rounds and the final payoffs are given by $(\widetilde{u}^b_J, \widetilde{u}^s_J)$. By backward induction, along the equilibrium path, in J-th round the starting intermediate payoff, $(\widehat{u}^b_{J-1}, \widehat{u}^s_{J-1})$, has to satisfy

$$F_r(\widehat{u}^b_{J-1}, \widehat{u}^s_{J-1}) = (\widetilde{u}^b_J, \widetilde{u}^s_J). \tag{105}$$

Now, when the output corresponding to $(\widetilde{u}^b_J, \widetilde{u}^s_J)$ is y^*, there can be multiple solution to (105), but the solution is unique otherwise.

Similar reasoning shows that for all $j = 1, ..., J-1$, (95) must hold. Lemma 9 implies that there is a unique solution to that except for $(\widehat{u}^b_{J-1}, \widehat{u}^s_{J-1})$. However, that payoff can be pinned down by the fact that buyer has to spend z/N in a simple SPE in round $J-1$.

Finally, when the output corresponding to $(\widetilde{u}^b_J, \widetilde{u}^s_J)$ is less than y^*, then $J = N$, and the solution to (105) is unique for all j. Since y^* is not achievable in any subgame, it follows that the SPE is unique.

Convergence to Gradual Nash Solution

We consider convergence of games with N even. The limit will be the same for N odd and hence we have convergence. By Proposition 7 the sequence of intermediate equilibrium payoffs at the end of each round is given by $\{(\widehat{u}^b_n, \widehat{u}^s_n)\}_{n=1}^N$ with $(u^b_0, u^s_0) = (0,0)$, and that $(\widehat{u}^b_n, \widehat{u}^s_n) = (\widetilde{u}^b_n, \widetilde{u}^s_n)$ for n even. Consider two bargaining rounds, $n-1$ and $n+1$, where n is a odd number. So, $(\widetilde{u}^b_{n-1}, \widetilde{u}^s_{n-1})$ and $(\widetilde{u}^b_{n+1}, \widetilde{u}^s_{n+1})$ are corresponding equilibrium intermediate payoffs.

Let $\Delta u^b = \widetilde{u}^b_{n+1} - \widetilde{u}^b_{n-1}$ (note that, however, $\widetilde{u}^b_{n+1} = \widetilde{u}^b_n$) denote the buyer's incremental payoffs (on the equilibrium path) in rounds $n-1$ and $n+1$, and $\Delta u^s = \widetilde{u}^s_{n+1} - \widetilde{u}^s_{n-1}$ (note that, however, $\widetilde{u}^s_n = \widetilde{u}^s_{n-1}$) denote the seller's incremental payoff (on the equilibrium path) in rounds $n-1$ and $n+1$. Similarly, let $\Delta z = 2z/N$. Note that for Δz we have increment of $2z/N$ corresponding to the jumps in asset payments in Δu^b and Δu^s. Then we have

$$H(\widetilde{u}^b_{n-1}, \widetilde{u}^s_{n-1};\ \frac{n-1}{N}z) = 0 \tag{106}$$

$$H(\widetilde{u}^b_{n-1} + \Delta u^b, \widetilde{u}^s_{n-1};\ \frac{n-1}{N}z + \frac{\Delta z}{2}) = 0 \tag{107}$$

$$H(\widetilde{u}^b_{n-1} + \Delta u^b, \widetilde{u}^s_{n-1} + \Delta u^s;\ \frac{n-1}{N}z + \Delta z) = 0. \tag{108}$$

According to (106), at the end of round $n-1$ the intermediate payoffs of the buyer and the seller are $\left(\widetilde{u}^b_{n-1}, \widetilde{u}^s_{n-1}\right)$ and they belong to the Pareto frontier such that $(n-1)z/N$ real balances are up for negotiation. According to (107), at the end of round n the intermediate payoffs are obtained by moving horizontally from the $(n-1)^{\text{th}}$ frontier to the n^{th} frontier (since n is odd). Hence, the seller's intermediate payoff is unchanged at $\widetilde{u}^s_{n-1}$ while the buyer's intermediate payoff increases by Δu^b. The amount of assets up for negotiation on the n^{th} frontier are nz/N. According to (108), at the end of round $n+1$ the intermediate payoffs are obtained by moving vertically from the n^{th} frontier to the $(n+1)^{\text{th}}$ frontier (since $n+1$ is even).

A first-order Taylor series expansion of (107) in the neighborhood of $(u^b, u^s, \tau) = \left(\widetilde{u}^b_{n-1}, \widetilde{u}^s_{n-1}, \frac{n-1}{N}z\right)$ yields:

$$H(\widetilde{u}^b_{n-1} + \Delta u^b, \widetilde{u}^s_{n-1};\ \frac{n}{N}z) = H_1 \Delta u^b + H_3 \frac{\Delta z}{2} + o(\Delta u^b) + o(\frac{1}{N}),$$

where $\lim_{N\to\infty} \frac{o(\Delta u^b)}{\Delta u^b} = \lim_{N\to\infty} N o(\frac{1}{N}) = 0$, we used that $H(\widetilde{u}^b_{n-1}, \widetilde{u}^s_{n-1};\ \frac{n-1}{N}z) = 0$ from (106), and the partial derivatives H_1, H_2, and H_3 are evaluated at $\left(\widetilde{u}^b_{n-1}, \widetilde{u}^s_{n-1}, \frac{n-1}{N}z\right)$. Similarly, a first-order Taylor series expansion of (108) yields

$$H(\widetilde{u}^b_{n-1} + \Delta u^b, \widetilde{u}^s_{n-1} + \Delta u^s;\ \frac{n+1}{N}z) = H_1 \Delta u^b + H_2 \Delta u^s + H_3 \Delta z + o(\Delta u^b) + o(\Delta u^s) + o(\frac{1}{N}),$$

where $\lim_{N\to\infty, n/N\to\tau} \frac{o(\Delta u^b)}{\Delta u^b} = \lim_{N\to\infty, n/N\to\tau} \frac{o(\Delta u^s)}{\Delta u^s} = \lim_{N\to\infty} N o(\frac{1}{N}) = 0$. Using that $H = 0$ for payoffs on the Pareto frontiers, we obtain that

$$\begin{aligned}
H_1 \Delta u^b + o(\Delta u^b) &= -H_3 \frac{\Delta z}{2} + o(\frac{1}{N}), \\
H_1 \Delta u^b + o(\Delta u^b) + H_2 \Delta u^s + o(\Delta u^s) &= -H_3 \Delta z + o(\frac{1}{N}), \\
o(\Delta u^b) + H_2 \Delta u^s + o(\Delta u^s) &= -H_3 \frac{\Delta z}{2} + o(\frac{1}{N}).
\end{aligned}$$

From the first one and rearranging terms, we obtain

$$\frac{\Delta u^b}{\Delta z} = -\frac{H_3}{2H_1} + \frac{o(\Delta u^b)}{H_1 \Delta z} + \frac{o(\frac{1}{N})}{H_1 \Delta z}.$$

Note that

$$\lim_{N\to\infty} \frac{o(\frac{1}{N})}{H_1 \Delta z} = \frac{o(\frac{1}{N})N}{H_1 z} = 0 \text{ and } \lim_{N\to\infty} \frac{o(\Delta u^b)}{H_1 \Delta z} = \lim_{N\to\infty} \frac{o(\Delta u^b)}{H_1 z \Delta u^b} (\Delta u^b N) = 0,$$

where

$$\Delta u^b N = (\widetilde{u}^b_{n+1} - \widetilde{u}^b_{n-1})N \in [[1 - v'(\widetilde{y}_{n+1})/u'(\widetilde{y}_{n+1})]z, [1 - v'(\widetilde{y}_{n-1})/u'(\widetilde{y}_{n-1})]z]$$

and hence it limit exists and is bounded away from zero by the concavity of the function $S()$. Thus,

$$\frac{\partial u^b}{\partial \tau} = \lim_{N\to\infty} \frac{\Delta u^b}{\Delta z} = -\frac{1}{2}\frac{H_3}{H_1} = -\frac{1}{2}\frac{\partial H/\partial \tau}{\partial H/\partial u^b}.$$

Similarly, combining these two equations and rearranging, we obtain

$$\frac{\Delta u^s}{\Delta z} = -\frac{H_3}{2H_2} + \frac{o(\Delta u^b)}{H_2 \Delta z} + \frac{o(\Delta u^s)}{H_2 \Delta z} + \frac{o(\frac{1}{N})}{H_2 \Delta z}.$$

By the same arguments, we have

$$\frac{\partial u^s}{\partial \tau} = \lim_{N\to\infty} \frac{\Delta u^s}{\Delta z} = -\frac{1}{2}\frac{H_3}{H_2} = -\frac{1}{2}\frac{\partial H/\partial \tau}{\partial H/\partial u^s}.$$

These correspond to (10) and (11).

Extensions

Negotiation with limited time

Here we introduce a time frame within which the negotiation can occur. Suppose that the two players are given a specific amount of time, $\overline{\tau}$, to negotiate their trades. Each unit of the asset takes $1/\delta$ units of time and hence the maximum amount of assets that can be traded is $\overline{\tau}\delta$. Our target is the continuous time model but here we provide a discrete time foundation. So suppose that there are M rounds of bargaining, and hence each round of bargaining takes $\eta = \overline{\tau}/M$ units of time and in each round at most $\delta\eta$ units of assets can be put up for negotiation.

Let z be the consumer's asset holding. If $z \geq \overline{\tau}\delta$, then the game is exactly the same as in the last section with asset holding $z' = \overline{\tau}\delta$. So suppose that $z < \overline{\tau}\delta$. For simplicity we assume that there exists N such that $N = \frac{z}{\delta\eta} < M$, and hence it takes exactly N rounds to negotiate the whole asset holdings, and at each round up to z/N units of assets can be negotiated. As before, we use (y_n, p_n) to denote cumulative offers accepted up to round n. We can define the intermediate payoffs as in (81)-(82).

Given these background assumptions, we now analyze the game. As before, we consider the case where the consumer makes the very last offer, at round M. We denote such a game by $(z, M, \overline{\tau}, \delta)$. The following is our proposition.

Proposition 13 *The game $(z, M, \overline{\tau}, \delta)$ has a unique SPE final payoffs that coincide with the final payoff of the game $[(u_0^b, u_0^s), N]$ with $u_0^b = 0 = u_0^s$ as constructed in Lemma 6, where $N = \frac{z}{\delta\eta}$.*

We prove this by induction. Indeed, if $N = M$, then the result follows directly from Lemma 6. Thus we shall prove this by induction on $M - N$. For this exercise we shall fix z and N, and increase M. Notice that for any M, once we reach round-$(M - N + 1)$, we are in the game $[(u_{M-N}^b, u_{M-N}^s), N]$, where (u_{M-N}^b, u_{M-N}^s) is the intermediate payoffs reached at the end of round-$(M - N)$. As mentioned, when $M - N = 0$, $(u_{M-N}^b, u_{M-N}^s) = (0, 0)$ and hence the equilibrium final payoffs are given by $(\tilde{u}_N^b, \tilde{u}_N^s)$ computed by (83)-(84).

Suppose that N is even. Now consider $M = N + 1$ and hence the consumer is the first to make the offer. Then, at the first stage, the producer can secure a final payoff of $\tilde{u}_N^s$ by rejecting any offer form the consumer. Moreover, we also know that in this game, any final payoff (u^b, u^s) must satisfy

$$H(u^b, u^s, z) \geq 0. \tag{109}$$

Since there is no other pair of final payoff (u^b, u^s) that satisfies both (109) and that $u^b > \tilde{u}_N^b$ and $u^s \geq \tilde{u}_N^s$, it follows that it is optimal for the consumer to offer $(0, 0)$ at the first stage, and hence $(\tilde{u}_N^b, \tilde{u}_N^s)$ is achievable; moreover, it is the unique equilibrium final payoff, as in any equilibrium we would have $u^s \geq \tilde{u}_N^s$ and $u^b \geq \tilde{u}_N^b$.

Suppose, by induction, that the result holds for some $M - 1$, $M > 0$. Then consider the game with M stages and suppose that M is even and hence the producer is the first to make the offer. By induction, we know that the consumer can secure a final payoff of $\tilde{u}_N^b$ by rejecting any offer form the producer. As before, we also know that in this game, any final payoff (u^b, u^s) must satisfy (109). The rest of the argument then follows.

Note that there are other SPEs sharing the same SPE payoffs. For example, it is also an SPE that they complete surplus-sharing bargaining in the initial N rounds and then there is no trade in the remaining $M - N$ rounds.

Asymmetric bargaining powers

Here we revise our game to support gradual Nash solution with asymmetric bargaining power, denoted by θ. The parameter θ affects the game as follows. We assume that the number of rounds is $2N$, and the producer is the one making the first offer and the consumer is making the last offer.

1. In each round $2n-1 \in \{1, 3, \ldots, 2N-1\}$, it is the producer's turn to make an offer, with asset transfer within the range $[0, (1-\theta)z/N]$; the consumer then decides to accept or reject the offer.

2. In each round $2n \in \{2, 4, \ldots, 2N\}$, it is the consumer's turn to make an offer, with asset transfer within the range $[0, \theta z/N]$; the producer the decides to accept or reject the offer.

Note that at the end of round $2n-1$, the maximum cumulative asset transfer is $[(n-1)+(1-\theta)]z/N$, and at the end of round $2n$, the maximum cumulative asset transfer is nz/N, for all $n = 1, ..., N$.

As before, to solve the game, we need to solve all possible subgames. Also, such subgame can still be characterized by $[(u_0^b, u_0^s), J]$, where (u_0^b, u_0^s) is the intermediate payoff at the beginning of the subgame and J is the number of remaining bargaining rounds.

Proposition 14 *Fix some $\theta \in [0,1]$. There exists a subgame perfect equilibrium (SPE) in each alternating-ultimatum offer game, and all SPE share the same final payoffs. When the output level corresponding to the final payoffs is less than y^*, the SPE is unique and is simple; otherwise, there is a unique simple SPE. Moreover, in any simple SPE, the intermediate payoffs, $\{(u_n^b, u_n^s)\}_{n=1,2,\ldots,2N}$, converge to the solution $(u^b(\tau), u^s(\tau))$ to the differential equations (17) and (18) as N approaches ∞ with $\tau = [(n-1)/2+(1-\theta)]z/N$ if n is odd and $\tau = nz/2N$ if n is even.*

Note that Proposition 1 is a special case of Proposition 14 with $\theta = 1/2$.

The proof follows exactly the same outline as that of Proposition XXX. In particular, we will use the same technique to compute the final payoffs for any subgame, but with necessary modification to accommodate the fact that the consumer has control over θ fraction of assets to be negotiated every two rounds. As before, we can denote an arbitrary subgame by $[(u_0^b, u_0^s), J]$ with $0 \le u_0^b + u_0^s < u(y^*) - v(y^*)$.

The final payoff is computed as follows. Define $\{(\widetilde{u}^b_j, \widetilde{u}^s_j)\}_{j=0}^J$ as $(\widetilde{u}^b_0, \widetilde{u}^s_0) = (u^b_0, u^s_0)$, and, for $j \geq 0$,

$$H(\widetilde{u}^b_{2j+1}, \widetilde{u}^s_{2j}, p_0 + \theta z/N + jz/N) = 0, \text{ and } \widetilde{u}^s_{2j+1} = \widetilde{u}^s_{2j}, \tag{110}$$

$$H(\widetilde{u}^b_{2j+1}, \widetilde{u}^s_{2j+2}, p_0 + (j+1)z/N) = 0, \text{ and } \widetilde{u}^b_{2j+2} = \widetilde{u}^b_{2j+1}, \tag{111}$$

where p_0 is given by (85). Below we show that the final equilibrium payoffs for the agents are given by $(\widetilde{u}^b_J, \widetilde{u}^s_J)$.

The logic behind this construction is exactly the same as the symmetric case, except for the fact that the consumer and the producer controls different shares of assets up for negotiation. In particular, when $J = 1$, the game $[(u^b_0, u^s_0), 1]$ is a standard take-it-or-leave-it offer game (with the consumer making the offer). Since the consumer can offer up to additional $\theta z/N$ units of assets, the final payoff is computed by a rightward shift to next Pareto frontier with intermediate payments $p_0 + \theta z/N$, as in (110) with $j = 0$. When $J = 2$, the producer makes the first offer and take the final payoff for consumer in case he rejects the offer as given. Note that with $J = 2$ the final Pareto frontier has intermediate payment of $p_0 + z/N$, as in (111) with $j = 0$.

To compute the intermediate payoffs, we first define the functions F_r and F_u analogous to (86) and (87):

$$F_r(u^b, u^s) = (u^{b\prime}, u^{s\prime}) \text{ such that } u^{s\prime s} \text{ and } H(u^{b\prime s}, p + \theta z/N), \tag{112}$$

$$F_u(u^b, u^s) = (u^{b\prime}, u^{s\prime}) \text{ such that } u^{b\prime b} \text{ and } H(u^b, u^{s\prime}, p + (1-\theta)z/N), \tag{113}$$

where p is given by (85). Now we are ready to explain how to compute intermediate payoffs. Consider a game $[(u^b_0, u^s_0), J]$ with J even. Using the same backward induction argument as in the symmetric case, if $(\widehat{u}^b_{J-1}, \widehat{u}^s_{J-1})$ is the equilibrium intermediate payoff for the agents at the end of round-$(J-1)$ bargaining, then

$$F_r(\widehat{u}^b_{J-1}, \widehat{u}^s_{J-1}) = (\widetilde{u}^b_J, \widetilde{u}^s_J). \tag{114}$$

As before, the solution would be obtained by a leftward shift, but, under θ, to the lower Pareto frontier with intermediate payment lowered by $\theta z/N$; that is,

$$H[\widehat{u}^b_{J-1}, \widetilde{u}^s_J, p_0 + Jz/2N - \theta z/N] = 0, \ \widehat{u}^s_{J-1} = \widetilde{u}^s_J. \tag{115}$$

Note that in this case, $(\widehat{u}^b_{J-1}, \widehat{u}^s_{J-1})$ and $(\widetilde{u}^b_{J-1}, \widetilde{u}^s_{J-1})$ do not lie on the same Pareto frontier unless $\theta = 1/2$.

In general, we can still use (95) to compute the equilibrium intermediate payoff at the end of round-$(J-j)$ bargaining, denoted by $(\widehat{u}^b_{J-j}, \widehat{u}^s_{J-j})$, with F_r and F_u defined by (112)-(113). Similar to the symmetric case, we define a second sequence, $\{(\overline{u}^b_{J-j}, \overline{u}^s_{J-j})\}_{j=0}^{J-1}$ as follows: $(\overline{u}^b_J, \overline{u}^s_J) = (\widetilde{u}^b_J, \widetilde{u}^s_J)$, and

$$H(\overline{u}^b_{J-j}, \overline{u}^s_{J-j+1}, p_0 + (J-j-1)z/2N + (1-\theta)z/N) = 0, \text{ and } \overline{u}^s_{J-j} = \overline{u}^s_{J-j+1} \text{ for } j \geq 1 \text{ odd}, \tag{116}$$

$$H(\overline{u}^b_{J-j+1}, \overline{u}^s_{J-j}, p_0 + (J-j)z/2N) = 0, \text{ and } \overline{u}^b_{J-j} = \overline{u}^b_{J-j+1} \text{ for } j \geq 1 \text{ even}. \tag{117}$$

Graphically, for j odd, $(\overline{u}^b_{J-j}, \overline{u}^s_{J-j})$ is obtained from $(\overline{u}^b_{J-j+1}, \overline{u}^s_{J-j+1})$ by moving toward left to the next lower Pareto frontier, with a decrease of incremental transfer of $\theta z/N$; for j even, $(\overline{u}^b_{J-j}, \overline{u}^s_{J-j})$ is obtained from $(\overline{u}^b_{J-j+1}, \overline{u}^s_{J-j+1})$ by moving downward to the next lower Pareto frontier, with a decrease of incremental transfer of $(1-\theta)z/N$. Note that $(\overline{u}^b_{J-1}, \overline{u}^s_{J-1}) = (\widehat{u}^b_{J-1}, \widehat{u}^s_{J-1})$ given by (115). Note also that, in contrast to the symmetric case, $(\widetilde{u}^b_{J-j}, \widetilde{u}^s_{J-j})$ is situated in the same Pareto frontier as $(\overline{u}^b_{J-j}, \overline{u}^s_{J-j})$ if and only if j is even; for j odd, $(\overline{u}^b_{J-j}, \overline{u}^s_{J-j})$ lies on a different frontier. To summarize, intermediate payoffs are then given by

$$(\widehat{u}^b_{J-j}, \widehat{u}^s_{J-j}) = (\overline{u}^b_{J-j}, \overline{u}^s_{J-j}) \text{ for } j \text{ odd,} \tag{118}$$

$$(\widehat{u}^b_{J-j}, \widehat{u}^s_{J-j}) = (\widetilde{u}^b_{J-j}, \widetilde{u}^s_{J-j}) \text{ for } j \text{ even.} \tag{119}$$

Moreover, with the appropriately modified argument adopted to θ, we can show that the corresponding offers to these intermediate payoffs are all feasible at each round.

Now we show that the intermediate payoffs converge to the same limit. As in the symmetric case, consider convergence of games with N even. The limit will be the same for N odd and hence we have convergence. By the above arguments we have that the sequence of intermediate equilibrium payoffs at the end of each round is given by $\{(\widehat{u}^b_n, \widehat{u}^s_n)\}_{n=1}^N$ with $(u^b_0, u^s_0) = (0,0)$, and that $(\widehat{u}^b_n, \widehat{u}^s_n) = (\widetilde{u}^b_n, \widetilde{u}^s_n)$ for n even. Consider two bargaining rounds, $2n$ and $2n+2$. So, $(\widetilde{u}^b_{2n}, \widetilde{u}^s_{2n})$ and $(\widetilde{u}^b_{2n+2}, \widetilde{u}^s_{2n+2})$ are corresponding equilibrium intermediate payoffs. Let $\Delta u^b = \widetilde{u}^b_{2n+2} - \widetilde{u}^b_{2n}$ denote the buyer's incremental payoffs (on the equilibrium path) in rounds $2n$ and $2n+2$, and $\Delta u^s = \widetilde{u}^s_{2n+2} - \widetilde{u}^s_{2n}$ denote the seller's incremental payoff (on the equilibrium path) in rounds $2n$ and $2n+2$. Let $\Delta z = z/N$ be the corresponding change in assets. Then we have

$$H(\widetilde{u}^b_{2n}, \widetilde{u}^s_{2n};\ nz/N) = 0 \tag{120}$$

$$H(\widetilde{u}^b_{2n} + \Delta u^b, \widetilde{u}^s_{2n}; \theta\Delta z + \frac{n}{N}z) = 0 \tag{121}$$

$$H(\widetilde{u}^b_{2n} + \Delta u^b, \widetilde{u}^s_{2n} + \Delta u^s;\ nz/N + \Delta z) = 0. \tag{122}$$

A first-order Taylor series expansion of (121) in the neighborhood of $(u^b, u^s, \tau) = \left(\widetilde{u}^b_n, \widetilde{u}^s_n, \frac{n}{N}z\right)$ yields:

$$H(\widetilde{u}^b_{2n} + \Delta u^b, \widetilde{u}^s_{2n};\ \frac{n}{N}z) = H_1\Delta u^b + H_3\theta\Delta z + o(\Delta u^b) + o(\frac{1}{N}),$$

where $\lim_{N\to\infty} \frac{o(\Delta u^b)}{\Delta u^b} = \lim_{N\to\infty} No(\frac{1}{N}) = 0$, we used that $H(\widetilde{u}^b_{2n}, \widetilde{u}^s_{2n};\ \frac{n}{N}z) = 0$ from (120), and the partial derivatives H_1, H_2, and H_3 are evaluated at $\left(\widetilde{u}^b_{2n}, \widetilde{u}^s_{2n}, \frac{n}{N}z\right)$. Similarly, a first-order Taylor series expansion of (122) yields

$$H(\widetilde{u}^b_{2n} + \Delta u^b, \widetilde{u}^s_{2n} + \Delta u^s;\ \frac{n+1}{N}z) = H_1\Delta u^b + H_2\Delta u^s + H_3\Delta z + o(\Delta u^b) + o(\Delta u^s) + o(\frac{1}{N}),$$

where $\lim_{N\to\infty}\frac{o(\Delta u^b)}{\Delta u^b}=\lim_{N\to\infty}\frac{o(\Delta u^s)}{\Delta u^s}=\lim_{N\to\infty}No(\frac{1}{N})=0$. Using that $H=0$ for payoffs on the Pareto frontiers, we obtain that

$$\begin{aligned}
H_1\Delta u^b+o(\Delta u^b) &= -H_3\theta\Delta z+o(\frac{1}{N}),\\
H_1\Delta u^b+o(\Delta u^b)+H_2\Delta u^s+o(\Delta u^s) &= -H_3\frac{z}{N}+o(\frac{1}{N}),\\
H_2\Delta u^s+o(\Delta u^s)+o(\Delta u^b) &= -(1-\theta)H_3\frac{z}{N}+o(\frac{1}{N})
\end{aligned}$$

From the first equation with rearranging, we obtain

$$\frac{\Delta u^b}{\Delta z}=-\theta\frac{H_3}{H_1}+\frac{o(\Delta u^b)}{H_1\Delta z}+\frac{o(\frac{1}{N})}{H_1\Delta z}.$$

Similarly, from the thrid equation with rearranging, we obtain

$$\frac{\Delta u^s}{\Delta z}=-(1-\theta)\frac{H_3}{H_2}+\frac{o(\Delta u^b)}{H_2\Delta z}+\frac{o(\Delta u^s)}{H_2\Delta z}+\frac{o(\frac{1}{N})}{H_2\Delta z}.$$

Thus, we have

$$\begin{aligned}
\frac{\partial u^b}{\partial \tau} &= \lim_{N\to\infty,n/N\to\tau}\frac{\Delta u^b}{\Delta z}=-\theta\frac{H_3}{H_1}=-\theta\frac{\partial H/\partial\tau}{\partial H/\partial u^b},\\
\frac{\partial u^s}{\partial \tau} &= \lim_{N\to\infty,n/N\to\tau}\frac{\Delta u^s}{\Delta z}=-(1-\theta)\frac{H_3}{H_2}=-(1-\theta)\frac{\partial H/\partial\tau}{\partial H/\partial u^s}.
\end{aligned}$$

Appendix C

Proof. Dual currency with exponential time constraint

Here we consider the dual currency economy with $\bar{\tau}$ exponentially distributed. From (41), and assuming an interior solution,

$$\begin{aligned}
i_1 &= \alpha\theta e^{-\lambda T_2}\ell[y(T_2)] && (123)\\
i_0-i_1 &= \alpha\theta\lambda\frac{(\delta_0-\delta_1)}{\delta_0}\int_{T_1}^{T_2}e^{-\lambda\tau}\ell[y(\tau)]d\tau, && (124)
\end{aligned}$$

where $T_1=a_0/\delta_0$, $T_2=a_0/\delta_0+a_1/\delta_1$, $y(T_2)=p^{-1}(a_0+a_1)$, and $y(\tau)=a_0+\delta_1(\tau-a_0/\delta_0)$. We have the following results.

Proposition 15 ***(Dual currency economy)*** *Suppose that $\delta_0>\delta_1$ and let i_0 be given. There exists a unique dual-currency steady-state equilibrium with for all $i_1\in(\underline{\iota},i_0)$ where $\underline{\iota}(\delta_0,\delta_1)<i_0$. Moreover, $\partial\underline{\iota}/\partial\delta_0<0$ and $\partial\underline{\iota}/\partial\delta_1>0$.*

Proof. Real that $T_2=\frac{a_0}{\delta_0}+\frac{a_1}{\delta_1}$. First note that there exists a unique $\widetilde{a}(i_1)$ such that $a_0=\widetilde{a}(i_1)$ and $a_1=0$ satisfy (123). For any given $a_0\in(0,\widetilde{a}(i_1))$, from (123) we can determine a unique $a_1=g(a_0;i_1)>0$

with $g(0;i_1) = \bar{a}(i_1) > 0$, and $\widetilde{a}(i_1) < \bar{a}(i_1)$, and $g' < 0$ for all such a_0. We can then rewrite the second equation as:

$$i_0 - i_1 = \alpha\theta\lambda \frac{(\delta_0 - \delta_1)}{\delta_0} e^{-\lambda \frac{a_0}{\delta_0}} \int_0^{\frac{g(a_0;i_1)}{\delta_1}} e^{-\lambda x} \left[\frac{u'\left[p^{-1}(a_0 + \delta_1 x)\right] - v'\left[p^{-1}(a_0 + \delta_1 x)\right]}{v'\left[p^{-1}(a_0 + \delta_1 x)\right]} \right] dx.$$

The right side is decreasing in a_0. There is a positive solution provided that $i_1 > \underline{\iota}$ where $\underline{\iota}$ solves

$$i_0 - \underline{\iota} = \alpha\theta\lambda \frac{(\delta_0 - \delta_1)}{\delta_0} \int_0^{\frac{\bar{a}(\underline{\iota})}{\delta_1}} e^{-\lambda x} \left[\frac{u'\left[p^{-1}(\delta_1 x)\right] - v'\left[p^{-1}(\delta_1 x)\right]}{v'\left[p^{-1}(\delta_1 x)\right]} \right] dx.$$

It is easy to check that the right side is increasing in δ_0 and decreasing in δ_1, hence $\partial\underline{\iota}/\partial\delta_0 < 0$ and $\partial\underline{\iota}/\partial\delta_1 > 0$. ■

Provided that the differential of inflation rates is not too large, there is coexistence of the two currencies. The differential consistent with a dual currency economy increases as currency 0 becomes more negotiable and decreases as currency 1 becomes more negotiable. Moreover, our model predicts that small trades are conducted with the most negotiable currency while larger trades are financed with both currencies.

Proposition 16 ***(Exchange rate)*** *In any dual-currency, steady-state equilibrium the nominal exchange rate is uniquely determined and can be expressed as $e_t = \bar{e}(\delta_0, \delta_1)A_{1,t}/A_{0,t}$ where $\bar{e}(\delta_0, \delta_1)$ increases with δ_0 but decreases with δ_1.*

Proof. The exchange rate can be expressed as $e_t \equiv (a_0/a_1) \times (A_{1,t}/A_{0,t})$. It can be checked from the proof of previous proposition that a_0 increases with δ_0 and decreases with δ_1. Moreover, $a_1 = g(a_0; i_1)$ decreases with a_0. ■

Endogenous search

So far we have endogenized the time required to negotiate the sale of assets. We now endogenize the time it takes to receive trading opportunities, $1/\alpha$. To do this, we introduce a participation decision on one side of the market. Suppose that producers of the DM good can choose to participate in the DM at some cost.[32] The measure of producers who participate is denoted n and the measure of DM matches is $\alpha(n)$ where $\alpha(n)$ is strictly concave. Assuming an interior solution n solves

$$\kappa = \frac{\alpha(n)}{n} \int_0^y e^{-\lambda \frac{z(x)}{\delta}} \frac{(1-\theta)\upsilon'(x)\left[u'(x) - \upsilon'(x)\right]}{\theta u'(x) + (1-\theta)\upsilon'(x)} dx. \tag{125}$$

The entry cost, κ, is equal to the expected surplus of the producer, which is equal to the matching probability, $\alpha(n)/n$, times the expected surplus from the negotiation. Entry requires $\theta < 1$ and it increases with y and δ.

[32] There are several ways to endogeneize the measure of trades taking place within a period: through costly entry (e.g., Rocheteau and Wright, 2005), endogenous search intensity (Lagos and Rocheteau, 2005), endogenous market composition (Rocheteau and Wright, 2009), endogenous asset acceptability (e.g., Lester et al., 2012).

From the previous subsection we can derive an equilibrium condition between y and α by substituting the market clearing spread, $s(Ad, \alpha)$, into the asset demand, $z^b(s, \alpha)$, and using the fact that $z^b = z(y)$. As α increases so does y. Hence, there is a positive relationship between y and n. As n approaches 0, s tends to 0 and $z(y) = \min\{(1+\rho)Ad/\rho, z(y^*)\}$. As n tends to infinity, α approaches 1 and $z(y)$ approaches a finite limit. The second relationship between n and y is given by (125). Provided that κ is not too large, an active equilibrium exists. Moreover, the model can generate multiple steady-state equilibria. At the high equilibrium the entry curve intersects the asset demand curve by below in the space (n, y). Hence, a reduction in κ that shifts the entry curve to the right leads to higher n and y and hence higher δ. This illustrates how search and negotiation frictions are complement.

We now compare equilibrium allocations to the constrained-efficient allocation in the absence of bargaining friction, λ goes to 0. The constrained-efficient allocation is a pair (y, n) that maximizes total welfare, $\alpha(n)\left[u(y) - \upsilon(y)\right] - \kappa n$. The solution is $y = y^*$ and $\alpha'(n)\left[u(y^*) - \upsilon(y^*)\right] = \kappa$.

Proposition 17 *Suppose there is free entry of producers and asymmetric gradual bargaining. An equilibrium allocation coincides with the constrained-efficient allocation if*

$$Ad \geq \frac{\rho}{1+\rho} \int_0^{y^*} \frac{u'(x)\,\upsilon'(x)}{\theta u'(x) + (1-\theta)\upsilon'(x)} dx \tag{126}$$

and

$$\frac{\alpha'(n)n}{\alpha(n)} = (1-\theta) \frac{\int_0^{y^*} \frac{\upsilon'(x)\left[u'(x) - \upsilon'(x)\right]}{\theta u'(x) + (1-\theta)\upsilon'(x)} dx}{\int_0^{y^*} \left[u'(x) - \upsilon'(x)\right] dx}. \tag{127}$$

Proof of Proposition 17. We verify that $s = 0$, $y = y^*$ and $n = n^*$is an equilibrium. Note that (126) ensures that $s = 0$ and $y = y^*$ solve (**??**) and (**??**), regardless of n. Now, (127) ensures that n^* solves (125). ■

The gradual Nash bargaining solution can implement the efficient allocation while the generalized bargaining solution cannot. The first best requires liquidity to be plentiful and a modified version of the Hosios condition to be satisfied. The bargaining power must be such that the effective producers' share is equal to the producers' contribution in the matching process. ■