Self-justified equilibria in dynamic economies with heterogenous agents*

Felix Kubler
DBF, University of Zurich
and Swiss Finance Institute
fkubler@gmail.com

September 25, 2017

Abstract

In this paper I introduce the concept of "self-justified" equilibrium as an alternative to rational expectations equilibrium in stochastic general equilibrium models heterogeneous agents. In a self-justified equilibrium agents' expectations can be described by finite sets and agents forecast future prices by some simple function that depends on this set. The expectations are correct (i.e. "self-justified") for a finite number of points in the endogeneous state space but it might lead to incorrect forecasts otherwise.

I consider a model with overlapping generations, stochastic production and idiosyncratic risk. Unlike rational expectations equilibria, self-justified equilibria always exist in this economy and they can be approximated numerically with standard methods. Error analysis for approximate solutions is straightforward and there exist approximate equilibria where forcasting errors are arbitrarily small.

*Preliminary and Incomplete

1 Introduction

The assumption of rational expectations and the use of recursive methods to analyze dynamic economic models has revolutionized financial economics, macroeconomics and public finance (see e.g. Ljungqvist and Sargent (2012)). Unfortunately, for stochastic general equilibrium models with heterogeneous agents rational expectations equilibria are generally not tractable. In these models, known sufficient conditions for the existence of stationary Markov equilibria are very restrictive (see Citanna and Siconolfi (2012) or Brumm, Kryczka and Kubler (2017)), computational methods to approximate these equilibria numerically are often ad hoc, and rigorous error analysis seems impossible. In this paper I develop an alternative to rational expectations equilibria and consider temporary equilibria with finitely self-justified expectations. Agents forecast future prices using continuous function of current endogenous variables. The key is that these functions are parameterized by a finite set of points and that they are simple in the sense that they can be easily evaluated. The solution might coincide with or be close to a rational expectations equilibrium if that is well behaved. However, the crucial advantages of self-justified equilibria is that they always exist and are easy to approximate numerically. Errors are attributed to agents and hence directly interpretable.

The basic idea of the approach is as follows. In a temporary equilibrium agents use current endogenous variables and the shock to forecast future prices and prices for commodities and assets in the current period ensure that markets clear. The forecasts are based on a finite set of expectations that are self-justified in the sense that in the temporary equilibrium they turn out to be correct. However, the agents 'extrapolate' from this finite set to form expectations using any possibe realization of current endogenous variables. In the temporary equilibrium these expectations might be far from correct and agents might make large mistakes. However, it turns out that in many simple examples agents do well with rule-of-thumb expectations which are computationally much simpler that rational expectations.

The concept does not require identical expectations or identical forecasts across agents. Different types of agents can have different expectations and different forecasting functions.

I illustrate the concept in a stochastic overlapping generations model with stochastic production. This is a abstract model which encompasses both models with neo-classical production (as e.g. in Krueger and Kubler (2004)) and Lucas-style asset pricing models (as e.g. in Storesletten et al. (2007)). Self-justified equilibria exist under standard conditions and can be approximated numerically with well-understood tools from numerical analysis. If one allows for approximate equilibria, it can be shown that agents forecasts can become arbitrarily accurate as the expectations-set becomes large. This latter result is relevant for numerical work, where rounding and truncation errors cannot be avoided.

There is a large and diverse body of work exploring deviations from rational expectation (see, e.g., Spear (1989), Sargent (1993), Guesnerie (2001,2005), Gabaix (2014), Adam et al. (2016)). Much of this work is motivated by insights from behavioral economics about agents' behavior or by the search for simple economic mechansism that enrich the observable implications of standard models. The motivation of this paper is rather different in that I want to develop a simple alternative to rational expectations that allows researchers to rigorously analyze stochastic dynamic models with heterogeneous agents.

As Sargent (1993) points out, "when implemented numerically ... rational expectations models impute more knowledge to the agent within the model ... than is possessed by an econometrician", and a sensible approach to relax rational expectations is "expelling rational agents from our model environment and replacing them with 'artificially intelligent 'agents how behave like econometricians."This is exactly the idea behind self-justified equilibria. In fact, many methods that are used to approximae rational expectations equilibria numerically (see Brumm, Kubler and Scheidegger (2017) for an overview) can be reinterpreted as methods that, in fact, compute self-justified equilibria. I will illustrate this point with simple examples below.

Applied dynamic general equilibrium modeling has been critized (rightly so) for its failure to take into account the large heterogeneity in tastes and technologies across agents. Part of its failure to be useful during the financial crises in 2008 might be attributed to this fact. The number of households that overborrowed was well known and data was available about their ability to service their mortgages in the event of an economic downturn. However, it seemed to complicated to incorporate this information in large-scale dynamic models because existing solution methods are not able to handle this amount of heterogeneity. Using the concept of self-justified equilbria, one can easily incorporate large scale heterogenity into general equilibrium models and potentially improve their usefulness for applied work.

The rest of the paper is organized as follows. In Section 2 the general economy is introduced. In Section 3 I define self-justified equilibria and show that they exist under general conditions. In Sections 4 and 5 I consider the two natural cases of forecasting from expectations: Collocation- and simulation-based methods. Section 6 presents results for approximate equilibria.

2 A general dynamic Markovian economy

In this section I describe an abstract a model of a Bewley-style production economy with overlapping generations. Brumm, Krycka and Kubler (2017) discuss a version of the model with infinitely lived agents. As they point out, there are two special cases of the model that play an important role in practice. In the first, a heterogeneous agent version of the Lucas (1978) asset pricing model, agents trade in several long-lived assets that are in unit net supply and pay exogenous positive dividends in terms of the single consumption good. In the second, a version of the Brock–Mirman stochastic growth model with heterogeneous agents, there is a single capital good that can be used in intraperiod production, together with labor, to produce the single consumption good. This good can be consumed or stored in a linear technology yielding one unit of the capital good in the subsequent period.

2.1 The model

I consider a Bewley-style overlapping generations model (see Bewley (1984)) with incomplete financial markets and a continuum of agents. Time is indexed by $t \in \mathbb{N}_0$. Aggregate shocks z_t realize in a finite set $\mathbf{Z} = \{1, \ldots, Z\}$, and follow a first-order Markov process with transition probability $\pi(z'|z)$. A history of aggregate shocks up to some date t (which is also referred to as a "date event") is denoted by $z^t = (z_0, z_1, \ldots, z_t)$. At each date event, H types of agents enter the economy and live for A periods. Within each type there is a continuum of agents that differ ex post by the realization of their idiosyncratic shocks. It is assumed that for each type h each agent within the type faces idiosyncratic shocks, $y_1, ..., y_A$, that have support in a finite set $\mathbf{Y}$ and are Markovian. I denote by $\eta_{ah}(y_{a+1}|y_a)$ the conditional probability of idiosyncratic shock y_{a+1} given that the current shock is y_a and that the current age is a, and I use $\eta_h(y^a)$ to denote the probability of a history of idiosyncratic shocks of agents of type h who were born at time $t - a + 1$ up to time t. I assume that the idiosyncratic shocks are independent of the aggregate shock, that they are identically distributed across agents within each type and age and, as in the construction in Proposition 2 in Feldman and Gilles (1985), that they "cancel out" in the aggregate, that is, the joint distribution of idiosyncratic shocks within a type ensures that at each history of aggregate shocks, z^t, for any $y^a \in \mathbf{Y}^a$ the fraction of agents of type h with history $y^a = (y_1, ..., y_a)$ is $\eta_h(y^a)$. This allows the focus on equilibria for which prices and aggregate quantities only depend on the history of aggregate shocks, z^t. I denote the set of all date events at time t by $\mathbf{Z}^t$ and, taking z_0 as fixed, I write $z^t \in \mathbf{Z}^t$ for any $t \in \mathbb{N}_0$ (including $t = 0$). At each z^t there are finitely many different agents actively trading (distinguishing themselves by age, type, and history of shocks), who are collected in a set $\mathbf{I}$. A specific agent at a given node z^t is denoted by $i = (h, y^a) \in \mathbf{I}$.

At each date event there are L perishable commodities, $l \in \mathbf{L} = \{1, ..., L\}$, available for consumption and production. The individual endowments are denoted by $e_{h,y^a}(z^t) \in \mathbb{R}^L_+$ and assumed to be time-invariant and measurable functions of the current aggregate and idiosyncratic shock. Each agent h has a time-separable expected utility function

$$U_{h,z^t}((x_{t+a})_{a=0}^{A-1}) = \mathbb{E}_0 \left[\sum_{a=0}^{A-1} u_{a,h} \left(y_a, z_{t+a}, x_{(h,z^t),t+a} \right) \right],$$

where $x_{(h,z^t),t+a} \in \mathbb{R}^L_+$ denotes the agent's (stochastic) consumption at date $t + a$.

It is useful to distinguish between intertemporal and intraperiod production. Intraperiod production is characterized by a measurable correspondence $\mathbf{F} : \mathbf{Z} \rightrightarrows \mathbb{R}^L$, where a production plan $f \in \mathbb{R}^L$ is feasible at shock z if $f \in \mathbf{F}(z)$. For simplicity (and without loss of generality) I assume throughout that each $\mathbf{F}(z)$ exhibits constant returns to scale so that ownership does not need to be specified.

Intertemporally each individual $i \in \mathbf{I}$ has access to J linear storage technologies, $j \in \mathbf{J} = \{1, ..., J\}$. At a node z each technology j is described by a column vector of inputs $g_j^0(z) \in \mathbb{R}_+^L$, and a vector-valued random variable of outputs in the subsequent period, $g_j^1(z') \in \mathbb{R}_+^L$, $z' \in \mathbf{Z}$. I write $G^0(z) = (g_1^0(z), \ldots, g_J^0(z))$ for the $L \times J$ matrix of inputs and $G^1(z') = (g_1^1(z'), \ldots, g_J^1(z'))$ for the $L \times J$ matrix of outputs. I denote by $\gamma_i(z^t) = (\gamma_{i1}(z^t), ..., \gamma_{iJ}(z^t))^\top \in \mathbb{R}_+^J$ the levels at which the linear technologies are operated at node z^t by agent $i = (h, y^a)$. Each period there are complete spot markets for the L commodities; prices are denoted by $p(z^t) = (p_1(z^t), ..., p_L(z^t))$, a row vector. It is assumed throughout that agents' utility is strictly increasing in the consumption of commodity 1 which is used as a numéraire, i.e. $p_1(z^t) = 1$ for all z^t. It simplifies notation to write $p \in \mathbb{R}_+^{L-1}$ to denote prices $(1, p_2, \ldots, p_L)$..

At $t = 0$ agents have some initial endowment in the capital goods which are denoted by $G_h^1(z_0)\gamma_{h,y^a}(z^{-1})$ for each agent (h, y^a).

2.2 Rational expectations equilibrium

In order to be able to compare self-justified equilibria to existing concepts, it is useful to formally define a (Rational expectations) sequential competitive equilibrium.

Given initial conditions $(G^1(z_0)\gamma_i(z^{-1}))_{i\in\mathbf{I}}$, we define a *sequential competitive equilibrium* to be a process of prices and choices,

$$\left(\bar{p}_t, (\bar{x}_{i,t}, \bar{\gamma}_{i,t},)_{i\in\mathbf{I}}, \bar{f}_t\right)_{t=0}^{\infty},$$

such that markets clear and agents optimize—that is to say, (A), (B), and (C) hold.

(A) Market clearing:

$$\sum_{i=(h,y^a)\in\mathbf{I}} \eta(y^a)(\bar{x}_h(z^t) + G^0(z_t)\bar{\gamma}_i(z^t) - e_i(z_t) - G^1(z_t)\bar{\gamma}_i(z^{t-1})) = \bar{f}(z^t), \quad \text{for all } z^t.$$

(B) Profit maximization:

$$\bar{f}(z^t) \in \arg\max_{y\in\mathbf{F}(z_t)} \bar{p}(z^t) \cdot y.$$

(C) Each agent (h, z^t) maximizes utility:

$$(\bar{x}_{h,t}, \bar{\alpha}_{h,t})_{t=0}^{\infty} \in \arg \max_{(x_{h,t},\alpha_{h,t})_{t=0}^{\infty} \geqslant 0} U_h((x_{h,t})_{t=0}^{\infty}) \text{ s.t.}$$

$$\bar{p}(z^t) \cdot \left(x_h(z^t) + A^0(z_t)\alpha_h(z^t) - e_h(z_t) - G^1(z_t)\alpha_h(z^{t-1})\right) \leqslant 0, \quad \text{for all } z^t.$$

The definition of equilibrium assumes rational expectations. In making their plans the agents know the process of future prices. This seems to be a strong requirement that is often justified by focusing on recursive equilibria where agents only need to know a polucy function and a transition function. The concept of a (rational expectations) recursive equilibrium, however, has three drawbacks: i) general existence can only be shown under strong conditions on the stochastic process (z_t) (see Brumm et al. (2017)), ii) the agents need to know and evaluate the measurable policy function and the measurable state transition. It is not possible to obtain results on continuity. iii) the agents need to observe the entire state – in economies with many agents or many capital-stocks this appears to be an unrealistically strong assumption.

3 Self justified equilibria

The agents in this paper deviate from rational expectations with respect to two aspects. First, it is assumed that they cannot evaluate (or store) arbitrary measurable functions. The agents approximate the equilibrium correspondence by "simple " functions. In the applications below I will consider polynomial and piecewise polynomial functions as well as Gaussian processes. Second, it is assumed that agents cannot fit their forecasts using a infinite amount of information. Expectations are characterized by a finite set and the approximating functions are parameterized by this set. In the applications below I consider interpolation as well as regression.

It is useful to define the endogenous state as the storage from the previous period across individuals. For this let $\mathbf{K} \subset \mathbb{R}^L_+$ with typical element

$$\kappa(z^t) = (\kappa_i(z^t))_{i \in \mathbf{I}} = \left(G^1(z_t)\gamma_{h,y^{a-1}}(z^{t-1})\right)_{(h,y^a)\in \mathbf{I}_{-A}}$$

and define the set of current choices and prices as $\mathbf{E}$, i.e.

$$((x_i, \gamma_i)_{i\in\mathbf{I}}, y, p) \in \mathbf{E} = \mathbb{R}^{I(L+J)}_+ \times \mathbb{R}^L \times \mathbb{R}^{L-1}_+.$$

Let $\mathbf{S} = \mathbf{Z} \times \mathbf{K}$ denote the (aggregate) state-space and let $\mathcal{O}$ denote the set of possible expectations, i.e. the set of finite subsets of $\mathbf{S} \times \mathbf{E}$. Each agent of type h is characterized by a set $\mathbf{O}_h \in \mathcal{O}$ and functions $\phi_{h,a} : \mathbf{S} \times \mathbf{E} \times \mathbf{Y} \times \mathbf{Z} \times \mathcal{O} \to \mathbb{R}^L_+$, $a = 1, \ldots, A$, that predicts future prices and marginal utilities, conditional on the future shock, on the basis of current state, current endogenous variables and the agent's expectations, $\mathbf{O}_h$.

Given any collection of functions $\bar{\phi} = (\bar{\phi}_{ah})_{h\in\mathbf{H},a=1,\ldots,A-1}$, $\bar{\phi}_{ha} : \mathbf{S} \times \mathbf{E} \times \mathbf{Y} \times \mathbf{Z} \times \mathcal{O} \to \mathbb{R}^L_+$ for all $h = 1, \ldots, H$, for each z, κ and each profile of expectations $(\mathbf{O}_h)_{h\in\mathbf{H}}$ the temporary equilibrium correspondence is defined as

$$
\begin{aligned}
\mathbf{N}(s,(\mathbf{O}_h)_{h\in\mathbf{H}}) \quad = \quad & \{\bar{w}=(\bar{x}_i,\bar{\gamma}_i)_{i\in\mathbf{I}},\bar{f},\bar{p})\in\mathbf{E}: \\
& (\bar{x}_{h,y^a},\bar{\gamma}_{h,y^a})\in\arg\max_{x_h\in\mathbb{R}^L_+,\gamma_h\in\mathbb{R}^J_+} u_{a,h}(y_a,z,x_h)+ \\
& \sum_{(y',z')\in\mathbf{Y}\times\mathbf{Z}} \pi(z'|z)\eta(y'|y)\phi_{a,h}(s,\bar{w},y',z',\mathbf{O}_h)G^1(z')\gamma_h \text{ s.t.} \\
& \qquad -\bar{p}\cdot(x_h+G^0(z)\gamma_h-e_{ah}(y,z)-\kappa_{h,y^a})\geqslant 0 \quad \text{for all } h,y^a\in\mathbf{I},\ a=1,\ldots,A, \\
& \bar{x}_{h,y^A}\in\arg\max_{x_i\in\mathbb{R}^L_+} u_{a,h}(y_a,z,x_h) \text{ s.t.} \\
& \qquad -\bar{p}\cdot(x_h-e_{ah}(y,z)-\kappa_{h,y^A})\geqslant 0 \quad \text{for all } h,y^A\in\mathbf{I}, \\
& \bar{f}\in\arg\max_{f\in\mathbf{F}(z)} \bar{p}\cdot f, \\
& \sum_{(h,y^a)\in\mathbf{I}} \eta(y^a)(\bar{x}_h+G^0_h(z)\bar{\gamma}_h-e_{ah}(y,z)-\kappa_{h,y^a})=\bar{f}\}.
\end{aligned}
$$

The basic idea of the concept of self-justified equilibria is that agents have knowledge of a finite number of points of the temporary equilibrium correspondence and use these to approximate the entire correspondence and for these forecasts. The known points one the correspondence are self-justified if the forecasts formed from these points leads to the correspondence. More formally,

DEFINITION 1 *Expectations* $(\mathbf{O}_h)_{h\in\mathbf{H}}$ *are self-justified if for each agent* $\bar{h}$ *each* $(s,\eta)\in\mathbf{O}_{\bar{h}}$

$$\eta\in\mathbf{N}(s,(\mathbf{O}_h)_{h\in\mathcal{H}}).$$

A self-justified equilibrium is then simply a temporary equilibrium arising from expectations $\mathbf{O}_{h\in\mathcal{H}}$. The formal definition is as follows.

DEFINITION 2 *A temporary equilibrium process,*

$$\left(\bar{p}_t\left(\bar{x}_{h,t},\bar{\gamma}_{h,t}\right)_{h\in\mathbf{H}},\bar{y}_t\right)_{t=0}^{\infty},$$

is self justified if there exist expectations $(\mathbf{O}_h)_{h\in\mathbf{H}}$ *such that*

$$\left((\bar{x}_h(z^t),\bar{\gamma}_h(z^t))_{h\in\mathbf{H}},\bar{f}(z^t),\bar{p}(z^t)\right)\in\mathbf{N}(z_t,\kappa(z^t),(\mathbf{O}_h)_{h\in\mathbf{H}})$$

for all z^t

3.1 Existence

To prove existence of a self-justified equilibrium one first needs to make standard assumptions on fundamentals not unlike those that ensure the existence of a rational expectations equilibrium.

Since I consider an economy with several commodities I want to allow for the fact that some commodities do not enter the utility functions of agents and some commodities, although their consumption provides utility, are not essential in that an agent might decide to consume zero of that commodity. Nevertheless, we need to assume that there is at least one commodity that is essential in the sense that independently of prices an agent will always consume positive amounts of that commodity. For simplicity we take the consumption space to be $\mathbf{X} = \mathbb{R}_{++} \times \mathbb{R}_{+}^{L-1}$, assuming that utility and marginal utility are well defined even if consumption of goods $2, \ldots, L$ are on the boundary.

The following assumptions on preferences, endowments and technologies are from Brumm et al. (2017).

Assumption 1

1. *Individual endowments in good 1 and aggregate endowments in all other goods are bounded above and bounded below away from zero—that is, there are* $\underline{e}, \overline{e} \in \mathbb{R}_{++}$ *such that for all shocks* z

$$\underline{e} < e_{h1}(z) < \overline{e} \text{ for all agents } h,$$

$$\underline{e} < \frac{1}{H} \sum_{h \in \mathbf{H}} e_{hl}(z) < \overline{e} \text{ for all goods } l = 2, \ldots, L.$$

2. *The Bernoulli functions* $u_{ah} : \mathbf{Y} \times \mathbf{Z} \times \mathbf{C} \to \mathbb{R}$, $h \in \mathbf{H}$, $a = 1, \ldots, A$, $y \in \mathbf{Y}$, $z \in \mathbf{Z}$, *are increasing, concave, and continuously differentiable in* x, *they are strictly increasing and strictly concave in* x_1, *and they satisfy a strong Inada condition: for any sequence* $x_1^n \to 0$, *we have* $\sup_{y \in \mathbf{Y}, z \in \mathbf{Z},, (x_2, \ldots, x_L) \in \mathbb{R}_{+}^{L-1}} u_h(z, (x_1^n, x_2, \ldots, x_L)) \to -\infty$. *Utility is bounded above: there exists a* $\bar{u}$ *such that* $u_h(z, x) \leqslant \bar{u}$ *for all* $h \in \mathbf{H}$, $z \in \mathbf{Z}$, $x \in \mathbf{C}$.

Assumption 2 *For each shock* z *the production set* $\mathbf{F}(z) \subset \mathbb{R}^L$ *is assumed to be closed, convex-valued, to contain* $\mathbb{R}_{-}^{L}$, *to exhibit constant returns to scale—that is,* $f \in \mathbf{F}(z) \Rightarrow \lambda f \in \mathbf{F}(z)$ *for all* $\lambda \geqslant 0$, *and to satisfy* $\mathbf{F}(z) \cap -\mathbf{F}(z) = \{0\}$. *In addition, production is bounded above: There is a* $\bar{\kappa} \in \mathbb{R}_{+}$ *so that for all* $\kappa \in \mathbf{K}^U$, $h \in \mathbf{H}$, $z \in \mathbf{Z}$, $l \in \mathbf{L}^K$, *and for all* $\gamma \in \mathbb{R}_{+}^{HJ}$

$$\sum_{h \in \mathbf{H}} \left(G^0(z)\gamma_h - \kappa_h - e_h(z) \right) \in \mathbf{F}(z) \Rightarrow \sup_{z'} \sum_{h \in \mathbf{H}} (e_{hl}(z') + \sum_{j \in \mathbf{J}} g_{jl}^1(z')\gamma_{hj}) \leqslant \max[\bar{\kappa}, \sum_{h \in \mathbf{H}} \kappa_{hl}].$$

As in Duffie et al. (1994) and Brumm et al. (2017), Assumption 1 implies that there is a $\underline{c} > 0$ such that, independently of prices, an agent will never choose consumption in commodity 1 that is below $\underline{c}$. The reason is that budget feasibility implies that an agent can always consume his or her endowments (the agent cannot sell them on financial markets in advance), and we therefore must have, for any shock z and for any x with $x_1 < \underline{c}$,

$$u_h(z, x) + \frac{\delta \overline{u}}{1 - \delta} < \frac{1}{1 - \delta} \inf_{z \in \mathbf{Z}} u_h(z, \underline{x}),$$

where $\bar{u}$ is the upper bound on Bernoulli utility and $\underline{x}_1 = \underline{\omega}, \underline{x}_l = 0, l = 2, \ldots, L$.

Assumption 2 ensures that the set $\mathbf{K}$ can be taken to be compact.

It is useful to define $\boldsymbol{\Xi}$ to be the set of storage decisions across agents, γ, that ensure that next period's endogenous state lies in $\mathbf{K}$:

$$\boldsymbol{\Xi} = \{\gamma \in \mathbb{R}_+^{HJ} : \left(G^1(z')\gamma_h\right)_{h\in\mathbf{H}} \in \mathbf{K} \text{ for all } z' \in \mathbf{Z}\}. \tag{1}$$

It is useful to decompose an agent h's expectations,

$$\mathbf{O}_h = ((s(h,1), w(h,1)), \ldots, (s(h,N), w(h,N))\},$$

into expectations about states,

$$\mathbf{O}_h^D = (s(h,1), \ldots, s(h,N)\},$$

and to consider a function $\hat{W} : \cup_{h\in\mathbf{H}}\mathbf{O}_h^D \to \mathbf{W}$ with $w(h,i) = W(s(h,i))$ for all $i = 1, \ldots, N$, $h \in \mathbf{H}$.

The following assumption on ϕ_{ah} suffices to ensure existence of self-justified equilibria.

Assumption 3 *The functions $\phi_{ah}(z, \kappa, w, y', z', \mathbf{O}_h)$ are continuous in κ, w and in $(w(h,1), \ldots, w(h,N_n))$ for all $z, y', z' \in \mathbf{Z} \times \mathbf{Y} \times \mathbf{Z}$. The functions are bounded below by 0 and, for given $\mathbf{O}_h$ uniformly bounded above.*

Existence of a self-justified equilibrium then simply reduces to the existence of a finite-dimensional fixed point, in partciluar - the first main result of this section is as follows.

Theorem 1 *Under Assumptions 1-3, given any expectations about states*

$$\mathbf{O}_h^D = ((s(h,1), \ldots, s(h,N_h)\}, \quad h \in \mathbf{H},$$

for all $h \in \mathbf{H}$

$$\hat{W}(s(h,i)) \in \mathbf{N}(s, (\mathbf{O}_h)_{h\in\mathbf{H}})$$

for all h, i.

To prove the result we decompose the economy into sub-economies for each $s(h,j)$ and introduce the standard agents' best repsponses for all these subnote. A slight technical difficulty arises as one needs to bound prices away from zero. The bounds are denoted by real positive numbers, β_x and β_p. For any $\beta_x > 0$, $\beta_p > 0$ sufficiently small and for any $s(i) = (z(i), \kappa(i)) \in \mathbf{O}^D$, define the following compact sets:

$$\widetilde{\mathbf{F}}(s(j)) = \{f \in \mathbf{F}(z) : f + \sum_{i\in\mathbf{I}} \eta(y^a)(e_h(y_a, z) + \kappa_i) \geqslant 0\},$$

$$\widetilde{\mathbf{X}}_{\beta_x}(s(j)) = \{x_i \in \mathbb{R}_+^L : \eta(y^a)\frac{1}{2}x_i - \sum_{i\in\mathbf{I}}(e_h(y_a, z) + \kappa_i) \in \mathbf{F}(z), x_{h1} \geqslant \beta_x\},$$

$$\mathbf{G} = \{\gamma_h \in \mathbb{R}_+^J : \eta(y^a)e_{il}(z') + \sum_{j\in\mathbf{J}} a_{hlj}^1(z')\gamma_{hj} \leqslant 2\bar{\kappa} \text{ for all } h \in \mathbf{H}, l \in \mathbf{L}, z' \in \mathbf{Z}\}.$$

Define the truncated price set $\Delta^{L-1}_{\beta_p} = \{p \in \mathbb{R}^L_+ : \sum_{l=1}^L p_l = 1, p_1 \geqslant \beta_p\}$ and for each agent $h = 1, \ldots, H$ the choice correspondence

$$\Phi^{i,j}_\eta : \Delta^{L-1}_\eta \times \mathbf{\Xi} \rightrightarrows \tilde{\mathbf{C}}(s) \times \mathbf{A}.$$

by

$$\Phi^h_\eta(p, \alpha^*) = \arg \max_{x_h \in \tilde{\mathbf{C}}(s), \alpha_h \in \mathbf{A}} E^{\bar{M}}_h(s, x_h, \alpha_h, \alpha^*) \text{ s.t.}$$
$$-p \cdot (x_h - e_h(z) - \kappa_h + A^0_h(z)\alpha_h) \geqslant 0$$

By a standard argument, the correspondence Φ is convex-valued, non-empty valued, and upper-hemicontinuous. Define the producer's best response $\Phi^{H+1}_\eta : \Delta^{L-1}_\eta \rightrightarrows \tilde{\mathbf{Y}}(s)$ by

$$\Phi^{H+1}_\eta(p) = \arg \max_{y \in \tilde{\mathbf{Y}}(s)} p \cdot y$$

and define a price player's best response,

$$\Phi^0_\eta : (\tilde{\mathbf{C}}(s) \times \mathbf{A})^H \times \tilde{\mathbf{Y}}(s) \rightrightarrows \Delta^{L-1}_\eta$$

by

$$\Phi^0_\eta((x_h, \alpha_h)_{h \in \mathbf{H}}, y) = \arg \max_{p \in \Delta^{L-1}_\eta} p \cdot \left(\sum_{h \in \mathbf{H}} (x_h - e_h(z) - \kappa_h + A^0_h(z)\alpha_h) - y \right).$$

It is easy to see that this correspondence is also upper-hemicontinuous, non-empty, and convex valued. Finally, define

$$\Phi^{H+2} : \mathbf{A}^H \rightrightarrows \mathbf{\Xi}$$

by

$$\Phi^{H+2}(\alpha) = \arg \min_{\alpha^* \in \mathbf{\Xi}} \|\alpha - \alpha^*\|_2.$$

By Kakutani's fixed-point theorem, the correspondence $\left(\bigtimes_{h=0}^{H+1} \Phi^h_\eta \right) \times \Phi^{H+2}$ has a fixed point, which we denote by $(\bar{x}, \bar{\alpha}, \bar{y}, \bar{\alpha}^*, \bar{p})$.

Since by budget feasibility we must have

$$\bar{p} \cdot \left(\sum_{h \in \mathbf{H}} (\bar{x}_h - e_h(z) - \kappa_h + G^0_h(z)\bar{\alpha}_h) - \bar{y} \right) \leqslant 0,$$

optimality of the price player implies that for sufficiently small $\eta > 0$ the upper bound imposed by requiring $x \in \tilde{\mathbf{C}}(s)$ and the upper bound on production will both never bind. Consumption solves the agent's problem for all $x \in \mathbf{C}$ and production maximizes profits among all $y \in \mathbf{Y}(z)$. In addition, Assumption 2 implies that the upper bound on each α_h cannot be binding and that in fact $\bar{\alpha} = \bar{\alpha}^*$.

Finally, there must be some $\epsilon > 0$ such that for all $\eta < \epsilon$ the fixed point must satisfy that $\bar{p}_1 \geqslant \epsilon$. This is true because all commodities must either be consumed, used as an input for intra-period production, or stored. If $\bar{p}_1 < \epsilon$ there must be some other commodity $l \neq 1$ with $\frac{\bar{p}_l}{\bar{p}_1} > \frac{1-\epsilon}{(L-1)\epsilon}$. But for sufficiently small $\epsilon > 0$ the (relative) price of this commodity is so high that it is not consumed—because marginal utility of good 1 is bounded away from zero in $\widetilde{\mathbf{C}}(s)$ and marginal utility of commodity l is finite (as utility is assumed to be continuously differentiable on $\mathbf{C} = \mathbf{R}_{++} \times \mathbf{R}_{+}^{L-1}$). Furthermore, good l can neither be used for (constant-returns-to-scale) intratemporal production, nor for (linear) storage—the agent who stores it could eventually increase his utility by selling a small fraction of this commodity and increasing his consumption of commodity 1. Therefore there is some $\epsilon > 0$ such that for $\eta < \epsilon$ the price player chooses a price with $\bar{p}_1 \geqslant \epsilon$ and a standard argument gives that

$$\sum_{h \in \mathbf{H}} (\bar{x}_h - e_h(z) - \kappa_h + A_h^0(z)\bar{\alpha}_h) = \bar{y}.$$

This proves the result. □

3.2 Approximate equilibria

In order to derive results on how well agents' can forecast future prices in self-justified equilibria it often turns out to be useful to consider approximate equilibria where agents make ϵ-mistakes in their choices. There are a variety of justifications for this, the only one that seems somewhat credible derives from the fact that compuational methods can only solve for approximate equilibria. However, it has to be kept in mind that the theoretical results in this paper will focus on one specific approximate equilibrium among many others and there is a priori no reason why the numerical methods should comptute that specific one. I will return to this issue when discussing specific examples.

For now it is useful to define the ϵ-equilibrium temporary equilibrium correspondence as follows.

$$\begin{aligned}
\mathbf{N}^\epsilon(s, (\mathbf{O}_h)_{h\in\mathbf{H}}) \quad = \quad & \{\bar{w} = ((\bar{x}_i, \bar{\gamma}_i)_{i\in\mathbf{I}}, \bar{f}, \bar{p}, \bar{U}) \in \mathbf{E} \times \mathbb{R}^I : \\
& \epsilon > \|\bar{U}_{y^a,h} - \left(u_{a,h}(y_a, z, x_h) + \sum_{(y',z')\in\mathbf{Y}\times\mathbf{Z}} \pi(z'|z)\eta(y'|y)\phi_{a,h}(s, \bar{w}, y', z', \mathbf{O}_h) G^1(z')\gamma_h \right)\| \\
& \bar{U} = \max_{x_h\in\mathbb{R}_+^L, \gamma_h\in\mathbb{R}_+^J} u_{a,h}(y_a, z, x_h) + \\
& \sum_{(y',z')\in\mathbf{Y}\times\mathbf{Z}} \pi(z'|z)\eta(y'|y)\phi_{a,h}(s, \bar{w}, y', z', \mathbf{O}_h) G^1(z')\gamma_h \text{ s.t.} \\
& \qquad -\bar{p}\cdot(x_h + G^0(z)\gamma_h - e_{ah}(y,z) - \kappa_{h,y^a}) \geqslant 0 \quad \text{for all } h, y^a \in \mathbf{I},\ a = 1, \ldots, A, \\
& \bar{f} \in \arg\max_{f\in\mathbf{F}(z)} \bar{p}\cdot f, \\
& \sum_{(h,y^a)\in\mathbf{I}} \eta(y^a)(\bar{x}_h + G_h^0(z)\bar{\gamma}_h - e_{ah}(y,z) - \kappa_{h,y^a}) = \bar{f}\}.
\end{aligned}$$

An ϵ equilibrium is then simply a sequence of values in the ϵ -equilibrium temporary equilibrium correspondence.

Clearly, in actual computations one can only expect to find ϵ-equilibria. If the set of these equilibria is large even for relativelty small values of ϵ it seems important to pin down how the errors arise. I will make this clear in the concrete examples below.

4 Interpolation

There is a certain sense where the choice of $\mathbf{O}_h^D$ and of the functions $\phi(.)$ should be guided by insights from numerical analysis. The problem of recovering unknown functions from finitely many function observations has a rich history. One possible method is called "interpolation"where an approximating function is chosen to coincide with the unknown function precisly at the known points.

Suppose we want to approximate $f : [-1,1]^d \to I\!R$ by interpolating it at points in $\mathcal{H} \subset [-1,1]^d$. It turns out that, in order to obtain a good approximation, both the choice of $\mathcal{H}$ and the interpolating function $\hat{f}$ are crucial. Krueger and Kubler (2003) use Smolyak's (1963) method which provides a general principle to construct good approximations for d-dimensional problems, based on approximations for the univariate case. Brumm Kubler and Scheidegger (2017) use adaptive sparse grids together with D-linear interpolation.

For the purposes of this paper, it is enough to consider an abstract method which requires a set of points $\mathcal{H} \subset [-1,1]^d$, together with function values $f(x_i)$ for all $i \in \mathcal{H}$ and produces a simple approximating function $\hat{f}(x; \mathcal{H}, f(\mathcal{H})$. It is without loss of generality to think of the approximating functions to be defined uniquely by the values of the unknown functions at all points in $\mathcal{H}$. With this the search for a self-justified equilibrium then reduces to solving a non-linear system of equations. At all points in $(\mathbf{O}_h^D)_{h \in \mathbf{H}}$ the system of equations (with the first order conditions of the maximizations problems taken to be equations, assuming interior solutions)

$$\hat{W}(s(h,i)) \in \mathbf{N}(s, (\mathbf{O}_h)_{h \in \mathbf{H}}) \text{ for all } h, i.$$

In other words, one searches for the points in the expectations set that ensure that the agents' forecasting function passes exactly through these points.

Obviously, this rasies the question who good the approximation is away from these points. This can be phrased in two ways. First, one can ask if actual bounds are available for a given number of points. It is clear that the answer to this will generally be negative. However, one can compute given paramteric examples and in these examples verify that forecasts can be very accurate.

The second question is how the model behaves if the number of points in the expectation-sets become very large. This question will be taken up in Section 4.2 below. For now a simple example is considered.

4.1 A numerical example

The following example is from Krueger and Kubler (2003) - however, in that paper the authors report errors in Euler equations for the approximated equilibrium. Here we use their solution method to compute self-justified temporary equilibria. For the reader's convenience I rewrite the model for the special case of a single capital stock and use the notation in Krueger and Kubler (2003):

The economy is populated by overlapping generations of agents that live for N periods. At each date-event z^t a representative household is born. As mentioned in the introduction we focus on the case where there is no within generation heterogeneity and households only distinguish themselves by the date-event of their birth so that a household is fully characterized by z^t. To simplify notation, we collect all households which are alive at some node z^t in a set $\mathcal{I}_{z^t}$ and denote a typical household by $i \in \mathcal{I}_{z^t}$. When there is no ambiguity about the identity of households we will index households simply by their time of birth.

In each period i of her life, an agent born at node z^s has non-negative, possibly stochastic labor endowment $l^i(z_t)$ which depends on the agents' age, $i = t - s + 1$ and on the current shock z_t alone. The price of the consumption good at each date event is normalized to one and at each date event z^t the household supplies her labor endowment inelastically for a market wage $w(z^t)$.

Let by c_t^s denote the consumption of an agent born at time s in period $t \geqslant s$.[1] Individuals value consumption according to

$$E_s\left[\sum_{t=s}^{s+N-1} \beta^{t-s} u(c_t^s)\right] \tag{2}$$

where $u : \mathbb{R}_{++} \to \mathbb{R}$ is assumed to be smooth, strictly increasing, strictly concave and to satisfy the Inada condition $\lim_{c \to 0} u'(c) = \infty$.

Households have access to a storage technology: they can use one unit of the consumption good to obtain one unit of the capital good next period. We denote the investment of household s into this technology by a_t^s. We do not restrict $a_t^s \geqslant 0$, because we want to permit households to borrow against future labor income. One possible interpretation of this assumption is that there is a bank which acts as an intermediary and which stores the capital good for all households, and each individual household can then borrow from this bank. At time t the household sells its capital goods accumulated from last period, a_{t-1}^s, to the firm for a market price $r_t > 0$. The budget constraint of household s in period $t \geqslant s$ is

$$c_t^s + a_t^s = r_t a_{t-1}^s + l^{t-s} w_t \tag{3}$$

We impose the restriction that in the last period of his life the agent is not allowed to borrow, $a_{s+N-1}^s \geqslant 0$, i.e. we rule out that households die in debt. Furthermore we assume that agents enter the economy without any assets, i.e. we assume that $a_{s-1}^s = 0$.

[1] Whenever there is no ambiguity we will use the notation $c_t^s = c_t^s(z^t)$ to denote consumption of an agent born at s at node z^t of the event tree. The notation for all other variables is to be understood correspondingly.

To start off the economy we assume that in period zero, there are N households of ages 1 through N who enter the period with given capital holdings $a^0_{-1}, \ldots, a^{-N+1}_{-1}$, where by assumption $a^0_{-1} = 0$

There is a single representative firm which in each period t uses labor and capital to produce the consumption good according to a constant returns to scale production function $f_t(K, L; z_t)$. Since firms make their decisions on how much capital to buy and how much labor to hire after the realization of the shock z_t they face no uncertainty and simply maximize current period profits.[2]

In the examples below we will always use the following parametric form for the production function.

$$f(K, L, z) = \eta(z)F(K, L) + K(1 - \delta(z)) \tag{4}$$

where $\eta(.)$ is the stochastic shock to productivity, where $\delta(.)$ can be interpreted as the (possibly) stochastic depreciation rate and where $F(.,.)$ is a Cobb-Douglas production function.

In this simple economy the only markets are spot markets for consumption, labor and capital, all of which are assumed to be perfectly competitive. It is not difficult to extend the model (but possibly difficult to compute its equilibrium) to include financial markets where J securities like bonds or options are traded. However, in order to focus on the main computational challenges we avoid unnecessary notation and additional prices and focus on the simplest possible asset structure.

DEFINITION 3 *A competitive equilibrium, given initial conditions* $z_0, (a^s_{-1})^0_{s=-N+1}$ *is a collection of choices for households* $(c^i_t, a^i_t)^{i \in \mathcal{I}_{zt}}$ *and for the representative firm* $\{K_t, L_t\}$ *as well as prices* $\{r_t, w_t\}$ *for all* $t = 0, \ldots \infty$. *such that*

1. *For all* $s = 0, ...,$, *given* $\{r_t, w_t\}^\infty_{t=0}$, *the choices* $\{c^s_t, a^s_t\}^{s+N-1}_{t=s}$ *maximize* (2), *subject to* (3)
2. *Given* r_t, w_t *the firm maximizes profits, i.e.*

$$(K_t, L_t) \in \arg \max_{K_t, L_t \geqslant 0} f(K_t, L_t, z_t) - r_t K_t - w_t L_t \tag{5}$$

3. *All markets clear: For all* t

$$\begin{aligned} L_t &= \sum_{i \in \mathcal{I}_{zt}} l^i_t \\ K_t &= \sum_{i \in \mathcal{I}_{zt}} a^i_{t-1} \end{aligned}$$

Note that by Walras law market clearing in the labor and capital market imply market clearing in the consumption goods market. Note furthermore that the assumptions on the parametric form

[2]We assume that households cannot convert capital goods back into consumption goods at the beginning of the period. This assumption is necessary to prevent households from consuming the capital at the beginning of the period instead of selling it to the firm in states where the net return to capital is negative. Alternatively one can assume that for all z^t, $r(z^t) \geqslant 1$, as is the case for our numerical examples below.

of the production function as well as concavity and differentiability of F imply that equilibrium prices satisfy

$$\begin{aligned} w(z^t) &= \eta(z_t) F_L(K(z^t), L(z^t)) \\ r(z^t) &= \eta(z_t) F_K(K(z^t), L(z^t)) + (1 - \delta(z_t)) \end{aligned}$$

For future reference, the Euler equation for consumption for any given generation s in node z^t reads as

$$\begin{aligned} u'(c_t^s(z^t)) &= \beta \sum_{z_{t+1} \in \mathcal{Z}} \Pi(z_{t+1}|z_t) r(z^t, z_{t+1}) u'(c_{t+1}^s(z^t, z_{t+1})) \\ &= \beta E_{z_t} u'(c_{t+1}^s(z^t, \tilde{z})) r(z^t, \tilde{z}) \end{aligned} \tag{6}$$

where E_{z_t} is the conditional expectation of $\tilde{z}$, conditional on z_t.

Aollowing our numerical algorithm described above we now approximate policy functions $\hat{\theta}$. Once these are obtained we choose some[3] initial conditions $(a_{-1}^0, \ldots, a_{-1}^{-N+1})$ and use these, together with a simulated path of shocks $\{z_t\}_{t=0}^T$ to generate approximated equilibrium allocations.

We use a sequence of N simulated aggregate capital stocks and the same simulated path of shocks $\{z_t\}_{t=N+1}^T$ to generate true equilibrium allocations, for initial conditions that are consistent with the initial conditions for the approximated equilibrium. We then compare these allocations to the true equilibrium allocations. For the evaluation we discard the first 1,000 of our 15,000 simulated observations, to avoid the influence of initial conditions on our results.

We parametrize the economy as follows. We set $\beta = 0.95^{60/N}$. We consider a Cobb-Douglas production function, $F(K, L) = K^\alpha L^{1-\alpha}$ and pick a capital share of $\alpha = 0.3$. The stochastic production shock is assumed to be *iid* across time and can take four values, i.e. $\mathcal{Z} = \{z_1, z_2, z_3, z_4\}$ with equal probability of $\frac{1}{4}$. The *iid* assumption is mainly made for expositional simplicity (the model has an analytical solution with serially correlated aggregate shocks as well). Experiments with positively correlated technology shocks deliver results that are similar to those reported in the text.

In order to shed some light on whether and to what extent the size of aggregate shocks matters and whether it is shocks to the return to capital or shocks to returns to labor that matter more for the quality of the approximation we consider four different cases for the values the productivity and depreciation shocks.

[3] In practice we use the steady state asset holdings for some deterministic steady state in which the aggregate shock is fixed permanently at an intermediate level.

Table 1: Specifications for Shocks				
	State 1	State 2	State 3	State 4
Case 1: η	0.95	1.05	0.95	1.05
Case 1: δ	0.7	0.7	0.7	0.7
Case 2: η	0.85	1.15	0.85	1.15
Case 2: δ	0.7	0.7	0.7	0.7
Case 3: η	0.95	1.05	0.95	1.05
Case 3: δ	0.5	0.5	0.9	0.9
Case 4: η	0.85	1.15	0.85	1.15
Case 4: δ	0.5	0.5	0.9	0.9

In Table 1 we list the parameters for the 4 cases. While it is obvious that for realistic calibrations the magnitude of the shocks has to depend on the length of a period (and therefore on the number of agents), we keep these four cases fixed throughout the paper. Our goal in this paper is to evaluate the proposed algorithm for a variety of different shocks.

In Table 2 we show how the number of generations N, the degree of the interpolating polynomial ρ (recall that the dimension of the state space is $N-1$ and that in the notation of Section 3 $\rho = 2^{q-d}$) as well as the magnitude of productivity and depreciation shocks influences the accuracy of our algorithm, as compared to the true equilibrium allocations (recall that the dimension d of the problem equals $d = N-1$). We implemented the algorithm for 2nd degree complete polynomials ($q = d+1$), 4th ($q = d+2$) and 8th ($q = d+3$) degree polynomials. We compute equilibria for $N = 3$, $N = 6$ and $N = 9$. For all examples we set the stopping tolerance of the time iteration method to $\tau = 10^{-6}$. As it turns out for this model specification, the results are numerically highly unstable for large N. This is caused by the fact, that agents have zero endowments for all but one period of their lives. We therefore do not consider $N > 10$ in this section.

The results are based on simulations of the economy for 14,000 periods (after discharging the initial 1,000 observations). What we report is the maximal error $\max_t \left|\frac{\hat{K}_t - K_t}{K_t}\right|$ in the aggregate capital stock $\hat{K}_t$ computed with our numerical approximation, as compared to the true analytical solution K_t.

Table 2: Maximal Errors in Allocations and average running times				
ρ, N	Case 1	Case 2	Case 3	Case 4
2,3	6.7 (-4)	2.2 (-3)	8.2(-3)	1.9 (-2)
4,3	1.1 (-5)	1.0 (-4)	3.4 (-4)	5.1 (-4)
8,3	4.2 (-6)	1.2 (-5)	4.9 (-5)	1.1 (-4)
2,6	6.3 (-2)	9.1 (-2)	1.2 (-1)	2.8 (-1)
4,6	3.9 (-4)	9.8 (-4)	2.3 (-3)	3.2 (-3)
8,6	7.4 (-5)	1.0 (-4)	3.5 (-4)	7.7 (-4)
2,9	9.8 (-2)	2.4 (-1)	5.1 (-1)	6.9 (-1)
4,9	1.1 (-3)	7.3 (-3)	2.0 (-2)	3.8 (-2)
8,9	6.7 (-4)	9.9 (-4)	3.1 (-3)	7.2 (-3)

We see that in almost all cases where $N = 3$ the quality of the approximation is excellent – a maximal error of 10^{-4} (in the table written as 1.0 (-4)) implies that if the true capital stock were always equal to 1 the approximated capital stock is never smaller than 0.9999 or larger than 1.0001. In general the quality of the approximation (but also obviously the running time of the algorithm) improves with the number of points. It is generally not advisable to only use polynomials of order 2 for the approximations.

It is interesting to note that the quality of the approximation decreases with the size of the technology shocks (compare errors for case 1 and 2, and for case 3 and 4); this is due to the fact that with a larger support of technology shocks the deterministic steady states lie further apart and hence the approximation occurs over a larger state space, which leads to poorer quality of the approximation. More importantly from a quantitative point of view, however, is that the approximation becomes significantly worse for economies for which stochastic returns to labor and to capital are imperfectly correlated (cases 3 and 4). In these cases the decision rules are more "curved" in the own asset holdings and thus cannot be approximated as well with Chebyshev polynomials as for the case with nonstochastic depreciation.

As the number of agents, N, increases, the quality of the approximation decreases quite rapidly.

4.2 Good approximations

While it is difficult to make statements about exact equilibria a simple argument shows that for approxiamte equilibria forecasts can be arbitrarily accurate as one allows for sufficiently many points in expectation-sets.

The following theorem makes this precise.

THEOREM 2 *For any $\epsilon > 0$ and any $(\mathbf{O})_{i\in\mathbf{I}})$ for any type $\bar{h}$ there exist$\phi_{a,h}(.)$ and $\mathbf{O}_{\bar{h}}$ as well as an ϵ temporary equilibrium sequence, such that agent $\bar{h}$'s forecasts are always within ϵ of the realized* *equilibrium values.*

To prove the result, observe... TBR

5 Regression

Clearly, in the presence of idiosyncratic shocks the above method is no longer feasible. The number of dimensions of the state space exceeds several millions and even with only a few points per dimension the problem is no longer feasible. In that situation it seems reasonable to assume that agents do not take the entire wealth-distribution into account to form their forecasts of next period equilibrium variables. One way to make this precise is to assume that agents try to find "an active subspace". Scheidegger and Bilonis (2017) show how active subspace method, applied to Gaussian processes, can be used in dynamic economic models. The basic idea is from Constantine et al. (2013)). They examine the problem of approximating a high dimensional function $f : \mathbb{R}^m \to \mathbb{R}$ by finding an appropriate matrix $m \times n$ matrix W (where n is much smaller than m) to be able to write

$$f(x) = F(Wy + W^R z) \sim g(y).$$

They provide a simple and (in variations applicable) construction for $g(.)$ and W and Scheidegger and Bilonis (2017) show that in many economic applications n can be taken to be very small relative to the true dimension of the state space.

In the model of this paper it seems reasonable that agents do not know the full state space and can only approximate functions of relatively low dimensions. An extreme case is, of course to assume that they do not take the wealth distribution into account at all. This is somewhat similar to the computational approach taken in Krusell and Smith (1997). In a model with productions, they assume simple laws of motion for aggregate capital and show that agents make small mistakes with these law of motions

5.1 A simple example

For the purposes of this paper, it suffices to consider a much simpler case. While the deviations from rationality can be large in this case, it illustrates the general idea and one does not require any additional technical methods.

Suppose that agents trade in a Lucas tree (and possibly financial markets). Production is trivial in the sense that a tree today is transformed into a tree tomorrow together with dividends.

Suppose for this purposes that the agents base their forecasts only on the current and next period's shock, i.e. the forecasting functions ϕ do not depend on endogenous variables at all. Agents simply assign the average price in that realizes in their expectations set to assign a price to each (z, z'). The agents then use cubic splines to approximate their policy functions numerically, taken as given the simplified evolution of prices.

It is useful to discuss this in a specific example. Assume that agent live for 12 periods ($A = 12$). For this case there are at most $2^{11} = 2048$ possible histories of idiosyncratic shocks per type active in financial markets and one can keep track of the wealth distribution among these agents exactly. For significantly larger A this obviously quickly becomes infeasible and one needs to approximate the wealth distribution itself. While it is now well understood how to do this efficiently (see e.g. Allais et al. (2014) for an overview), any approximation scheme will create another source of numerical error. Since the focus lies on evaluating the performance of the optimization approach I limit myself to cases where this additional error is absent and the wealth distribution is exact.

The agents' problems are solved accurately enough to ensure that the maximal relative errors in Euler equations are below 0.5×10^{-4}. Given that these are problems with occasionally binding constraints, a higher accuracy is difficult to obtain. This limits the search for ϵ-equilibria to the same order of magnitude. Two kinds of computational experiments are performed. First I discuss examples with several types and two assets but simple calibration for endowment shocks. Then I examine some simpler economies with a single tree but with more complex endowment processes. In all examples it is assumed that all agents maximize time-separable expected utility,

$$U_{h,z^t}(\vec{x}) = E_{z^t} \sum_{a=1}^{A} u_{a,h}(\vec{x}_a),$$

and that per-period utility is CRRA with coefficient of risk aversion $\sigma \neq 1$ and a discount factor β, that is,

$$u_{ah}(x) = \beta^a \frac{x^{1-\sigma}}{1-\sigma} \text{ for all } a, h.$$

In this first example it is assumed that there are two types of agent ($H = 2$) that distinguish themselves only by their trading constraints. The fraction of agent 1 is denoted by λ. There are two aggregate shocks and two idiosyncratic shocks. There are $J = 2$ assets; the first one is a Lucas tree with dividends $d(z)$ and the second one is a one-period Arrow security paying $f(z, q) = q + d(z)$ if the aggregate shock is $z = 1$ (and zero otherwise). The Lucas tree cannot be shorted and the Arrow security can only be shorted against the Lucas tree. Agents of type 1 can trade in both securities, while agents of type 2 can only trade in the tree.

Throughout this example it is assumed that $\beta = 1$; that both idiosyncratic and aggregate shocks are i.i.d., and that $\pi(z = 1) = \pi(z = 2) = \frac{1}{2}$ and $\eta_{ah}(y = 1) = \eta_{ah}(y = 2) = \frac{1}{2}$ for all (a, h). It is assumed that the life-cycle profile for aggregate endowments per cohort is as follows;

$$\bar{e} = (\bar{e}^1, \ldots, \bar{e}^{12}) = (0.25, 0.3, 0.35, 0.4, 0.45, 0.5, 0.5, 0.5, 0.5, 0.2, 0.2, 0.2), \quad \bar{d} = 1.$$

Define $d(y) = \mathfrak{d}(y)\bar{d}$ and

$$e_{ah}(y, z) = \begin{cases} \bar{e}^a \cdot \mathfrak{e}^y(y) \cdot \mathfrak{e}^z(z) & a = 1, \ldots, 9 \\ \bar{e}^a & a = 10, 11, 12. \end{cases}$$

Note that this deviates from the notation above and it is now assumed that agents of age $a = 1$ are subject to the idiosyncratic shock.

For this example, risk aversion, the shocks $\mathfrak{d}, \mathfrak{e}^y$, and $\mathfrak{e}^z$ as well as λ and σ are varied. For all experiments in this example $N = 2$; that is, there are only two possible prices per shock (that depend on the previous aggregate shock). Table 1 shows that both average errors as well as maximal errors are low for a wide variety of these parameters.

σ	$\mathfrak{d}$	$\mathfrak{e}^y$	$\mathfrak{e}^z$	λ	Avg. err.	Max. err.
2	(0.9,1)	(0.75, 1.25)	(0.9,1)	0.5	2.3E-04	5.4E-04
3	(0.9,1)	(0.75, 1.25)	(0.9,1)	0.5	4.1E-04	9.4E-04
4	(0.9,1)	(0.75, 1.25)	(0.9,1)	0.5	6.4E-04	1.4E-03
2	(0.85,1)	(0.75, 1.25)	(0.9,1)	0.5	6.7E-04	1.5E-03
2	(0.95,1)	(0.75, 1.25)	(0.9,1)	0.5	2.1E-04	5.0E-04
2	(1,1)	(0.75, 1.25)	(0.9,1)	0.5	6.3E-04	1.5E-03
2	(0.85,1)	(0.75, 1.25)	(0.85,1)	0.5	3.6E-04	8.6E-04
2	(0.95,1)	(0.75, 1.25)	(0.95,1)	0.5	1.6E-04	2.6E-04
2	(0.9,1)	(0.5, 1.5)	(0.9,1)	0.5	5.7E-04	1.3E-03
2	(0.9,1)	(0.25, 1.75)	(0.9,1)	0.5	8.2E-04	2.0E-03
2	(0.9,1)	(0.75, 1.25)	(0.9,1)	0.1	2.5E-04	6.0E-04
2	(0.9,1)	(0.75, 1.25)	(0.9,1)	0.9	4.7E-04	1.2E-03

Table 3: Errors for various specifications

As can be seen from Table 1, only two prices per aggregate shock suffice to obtain good approximations for a wide variety of parameters. Computing the welfare losses of agents in the bounded rationality interpretation leads to similar errors. Agents' errors in consumption choices rarely exceed 0.001 percent of their planned consumption.

Of course, this example is oversimlified and in many more complex examples one would certainly not expect simple forecasts to fare as well. The example is simply presented to show the basic forces at work.

5.2 Stable ϵ-equilibria

TBR

6 Conclusion

TBR

References

[1] Bewley, T. (1984), “Fiscal and Monetary Policy in a General Equilibrium Model”, Cowles Foundation Discussion Paper 690.

[2] Brock, W. and L. Mirman (1972), “Optimal Economic Growth and Uncertainty: The Discounted Case,” Journal of Economic Theory, 4, 479–513.

[3] Brumm, J. , D. Kryczka and F. Kubler, “Recursive Equilibria in Dynamic Economies with Stochastic Production”, (2017), *Econometrica*, forthcoming.

[4] Brumm, J., F. Kubler and S. Scheidegger, “Computing equilibria in dynamic stochastic macro-models with heterogeneous agents”, (2017), in *Advances in Economics and Econometrics: Theory and Applications, Eleventh World Congress.*

[5] Citanna, A. and P. Siconolfi (2012), “Recursive equilibria in stochastic OLG economies: Incomplete markets,” *Journal of Mathematical Economics*, 48, 322–337.

[6] Duffie, D., J. Geanakoplos, A. Mas-Colell, and A. McLennan (1994), “Stationary Markov Equilibria,” *Econometrica*, 62, 745–781.

[7] Feng, Z., J. Miao, A. Peralta-Alva and M. Santos (2014), “Numerical Simulation of Non-optimal Dynamic Equilibrium Models,” *International Economic Review*, 55, 83–111.

[8] Heaton, J. and D. Lucas (1996), “Evaluating the effects of incomplete markets on risk sharing and asset prices,” *Journal of Political Economy,* 104, 443–487.

[9] Krueger, D. and F. Kubler, “Computing equilibrium in OLG models with stochastic production”, (2004), *Journal of Economic Dynamics and Control*, 28, 1411-1436.

[10] Krusell, P. and A. Smith (1998), “Income and Wealth Heterogeneity in the Macroeconomy,” *Journal of Political Economy*, 106, 867–896.

[11] Kubler, F. and H.M. Polemarchakis (2004), “Stationary Markov Equilibria for Overlapping Generations,” *Economic Theory*, 24, 623–643.

[12] Kubler, F. and K. Schmedders (2003), “Stationary Equilibria in Asset-Pricing Models with Incomplete Markets and Collateral,” *Econometrica*, 71, 1767–1795.

[13] Lucas, R. E (1978), “Asset prices in an exchange economy,” *Econometrica,* 46, 1429–1446.

[14] Ljungqvist, L. and T.J. Sargent, (2012), *Recursive Macroeconomic Theory.* MIT Press.

[15] Sargent, T.J., (2002), *Bounded Rationality in Macroeconomics*

[16] Scheidegger, S., Bilionis, I. (2017). "Machine learning for high-dimensional dynamic stochastic economies", working paper, University of Zurich.

[17] Storesletten, K., C. I. Telmer, and A. Yaron (2007). "Asset pricing with idiosyncratic risk and overlapping generations."*Review of Economic Dynamics*, 10, 519 548.