Information and Optimal Trading Strategies with Dark Pools *

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Abstract

We study how asymmetric information affects market participants' choice of trading venue (either an exchange or dark pool), and the optimal submission strategies in a sequential trading game. The exchange is organized as a fully transparent limit order book, and the dark pool is an opaque venue where orders are continuously executed at the midpoint of the bid and ask prices that prevail in the exchange. We find that, when the limit order book conveys no information, rational uninformed traders never trade in the dark pool due to price risk. However, price risk may be reduced when the information in the book induces an uninformed buyer (seller) to believe that the value of the asset is high (low) since the order was previously submitted by an informed buyer (seller). Adding a dark pool alongside an exchange may divert the informed trader from the exchange to the dark pool if the execution risk in the dark is sufficiently low. An uninformed trader only goes to the dark if the limit order book is sufficiently informative and price risk is low. We show that adding a dark pool alongside an exchange reduces price informativeness and increases the expected welfare of rational traders. Its effects on market liquidity and trading volume depend on stock market characteristics since these determine whether traders supply, demand or do not provide liquidity in the exchange when the dark pool is unavailable.

Keywords: trading venues, dark liquidity, limit order book, price risk, adverse selection

JEL codes: G12, G14, G18

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1 Introduction

In today's financial markets, traders have access to different types of trading venues, which differ in their level of transparency. In addition to the traditional exchanges (lit markets), traders also have access to dark pools - "trading venues or mechanisms containing anonymous, non-displayed liquidity that is available for execution " (Banks, 2014). Dark pools grew as a result of technological innovations and Reg NMS regulation in 2005.[1] Their current consolidated trading volume in US equity markets is around 14%, while in European equity markets is around 9.1% (Rosenblatt Securities Inc.). Our paper studies how the existence of a dark pool affects traders' optimal submission strategies in a sequential trading game with asymmetric information. Informed and uninformed traders face a simultaneous choice of trading venue and order type in the exchange (market order or limit order) or refrain from trading. We model the competition between an exchange (that is organized as a limit order book) and a dark pool in the presence of asymmetric information. Our model therefore allows us to understand the role played by information in the price discovery process and the strategic behavior of traders in the presence of asymmetric information but also, unlike other papers in the literature, we study the optimal venue and type of order decision, and how the leakage of information affects this decision.

Our main finding is that adding a dark pool alongside an exchange decreases price informativeness in the exchange in the first period and improves expected profits of both informed and uninformed traders. The effect on expected inside spread and trading volume depends on stock market characteristics such as liquidity, volatility, adverse selection, tick size etc. In addition, adding a dark pool alongside an exchange may switch the optimal strategies of each type of traders. Thus, in the first period, the informed trader may divert from the exchange to the dark pool if the execution risk in the dark pool is sufficiently low. The uninformed trader does not go to the dark pool when the limit order book contains no information since price risk is too high. However, adding a dark pool alongside an exchange may switch the optimal strategy of the uninformed trader from no trade to placing a limit order in the exchange due to the reduction in adverse selection. An uninformed trader only goes to the dark when the limit order book is sufficiently informative and when price risk is low. The stock market characteristics determine also if there is or not segmentation of the informed and uninformed order flow in the two venues. These results are very important given the current policy debate on the impact of the dark pools on price discovery, market liquidity and order flow segmentation.

Our model reflects the main characteristics of today's financial markets. The exchange is organized as a fully transparent limit order book with a discrete price grid. Despite the fact that there exist many types of dark pools, our modeling captures two of their main features: (1) no pre-trade transparency. Dark pools are completely opaque in the sense that do not quote the liquidity that is available and this makes execution uncertain; (2) they do not determine prices and derive their

[1] The policy debate on the regulation of dark pools is currently very active (see, for example, U.S. Securities and Exchange Commission, 2015).

price from those prevailing in the exchange as the midpoint between the best bid and ask prices at any point in time (if the order is executed). This type of pricing is typical of dark pools which are owned by agency brokers or exchanges and represent 57% of the consolidated dark trading volume (Buti et al. 2017). Traders may be of two different types: rational or liquidity traders. Rational traders strategically choose whether or not to trade, and if they trade they simultaneously choose the venue and the type of order (in the exchange both market and limit orders are available) given their information at each point in time. Rational traders are (privately) informed if they know the liquidation value of the asset, and (privately) uninformed traders if they only know the distribution of the liquidation value of the asset conditional on the information provided by the book. Since the limit order book is fully transparent, information in the book is public and available to all types of traders.

The timing of the model is as follows. First, the liquidation value of the asset is realized. Second, there are two periods of trading. In each period, a new trader may arrive to the market. Rational traders observe the state of the limit order book and strategically choose an order submission strategy that maximizes expected profits given the information set at each period. Liquidity traders always trade based on their exogenous liquidity needs, and only submit market orders to the exchange to ensure execution. Third, if in the first period a trader had submitted a dark pool order and this order was not executed in the dark, then the order returns to the exchange. Fourth, the liquidation value of the asset is made public and the trading game is over. Since our model can be represented by a sequential game of incomplete information, the equilibrium concept used is the Perfect Bayesian Equilibrium. To the best of our knowledge, we are the first to model the competition between an exchange that is organized as a limit order book and a dark pool in the presence of asymmetric information.

To understand the effects of adding a dark pool alongside the exchange, we first discuss the equilibrium in the benchmark model where traders do not have access to the dark pool.[2] We solve the model backwards. In the last trading period, limit orders are not chosen since they will not be executed. An informed trader always chooses a market order since it gives positive profits. An uninformed trader obtains information about the state of the book and updates his beliefs about the value of the asset. An uninformed trader selects a market order if he strongly believes that an informed trader in the previous period had chosen an order of the same direction, which indicates that the value of the asset is favorable. Otherwise, an uninformed trader refrains from trading. In the first trading period, an informed trader chooses a market order over a limit order when he prefers immediacy to the potential price improvement provided by a limit order. An uninformed trader selects a limit order instead of not trading when there is a high probability that his order will be executed against the order of a liquidity trader instead of an informed trader (low adverse selection). If his limit order is executed against a market order of the opposite sign submitted by an informed trader, then it reveals that the value of the asset is disadvantageous (high adverse selection). Even

[2]In the benchmark model the third stage described in the previous paragraph is dropped.

though the optimal strategies profiles are unique in the first trading period, multiple equilibria may exist in the case that in the first trading period informed traders choose market orders and uninformed traders choose not to trade.

Adding a dark pool not only enlarges traders' strategies set but also may induce a substitution of trading venue, order type, and market participation in relation to when the dark pool is unavailable. In the second round of trading, the optimal trading strategy depends on the state of the LOB and the assessment of the expected profits. An informed trader chooses between a market order or a dark order depending on the trade-off between price improvement and execution risk in the dark versus immediate execution. In the second round of trading, uninformed traders refrain from trading if the LOB is sufficiently uninformative. However, if the LOB indicates the value of the asset is favorable, uninformed traders choose between market orders and dark orders. Whether there is segmentation of the order flow in the second round of trading (equilibria where informed and uninformed participate both in the lit and dark or equilibria where only one type of traders participates in the dark) depends on stock characteristics.

In the first trading period, even if execution risk in the dark pool is high, informed traders tend to replace market orders by dark orders when they can take advantage of the price improvement. The expected profit from submitting a dark order is the largest when the discount factor is high and when there is a small probability that the market moves against the trader. As the execution risk in the dark reduces, strategy profiles in which informed traders submit market orders cannot be anymore an equilibrium of the game, and informed traders decide to go to the dark. Despite of the fact that uninformed traders do not select to go to the dark pool when the limit order book is uninformative (due to adverse selection and price risk in the first trading period), adding a dark pool alongside an exchange affects the optimal submission strategies of the uninformed trader even in the first period. Uninformed traders may switch from no trade to trade in the exchange using limit orders. This is because the mere existence of the dark market offers the possibility to informed traders in the second period to migrate from the exchange to the dark, and consequently reduce the adverse selection which induces uninformed traders to participate in the market submitting limit orders. Despite the fact that optimal strategies profiles are unique in the first trading period, multiple equilibria may exist unless in the first trading period informed traders choose dark pool orders and uninformed traders choose limit orders.

In terms of market quality and welfare, we find that the addition of a dark pool alongside an exchange decreases price informativeness but it increases the expected profits of rational traders (both informed and uninformed). The effect of adding a dark pool on the expected inside spread and trading volume depends on stock market characteristics. These characteristics determine if a trader who migrates to the dark pool was a supplier or a consumer of liquidity in the exchange (when the dark pool was unavailable) and therefore whether the migration to the dark pool has a positive or negative effect on market liquidity and trading volume. These theoretical results help reconcile the positive and negative effects the existence of dark pools has on the market performance of competing exchanges found in the empirical studies, and provides new empirical predictions for cross-sectional analysis.

Our paper is closely related to the theoretical analysis which analyzes the effects of adding a dark pool alongside an exchange on market performance. This literature shows that the impact of the dark pool on price discovery is ambiguous. On the one hand, there is a strand of literature that studies the competition between a dealer market and a dark pool (or crossing network) in the presence of asymmetric information. Zhu (2014), using a Glosten and Milgrom (1985) type model, finds that adding a dark pool alongside an exchange concentrates price-relevant information into the exchange and improves price discovery. This is due to the fact that a continuum of informed traders receive the same perfect signal and trade simultaneously on the same signal.[3] Therefore, when they submit orders to the dark pool, their execution probability reduces as all submit orders on the same side. Thus, Zhu (2014) shows that the informed traders submit orders to the dark pool only if the uncertainty of the asset value is very high. Ye (2011), using a Kyle (1985) type model, finds that a dark pool reduces price discovery and volatility. Note that these two models impose a different market structure: Ye (2011) only allows the informed trader to select trading venue, while Zhu (2014) allows both informed and liquidity traders to select the venue to trade.

Competition between dealer markets and other forms of exchange, such as passive crossing networks (similar to dark pools), and in the presence of asymmetric information has also been analyzed by Hendershott and Mendelson (2000) and Degryse et al. (2009). Hendershott and Mendelson (2000) in a static setup find that a crossing network imposes positive liquidity externalities and negative crowding externalities on each other and therefore have ambiguous effects on spread. Degryse et al. (2009) show that the same positive and negative externalities are preserved in a dynamic setup and analyze how welfare and the order flow dynamics depend on the degree of market transparency.

On the other hand, Buti et al. (2017) model competition between a fully transparent limit order book and a dark pool without asymmetric information regarding the asset value. They show that the welfare effects of adding a dark pool are negative if the initial book is illiquid, while when book liquidity increases, large traders are better off and small traders are worst off. They also find that the market share of the dark pool is higher when the depth of the limit order book is high, when the spread of the limit order book is narrow, when the tick size is large, and when traders seek protection from price impact.

We build a limit order book model with asymmetric information that complements the models of Zhu (2014) and Buti et al. (2017). We model a limit order book model where traders have asymmetric information and common valuation, while Buti et al. (2017) consider a model of symmetric information and private valuations. The existence of private information in our model permits us to analyse the informational role of prices, the adverse selection problem and their impact on the segmentation of the order flow, as well as to study the traders'dynamic strategies as in Buti et al. (2017). Compared with Zhu (2014) who models competition between a dealer market (where

[3]Note that Zhu (2014) models a market with a large number of informed strategic traders. As Kyle (1989) points out this does not reflect the situation of real financial markets.

traders submit market orders) and a dark pool with asymmetric information, we model a limit order book where traders can submit both market and limit orders to the exchange. Moreover, trading takes place sequentially in our model and this together with the different market structure leads to exactly opposite results in terms of price discovery. Note that in our model the segmentation of the order flow arises endogenously and it depends on the stock market characteristics. We show that when the order book is not informative the informed traders might migrate to the dark pool while uninformed stay in the lit market and therefore the informed traders do not contribute to the price discovery in the lit market.

Our research is also related to two broader strands of literature that the previously mentioned papers tackle: competition between integrated and segmented markets, and traders' optimal order submission strategies. On the one hand, early theoretical papers involving multiple trading venues (see, for instance, Pagano, 1989; and Chowdry and Nanda, 1991) show that there is a natural tendency towards agglomeration since liquidity increases due to scale, and it is beneficial. This tendency may be offset by the presence of frictions, trading costs, informational barriers or regulatory obstacles. Since market fragmentation is associated with the surge of venues with different degrees of transparency, a part of this literature relates to fragmentation and market transparency.[4] Concerning the visibility of market quotes, for instance, Biais (1993) and Frutos and Manzano (2002) compare centralized and fragmented markets and show that the ability to observe price setters' quotes affects spreads and market participants' welfare. In relation to the disclosure of post-trade information, Madhavan (1995) shows that delaying disclosure benefits large traders who place multiple trades. Frutos and Manzano (2005) in a two-stage trade model show that opaqueness increases competition among dealers to attract valuable order flow, leading to better prices for investors in the first period, while the effect on market participants' welfare in second period is ambiguous.

On the other hand, there is also a large literature that studies limit order markets and traders' optimal order submission strategies with models of asymmetric information . The early static models of limit order markets assume that informed traders only use market orders, while uninformed traders or liquidity traders only use limit orders. Angel (1994), Easley and O'Hara (1991) and Harris (1998) model an informed investor's order placement strategy in choosing between market and limit orders. They argue that informed traders are more likely to use market orders and rarely use limit orders. Also, Glosten (1994), Rock (1996), Seppi (1997), and Biais et al. (2000) argue that informed traders prefer market orders to profit from their private information. In contrast, Chakravarty and Holden (1995) and Kaniel and Liu (2006), who analyze informed traders' choice between limit orders and market orders, find that informed traders may prefer limit orders since they may actually convey less information than market orders. Finally, Parlour (1998), Foucault (1999), Foucault, Kadan and Kandel (2005), Goettler et al. (2009), Rosu (2009), Van Achter (2008), and Rosu (2012) introduce dynamic models that allow informed and uninformed traders to determine their optimal choice of order type in limit order markets. Empirical studies find that both informed

[4] See also Gomber et al. (2016) for a review of the consolidation versus fragmentation of markets.

and uninformed traders use a mixture of market orders and limit orders (see for example Biais et al. 1995; Kavajecz and Odders-White 2004; Anand et al., 2005). Bloomfield et al. (2005) conduct a laboratory experiment and find that informed traders use both market and limit orders. They use market orders earlier in the trading period to profit from their private information, and then (as the prices get closer to true value) they use their private information to switch to limit orders to earn the bid-ask spread. Uninformed traders use limit orders early but then switch to market orders to meet their liquidity targets (they are liquidity traders). They also find that informed traders' submission patterns are more sensitive to changes in market transparency. We build on this literature by developing a model in which rational traders can decide simultaneously the venue in which they want to trade and their optimal order submission strategy.

The paper is organized as follows. Section 2 presents the model. Section 3 analyzes the equilibrium in benchmark model without dark pool. Section 4 presents the equilibrium in the full model where rational traders have access to the dark pool. Section 5 analyzes how welfare and market quality change when we add a dark pool alongside an exchange and section 6 provides the empirical implications of these results. Section 7 concludes. Proofs are presented in the Appendices.

2 Model

We consider an economy in which a single risky asset is traded. The liquidation value of the asset, $\widetilde{v}$, may take two values, $v \in \{v^H, v^L\}$, with equal probabilities, μ is the unconditional mean of $\widetilde{v}$, and $\sigma > 0$ its standard deviation. The asset may be traded in two venues: an exchange or a dark pool.

The exchange is organized as a limit order book (hereafter LOB). We assume that the initial LOB has two prices on each side of the book: A_1^1, A_1^2, B_1^1, B_1^2, such that $v^L < B_1^2 < B_1^1 < A_1^1 < A_1^2 < v^H$. We assume that prices are placed on a grid and that the following relationships hold:

$$\begin{aligned} A_1^1 &= \mu + k_1\tau,\ A_1^2 = \mu + k_2\tau,\ v^H = \mu + k_3\tau, \\ B_1^1 &= \mu - k_1\tau,\ B_1^2 = \mu - k_2\tau,\ v^L = \mu - k_3\tau, \end{aligned}$$

with $1 \leq k_1 < k_2 < k_3$, where k_1 and k_2 are natural numbers, and τ is the tick size (i.e., the minimum price increment that traders are allowed to quote over the existing price). Note that $k_3\tau = \sigma$, and therefore, k_3 is a real number. For simplicity we assume that the depth of the LOB at each bid and ask price is equal to 1, and that the LOB follows price and time priority rules.[5] The LOB is fully transparent (i.e., all the information in the LOB is available to all market participants at any point in time). There are no transaction costs.

The dark pool is completely opaque in the sense that an order submitted to the dark pool is not observable to anyone but the trader who submitted it. We assume that the dark pool has

[5]First, the order with the best price is executed. Second, among the orders with the same price, they are executed in order of arrival.

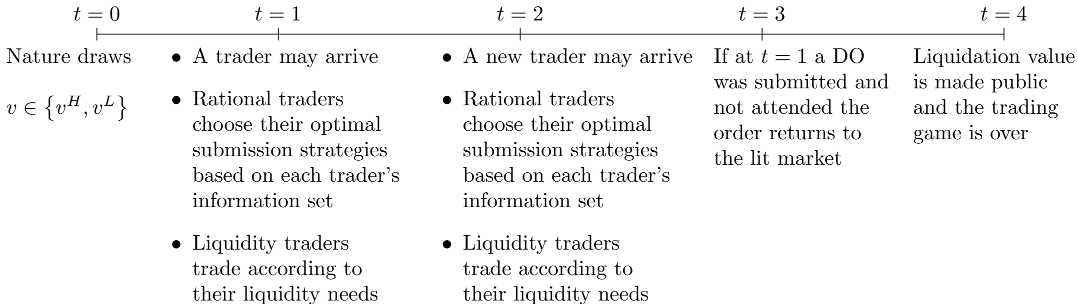

Figure 1: Time line of the trading game when traders have access to the dark pool

an execution probability $\theta \in [0, 1]$ that is exogenous and does not change in time. If an order is submitted to the dark pool and it is attended at time t then the execution price is equal to the midpoint of the exchange at time t: $\dfrac{A_t^1 + B_t^1}{2}$. If the order is *not* attended in the dark pool then it returns to the exchange at $t + 2$.[6]

The sequence of events described also in Figure (1) is as follows:

Date t=0: The liquidation value of the asset $\widetilde{v}$ is realized.

Dates t=1, 2: In each date, a *new* trader may arrive to the market and may either trade 0 or 1 unit of the asset. An informed trader observes the liquidation value of the asset. The state of the LOB at the beginning of each date is public information. A rational trader chooses an order submission strategy that maximizes his expected profits given the information set at each date. Liquidity traders always trade based on their exogenous liquidity needs. All traders may trade one unit of the asset.

Date t=3: If at $t = 1$ a rational trader submitted a dark pool order and this order was not attended, then the order returns to the exchange as a market order.

Date t=4: The liquidation value of the asset is made public and the trading game is over.

Figure 2 illustrates the tree of events related to the first trading period.[7] There are two possible types of traders: rational traders or liquidity traders. All traders are risk neutral. A rational trader arrives to the market with probability $\lambda > 0$, a liquidity trader arrives with probability $\eta > 0$, and no trader arrives with probability $1 - \lambda - \eta \geq 0$. Rational traders may be either (privately) informed if they have perfect information about the liquidation value of the asset (with probability π), or (privately) uninformed if they only know the distribution of the liquidation value of the asset (with probability $1 - \pi$). An informed trader buys whenever he observes $v = v^H$ (henceforth IH), and sells whenever he observes $v = v^L$ (henceforth IL). An uninformed trader is a buyer with

[6]Our assumptions are motivated by agency-broker or exchanged-owned dark pools. In these dark pools, price discovery does not take place since, if an order is executed, the price is equal to the midpoint of the National Best Bid and Offer (in this model the lit exchange).

[7]A similar tree of events for the second trading period could be drawn.

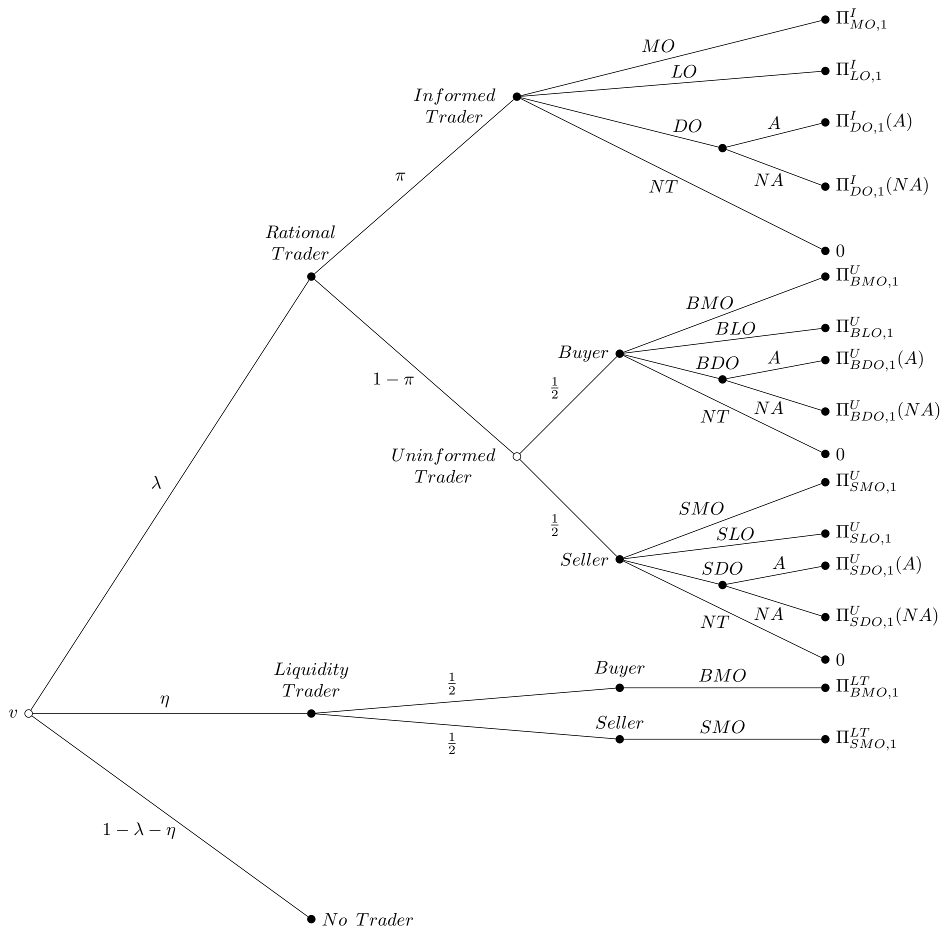

Figure 2: Tree of events of the first trading period.

probability $\dfrac{1}{2}$ (henceforth UB) and a seller with probability $\dfrac{1}{2}$ (henceforth US). There is also a discount factor, $\delta \in [0, 1]$, that is common across traders and periods.

Liquidity traders buy (with probability $\frac{1}{2}$) or sell (with probability $\frac{1}{2}$) for liquidity or hedging needs. Hence, in order to ensure immediate execution they trade in the exchange and always submit market orders. Rational traders trade up to one unit of the risky asset and choose an order submission strategy that maximizes their expected profits conditional on their information set at each date, I_t, which includes information about the liquidation value of the asset and about the state of the LOB. Rational traders simultaneously select whether not to trade (NT) or to trade, the trading venue (exchange or dark pool, DO), and the order type in the exchange (market order, MO, or limit order, LO). Consequently, the possible strategies of a rational trader (both informed and uninformed) are

$$\mathbb{O}_D = \{MO,\ LO,\ DO,\ NT\}\,, \tag{1}$$

where B in front of an order type denotes a buy order and S denotes a sell order. Note that the direction of trade for informed traders is endogenous since it depends on their private information.

For each possible order type, we next examine its characteristics and the associated profits for a rational trader that submits it. Internet Appendix I describes the expected profits of all traders, at all times, for all the possible states of the LOB.[8] Denote the profits of a particular order as $\Pi^R_{\mathcal{O},t}$, where superscript R denotes that the order comes from a rational trader (either informed, I, or uninformed, U); subscript $\mathcal{O}$ is the order type $\mathcal{O} \in \mathbb{O}_D$ defined in (1); and subscript t is the date when the order is submitted. If profits are strictly positive then a rational trader can choose between a MO or LO, and if available, or a DO. Otherwise, he chooses not to trade (NT).

- Market order (MO): Market orders are executed immediately at the given best available ask/bid prices. The expected profits of a buy market order at date t are

$$\mathbb{E}\left(\Pi^R_{BMO,t}|I_t\right) = \mathbb{E}\left(\widetilde{v}|I_t\right) - A^1_t,$$

  and the expected profits of a sell market order are

$$\mathbb{E}\left(\Pi^R_{SMO,t}|I_t\right) = B^1_t - \mathbb{E}\left(\widetilde{v}|I_t\right).$$

- Limit orders (LO): A limit order that improves the current market price may be executed in the next period if a market order of the opposite sign hits the limit order. Therefore, the expected profits from a limit order are discounted by δ. Thus, limit orders provide better prices than market orders but exhibit execution risk. We assume that a LO always improves the price by one tick because: (i) it is never optimal for the trader to improve the price by more than one tick since it reduces his profits. (ii) it is never optimal for the trader to submit a non-improving LO since the order is not executed (the order goes to the queue, and due to the time priority, it is not executed), and the trader obtains zero profits. Hence, the expected profits of a buy limit order at date t are

$$\mathbb{E}\left(\Pi^R_{BLO,t}|I_t\right) = p^R_{BLO,t}\left(I_t\right)\delta(\mathbb{E}\left(\widetilde{v}|I_t\right) - B^1_t - \tau),$$

  and of a sell limit order at date t is

$$\mathbb{E}\left(\Pi^R_{SLO,t}|I_t\right) = p^R_{SLO,t}\left(I_t\right)\delta(A^1_t - \tau - \mathbb{E}\left(\widetilde{v}|I_t\right)),$$

  where $p^R_{BLO,t}$ ($p^R_{SLO,t}$) is the probability of execution of a buy (sell) LO submitted by a rational trader at time t, respectively.

- Dark orders (DO): With probability θ an order submitted to the dark pool is attended (executed), and with probability $(1-\theta)$ it is not attended. Since no new trader arrives in the

[8] The Internet Appendix I is available from the authors upon request.

market at $t = 3$, 4, an order that returns to the exchange from dark pool at $t+2$ will be either a MO or NT since the probability of execution of a LO at $t = 3$ is equal to 0. Therefore, the expected profits of a buy dark order submitted at time t are:

$$\mathbb{E}\left(\Pi^R_{BDO,t}|I_t\right) = \theta\left(\mathbb{E}\left(\widetilde{v}|I_t\right) - \frac{A^1_t + B^1_t}{2}\right) + \max\left\{(1-\theta)\delta^2\mathbb{E}\left(\Pi^R_{BMO,t+2}|I_t\right), 0\right\}$$

and for a sell dark order submitted at time t, the expected profits equal

$$\mathbb{E}\left(\Pi^R_{SDO,t}|I_t\right) = \theta\left(\frac{A^1_t + B^1_t}{2} - \mathbb{E}\left(\widetilde{v}|I_t\right)\right) + \max\left\{(1-\theta)\delta^2\mathbb{E}\left(\Pi^R_{SMO,t+2}|I_t\right), 0\right\}.$$

- No trade (NT): A trader refrains from participating in the market which leads to zero profits at time t: $\mathbb{E}\left(\Pi^R_{NT,t}|I_t\right) = 0$.

In case of equality of profits, we assume that a MO dominates LO and DO; and a LO dominates DO. If the expected profits of a MO are null, a rational trader refrains from trading.

Our model can be represented by a sequential game of incomplete information. The equilibrium definition used is as follows.

Definition 1 *A Perfect Bayesian Equilibrium (henceforth PBE) of the trading game is a strategy profile for all rational traders and belief system about other traders types at all information sets such that:*

i) Sequential Rationality: Given the belief system, at each information set each trader's strategy specifies an optimal order that maximizes traders' expected profits given his beliefs and the strategies of other traders.

ii) Consistent beliefs: Given the strategy profile, the beliefs are consistent with Bayes rule (when appropriate).

In what follows, we focus on symmetric Perfect Bayesian Equilibria in pure strategies. A symmetric equilibrium refers to a situation where buyers and sellers with the same information (i.e, informed or uninformed) choose the same type of order in the first round of trading (except from the direction of trade).

3 Equilibrium in the benchmark model without dark pool

We first consider the benchmark model without a dark pool (ND) where the available orders are: $\mathbb{O}_{ND} = \{MO, LO, NT\}$. The sequence of events can be seen in Figure (3). Note that the difference with respect to the timeline in Figure (1) when the dark pool is not available is that the liquidation value of the asset is revealed at $t = 3$.

We define Ω_o and Γ_o as the probability that an informed trader and uninformed trader at $t = 1$ choose an order $\mathcal{O} \in \mathbb{O}_{ND}$, where $o = 0$ corresponds to a NT order; $o = 1$ to a MO; $o = 2$ to a LO;

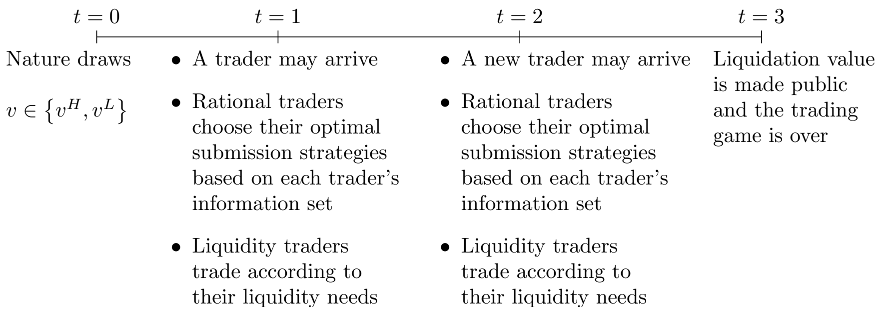

Figure 3: Time line of the trading game when traders do not have access to the dark pool

and such that $\sum_{o=0}^{2} \Omega_o = 1$, and $\sum_{o=0}^{2} \Gamma_o = 1$. We also define by $\mathbb{B}$ the set of all possible states of the LOB at the end of the first trading period and by $\mathcal{B}_1 \in \mathbb{B}$ a possible state of the book such that

$$\mathcal{B}_1 = \begin{cases} \varnothing, & \text{if the best prices in the book are } (A_1^1, B_1^1) \\ BMO, & \text{if the best prices in the book are } (A_1^2, B_1^1) \\ BLO, & \text{if the best prices in the book are } (A_1^1, B_1^1 + \tau) \\ SMO, & \text{if the best prices in the book are } (A_1^1, B_1^2) \\ SLO, & \text{if the best prices in the book are } (A_1^1 - \tau, B_1^1). \end{cases}$$

Note that the state of the book $\mathcal{B}_1 = \varnothing$, can be obtained either because no trader arrived or a trader arrived but he decided not to trade, while the other states of the book are uniquely determined by the traders'actions at $t = 1$.

We solve the game backwards. At $t = 2$, the expected profits for an informed buyer and seller are summarized in Table *1,* while Table *2* presents the expected profits of an uninformed buyer and seller.

	IH			IL		
	BMO	BLO	NT	SMO	SLO	NT
(A_1^1, B_1^1)	$(k_3 - k_1)\,\tau$	0	0	$(k_3 - k_1)\,\tau$	0	0
(A_1^2, B_1^1)	$(k_3 - k_2)\,\tau$	0	0	$(k_3 - k_1)\,\tau$	0	0
$(A_1^1, B_1^1 + \tau)$	$(k_3 - k_1)\,\tau$	0	0	$(k_3 - k_1 + 1)\,\tau$	0	0
(A_1^1, B_1^2)	$(k_3 - k_1)\,\tau$	0	0	$(k_3 - k_2)\,\tau$	0	0
$(A_1^1 - \tau, B_1^1)$	$(k_3 - k_1 + 1)\,\tau$	0	0	$(k_3 - k_1)\,\tau$	0	0

Table 1: Expected profits of an informed buyer (IH) and an informed seller (IL) at $t = 2$ when traders do not have access to the dark pool.

	UB			US		
	BMO	BLO	NT	SMO	SLO	NT
(A_1^1, B_1^1)	$-k_1\tau$	0	0	$-k_1\tau$	0	0
(A_1^2, B_1^1)	$(Xk_3 - k_2)\,\tau$	0	0	$-(k_1 + Xk_3)\,\tau$	0	0
$(A_1^1, B_1^1 + \tau)$	$(Yk_3 - k_1)\,\tau$	0	0	$-(k_1 - 1 + Yk_3)\,\tau$	0	0
(A_1^1, B_1^2)	$-(Xk_3 + k_1)\,\tau$	0	0	$(Xk_3 - k_2)\,\tau$	0	0
$(A_1^1 - \tau, B_1^1)$	$-(Yk_3 + k_1 - 1)\,\tau$	0	0	$(Yk_3 - k_1)\,\tau$	0	0

Table 2: Expected profits of an uninformed buyer (UB) and an uninformed seller (US) at $t = 2$ when traders do not have access to the dark pool.

Note that at $t = 2$ the expected profits of each strategy depend on the state of the LOB (which on its turn depends on the chosen strategy at $t = 1$). Uniformed traders at $t = 2$ form beliefs about the strategies and type of player in $t = 1$. Thus, we define the uninformed traders' belief at $t = 2$ about the probability that the MO observed in the LOB was submitted by an informed trader as

$$X = \frac{\lambda\pi\Omega_1}{\eta + \lambda\pi\Omega_1 + \lambda\,(1 - \pi)\,\Gamma_1}. \tag{2}$$

Similarly, we define the uninformed traders' belief at $t = 2$ about the probability that the LO (observed in the LOB) was submitted by an informed trader as

$$Y = \frac{\pi\Omega_2}{\pi\Omega_2 + (1 - \pi)\,\Gamma_2}. \tag{3}$$

By comparing the expected profits of rational traders in $t = 2$ we obtain the following lemma.

Lemma 1 *In equilibrium the following results hold:*

- *at $t = 2$ an informed trader always submits a MO.*
- *at $t = 2$ an uninformed trader may submit either MO or NT, but never chooses LO. The optimal strategy for an uninformed trader is presented in Table 3:*

An informed trader at $t = 2$ always chooses MO since it generates positive expected profits, while the expected profits of LO or NT are always null. An uninformed trader never chooses LO since the probability of execution is 0, because no new orders arrive at $t = 3$ and, hence, the expected profits are null. An uninformed trader's choice at $t = 2$ depends on the state of the LOB since it reveals information. The choice will be either MO or NT. Without loss of generality, let us focus on an uninformed buyer at $t = 2$.[9] When the state of the LOB conveys no information

[9] The argument for an uninformed seller follows since there exists the following symmetry: If the state of the LOB is (A_1^1, B_1^1), uninformed sellers and buyers always make the same choice. If the state of the LOB is (A_1^2, B_1^1), uninformed sellers choose the same as uninformed buyers when the LOB is (A_1^1, B_1^2). If the state of the LOB is $(A_1^1, B_1^1 + \tau)$ then uninformed sellers choose the same as uninformed buyers when the state of the LOB is $(A_1^1 - \tau, B_1^1)$.

State of the Book	*UB*	*US*
(A_1^1, B_1^1)	NT	NT
(A_1^2, B_1^1)	$\begin{cases} MO & \text{if } X > \dfrac{k_2}{k_3} \\ NT & \text{if } X \leq \dfrac{k_2}{k_3} \end{cases}$	NT
$(A_1^1, B_1^1 + \tau)$	$\begin{cases} MO & \text{if } Y > \dfrac{k_1}{k_3} \\ NT & \text{if } Y \leq \dfrac{k_1}{k_3} \end{cases}$	NT
(A_1^1, B_1^2)	NT	$\begin{cases} MO & \text{if } X > \dfrac{k_2}{k_3} \\ NT & \text{if } X \leq \dfrac{k_2}{k_3} \end{cases}$
$(A_1^1 - \tau, B_1^1)$	NT	$\begin{cases} MO & \text{if } Y > \dfrac{k_1}{k_3} \\ NT & \text{if } Y \leq \dfrac{k_1}{k_3} \end{cases}$

Table 3: Optimal trading strategies of an uniformed buyer (UB) and seller (US) at $t = 2$ when traders do not have access to the dark pool.

(i.e., (A_1^1, B_1^1)) then the optimal choice is NT since the expected profits of MO are negative. If the LOB reveals that at $t = 1$ there has been a BMO or BLO (i.e., (A_1^2, B_1^1) or $(A_1^1, B_1^1 + \tau)$), then the uninformed buyer at $t = 2$ chooses BMO if his belief that order came from an informed trader at $t = 1$ is sufficiently strong (i.e., X or Y is sufficiently large, respectively) so that expected profits are positive. Otherwise, the uninformed trader refrains from trading. In addition, if the state of the LOB is either (A_1^1, B_1^2) or $(A_1^1 - \tau, B_1^1)$ then it reveals that the trader at $t = 1$ submitted a SMO or SLO, respectively, which implies that uninformed trader's expected profits of submitting a BMO at $t = 2$ are negative.

At $t = 1$, expected profits of an informed and uniformed trader are presented in Table *4* and Table *5*, respectively.

IH	*IL*	*Expected Profits*
BMO	SMO	$(k_3 - k_1)\,\tau$
BLO	SLO	$\dfrac{\delta\eta}{2}\,(k_3 + k_1 - 1)\,\tau$
NT	NT	0

Table 4: Expected profits of an informed buyer (IH) and seller (IL) at $t = 1$ when traders do not have access to the dark pool.

UB	US	*Expected Profits*
BMO	SMO	$-k_1\tau$
BLO	SLO	$\frac{\delta}{2}\left(\left(\lambda\pi+\eta\right)\left(k_1-1\right)-\lambda\pi k_3\right)\tau$
NT	NT	0

Table 5: Expected profits of an uninformed buyer (UB) and seller (US) at $t = 1$ when traders do not have access to the dark pool.

The following lemma presents the informed and uninformed traders' optimal strategies at $t = 1$.

Lemma 2 *In equilibrium the following results hold:*

- *at $t = 1$ an informed trader never chooses NT.*
- *at $t = 1$ an uniformed trader never chooses a MO.*

At $t = 1$, an informed trader never chooses NT since it is always dominated by at least a MO and, hence, an informed trader may either choose MO or LO. In contrast, an uninformed trader at $t = 1$ never chooses a MO since its expected profits are negative, and it is always dominated by at least the NT strategy and, consequently, an uninformed trader at $t = 1$ may either choose LO or NT. Hence, the candidate strategy profiles at $t = 1$ that can be sustained as a symmetric PBE are:

$$(BMO,\ SMO,\ BLO,\ SLO),\quad (BMO,\ SMO,\ NT,\ NT),$$
$$(BLO,\ SLO,\ BLO,\ BLO),\quad (BLO,\ SLO,\ NT,\ NT),$$

where the two first components correspond to strategies of informed traders at $t = 1$ (IH and IL, respectively) and the two last components correspond to strategies of uninformed traders at $t = 1$ (UB and US, respectively).

We are now in a position to characterize the PBE of the reduced trading game where the dark pool is not available.

Proposition 1 *If $k_1 > 1$, then a PBE of the game is as follows:*

- *$\mathcal{E}_1^{ND}$: (BMO, SMO, BLO, SLO) is the optimal strategy profile at $t = 1$ if*

Conditions
$k_3 - k_1 \geq \delta\frac{\eta}{2}\left(k_3 + k_1 - 1\right)$ *and* $\left(\lambda\pi+\eta\right)\left(k_1-1\right)-\lambda\pi k_3 > 0.$

The beliefs of an uninformed trader at $t = 2$ are: $X = \dfrac{\lambda\pi}{\eta + \lambda\pi}$ and $Y = 0$. The optimal strategy of an informed trader at $t = 2$ is to choose MO for all possible states of the LOB, and *the optimal strategy of an uninformed trader at $t = 2$ is to choose NT for all possible states of the LOB.*

- $\mathcal{E}_2^{ND}$: *(BMO, SMO, NT, NT) is the optimal strategy profile at $t = 1$ if*

Conditions
$k_3 - k_1 \geq \delta\frac{\eta}{2}(k_3 + k_1 - 1)$ *and* $0 \geq (\lambda\pi + \eta)(k_1 - 1) - \lambda\pi k_3.$

The beliefs of an uninformed trader at $t = 2$ are: $X = \dfrac{\lambda\pi}{\eta + \lambda\pi}$ and $Y = p \in [0, 1]$. The optimal strategy of an informed trader at $t = 2$ is to choose MO for all possible states of the LOB, and the optimal strategy of an uninformed trader at $t = 2$ is described in Table A.2 of Appendix A.

- $\mathcal{E}_3^{ND}$: *(BLO, SLO, BLO, BLO) is the optimal strategy profile at $t = 1$ if*

Conditions
$\delta\frac{\eta}{2}(k_3 + k_1 - 1) > k_3 - k_1$ *and* $(\lambda\pi + \eta)(k_1 - 1) - \lambda\pi k_3 > 0.$

The beliefs of an uninformed trader at $t = 2$ are: $X = 0$ and $Y = \pi$. The optimal strategy of an informed trader at $t = 2$ is to choose MO for all possible states of the LOB, and the optimal strategy of an uninformed trader at $t = 2$ is described in Table A.3 of Appendix A.

- $\mathcal{E}_4^{ND}$: *(BLO, SLO, NT, NT) is the optimal strategy profile at $t = 1$ if*

Conditions
$\delta\frac{\eta}{2}(k_3 + k_1 - 1) > k_3 - k_1$ *and* $0 \geq (\lambda\pi + \eta)(k_1 - 1) - \lambda\pi k_3.$

The beliefs of an uninformed trader at $t = 2$ are: $X = 0$ and $Y = 1$. The optimal strategy of an informed trader at $t = 2$ is to choose MO for all possible states of the LOB, and the optimal strategy of an uninformed trader at $t = 2$ is described in Table A.4 of Appendix A.[10]

Some of the features of Proposition 1 are illustrated in Figure 4. The left panel shows the optimal strategy of a trader that arrives at $t = 1$ when there is no access to the dark pool as a function of the probability that a liquidity trader arrives, η, and the discount factor, δ, if there is low adverse selection (measured as the probability of arrival of an informed trader). The right panel presents the case of high adverse selection. In addition, the next corollary summarizes under which conditions informed and uninformed traders at $t = 1$ find an order type relatively more attractive than the alternative, keeping the other factors fixed.

[10] Proposition *4* in Appendix A characterizes the PBE when $k_1 = 1$.

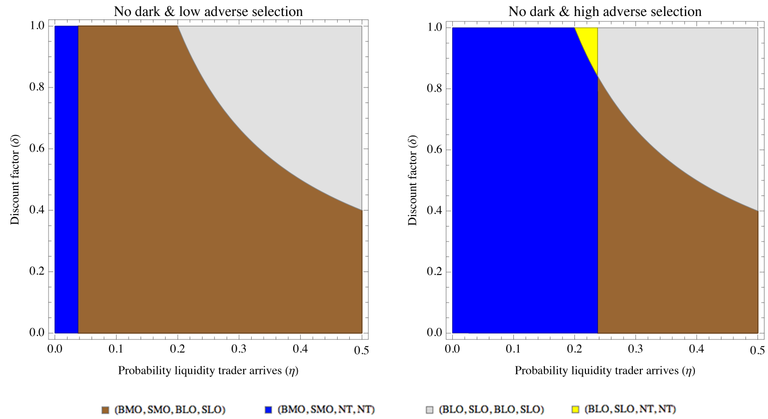

Figure 4: Optimal strategies at $t = 1$ without dark pool. Parameters values: $k_1 = 2$, $k_2 = 3$, $k_3 = 4$, $\lambda = 0.5$. Left Panel $\pi = 0.15$, Right Panel $\pi = 0.95$.

Corollary 1 *At $t = 1$, when there is no access to the dark pool:*

- *An informed trader finds that expected profits of a MO increase in relation to those of a LO when σ increases, when k_1, δ, η or τ decreases and the choice between a MO or LO does not depend on λ nor π.*
- *An uninformed trader finds that expected profits of a LO increase in relation to those of a NT when k_1, η, δ or τ increases and when π or λ or σ decreases.*

The optimal strategy profiles at $t = 1$ depend on trader characteristics (δ) and stock market characteristics (k_1, σ, π, τ, λ, η). Notice that we can understand $1/k_1$ as a measure of liquidity; σ as a measure of the asset's volatility; and the probability that the trader is informed, π, as a measure of adverse selection. The optimal strategy profiles at $t = 1$ described in Proposition 1 are the result of combining how trader (δ) and stock market characteristics ($k_1, \sigma, \tau, \lambda, \pi, \eta$) impact the optimal order choice for an informed and an uninformed trader. These are described below.

Let us discuss how trader characteristics impact the optimal choice of strategy profile at $t = 1$. Informed traders choose between MO and LO. If the probability that a liquidity trader arrives is not too large, then an informed trader at $t = 1$ chooses a MO since execution is guaranteed, otherwise he chooses a LO. This is because, as η increases the relative attractiveness of a LO of an informed trader raises since there is a higher probability of execution. Uninformed traders select between NT and participating with LO. When adverse selection is high, the uninformed trader chooses NT: if the LO is executed due to a MO of the opposite sign submitted by informed trader at $t = 2$, then it reveals that the value of the asset is low (if a BLO has been submitted) and high (if a SLO has been submitted). Otherwise, the uninformed trader chooses LO. In contrast, notice that the profits of an informed trader do not depend on the probability that an informed trader arrives ($\lambda\pi$) in the next trading period. This is because the probability of execution of a LO submitted by an informed trader only depends on the probability that a liquidity trader arrives in the next trading period, and that the liquidity trader submits a MO of a different sign with respect to the initial LO.[11] Furthermore, notice that a LO is attractive when the discount factor is high since traders do not give a high value to immediacy.

In terms of stock market characteristics, low liquidity, high volatility or low tick size (high k_1, high σ, low τ respectively) foster that LO are less attractive since the potential increment in profits does not compensate for the execution risk. Hence, if the asset has low liquidity, high volatility or low tick size, then an uninformed trader prefers NT to LO, and an informed trader prefers MO to LO.

4 Equilibrium in a model with dark pool

We next consider a model where rational traders have both access to the exchange and to the dark pool and can submit the orders in (1). We define Ω_3 and Γ_3 as the probability that an informed trader and uninformed trader at $t = 1$ choose a DO, and such that $\sum_{o=0}^{3} \Omega_o = 1$, and $\sum_{o=0}^{3} \Gamma_o = 1$. Note that the set of the possible states of the book is the same as in the case there is no dark pool but the state of the book $\mathcal{B}_1$=$\varnothing$, can be obtained in this case either because no trader arrived or a trader arrived and decided not to trade or because a trader arrived and he submitted a DO.

We solve the model backwards. At $t = 2$ the expected profits of each strategy depend on the state of the LOB. Additionally, uninformed traders form beliefs about the strategies that have been chosen at $t = 1$. Let X and Y be defined as in (2) and (3), respectively, and Z denote the uninformed trader's belief at $t = 2$ about the probability that a DO was submitted by an informed, which is equal to

$$Z = \frac{\pi\Omega_3}{\pi\Omega_3 + (1-\pi)\,\Gamma_3}. \tag{4}$$

As in the case when the dark pool was not available, and without loss of generality, we will focus on the expected profits for an informed buyer at $t = 2$, as summarized in Table *6* below.

[11] An informed trader that submits a LO at $t = 1$ knows that the LO will not be executed in the next trading period against an order submitted by an informed trader since an informed trader at $t = 2$ chooses an order of the same sign as the initial order. In addition, an informed trader at $t = 1$ correctly predicts that an uninformed trader at $t = 2$ never submits a MO of the opposite sign as the informed trader at $t = 1$.

IH	BMO	BDO	BLO	NT
(A_1^1, B_1^1)	$(k_3 - k_1)\,\tau$	$\theta k_3 \tau$	$P_I \delta\,(k_1 + k_3 - 1)\,\tau$	0
(A_1^2, B_1^1)	$(k_3 - k_2)\,\tau$	$\theta\left(k_3 - \frac{k_2-k_1}{2}\right)\tau$	0	0
$(A_1^1, B_1^1 + \tau)$	$(k_3 - k_1)\,\tau$	$\theta\left(k_3 - \frac{1}{2}\right)\tau$	0	0
(A_1^1, B_1^2)	$(k_3 - k_1)\,\tau$	$\theta\left(k_3 + \frac{k_2-k_1}{2}\right)\tau$	0	0
$(A_1^1 - \tau, B_1^1)$	$(k_3 - k_1 + 1)\,\tau$	$\theta\left(k_3 + \frac{1}{2}\right)\tau$	0	0

Table 6: Expected profits of an informed buyer (IH) at $t = 2$

where P_I is the probability of execution of a limit order placed by an informed trader at $t = 2$ conditional on the fact that there is no change in the LOB during the first trading period, and equals

$$P_I = p_{BLO,2}^{IH}\,(\mathcal{B}_1 = \varnothing) = \frac{\lambda(1-\theta)\frac{1-\pi}{2}\Gamma_3}{1 - \lambda - \eta + \lambda\left(\pi\Omega_3 + (1-\pi)(\Gamma_0 + \Gamma_3)\right)}.$$

Similarly, the expected profits of an uninformed buyer at $t = 2$ are summarized in Table *7*.

UB	BMO	BDO	BLO	NT
(A_1^1, B_1^1)	$-k_1\tau$	0	$P_U \delta\,(k_1 - Zk_3 - 1)\,\tau$	0
(A_1^2, B_1^1)	$(Xk_3 - k_2)\,\tau$	$\theta\left(Xk_3 - \frac{k_2-k_1}{2}\right)\tau$	0	0
$(A_1^1, B_1^1 + \tau)$	$(Yk_3 - k_1)\,\tau$	$\theta\left(Yk_3 - \frac{1}{2}\right)\tau$	0	0
(A_1^1, B_1^2)	$-\,(Xk_3 + k_1)\,\tau$	$-\theta\left(Xk_3 - \frac{k_2-k_1}{2}\right)\tau$	0	0
$(A_1^1 - \tau, B_1^1)$	$-\,(Yk_3 + k_1 - 1)\,\tau$	$-\theta\left(Yk_3 - \frac{1}{2}\right)\tau$	0	0

Table 7: Expected profits of an uninformed buyer (UB) at $t = 2$

where P_U is the probability of execution of a limit order placed by an uninformed trader at $t = 2$ given that there are no changes in prices in the LOB during the first trading period, and equals

$$P_U = p_{BLO,2}^{UB}\,(\mathcal{B}_1 = \varnothing) = \frac{\frac{1}{2}\lambda(1-\theta)(\pi\Omega_3 + (1-\pi)\Gamma_3)}{1 - \lambda - \eta + \lambda(\pi\Omega_3 + (1-\pi)(\Gamma_0 + \Gamma_3))}.$$

At $t = 1$ the expected profits of an informed IH and an uniformed buyer UB are summarized in Table *8* and Table *9*, respectively.[12]

Notice that the expected profits of a BDO submitted by an uninformed trader at $t = 1$ can be rewritten as

$$\theta{\cdot}0 + \max\left\{(1-\theta)\delta^2\left(-k_1\tau + \frac{\lambda}{2}\left(\pi I_{SLO,2}^{I,L,\mathcal{B}_1=\varnothing} + (1-\pi) I_{SLO,2}^{U,S,\mathcal{B}_1=\varnothing}\right)\tau - \left(\frac{\lambda\pi}{2} I_{BMO,2}^{I,H,\mathcal{B}_1=\varnothing} + \frac{\eta}{2}\right)(k_2 - k_1)\,\tau\right), 0\right\}.$$

This expression indicates that if a BDO gets executed at $t = 1$, then its expected profits are zero.

[12]Notice that due to the symmetry of the game, the expected profits of the informed IL trader and uninformed seller US are the same as the ones displayed in Tables *8* and *9*, respectively.

IH	*Expected Profits at* $t = 1$
BMO	$(k_3 - k_1)\,\tau$
BLO	$\frac{\eta\delta}{2}\,(k_3 + k_1 - 1)\,\tau$
BDO	$\theta k_3 \tau + \max\left\{(1-\theta)\delta^2 \left(\lambda \frac{(1-\pi)}{2} I_{SLO,2}^{US,\mathcal{B}_1=\varnothing} + \right.\right.$ $\left.\left. +(k_3 - k_1) - (k_2 - k_1)\left(\lambda\pi I_{BMO,2}^{IH,\mathcal{B}_1=\varnothing} + \frac{\eta}{2}\right)\right)\tau, 0\right\}$
NT	0

Table 8: Expected profits of an informed buyer (IH) at $t = 1$

UB	*Expected Profits at* $t = 1$
BMO	$-k_1\tau$
BLO	$\frac{\delta}{2}\left(\eta(k_1 - 1) - \lambda\pi I_{SMO,2}^{IL,\mathcal{B}_1=BLO}(k_3 - k_1 - 1)\right)\tau$
BDO	$\max\left\{(1-\theta)\delta^2\left(\frac{\lambda}{2}(\pi I_{SLO,2}^{IL,\mathcal{B}_1=\varnothing} + (1-\pi) I_{SLO,2}^{US,\mathcal{B}_1=\varnothing}) + \right.\right.$ $\left.\left. +(\frac{\lambda\pi}{2} I_{BMO,2}^{IH,\mathcal{B}_1=\varnothing} + \frac{\eta}{2})(k_1 - k_2) - k_1\right)\tau, 0\right\}$
NT	0

Table 9: Expected profits of an uninformed buyer (UB) at $t = 1$

Otherwise (if the order is not attended executed at $t = 1$ and returns to the market at $t = 3$), the expected profits are given by the second summand of the previous formula. The expected profits depend on whether a trader who returns to the market decides to submit a MO or NT. If he submits a MO then the profit consists of the product of the probability of no execution in the dark pool (i.e., $1-\theta$) and the squared discount factor (i.e., δ^2) and by the expected profits of a BMO at t=1 for an UB (i.e., $-k_1\tau$) adjusted by two terms. The first one, $\frac{\lambda}{2}\left(\pi I_{SLO,2}^{IL,\mathcal{B}_1=\varnothing} + (1-\pi) I_{SLO,2}^{US,\mathcal{B}_1=\varnothing}\right)\tau$, shows the increase in the expected profits a UB due to the possibility that at $t = 2$ a new trader arrives and submits a SLO. The second one, $-\left(\frac{\lambda}{2}\pi I_{BMO,2}^{IH,\mathcal{B}_1=\varnothing} + \frac{\eta}{2}\right)(k_2 - k_1)\,\tau$, shows the decrease in expected profits of a UB due to the possibility that a trader at $t = 2$ submits a BMO and, consequently, the MO that arrives at $t = 3$ from the dark pool is executed at a worse price. Notice that the increase in expected profits due to the potential arrival of a SLO at $t = 2$ does not compensate for the negative profits of a MO and, as we have point out, these losses might be even greater in case that a BMO is submitted at $t = 2$. If he selects NT then the expected profits equal zero.

By comparing the expected profits of each of the possible strategies for each type of rational trader at $t = 2$, Lemma 3 states the strategies that are dominated and, hence, never chosen by a rational player.

Lemma 3 *In equilibrium the following results hold:*

- *at $t = 2$ an informed trader never chooses a LO or NT. The optimal strategy depends on the value of the parameters as explained in Table B.1 in Appendix B.*

- *at $t = 2$ an uninformed trader at $t = 2$ never chooses a LO. The optimal strategy depends on the value of the parameters as explained in Table B.2 in Appendix B.*

An informed trader at $t = 2$ may choose between MO or DO, and never chooses NT or LO since NT is always dominated by MO, and LO is never executed since: a) if the LOB has changed, then no MO can arrive at $t = 3$; b) if the LOB has not changed, then BLO (SLO) can only be executed if an uninformed seller (buyer) at $t = 1$ chooses DO, but as we have discussed this cannot occur in equilibrium.

An uninformed trader at $t = 2$ might choose between MO, DO or NT order, but never LO since: a) if the LOB has changed, then no MO arrives at $t = 3$ and, hence, it has zero probability of execution; b) if the LOB has not changed, then the LO can only be executed if a trader at $t = 1$ has chosen a DO. However, as we have explained after Table 9, we know that it is never optimal for an uninformed trader at $t = 1$ to choose a DO. Hence the trader at $t = 2$ forms the correct beliefs that, if a LO is executed at $t = 3$, it must have come from an informed trader at $t = 1$ with probability 1. But this information reveals to the uninformed buyer (seller) that the value of the asset must be low (high) and, hence, expected profits of a LO are negative.

By comparing the expected profits for each type of orders at $t = 1$, we find the optimal strategies chosen by informed and uninformed traders.

Lemma 4 *In equilibrium the following results hold:*

- *at $t = 1$ an informed trader never chooses NT.*
- *at $t = 1$ an uniformed trader never chooses a MO or a DO.*

An informed trader at $t = 1$ chooses among MO, LO or DO, but never NT since it is always dominated by at least a MO. An uninformed trader at $t = 1$ may choose between LO or NT since the expected profits of a MO are negative, and also the expected profits of a DO as explained after Table 9.

Hence, the candidate strategy profiles at $t = 1$ that can be sustained as a PBE are:

$$(BMO, SMO, BLO, SLO), \quad (BMO, SMO, NT, NT), \quad (BLO, SLO, BLO, BLO),$$
$$(BLO, SLO, NT, NT), \quad (BDO, SDO, BLO, SLO), \quad (BDO, SDO, NT, NT),$$

where, as before, the two first components correspond to strategies of informed traders at $t = 1$ (IH and IL, respectively) and the two last components correspond to strategies of uninformed traders at $t = 1$ (UB and US, respectively).

The PBE of the trading game where rational traders have access to a dark pool is characterized as follows.

Proposition 2 *If $k_1 > 1$, then a PBE of the game is as follows:*

- $\mathcal{E}_1^D$: (BMO, SMO, BLO, SLO) *is the optimal strategy profile at* $t = 1$ *if*

Conditions	
$\theta \leq \frac{k_3 - k_1}{k_3}$	$k_3 - k_1 \geq \frac{\eta}{2}\delta\left(k_3 + k_1 - 1\right)$, $k_3 - k_1 \geq \theta k_3 + (1-\theta)\delta^2\left(k_3 - k_1 - (k_2 - k_1)\left(\lambda\pi + \frac{\eta}{2}\right)\right)$, *and* $(\lambda\pi + \eta)(k_1 - 1) - \lambda\pi k_3 > 0$.

The beliefs of an uninformed trader at $t = 2$ *are:* $X = \dfrac{\lambda\pi}{\eta + \lambda\pi}$, $Y = 0$ *and* $Z = q \in [0,1]$. *The optimal strategy of an informed trader and an uninformed at* $t = 2$ *are described in Tables B.1 and B.5 of Appendix B, respectively.*

- $\mathcal{E}_2^D$: (BMO, SMO, NT, NT) *is the optimal strategy profile at* $t = 1$ *if*

Conditions	
$\theta \leq \frac{k_3 - k_1}{k_3}$	$k_3 - k_1 \geq \frac{\eta}{2}\delta\left(k_3 + k_1 - 1\right)$, $k_3 - k_1 \geq \theta k_3 + (1-\theta)\delta^2\left((k_3 - k_1) - (k_2 - k_1)\left(\lambda\pi + \frac{\eta}{2}\right)\right)$, *and* $0 \geq (\lambda\pi + \eta)(k_1 - 1) - \lambda\pi k_3$.

The beliefs of an uninformed trader at $t = 2$ *are:* $X = \dfrac{\lambda\pi}{\eta + \lambda\pi}$, $Y = p \in [0,1]$ *and* $Z = q \in [0,1]$. *The optimal strategy of an informed trader and an uninformed at* $t = 2$ *are described in Tables B.1 and B.6 of Appendix B, respectively.*

- $\mathcal{E}_3^D$: (BLO, SLO, BLO, BLO) *is the optimal strategy profile at* $t = 1$ *if*

Conditions	
$\theta \leq \frac{k_3 - k_1}{k_3}$	$\frac{\eta}{2}\delta\left(k_3 + k_1 - 1\right) \geq \theta k_3 + (1-\theta)\delta^2\left((k_3 - k_1) - (k_2 - k_1)\left(\lambda\pi + \frac{\eta}{2}\right)\right)$, $\frac{\eta}{2}\delta\left(k_3 + k_1 - 1\right) > k_3 - k_1$, *and* $(\lambda\pi + \eta)(k_1 - 1) - \lambda\pi k_3 > 0$.
$\frac{k_3 - k_1}{k_3} < \theta \leq \frac{k_3 - k_1 + 1}{k_3 + \frac{1}{2}}$	$\frac{\eta}{2}\delta\left(k_3 + k_1 - 1\right) \geq \theta k_3 + (1-\theta)\delta^2\left((k_3 - k_1) - (k_2 - k_1)\left(\frac{\eta}{2}\right)\right)$ *and* $(\lambda\pi + \eta)(k_1 - 1) - \lambda\pi k_3 > 0$.
$\frac{k_3 - k_1 + 1}{k_3 + \frac{1}{2}} < \theta$	$\frac{\eta}{2}\delta\left(k_3 + k_1 - 1\right) \geq \theta k_3 + (1-\theta)\delta^2\left((k_3 - k_1) - (k_2 - k_1)\left(\frac{\eta}{2}\right)\right)$.

The beliefs of an uninformed trader at $t = 2$ *are:* $X = 0$, $Y = \pi$ *and* $Z = q \in [0,1]$. *The optimal strategy of an informed trader and an uninformed at* $t = 2$ *are described in Tables B.1 and B.7 of Appendix B, respectively.*

- $\mathcal{E}_4^D$: (BLO, SLO, NT, NT) *is the optimal strategy profile of a trader at* $t = 1$ *if*

Conditions	
$\theta \leq \frac{k_3 - k_1}{k_3}$	$\frac{\eta}{2}\delta\left(k_3 + k_1 - 1\right) \geq \theta k_3 + (1-\theta)\delta^2\left(\left(k_3 - k_1\right) - \left(k_2 - k_1\right)\left(\lambda\pi + \frac{\eta}{2}\right)\right)$, $\frac{\eta}{2}\delta\left(k_3 + k_1 - 1\right) > \left(k_3 - k_1\right)$, *and* $0 \geq \left(\lambda\pi + \eta\right)\left(k_1 - 1\right) - \lambda\pi k_3$.
$\frac{k_3 - k_1}{k_3} < \theta \leq \frac{k_3 - k_1 + 1}{k_3 + \frac{1}{2}}$	$\frac{\eta}{2}\delta\left(k_3 + k_1 - 1\right) \geq \theta k_3 + (1-\theta)\delta^2\left(\left(k_3 - k_1\right) - \left(k_2 - k_1\right)\left(\frac{\eta}{2}\right)\right)$ *and* $0 \geq \left(\lambda\pi + \eta\right)\left(k_1 - 1\right) - \lambda\pi k_3$.

The beliefs of an uninformed trader at $t = 2$ *are:* $X = 0$, $Y = 1$ *and* $Z = q \in [0, 1]$. *The optimal strategy of an informed trader and an uninformed at* $t = 2$ *are described in Tables B.1 and B.8 of Appendix B, respectively.*

- $\mathcal{E}_5^D$: (BDO, SDO, BLO, SLO) *is the optimal strategy profile of a trader at* $t = 1$ *if*

Conditions	
$\theta \leq \frac{k_3 - k_1}{k_3}$	$\theta k_3 + (1-\theta)\delta^2\left(k_3 - k_1 - \left(k_2 - k_1\right)\left(\lambda\pi + \frac{\eta}{2}\right)\right) > k_3 - k_1$, $\theta k_3 + (1-\theta)\delta^2\left(k_3 - k_1 - \left(k_2 - k_1\right)\left(\lambda\pi + \frac{\eta}{2}\right)\right) > \frac{\eta}{2}\delta\left(k_3 + k_1 - 1\right)$, *and* $(\lambda\pi + \eta)(k_1 - 1) - \lambda\pi k_3 > 0$.
$\frac{k_3 - k_1}{k_3} < \theta \leq \frac{k_3 - k_1 + 1}{k_3 + \frac{1}{2}}$	$\theta k_3 + (1-\theta)\delta^2\left(k_3 - k_1 - \left(k_2 - k_1\right)\left(\frac{\eta}{2}\right)\right) > \frac{\eta}{2}\delta\left(k_3 + k_1 - 1\right)$ *and* $((\lambda\pi + \eta)(k_1 - 1) - \lambda\pi k_3) > 0$.
$\frac{k_3 - k_1 + 1}{k_3 + \frac{1}{2}} < \theta$	$\theta k_3 + (1-\theta)\delta^2\left(k_3 - k_1 - \left(k_2 - k_1\right)\left(\frac{\eta}{2}\right)\right) > \frac{\eta}{2}\delta\left(k_3 + k_1 - 1\right)$.

The beliefs of an uninformed trader at $t = 2$ *are:* $X = 0$, $Y = 0$ *and* $Z = 1$. *The optimal strategy of an informed trader and an uninformed at* $t = 2$ *are described in Tables B.1 and B.9 of Appendix B, respectively.*

- $\mathcal{E}_6^D$: (BDO, SDO, NT, NT) *is the optimal strategy profile of a trader at* $t = 1$ *if*

Conditions	
$\theta \leq \frac{k_3 - k_1}{k_3}$	$\theta k_3 + (1-\theta)\delta^2\left(\left(k_3 - k_1\right) - \left(k_2 - k_1\right)\left(\lambda\pi + \frac{\eta}{2}\right)\right) > \left(k_3 - k_1\right)$, $\theta k_3 + (1-\theta)\delta^2\left(\left(k_3 - k_1\right) - \left(k_2 - k_1\right)\left(\lambda\pi + \frac{\eta}{2}\right)\right) > \frac{\eta}{2}\delta\left(k_3 + k_1 - 1\right)$, $0 \geq \left(\lambda\pi + \eta\right)\left(k_1 - 1\right) - \lambda\pi k_3$.
$\frac{k_3 - k_1}{k_3} < \theta \leq \frac{k_3 - k_1 + 1}{k_3 + \frac{1}{2}}$	$\theta k_3 + (1-\theta)\delta^2\left(\left(k_3 - k_1\right) - \left(k_2 - k_1\right)\left(\frac{\eta}{2}\right)\right) > \frac{\eta}{2}\delta\left(k_3 + k_1 - 1\right)$, $0 \geq \left(\lambda\pi + \eta\right)\left(k_1 - 1\right) - \lambda\pi k_3$.

The beliefs of an uninformed trader at $t = 2$ *are:* $X = 0$, $Y = p \in [0, 1]$ *and* $Z = 1$. *The optimal strategy of an informed trader and an uninformed at* $t = 2$ *are described in Tables B.1 and B.10 of Appendix B, respectively.* [13]

[13] Proposition *5* in Appendix B characterizes the PBE when $k_1 = 1$.

Proposition 2 shows that having access to a dark pool changes the optimal submission strategy profiles at $t = 1$ for informed and uninformed traders. In addition, Proposition 2 (and Appendix B) characterize the optimal trading strategies for informed and uninformed traders at $t = 2$. Without loss of generality, let us focus on the optimal strategy of a buyer at $t = 2$. Informed traders at $t = 2$ submit BMO for all states of the LOB when the execution risk in the dark is high ($\theta \leq \frac{k_3 - k_2}{k_3 - \frac{k_2 - k_1}{2}}$) and submit BDO for all the states of the LOB when the execution risk in the dark is low ($\theta > \frac{k_3 - k_1 + 1}{k_3 + \frac{1}{2}}$). As the execution risk in the dark lowers, informed traders replace BMO by BDO in the following order of states of the LOB: $(A_1^2, B_1^1), (A_1^1, B_1^2), (A_1^1, B_1^1), (A_1^1, B_1^1 + \tau), (A_1^1 - \tau, B_1^1)$. This is because, when a BMO had been submitted at $t = 1$ (state of the LOB : (A_1^2, B_1^1)), then the gain from another BMO is small in relation to a BDO for high execution risk, while when a SLO had been previously submitted at $t = 1$ (state of the LOB : $(A_1^1 - \tau, B_1^1)$) then the gain from another BMO is large in relation to a BDO for low execution risk.

The optimal strategy of an uninformed trader at $t = 2$ critically depends on the set of the uninformed trader's beliefs at $t = 2$, X, Y, Z, about the probability that a MO, LO, DO order, respectively, was submitted by an informed trader at $t = 1$. These beliefs are different given the optimal strategy profile at $t = 1$. When the state of the LOB contains no information, i.e., (A_1^1, B_1^1), then uninformed traders at $t = 2$ submit NT orders since the expected profits of a MO are negative, and the profits of a DO are zero because the mid-point price is equal to the unconditional expected value of the asset, μ. When the state of the LOB indicates that a BMO (SMO) had been submitted at $t = 1$, with state of the LOB : (A_1^2, B_1^1) ((A_1^1, B_1^2)) and it is optimal that informed traders submit this strategy profile (i.e., in $\mathcal{E}_1^D$ and $\mathcal{E}_2^D$), then uniformed buyers at $t = 2$ may submit BMO, BDO or NT (BDO or NT) depending on which of the expected profits is higher. In all other possible equilibrium strategy profiles, uninformed buyers prefer NT (BDO). When the state of the LOB indicates that a BLO (SLO) had been submitted at $t = 1$, with state of the LOB : $(A_1^1, B_1^1 + \tau)$ ($(A_1^1 - \tau, B_1^1)$), and it is optimal that uninformed traders submit this strategy profile (i.e., in $\mathcal{E}_1^D$ and $\mathcal{E}_5^D$), then uniformed buyers at $t = 2$ prefer NT (BDO). Otherwise, uninformed traders prefer BMO, BDO or NT (BDO or NT) depending on which of the expected profits is higher.

Figures 5 and 6 show the optimal strategy of a trader that arrives at $t = 1$ when a dark pool is available for different levels of execution risk in the dark pool (lowest in the upper right graph and highest in the lower right graph) as a function of the discount factor and the probability that a liquidity trader arrives for markets with low and high adverse selection, respectively.

We observe that when the execution risk is high (low θ) the optimal strategy of the uninformed trader does not change in relation to when the dark pool is unavailable. Adding a dark pool alongside the exchange also changes the optimal strategy of an uninformed trader at $t = 1$ even if an uninformed trader never goes to the dark. When the execution risk in the dark is low, uninformed traders may switch from NT to using LO in the exchange. This is because the mere existence of the dark market offers the possibility for informed traders at $t = 2$ to migrate from the exchange to the dark and, consequently, adverse selection is reduced in the exchange. This induces uninformed

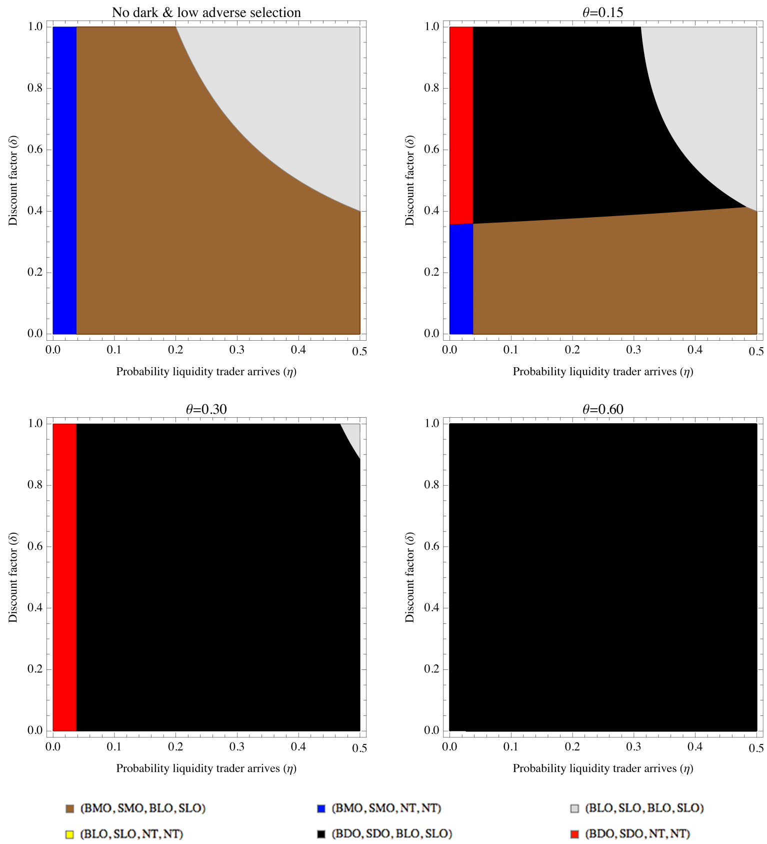

Figure 5: Optimal strategies at $t = 1$ with dark pool as a function of the probability that a liquidity trader arrives, η, and the discount factor, δ. Parameter values: $k_1 = 5$, $k_2 = 6$, $k_3 = 7$, $\lambda = 0.5$, $\pi = 0.15$. Values for the execution probability in the dark, θ, are specified above each graph.

traders to switch from NT to LO. An informed trader's optimal strategy at $t = 1$ changes when the dark pool is available: as the probability of execution in the dark increases, informed traders gradually replace MO by DO, and as θ is even larger they also replace LO by DO.

Proposition 2 indicates that the conditions for a PBE to exist depend on how the execution risk in the dark pool, θ, compares to two cutoffs related to the asset's volatility σ and liquidity $(1/k_1)$ since these influence the prices of the LOB and the possible realizations of the liquidation value. Specifically, strategy profiles where an informed trader at $t = 1$ submits a MO cannot be part of a PBE if $\dfrac{k_3 - k_1}{k_3} < \theta$ since these strategies are dominated by LO or DO because prices are more attractive given their execution risk. Additionally, strategy profiles where an uninformed trader at $t = 1$ chooses NT cannot be equilibrium strategy profiles at $t = 1$ if $\dfrac{k_3 - k_1 + 1}{k_3 + \frac{1}{2}} < \theta$. Hence, if the execution probability in the dark, θ, changes continuously, then optimal strategy profiles at $t = 1$ may present discontinuities. Second, even though optimal strategies profiles are unique at $t = 1$

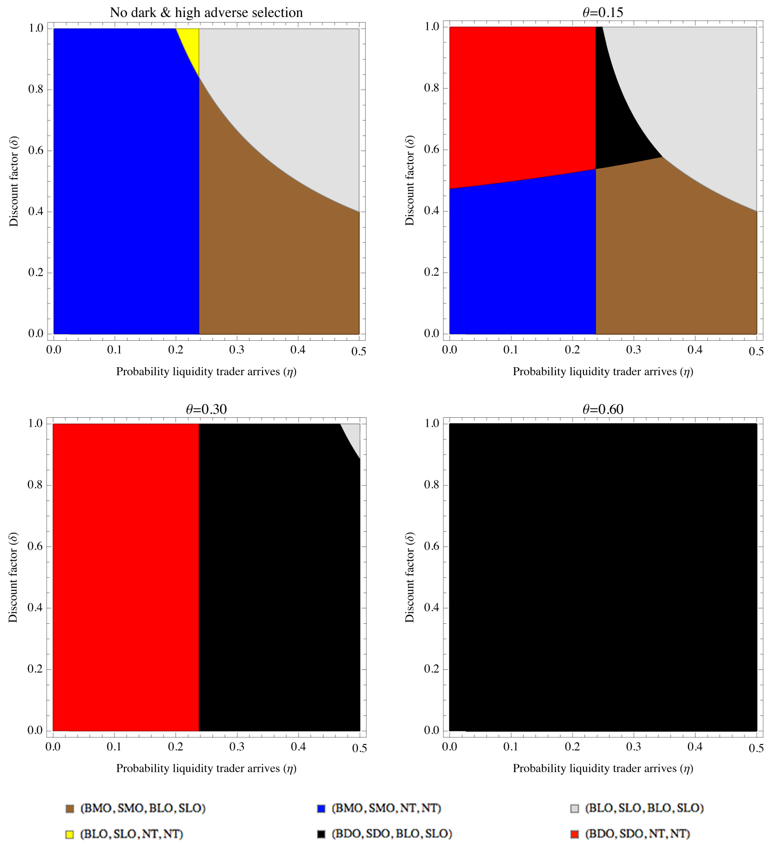

Figure 6: Optimal strategies at $t = 1$ with dark pool as a function of the probability that a liquidity trader arrives, η, and the discount factor, δ. Parameter values: $k_1 = 5$, $k_2 = 6$, $k_3 = 7$, $\lambda = 0.5$, $\pi = 0.95$. Values for the execution probability in the dark, θ, are specified above each graph.

for given parameter values, multiple equilibria may exist except when (BDO, SDO, BLO, SLO) is optimal. Multiple equilibria exist since there is a continuum of uninformed trader's beliefs at $t = 2$ that can sustain the PBE. However, when (BDO, SDO, BLO, SLO) is optimal at $t = 1$, the beliefs of uninformed traders at $t = 2$ are uniquely determined since $X = 0$, $Y = 0$ and $Z = 1$.

We analyze which PBE exists in a world with almost no adverse selection (i.e., if π is low enough).

Corollary 2 *Suppose that $k_1 > 1$ and that π is low enough. At $t = 1$, the strategy profiles that are part of the PBE are the following: (BMO, SMO, BLO, SLO), (BLO, SLO, BLO, SLO), and (BDO, SDO, BLO, SLO).*

The intuition is as follows. When the probability that an informed trader arrives is low enough, uninformed traders realize that they are very likely to trade with liquidity traders instead of informed traders and have higher expected profits with LO than with NT. In contrast, when π is high, then adverse selection is high because if the LO is executed it is very likely that it is because a MO of the opposite sign has been submitted by an informed trader. Then this reveals that the value of the asset is low (if a BLO was submitted) and high (if a SLO was submitted). Hence, due to adverse selection, uninformed traders do not participate in the market since the probability of obtaining negative profits is higher.

5 Market quality and welfare analysis

In the presence of asymmetric information, we study how market quality and welfare are affected by the dark pool. Hence, we compare several measures of market quality and welfare when the dark pool is available (Section 4) to when it is unavailable (Section 3). The measures of market quality that we consider are: expected inside spread, expected traded volume, and expected price informativeness.

We denote by $\mathcal{E}_i^a$ an equilibrium, where $a = D$, ND indicates whether we consider that the dark pool (D) is available or unavailable (ND). Note that if the dark pool is available we have six equilibria $\mathcal{E}_i^D$ with $i = 1, ..., 6$, while if the dark pool is not available we have four equilibria $\mathcal{E}_i^{ND}$ for $i = 1, ..., 4$.

We proceed as follows for market quality and welfare at $t = 1$. First, we compute the market quality and welfare measures for each of the possible strategy profiles when the dark pool is not available, $\mathcal{E}_i^{ND}$ for $i = 1, ..., 4$. Second, when traders have access to the dark pool, we assume that θ is uniformly distributed in the interval $[0, 1]$. Then, starting from each of the four possible equilibria when traders do not have access to the dark pool ($\mathcal{E}_i^{ND}$ for $i = 1, ..., 4$), we find the optimal strategy profile at $t = 1$ when traders have access to the dark pool for all the values of θ. For example, starting from strategy (BMO, SMO, NT, NT), we find the following cutoffs of θ for which traders switch their optimal trading strategy when the dark pool is available. We define first,

$$\overline{\theta}_1 \equiv \frac{k_3 - k_1 - \delta^2 \left(k_3 - k_1 + (k_1 - k_2)\left(\frac{1}{2}\eta + \pi\lambda\right)\right)}{\left(k_3 - \delta^2 \left(k_3 - k_1 + (k_1 - k_2)\left(\frac{1}{2}\eta + \pi\lambda\right)\right)\right)},$$

and write therefore:

Optimal Trading Strategy at $t = 1$ starting from (BMO, SMO, NT, NT)		
(BMO, SMO, NT, NT)	if	$\theta \leq \overline{\theta}_1$
(BDO, SDO, NT, NT)	if	$\overline{\theta}_1 < \theta \leq \dfrac{k_3 - k_1 + 1}{k_3 + \frac{1}{2}}$
(BDO, SDO, BLO, SLO)	if	$\dfrac{k_3 - k_1 + 1}{k_3 + \frac{1}{2}} < \theta.$

Third, we find the *average market quality measure and welfare over all the possible values of θ* at the end of the first trading period ($t = 1$) when the dark pool is available, given that the optimal trading strategy when the dark pool was not available, $D; \mathcal{E}_i^{ND}$, as:

$$\mathbb{E}_0 \left(y_1^{D;\mathcal{E}_i^{ND}} \right) = \int_0^1 y_1^{D,e(\theta;\mathcal{E}_i^{ND})} d\theta,$$

where $y_1 \in \left\{ S_1^{D;\mathcal{E}_i^{ND}}, V_{EX;1}^{D;\mathcal{E}_i^{ND}}, V_{DP;1}^{D;\mathcal{E}_i^{ND}}, V_{T;1}^{D;\mathcal{E}_i^{ND}}, PI_1^{D;\mathcal{E}_i^{ND}}, W_1^{D;\mathcal{E}_i^{ND}} \right\}$, and where S stands for the inside spread, V_{EX} the volume traded in the exchange, V_{DP} the volume traded in the dark pool, V_T the total traded volume, PI for price informativeness, and W for welfare, and superscript $D, e(\theta; \mathcal{E}_i^{ND})$ indicates that the variable corresponds to the associated equilibrium strategy profile at $t = 1$ when rational traders have access to a dark pool, the probability of execution in the dark pool is θ and the other parameters are such that when there is no access to the dark pool the equilibrium strategy profile at $t = 1$ is $\mathcal{E}_i^{ND}$, with $i = 1, ..., 4$.

We denote by $\mathbb{E}_0 \left(S_1^{\mathcal{E}_i^a} \right)$ the *expected inside spread*, where subscript 1 denotes that the measure is computed at the end of the first trading period, $t = 1$, for a given equilibrium $\mathcal{E}_i^a$ and define it as $\mathbb{E}_0 \left(S_1^{\mathcal{E}_i^a} \right) = \sum_{\mathcal{B}_1 \in \mathbb{B}} \Pr_1^{\mathcal{E}_i^a}(\mathcal{B}_1) S(\mathcal{B}_1)$, where $\mathcal{B}_1 \in \mathbb{B}$ is a possible state of the book and $a \in \{D, ND\}$.

Denote $\mathbb{E}_0 \left(V_{EX,1}^{\mathcal{E}_i^a} \right)$ as the *expected traded volume* in the exchange (subscript EX) in the first trading period for a given equilibrium $\mathcal{E}_i^a$ and define it as: $\mathbb{E}_0 \left(V_{EX,1}^{\mathcal{E}_i^a} \right) = \sum_{\mathcal{B}_1 \in \mathbb{B}} \Pr_1^{\mathcal{E}_i^a}(\mathcal{B}_1) I_{EX,1}^{\mathcal{E}_i^a}(\mathcal{B}_1)$, where $I_{EX,1}^{\mathcal{E}_i^a}(\mathcal{B}_1) = 1$, if an order has been executed in the exchange during the first trading period when the best prices at the end of the period are $\mathcal{B}_1$, and $I_{EX,1}^{\mathcal{E}_i^a}(\mathcal{B}_1) = 0$ otherwise. Analogously, the expected traded volume in the dark pool at $t = 1$ for a given equilibrium $\mathcal{E}_i^a$ is measured as $\mathbb{E}_0 \left(V_{DP,1}^{\mathcal{E}_i^a} \right) = \sum_{\mathcal{B}_1 \in \mathbb{B}} \Pr_1^{\mathcal{E}_i^a}(\mathcal{B}_1) I_{DP,1}^{\mathcal{E}_i^a}(\mathcal{B}_1)$, where $I_{DP,1}^{\mathcal{E}_i^a}(\mathcal{B}_1) = 1$ if an order is executed in the dark pool at $t = 1$ when the best prices at the end of the period are $\mathcal{B}_1$, and $I_{DP,1}^{\mathcal{E}_i^a}(\mathcal{B}_1) = 0$ otherwise. Therefore, the total expected traded volume is: $\mathbb{E}_0 \left(V_{T,1}^{\mathcal{E}_i^a} \right) = \mathbb{E}_0 \left(V_{EX,1}^{\mathcal{E}_i^a} \right) + \mathbb{E}_0 \left(V_{DP,1}^{\mathcal{E}_i^a} \right)$.

Denote $\mathbb{E}_0 \left(PI_1^{\mathcal{E}_i^a} \right)$ as *expected price informativeness* at the end of the first trading period for a given equilibrium $\mathcal{E}_i^a$ and define it as: $\mathbb{E}_0 \left(PI_1^{\mathcal{E}_i^a} \right) = var\,(v) - \sum_{\mathcal{B}_1 \in \mathbb{B}} \Pr_1^{\mathcal{E}_i^a}(\mathcal{B}_1) var(v|\mathcal{B}_1, \mathcal{E}_i^a)$, where $var\,(v) = \tau^2 k_3^2 = \sigma^2$, and $var(v|\mathcal{B}_1, \mathcal{E}_i^a) = \mathbb{E}(v^2|\mathcal{B}_1, \mathcal{E}_i^a) - (\mathbb{E}(v|\mathcal{B}_1, \mathcal{E}_i^a))^2$ is the conditional variance of the value of the asset given the state of the book at the end of the period, $\mathcal{B}_1$. Notice that the state of the book at the end of the first period will be different for each equilibrium considered.

Denote $\mathbb{E}_0 \left(W_{1,type}^{\mathcal{E}_i^a} \right)$ as *expected welfare*, where subscript 1 denotes that it is at the end of the first trading period, where subscript *type* denotes whether the trader is informed (I), uninformed (U), or liquidity trader, and where the superscript indicates the corresponding equilibrium $\mathcal{E}_i^a$. We define expected welfare as ex-ante expected profits and, hence, $\mathbb{E}_0 \left(W_{1,type}^{\mathcal{E}_i^a} \right) = \mathbb{E} \left(\Pi_{1,type}^{\mathcal{E}_i^a} \right)$.

The next proposition finds the effects of adding a dark pool alongside an exchange on market quality and welfare.

Proposition 3 *In relation to when there is no access to the dark pool, adding a dark pool alongside an exchange causes the following effects on the market quality parameters at $t = 1$:*

- *Expected inside spread:*
  - *Decreases if the initial equilibrium strategies are (BMO, SMO, BLO, SLO) or (BMO, SMO, NT, NT).*
  - *Increases if the initial strategy is (BLO, SLO, BLO, BLO).*
  - *For (BLO, SLO, NT, NT), the change in the expected inside spread depends on the value of π. If π is sufficiently large, then the inside spread increases due to the presence of the dark pool; if π is sufficiently small, then the opposite occurs.*
- *Expected trading volume:*
  - *Decreases the expected total trading volume and also decreases the expected trading volume in the exchange if (BMO, SMO, BLO, SLO) or (BMO, SMO, NT, NT).*
  - *Increases the expected total trading volume but there is no change in the expected trading volume in the exchange if (BLO, SLO, NT, NT) and (BLO, SLO, BLO, SLO)*
- *Expected price informativeness is always lower due to the existence of the dark pool.*
- *Expected welfare of rational traders is always higher due to the existence of the dark pool, while the expected welfare for liquidity traders is the same with or without the existence of the dark pool.*

Proposition 3 shows at $t = 1$ some market quality parameters depend on trader or stock market characteristics (i.e., expected inside spread and expected trading volume), while others are unambiguously determined (i.e., expected price informativeness and expected welfare).

Due to the existence of the dark pool *expected welfare* of each type of market participant at $t = 1$ is not lower than when the dark pool is unavailable. The ex-ante expected profits of rational traders (informed or uninformed) are always higher with the dark pool compared to when the dark pool is unavailable. Informed traders choose DO when the price improvement outweights the execution risk in the dark and, hence, equilibrium profits of informed traders are higher with dark pool access. In addition, uninformed traders also benefit from the existence of the dark pool since there is a reduction of adverse selection in the exchange when θ is high since: (1) if uninformed traders choose LO when there is no access to the dark then uninformed traders continue to submit LO when the dark becomes available. However, when θ is high, the profits of uninformed traders are higher due to the reduction of adverse selection in the exchange; (2) if uninformed traders choose NT when there is no access to the dark then uninformed traders switch to LO when the dark becomes available and θ is sufficiently large. Hence, the average profits of uninformed traders will be higher when there is access to the dark pool. Liquidity traders obtain the same expected profits at $t = 1$ with or without the existence of the dark pool since they only trade according to liquidity needs and do not optimally choose the trading strategy. However, at $t = 2$ the expected profits of liquidity traders will be affected by the presence of the dark pool, since the profit depends directly on the spread at the end of period $t = 1$.

Due to the existence of the dark pool, *expected price informativeness* at $t = 1$ is always lower compared to when traders do not have access to the dark pool since informed traders migrate to the dark when θ is sufficiently large. Therefore, on average, the LOB contains less information with the dark pool than without it.

Adding a dark pool alongside an exchange has an ambiguous effect on the *expected inside spread* at $t = 1$, which depends on both trader and stock market characteristics. By Corollary 1, we find that if the asset's volatility is high and liquidity is low, or if the probability that a noise trader arrives is low, or if the discount factor is low, then the expected inside spread decreases. This is because, when the dark pool is available, informed traders switch from MO to DO if θ is sufficiently high and, hence, expected inside spread decreases since the DO does not change the state of the LOB. Uninformed traders either do not switch their optimal trading strategy (if a LO is submitted) or switch from NT to LO, which decreases the inside spread. In contrast, the expected inside spread increases when the asset's volatility is low and liquidity is high, or if the probability that a noise trader arrives is high, or if the discount factor is high, and adverse selection is low. In contrast, if adverse selection is higher, and the market conditions are such that the optimal trading strategy at $t = 1$ is (BLO, SLO, NT, NT), then the change in the expected inside spread depends on the degree of adverse selection. If adverse selection is sufficiently high then the inside spread increases, while the converse occurs if adverse selection is sufficiently low. This is because, when θ is sufficiently large, the informed switch from LO to DO which increases the inside spread; but the uninformed switch from NT to LO, which decreases the inside spread. Depending on the value of π, one effect might dominate the other.

The effect of adding a dark pool in terms of *expected trading volume* at $t = 1$ depends on the trader and stock market characteristics which influence the choice of the informed trader in the first round of trading. Specifically, if the asset's volatility is high and liquidity is low, or if the probability that a noise trader arrives is low, or if the discount factor is low then the total expected trading volume and also expected trading volume in the exchange decrease. This is because informed traders' orders migrate to the dark pool if θ is sufficiently large, which reduces the expected traded volume in the exchange but also the total expected trading volume since orders that are submitted to the dark pool do not execute at $t = 1$ with probability $1 - \theta$. However, if the asset's volatility is low and liquidity is high, or if the probability that a noise trader arrives is high, or if the discount factor is high then the total expected trading volume increases, but expected trading volume in the exchange remains the same. This occurs since the informed trader migrates to the dark pool when θ is sufficiently high, thus augmenting the total expected trading volume in relation to when the informed submits a LO. The potential switch of the uninformed trader's optimal strategy from NT to LO does not affect the expected volume traded in the exchange at $t = 1$, and hence, the expected total trading volume remains the same.

6 Empirical Implications

In this section we discuss the empirical implications of our model and whether the findings of the empirical literature are consistent with our predictions. These predictions are relevant for the current policy and regulatory debate regarding the effects of dark trading on price discovery, market liquidity and fragmentation of the order flow. The comparative statics that we perform in Section 5 give us two types of predictions: some are true unconditional on the stock market characteristics, while others are true conditional on stock market characteristics. Few of these new cross-sectional empirical predictions are extremely important because they were not previously explored in the previous theoretical models of dark pools, since those models were not delving into the effect of information on the decision to supply or demand liquidity in LOB or migrate to the dark pools.

Our model allows us to derive empirical predictions on the effects of adding a dark pool to a lit exchange based on the comparative statics at $t = 1$. First of all, there is an empirical prediction that is at the heart of the regulatory debate about whether dark pools increase or reduce price discovery.

Prediction 1. *Adding a dark pool alongside an exchange decreases the informativeness of prices.*

Note that our prediction is similar to the theoretical prediction of Ye (2012) and contrary of the one of Zhu (2014) who consider adding a dark pool to a dealer market. The differences between the Ye (2012) and Zhu (2014) are driven by the availability of both informed and uninformed (liquidity) traders to place orders in both the lit and dark venues. Our model features both informed and uninformed traders who can place orders in the two venues but with a different market structure. The possibility to place both market and limit orders changes the order of preferences for the rational traders depending on adverse selection in the LOB (which is a result of the presence of asymmetric information). In addition, since both market and limit order reveal information (Kaniel and Liu, 2006), dark trading may induce a negative effect on price discovery. Our results are in line with the empirical results of Hendershott and Jones (2005), Hatheway et al. (2017), Weaver (2014) and Comerton-Forde and Putniņš (2015) (this last paper only for stocks in which the dark pool trading is large).

Second, our model predicts that adding a dark pool alongside an exchange may increase or decrease market liquidity (measured in our model as the inside spread). Thus, as explained in Proposition 3, it depends whether the trader who migrates to the dark was a supplier or a consumer of liquidity in the exchange. Thus trading in the dark pool can have a positive impact on the spread (reducing it) when the trader who migrates to dark was a consumer of liquidity or a negative impact on spread when he was a supplier of liquidity in the stock exchange. Note that our model without dark pool implies that the willingness of a trader to demand or supply liquidity depends on stock market-characteristics such as liquidity, volatility, adverse selection or tick size.

Prediction 2. Cross-section variation with respect to market liquidity and market volatility.

Ceteris paribus, adding a dark pool to a LOB

- *where a low liquidity/low volatility stock is traded has a negative impact on inside spread.*
- *where a midlle liquidity stock/ middle volatility stock is traded has a positive or a negative impact on inside spread depending on the level of adverse selection in that market.*
- *where a high liquidity stock/high volatility stock is traded has a positive impact on inside spread.*

Our theoretical results potentially reconcile the mixed empirical results previously found in the literature. Thus, there are several studies that show that high levels of dark pool trading decreases market liquidity (Degryse et al., 2014; Hatheway et al., 2014; Kwan et al., 2014; Nimalendran and Ray, 2014; Weaver, 2014) while other studies show that dark pool trading increases market liquidity (Buti et al., 2011; Gresse, 2006; Aquilina et al., 2017). Finally, Foley and Putniņš (2016) show that mid-point dark trading in Canadian market does not affect market liquidity.

Note that these papers are developped using very different datasets and this implies that the research questions, the type of data and the regulatory environments are very different. As a result, most of the empirical papers suggest that these differences are determined by the market structure and the financial market regulation that governs a particular market. Interestingly, our model predicts that adding a dark pool can have both a negative and a positive effect on the market performance of the LOB even if the market structure and the regulatory environment are exactly the same. Our results show therefore that cross-sectional characteristics such as market liquidity, volatility or adverse selection are a possible explanation of this heterogeneity in results.

Prediction 3. Cross-section variation with respect to tick size.

Ceteris paribus, adding a dark pool to a LOB

- *where a low tick size stock is traded has a positive impact on inside spread.*
- *where a middle tick size stock is traded has a positive or negative impact on inside spread depending on the level of adverse selection in that market.*
- *where a high tick size stock is traded has a negative impact on inside spread.*

Notice that when the tick size is high, the market liquidity is low when there is no dark pool. So, adding a dark pool increases the expected inside spread, as the traders who before provided liquidity by submiting limit orders now prefer to migrate to the dark. This result is similar to the one of Buti et al. (2015) who show that allowing dark orders to "queue-jump" displayed orders reduces traders' willingness to display limit orders on competing lit markets. These results are consistent with Buti et al. (2011) and Kwan et al. (2015) who show that when spreads on traditional exchanges are constrained by minimum pricing increments, traders have incentives to migrate toward dark trading venues since the execution risk in the dark is lower than the execution risk of the limit orders in the exchange. Our results are similar, but the mechanism is different from the one explained by Buti et al. (2011) and Kwan et al. (2015). In our case the tick size does not affect the execution probability but the profits obtained in case of execution. Placing an improving limit order in the case the tick size is large reduces the profits of the limit order and therefore, the traders migh thave more incentives to go to the dark pool.

Prediction 4. Cross-section variation with respect to the noise and adverse selection in the lit market.

Ceteris paribus, adding a dark pool to a LOB

- *when the probability of a liquidity trader to arrive is low has a negative impact on inside spread.*
- *when the probability of a liquidity trader to arrive is high has a positive impact on inside spread when the adverse selection is low and either a positive or a negative impact when the adverse selection is high.*

Aquilina et al. (2017) study the other direction of causality: how dark trading affects adverse selection in the lit market. Future empirical studies could analyse using a natural experiment how different levels of adverse selection in the market (measured for example by the probability of informed trading PIN) affect the spread after the introduction of a dark pool.

Prediction 5. Cross-section variation with respect to immediacy.

- *Adding a dark pool to a LOB when investors are characterized by high immediacy (low δ) has a negative impact on inside spread (all else equal).*
- *Adding a dark pool to a LOB when investors are characterized by low immediacy (high δ) has a positive impact on inside spread when the adverse selection is low and either positive or negative impact when the adverse selection is high.*

7 Conclusions

In this paper we study market participants' simultaneous strategic choice of trading venue and order type when traders have access to a dark pool and to an exchange (lit market) that is organized as a limit order book (LOB). We model the exchange as a fully transparent LOB, while the dark pool is an opaque market where orders are executed at the midpoint between the best bid and ask prices prevailing in the exchange. We build a multi-period model that allows us to understand the interaction of the LOB with the dark pool in the presence of asymmetric information. We characterize the Perfect Bayesian Equilibria of the trading game.

We find that adding a dark pool alongside an exchange may shift the optimal strategies of each type of rational trader. In the first period, an uninformed trader may switch from *no trade* to submitting a limit order in the exchange due to the reduction of adverse selection, while the informed trader's strategy diverts from the exchange to the dark pool when the execution risk in the dark pool is sufficiently low. However, uninformed traders may trade in the dark once they have learnt that the value of the asset is favorable from observing the state of the LOB. In addition, we find that due to adverse selection, uninformed traders prefer not to trade with informed traders. The optimal strategy of an informed trader in the first period reveals information to uninformed traders about the value of the asset. Our findings show that, even if execution risk in the dark pool is high, informed traders tend to replace market orders by dark orders when they can take advantage of the price improvement.

We also show that adding a dark pool alongside exchange increases the expected welfare of the rational market participants and reduces the expected price informativeness at $t = 1$. The effect on market liquidity and expected trading volume depend crucially on the stock market characteristics (i.e., liquidity, volatility, adverse selection) and trader characteristics (i.e., discount factor) as these determine if a trader who migrates to the dark pool is a supplier, consumer or does not provide liquidity in the exchange when the dark pool is unavailable. Thus our results help reconcile the positive and negative effects of dark pools on market quality previously found in the empirical studies. Our model provides new testable predictions for cross-sectional analysis. In addition, we provide the asset and trader conditions under which we should expect to observe segmentation of the informed-uninformed order flow. The findings of our paper call for the development of further empirical and experimental work which study the role of information in the competition between a dark pool and an exchange for liquidity.

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8 Appendices

A Model without dark pool

Proof of Proposition 1. The procedure we follow to check if a particular strategy profile constitutes a PBE is as follows:

1. Specify a strategy profile for rational traders at $t = 1$.
2. Update the beliefs of the uninformed trader at $t = 2$ using Bayes' rule at all information sets, whenever possible.
3. Given their beliefs, find the optimal response for the traders at $t = 2$.
4. Given the optimal response of traders at $t = 2$, find the optimal action for rational traders at $t = 1$.
5. Check if the optimal strategy profile for the traders at $t = 1$ coincide with the profile suggested in step 1.

We apply the procedure outlined above to check when each possible strategy profile can be an equilibrium.

$\mathcal{E}_1^{ND}$**:** (BMO, SMO, BLO, SLO)

First step. In this case $\Omega_0 = 0$, $\Omega_1 = 1$, $\Omega_2 = 0$, $\Gamma_0 = 0$, $\Gamma_1 = 0$, and $\Gamma_2 = 1$.

Second step. Using Bayes' rule we obtain that $X = \dfrac{\lambda\pi}{\eta + \lambda\pi}$ and $Y = 0$.

Third step. Applying Lemma 1, we know that at $t = 2$ the optimal strategy of informed traders is to choose a MO, while the optimal strategy of the uninformed trader is as follows:

State of the book	UB	US
(A_1^1, B_1^1)	NT	NT
(A_1^2, B_1^1)	$\begin{cases} MO & \text{if } \dfrac{\lambda\pi}{\eta+\lambda\pi} > \dfrac{k_2}{k_3} \\ NT & \text{if } \dfrac{\lambda\pi}{\eta+\lambda\pi} \leq \dfrac{k_2}{k_3} \end{cases}$	NT
$(A_1^1, B_1^1 + \tau)$	NT	NT
(A_1^1, B_1^2)	NT	$\begin{cases} MO & \text{if } \dfrac{\lambda\pi}{\eta+\lambda\pi} > \dfrac{k_2}{k_3} \\ NT & \text{if } \dfrac{\lambda\pi}{\eta+\lambda\pi} \leq \dfrac{k_2}{k_3} \end{cases}$
$(A_1^1 - \tau, B_1^1)$	NT	NT

Table A.1: Optimal responses of uninformed traders at $t = 2$ when the strategy profile at $t = 1$ is (BMO, SMO, BLO, SLO).

Fourth step. Given the optimal response of traders at $t = 2$, we find the optimal action for all rational traders at $t = 1$.

Informed traders. If they choose a MO, their expected profits equal $(k_3 - k_1)\,\tau$. If, instead, they deviate towards a LO, as they anticipate that uninformed traders choose NT, their expected profits of a LO are $\delta\frac{\eta}{2}\,(k_3 + k_1 - 1)\,\tau$. Hence, informed traders at $t = 1$ have no incentives to deviate from the prescribed strategy profile whenever

$$(k_3 - k_1)\,\tau \geq \delta\frac{\eta}{2}\,(k_3 + k_1 - 1)\,\tau.$$

Uninformed traders. If they behave as the prescribed profile (LO), then they obtain

$$\frac{\delta}{2}\,(\eta(k_1 - 1) - \lambda\pi\,(k_3 - (k_1 - 1)))\,\tau,$$

given that they anticipate that uninformed traders at $t = 2$ will choose NT. If, instead, they deviate choosing NT, then they obtain zero profits. Therefore, uninformed traders at $t = 1$ have no incentives incentive to deviate from the prescribed strategy if and only if

$$\eta(k_1 - 1) - \lambda\pi\,(k_3 - (k_1 - 1)) > 0. \tag{5}$$

Fifth step. From the previous two inequalities, nobody at $t = 1$ has unilateral incentives to deviate from (BMO, SMO, BLO, SLO) whenever

$$k_3 - k_1 \geq \delta\frac{\eta}{2}\,(k_3 + k_1 - 1) \text{ and}$$
$$(\lambda\pi + \eta)\,(k_1 - 1) - \lambda\pi k_3 > 0.$$

The previous inequality implies that the optimal strategy of an uninformed trader at $t = 2$ is to choose NT for all possible states of the book.

$\mathcal{E}_2^{ND}$**:** (BMO, SMO, NT, NT)

First step. In this case $\Omega_0 = 0$, $\Omega_1 = 1$, $\Omega_2 = 0$, $\Gamma_0 = 1$, $\Gamma_1 = 0$, and $\Gamma_2 = 0$.

Second step. Using Bayes's rule we obtain that $X = \dfrac{\lambda\pi}{\eta + \lambda\pi}$ and Y is undetermined $Y \in [0, 1]$ (as Bayes's rule implies $Y = \frac{0}{0}$).

Third step. Applying Lemma 1, we know that at $t = 2$ the optimal strategy of informed traders is to choose a MO, while the optimal strategy of uninformed trader is as follows:

State of the book	UB	US
(A_1^1, B_1^1)	NT	NT
(A_1^2, B_1^1)	$\begin{cases} MO & \text{if } \dfrac{\lambda\pi}{\eta+\lambda\pi} > \dfrac{k_2}{k_3} \\ NT & \text{if } \dfrac{\lambda\pi}{\eta+\lambda\pi} \leq \dfrac{k_2}{k_3} \end{cases}$	NT
$(A_1^1, B_1^1+\tau)$	$\begin{cases} MO & \text{if } Y > \dfrac{k_1}{k_3} \\ NT & \text{if } Y \leq \dfrac{k_1}{k_3} \end{cases}$	NT
(A_1^1, B_1^2)	NT	$\begin{cases} MO & \text{if } \dfrac{\lambda\pi}{\eta+\lambda\pi} > \dfrac{k_2}{k_3} \\ NT & \text{if } \dfrac{\lambda\pi}{\eta+\lambda\pi} \leq \dfrac{k_2}{k_3} \end{cases}$
$(A_1^1-\tau, B_1^1)$	NT	$\begin{cases} MO & \text{if } Y > \dfrac{k_1}{k_3} \\ NT & \text{if } Y \leq \dfrac{k_1}{k_3} \end{cases}$

Table A.2: Optimal responses of uninformed traders at $t = 2$ when the strategy profile at $t = 1$ is (BMO, SMO, NT, NT).

Fourth step. Given the optimal response of traders at $t = 2$, we find the optimal action of rational traders at $t = 1$.

Informed traders. If they choose a MO, they obtain $(k_3 - k_1)\,\tau$. If, instead, they deviate towards a LO, as they anticipate that uninformed traders will not choose a MO of different sign at $t = 2$, then their expected profits will be: $\delta\frac{\eta}{2}\,(k_3 + k_1 - 1)\,\tau$. Hence, informed traders at $t = 1$ have no incentives to deviate from the prescribed strategy profile whenever

$$k_3 - k_1 \geq \delta\frac{\eta}{2}\,(k_3 + k_1 - 1)\,.$$

Uninformed traders. If they behave as the prescribed profile (NT), then they obtain 0. If, instead, they deviate choosing a LO, as they anticipate that uninformed traders will not choose a MO of different sign at $t = 2$, their expected profits will be

$$\frac{\delta}{2}\,(\eta(k_1 - 1) - \lambda\pi\,(k_3 - (k_1 - 1)))\,\tau.$$

Therefore, uninformed traders at $t = 1$ have no incentives to deviate from the prescribed strategy if and only if

$$0 \geq (\lambda\pi + \eta)\,(k_1 - 1) - \lambda\pi k_3. \tag{6}$$

Fifth step. No trader at $t = 1$ has unilateral incentives to deviate from (BMO, SMO, NT, NT) if and only if

$$k_3 - k_1 \geq \delta \frac{\eta}{2} (k_3 + k_1 - 1) \text{ and}$$
$$0 \geq (\lambda\pi + \eta)(k_1 - 1) - \lambda\pi k_3.$$

$\mathcal{E}_3^{ND}$**:** (BLO, SLO, BLO, BLO)
First step. In this case $\Omega_0 = 0$, $\Omega_1 = 0$, $\Omega_2 = 1$, $\Gamma_0 = 0$, $\Gamma_1 = 0$, and $\Gamma_2 = 1$.
Second step. Using Bayes's rule we obtain that $X = 0$ and $Y = \pi$.
Third step. Applying Lemma 1, we know that at $t = 2$ the optimal strategy for informed traders is to choose a MO, while for the uninformed trader is as follows:

State of the book	**UB**	**US**
(A_1^1, B_1^1)	NT	NT
(A_1^2, B_1^1)	NT	NT
$(A_1^1, B_1^1 + \tau)$	$\begin{cases} MO & \text{if } \pi > \frac{k_1}{k_3} \\ NT & \text{if } \pi \leq \frac{k_1}{k_3} \end{cases}$	NT
(A_1^1, B_1^2)	NT	NT
$(A_1^1 - \tau, B_1^1)$	NT	$\begin{cases} MO & \text{if } \pi > \frac{k_1}{k_3} \\ NT & \text{if } \pi \leq \frac{k_1}{k_3} \end{cases}$

Table A.3: Optimal responses of uninformed traders at $t = 2$ when the strategy profile at $t = 1$ is (BLO, SLO, BLO, BLO).

Fourth step. Given the optimal response of traders at $t = 2$, we find the optimal action for the rational traders at $t = 1$.

Informed traders. If they choose a LO, as they anticipate that uninformed traders will not choose a MO of different sign at $t = 2$, then their expected profits will be: $\delta\frac{\eta}{2}(k_3 + k_1 - 1)\tau$. If, instead, they choose a market order, then they obtain $(k_3 - k_1)\tau$. Hence, at $t = 1$ informed traders have no incentives to deviate from the prescribed strategy profile whenever

$$\delta \frac{\eta}{2} (k_3 + k_1 - 1) > k_3 - k_1.$$

Uninformed traders. If they select a LO, their expected profits are $\frac{\delta\tau}{2}(\eta(k_1 - 1) - \lambda\pi(k_3 - (k_1 - 1)))$ since they anticipate that traders will not choose a MO of different sign at $t = 2$. If, instead, they choose NT, they obtain null profits. Therefore, at $t = 1$ uninformed traders have no incentives to deviate from the prescribed strategy if and only if

$$(\lambda\pi + \eta)(k_1 - 1) - \lambda\pi k_3 > 0. \tag{7}$$

Fifth step. No trader at $t = 1$ has unilateral incentives to deviate from (BLO, SLO, BLO, SLO) if and only if

$$\delta\frac{\eta}{2}(k_3 + k_1 - 1) > k_3 - k_1 \text{ and}$$
$$(\lambda\pi + \eta)(k_1 - 1) - \lambda\pi k_3 > 0.$$

$\mathcal{E}_4^{ND}$**:** (BLO, SLO, NT, NT)
First step. In this case $\Omega_0 = 0$, $\Omega_1 = 0$, $\Omega_2 = 1$, $\Gamma_0 = 1$, $\Gamma_1 = 0$, and $\Gamma_2 = 0$.
Second step. Using Bayes's rule we obtain that $X = 0$ and $Y = 1$.
Third step. Applying Lemma 1, we know that at $t = 2$ the optimal strategy for informed traders is to choose a MO, while for the uninformed trader is as follows:

State of the book	UB	US
(A_1^1, B_1^1)	NT	NT
(A_1^2, B_1^1)	NT	NT
$(A_1^1, B_1^1 + \tau)$	MO	NT
(A_1^1, B_1^2)	NT	NT
$(A_1^1 - \tau, B_1^1)$	NT	MO

Table A.4: Optimal responses of uninformed traders at $t = 2$ when the strategy profile at $t = 1$ is (BLO, SLO, NT, NT).

Fourth step. Given the optimal response of traders at $t = 2$, find the optimal action for the rational traders at $t = 1$.

- *Informed traders.* As they anticipate that uninformed traders will not choose a MO of different sign at $t = 2$, then their expected profits are: $\delta\frac{\eta}{2}(k_3 + k_1 - 1)\tau$. If, instead, they deviate towards a MO, they obtain $(k_3 - k_1)\tau$. Hence, informed traders have no incentives to deviate from the prescribed strategy profile whenever

$$\delta\frac{\eta}{2}(k_3 + k_1 - 1)\tau > (k_3 - k_1)\tau.$$

- *Uninformed traders.* If they chooses NT, they obtain zero profits. If, instead, an uninformed buyer (seller) deviate towards a BLO (SLO), then he obtains $\frac{\delta\tau}{2}((\lambda\pi + \eta)(k_1 - 1) - \lambda\pi k_3)$. Hence, uninformed traders have no incentive to deviate from the prescribed strategy profile whenever

$$0 \geq \frac{\delta}{2}(\eta(k_1 - 1) - \lambda\pi(k_3 - (k_1 - 1))). \tag{8}$$

Fifth step. Nobody at $t = 1$ has unilateral incentives to deviate from (BLO, SLO, NT, NT) whenever

$$\delta\frac{\eta}{2}(k_3 + k_1 - 1) > (k_3 - k_1)$$
$$0 \geq \frac{\delta}{2}\left(\eta(k_1 - 1) - \lambda\pi\left(k_3 - (k_1 - 1)\right)\right).$$

■

Lemma 5 *When there is no dark pool and $k_1 = 1$ the uninformed trader at $t = 1$ always chooses NT.*

Proof. The previous lemma implies that the strategies (BMO, SMO, BLO, SLO) and (BLO, SLO, BLO, BLO) cannot be equilibria of the game. ■

Proposition 4 *If $k_1 = 1$, then a PBE of the game is as follows:*

- (BMO, SMO, NT, NT) *is the optimal strategy profile for traders at $t = 1$ if*

  $$k_3 - 1 \geq \delta\frac{\eta}{2}k_3.$$

  The beliefs of uninformed traders at $t = 2$ are: $X = \dfrac{\lambda\pi}{\eta + \lambda\pi}$ and $Y = p \in [0, 1]$. The optimal strategy of informed traders at $t = 2$ is to choose MO for all possible states of the book and the optimal strategies of uninformed traders at $t = 2$ are described in Table A.2.

- (BLO, SLO, NT, NT) *is the optimal strategy profile for traders at $t = 1$ if*

  $$\delta\left(\frac{\eta}{2} + \frac{(1-\pi)\lambda}{2}\right)k_3 > k_3 - 1.$$

  The beliefs of uninformed traders at $t = 2$ are: $X = 0$ and $Y = 1$. The optimal strategy of informed traders at $t = 2$ is to choose MO for all possible states of the book and the optimal strategies of uninformed traders at $t = 2$ are described in Table A.4.

Proof of Proposition 4. Note that when we replace $k_1 = 1$ in the Proof of Proposition 1 the conditions (5) and (7) are never satisfied and therefore the strategies (BMO, SMO, BLO, SLO) and (BLO, SLO, BLO, BLO) cannot be part of an equilibrium of the game.

Moreover when $k_1 = 1$, the conditions (6) and (8) are always satisfied. ■

B Model with dark pool

Proof of Lemma 3. By simply inspection of the payoff in Table (6) can be seen that the informed buyers at $t = 2$ never choose NT because this order is dominated by placing a MO. Notice also that the uninformed traders never select a DO and therefore $\Gamma_3 = 0$ which implies

$$P_I = p^{IH}_{BLO,2}\left(\mathcal{B}_1 = \varnothing\right) = p^{IL}_{SLO,2}\left(\mathcal{B}_1 = \varnothing\right) = 0.$$

Consequently, the informed traders never choose a LO at $t = 2$, since this order is also dominated by a MO.

Let us determine next the optimal strategy for each trader. Depending on the values of the parameters we have 6 possible cases for the informed trader and 16 for the uninformed trader.

First let us first focus on the informed traders. Note that since $k_3 > k_2 > k_1 \geq 1$, the following inequalities hold

$$\frac{k_3 - k_2}{k_3 - \frac{k_2 - k_1}{2}} < \frac{k_3 - k_1}{k_3 + \frac{k_2 - k_1}{2}} < \frac{k_3 - k_1}{k_3} < \frac{k_3 - k_1}{k_3 - \frac{1}{2}} < \frac{k_3 - k_1 + 1}{k_3 + \frac{1}{2}}.$$

We define by

$$\begin{aligned} \theta_X &\equiv \frac{Xk_3 - k_2}{Xk_3 - \frac{k_2 - k_1}{2}} \\ \theta_Y &\equiv \frac{Yk_3 - k_1}{Yk_3 - \frac{1}{2}}. \end{aligned}$$

The optimal strategies of the informed traders are:

Condition	**Optimal Strategies of Informed Traders at t=2**		
	State of the Book	**IH**	**IL**
Case I_1 $\theta \leq \frac{k_3-k_2}{k_3-\frac{k_2-k_1}{2}}$	(A_1^1, B_1^1)	BMO	SMO
	(A_1^2, B_1^1)	BMO	SMO
	$(A_1^1, B_1^1+\tau)$	BMO	SMO
	(A_1^1, B_1^2)	BMO	SMO
	$(A_1^1-\tau, B_1^1)$	BMO	SMO
Case I_2 $\frac{k_3-k_2}{k_3-\frac{k_2-k_1}{2}} < \theta$ $\leq \frac{k_3-k_1}{k_3+\frac{k_2-k_1}{2}}$	(A_1^1, B_1^1)	BMO	SMO
	(A_1^2, B_1^1)	BDO	SMO
	$(A_1^1, B_1^1+\tau)$	BMO	SMO
	(A_1^1, B_1^2)	BMO	SDO
	$(A_1^1-\tau, B_1^1)$	BMO	SMO
Case I_3 $\frac{k_3-k_1}{k_3+\frac{k_2-k_1}{2}} < \theta$ $\leq \frac{k_3-k_1}{k_3}$	(A_1^1, B_1^1)	BMO	SMO
	(A_1^2, B_1^1)	BDO	SDO
	$(A_1^1, B_1^1+\tau)$	BMO	SMO
	(A_1^1, B_1^2)	BDO	SDO
	$(A_1^1-\tau, B_1^1)$	BMO	SMO
Case I_4 $\frac{k_3-k_1}{k_3} < \theta$ $\leq \frac{k_3-k_1}{k_3-\frac{1}{2}}$	(A_1^1, B_1^1)	BDO	SDO
	(A_1^2, B_1^1)	BDO	SDO
	$(A_1^1, B_1^1+\tau)$	BMO	SMO
	(A_1^1, B_1^2)	BDO	SDO
	$(A_1^1-\tau, B_1^1)$	BMO	SMO
Case I_5 $\frac{k_3-k_1}{k_3-\frac{1}{2}} < \theta$ $\leq \frac{k_3-k_1+1}{k_3+\frac{1}{2}}$	(A_1^1, B_1^1)	BDO	SDO
	(A_1^2, B_1^1)	BDO	SDO
	$(A_1^1, B_1^1+\tau)$	BDO	SMO
	(A_1^1, B_1^2)	BDO	SDO
	$(A_1^1-\tau, B_1^1)$	BMO	SDO
Case I_6 $\frac{k_3-k_1+1}{k_3+\frac{1}{2}} < \theta$	(A_1^1, B_1^1)	BDO	SDO
	(A_1^2, B_1^1)	BDO	SDO
	$(A_1^1, B_1^1+\tau)$	BDO	SDO
	(A_1^1, B_1^2)	BDO	SDO
	$(A_1^1-\tau, B_1^1)$	BDO	SDO

Table B.1: Optimal Strategies of Informed Traders at $t = 2$

<table>
<tr><th>Condition</th><th colspan="3">Optimal Strategies of Uninformed Traders at $t = 2$</th></tr>
<tr><th></th><th>State of the Book</th><th>UB</th><th>US</th></tr>
<tr><td>Case U.1.1
$Xk_3 < \frac{k_2-k_1}{2} < k_2$
$Yk_3 < \frac{1}{2}(< k_1)$</td><td>$(A_1^1, B_1^1)$
(A_1^2, B_1^1)
$(A_1^1, B_1^1 + \tau)$
(A_1^1, B_1^2)
$(A_1^1 - \tau, B_1^1)$</td><td>NT
NT
NT
BDO
BDO</td><td>NT
SDO
SDO
NT
NT</td></tr>
<tr><td>Case U.1.2
$Xk_3 < \frac{k_2-k_1}{2} < k_2$
$Yk_3 = \frac{1}{2}(< k_1)$</td><td>$(A_1^1, B_1^1)$
(A_1^2, B_1^1)
$(A_1^1, B_1^1 + \tau)$
(A_1^1, B_1^2)
$(A_1^1 - \tau, B_1^1)$</td><td>NT
NT
NT
BDO
NT</td><td>NT
SDO
NT
NT
NT</td></tr>
<tr><td>Case U.1.3
$Xk_3 < \frac{k_2-k_1}{2} < k_2$
$\frac{1}{2} < Yk_3 \leq k_1$</td><td>$(A_1^1, B_1^1)$
(A_1^2, B_1^1)
$(A_1^1, B_1^1 + \tau)$
(A_1^1, B_1^2)
$(A_1^1 - \tau, B_1^1)$</td><td>NT
NT
BDO
BDO
NT</td><td>NT
SDO
NT
NT
SDO</td></tr>
<tr><td>Case U.1.4
$Xk_3 < \frac{k_2-k_1}{2} < k_2$
$k_1 < Yk_3$</td><td>(A_1^1, B_1^1)
(A_1^2, B_1^1)
$(A_1^1, B_1^1 + \tau)$
(A_1^1, B_1^2)
$(A_1^1 - \tau, B_1^1)$</td><td>NT
NT
$\begin{cases} BDO & \theta > \theta_Y \\ BMO & \theta \leq \theta_Y \end{cases}$
BDO
NT</td><td>NT
SDO
NT
NT
$\begin{cases} SDO & \theta > \theta_Y \\ SMO & \theta \leq \theta_Y \end{cases}$</td></tr>
</table>

Table B.2: Optimal Strategies of Uninformed Traders at $t = 2$

Condition	Optimal Strategies of Uninformed Traders at $t=2$		
	State of the Book	**UB**	**US**
Case U.2.1 $Xk_3 = \frac{k_2-k_1}{2} < k_2$ $Yk_3 < \frac{1}{2}(< k_1)$	(A_1^1, B_1^1) (A_1^2, B_1^1) $(A_1^1, B_1^1+\tau)$ (A_1^1, B_1^2) $(A_1^1-\tau, B_1^1)$	NT NT NT NT BDO	NT NT SDO NT NT
Case U.2.2 $Xk_3 = \frac{k_2-k_1}{2} < k_2$ $Yk_3 = \frac{1}{2}(< k_1)$	(A_1^1, B_1^1) (A_1^2, B_1^1) $(A_1^1, B_1^1+\tau)$ (A_1^1, B_1^2) $(A_1^1-\tau, B_1^1)$	NT NT NT NT NT	NT NT NT NT NT
Case U.2.3 $Xk_3 = \frac{k_2-k_1}{2} < k_2$ $\frac{1}{2} < Yk_3 \leq k_1$	(A_1^1, B_1^1) (A_1^2, B_1^1) $(A_1^1, B_1^1+\tau)$ (A_1^1, B_1^2) $(A_1^1-\tau, B_1^1)$	NT NT BDO NT NT	NT NT NT NT SDO
Case U.2.4 $Xk_3 < \frac{k_2-k_1}{2} < k_2$ $k_1 < Yk_3$	(A_1^1, B_1^1) (A_1^2, B_1^1) $(A_1^1, B_1^1+\tau)$ (A_1^1, B_1^2) $(A_1^1-\tau, B_1^1)$	NT NT $\begin{cases} BDO & \theta > \theta_Y \\ BMO & \theta \leq \theta_Y \end{cases}$ NT NT	NT NT NT NT $\begin{cases} SDO & \theta > \theta_Y \\ SMO & \theta \leq \theta_Y \end{cases}$

Optimal Strategies of Uninformed Traders at $t = 2$ (Continuation)

Condition	Optimal Strategies of Uninformed Traders at $t = 2$		
	State of the Book	**UB**	**US**
Case U.3.1 $\frac{k_2-k_1}{2} < Xk_3 \leq k_2$ $Yk_3 < \frac{1}{2}$	(A_1^1, B_1^1) (A_1^2, B_1^1) $(A_1^1, B_1^1 + \tau)$ (A_1^1, B_1^2) $(A_1^1 - \tau, B_1^1)$	NT BDO NT NT BDO	NT NT SDO SDO NT
Case U.3.2 $\frac{k_2-k_1}{2} < Xk_3 \leq k_2$ $Yk_3 = \frac{1}{2}$	(A_1^1, B_1^1) (A_1^2, B_1^1) $(A_1^1, B_1^1 + \tau)$ (A_1^1, B_1^2) $(A_1^1 - \tau, B_1^1)$	NT BDO NT NT NT	NT NT NT SDO NT
Case U.3.3 $\frac{k_2-k_1}{2} < Xk_3 \leq k_2$ $\frac{1}{2} < Yk_3 \leq k_1$	(A_1^1, B_1^1) (A_1^2, B_1^1) $(A_1^1, B_1^1 + \tau)$ (A_1^1, B_1^2) $(A_1^1 - \tau, B_1^1)$	NT BDO BDO NT NT	NT NT NT SDO SDO
Case U.3.4 $\frac{k_2-k_1}{2} < Xk_3 \leq k_2$ $k_1 < Yk_3$	(A_1^1, B_1^1) (A_1^2, B_1^1) $(A_1^1, B_1^1 + \tau)$ (A_1^1, B_1^2) $(A_1^1 - \tau, B_1^1)$	NT BDO $\begin{cases} BDO & \theta > \theta_Y \\ BMO & \theta \leq \theta_Y \end{cases}$ NT NT	NT NT NT SDO $\begin{cases} SDO & \theta > \theta_Y \\ SMO & \theta \leq \theta_Y \end{cases}$

Optimal Strategies of Uninformed Traders at $t = 2$ (Continuation)

Condition	Optimal Strategies of Uninformed Traders at $t = 2$		
	State of the Book	**UB**	**US**
Case U.4.1	(A_1^1, B_1^1)	NT	NT
	(A_1^2, B_1^1)	$\begin{cases} BDO & \text{if } \theta > \theta_X \\ BMO & \text{if } \theta \leq \theta_X \end{cases}$	NT
$k_2 < Xk_3$	$(A_1^1, B_1^1 + \tau)$	NT	SDO
$Yk_3 < \frac{1}{2}$	(A_1^1, B_1^2)	NT	$\begin{cases} SDO & \text{if } \theta > \theta_X \\ SMO & \text{if } \theta \leq \theta_X \end{cases}$
	$(A_1^1 - \tau, B_1^1)$	BDO	NT
Case U.4.2	(A_1^1, B_1^1)	NT	NT
	(A_1^2, B_1^1)	$\begin{cases} BDO & \text{if } \theta > \theta_X \\ BMO & \text{if } \theta \leq \theta_X \end{cases}$	NT
$k_2 < Xk_3$	$(A_1^1, B_1^1 + \tau)$	NT	NT
$Yk_3 = \frac{1}{2}$	(A_1^1, B_1^2)	NT	$\begin{cases} SDO & \text{if } \theta > \theta_X \\ SMO & \text{if } \theta \leq \theta_X \end{cases}$
	$(A_1^1 - \tau, B_1^1)$	NT	NT
Case U.4.3	(A_1^1, B_1^1)	NT	NT
	(A_1^2, B_1^1)	$\begin{cases} BDO & \text{if } \theta > \theta_X \\ BMO & \text{if } \theta \leq \theta_X \end{cases}$	NT
$k_2 < Xk_3$	$(A_1^1, B_1^1 + \tau)$	BDO	NT
$\frac{1}{2} < Yk_3 \leq k_1$	(A_1^1, B_1^2)	NT	$\begin{cases} SDO & \text{if } \theta > \theta_X \\ SMO & \text{if } \theta \leq \theta_X \end{cases}$
	$(A_1^1 - \tau, B_1^1)$	NT	SDO
Case U.4.4	(A_1^1, B_1^1)	NT	NT
	(A_1^2, B_1^1)	$\begin{cases} BDO & \text{if } \theta > \theta_X \\ BMO & \text{if } \theta \leq \theta_X \end{cases}$	NT
$k_2 < Xk_3$	$(A_1^1, B_1^1 + \tau)$	$\begin{cases} BDO & \text{if } \theta > \theta_Y \\ BMO & \text{if } \theta \leq \theta_Y \end{cases}$	NT
$k_1 < Yk_3$	(A_1^1, B_1^2)	NT	$\begin{cases} SDO & \text{if } \theta > \theta_X \\ SMO & \text{if } \theta \leq \theta_X \end{cases}$
	$(A_1^1 - \tau, B_1^1)$	NT	$\begin{cases} SDO & \text{if } \theta > \theta_Y \\ SMO & \text{if } \theta \leq \theta_Y \end{cases}$

Optimal Strategies of Uninformed Traders at $t = 2$ (Continuation)

$\blacksquare$

Proof of Proposition 2. Because of the symmetry of the model, without any loss of generality, at $t = 1$ we focus on buyers. We present the proof for one of the possible strategy profile at $t = 1$ that yields an equilibrium. The proofs of all the other 5 equilibria can be obtained on request from the authors. Note that in all equilibria the optimal responses of informed traders at $t = 2$ are given in Table B.1.

$\mathcal{E}_1^D$**:** (BMO, SMO, BLO, SLO)

First step. In this case $\Omega_0 = 0, \Omega_1 = 1, \Omega_2 = 0, \Omega_3 = 0, \Gamma_0 = 0, \Gamma_1 = 0$, $\Gamma_2 = 1$, and $\Gamma_3 = 0$.

Second step. Using Bayes's rule,

$$X = \frac{\lambda\pi}{\eta + \lambda\pi}, \, Y = 0 \text{ and } Z = q \in [0, 1] \, .$$

Third step. Using steps 1 and 2, the expected profits of uninformed traders at $t = 2$ are given by

UB	**BMO**	**BDO**	**BLO**	**NT**
(A_1^1, B_1^1)	$-k_1\tau$	0	0	0
(A_1^2, B_1^1)	$\left(\frac{\lambda\pi}{\eta+\lambda\pi}k_3 - k_2\right)\tau$	$\theta\left(\frac{\lambda\pi}{\eta+\lambda\pi}k_3 - \frac{k_2-k_1}{2}\right)\tau$	0	0
$(A_1^1, B_1^1 + \tau)$	$-k_1\tau$	$-\frac{\theta}{2}\tau$	0	0
(A_1^1, B_1^2)	$-\left(\frac{\lambda\pi}{\eta+\lambda\pi}k_3 + k_1\right)\tau$	$-\theta\left(\frac{\lambda\pi}{\eta+\lambda\pi}k_3 - \frac{k_2-k_1}{2}\right)\tau$	0	0
$(A_1^1 - \tau, B_1^1)$	$-(k_1 - 1)\tau$	$\frac{\theta}{2}\tau$	0	0

Table B.3: Expected profits for an uninformed buyer at $t = 2$ when the strategy profile at $t = 1$ is (BMO, SMO, BLO, SLO).

US	**SMO**	**SDO**	**SLO**	**NT**
(A_1^1, B_1^1)	$-k_1\tau$	0	0	0
(A_1^2, B_1^1)	$-\left(k_1 + \frac{\lambda\pi}{\eta+\lambda\pi}k_3\right)\tau$	$-\theta\left(\frac{\lambda\pi}{\eta+\lambda\pi}k_3 - \frac{k_2-k_1}{2}\right)\tau$	0	0
$(A_1^1, B_1^1 + \tau)$	$-(k_1 - 1)\tau$	$\frac{\theta\tau}{2}$	0	0
(A_1^1, B_1^2)	$\left(\frac{\lambda\pi}{\eta+\lambda\pi}k_3 - k_2\right)\tau$	$\theta\left(\frac{\lambda\pi}{\eta+\lambda\pi}k_3 - \frac{k_2-k_1}{2}\right)\tau$	0	0
$(A_1^1 - \tau, B_1^1)$	$-k_1\tau$	$-\frac{\theta\tau}{2}$	0	0

Table B.4: Expected profits for an uninformed seller at $t = 2$ when the strategy profile at $t = 1$ is (BMO, SMO, BLO, SLO).

Hence, the optimal responses of uninformed traders are:

State of the book	**UB**	**US**
(A_1^1, B_1^1)	NT	NT
(A_1^2, B_1^1)	$\begin{cases} NT & \text{if } Xk_3 \leq \frac{k_2-k_1}{2} \\ BDO & \begin{cases} \text{if } \frac{k_2-k_1}{2} < Xk_3 \leq k_2 \text{ or} \\ \text{if } k_2 < Xk_3 \text{ and } \theta > \theta_X \end{cases} \\ BMO & \text{if } k_2 < Xk_3 \text{ and } \theta \leq \theta_X \end{cases}$	$\begin{cases} SDO \text{ if } Xk_3 < \frac{k_2-k_1}{2} \\ NT \quad \text{if } \frac{k_2-k_1}{2} \leq Xk_3 \end{cases}$
$(A_1^1, B_1^1 + \tau)$	NT	SDO
(A_1^1, B_1^2)	$\begin{cases} BDO \text{ if } Xk_3 < \frac{k_2-k_1}{2} \\ NT \quad \text{if } \frac{k_2-k_1}{2} \leq Xk_3 \end{cases}$	$\begin{cases} NT & \text{if } Xk_3 \leq \frac{k_2-k_1}{2} \\ SDO & \begin{cases} \text{if } \frac{k_2-k_1}{2} < Xk_3 \leq k_2 \text{ or} \\ \text{if } k_2 < Xk_3 \text{ and } \theta > \theta_X \end{cases} \\ SMO & \text{if } k_2 < Xk_3 \text{ and } \theta \leq \theta_X \end{cases}$
$(A_1^1 - \tau, B_1^1)$	BDO	NT

Table B.5: Optimal responses of uninformed traders at $t = 2$ when the strategy profile at $t = 1$ is (BMO, SMO, BLO, SLO).

Fourth step. Given the optimal responses of rational traders at $t = 2$, we find the optimal action for the rational traders at $t = 1$ in each of the 6 cases. However, given the nature of this particular equilibrium, we can group cases into the following and analyze them:

Case $I_1 + I_2 + I_3 : \theta \leq \frac{k_3-k_1}{k_3}$

- *Informed traders*

Consider an informed buyer at $t = 1$. If he chooses a BMO, then he obtains

$$\mathbb{E}\left(\Pi_{BMO,1}^{IH}\right) = (k_3 - k_1)\,\tau.$$

If instead he deviates towards a BLO, then in the next period the prices will be $(A_1^1, B_1^1 + \tau)$ and, then, he anticipates the following behavior for potential sellers at $t = 2$:

1. if there is an uninformed seller, then he will choose SDO, and

2. if there is a liquidity seller, then he will place a SMO.

Therefore, the BLO at $t = 1$ will only be executed if in the next period there is a liquidity seller. Thus, the corresponding expected profits are given by

$$\mathbb{E}\left(\Pi_{BLO,1}^{IH}\right) = \frac{\eta}{2}\delta\,(k_3 + k_1 - 1)\,\tau.$$

If instead he deviates towards a BDO, he knows that in the next period the prices in the book will not change. In this case, he anticipates the following behavior for traders at $t = 2$:

1. if there is an informed trader, then he will be a buyer and will choose a BMO,
2. if there is an uninformed buyer, then he will choose a NT, and
3. if there is an uninformed seller, then he will choose a NT.

Thus, Table 8 implies that

$$\mathbb{E}\left(\Pi^{IH}_{BDO,1}\right) = \theta k_3 \tau + (1-\theta)\delta^2 \left(k_3 - k_1 - (k_2 - k_1)\left(\lambda\pi + \frac{\eta}{2}\right)\right)\tau,$$

since $I^{US,\mathcal{B}_1=\varnothing}_{SLO,2} = 0$ and $I^{IH,\mathcal{B}_1=\varnothing}_{BMO,2} = 1$. Hence, informed traders at $t = 1$ have no incentives to deviate from the prescribed strategy profile whenever

$$k_3 - k_1 \geq \frac{\eta}{2}\delta\left(k_3 + k_1 - 1\right) \text{ and}$$
$$k_3 - k_1 \geq \theta k_3 + (1-\theta)\delta^2\left((k_3 - k_1) - (k_2 - k_1)\left(\lambda\pi + \frac{\eta}{2}\right)\right).$$

- *Uninformed traders*

Consider an uninformed buyer at $t = 1$. If instead he deviates towards a BLO, then in the next period the prices will be $(A^1_1, B^1_1 + \tau)$ and, then, he anticipates the following behavior for potential sellers at $t = 2$:

1. if there is an informed seller, then he chooses SMO,
2. if there is an uninformed seller, then he will choose SDO, and
3. if there is a liquidity seller, then he will set SMO.

Therefore, the BLO will be executed only with a liquidity seller or with a informed seller. Hence, the corresponding expected profits will be

$$\mathbb{E}\left(\Pi^{UB}_{BLO,1}\right) = \frac{\delta}{2}\left((\lambda\pi + \eta)(k_1 - 1) - \lambda\pi k_3\right)\tau,$$

since $I^{IL,\mathcal{B}_1=BLO}_{SMO,2} = 1$.

If instead, the uninformed buyer deviates and chooses BDO, then he knows that at $t = 2$ the prices in the book will not change. In this case, he anticipates the following behavior for traders at $t = 2$:

1. if there is an informed buyer, then he will choose a BMO,

2. if there is an informed seller, then he will choose a SMO,

3. if there is an uninformed buyer, then he will choose NT, and

4. if there is an uninformed seller, then he will choose NT.

Thus, Table 9 implies that

$$\mathbb{E}\left(\Pi^{UB}_{BDO,1}\right) = (1-\theta)\delta^2\left(\frac{\lambda}{2}\pi\left(k_1-k_2\right)+\frac{\eta}{2}\left(k_1-k_2\right)-k_1\right)\tau < 0,$$

since $I^{IL,\mathcal{B}_1=\varnothing}_{SLO,2} = I^{US,\mathcal{B}_1=\varnothing}_{SLO,2} = 0$ and $I^{IH,\mathcal{B}_1=\varnothing}_{BMO,2} = 1$.

If instead, the uninformed buyer deviates and chooses NT, then he will obtain

$$\mathbb{E}\left(\Pi^{UB}_{NT,1}\right) = 0.$$

Hence, he will not have incentives to deviate if

$$\left(\lambda\pi+\eta\right)\left(k_1-1\right)-\lambda\pi k_3 > 0.$$

Fifth step. From step 4, nobody at $t=1$ has unilateral incentives to deviate whenever

$$\begin{gathered}
k_3-k_1 \geq \frac{\eta}{2}\delta\left(k_3+k_1-1\right), \\
k_3-k_1 \geq \theta k_3+(1-\theta)\delta^2\left(k_3-k_1-\left(k_2-k_1\right)\left(\lambda\pi+\frac{\eta}{2}\right)\right), \text{ and} \\
\left(\lambda\pi+\eta\right)\left(k_1-1\right)-\lambda\pi k_3 > 0.
\end{gathered}$$

Case $I_4+I_4+I_6 : \frac{k_3-k_1}{k_3} < \theta$

- *Informed traders*

Consider an informed buyer at $t=1$. If he chooses a BMO, then he obtains

$$\mathbb{E}\left(\Pi^{IH}_{BMO,1}\right) = \left(k_3-k_1\right)\tau.$$

If instead he deviates towards a BDO, he knows that in the next period the prices in the book will not change. In this case, he anticipates the following behavior for rational traders at $t=2$:

1. if there is an informed trader, then it will be a buyer and will choose a BDO,

2. if there is an uninformed buyer, then he will choose a NT, and

3. if there is an uninformed seller, then he will choose a NT.

Therefore, it follows that

$$\mathbb{E}\left(\Pi_{BDO,1}^{IH}\right) = \theta k_3 \tau + (1-\theta)\delta^2 \left(k_3 - k_1 - (k_2 - k_1)\frac{\eta}{2}\right)\tau,$$

since $I_{SLO,2}^{US,\mathcal{B}_1=\varnothing} = 0$ and $I_{BMO,2}^{IH,\mathcal{B}_1=\varnothing} = 0$.

However, since $\frac{k_3-k_1}{k_3} < \theta$, then

$$\theta k_3 \tau + (1-\theta)\delta^2 \left(k_3 - k_1 - (k_2 - k_1)\frac{\eta}{2}\right)\tau > (k_3 - k_1)\,\tau$$

is always satisfied and, hence, in this case we conclude that in this case there is no equilibrium in which (BMO, SMO, BLO, SLO) is the strategy profile chosen at $t = 1$.

Fifth step. Based on the above, nobody at $t = 1$ has unilateral incentives to deviate whenever

$$\theta \leq \frac{k_3 - k_1}{k_3},$$
$$k_3 - k_1 \geq \delta\frac{\eta}{2}\left(k_3 + k_1 - 1\right),$$
$$k_3 - k_1 \geq \theta k_3 + (1-\theta)\delta^2 \left(k_3 - k_1 - (k_2 - k_1)\left(\lambda\pi + \frac{\eta}{2}\right)\right), \text{ and}$$
$$\left(\lambda\pi + \eta\right)\left(k_1 - 1\right) - \lambda\pi k_3 > 0.$$

$\mathcal{E}_2^D$**:** (BMO, SMO, NT, NT)

Following the same procedure we obtain that in this case $Y = p \in [0,1]$, $X = \dfrac{\lambda\pi}{\eta + \lambda\pi}$ and the optimal responses of uninformed traders are:

State of the book	**UB**	**US**
(A_1^1, B_1^1)	NT	NT
(A_1^2, B_1^1)	$\begin{cases} NT & \text{if } Xk_3 \le \frac{k_2-k_1}{2} \\ BDO & \begin{cases} \text{if } \frac{k_2-k_1}{2} < Xk_3 \le k_2 \text{ or} \\ \text{if } k_2 < Xk_3 \text{ and } \theta > \theta_X \end{cases} \\ BMO & \text{if } k_2 < Xk_3 \text{ and } \theta \le \theta_X \end{cases}$	$\begin{cases} SDO & \text{if } Xk_3 < \frac{k_2-k_1}{2} \\ NT & \text{if } \frac{k_2-k_1}{2} \le Xk_3 \end{cases}$
$(A_1^1, B_1^1 + \tau)$	$\begin{cases} NT & \text{if } Y \le \frac{1}{2k_3} \\ BDO & \begin{cases} \text{if } \frac{1}{2k_3} < Y < \frac{k_1}{k_3} \text{ or} \\ \text{if } Y \ge \frac{k_1}{k_3} \text{ and } \theta > \theta_Y \end{cases} \\ BMO & \text{if } Y \ge \frac{k_1}{k_3} \text{ and } \theta \le \theta_Y \end{cases}$	$\begin{cases} SDO & \text{if } Y < \frac{1}{2k_3} \\ NT & \text{if } \frac{1}{2k_3} \le Y \end{cases}$
(A_1^1, B_1^2)	$\begin{cases} BDO & \text{if } Xk_3 < \frac{k_2-k_1}{2} \\ NT & \text{if } \frac{k_2-k_1}{2} \le Xk_3 \end{cases}$	$\begin{cases} NT & \text{if } Xk_3 \le \frac{k_2-k_1}{2} \\ SDO & \begin{cases} \text{if } \frac{k_2-k_1}{2} < Xk_3 \le k_2 \text{ or} \\ \text{if } k_2 < Xk_3 \text{ and } \theta > \theta_X \end{cases} \\ SMO & \text{if } k_2 < Xk_3 \text{ and } \theta \le \theta_X \end{cases}$
$(A_1^1 - \tau, B_1^1)$	$\begin{cases} BDO & \text{if } Y < \frac{1}{2k_3} \\ NT & \text{if } \frac{1}{2k_3} \le Y \end{cases}$	$\begin{cases} NT & \text{if } Y \le \frac{1}{2k_3} \\ SDO & \begin{cases} \text{if } \frac{1}{2k_3} < Y < \frac{k_1}{k_3} \text{ or} \\ \text{if } Y \ge \frac{k_1}{k_3} \text{ and } \theta > \theta_Y \end{cases} \\ SMO & \text{if } Y \ge \frac{k_1}{k_3} \text{ and } \theta \le \theta_Y \end{cases}$

Table B.6: Optimal responses of uninformed traders at $t = 2$ when the strategy profile at $t = 1$ is (BMO, SMO, NT, NT).

$\mathcal{E}_3^D$**:** (BLO, SLO, BLO, BLO)

Following the same procedure we obtain that in this case $Y = \pi$ and the optimal responses of uninformed traders are:

State of the book	**UB**	**US**
(A_1^1, B_1^1)	NT	NT
(A_1^2, B_1^1)	NT	SDO
$(A_1^1, B_1^1 + \tau)$	$\begin{cases} NT & \text{if } Y \leq \frac{1}{2k_3} \\ BDO & \begin{cases} \text{if } \frac{1}{2k_3} < Y < \frac{k_1}{k_3} \text{ or} \\ \text{if } Y \geq \frac{k_1}{k_3} \text{ and } \theta > \theta_Y \end{cases} \\ BMO & \text{if } Y \geq \frac{k_1}{k_3} \text{ and } \theta \leq \theta_Y \end{cases}$	$\begin{cases} SDO & \text{if } Y < \frac{1}{2k_3} \\ NT & \text{if } \frac{1}{2k_3} \leq Y \end{cases}$
(A_1^1, B_1^2)	BDO	NT
$(A_1^1 - \tau, B_1^1)$	$\begin{cases} BDO & \text{if } Y < \frac{1}{2k_3} \\ NT & \text{if } \frac{1}{2k_3} \leq Y \end{cases}$	$\begin{cases} NT & \text{if } Y \leq \frac{1}{2k_3} \\ SDO & \begin{cases} \text{if } \frac{1}{2k_3} < Y < \frac{k_1}{k_3} \text{ or} \\ \text{if } Y \geq \frac{k_1}{k_3} \text{ and } \theta > \theta_Y \end{cases} \\ SMO & \text{if } Y \geq \frac{k_1}{k_3} \text{ and } \theta \leq \theta_Y \end{cases}$

Table B.7: Optimal responses of uninformed traders at $t = 2$ when the strategy profile at $t = 1$ is (BLO, SLO, BLO, BLO).

$\mathcal{E}_4^D$**:** (BLO, SLO, NT, NT)

Following the same procedure we obtain that in this case $Y = 1$ and the optimal responses of uninformed traders are:

State of the book	UB	US
(A_1^1, B_1^1)	NT	NT
(A_1^2, B_1^1)	NT	SDO
$(A_1^1, B_1^1+\tau)$	$\begin{cases} BDO \text{ if } \theta > \frac{k_3-k_1}{k_3-\frac{1}{2}} \\ BMO \text{ if } \theta \leq \frac{k_3-k_1}{k_3-\frac{1}{2}} \end{cases}$	NT
(A_1^1, B_1^2)	BDO	NT
$(A_1^1-\tau, B_1^1)$	NT	$\begin{cases} SDO \text{ if } \theta > \frac{k_3-k_1}{k_3-\frac{1}{2}} \\ SMO \text{ if } \theta \leq \frac{k_3-k_1}{k_3-\frac{1}{2}} \end{cases}$

Table B.8: Optimal responses of uninformed traders at $t = 2$ when the strategy profile at $t = 1$ is (BLO, SLO, NT, NT).

$\mathcal{E}_5^D$**:** (BDO, SDO, BLO, SLO)

Following the same procedure we obtain that in this case the optimal responses of uninformed traders are:

State of the book	UB	US
(A_1^1, B_1^1)	NT	NT
(A_1^2, B_1^1)	NT	SDO
$(A_1^1, B_1^1+\tau)$	NT	SDO
(A_1^1, B_1^2)	BDO	NT
$(A_1^1-\tau, B_1^1)$	BDO	NT

Table B.9: Optimal responses of uninformed traders at $t = 2$ when the strategy profile at $t = 1$ is (BDO, SDO, BLO, SLO).

$\mathcal{E}_6^D$**:** (BDO, SDO, NT, NT)

Following the same procedure we obtain that in this case $Y = p \in [0, 1]$ and the optimal responses of uninformed traders are:

State of the book	**UB**	**US**
(A_1^1, B_1^1)	NT	NT
(A_1^2, B_1^1)	NT	SDO
$(A_1^1, B_1^1 + \tau)$	$\begin{cases} NT & \text{if } Y \leq \frac{1}{2k_3} \\ BDO & \begin{cases} \text{if } \frac{1}{2k_3} < Y < \frac{k_1}{k_3} \text{ or} \\ \text{if } Y \geq \frac{k_1}{k_3} \text{ and } \theta > \theta_Y \end{cases} \\ BMO & \text{if } Y \geq \frac{k_1}{k_3} \text{ and } \theta \leq \theta_Y \end{cases}$	$\begin{cases} SDO & \text{if } Y < \frac{1}{2k_3} \\ NT & \text{if } Y \geq \frac{1}{2k_3} \end{cases}$
(A_1^1, B_1^2)	BDO	NT
$(A_1^1 - \tau, B_1^1)$	$\begin{cases} BDO & \text{if } Y < \frac{1}{2k_3} \\ NT & \text{if } Y \geq \frac{1}{2k_3} \end{cases}$	$\begin{cases} NT & \text{if } Y \leq \frac{1}{2k_3} \\ SDO & \begin{cases} \text{if } \frac{1}{2k_3} < Y < \frac{k_1}{k_3} \text{ or} \\ \text{if } Y \geq \frac{k_1}{k_3} \text{ and } \theta > \theta_Y \end{cases} \\ SMO & \text{if } Y \geq \frac{k_1}{k_3} \text{ and } \theta \leq \theta_Y \end{cases}$

Table B.10: Optimal responses of uninformed traders at $t = 2$ when the strategy profile at $t = 1$ is (BDO, SDO, NT, NT).

■

Proposition 5 *If $k_1 = 1$, then a PBE of the game is as follows.*

- *(BMO, SMO, NT, NT) is the optimal strategy profile for traders at $t = 1$ if*

Conditions	
$\theta \leq \frac{k_3 - k_1}{k_3}$	$k_3 - 1 \geq \frac{\eta}{2}\delta k_3$, $k_3 - 1 \geq \theta k_3 + (1-\theta)\delta^2 \left(k_3 - 1 - (k_2 - 1)\left(\lambda\pi + \frac{\eta}{2}\right)\right)$.
$\frac{k_3 - k_1}{k_3} < \theta$	*Not an equilibrium.*

The beliefs of uninformed traders at $t = 2$ are: $X = \dfrac{\lambda\pi}{\eta + \lambda\pi}$, $Y = p \in [0, 1]$ and $Z = q \in [0, 1]$. The optimal strategy for uninformed and informed traders at $t = 2$ are described in Tables B.6 and B.1 of Appendix B, respectively.

- (BLO, SLO, NT, NT) *is the optimal strategy profile for traders at* $t = 1$ *if*

Conditions	
$\theta \leq \frac{k_3-k_1}{k_3}$	$\frac{\eta}{2}\delta k_3 \geq \theta k_3 + (1-\theta)\delta^2\left((k_3-1)-(k_2-1)\left(\lambda\pi+\frac{\eta}{2}\right)\right)$, $\frac{\eta}{2}\delta k_3 > (k_3-1)$.
$\frac{k_3-k_1}{k_3} < \theta \leq \frac{k_3-k_1+1}{k_3+\frac{1}{2}}$	$\frac{\eta}{2}\delta k_3 \geq \theta k_3 + (1-\theta)\delta^2\left((k_3-1)-(k_2-1)\left(\frac{\eta}{2}\right)\right)$.
$\frac{k_3-k_1+1}{k_3+\frac{1}{2}} < \theta$	$\delta\frac{\eta}{2}k_3 \geq \theta k_3 + (1-\theta)\delta^2\left((k_3-1)-\frac{\eta}{2}(k_2-1)\right)$.

The beliefs of uninformed traders at $t = 2$ *are:* $X = 0$*,* $Y = 1$ *and* $Z = q \in [0,1]$*. The optimal strategy for uninformed and informed traders at* $t = 2$ *are described in Tables B.8 and B.1 of Appendix B, respectively.*

- (BDO, SDO, NT, NT) *is the optimal strategy profile for traders at* $t = 1$ *if*

Conditions	
$\theta \leq \frac{k_3-k_1}{k_3}$	$\theta k_3 + (1-\theta)\delta^2\left((k_3-1)-(k_2-1)\left(\lambda\pi+\frac{\eta}{2}\right)\right) > (k_3-1)$, $\theta k_3 + (1-\theta)\delta^2\left((k_3-1)-(k_2-1)\left(\lambda\pi+\frac{\eta}{2}\right)\right) > \frac{\eta}{2}\delta k_3$.
$\frac{k_3-k_1}{k_3} < \theta \leq \frac{k_3-k_1+1}{k_3+\frac{1}{2}}$	$\theta k_3 + (1-\theta)\delta^2\left((k_3-1)-(k_2-1)\left(\frac{\eta}{2}\right)\right) > \frac{\eta}{2}\delta k_3$.
$\frac{k_3-k_1+1}{k_3+\frac{1}{2}} < \theta$	$\theta k_3 + (1-\theta)\delta^2\left((k_3-1)-(k_2-1)\left(\frac{\eta}{2}\right)\right) > \frac{\eta}{2}\delta k_3$.

The beliefs of uninformed traders at $t = 2$ *are:* $X = 0$*,* $Y = p \in [0,1]$ *and* $Z = 1$*. The optimal strategy for uninformed and informed traders at* $t = 2$ *are described in Tables B.10 and B.1 of Appendix B, respectively.*

Proof. The Corollary follows immediately by taking the limit π goes to zero in the conditions that guarantee the existence of the PBE stated in Proposition 2. ■