Affine Term Structure Modeling and Macroeconomic Risks at the Zero Lower Bound

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Abstract

We propose the first affine pricing model capturing the joint dynamics of macroeconomic variables and the yield curve while being consistent with the zero lower bound. Our model-implied short-rate distribution has a probability-mass at its lower bound and depends on quadratic combinations of Gaussian macroeconomic and yield-specific factors. With a standard pricing kernel, interest rates and their forecasts are closed-form functions of the macroeconomy. Our empirics investigate the pricing of inflation risks in nominal U.S. rates. We show that the recent crisis triggers substantial short-term deflation fears, so keeping interest rates low is beneficial for the macroeconomy and investors' utility.

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Key-words: Affine Term Structure Model, Zero Lower Bound, QTSM, TIPS, Liftoff Probabilities, Inflation Risk Premia.

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1 Introduction

Understanding the joint dynamics of macroeconomic variables and the yield curve is of paramount importance for both policy makers and investors. Macro-finance asset pricing models of the term structure are a natural way of capturing them. Two main principles usually drive the underlying assumptions in these models. First, the joint dynamics should allow to easily derive policy implications such as forecasts or impulse-response functions. Second, the joint dynamics should provide a simple formulation of asset prices as a function of the macroeconomic risk factors using no-arbitrage. The class of affine term structure models (ATSM) is exactly fitted for these two goals. The canonical ATSM formulation following Ang and Piazzesi (2003) expresses the yield curve as an affine function of both macroeconomic and latent factors whose evolution is driven by - possibly heteroskedastic - Gaussian shocks. The resulting conditional distribution of the yield curve is then Gaussian, placing the nominal interest rates on a positive and negative statistical support.

Whereas the Gaussianity assumption is perceived as reasonably valid when interest rates are high, it is strongly at odds with no-arbitrage theory when the policy rate approaches zero. Because investors can always hold cash for a zero nominal return, interest rates cannot theoretically go below the zero lower bound (ZLB).[2] A model violating this restriction can be extremely harmful from a policy perspective since all model-implied economic quantities during the ZLB are likely to be highly biased due to the misspecification error. Consistent affine pricing models hence require the departure from conditional Gaussianity to reproduce the non-negativity of interest rates and the persistence of the ZLB state. Recent works have thus considered non-Gaussian ATSMs where the yield curve is driven by non-negative latent factors only. However, these models are incapable of including macroeconomic variables that may be negative, leading to a trade-off between ZLB-consistency and macro-finance purposes.

In this paper, we provide a new way of modeling both nominal and real yield curves in an affine framework, which allows for the presence of observable macroeconomic variables and is consistent with the zero lower bound. Our ATSM formulation is motivated by the investigation of the macroeconomic risk premia embedded in nominal securities during the ZLB period. In particular, a key driver of nominal rates is expected inflation and the associated inflation risk premium, which we explore in the empirical application. These quantities can guide policy-makers in the assessment of the harm done to the investors' utilities regarding the central bank reaction to macroeconomic shocks and the timing of the raise of interest rates.

We start from the same formulation as the canonical ATSM and specify the joint dynamics

[2] More generally, if the cost of holding cash is positive, nominal interest rates cannot go below minus this holding cost.

of macroeconomic variables and yield-specific latent factors in a standard Gaussian VAR(1). These variables are gathered in our vector of endogenous risk factors. The nominal short-rate specification constitutes the main novelty of our framework. Conditionally on the current and past factor values, the short-term interest rate is gamma-distributed, its shape being equal to a Poisson draw whose intensity is a positive linear-quadratic combination of current factor values. This so-called *gamma-zero* distribution defines a process able to mimic the specific features of the short-rate at the ZLB: it is non-negative and can reach zero and stay there for extended periods of time before bouncing back to positive values. In particular, the probabilities of escaping the ZLB are functions of current and past macroeconomic shocks. This specification can be interpreted as a quadratic Taylor-type rule where the loadings on macroeconomic variables are varying over time. This representation emphasizes the presence of shifts in monetary policy, as in Ang et al. (2011).

To obtain the entire yield curve dynamics, we express the nominal log-pricing kernel as a quadratic function of the risk factors and a linear function of the short-term nominal rate, where the investors' prices of risk are time-varying *à la* Duffee (2002). This particular pricing kernel formulation has two important implications. First, it is extremely flexible and allows for possibly non-monotonic risk perception regarding macroeconomic shocks. As a result, our model is able to obtain positive risk premia associated with high inflation and deflation at the same time, an absent feature of most existing ATSMs. Second, we show that the distributions of the risk factors under the pricing measure are the same as under the physical measure, with shifted parameters.

Our main modeling result is that these particular dynamics define an ATSM and the nominal yield curve is obtained as closed-form function of macroeconomic variables. As long as the inflation rate is included in the set of observables, we show that the model also produces closed-form interest rate formulas for inflation-indexed bonds. We obtain this affine property expressing the model with the *extended vector of factors*, which gathers the short-term nominal rate and the linear and quadratic combinations of the vector of risk factors. Due to the properties of linear-quadratic Gaussian processes and of the gamma-zero distribution, this extended vector is an affine process under both the physical and pricing measures.[3] An important by-product of this result is that macroeconomic forecasts, nominal and real interest rate forecasts, non-linear impulse-response functions and probabilities of staying at the ZLB (or escaping the ZLB, the *liftoff*) are available as closed-form functions of macroeconomic variables.

We study the empirical performance and economic implications of the model using monthly U.S. data from January 1990 to March 2015. The model incorporates the year-

[3]Since we consider quadratic combinations of Gaussian processes, the model is also a quadratic term structure model in the sense of Ahn et al. (2002).

on-year inflation rate and is estimated to fit both the nominal and the real (TIPS) term structures of interest rates. Using the real yield curve in a term structure model improves the estimation of inflation expectations and makes it easy to isolate the inflation risk components in nominal interest rates. To better pin down the macroeconomic and financial dynamics, we also fit expectations of inflation and expectations on long-term nominal interest rates at different horizons as measured by surveys of professional forecasters and proxies of ZLB probabilities as measured by primary dealer surveys. Our framework is the only existing model to be able to easily incorporate and fit the latter proxies without destroying the affine structure of the state-space model.

We estimate the model with three latent variables that we filter using the quadratic Kalman filter (QKF) quasi maximum likelihood algorithm of Monfort, Renne, and Roussellet (2015). This method mostly relies on the fact that all our observable variables are affine functions of the extended vector of factors, whose total dynamics can be expressed with a semi-strong affine VAR(1) formulation. This estimation method is simple, fast, and can easily accommodate missing data. As a first performance assessment, we show that the model is able to reproduce the time series of nominal and real term structures of interest rates with an average error of 8bps and 12bps respectively. This fit is particularly high with respect to the small number of latent yield factors, emphasizing the flexibility of the model. Second, we show that the model-implied marginal term structures of levels and volatilities are reasonable compared to the data. Last, we extend the classical Campbell and Shiller (1991) regressions to assess the model's ability to both reproduce deviations from the expectation hypothesis and predict excess returns, for both nominal and real term structures. Comparing the model-implied regressions with the data counterparts, we cannot reject that the one-year excess returns and predicted excess returns are consistently reproduced by the model for nearly all maturities.[4] Contrary to yield-only models with positive factors, our framework is able to produce both reliable time-series properties, moments and risk premia estimates.

Our main economic motivation in the application is to study the sign and size of inflation risk premium and its interaction with interest rate shocks during the ZLB. First, we emphasize the importance of a U-shaped pricing kernel by calculating the physical and risk-neutral conditional probabilities of deflation and high inflation (above 4%). We show that both series of risk-neutral probabilities are consistently above their physical counterparts showing that investors fear both low and high inflation shocks at the same time. We then provide the decomposition of nominal interest rates in expected real rates, expected inflation, real term premia and inflation risk premia. We show that the inflation risk premium changes sign over

[4] In other words, the model is consistent with conditions LPY-I and LPY-II, formulated by Dai and Singleton (2002).

time, reflecting investors changing views about inflation shocks. In particular, short-term inflation risk premia becomes consistently negative in the ZLB, showing that investors require a compensation to insure against negative inflation shocks. This premium is economically significant as it reaches a maximum of 4.3% when the ZLB starts binding. These short-term inflation premia fluctuations are however largely offset by the real term premium, producing a very low nominal risk premium component associated with low nominal risk. In comparison, the long-term inflation risk premium is low and fluctuating between -75bps and 75bps at the ZLB, emphasizing investors' confidence in monetary policy long-term effectiveness. Long-term nominal risk premia are therefore mostly driven by real term premia.

We use our estimated joint dynamics to quantify the premium associated with staying at the ZLB, or lifting-off. To first assess the real effects of lifting off, we quantify the interactions between monetary policy and the inflation rate during the zero lower bound. Using an impulse-response analysis, we show that the liftoff can have very detrimental effects on the inflation path and increase dramatically short-term deflation fears. We find that lifting-off from the ZLB by a 10bps initial increase in the short-term nominal rate is very detrimental, translating into -22bps on the inflation rate after one year, and a fall of the short-term inflation risk premium by about -120bps, compared to only -5bps and -30bps respectively for the same increase in normal times. In turn, a 35bps inflation shock raises the short-term nominal rate by about 7bps in normal times but has virtually no impact at the ZLB, emphasizing the will of the central bank to let inflation run up. These detrimental effects of lifting-off are further investigated computing both objective and risk-neutral ZLB and liftoff probabilities. We find that the one-year ZLB risk premium is mostly negative, showing that the ZLB is perceived as a good outcome given the state of the economy. Consistently with this result, we show that the liftoff risk premium becomes negative only after a certain horizon, underlying investors' fears about lifting off *too soon* during the ZLB period.

The remainder of the paper is organized as follows. Section 2 describes the related literature. Section 3 presents the formulation and the properties of the term structure model. We present a general estimation method in Section 4. Section 5 details the data and the identification constraints, while Section 6 focuses on a first analysis of the model in terms of time-series fit, moments and predictability of excess returns. Section 7 explores the economic implications of the model in terms of inflation risk. Section 8 concludes.

2 Literature Review

After their introduction by Duffie and Kan (1996) and Duffie and Singleton (1997), affine term structure models have been very popular. The class of conditionally Gaussian affine models makes it easy to introduce macroeconomic variables in the analysis, as the economic

theory on monetary policy suggests (see e.g. Taylor (1993) and Ang et al. (2004) for Taylor rules in asset pricing models).[5] Ang and Piazzesi (2003) are the first to introduce a no-arbitrage affine model which contains both real activity and inflation as well as unobservable factors to price the term structure, pioneering the so-called macro-finance asset pricing models. Diebold et al. (2005) and Diebold et al. (2006) provide an early summary of the literature, showing that the VAR structure of a macro-finance model allows all traditional policy analysis such as forecasting, impulse-response function computation or risk premia decomposition (see also Dewachter and Lyrio (2006)). Ang et al. (2006) show that the no-arbitrage restrictions employed in a macro-finance model improve the identification of the macroeconomic variables dynamics. Rudebusch and Wu (2008) and Bikbov and Chernov (2010) help relating the level, slope and curvature factors of interest rates to macroeconomic variables. More recently, the literature has focused on whether macroeconomic factors were significantly priced in the interest rates, leading to the so-called spanning puzzle (see e.g. Joslin et al. (2014) or Bauer and Rudebusch (2015)).

A second class of affine models tackles the non-negativity of interest rates. The model of Cox et al. (1985) (CIR henceforth) has positive factors and allows to obtain closed-form positive yield curve estimates (see also its discrete-time formulation by Gouriéroux and Jasiak (2006) and Dai et al. (2010)). However, as shown by Dai and Singleton (2002) and Backus et al. (2001), these processes have difficulties to reproduce the moments of the term structure and a reliable term premium. A second approach consists in extending the Gaussian affine framework in a Gaussian quadratic framework as in e.g. Leippold and Wu (2002, 2007). The short-term interest rate is given by a quadratic combination of factors following a Gaussian VAR, and it preserves the closed-formedness of pricing formulas (see Cheng and Scaillet (2007)). Positivity of the term structure is easily imposed in this framework (see e.g. Gouriéroux and Sufana (2011) or Dubecq et al. (2016)) as well as including macroeconomic variables (see Ang et al. (2011) or Campbell et al. (2016)).[6] However, the positive affine models all treat zero as a reflecting barrier and are not able to generate enough stickiness at the zero lower bound except the model of Monfort et al. (2016). In the latter, the risk factors are positive affine processes with a zero point mass but their formulation does not allow for the introduction of observable real-valued macroeconomic factors.

In the last decade, the literature on modeling the term structure at the zero lower bound has been rapidly growing. A large number of authors have focused on the so-called shadow-rate model (or Black (1995) model, SR henceforth), such as e.g. Kim and Singleton (2012) or Krippner (2013). In this approach, the effective short-rate is the maximum of zero and a Gaussian random variable called shadow-rate. SR models can be yield-only (see for instance

[5] See also Hordahl et al. (2006) or Creal and Wu (2016) for macro-finance models that relate to a structural macroeconomic formulation.

[6] Note that these two latter papers cannot or do not impose the positivity of interest rates.

Lemke and Vladu (2016), or Andreasen and Meldrum (2015)) or incorporate macroeconomic variables (see e.g. Bauer and Rudebusch (2016) or Jackson (2014)). The main drawback of the SR model is that it is not affine hence does not produce closed-form pricing formulas. This often leads to complexity in terms of estimation when more than two factors are included and current methods are either approximate (see e.g. Kim and Priebsch (2013), Priebsch (2013), Wu and Xia (2016) or Christensen and Rudebusch (2015)) or involve computationally intensive algorithms (see Andreasen and Meldrum (2011) or Pericoli and Taboga (2015)).[7] Most papers focus on the estimation of the risk premium in the ZLB or the effectiveness of monetary policy measures (see for example Hamilton and Wu (2012)).

Our empirical application contributes to the vast literature on inflation risk and the term structure of interest rates. No-arbitrage asset pricing models of nominal and inflation-indexed securities have been developed by numerous authors, starting with U.K. and European data as Barr and Campbell (1997), Evans (1998), or Anderson and Sleath (2001) for instance.[8]

Most papers have been focused on finding the relative size of real term premium and inflation risk premium in nominal yields (see Campbell and Viceira (2001)). Buraschi and Jiltsov (2005), Hordahl and Tristani (2012) build macroeconomic-motivated asset pricing models leading to an affine formulation, while Campbell et al. (2009) and Hsu et al. (2014) consider consumption-based pricing models. More reduced-form models have also been considered by for instance Grischenko and Huang (2013), Abrahams et al. (2016), or D'Amico et al. (2014) who directly build on Gaussian affine models of the term structure on U.S. data, or Garcia and Werner (2010) and Joyce, Lildholdt, and Sorensen (2010) who respectively use a three-factor Gaussian affine model on Eurozone data and a four-factor Gaussian affine model on U.K. data. Ang et al. (2008) and Chernov and Mueller (2012) add inflation surveys to better pin down inflation expectations. Adrian and Wu (2009), Haubrich et al. (2012) and Campbell et al. (2016) add volatility factors that drive the variability of the term structure but leave the conditional Gaussianity assumption intact.

Other studies exploit the U.S. inflation-indexed bonds (TIPS) specificities and develop pricing models to back out inflation densities or deflation probabilities. Grischenko et al. (2011) exploit the fact that the TIPS has an embedded deflation option to derive deflation probabilities and the associated risk premium. Christensen et al. (2012, 2016) develop arbitrage-free four-factor models with and without volatility factors to reproduce both nom-

[7] Alternative approaches have also been developed to enforce the zero lower bound. Filipovic, Larsson, and Trolle (2016) develop the linear-rational term structure model and Feunou, Fontaine, and Le (2015) model directly the price of bonds in a nearly arbitrage-free framework. Renne (2016) uses a term structure model where the short-rate can reach discrete positive states.

[8] Other approaches have been employed to quantify the size of the inflation risk premium. Fama (1976, 1990) constitutes its first attempts with linear regressions. Campbell and Shiller (1996) study the properties of inflation-linked securities before they were introduced in the U.S. and Wright (2011) uses panel data regressions to assess the size of international inflation premia.

inal and real term structures and price deflation risk in the U.S.. Fleckenstein et al. (2014a) develop a three-factor model for inflation to price inflation-indexed swaps and options and derive the term structure of deflation risk premium. Last, Kitsul and Wright (2013) develop asset pricing models of TIPS or inflation options to back out inflation conditional densities and inflation risk premium.

With the exception of Carriero et al. (2015) who employ a SR model on nominal and real term structures, all the aforementioned work neglect the consistency with the zero lower bound. This has become paramount regarding the recent period of low interest rates. We hereby propose a solution to this inconsistency.

3 The Model

3.1 Macroeconomic and yields joint dynamics

Let $M_t \in \mathbb{R}^{n_M}$ be a set of observable macroeconomic variables and $Z_t \in \mathbb{R}^{n_Z}$ be a set of latent yield-related risk factors, respectively. Their joint dynamics are given by a standard Gaussian VAR(1) of the following form:

$$\begin{pmatrix} M_t \\ Z_t \end{pmatrix} = \begin{pmatrix} \mu_M \\ \mu_Z \end{pmatrix} + \begin{pmatrix} \Phi_M & \Phi_{M,Z} \\ \Phi_{Z,M} & \Phi_Z \end{pmatrix} \begin{pmatrix} M_{t-1} \\ Z_{t-1} \end{pmatrix} + \begin{pmatrix} \Sigma_M & 0 \\ 0 & I_{n_Z} \end{pmatrix}^{1/2} \varepsilon_t \,, \qquad (1)$$

where ε_t is a zero-mean unit-variance-covariance Gaussian white-noise. Denoting by X_t a size-n vector such that $X_t = (M_t', Z_t')'$, $(n = n_Z + n_M)$, we can write the dynamics in compact form as:

$$X_t = \mu + \Phi X_{t-1} + \Sigma^{1/2} \varepsilon_t \,, \qquad (2)$$

with adequate sizes for μ, Φ and Σ.[9] At time t, economic agents can invest in a one-period risk-less nominal zero-coupon bond providing a known interest rate between t and $t+1$ denoted by r_t. For the sake of generality, we authorize the lower bound of the short-rate to be different from zero, such that:

$$r_t = \delta_0 + \delta \mathfrak{r}_t \,, \quad \delta_0 \in \mathbb{R} \qquad (3)$$

where δ_0 is a parameter representing the value of the lower bound. In the following, we refer to $\mathfrak{r}_t$ as the scaled short-term interest rate. We express $\mathfrak{r}_t$ dynamics using a Poisson mixing

[9] We do not discuss the identification of $\Sigma^{1/2}$. Indeed, since ε_t is a Gaussian shock, the square-root of Σ is only set-identified and can be obtained using for instance zero, sign or long-run restrictions. This matter is however beyond the scope of this paper.

variable denoted by P_t:[10]

$$\begin{cases} P_t|(\underline{X_t}, \underline{\mathfrak{r}_{t-1}}) & \sim & \mathcal{P}\left(\theta_0 + \theta' X_t + X_t' \Theta X_t + \beta \mathfrak{r}_{t-1}\right) \\ \mathfrak{r}_t|(\underline{X_t}, \underline{\mathfrak{r}_{t-1}}, P_t) & \sim & \text{Gamma}\left(P_t, \varsigma\right) , \end{cases} \tag{4}$$

where $(\underline{X_t}, \underline{\mathfrak{r}_{t-1}}) = \{X_t, (\mathfrak{r}_{t-1}, X_{t-1}, \mathfrak{r}_{t-2}, X_{t-2}, \ldots)\}$ is the present and values states of the risk factors and the past of the scaled interest rate, P_t is the shape parameter of the Gamma distribution[11], $\varsigma > 0$ is a scaling parameter, $\beta \geqslant 0$, θ_0 is a constant, θ is a vector of size n and Θ is a positive symmetric $(n \times n)$ matrix.[12] Integrating out P_t, we obtain the so-called conditional gamma-zero distribution (see Monfort, Pegoraro, Renne, and Roussellet (2016)). We rewrite System (4) as:

$$\mathfrak{r}_t|(\underline{X_t}, \underline{\mathfrak{r}_{t-1}}) \sim \text{Gamma}_0\left(\theta_0 + \theta' X_t + X_t' \Theta X_t + \beta \mathfrak{r}_{t-1}, \varsigma\right)$$

The gamma-zero distribution is particularly fitted to represent the behavior of interest rates during the zero lower bound. First, it allows us to define a short-term nominal rate that has a lower bound given by δ_0, as implied by no-arbitrage with cash. The first argument $(\theta_0 + \theta' X_t + X_t' \Theta X_t + \beta \mathfrak{r}_{t-1})$ is an intensity and must be non-negative for the Poisson distribution to be well-defined. For that matter it is sufficient that $\theta_0 \geqslant \frac{1}{4}\theta' \Theta^{-1} \theta$. Second, when $P_t = 0$ the scaled short-rate distribution collapses to a Dirac mass at zero. When $P_t = 0$ for several consecutive periods, the short-rate stays at its lower bound, which reproduces the persistence of the ZLB period. Importantly, the conditional Laplace transform of $\mathfrak{r}_t$ given the current factor values and the past is easily expressed as:

$$\forall u_r < \frac{1}{\varsigma}, \quad \mathbb{E}\left[\exp\left(u_r \mathfrak{r}_t\right) | \underline{X_t}, \underline{\mathfrak{r}_{t-1}}\right] = \exp\left[\frac{u_r \varsigma}{1 - u_r \varsigma}\left(\theta_0 + \theta' X_t + X_t' \Theta X_t + \beta \mathfrak{r}_{t-1}\right)\right] , \tag{5}$$

and the conditional mean and variances of the short-rate are given by:

$$\begin{array}{rcl} \mathbb{E}\left[r_t | \underline{X_t}, \underline{\mathfrak{r}_{t-1}}\right] & = & \delta_0 + \delta_\varsigma\left(\theta_0 + \theta' X_t + X_t' \Theta X_t + \beta \mathfrak{r}_{t-1}\right) \\ \mathbb{V}\left[r_t | \underline{X_t}, \underline{\mathfrak{r}_{t-1}}\right] & = & 2\delta^2 \varsigma^2\left(\theta_0 + \theta' X_t + X_t' \Theta X_t + \beta \mathfrak{r}_{t-1}\right) \end{array} \tag{6}$$

Further details on the gamma-zero distribution are provided in Appendix A.1.

[10] Since P_t is a mixing variable, it has no economic interpretation.

[11] The conditional density of $\mathfrak{r}_t$ given $(\underline{X_t}, \underline{\mathfrak{r}_{t-1}}, P_t)$ is given by: $g_t(x) = \frac{1}{\Gamma(P_t)\varsigma^{P_t}} x^{P_t - 1} \exp\left(-\frac{x}{\varsigma}\right)$, where $\Gamma(\cdot)$ is the standard Gamma function.

[12] Note that Θ does not need to be a strictly positive matrix in all cases. If $\theta = 0$, it is sufficient to have $\theta_0 \geqslant 0$ and Θ semi-positive definite.

3.2 Underlying monetary policy decisions

The short-rate dynamics allows for a more conventional interpretation in terms of the underlying monetary policy reaction function. Using equation (6) we write the following decomposition:

$$\begin{aligned} r_t &= \mathbb{E}\left(r_t|\underline{X_t}, \underline{\mathfrak{r}_{t-1}}\right) + \upsilon_t \quad \text{with} \quad \mathbb{E}(\upsilon_t|\underline{X_t}, \underline{r_{t-1}}) = 0 \\ &= \delta_0 + \delta_\varsigma \left[\theta_0 + (\theta_M + 2\Theta_{M,Z} Z_t + \Theta_M M_t)' M_t + \theta_Z' Z_t + Z_t' \Theta_Z Z_t + \beta \mathfrak{r}_{t-1}\right] + \upsilon_t \\ &=: (\delta_0 + \delta_\varsigma \theta_0 - \delta_0 \varsigma \beta) + \underbrace{\varsigma \beta r_{t-1}}_{\text{smoothing}} + \underbrace{b(M_t, Z_t)' M_t}_{\text{response to macro shocks}} + \underbrace{c(Z_t) + \upsilon_t}_{\text{monetary policy shocks}} , \end{aligned} \tag{7}$$

where the subscripts $(.)_M$ and $(.)_Z$ are explicit notations for the partitions of θ and Θ. Equation (7) can be seen as a Taylor-type rule where the loadings on macroeconomic variables are time-varying Gaussian variables, as in e.g. Ang et al. (2011). $b(M_t, Z_t)$ therefore represents the central bank response to macroeconomic shocks. In contrast, both $c(Z_t)$ and υ_t are monetary policy shocks that are conditionally uncorrelated with macroeconomic shocks given the past. The former can be persistent if Z_t is persistent, whereas the latter is non-persistent. Note however that the distributions of the monetary policy shocks $c(Z_t)$ and υ_t are non-Gaussian, resulting in a crucial difference with the standard Taylor-type rule case.[13]

3.3 Nominal pricing kernel and risk-neutral dynamics

Between $t-1$ and t, economic agents discount payoffs with the nominal pricing kernel $m_{t-1,t}$ (or stochastic discount factor, SDF henceforth). The SDF is specified as an exponential-quadratic function of $(X_t, \mathfrak{r}_t)$ with time-varying prices of risk.

$$\begin{aligned} m_{t-1,t} &= \exp\Big\{ - r_{t-1} + \Lambda_{t-1}' X_t + X_t' \Lambda_X X_t + \Lambda_r \mathfrak{r}_t \\ &\quad - \log \mathbb{E}\left[\exp\left(\Lambda_{t-1}' X_t + X_t' \Lambda_X X_t + \Lambda_r \mathfrak{r}_t\right) \left|\underline{X_{t-1}}, \underline{\mathfrak{r}_{t-1}}\right.\right] \Big\} , \end{aligned} \tag{8}$$

where the expectation term is the convexity adjustment such that $\mathbb{E}(m_{t-1,t}|\underline{X_{t-1}}, \underline{\mathfrak{r}_{t-1}}) = \exp(-r_{t-1})$. The linear prices of risk Λ_{t-1} are given by an affine function of the past risk factors X_{t-1} (see Duffee (2002)):

$$\Lambda_{t-1} =: \lambda_0 + \lambda X_{t-1} . \tag{9}$$

[13] This short-term interest rate specification should not be interpreted as a structural monetary policy reaction function. As noted by Backus et al. (2015), the identification of the structural Taylor rule parameters can be difficult in the affine framework. We therefore interpret the present specification as a reduced-form for the short-term interest rate dynamics.

This SDF specification allows for a very simple derivation of risk-neutral $\mathbb{Q}$-dynamics of $(X_t, \mathfrak{r}_t)$. In particular, the form of Equations (8) and (9) preserves the same class of probability distributions under the risk-neutral measure.

Proposition 3.1 *X_t follows a Gaussian VAR under the risk-neutral measure.*

$$X_t = \mu^{\mathbb{Q}} + \Phi^{\mathbb{Q}} X_{t-1} + \Sigma^{\mathbb{Q}^{1/2}} \varepsilon_t^{\mathbb{Q}}, \tag{10}$$

where $\varepsilon_t^{\mathbb{Q}}$ is a zero-mean unit-variance Gaussian white noise, and $\mu^{\mathbb{Q}}$, $\Phi^{\mathbb{Q}}$ and $\Sigma^{\mathbb{Q}}$ are given by:

$$\begin{cases} \mu^{\mathbb{Q}} &= \Sigma^{\mathbb{Q}} \left(\lambda_0 + \dfrac{\Lambda_r \varsigma}{1 - \Lambda_r \varsigma} \theta + \Sigma^{-1} \mu \right), \\ \Phi^{\mathbb{Q}} &= \Sigma^{\mathbb{Q}} \left(\lambda + \Sigma^{-1} \Phi \right), \\ \Sigma^{\mathbb{Q}} &= \left(\Sigma^{-1} - 2 \dfrac{\Lambda_r \varsigma}{1 - \Lambda_r \varsigma} \Theta - 2 \Lambda_X \right)^{-1}, \end{cases} \tag{11}$$

whenever Λ_X and Λ_r define non-negative eigenvalues for the matrix $\Sigma^{\mathbb{Q}}$.

Proof See Appendix A.2. ■

Additional flexibility appears compared to the standard Gaussian ATSM. First, when either Λ_r or Λ_X are different from zero, the conditional variance of the Gaussian VAR is different under the physical and the risk-neutral measure, an absent feature of most ATSMs. Economic agents price shocks increasing the short-term interest rate higher than expected at $t-1$ with a price Λ_r. This translates into a premium associated with high factor values ($\mu^{\mathbb{Q}} > \mu$) and with the factors variance ($\Sigma^{\mathbb{Q}} > \Sigma$). Second, agents value large unexpected shocks making the risk factors X_t deviate from their mean with price Λ_X and Λ_{t-1}. In essence, Λ_X can be seen as the price of variance-covariance risk, driving an additional wedge between $\Sigma^{\mathbb{Q}}$ and Σ.

Similar transition formulas between the physical and risk-neutral measures can be derived for the short-term interest rate dynamics.

Proposition 3.2 *$\mathfrak{r}_t$ is conditionally gamma-zero distributed given $(\underline{X_t}, \underline{\mathfrak{r}_{t-1}})$ under the risk-neutral measure.*

$$\mathfrak{r}_t | (\underline{X_t}, \underline{\mathfrak{r}_{t-1}}) \overset{\mathbb{Q}}{\sim} \text{Gamma}_0 \left(\theta_0^{\mathbb{Q}} + \theta^{\mathbb{Q}'} X_t + X_t' \Theta^{\mathbb{Q}} X_t + \beta^{\mathbb{Q}} \mathfrak{r}_{t-1}, \varsigma^{\mathbb{Q}} \right), \tag{12}$$

where the risk-neutral parameters are given by:

$$\theta_0^{\mathbb{Q}} = \frac{\theta_0}{1 - \Lambda_r \varsigma}, \ \theta^{\mathbb{Q}} = \frac{1}{1 - \Lambda_r \varsigma} \theta, \ \Theta^{\mathbb{Q}} = \frac{1}{1 - \Lambda_r \varsigma} \Theta, \ \beta^{\mathbb{Q}} = \frac{\beta}{1 - \Lambda_r \varsigma}, \ \varsigma^{\mathbb{Q}} = \frac{\varsigma}{1 - \Lambda_r \varsigma}. \tag{13}$$

Proof see Appendix A.2. ■

A positive Λ_r drives a positive discrepancy between risk-neutral and physical parameters, shifting all risk-neutral moments of the short-term interest rate upwards.[14]

3.4 The affine property of the model

Affine term structure models (ATSM) are a very convenient class of models since they allow to obtain closed-form interest rate formulas for zero-coupon bonds (see e.g. Duffie and Kan (1996) or Dai and Singleton (2000)). A model verifies the affine property if the process gathering the risk-factors and the scaled short-term interest rate has a risk-neutral conditional Laplace transform given its past which is an exponential-affine function of its past (see Darolles et al. (2006)). In this section, we show that the model is an ATSM.

Let $f_t = [X_t', \text{Vec}(X_t X_t'), \mathfrak{r}_t]'$ be the *extended vector of factors* (see Cheng and Scaillet (2007)). The conditional Laplace transform of f_t given its past under the risk-neutral measure is denoted by:

$$\phi_{t-1}^{\mathbb{Q}}(u) := \mathbb{E}^{\mathbb{Q}}\left[\exp(u' f_t)\middle|\underline{f_{t-1}}\right] \quad \text{where} \quad u = [u_x', \text{Vec}(U_x)', u_r]' \ .$$

Proposition 3.3 *(f_t) is an affine process under the risk-neutral measure, so the risk-neutral conditional Laplace transform of f_t given its past is exponential-affine. Its closed-form expression is given by:*

$$\phi_{t-1}^{\mathbb{Q}}(u) = \exp\left\{\mathbb{A}^{\mathbb{Q}}(u) + \mathbb{B}^{\mathbb{Q}'}(u) X_{t-1} + X_{t-1}' \mathbb{C}^{\mathbb{Q}}(u)\, X_{t-1} + \mathbb{D}^{\mathbb{Q}}(u)\mathfrak{r}_{t-1}\right\} \ ,$$

where the loadings $\mathbb{A}^{\mathbb{Q}}(u)$, $\mathbb{B}^{\mathbb{Q}}(u)$, $\mathbb{C}^{\mathbb{Q}}(u)$ and $\mathbb{D}^{\mathbb{Q}}(u)$ are given by:

$$\begin{aligned}
\mathbb{A}^{\mathbb{Q}}(u) &= \frac{u_r \varsigma^{\mathbb{Q}} \theta_0^{\mathbb{Q}}}{1 - u_r \varsigma^{\mathbb{Q}}} - \frac{1}{2}\log\left|I_n - 2\Sigma^{\mathbb{Q}}\left(U_x + \frac{u_r\varsigma^{\mathbb{Q}}}{1-u_r\varsigma^{\mathbb{Q}}}\Theta^{\mathbb{Q}}\right)\right| \\
&+ \mu^{\mathbb{Q}'}\left(U_x + \frac{u_r\varsigma^{\mathbb{Q}}}{1-u_r\varsigma^{\mathbb{Q}}}\Theta^{\mathbb{Q}}\right)\left[I_n - 2\Sigma^{\mathbb{Q}}\left(U_x + \frac{u_r\varsigma^{\mathbb{Q}}}{1-u_r\varsigma^{\mathbb{Q}}}\Theta^{\mathbb{Q}}\right)\right]^{-1}\mu^{\mathbb{Q}} \\
&+ \left(u_x + \frac{u_r\varsigma^{\mathbb{Q}}}{1-u_r\varsigma^{\mathbb{Q}}}\theta^{\mathbb{Q}}\right)'\left[I_n - 2\Sigma^{\mathbb{Q}}\left(U_x + \frac{u_r\varsigma^{\mathbb{Q}}}{1-u_r\varsigma^{\mathbb{Q}}}\Theta^{\mathbb{Q}}\right)\right]^{-1}\left[\mu^{\mathbb{Q}} + \frac{1}{2}\Sigma^{\mathbb{Q}}\left(u_x + \frac{u_r\varsigma^{\mathbb{Q}}}{1-u_r\varsigma^{\mathbb{Q}}}\theta^{\mathbb{Q}}\right)\right] \\
\mathbb{B}^{\mathbb{Q}'}(u) &= \left[\left(u_x + \frac{u_r\varsigma^{\mathbb{Q}}}{1-u_r\varsigma^{\mathbb{Q}}}\theta^{\mathbb{Q}}\right)' + 2\mu^{\mathbb{Q}'}\left(U_x + \frac{u_r\varsigma^{\mathbb{Q}}}{1-u_r\varsigma^{\mathbb{Q}}}\Theta^{\mathbb{Q}}\right)\right]\left[I_n - 2\Sigma^{\mathbb{Q}}\left(U_x + \frac{u_r\varsigma^{\mathbb{Q}}}{1-u_r\varsigma^{\mathbb{Q}}}\Theta^{\mathbb{Q}}\right)\right]^{-1}\Phi^{\mathbb{Q}} \\
\mathbb{C}^{\mathbb{Q}}(u) &= \Phi^{\mathbb{Q}'}\left(U_x + \frac{u_r\varsigma^{\mathbb{Q}}}{1-u_r\varsigma^{\mathbb{Q}}}\Theta^{\mathbb{Q}}\right)\left[I_n - 2\Sigma^{\mathbb{Q}}\left(U_x + \frac{u_r\varsigma^{\mathbb{Q}}}{1-u_r\varsigma^{\mathbb{Q}}}\Theta^{\mathbb{Q}}\right)\right]^{-1}\Phi^{\mathbb{Q}} \\
\mathbb{D}^{\mathbb{Q}}(u) &= \frac{u_r\varsigma^{\mathbb{Q}}}{1-u_r\varsigma^{\mathbb{Q}}}\beta^{\mathbb{Q}} .
\end{aligned}$$

[14] Interestingly, Equations (12) and (13) also imply a different Taylor-type rule under the risk neutral measure. Similarly to Equation (7), we have: $r_t = \delta_0 + \frac{(\varsigma\theta_0 - \delta_0\varsigma\beta) + \varsigma\beta r_{t-1} + b(M_t, Z_t)' M_t + c(Z_t)}{(1-\Lambda_r\varsigma)^2} + \upsilon_t^{\mathbb{Q}}$ where the risk-neutral response to the macroeconomic variables is exactly proportional to the physical one by a factor $\frac{1}{(1-\Lambda_r\varsigma)^2}$.

Proof See Appendix A.2. ■

$\phi^{\mathbb{Q}}_{t-1}(u)$ is therefore an exponential-affine function of f_{t-1}. Since the class of distributions of $(X_t, \mathfrak{r}_t)$ are the same under the physical and the risk-neutral measures (see previous Section), the properties of f_t are the similar under the physical measure.

Proposition 3.4 *(f_t) is an affine process under the physical measure, and the physical conditional Laplace transform of f_t given its past is exponential-affine with a closed-form expression given by:*

$$\phi_{t-1}(u) := \mathbb{E}\left[\exp(u' f_t)\middle|\underline{f_{t-1}}\right] = \exp\left\{\mathbb{A}(u) + \mathbb{B}'(u)X_{t-1} + X'_{t-1}\mathbb{C}(u)\,X_{t-1} + \mathbb{D}(u)\mathfrak{r}_{t-1}\right\}\,,$$

where the loadings $\mathbb{A}(u)$, $\mathbb{B}(u)$, $\mathbb{C}(u)$ and $\mathbb{D}(u)$ are given by the same recursions as $\mathbb{A}^{\mathbb{Q}}(u)$, $\mathbb{B}^{\mathbb{Q}}(u)$, $\mathbb{C}^{\mathbb{Q}}(u)$ and $\mathbb{D}^{\mathbb{Q}}(u)$ respectively, plugging the physical parameters instead of the risk-neutral ones.

Corollary 3.4.1 *(f_t) has a semi-strong affine VAR(1) representation under $\mathbb{P}$. Its first-two conditional moments given its past are hence affine functions of f_{t-1}, and its marginal mean and covariance matrix can be obtained in closed-form. The dynamics of f_t can be expressed as:*

$$f_t =: \Psi_0 + \Psi f_{t-1} + \left[\mathrm{Vec}^{-1}\left(\Omega_0 + \Omega f_{t-1}\right)\right]^{1/2}\xi_t\,, \tag{14}$$

where ξ_t is a martingale difference with zero mean and unit variance, and exact formulas for Ψ_0, Ψ, Ω_0 and Ω depend explicitly on μ, Φ, Σ, θ_0, θ, Θ, β and ς:

$$\begin{aligned}
\Psi_0 &= \left.\frac{\partial \mathbb{A}(u)}{\partial u}\right|_{u=0}, \quad \Psi = \left.\frac{\partial\left[\mathbb{B}'(u), \mathrm{Vec}(\mathbb{C}(u))', \mathbb{D}(u)\right]'}{\partial u}\right|_{u=0} \\
\Omega_0 &= \left.\frac{\partial^2 \mathbb{A}(u)}{\partial u \partial u'}\right|_{u=0}, \quad \Omega = \sum_{i=1}^{n} \mathrm{Vec}\left(\left.\frac{\partial^2\left[\mathbb{B}'(u), \mathrm{Vec}(\mathbb{C}(u))', \mathbb{D}(u)\right]'_i}{\partial u \partial u'}\right|_{u=0}\right) \times e'_i\,,
\end{aligned}$$

where e_i is the i^{th} column of identity matrix I_n. Explicit formulas of these derivatives can be found in Appendix A.3.

Proof See Appendix A.3. ■

Since the f_t is an affine process under the physical measure, the forecasts of the factor values are simply expressed with a closed-form affine expression. Using the semi-strong VAR representation of Equation (14), we obtain the following proposition.

Proposition 3.5 *The process (f_t) is stationary if and only if the eigenvalues of the matrix Φ are lower than 1 in modulus and $|\beta_\varsigma| < 1$. The first two conditional moments of the factors k periods ahead are affine functions of the current value of the factors f_t and are given by:*[15]

$$\begin{aligned}
\mathbb{E}\left(f_{t+k}\middle|\underline{f_t}\right) &= (I_{n+n^2+1} - \Psi)^{-1}(I_{n+n^2+1} - \Psi^k)\Psi_0 + \Psi^k f_t \\
\mathrm{Vec}\left[\mathbb{V}\left(f_{t+k}\middle|\underline{f_t}\right)\right] &= \sum_{i=0}^{k-1} (\Psi \otimes \Psi)^i \left(\Omega_0 + \Omega\left[(I_{n+n^2+1} - \Psi)^{-1}(I_{n+n^2+1} - \Psi^{k-i-1})\Psi_0 + \Psi^{k-i-1} f_t\right]\right) .
\end{aligned}$$

Provided stationarity, the first two marginal moments of f_t are given by:

$$\begin{aligned}
\mathbb{E}\left(f_t\right) &= (I_{n+n^2+1} - \Psi)^{-1}\Psi_0 \\
\mathrm{Vec}\left[\mathbb{V}\left(f_t\right)\right] &= \left(I_{(n+n^2+1)^2} - \Psi \otimes \Psi\right)^{-1}\left[\Omega_0 + \Omega(I_{n+n^2+1} - \Psi)^{-1}\Psi_0\right] .
\end{aligned}$$

Proof See Appendix A.4. ■

3.5 Pricing nominal zero-coupon bonds

Nominal zero-coupon bonds are securities that deliver one unit of cash at maturity date. Let us denote by $B(t,h)$ and $R(t,h) = -h^{-1}\log B(t,h)$ respectively the price and continuously compounded interest rate of a nominal zero-coupon bond at time t, with residual maturity h. Standard no-arbitrage arguments imply:

$$B(t,h) = \mathbb{E}^{\mathbb{Q}}\left[\exp(-r_t)B(t+1,h-1)\middle|\underline{f_t}\right] = \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\sum_{i=0}^{h-1} r_{t+i}\right)\middle|\underline{f_t}\right] . \tag{15}$$

When δ_0 is non-negative, the term in the exponential is always negative. Bond prices at all maturities are hence constrained between 0 and 1, such that the associated interest rates at all maturities are always positive. Since the model is an ATSM, the following proposition is immediately obtained.

Proposition 3.6 *$B(t,h)$ is an exponential-affine function of f_t and the associated interest rate $R(t,h)$ is affine in f_t.*

$$R(t,h) = -h^{-1}\left(A_h + B_h' X_t + X_t' C_h X_t + D_h \mathfrak{r}_t\right) =: a_h + \mathcal{B}_h' f_t , \tag{16}$$

where $a_h = -A_h/h$ and $\mathcal{B}_h = [-B_h'/h, \mathrm{Vec}(-C_h/h)', -D_h/h]'$, and the explicit recursive

[15] The conditional moments formulas are given with the use of the matrix $(I_{n+n^2+1} - \Psi)^{-1}$ which is only invertible if the system is stationary. Note that the stationarity assumption is however not necessary and the same formulas can be expressed in the form of truncated sums.

expressions for computing A_h, B_h and C_h are given by:

$$\begin{aligned} A_h &= A_{h-1} - \delta_0 + \mathbb{A}^{\mathbb{Q}}\left(\left[B_{h-1}', \text{Vec}\left(C_{h-1}\right)', -D_{h-1}\right]'\right) \\ B_h &= \mathbb{B}^{\mathbb{Q}}\left(\left[B_{h-1}', \text{Vec}\left(C_{h-1}\right)', -D_{h-1}\right]'\right) \\ C_h &= \mathbb{C}^{\mathbb{Q}}\left(\left[B_{h-1}', \text{Vec}\left(C_{h-1}\right)', -D_{h-1}\right]'\right) \\ D_h &= \mathbb{D}^{\mathbb{Q}}\left(\left[B_{h-1}', \text{Vec}\left(C_{h-1}\right)', -D_{h-1}\right]'\right). \end{aligned} \tag{17}$$

Proof Straightforward computation of no-arbitrage relationship (15). ■

The unspanned macroeconomic factors model of Joslin, Priebsch, and Singleton (2014) is a nested specification of the model presented above. Three constraints need to be imposed. Macroeconomic variables should not intervene in the short-term nominal interest rate specification. This is easily obtained imposing that Equation (4) simplifies to $\mathfrak{r}_t|(\underline{X_t}, \underline{\mathfrak{r}_{t-1}}) \sim \text{Gamma}_0\left(\theta_0 + \theta_Z' Z_t + Z_t' \Theta_Z Z_t + \beta \mathfrak{r}_{t-1}, \varsigma\right)$. The second constraint is that the macroeconomic variables and the yield factors must be conditionally independent under the risk-neutral measure, that is $\Sigma_{M,Z}^{\mathbb{Q}} = 0$. This is easy to impose via linear constraints on Λ_X. Last, the macroeconomic variables M_t do not Granger-cause the yield factors Z_t under the risk neutral measure. Since $\Phi^{\mathbb{Q}} = \Sigma^{\mathbb{Q}}(\lambda + \Sigma^{-1}\Phi)$, imposing the bottom-left block of $\Phi^{\mathbb{Q}}$ to be equal to zero is easy via linear constraints on the price of risk λ. As in the unspanned risk literature, the macroeconomic variables would not be priced in nominal interest rates but would help predict and be predicted by yield factors whenever Φ is unconstrained.

3.6 Pricing inflation-indexed zero-coupon bonds

Inflation-indexed zero-coupon bonds are securities that deliver a payment at maturity which is equal to the compounded inflation between the inception date and the maturity date. They can be seen as inflation hedges in nominal terms, or risk-less investments in real terms. Let us denote by $B^*(t,h)$ the price in dollars of an inflation-indexed zero-coupon bond issued at time t and maturing at $t+h$. The reference price index used to compute inflation-indexed payments is denoted by CPI_t. The inflation rate between t and $t+1$ is denoted by π_{t+1} and is equal to $\log(CPI_{t+1}/CPI_t)$. Standard no-arbitrage arguments imply:

$$\begin{aligned} B^*(t,h) &= \mathbb{E}^{\mathbb{Q}}\left[\exp(-r_t + \pi_{t+1}) B^*(t+1, h-1) \middle| \underline{f_t}\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\sum_{i=0}^{h-1}(r_{t+i} - \pi_{t+i+1})\right) \middle| \underline{f_t}\right], \end{aligned} \tag{18}$$

For inflation-indexed bonds, we use the term *interest rate* to designate to the *ex-ante* real interest rate – denoted by $R_a^*(t,h)$ – and which is given by $R_a^*(t,h) = -\frac{1}{h}\log B^*(t,h)$. We hereby consider versions of the model where the inflation rate is included in the set of observable macroeconomic variables M_t.[16]

Proposition 3.7 *The price of inflation-indexed bonds are given by an exponential-affine function of f_t and the associated interest rate is affine in f_t whenever π_t is in M_t.*

$$R_a^*(t,h) \;=\; -h^{-1}\left(A_h^* + B_h^{*'}X_t + X_t' C_h^* X_t + D_h^* \mathfrak{r}_t\right) =: a_h^* + \mathcal{B}_h^{*'} f_t\,, \tag{19}$$

where $a_h^* = -A_h^*/h$ and $\mathcal{B}_h^* = (-B_h^{*'}/h,\, -\mathrm{Vec}(C_h^*/h)',\, -D_h^*/h)'$, and the explicit recursive expressions for computing A_h^*, B_h^*, C_h^* and D_h^* are given by:

$$\begin{aligned}
A_h^* &= A_{h-1}^* - \delta_0 + \mathbb{A}^{\mathbb{Q}}\left(\left[\left(B_{h-1}^* + e_\pi\right)', \mathrm{Vec}\left(C_{h-1}^*\right)', -D_{h-1}^*\right]'\right)\\
B_h^* &= \mathbb{B}^{\mathbb{Q}}\left(\left[\left(B_{h-1}^* + e_\pi\right)', \mathrm{Vec}\left(C_{h-1}^*\right)', -D_{h-1}^*\right]'\right)\\
C_h^* &= C_h\\
D_h^* &= D_h\,,
\end{aligned}$$

e_π being a selection vector of the inflation rate in the vector X_t.

Proof See Appendix A.5. ■

3.7 Forecasting macroeconomic and financial variables and predicting the liftoff

In the previous section, we have shown that the interest rates of nominal and real securities are affine functions if the extended vector of factors f_t. Using the formulas for the conditional moments of the factors (Proposition 3.5), the forecasts of any observable variable is simply expressed with a closed-form affine expression.

Proposition 3.8 *The k-period ahead optimal forecast of macroeconomic variables, nominal and real yields at any maturity are affine functions of f_t. In the same fashion, the optimal covariance forecast of macroeconomic variables, nominal and real yields at any maturity are affine functions of f_t. The detailed formulas are available in Appendix A.3.*

[16] Another option would be to consider π_t as a quadratic combination of latent variables and filtering it from the data, as in e.g. Abrahams et al. (2016).

Proof Both macroeconomic and financial variables are affine combinations of f_t. All their future values are hence affine in f_{t+k}. Using Proposition 3.5, their optimal level and covariance forecasts are affine in f_t. ■

The model also provides a natural framework for analyzing the so-called liftoff probabilities. These probabilities are associated with the event that the economy goes out of the zero lower bound after k periods, that is $\{r_{t+1:t+k} = \delta_0,\, r_{t+k+1} > \delta_0\}$. In this section, we exploit the properties of the gamma-zero distribution to derive closed-form expressions for the conditional probabilities to be at zero in the future. We mostly use the results provided in Monfort et al. (2016).

Let us denote by $\varphi_t(u_1, \ldots, u_k)$ and $\varphi_t^{\mathbb{Q}}(u_1, \ldots, u_k)$ the physical and risk-neutral multi-horizon conditional Laplace transform of f_{t+k} given $\underline{f_t}$ respectively, where $(u_0, \ldots, u_k)$ are vectors of size $n + n^2 + 1$.

$$\varphi_t(u_1, \ldots, u_k) = \mathbb{E}\left[\exp\left(\sum_{i=1}^{k} u_i' f_{t+i}\right) \Big| \underline{f_t}\right] \quad \text{and} \quad \varphi_t^{\mathbb{Q}}(u_1, \ldots, u_k) = \mathbb{E}^{\mathbb{Q}}\left[\exp\left(\sum_{i=1}^{k} u_i' f_{t+i}\right) \Big| \underline{f_t}\right].$$

It is well known that these expressions are available in closed-form for all arguments when f_t is an affine process under the physical and the risk-neutral measures, and are exponential-affine functions of f_t computable with closed-from recursions (See Appendix A.5 for the formulas).

Proposition 3.9 *Let $u_v = (0, \ldots, 0, v)'$, $u_v \in \mathbb{R}^{n+n^2+1}$, v being a real number. The probability for the short-rate to be equal to its lower bound for k periods is given by the following expression:*

$$\mathbb{P}\left(r_{t+1:t+k} = \delta_0 \big| \underline{f_t}\right) = \lim_{v \to -\infty} \varphi_t(u_v, \ldots, u_v) \quad \textit{and} \quad \mathbb{Q}\left(r_{t+1:t+k} = \delta_0 \big| \underline{f_t}\right) = \lim_{v \to -\infty} \varphi_t^{\mathbb{Q}}(u_v, \ldots, u_v).$$

These probabilities are exponential-affine functions of f_t by continuity, and we write:

$$\mathbb{P}\left(r_{t+1:t+k} = \delta_0 \big| \underline{f_t}\right) =: \exp\left(\mathcal{D}_{0,k} + \mathcal{D}_k' f_t\right) \quad \textit{and} \quad \mathbb{Q}\left(r_{t+1:t+k} = \delta_0 \big| \underline{f_t}\right) =: \exp\left(\mathcal{D}_{0,k}^{\mathbb{Q}} + \mathcal{D}_k^{\mathbb{Q}'} f_t\right),$$

where the loadings are detailed in Appendix A.5.

Proof See Monfort, Pegoraro, Renne, and Roussellet (2016). ■

Corollary 3.9.1 *The liftoff probabilities are given by:*

$$\begin{aligned}
\mathbb{P}\left(r_{t+1:t+k} = \delta_0, r_{t+k+1} > \delta_0 \big| \underline{f_t}\right) &= \exp\left(\mathcal{D}_{0,k} + \mathcal{D}_k' f_t\right) - \exp\left(\mathcal{D}_{0,k+1} + \mathcal{D}_{k+1}' f_t\right) \\
\mathbb{Q}\left(r_{t+1:t+k} = \delta_0, r_{t+k+1} > \delta_0 \big| \underline{f_t}\right) &= \exp\left(\mathcal{D}_{0,k}^{\mathbb{Q}} + \mathcal{D}_k^{\mathbb{Q}'} f_t\right) - \exp\left(\mathcal{D}_{0,k+1}^{\mathbb{Q}} + \mathcal{D}_{k+1}^{\mathbb{Q}'} f_t\right)
\end{aligned}$$

Proof It is easily shown that $\mathbb{1}\{r_{t+1:t+k} = \delta_0, r_{t+k+1} > \delta_0\} = \mathbb{1}\{r_{t+1:t+k} = \delta_0\} - \mathbb{1}\{r_{t+1:t+k+1} = \delta_0\}$. The result follows immediately. ■

This gives us a simple expression to compute probabilities to be at the ZLB during k periods exactly or the associated liftoff probabilities. It is obvious from the previous proposition that the log of the probabilities to stay at zero are affine functions of f_t.

Proposition 3.10 *The logarithm of the risk-neutral conditional probability that the short-rate is equal to zero correspond to the (scaled) ex-ante return $R_{zlb}(t,k)$ of a synthetic ZLB-insurance bond whose payoff is equal to $\exp(r_t + (k-1)\delta_0)\,\mathbb{1}\{r_{t+1:t+k} = \delta_0\}$ at period $t+k$. The physical counterpart is the (scaled) expected hypothesis component $R^{EH}_{zlb}(t,k)$ of this return. Both these quantities are affine in f_t, such that:*

$$R_{zlb}(t,k) = -k^{-1}\left(\mathcal{D}^{\mathbb{Q}}_{0,k} + \mathcal{D}^{\mathbb{Q}'}_{k} f_t\right) \quad \textit{and} \quad R^{EH}_{zlb}(t,k) = -k^{-1}\left(\mathcal{D}_{0,k} + \mathcal{D}'_k f_t\right) \tag{20}$$

Proof The price of such a bond is given by:

$$\begin{aligned} B_{zlb}(t,k) &= \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\sum_{i=0}^{k-1} r_{t+i}\right)\exp\left[r_t + (k-1)\delta_0\right]\mathbb{1}\{r_{t+1:t+k} = \delta_0\}\big|\underline{f_t}\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left(\mathbb{1}\{r_{t+1:t+k} = \delta_0\}\big|\underline{f_t}\right) = \mathbb{Q}\left(r_{t+1:t+k} = \delta_0\big|\underline{f_t}\right). \end{aligned}$$

The same argument applies for the physical measure. ■

3.8 Performing an impulse response analysis

The affine structure of the model makes it easy to perform an impulse response analysis. All the variables considered in this section can be expressed as linear combinations of f_t components. Let us consider the impact of a shock of size s of variable v_2 on variable v_1, where $v_1 = e'_{v_1} f_t$ and $v_2 = e'_{v_2} f_t$, with e_{v_1} and e_{v_2} vectors weighting and selecting the right entries of f_t depending on the variables of interest. Let us also denote by $\mathcal{E}_v = (e_{v_3}, \cdots, e_{v_q})$ the matrix of $(q-2)$ weighting vectors that define variables $v_j = e'_{v_j} f_t$ that we do not want to shock at the initial period. The impulse response at horizon k, denoted by $\mathcal{I}^{v_2 \to v_1}_{t,k}$ is given by:

$$\begin{aligned} \mathcal{I}^{v_2\to v_1}_{t,k} &= \mathbb{E}\left(e'_{v_1} f_{t+k}\Big|\underline{f_{t-1}},\, e'_{v_2}[f_t - \mathbb{E}(f_t|\underline{f_{t-1}})] = s,\, \mathcal{E}'_v[f_t - \mathbb{E}(f_t|\underline{f_{t-1}})] = 0\right) \\ &- \mathbb{E}\left(e'_{v_1} f_{t+k}\big|\underline{f_{t-1}},\, e'_{v_2}[f_t - \mathbb{E}(f_t|\underline{f_{t-1}})] = 0,\, \mathcal{E}'_v[f_t - \mathbb{E}(f_t|\underline{f_{t-1}})] = 0\right). \end{aligned} \tag{21}$$

Proposition 3.11 *The impulse response function* $\mathcal{I}_{t,k}^{v_2 \to v_1}$ *is given by:*

$$\begin{aligned}\mathcal{I}_{t,k}^{v_2 \to v_1} &= e_{v_1}' \Psi^k \Big[\mathbb{E}\left(f_t \middle| \underline{f_{t-1}},\, e_{v_2}'[f_t - \mathbb{E}(f_t|\underline{f_{t-1}})] = s,\, \mathcal{E}_v'[f_t - \mathbb{E}(f_t|\underline{f_{t-1}})] = 0 \right) \\ &- \mathbb{E}\left(f_t \middle| \underline{f_{t-1}},\, e_{v_2}'[f_t - \mathbb{E}(f_t|\underline{f_{t-1}})] = 0,\, \mathcal{E}_v'[f_t - \mathbb{E}(f_t|\underline{f_{t-1}})] = 0 \right) \Big], \quad (22)\end{aligned}$$

which only requires filtered values of the factor f_t *given initial and observable conditions.*

Proof The semi-strong VAR form of Equation (14) directly gives the result. ■

Corollary 3.11.1 *Assuming conditions to identify* $\Sigma^{1/2}$ *(that is* $\Sigma_M^{1/2}$*) have been specified, the impulse response function of any variable* v_1 *to a "structural" macroeconomic shock on* $M_{j,t} = e_j' X_t$ *(with* $e_j = [0, \dots, 0, 1, 0 \dots, 0]'$ *of size* n*) is defined by* $e_{v_2} = (e_j', 0, \cdots, 0)'$*,* $\mathcal{E}_v$ *is a matrix of size* $((n + n^2 + 1) \times n)$ *selecting* Z_t *in* f_t*, such that the conditioning set is given by* $X_t = \mu + \Phi X_{t-1} + \Sigma^{1/2}(se_j)$*. The IRF writes:*

$$\mathcal{I}_{t,k}^{v_2 \to v_1} = e_{v_1}' \Psi^k \begin{bmatrix} \Sigma^{1/2} & 0 \\ \Gamma_{t-1}\Sigma^{1/2} & \Sigma^{1/2} \otimes \Sigma^{1/2} \\ \varsigma\theta' & \varsigma \mathrm{Vec}(\Theta)' \end{bmatrix} \begin{bmatrix} se_j \\ s^2 \mathrm{Vec}\left(e_j e_j'\right) \end{bmatrix}, \quad (23)$$

where $\Gamma_{t-1} = [I_n \otimes (\mu + \Phi X_{t-1}) + (\mu + \Phi X_{t-1}) \otimes I_n]$*.*

Proof Straightforward calculation of the quantities $\mathbb{E}\left(f_t \middle| \underline{f_{t-1}},\, X_t = \mu + \Phi X_{t-1} + \Sigma^{1/2}(se_j) \right)$ and $\mathbb{E}\left(f_t \middle| \underline{f_{t-1}},\, X_t = \mu + \Phi X_{t-1} \right)$. ■

Since the model is non-linear, the shape and amplitude of the IRFs depend on the initial condition f_{t-1} for any variable being a function of $\mathrm{Vec}(X_t X_t')$ or $\mathfrak{r}_t$. However, since Ψ is block lower-triangular and the first block is given by Φ (see Appendix A.3), the effect of a macroeconomic shock on another macroeconomic variable j' is given by the usual expression $e_{j'}' \Phi^k \Sigma^{1/2} s e_j$.

The average IRF can be computed in two different ways. First, we can apply Formula (22) to the initial condition $f_{t-1} = \left[\bar{X}', \mathrm{Vec}\left(\bar{X}\bar{X}'\right)', \bar{\mathfrak{r}} \right]'$, where $\bar{X} := \mathbb{E}(X_t)$ and $\bar{\mathfrak{r}} = \mathbb{E}(\mathfrak{r}_t)$.[17] Second, we can simulate many initial conditions f_{t-1} using its marginal distribution, compute the IRFs using Formula (22) for each initial condition, and average over the responses. The two approaches are not equivalent since they flip the order of integration (see for example Gallant et al. (1993) or Koop et al. (1996)).

[17] It is worth mentioning that the initial condition $f_{t-1} = [\bar{X}', \mathrm{Vec}(\bar{X}\bar{X}')', \varsigma(\theta_0 + \theta'\bar{X} + \bar{X}'\Theta\bar{X})]'$ is different from $f_{t-1} = \mathbb{E}(f_{t-1})$ since $\mathbb{E}[XX'] \neq \mathbb{E}(X)\mathbb{E}(X')$. However, once conditioning by $X_{t-1} = \bar{X}$, it follows directly that $\mathrm{Vec}(X_{t-1}X_{t-1}') = \mathrm{Vec}(\bar{X}\bar{X}')$ with probability one.

4 Estimation Strategy

This section deals with the estimation procedure for the model presented above. We present hereafter a general method that can encompass any number of latent and observable factors, as well as observable variables that are affine functions of f_t. We first provide the state-space representation of the model before describing the filtering algorithm allowing to estimate the parameters and evaluate the most probable values of the factors at the same time.

4.1 The state-space representation

Throughout this section, we consider measurement variables that are affine functions of the extended vector of factors f_t. Macroeconomic variables included in M_t are obviously contained in this set since they correspond to one component of X_t, hence of f_t. Building on the results of the previous section, the set of observable variables may also include, among others, nominal and real yields of zero-coupon bonds of any maturity (resp. $R(t,h)$ and $R_a^*(t,h)$), survey data on k period ahead expected future rates or macroeconomic variables (denoted by $S_t^{(k)}$), or the log-probability of being in ZLB for k periods, $R_{zlb}^{EH}(t,k)$. We gather all these observables at time t in a single vector denoted by Y_t.

$$Y_t = \left[R(t,\mathcal{H})'\,,\, R_a^*(t,\mathcal{H}^*)'\,,\, M_t'\,,\, S_t^{(\mathcal{K})'}\,,\, R_{zlb}^{EH}(t,\mathcal{K}_{zlb})'\right]'\,,$$

where $\mathcal{H}$ and $\mathcal{H}^*$ are respectively the set of nominal and real yields maturities, $\mathcal{K}$ is the horizon of the survey data, and $\mathcal{K}_{zlb}$ is the horizon of the ZLB log-probability. A standard assumption in factor models is that observable variables $Y_t \in \mathbb{R}^m$ are measured with i.i.d. Gaussian errors that we denote by $\eta_t \in \mathbb{R}^m$.

Putting together the transition equation of the model given by Equation (14) and our assumptions on the observable variables, we obtain a linear state-space model where the factors are conditionally non-Gaussian and heteroskedastic.

$$\begin{aligned} f_t &= \Psi_0 + \Psi f_{t-1} + \left[\mathrm{Vec}^{-1}(\Omega_0 + \Omega f_{t-1})\right]^{1/2} \xi_t \\ Y_t &=: \mathcal{A} + \mathcal{B}' f_t + \Sigma_\eta^{1/2} \eta_t\,, \end{aligned} \tag{24}$$

where $\mathcal{A} \in \mathbb{R}^m$ and $\mathcal{B} \in \mathbb{R}^{(n+n^2+1)\times m}$ stack respectively the intercepts and the loadings of the different observables, η_t stacks the measurement errors in a zero-mean unit-covariance matrix Gaussian vector, and $\Sigma_\eta^{1/2}$ is the matrix containing the measurement errors standard deviations. Some of these variables can be assumed to be measured without errors if the corresponding rows of $\Sigma_\eta^{1/2}$ are equal to zero.

4.2 The filtering method

Since some components of f_t are unobservable, we resort to filtering techniques to estimate the model and evaluate the factor values. The measurement equation (24) is an affine function of f_t, that is a linear-quadratic combination of X_t and a linear function of $\mathfrak{r}_t$. The Quadratic Kalman Filter (QKF) developed by Monfort et al. (2015) is particularly fitted to this class of models. The original filtering algorithm has been applied to state-space models where the transition dynamics are given by a Gaussian VAR and the measurement equations are linear-quadratic. This algorithm is slightly modified to incorporate $\mathfrak{r}_t$ (which is non-Gaussian) and is detailed below.

Since the state-space model expressed with respect to f_t is affine, we can apply the Kalman filter algorithm. Using the notations $f_{t|t-1} = \mathbb{E}(f_t|\underline{Y_{t-1}})$, $P_{t|t-1} = \mathbb{V}(f_t|\underline{Y_{t-1}})$, $f_{t|t} = \mathbb{E}(f_t|\underline{Y_t})$, $Y_{t|t-1} = \mathbb{E}(Y_t|\underline{Y_{t-1}})$, $M_{t|t-1} = \mathbb{V}(Y_t|\underline{Y_{t-1}})$, $P_{t|t} = \mathbb{V}(f_t|\underline{Y_t})$, the steps in the algorithm are the following. Initialize the filter at $f_{0|0} = \mathbb{E}(f_t)$ and $P_{0|0} = \mathbb{V}(f_t)$ (see Proposition 3.5). Then, for each period t, predict the latent:

$$\begin{aligned} f_{t|t-1} &= \Psi_0 + \Psi f_{t-1|t-1} \\ P_{t|t-1} &= \Psi P_{t-1|t-1}\Psi' + \mathrm{Vec}^{-1}(\Omega_0 + \Omega f_{t-1|t-1})\,, \end{aligned}$$

predict the observable:

$$\begin{aligned} Y_{t|t-1} &= \mathcal{A} + \mathcal{B}' f_{t|t-1} \\ M_{t|t-1} &= \mathcal{B}' P_{t|t-1}\mathcal{B} + \Sigma_\eta\,, \end{aligned}$$

update the prediction of the latent:

$$\begin{aligned} f_{t|t} &= f_{t|t-1} + P_{t|t-1}\mathcal{B}M_{t|t-1}^{-1}\left(Y_t - Y_{t|t-1}\right) \\ P_{t|t} &= P_{t|t-1} - P_{t|t-1}\mathcal{B}M_{t|t-1}^{-1}\mathcal{B}'P_{t|t-1}\,, \end{aligned}$$

and compute the quasi log-likelihood assuming that the conditional distribution of Y_t given $\underline{Y_{t-1}}$ is Gaussian with mean $Y_{t|t-1}$ and variance $M_{t|t-1}$.

$$\mathcal{L}_t = -\frac{1}{2}\left[m\log(2\pi) + \log|M_{t|t-1}| + (Y_t - Y_{t|t-1})'M_{t|t-1}^{-1}(Y_t - Y_{t|t-1})\right]\,.$$

In order to be consistent with the theoretical properties of the processes, two corrections are applied to the filtered values after storing the results. First, if the components of $\mathfrak{r}_{t|t}$ are negative, they are set to zero. Second, the filtered values of $\mathrm{Vec}(XX')_{t|t}$ are imposed to be exactly equal to $\mathrm{Vec}(X_{t|t}X'_{t|t})$.

As for the standard Kalman filter, the QKF provides a convenient way to handle missing data. One just has to adjust the size of the parameters in the measurement equations to predict only the variables that are observed. The measurement equation rewrites

$$Y_t^{(obs)} = E_t \left(\mathcal{A} + \mathcal{B}' f_t + \Sigma_\eta^{1/2} \eta_t \right) =: \mathcal{A}_t + \mathcal{B}_t' f_t + \Sigma_{\eta,t}^{1/2} \eta_t \,,$$

where $Y_t^{(obs)} \in \mathbb{R}^{m_t}$ is the subset of variables of Y_t that is observed, and E_t is a matrix selecting the corresponding rows. The prediction and update states remain the same using the adjusted parameters. The QKF finally provides a natural procedure to obtain the IRF once the model is estimated.

Proposition 4.1 *Let $\widetilde{\mathcal{B}} = (e_{v_2}, \mathcal{E}_v)$ as defined in Section 3.8. The computable version of the IRF of Equation (22) is given by:*

$$\mathcal{I}_{t,k}^{v_2 \to v_1} = e_{v_1}' \Psi^k \left[\text{Vec}^{-1}(\Omega_0 + \Omega f_{t-1}) \widetilde{\mathcal{B}} \left(\widetilde{\mathcal{B}}' \left[\text{Vec}^{-1}(\Omega_0 + \Omega f_{t-1}) \right] \widetilde{\mathcal{B}} \right)^{-1} \begin{pmatrix} s \\ \\ 0 \end{pmatrix} \right] . \tag{25}$$

Again, the terms in the bracket are slightly modified such that $\text{Vec}(XX')_{t|t} = \text{Vec}(X_{t|t} X_{t|t}')$ *and* $\mathfrak{r}_{t|t} \geqslant 0$.

Proof We consider that the initial conditions f_{t-1} and the shocks are known without errors, so $P_{t-1,t-1} = 0$ and $\Sigma_\eta = 0$ in this case. Replacing the unknown quantities in Equation (22) by the values given by the QKF, the result is immediately obtained. ■

5 Data and estimation constraints

5.1 The data

We consider monthly U.S. data from January 1990 to March 2015.[18] The starting date is determined to avoid issues related to the Volcker period. We first extract monthly nominal zero-coupon yields are from Gurkaynak et al. (2007) for maturities of 1, 2, 3, 5, 7, and 10 years. We add the one-month nominal interest rate series taken from Bloomberg.[19] Second, following Haubrich et al. (2012), we compute liquidity-adjusted synthetic yields for inflation-linked bonds using zero-coupon inflation swap rates obtained from Bloomberg for

[18] Considering monthly end-of-month data avoids issues related to the CPI interpolation for the computation of TIPS payoffs.

[19] The one-month rate is available under the ticker <*GB1M Index*>.

maturities of 1, 2, 3, 5, 7, and 10 years.[20] The synthetic TIPS yields are obtained as the difference between the nominal yields and the inflation swap rates at the same maturities.[21] Due to data limitations, the inflation-linked series start in July 2004. We also treat the months in the direct aftermath of Lehman failure – from September 2008 to February 2009 – as missing data since most movements on the TIPS interest rates during this period can likely be attributed to the large disruption of the inflation-indexed market (see for instance D'Amico et al. (2014)). As for the macroeconomic variables, we consider the year-on-year inflation rate at the monthly frequency, computed from the CPI-U series of the BLS database.[22] We follow Kim and Orphanides (2012) and Chernov and Mueller (2012) adding two sets of survey forecasts in the observable variables. We obtain series of expected average inflation over the next 1 and 10 years and nominal yields forecasts for the 10-year maturity, respectively 3-months and 1-year ahead from the Philadelphia Fed database. All these surveys are quarterly. Last, we gather data from the primary dealer survey conducted on by the New York Fed. We collect information starting in January 2011 concerning the probabilities of seeing no interest rate increase by the Fed between each date and one year ahead. Details on these computations are provided in Appendix A.6. Time series and standard descriptive statistics of interest rates and inflation are presented on Figure 1 and in Table 2. Survey and probability series are represented on Figure 4.

[Insert Figure 1 Table 2 about here.]

The nominal interest rates are very persistent at all maturities and are upward sloping with maturity on average, from 2.9% to 5.1%. Nominal interest rate standard deviations are slightly decreasing with maturity. Their time-series show a globally decreasing behavior up to the recent zero lower bound period where the one-month interest rate is virtually zero from mid-2009 on (see Figure 1). Real interest rates are also very persistent. They are upward sloping on average, starting with negative mean values at short maturities since they are

[20] Swap rates are available under the ticker $<USSWITx\ Crncy>$, where x stands for the maturity. Swap interest rates are not available as continuously compounded rates and must be transformed. The continuously compounded yield is obtained as: $\kappa(t,h) = \log(1+\tilde{\kappa}(t,h))$ where $\tilde{\kappa}(t,h)$ is the quoted swap rate on Bloomberg terminal.

[21] Christensen and Gillan (2012) note that though not free from liquidity risk, inflation swaps are less likely to be affected by liquidity issues compared to TIPS (see also Fleckenstein et al. (2014a)). For papers who focus on extracting the liquidity risk from TIPS data, see for instance Sack and Elasser (2004), Shen (2006), Gurkaynak et al. (2010), Grischenko and Huang (2013), Pflueger and Viceira (2013) or D'Amico et al. (2014). Fleckenstein et al. (2014b) note that the TIPS bonds were also subject to large mispricing during the crisis.

[22] To be consistent with the reference price index of the inflation-indexed securities, the realized inflation series is lagged of 3 months. We do not see this as a caveat since the information available at date t is closer to the realized inflation rate of the reference index rather than to the real-time realized inflation rate due to the publication lag of the different price indices. Taking the lagged inflation rate is hence more consistent with the information set available by the representative agent. For long enough maturities, this difference is likely to be negligible.

not constrained by the zero lower bound. Excluding the aftermath of the Lehman failure the mean real yield curve becomes more negative, from -0.22% at the 1-year to 0.86% at the 10-year maturity. The real interest rate standard deviations are lower than the nominal ones, but are also decreasing with maturity.

5.2 Identification and estimation constraints

We consider three latent yield factors Z_t ($n_Z = 3$). Hence X_t is a four-dimensional vector, and f_t is a vector of size 21. The identification of the factors and the physical dynamics is obtained with the sufficient conditions that $\mu_Z = 0$, that the bottom-right block of Φ, Φ_Z is lower triangular and that the scale parameter $\varsigma = 1$ (see Appendix B.2 for a proof). For parsimony reasons, several additional constraints are imposed on the parameters. First, $\Phi^{\mathbb{Q}}$ is imposed to have the same sparse structure as Φ. Second, we set the quadratic price of risk components to zero except the one associated with the inflation rate, so $\Lambda_{X,\Pi}$ is the only entry of Λ_X that is different from zero. For the short-term nominal interest rate physical dynamics, we impose that $\theta = 0$. While it would be possible to estimate the components of θ, this creates numerical issues whenever Θ is close to being semi positive-definite. By setting $\theta = 0$, we immediately obtain $\theta_0 = 0$ and ensure the non-negativity of the quadratic combination $\theta_0 + \theta' X_t + X_t' \Theta X_t$. The measurement errors are uncorrelated and have the same variance ($\Sigma_\eta = \sigma_\eta I_m$) and all the standard deviations of the survey measurement errors are calibrated to the average forecaster disagreement.[23] Last, to better pin down the physical parameters, we impose that the marginal mean of the short-rate is equal to its sample mean and that its autocorrelation is equal to 0.97.

We perform a first estimation with the previous constraints and set all non-significant parameters to zero for a second round. We obtain notably that the price of interest risk Λ_r is not significantly different from zero and that Z_1 is autonomous under the physical measure.

6 Estimation results and fitting properties

6.1 Parameter estimates and factor values

The estimated parameters are presented on Tables 3 and 4, gathering respectively the parameters for the joint dynamics of the factors X_t, and the parameters for the short-rate dynamics, the market prices of risk, and the measurement errors.

[23] In the data, only the interquartile range of forecasters answer is provided. Assuming the distribution among forecasters is Gaussian, in order to obtain a quantity comparable to a standard deviation, we divide the average interquartile range over the whole sample by $2 \times F^{-1}(0.75)$ where $F(\bullet)$ is the c.d.f of a normalized Gaussian distribution. Indeed, for any $\omega \sim \mathcal{N}(0, \sigma)$, if $F_\omega(\bullet)$ is the c.d.f of ω, we have $F_\omega^{-1}(0.75) - F_\omega^{-1}(0.25) = \sigma[F^{-1}(0.75) - F^{-1}(0.25)] = 2F^{-1}(0.75)\sigma$.

[Insert Tables 3 and 4 about here.]

All parameters are highly significant. Inflation and all yield factors are persistent, and Φ possesses diagonal terms comprised between 0.90 and 0.99 under the physical measure. Although Z_1 is autonomous under the physical measure, it is significantly caused by Z_3 under the risk-neutral one. In addition, Z_1 helps the pricing of both nominal and real rates since it appears significantly in the short-rate dynamics through the matrix Θ (see Table 4). The estimates for the short-rate dynamics show a persistence (or smoothing) parameter β of 0.85, but its persistence is also influenced by the persistence of the factors X_t. Last, the quadratic price of risk parameter $\Lambda_{X,\Pi}$ is significantly positive. The conditional variance of inflation is thus higher under the risk-neutral measure compared to the physical one – 0.55 against 0.14 – indicating investors fears for inflation variance.

[Insert Figure 2 and Figure 3 about here.]

Figure 2 presents the filtered factors. The first factor is the realized year-on-year inflation rate whereas the yield factors are evaluated by the filter so as to adjust to the observables. Figure 3 plots the normalized factor loadings of nominal and real interest rates with respect to maturity. The loadings of inflation are positive at the short end and slightly decreasing with maturity. Looking only at the linear loadings on the left and middle graphs of Figure 3, we observe that Z_1 has the same loadings as a level factor, Z_2 looks like a slope and Z_3 looks like a curvature factor for both the nominal and real yield curves (see Litterman and Scheinkman (1991)). However, the role of the factors is distorted by the quadratic loadings, which are dominant for Z_3. Such an interpretation of the factors is therefore difficult and can be inconsistent with their time series properties of Figure 2. For example, Z_1 resembles an economic activity indicator or a slope factor in terms of time-series.

6.2 Goodness of fit

Using the filtered factor series, it is easy to reconstruct the short-term interest rate r_t series along with its 95% confidence bounds (not presented here for the interest of space). This allows us to determine the starting date of the zero lower bound period as the first date when the lower confidence bound of $\mathfrak{r}_{t|t}$ reaches 0, that is from September 2008 on. Hence, every reference to the *zero lower bound period* is considered from September 2008 to the end of the estimation sample. We also reconstruct the rest of the nominal and the real yield curves and the filtered survey data, and compute the associated fitting errors. The SPF and primary dealer survey data and model-implied series are presented on Figure 4 and the RMSEs for the yield curves are shown in Table 5.

[Insert Table 5 and Figure 4 about here.]

The model is able to provide both a reasonable fit on the survey data, consistently with the fairly large forecasters disagreement, and an impressive fit on both the nominal and the real yield curve with only 3 unobservable factors. RMSEs range from 6.1bps to 9.3bps for nominal rates and from 8.4bps to 16.4bps for real rates (see Table 5). This is partly linked to the rich linear-quadratic model formulated in the previous section, which – as documented by Leippold and Wu (2007) – fits the data more efficiently than a pure linear model with the same number of factors.

[Insert Figure 5 about here.]

The model-implied marginal first-two moments of the observables are presented on Figure 5. The average term structure of nominal yields produced by the model is upward sloping and is slightly higher than the data counterpart, from 3.6% at one-year maturity to 5.9% at ten-year maturity, compared to [3.37%, 5.10%] for the data (see panel (a.1) of Figure 5). The model-implied mean of TIPS yields and breakeven inflation rates is higher than the data counterpart, mostly due to the short observation sample of these quantities. Once again, the estimates are economically significant and plausible. Panel (b) of Figure 5 performs the same comparison for the marginal volatility of the observables. For nominal yields, TIPS yields and breakeven inflation rates, the model-implied volatilities are higher than the empirical counterparts, emphasizing the high marginal variance implied by the high persistence and the decreasing trend of yields. Again, the model-implied volatility of TIPS is higher than the data estimates due to data limitations. Despite these differences, the estimates are of reasonable order of magnitude. This feature is often not reproduced by gamma-type models in which the marginal volatilities estimates tend to explode. In the end, our model seems to be able to produce reasonable time-series and moment properties of the nominal and real interest rate data.

6.3 The predictability of excess returns

A well-know limitation of models built on gamma-type processes is that they are usually unable to reproduce both moments of the interest rate data and to provide reasonable predicted excess return estimates (see e.g. Dai and Singleton (2002) or Backus et al. (2001)). We investigate the latter in this section.

The excess returns of any bond for k-holding periods can be defined as the return of a strategy consisting in buying the bond at time t and selling it at time $t+k$, minus the risk-less interest rate of maturity k. This k-period risk-less rate is equal to $R(t,k)$ in the nominal world and $R_a^*(t,k)$ in the real world.

Lemma 6.1 *Let us denote by $\mathcal{R}_{S,t+k}$ the nominal returns of a strategy S between t and $t+k$. The nominal and real excess returns of this strategy are respectively given by:*

$$\begin{aligned}\mathcal{XR}^{(N)}_{S,t+k} &= \mathcal{R}_{S,t+k} - R(t,k)\\ \mathcal{XR}^{(\pi)}_{S,t+k} &= \mathcal{XR}^{(N)}_{S,t+k} + R(t,k) - R^*_a(t,k) - \sum_{i=1}^{k} \pi_{t+i}\,,\end{aligned}$$

where $R(t,k) - R^*_a(t,k)$ is the so-called breakeven inflation rate of maturity k. The expected real excess returns are therefore equal to the expected nominal excess returns plus the k-period ahead inflation risk premium.

Proposition 6.1 *The k-period nominal excess returns of nominal bonds and real excess returns of TIPS are affine functions of $\underline{f_{t+k}}$ and are written:*

$$\mathcal{XR}^{(N)}_{R,t+k} = \frac{h-k}{k}\left[R(t,h) - R(t+k,h-k)\right] + R(t,h) - R(t,k) \tag{26}$$

$$\mathcal{XR}^{(\pi)}_{R^*_a,t+k} = \frac{h-k}{k}\left[R^*_a(t,h) - R^*_a(t+k,h-k)\right] + R^*_a(t,h) - R^*_a(t,k)\,. \tag{27}$$

The real excess returns of nominal bonds and nominal excess returns of TIPS can be easily obtained using Lemma 6.1.

Corollary 6.1.1 *The nominal and real expected excess returns of nominal bonds and TIPS at date t are affine functions of f_t computable in closed-form.*

Proof See Appendix B.3. ■

These excess returns computations can be used to test whether the model is able to reproduce the deviations from the expectation hypothesis consistently with the data, and whether the model-implied predictions of excess returns are reasonable. These two tests are respectively called LPY-I and LPY-II in the terminology of Dai and Singleton (2002). Both LPY-I and LPY-II reformulates the excess returns in the form of the well-known Campbell and Shiller (1991) regressions (CS henceforth).

Proposition 6.2 *Four versions of the CS regressions can be written with nominal bonds*

and TIPS. Denoting $\pi_{t,t+k}$ the cumulated inflation between t and $t+k$:[24]

$$R(t+k,h-k) - R(t,h) = \omega_{k,h} + \phi_{k,h}\frac{k\left(R(t,h) - R(t,k)\right)}{h-k} + \epsilon_{t+k,h} \tag{28}$$

$$R(t+k,h-k) - R(t,h) + \frac{\pi_{t,t+k}}{h-k} = \omega_{k,h} + \phi_{k,h}\frac{k\left(R(t,h) - R^*_a(t,k)\right)}{h-k} + \epsilon_{t+k,h} \tag{29}$$

$$R^*_a(t+k,h-k) - R^*_a(t,h) - \frac{\pi_{t,t+k}}{h-k} = \omega^*_{k,h} + \phi^*_{k,h}\frac{k\left(R^*_a(t,h) - R(t,k)\right)}{h-k} + \epsilon^*_{t+k,h} \tag{30}$$

$$R^*_a(t+k,h-k) - R^*_a(t,h) = \omega^*_{k,h} + \phi^*_{k,h}\frac{k\left(R^*_a(t,h) - R^*_a(t,k)\right)}{h-k} + \epsilon^*_{t+k,h}\,. \tag{31}$$

All model-implied intercepts and slopes $\omega_{k,h}$, $\omega^*_{k,h}$, $\phi_{k,h}$, and $\phi^*_{k,h}$ are computable in closed-form.

Proof Straightforward application of Corollary 6.1.1. See Appendix B.3 for the coefficients formulas. ■

The usual formulations (28) and (31) should be thought as representing nominal and real term premium only, whereas Equations (29) and (30) look at inflation risk premia jointly with nominal and real term premia respectively. If the expectation hypothesis was holding true, intercept and slopes would all be respectively equal to 0 and 1 and the corresponding excess return would average to zero. However, since the expectation hypothesis is largely violated in practice, the current slope of nominal/real interest rates can predict future excess returns. In practice, we consider $k = 12$ months. Testing LPY-I consists in estimating regressions (28-31) on the data for maturities ranging from 1 to 10 years, and comparing the estimated regression coefficients to the model-implied ones.[25] Testing LPY-II consists in performing the same regressions on the data adding the corresponding model-implied expected excess returns series on the right-hand side of the regression. Adding the expected excess return should in theory correct the deviations from the expectation hypothesis.[26] A consistent model should be able to produce $\phi_{k,h}$ coefficients non significantly different from 1. Results of these regressions are respectively provided on Figure 6 and 7.

[Insert Figures 6 and 7 about here.]

[24] Note that Haubrich et al. (2012) also perform a similar exercise but they do not get a formulation with the realized inflation on the left-hand side. In essence, they obtain regressions (28) and (31). Evans (1998) formulates a slightly different regression with the Equation (20) of his paper. He expresses the expectation hypothesis equating the expected nominal excess returns of TIPS with the expected nominal excess returns of nominal bonds. As such, his formulation can be thought as a combination of Equations (28) and (30).

[25] To obtain the yields of nominal bonds and TIPS at all maturities for the whole sample period, we use the model-implied yield series reconstructed from the filtered factors and omit the measurement errors.

[26] We add the series of expected excess returns to the regressor so that we still estimate one regression slope.

For all CS regressions testing LPY-I, the model-implied slopes lie inside the Newey-West 95% confidence bounds, the only exception being observed for real excess returns of nominal bonds for a few long maturities. For both nominal excess returns of nominal bonds and real excess returns of TIPS (top-left and bottom-right graphs of Figure 6), model-implied regression slopes are very close to those obtained with the data. we thereby provide evidence that the pricing of both inflation risk, nominal and real interest rates are consistent with the LPY-I condition. Focusing on the LPY-II results of Figure 7, we observe that the unit values lie inside the 95% Newey-West confidence intervals of the CS regressions, the only exception being again real excess returns of nominal bonds at the long-end.[27] Most slopes are statistically equal to one, indicating a strong capacity of the model to jointly reproduce the behavior of both inflation risk premia, real term premia, and nominal risk premia.

7 Inflation risk and the zero lower bound

In this section, we use the estimated model to explore the investors' views on inflation risks, especially during the ZLB and its relationship with the raise of interest rates. We begin by emphasizing the U-shaped form of the pricing kernel and present the risk premium associated with high inflation and deflation risks. We then distinguish the compensation for both inflation risk and for real interest risk in nominal rates and their time-series behavior. Last, we calculate the impact of lifting off at the zero lower bound in terms of the path of inflation, the path of interest rates, and the risk premium associated to that situation.

7.1 U-shaped pricing kernel and inflation fears

Kitsul and Wright (2013) document non-monotonic pricing kernels regarding inflation risk. In Section 3.3, we emphasized that our specification allows for a U-shaped nominal pricing kernel as a function of the Gaussian yield factors Z_t and of the macroeconomic variables M_t. In this section, we study what this specification implies for the pricing of inflation risks.

U-shaped pricing kernels essentially indicate investors double-sided fears of inflation outcomes, that is fears of both positive and negative deviations from expectations. To illustrate this effect, we use our model to compute the shape of the nominal pricing kernel for different factor values and horizons. Figure 8 presents these physical and risk-neutral inflation conditional pdfs for one-month, one-year and ten-year horizons.

[Insert Figure 8 about here.]

[27] Note that these results rely on the filtered values of the real interest rates on the whole sample, which gather uncertainty around their estimates. This analysis does not consider this source of uncertainty which would produce even larger confidence bands in the figures, potentially solving this inconsistency.

Panel (a) presents the conditional densities starting from $X_t = \mathbb{E}(X_t)$ whereas panel (b) starts from the average ZLB factor values $X_t = (T-\tau)^{-1}\sum_{i=\tau}^{T} X_i$, τ being the starting date of the ZLB. Even for the one-month horizon, the risk-neutral inflation densities exhibit different means and variances than their physical counterparts. This is usually absent from ATSMs with exponential-affine pricing kernels. As a result, the $\mathbb{Q}/\mathbb{P}$ density ratio – which is proportional to the (projected) nominal pricing kernel – has a U-shape in both panels.[28] From panel (a) to panel (b), the change of monotonicity varies as a function of the states. Increasing the horizon enlarges conditional variances of both distributions, resulting in flatter density ratios. This U-shaped pattern reflects investors fears of both low and high inflation for all horizons.

We can also observe this effect computing one-year ahead conditional probabilities of the year-on-year inflation rate being bigger than 4% (*High inflation*) and one-year ahead conditional deflation probabilities under both the physical and the risk-neutral measure. The discrepancy between risk-neutral and physical estimates provides the risk-premium associated with each of these events at every point in time. Results are provided on Figure 9.

[Insert Figure 9 about here.]

Given the rare occurrence of the year-on-year inflation going negative in our sample, the ex-ante conditional deflation physical probabilities are fairly low, from virtually zero to 8% at their highest just after the economy hits the ZLB. The risk-neutral deflation probabilities are more volatile, and consistently higher than their physical counterparts. They peak at several times during the estimation sample, rising to 35% in November 1998, 95% when the economy hits the ZLB, and to 44% in December 2014, corresponding to periods when the inflation rates went below 2%. These differences between risk-neutral and physical estimates reflect a high and variable deflation risk premium during the sample, reaching its highest during the ZLB period. On the right panel of Figure 9, we observe physical probabilities of high inflation that are globally decreasing during the estimation sample, following the downward trend of inflation expectations of the last 25 years. In 1990, these probabilities are as high as 60% but reach virtually zero during the ZLB emphasizing the low inflation expectations during the crisis. Again, the risk-neutral probabilities of high inflation are nearly always higher and more volatile than their physical counterparts, peaking during the 1990 recession and the burst of the *dotcom-bubble*. These differences also translate into significantly positive high inflation risk premia. These double-sided risk premia illustrate the effect of the U-shaped pricing kernel on inflation risk and the non-monotonicity of investors fears with respect to inflation.

[28] Since the ratio is only proportional to the nominal pricing kernel, their size should not be economically interpreted.

7.2 Inflation and real risk premia decomposition

Considering jointly inflation and real interest rate data allows to decompose nominal interest rates at each maturity in four different parts and investigate the evolution of inflation risks further: the expected compounded real interest rate, the expected inflation up to maturity, real risk premia and inflation risk premia.[29] Nominal bonds indeed contain inflation risk since the real return of nominal bonds decreases when inflation turns out to be higher than expected. The inflation premium associated with the latter event is positive (resp. negative) if investors require a higher return for providing insurance against positive (resp. negative) inflation shocks. We present the model-implied marginal decomposition and the time-series of the various components of the nominal interest rates on Figures 5 and 10 respectively.

[Insert Figure 10 about here.]

Consistently with the existing literature and the violations of the expectation hypothesis, the model-implied nominal risk premium is time-varying and upward sloping with maturity (see top-left graph of Figure 5). The 1-year nominal yield is very close to zero during the ZLB, as well as its risk premia component whereas most of the 10-year yield fluctuations during the ZLB are related to risk premia (see first row of Figure 10). The second row of Figure 10 shows significantly time-varying real term premia components both at the 1-year and at the 10-year maturity which are increasing with maturity on average. During the ZLB period, the expected real rates fall far into negative territory with minimum values of -2% and -1.3% for the 1- and 10-year maturities, producing negative real interest rates at the short-end. The real term premia is positive, reaching an all times high of 4% and 3% during the crisis for the 1- and 10-year maturities respectively. Again, the real term premium drives most fluctuations of the 10-year real interest rate during the ZLB.

The short- and long-term inflation risk premium is whipsawing around zero and changes sign during most of the sample. When entering the ZLB, the premium becomes largely negative reflecting immediate fears of a deflation spiral. At the 1-year maturity, inflation risk premia stay in the negative territory during the ZLB, between -430bps and small negative values. Thus, when the ZLB starts binding, investors require a premium of 4.3% to provide insurance against negative inflation shocks during the next year. Conversely, the 10-year inflation premia component comes back to close to zero values soon after October 2008 where

[29] The methodology is as follows. First, we obtain the nominal expected component calculating the pricing formulas for nominal bonds and imposing all prices of risk λ_0, λ and Λ_X to be equal to zero. The pricing formulas are easily obtained using Equation (17) and replacing the risk-neutral parameters by the physical ones. The nominal risk premium is the spread between the model-implied nominal yields and their expected components. The same method can be applied to get the real rates decomposition in expected real rates and real term premia. Last, expected inflation is given by the difference between nominal and real expected components, and inflation risk premia as the difference between nominal and real term premia.

an all times low of -91bps is observed. In comparison, the expected inflation component is slowly moving around 2.5% so most fluctuations in the inflation component comes from inflation risk premia, for every maturity. For the 10-year maturity, the inflation risk premia component is overall very small, fluctuating between -75bps and 75bps. This implies that long-term inflation expectations are well-anchored, and that there is low uncertainty around the level of the 10-year ahead compounded inflation rate. Economic agents are thus confident that the central bank will stabilize inflation in the long-run whereas short-term concerns produce sizable inflation risk premia at the one-year maturity.

In comparison with the literature, our long-term inflation risk premia estimates are broadly in line with those of Abrahams et al. (2016), Fleckenstein et al. (2014a) and D'Amico et al. (2014). Our short-term inflation premia estimates are however more volatile than the ones of the latter, which are extremely small in magnitude. Our estimates differ more largely from Haubrich et al. (2012). Their model imposes the risk premia estimates to be functions of the conditional volatility of interest rates only. This produces long-term real term premia and inflation risk premia that are roughly constant over time and consistently positive. They however do not explicitly include the zero lower bound constraint on nominal interest rates in the estimation, which can bias the expected component estimates of nominal yields.

7.3 The liftoff and the real economy

We use the model to explore the implications of the liftoff and its impact on the real economy and on the risk premia. We consider two exercises. We first compute impulse-response functions of monetary policy shocks and inflation shocks in and out of the ZLB. We compare the impact of an interest rate increase in normal times and during the ZLB with respect to inflation and the associated risk premia. Second, we calculate the conditional ZLB and liftoff probabilities and the associated risk premia.

In Section 3.8, we develop the methodology to perform impulse-response analysis. We apply this methodology in this section by studying the effects of an inflation shock and a monetary policy shock. The former is reflected by a shock on Π_t that does not impact Z_t contemporaneously, while the latter is a shock on the components of Z_t that does not impact Π_t contemporaneously.[30] The size of the inflation shock is calibrated to the conditional standard deviation of inflation and the size of the monetary policy shock is 10bps. We apply these shocks to two different initial conditions f_{t-1}. The first set of impulse-responses is computed at the "steady-state" $f_{t-1} = \left[\bar{X}, \text{Vec}\left(\bar{X}\bar{X}'\right)', \bar{\mathfrak{r}}\right]'$, $\bar{X} = \mathbb{E}(X_t)$ and $\bar{\mathfrak{r}} = \mathbb{E}(\mathfrak{r}_t)$ and

[30] We impose that the monetary policy shock is fully reflected by a shock in the latent yield variables Z_t. In the light of the decomposition performed in Equation (7), this implies that the value of v_t is imposed to zero. In the conditioning set of Equation (22), we therefore impose that $\Pi_t = \mathbb{E}_{t-1}(\Pi_t)$, $r_t = \mathbb{E}_{t-1}(r_t) + s$ and $c(Z_t) = \mathbb{E}_{t-1}(r_t) + s$, s being the size of the shock.

is presented on Figure 11. The second set of impulse-responses is computed at the ZLB, setting $f_{t-1} = \left[\bar{X}_{zlb}, \text{Vec}\left(\bar{X}_{zlb}\bar{X}'_{zlb}\right)', \bar{\mathfrak{r}}_{zlb}\right]'$, where $\bar{X}_{zlb} = (T-\tau)^{-1}\sum_{i=\tau}^{T} X_i$ and $\bar{\mathfrak{r}}_{zlb} = (T-\tau)^{-1}\sum_{i=\tau}^{T}\mathfrak{r}_i$, τ being the beginning date of the ZLB period and T being the final date of the sample. The results are presented on Figure 12.

[Insert Figures 11 and 12 about here.]

At the steady-state, the upward monetary policy shock is very persistent and the one-month interest rate is still 10bps above its steady-state value after 10 years. This results in an immediate and persistent increase of the 10-year nominal rate of about 20bps despite a decrease of the associated risk premia of about 10bps (see panels (a.1-2) of Figure 11). This monetary policy tightening has only a small negative effect of -5bps on the inflation rate after 12 months. This result is consistent with the IRFs obtained with FAVAR approaches of Bernanke et al. (2005) or Wu and Xia (2016), where the authors find that the effect of a monetary policy shock has a small or non-significant impact on the CPI index. Short-term and long-term inflation risk premia drop respectively by 30bps and 10bps, reflecting slightly higher deflation fears. On panel (b) of Figure 11, we observe that the effect of a 35bps inflation shock results in a 2bps to 7bps increase in the short-term nominal rate and a flattening of the yield curve due to a smaller increase of the long-term nominal rate. The effect on inflation risk premia is overall small and negative, and dies after 2 to 3 years. This shows that long-term inflation expectations are still well-anchored following an inflation shock.

The effects are completely different starting in the ZLB period. The initial 10bps monetary policy shock has a high instantaneous impact on the 10-year yield driven by an increase of nominal expectations of more than 100bps. The contraction of inflation after the monetary policy shock is about -22bps after one year, driving high fears of long-term deflation. The one-year inflation risk premium decreases by nearly 120bps and the ten-year by 40bps. The impact of lifting-off can hence be very detrimental with respect to stabilizing inflation and the associated premium. Looking at the panel (b) of Figure 12, we see that a 35bps inflation shock has virtually no effect on the nominal yield curve, consistently with the fact that the central bank is trying to restore stable long-term inflation and is stuck at the ZLB. These results emphasize the time-varying nature of responses to economic and financial shocks and the possible detrimental effects of lifting off on the real economy.

These time-varying effects of the liftoff on the real economy imply time-varying views on the liftoff for the investors. We ask how the liftoff is perceived by looking at the risk premium associated to such an event. On Figure 13 we plot the times series of one-year ahead physical and risk-neutral probabilities of being stuck at the ZLB (left column) and

the liftoff probabilities with respect to the horizon in April 2010 and January 2011 (right column). All quantities are presented with the associated risk premium.

[Insert Figure 13 about here.]

The ZLB physical probabilities reproduce the primary dealer survey data and the risk-neutral probabilities evolve mostly below their physical counterparts during the ZLB period. Except for a few months, this indicates a negative risk premium for staying at the ZLB for one year, so investors view staying at the ZLB for a year as a good outcome in terms of utility. Even when the physical probabilities of staying at the ZLB begin to decrease in 2014, the risk-neutral probabilities follow the same pattern and the risk premium increases while staying negative.

Last, we focus on the liftoff probabilities presented on the right panel of Figure 13. April 2010 is the date when one-year ZLB physical probabilities plunges before going back to close to one values. The most probable liftoff date is then on June 2010 with a 20% probability. However, the risk premium estimates show that lifting-off before January 2011 was perceived as a bad outcome, consistently with the negative risk premium associated with the one-year ahead ZLB probabilities at the same time. On January 2011, the most probable liftoff date has been pushed back to January 2012 with a probability of 6% only. Again, the risk premium associated with lifting-off before March 2012 is positive and perceived as a bad outcome from the investors point of view. Lifting-off too early would have represented a bad outcome because of its detrimental effect on the real economy and on the interest rates, while lifting-off too late seemed to be preferred.

8 Conclusion

In this paper, we provide a new way of modeling both nominal and real yield curves in an affine framework, which allows for the presence of observable macroeconomic variables and is consistent with the zero lower bound. Relying on a combination of quadratic term structure models and the gamma-zero distribution, the model is able to generate a short-term nominal rate stuck at the zero lower bound for several periods. We show that the short-term interest rate specification can have a convenient economic interpretation in terms of a time-varying Taylor-type rule. We show that the model is an ATSM such that it provides closed-form formulas for nominal and real interest rates, interest rate forecasts, macroeconomic forecasts, impulse-response functions, and liftoff probabilities under both physical and risk-neutral measure.

The relevance of this new framework is explored with an empirical application on U.S. data to study the interactions between inflation and the monetary policy in and out of the

zero lower bound. We first provide evidence that the model delivers a good fit in terms of moments, time series properties, and predictability of excess returns. Second, we explore its implications for risk premia estimates. We provide evidence that high inflation and deflation fears arise at the same time due to the particular U-shaped structure of the nominal pricing kernel. During the ZLB period, inflation risk premia become negative at the short-end of the yield curve, while staying closer to zero at longer horizons, emphasizing the horizon-dependent deflation risk aversion. We last study the effect and the cost of lifting-off. While an increase of the short-term interest rate has little impact on inflation during normal times, it severely and negatively impacts inflation views during the ZLB. This pushes the fear of lifting-off to change over time. As such, the model provides a convenient tool for policy-makers to monitor not only market views on the timing of interest rate increases, but also on the harm done to economic agents when the liftoff occurs.

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A Appendix

A.1 The Gamma-zero (γ_0) distribution

The gamma-zero autoregressive process was introduced by Monfort, Pegoraro, Renne, and Roussellet (2016) as a generalization of the autoregressive gamma process of Gouriéroux and Jasiak (2006). Its construction is summarized hereafter.

Let $\mathcal{I}_t = \mathcal{I}(X_t)$ be a non-negative process which is a function of the risk factors X_t and $\mathfrak{r}_t$, and P_t be a Poisson variable with intensity $\mathcal{I}_t$. $\mathfrak{r}_t$ is conditionally gamma-zero distributed if:

$$P_t|\underline{X_t}, \underline{\mathfrak{r}_{t-1}} \sim \mathcal{P}(\mathcal{I}(\underline{X_t}, \underline{\mathfrak{r}_{t-1}})) \quad \text{and} \quad \mathfrak{r}_t|P_t \sim \gamma_{P_t}(\varsigma) \,, \tag{32}$$

that is, conditionally on the Poisson mixing variable, $\mathfrak{r}_t$ has a gamma distribution of shape (or degree of freedom) parameter P_t and a scale parameter ς. When $P_t = 0$, the conditional distribution of $\mathfrak{r}_t$ converges to a Dirac point mass at zero. Integrating with respect to P_t, we obtain the conditional distribution of $\mathfrak{r}_t$ given X_t that we call gamma-zero, encompassing a zero point mass. In this paper, the intensity $\mathcal{I}_t$ is given by a quadratic combination of the Gaussian vector X_t, that is:

$$\mathcal{I}_t = \theta_0 + \theta' X_t + X_t' \Theta X_t + \beta \mathfrak{r}_{t-1} \,.$$

The conditional distribution of $\mathfrak{r}_t$ given X_t and its past can be expressed with its conditional Laplace transform:

$$\mathbb{E}\left[\exp(u_r \mathfrak{r}_t) \middle| \underline{X_t}, \underline{\mathfrak{r}_{t-1}}\right] = \exp\left(\frac{u_r \varsigma}{1 - u_r \varsigma}(\theta_0 + \theta' X_t + X_t' \Theta X_t + \beta \mathfrak{r}_{t-1})\right) \,, \tag{33}$$

which is an exponential-quadratic function of X_t and exponential-linear in $\mathfrak{r}_{t-1}$.

A.2 Affine $\mathbb{P}$-property and risk neutral dynamics of f_t

Define $u = [u_x', \mathrm{Vec}(U_x)', u_r']'$, where the blocks have respective size n, n^2 and 1. We first introduce the following Lemma.

Lemma A.1 *The conditional Laplace transform of* $[X_t', \mathrm{Vec}(X_t X_t')']'$ *given its past is given*

by:

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(u_x'X_t + X_t'U_xX_t\right)\left|\underline{X_{t-1}}\right.\right] \\
= \;& \exp\left\{u_x'(I_n - 2\Sigma U_x)^{-1}\left(\mu + \frac{1}{2}\Sigma u_x\right) + \mu' U_x (I_n - 2\Sigma U_x)^{-1}\mu - \frac{1}{2}\log\left|I_n - 2\Sigma U_x\right|\right. \\
+ \;& \left.(u_x + 2U_x\mu)'(I_n - 2\Sigma U_x)^{-1}\Phi X_{t-1} + X_{t-1}'\Phi' U_x (I_n - 2\Sigma U_x)^{-1}\Phi X_{t-1}\right\}
\end{aligned}$$

Proof See Cheng and Scaillet (2007). ■

Let us now calculate the conditional Laplace transform of f_t given $\underline{f_{t-1}}$.

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(u'f_t\right)\left|\underline{f_{t-1}}\right.\right] \\
= \;& \mathbb{E}\left[\exp\left(u_x'X_t + X_t'U_xX_t + u_r\mathfrak{r}_t\right)\left|\underline{f_{t-1}}\right.\right] \\
= \;& \mathbb{E}\left\{\mathbb{E}\left[\exp\left(u_x'X_t + X_t'U_xX_t + u_r\mathfrak{r}_t\right)\left|\underline{f_{t-1}}, X_t\right.\right]\left|\underline{f_{t-1}}\right.\right\} \\
= \;& \mathbb{E}\left[\exp\left\{u_x'X_t + X_t'U_xX_t + \frac{u_r\varsigma}{1-u_r\varsigma}\left[\theta_0 + \theta'X_t + X_t'\Theta X_t + \beta\mathfrak{r}_{t-1}\right]\right\}\left|\underline{f_{t-1}}\right.\right], \\
= \;& \exp\left(\frac{u_r\varsigma}{1-u_r\varsigma}(\theta_0 + \beta\mathfrak{r}_{t-1})\right)\mathbb{E}\left[\exp\left\{\left(u_x + \frac{u_r\varsigma}{1-u_r\varsigma}\theta\right)'X_t + X_t'\left(U_x + \frac{u_r\varsigma}{1-u_r\varsigma}\Theta\right)X_t\right\}\left|\underline{f_{t-1}}\right.\right],
\end{aligned}$$

which is obtained using the law of iterated expectations and the conditional Laplace transform of $\mathfrak{r}_t$ given X_t (see Equation (33)). We hence obtain the conditional Laplace transform of $[X_t', \mathrm{Vec}(X_tX_t')']'$ applied in the two arguments $\left[\left(u_x + \frac{u_r\varsigma}{1-u_r\varsigma}\theta\right)'; \mathrm{Vec}\left(U_x + \frac{u_r\varsigma}{1-u_r\varsigma}\Theta\right)'\right]'$. Using Lemma A.1, we have:

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(u'f_t\right)\left|\underline{f_{t-1}}\right.\right] \\
= \;& \exp\left\{\frac{u_r\varsigma}{1-u_r\varsigma}(\theta_0 + \beta\mathfrak{r}_{t-1}) + \left(u_x + \frac{u_r\varsigma}{1-u_r\varsigma}\theta\right)'\left[I_n - 2\Sigma\left(U_x + \frac{u_r\varsigma}{1-u_r\varsigma}\Theta\right)\right]^{-1}\left[\mu + \frac{1}{2}\Sigma\left(u_x + \frac{u_r\varsigma}{1-u_r\varsigma}\theta\right)\right]\right. \\
+ \;& \mu'\left(U_x + \frac{u_r\varsigma}{1-u_r\varsigma}\Theta\right)\left[I_n - 2\Sigma\left(U_x + \frac{u_r\varsigma}{1-u_r\varsigma}\Theta\right)\right]^{-1}\mu - \frac{1}{2}\log\left|I_n - 2\Sigma\left(U_x + \frac{u_r\varsigma}{1-u_r\varsigma}\Theta\right)\right| \\
+ \;& \left[\left(u_x + \frac{u_r\varsigma}{1-u_r\varsigma}\theta\right)' + 2\mu'\left(U_x + \frac{u_r\varsigma}{1-u_r\varsigma}\Theta\right)\right]\left[I_n - 2\Sigma\left(U_x + \frac{u_r\varsigma}{1-u_r\varsigma}\Theta\right)\right]^{-1}\Phi X_{t-1} \\
+ \;& \left.X_{t-1}'\Phi'\left(U_x + \frac{u_r\varsigma}{1-u_r\varsigma}\Theta\right)\left[I_n - 2\Sigma\left(U_x + \frac{u_r\varsigma}{1-u_r\varsigma}\Theta\right)\right]^{-1}\Phi X_{t-1}\right\}. \qquad (34)
\end{aligned}$$

This conditional Laplace transform is obviously an exponential-quadratic function of X_{t-1} and an exponential-linear function of $\mathfrak{r}_{t-1}$, that is by extension an exponential-affine function of f_{t-1}. (f_t) is therefore an affine process under the physical measure.

To derive the risk-neutral conditional Laplace transform of f_t given $\underline{f_{t-1}}$, we use the transition formulas provided in Roussellet (2015), Chapter 4. Using the block recursive affine structure of f_t, the risk-neutral conditional Laplace transform of r_t given X_t and $\underline{f_{t-1}}$ is given by:

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\left(\exp\{u_r'\mathfrak{r}_t\}\middle|X_t,\,\underline{f_{t-1}}\right) &= \frac{\mathbb{E}\left(\exp\left\{[u_r+\Lambda_r]'\,\mathfrak{r}_t\right\}\middle|X_t,\,\underline{f_{t-1}}\right)}{\mathbb{E}\left(\exp\left\{\Lambda_r'\mathfrak{r}_t\right\}\middle|X_t,\,\underline{f_{t-1}}\right)} \\ &= \exp\left\{\left(\frac{(u_r+\Lambda_r)\varsigma}{1-(u_r+\Lambda_r)\varsigma}-\frac{\Lambda_r\varsigma}{1-\Lambda_r\varsigma}\right)(\theta_0+\theta'X_t+X_t'\Theta X_t+\beta\mathfrak{r}_{t-1})\right\},\end{aligned} \tag{35}$$

where $\mathbb{E}^{\mathbb{Q}}(\cdot)$ is the expectation operator under the risk-neutral measure. The difference of ratios can be simplified as follows.

$$\begin{aligned}\frac{(u_r+\Lambda_r)\varsigma}{1-(u_r+\Lambda_r)\varsigma}-\frac{\Lambda_r\varsigma}{1-\Lambda_r\varsigma} &= \frac{(1-\Lambda_r\varsigma)\,(u_r+\Lambda_r)\varsigma-[1-(u_r+\Lambda_r)\varsigma]\,\Lambda_r\varsigma}{[1-\Lambda_r\varsigma]\,[1-(u_r+\Lambda_r)\varsigma]} \\ &= \varsigma\frac{u_r-\Lambda_r u_r\varsigma+u_r\Lambda_r\varsigma}{[1-\Lambda_r\varsigma]\,[1-(u_r+\Lambda_r)\varsigma]} \\ &= \frac{u_r\varsigma}{[1-\Lambda_r\varsigma]\,[1-(u_r+\Lambda_r)\varsigma]}.\end{aligned}$$

Define now $\varsigma^{\mathbb{Q}}=\dfrac{\varsigma}{1-\Lambda_r\varsigma}$, that is $\varsigma=\dfrac{\varsigma^{\mathbb{Q}}}{1+\Lambda_r\varsigma^{\mathbb{Q}}}$. We obtain:

$$\begin{aligned}\frac{u_r\varsigma}{1-(u_r+\Lambda_r)\varsigma} &= \frac{u_r\frac{\varsigma^{\mathbb{Q}}}{1+\Lambda_r\varsigma^{\mathbb{Q}}}}{1-(u_r+\Lambda_r)\frac{\varsigma^{\mathbb{Q}}}{1+\Lambda_r\varsigma^{\mathbb{Q}}}} \\ &= \frac{1+\Lambda_r\varsigma^{\mathbb{Q}}}{1-u_r\varsigma^{\mathbb{Q}}}\times\frac{u_r\varsigma^{\mathbb{Q}}}{1+\Lambda_r\varsigma^{\mathbb{Q}}} \\ &= \frac{u_r\varsigma^{\mathbb{Q}}}{1-u_r\varsigma^{\mathbb{Q}}}.\end{aligned}$$

Hence the conditional Laplace transform of Equation (35) is given by:

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\left(\exp\{u_r'\mathfrak{r}_t\}\middle|X_t,\,\underline{f_{t-1}}\right) &= \exp\left\{\frac{u_r\varsigma^{\mathbb{Q}}}{1-u_r\varsigma^{\mathbb{Q}}}\times\frac{\theta_0+\theta'X_t+X_t'\Theta X_t+\beta\mathfrak{r}_{t-1}}{1-\Lambda_r\varsigma}\right\} \\ &=: \exp\left\{\frac{u_r\varsigma^{\mathbb{Q}}}{1-u_r\varsigma^{\mathbb{Q}}}\left(\theta_0^{\mathbb{Q}}+\theta^{\mathbb{Q}'}X_t+X_t'\Theta^{\mathbb{Q}}X_t+\beta^{\mathbb{Q}}\mathfrak{r}_{t-1}\right)\right\}.\end{aligned}$$

$\mathfrak{r}_t$ is therefore conditionally gamma-zero distributed given X_t and its past, where the risk-neutral parameters are given by:

$$\theta_0^{\mathbb{Q}} = \frac{\theta_0}{1-\Lambda_r\varsigma},\quad \theta^{\mathbb{Q}}=\frac{\theta}{1-\Lambda_r\varsigma},\quad \Theta^{\mathbb{Q}}=\frac{\Theta}{1-\Lambda_r\varsigma},\quad \beta^{\mathbb{Q}}=\frac{\beta}{1-\Lambda_r\varsigma},\quad \varsigma^{\mathbb{Q}}=\frac{\varsigma}{1-\Lambda_r\varsigma}$$

We turn now to the computation of the risk-neutral conditional Laplace transform of $(X_t', \text{Vec}(X_t X_t')')'$ given $\underline{f_{t-1}}$. Again, using the property in Roussellet (2015) Chapter 4, we have:

$$\mathbb{E}^{\mathbb{Q}}\left(\exp\left\{u_x' X_t + X_t' U_x X_t\right\} \middle| \underline{f_{t-1}}\right) = \frac{\mathbb{E}\left[\exp\left\{(u_x + \widetilde{\Lambda}_{t-1})' X_t + X_t'(U_x + \widetilde{\Lambda}_r) X_t\right\} \middle| \underline{f_{t-1}}\right)}{\mathbb{E}\left[\exp\left\{\widetilde{\Lambda}_{t-1}' X_t + X_t'(U_x + \widetilde{\Lambda}_r) X_t\right\} \middle| \underline{f_{t-1}}\right)},$$

where $\widetilde{\Lambda}_{t-1}$ and $\widetilde{\Lambda}_r$ are given by:

$$\widetilde{\Lambda}_{t-1} = \lambda_0 + \theta \frac{\Lambda_r \varsigma}{1 - \Lambda_r \varsigma} + \lambda X_{t-1}, \quad \widetilde{\Lambda}_r = \frac{\Lambda_r \varsigma}{1 - \Lambda_r \varsigma} \Theta + \Lambda_X.$$

The transition between the physical and risk-neutral dynamics of X_t are as if the SDF was exponential-quadratic, with adjusted prices of risk $\widetilde{\Lambda}_{t-1}$ and $\widetilde{\Lambda}_r$. Since $\widetilde{\Lambda}_r$ the price associated to $\text{Vec}(X_t X_t')$ is constant through time, we can rely on the results of Monfort and Pegoraro (2012). We obtain that X_t follows a Gaussian VAR(1) under the risk-neutral measure and:

$$X_t = \mu^{\mathbb{Q}} + \Phi^{\mathbb{Q}} X_{t-1} + \Sigma^{\mathbb{Q}^{1/2}} \varepsilon_t^{\mathbb{Q}},$$

where $\varepsilon_t^{\mathbb{Q}}$ is a zero-mean unit-variance Gaussian white noise, and $\mu^{\mathbb{Q}}$, $\Phi^{\mathbb{Q}}$ and $\Sigma^{\mathbb{Q}}$ are given by:

$$\begin{cases} \mu^{\mathbb{Q}} = \Sigma^{\mathbb{Q}} \left(\lambda_0 + \theta \frac{\Lambda_r \varsigma}{1 - \Lambda_r \varsigma} + \Sigma^{-1} \mu\right), \\ \Phi^{\mathbb{Q}} = \Sigma^{\mathbb{Q}} \left(\lambda + \Sigma^{-1} \Phi\right), \\ \Sigma^{\mathbb{Q}} = \left(\Sigma^{-1} - 2 \frac{\Lambda_r \varsigma}{1 - \Lambda_r \varsigma} \Theta - 2 \Lambda_X\right)^{-1}. \end{cases}$$

Since $\mathfrak{r}_t$ is conditionally gamma-zero given X_t and that X_t follows a VAR(1) under the risk-neutral measure, the class of distributions are the same under the physical and the risk-neutral measure. Trivially transforming Formula 34, the risk-neutral Laplace transform of

f_t given $\underline{f_{t-1}}$ is given by:

$$
\begin{aligned}
& \mathbb{E}^{\mathbb{Q}}\left[\exp\left(u' f_t\right)\left|\underline{f_{t-1}}\right.\right] \\
= \; & \exp\left\{\frac{u_r \varsigma^{\mathbb{Q}}}{1-u_r\varsigma^{\mathbb{Q}}}\left(\theta_0^{\mathbb{Q}}+\beta^{\mathbb{Q}}\mathfrak{r}_{t-1}\right)-\frac{1}{2}\log\left|I_n-2\Sigma^{\mathbb{Q}}\left(U_x+\frac{u_r\varsigma^{\mathbb{Q}}}{1-u_r\varsigma^{\mathbb{Q}}}\Theta^{\mathbb{Q}}\right)\right|\right. \\
+ \; & \left(u_x+\frac{u_r\varsigma^{\mathbb{Q}}}{1-u_r\varsigma^{\mathbb{Q}}}\theta^{\mathbb{Q}}\right)'\left[I_n-2\Sigma^{\mathbb{Q}}\left(U_x+\frac{u_r\varsigma^{\mathbb{Q}}}{1-u_r\varsigma^{\mathbb{Q}}}\Theta^{\mathbb{Q}}\right)\right]^{-1}\left[\mu^{\mathbb{Q}}+\frac{1}{2}\Sigma^{\mathbb{Q}}\left(u_x+\frac{u_r\varsigma^{\mathbb{Q}}}{1-u_r\varsigma^{\mathbb{Q}}}\theta^{\mathbb{Q}}\right)\right] \\
+ \; & \mu^{\mathbb{Q}'}\left(U_x+\frac{u_r\varsigma^{\mathbb{Q}}}{1-u_r\varsigma^{\mathbb{Q}}}\Theta^{\mathbb{Q}}\right)\left[I_n-2\Sigma^{\mathbb{Q}}\left(U_x+\frac{u_r\varsigma^{\mathbb{Q}}}{1-u_r\varsigma^{\mathbb{Q}}}\Theta^{\mathbb{Q}}\right)\right]^{-1}\mu^{\mathbb{Q}} \\
+ \; & \left[\left(u_x+\frac{u_r\varsigma^{\mathbb{Q}}}{1-u_r\varsigma^{\mathbb{Q}}}\theta^{\mathbb{Q}}\right)'+2\mu^{\mathbb{Q}'}\left(U_x+\frac{u_r\varsigma^{\mathbb{Q}}}{1-u_r\varsigma^{\mathbb{Q}}}\Theta^{\mathbb{Q}}\right)\right]\left[I_n-2\Sigma^{\mathbb{Q}}\left(U_x+\frac{u_r\varsigma^{\mathbb{Q}}}{1-u_r\varsigma^{\mathbb{Q}}}\Theta^{\mathbb{Q}}\right)\right]^{-1}\Phi^{\mathbb{Q}}X_{t-1} \\
+ \; & \left. X_{t-1}'\Phi^{\mathbb{Q}'}\left(U_x+\frac{u_r\varsigma^{\mathbb{Q}}}{1-u_r\varsigma^{\mathbb{Q}}}\Theta^{\mathbb{Q}}\right)\left[I_n-2\Sigma^{\mathbb{Q}}\left(U_x+\frac{u_r\varsigma^{\mathbb{Q}}}{1-u_r\varsigma^{\mathbb{Q}}}\Theta^{\mathbb{Q}}\right)\right]^{-1}\Phi^{\mathbb{Q}}X_{t-1}\right\}. \qquad (36)
\end{aligned}
$$

This conditional Laplace transform is also exponential-quadratic in X_{t-1} and exponential-linear in $\mathfrak{r}_{t-1}$, that is an exponential-affine function of f_{t-1}. (f_t) is therefore an affine process under the risk-neutral measure.

A.3 Conditional moments of f_t

From Cheng and Scaillet (2007) and using the same notations as in Monfort, Renne, and Roussellet (2015), the conditional first two moments of $(X_t', \mathrm{Vec}(X_t X_t')')'$ given the past can be expressed as:

$$
\begin{aligned}
\mathbb{E}\left[\begin{pmatrix} X_t \\ \mathrm{Vec}(X_t X_t') \end{pmatrix}\left|\underline{f_{t-1}}\right.\right] &= \begin{pmatrix} \mu \\ \mathrm{Vec}(\mu\mu'+\Sigma) \end{pmatrix} + \left(\begin{array}{c|c} \Phi & 0 \\ \hline \mu\otimes\Phi+\Phi\otimes\mu & \Phi\otimes\Phi \end{array}\right)\begin{pmatrix} X_{t-1} \\ \mathrm{Vec}(X_{t-1}X_{t-1}') \end{pmatrix} \\
\mathbb{V}\left[\begin{pmatrix} X_t \\ \mathrm{Vec}(X_t X_t') \end{pmatrix}\left|\underline{f_{t-1}}\right.\right] &= \left(\begin{array}{c|c} \Sigma & \Sigma\Gamma_{t-1}' \\ \hline \Gamma_{t-1}\Sigma & \Gamma_{t-1}\Sigma\Gamma_{t-1}'+(I_{n^2}+K_n)(\Sigma\otimes\Sigma) \end{array}\right).
\end{aligned}
$$

where $\otimes$ is the standard Kronecker product, $\Gamma_{t-1}=[I_n\otimes(\mu+\Phi X_{t-1})+(\mu+\Phi X_{t-1})\otimes I_n]$, and K_n is the $(n^2\times n^2)$ commutation matrix.

Using the law of iterated expectations and the conditional first two moments of $\mathfrak{r}_t$ given X_t,

we have:

$$\begin{aligned}
\mathbb{E}\left[\mathfrak{r}_t\middle|\underline{f_{t-1}}\right] &= \varsigma\left(\theta_0+\theta'\mathbb{E}(X_t|\underline{f_{t-1}})+\text{Vec}(\Theta)'\mathbb{E}[\text{Vec}(X_tX_t')|\underline{f_{t-1}}]+\beta\mathfrak{r}_{t-1}\right)\\
&= \varsigma\Bigg(\theta_0+\beta\mathfrak{r}_{t-1}+\theta'\left(\mu+\Phi X_{t-1}\right)\\
&+ \text{Vec}(\Theta)'\left[\text{Vec}(\mu\mu'+\Sigma)+\left(\mu\otimes\Phi+\Phi\otimes\mu\right)X_{t-1}+(\Phi\otimes\Phi)\text{Vec}(X_{t-1}X_{t-1}')\right]\Bigg)\\
\mathbb{V}\left[\mathfrak{r}_t\middle|\underline{f_{t-1}}\right] &= \mathbb{E}\left(\mathbb{V}\left[\mathfrak{r}_t\middle|\underline{f_{t-1}},X_t\right]\middle|\underline{f_{t-1}}\right)+\mathbb{V}\left(\mathbb{E}\left[\mathfrak{r}_t\middle|\underline{f_{t-1}},X_t\right]\middle|\underline{f_{t-1}}\right)\\
&= 2\varsigma\mathbb{E}\left[\mathfrak{r}_t\middle|\underline{f_{t-1}}\right]+\mathbb{V}\left(\varsigma\left[\theta'X_t+X_t'\Theta X_t\right]\middle|\underline{f_{t-1}}\right)\\
&= 2\varsigma^2\Bigg(\theta_0+\beta\mathfrak{r}_{t-1}+\theta'\left(\mu+\Phi X_{t-1}\right)\\
&+ \text{Vec}(\Theta)'\left[\text{Vec}(\mu\mu'+\Sigma+\left(\mu\otimes\Phi+\Phi\otimes\mu\right)X_{t-1}+(\Phi\otimes\Phi)\text{Vec}(X_{t-1}X_{t-1}')\right]\Bigg)\\
&+ \varsigma^2\Bigg(\theta'\Sigma\theta+2\text{Vec}(\Theta)'\Gamma_{t-1}\Sigma\theta+\text{Vec}(\Theta)'\left[\Gamma_{t-1}\Sigma\Gamma_{t-1}'+(I_{n^2}+K_n)(\Sigma\otimes\Sigma)\right]\text{Vec}(\Theta)\Bigg).
\end{aligned}$$

The conditional covariance is given by:

$$\begin{aligned}
\mathbb{C}ov\left[\begin{pmatrix}X_t\\ \text{Vec}(X_tX_t')\end{pmatrix},\mathfrak{r}_t\middle|\underline{f_{t-1}}\right] &= \mathbb{C}ov\left[\begin{pmatrix}X_t\\ \text{Vec}(X_tX_t')\end{pmatrix},\varsigma\left(\theta_0+\theta'X_t+X_t'\Theta X_t+\beta\mathfrak{r}_{t-1}\right)\middle|\underline{f_{t-1}}\right]\\
&= \varsigma\begin{pmatrix}\Sigma\left[\theta+\Gamma_{t-1}'\text{Vec}(\Theta)\right]\\ \Gamma_{t-1}\Sigma\left[\theta+\Gamma_{t-1}'\text{Vec}(\Theta)\right]+(I_{n^2}+K_n)(\Sigma\otimes\Sigma)\text{Vec}(\Theta)\end{pmatrix}.
\end{aligned}$$

In the end, putting the previous results together, we obtain the transition equation in the form of Equation (14) with parameters given by:

$$\Psi_0=\begin{pmatrix}\mu\\ \text{Vec}(\mu\mu'+\Sigma)\\ \varsigma\left(\theta_0+\theta'\mu+\text{Vec}(\Theta)'\text{Vec}(\mu\mu'+\Sigma)\right)\end{pmatrix},$$

$$\Psi=\left(\begin{array}{c|c|c}\Phi & 0 & 0\\ \hline \mu\otimes\Phi+\Phi\otimes\mu & \Phi\otimes\Phi & 0\\ \hline \varsigma\left(\theta'\Phi+\text{Vec}(\Theta)'[\mu\otimes\Phi+\Phi\otimes\mu]\right) & \varsigma\left(\Phi\otimes\Phi\right) & \varsigma\beta\end{array}\right),$$

and,

$$\text{Vec}^{-1}(\Omega_0 + \Omega f_{t-1}) =$$

$$\left(\begin{array}{c|c|l} \Sigma & \Sigma\Gamma_{t-1}' & \varsigma\Sigma\left(\theta + \Gamma_{t-1}'\text{Vec}(\Theta)\right) \\ \hline & \Gamma_{t-1}\Sigma\Gamma_{t-1}' & \varsigma\Gamma_{t-1}\Sigma\left[\theta + \Gamma_{t-1}'\text{Vec}(\Theta)\right] \\ & +(I_{n^2} + K_n)(\Sigma\otimes\Sigma) & \quad + \quad \varsigma(I_{n^2} + K_n)(\Sigma\otimes\Sigma)\text{Vec}(\Theta) \\ \hline & & 2\varsigma^2\Big(\theta_0 + \beta\mathfrak{r}_{t-1} + \theta'\left(\mu + \Phi X_{t-1}\right) \\ & & +\text{Vec}(\Theta)'\left[\text{Vec}(\mu\mu' + \Sigma + (\mu\otimes\Phi + \Phi\otimes\mu)X_{t-1} + (\Phi\otimes\Phi)\text{Vec}(X_{t-1}X_{t-1}')\right]\Big) \\ & & +\varsigma^2\Big(\theta'\Sigma\theta + 2\text{Vec}(\Theta)'\Gamma_{t-1}\Sigma\theta + \text{Vec}(\Theta)'\left[\Gamma_{t-1}\Sigma\Gamma_{t-1}' + (I_{n^2} + K_n)(\Sigma\otimes\Sigma)\right]\text{Vec}(\Theta)\Big) \end{array}\right).$$

A.4 Forecasts and marginal moments with a semi-strong VAR formulation

From Equation (14), we have:

$$f_t = \Psi_0 + \Psi f_{t-1} + \left[\text{Vec}^{-1}\left(\Omega_0 + \Omega f_{t-1}\right)\right]^{1/2}\xi_t\,,$$

where ξ_t is a martingale difference with zero mean and unit variance. Assuming that f_t is stationary so that $(I_{n+n^2+1} - \Psi)^{-1}$ exists, we have:

$$\begin{aligned} \mathbb{E}\left(f_{t+k}|\underline{f_t}\right) &= \mathbb{E}\left(\Psi_0 + \Psi f_{t+k-1} + \left[\text{Vec}^{-1}\left(\Omega_0 + \Omega f_{t+k-1}\right)\right]^{1/2}\xi_{t+k}|\underline{f_t}\right) \\ &= \Psi_0 + \Psi\mathbb{E}\left(f_{t+k-1}|\underline{f_t}\right) \\ &= \sum_{i=0}^{k-1}\Psi^i\Psi_0 + \Psi^k f_t \end{aligned}$$

Replacing the sum by the following formula, we obtain the desired result:

$$\sum_{i=0}^{k-1}\Psi^i = \sum_{i=0}^{+\infty}\Psi^i - \sum_{i=k}^{+\infty}\Psi^i = \left(\sum_{i=0}^{+\infty}\Psi^i\right)\left(I_{n+n^2+1} - \Psi^k\right) = \left(I_{n+n^2+1} - \Psi\right)^{-1}\left(I_{n+n^2+1} - \Psi^k\right)\,.$$

Taking the limit when k tends to infinity, Ψ^k goes to zero and we obtain the marginal mean. For the conditional variance, we have:

$$\begin{aligned} \mathbb{V}\left(f_{t+k}|\underline{f_t}\right) &= \mathbb{V}\left[\mathbb{E}\left(f_{t+k}|\underline{f_{t+k-1}}\right)|\underline{f_t}\right] + \mathbb{E}\left[\mathbb{V}\left(f_{t+k}|\underline{f_{t+k-1}}\right)|\underline{f_t}\right] \\ &= \mathbb{V}\left(\Psi f_{t+k-1}|\underline{f_t}\right) + \mathbb{E}\left[\text{Vec}^{-1}\left(\Omega_0 + \Omega f_{t+k-1}\right)|\underline{f_t}\right] \\ &= \Psi\mathbb{V}\left(f_{t+k-1}|\underline{f_t}\right)\Psi' + \text{Vec}^{-1}\left[\Omega_0 + \Omega\mathbb{E}\left(f_{t+k-1}|\underline{f_t}\right)\right]\,. \end{aligned}$$

Therefore,

$$\begin{aligned}\mathrm{Vec}\left[\mathbb{V}\left(f_{t+k}|\underline{f_t}\right)\right] &= (\Psi\otimes\Psi)\mathrm{Vec}\left[\mathbb{V}\left(f_{t+k-1}|\underline{f_t}\right)\right]+\\ &\quad\left[\Omega_0+\Omega\left\{\left(I_{n+n^2+1}-\Psi\right)^{-1}\left(I_{n+n^2+1}-\Psi^{k-1}\right)\Psi_0+\Psi^{k-1}f_t\right\}\right].\end{aligned}$$

A simple recursion gives the desired result, that is:

$$\mathrm{Vec}\left[\mathbb{V}\left(f_{t+k}\middle|\underline{f_t}\right)\right]=\sum_{i=0}^{k-1}(\Psi\otimes\Psi)^i\left(\Omega_0+\Omega\left[(I_{n+n^2+1}-\Psi)^{-1}(I_{n+n^2+1}-\Psi^{k-i-1})\Psi_0+\Psi^{k-i-1}f_t\right]\right).$$

For the marginal variance, again using the law of total variance we have:

$$\begin{aligned}\mathbb{V}(f_t) &= \mathbb{V}\left[\mathbb{E}\left(f_t|\underline{f_{t-1}}\right)\right]+\mathbb{E}\left[\mathbb{V}\left(f_t|\underline{f_{t-1}}\right)\right]\\ &= \Psi\mathbb{V}(f_t)\Psi'+\mathrm{Vec}^{-1}\left[\Omega_0+\Omega\mathbb{E}(f_t)\right],\end{aligned}$$

and using the vec operator, we get the desired result.

A.5 Multi-horizon Laplace transform of f_t

Since the one-period ahead conditional risk-neutral Laplace transform of f_t given $\underline{f_{t-1}}$ is exponential-affine in f_{t-1}, it is well-known that the conditional multi-horizon risk-neutral Laplace transform of $(f_t,\ldots,f_{t+k})$ is also exponential-affine in f_{t-1} (see e.g. Darolles, Gourieroux, and Jasiak (2006)). Using the notation:

$$\mathbb{E}^{\mathbb{Q}}\left[\exp(u'f_t)\left|\underline{f_{t-1}}\right.\right]=:\exp\left\{\mathbb{A}^{\mathbb{Q}}(u)+\mathbb{B}^{\mathbb{Q}'}(u)X_{t-1}+X_{t-1}'\mathbb{C}^{\mathbb{Q}}(u)X_{t-1}+\mathbb{D}^{\mathbb{Q}}(u)\mathfrak{r}_{t-1}\right\},$$

with:

$$\begin{aligned}\mathbb{A}^{\mathbb{Q}}(u) &= \frac{u_r\varsigma^{\mathbb{Q}}\theta_0^{\mathbb{Q}}}{1-u_r\varsigma^{\mathbb{Q}}}-\frac{1}{2}\log\left|I_n-2\Sigma^{\mathbb{Q}}\left(U_x+\frac{u_r\varsigma^{\mathbb{Q}}}{1-u_r\varsigma^{\mathbb{Q}}}\Theta^{\mathbb{Q}}\right)\right|\\ &+ \left(u_x+\frac{u_r\varsigma^{\mathbb{Q}}}{1-u_r\varsigma^{\mathbb{Q}}}\theta^{\mathbb{Q}}\right)'\left[I_n-2\Sigma^{\mathbb{Q}}\left(U_x+\frac{u_r\varsigma^{\mathbb{Q}}}{1-u_r\varsigma^{\mathbb{Q}}}\Theta^{\mathbb{Q}}\right)\right]^{-1}\left[\mu^{\mathbb{Q}}+\frac{1}{2}\Sigma^{\mathbb{Q}}\left(u_x+\frac{u_r\varsigma^{\mathbb{Q}}}{1-u_r\varsigma^{\mathbb{Q}}}\theta^{\mathbb{Q}}\right)\right]\\ &+ \mu^{\mathbb{Q}'}\left(U_x+\frac{u_r\varsigma^{\mathbb{Q}}}{1-u_r\varsigma^{\mathbb{Q}}}\Theta^{\mathbb{Q}}\right)\left[I_n-2\Sigma^{\mathbb{Q}}\left(U_x+\frac{u_r\varsigma^{\mathbb{Q}}}{1-u_r\varsigma^{\mathbb{Q}}}\Theta^{\mathbb{Q}}\right)\right]^{-1}\mu^{\mathbb{Q}}\\ \mathbb{B}^{\mathbb{Q}'}(u) &= \left[\left(u_x+\frac{u_r\varsigma^{\mathbb{Q}}}{1-u_r\varsigma^{\mathbb{Q}}}\theta^{\mathbb{Q}}\right)'+2\mu^{\mathbb{Q}'}\left(U_x+\frac{u_r\varsigma^{\mathbb{Q}}}{1-u_r\varsigma^{\mathbb{Q}}}\Theta^{\mathbb{Q}}\right)\right]\left[I_n-2\Sigma^{\mathbb{Q}}\left(U_x+\frac{u_r\varsigma^{\mathbb{Q}}}{1-u_r\varsigma^{\mathbb{Q}}}\Theta^{\mathbb{Q}}\right)\right]^{-1}\Phi^{\mathbb{Q}}\\ \mathbb{C}^{\mathbb{Q}}(u) &= \Phi^{\mathbb{Q}'}\left(U_x+\frac{u_r\varsigma^{\mathbb{Q}}}{1-u_r\varsigma^{\mathbb{Q}}}\Theta^{\mathbb{Q}}\right)\left[I_n-2\Sigma^{\mathbb{Q}}\left(U_x+\frac{u_r\varsigma^{\mathbb{Q}}}{1-u_r\varsigma^{\mathbb{Q}}}\Theta^{\mathbb{Q}}\right)\right]^{-1}\Phi^{\mathbb{Q}}\\ \mathbb{D}^{\mathbb{Q}}(u) &= \frac{u_r\varsigma^{\mathbb{Q}}}{1-u_r\varsigma^{\mathbb{Q}}}\beta^{\mathbb{Q}},\end{aligned}$$

We obtain:

$$\begin{aligned}\varphi^{\mathbb{Q}}_{t-1}(u_0,\ldots,u_k) &= \mathbb{E}^{\mathbb{Q}}\left[\exp\left(\sum_{i=0}^{k} u_i' f_{t+i}\right)\Big|\underline{f_{t-1}}\right]\\ &= \exp\Bigg(\mathbb{A}^{\mathbb{Q}}_k(u_0,\ldots,u_k)+\mathbb{B}^{\mathbb{Q}'}_k(u_0,\ldots,u_k)X_{t-1}\\ &\quad + X'_{t-1}\mathbb{C}^{\mathbb{Q}}_k(u_0,\ldots,u_k)\,X_{t-1}+\mathbb{D}^{\mathbb{Q}}_k(u_0,\ldots,u_k)\,\mathfrak{r}_{t-1}\Bigg),\end{aligned}$$

where:

$$\begin{aligned}\mathbb{A}^{\mathbb{Q}}_k(u_0,\ldots,u_k) &:= \mathbb{A}^{\mathbb{Q}}_{k,k}(u_0,\ldots,u_k)\\ \mathbb{B}^{\mathbb{Q}}_k(u_0,\ldots,u_k) &:= \mathbb{B}^{\mathbb{Q}}_{k,k}(u_0,\ldots,u_k)\\ \mathbb{C}^{\mathbb{Q}}_k(u_0,\ldots,u_k) &:= \mathbb{C}^{\mathbb{Q}}_{k,k}(u_0,\ldots,u_k)\\ \mathbb{D}^{\mathbb{Q}}_k(u_0,\ldots,u_k) &:= \mathbb{D}^{\mathbb{Q}}_{k,k}(u_0,\ldots,u_k),\end{aligned}$$

with initial conditions $\mathbb{A}^{\mathbb{Q}}_{k,1}(u_0,\ldots,u_k)=\mathbb{A}^{\mathbb{Q}}(u_k)$, $\mathbb{B}^{\mathbb{Q}}_{k,1}(u_0,\ldots,u_k)=\mathbb{B}^{\mathbb{Q}}(u_k)$, $\mathbb{C}^{\mathbb{Q}}_{k,1}(u_0,\ldots,u_k)=\mathbb{C}^{\mathbb{Q}}(u_k)$ and $\mathbb{D}^{\mathbb{Q}}_{k,1}(u_0,\ldots,u_k)=\mathbb{D}^{\mathbb{Q}}(u_k)$, and $\forall i\in\{2,\ldots,k\}$,

$$\begin{aligned}\mathbb{A}^{\mathbb{Q}}_{k,i}(u_0,\ldots,u_k) &= \mathbb{A}^{\mathbb{Q}}_{k,i-1}(u_0,\ldots,u_k)\\ &\quad + \mathbb{A}^{\mathbb{Q}}\left(u_{k-i+1}+\left[\mathbb{B}^{\mathbb{Q}'}_{k,i-1}(u_0,\ldots,u_k),\mathrm{Vec}\left(\mathbb{C}^{\mathbb{Q}}_{k,i-1}(u_0,\ldots,u_k)\right)',\mathbb{D}^{\mathbb{Q}}_{k,i-1}(u_0,\ldots,u_k)\right]'\right)\\ \mathbb{B}^{\mathbb{Q}}_{k,i}(u_0,\ldots,u_k) &= \mathbb{B}^{\mathbb{Q}}\left(u_{k-i+1}+\left[\mathbb{B}^{\mathbb{Q}'}_{k,i-1}(u_0,\ldots,u_k),\mathrm{Vec}\left(\mathbb{C}^{\mathbb{Q}}_{k,i-1}(u_0,\ldots,u_k)\right)',\mathbb{D}^{\mathbb{Q}}_{k,i-1}(u_0,\ldots,u_k)\right]'\right)\\ \mathbb{C}^{\mathbb{Q}}_{k,i}(u_0,\ldots,u_k) &= \mathbb{C}^{\mathbb{Q}}\left(u_{k-i+1}+\left[\mathbb{B}^{\mathbb{Q}'}_{k,i-1}(u_0,\ldots,u_k),\mathrm{Vec}\left(\mathbb{C}^{\mathbb{Q}}_{k,i-1}(u_0,\ldots,u_k)\right)',\mathbb{D}^{\mathbb{Q}}_{k,i-1}(u_0,\ldots,u_k)\right]'\right)\\ \mathbb{D}^{\mathbb{Q}}_{k,i}(u_0,\ldots,u_k) &= \mathbb{D}^{\mathbb{Q}}\left(u_{k-i+1}+\left[\mathbb{B}^{\mathbb{Q}'}_{k,i-1}(u_0,\ldots,u_k),\mathrm{Vec}\left(\mathbb{C}^{\mathbb{Q}}_{k,i-1}(u_0,\ldots,u_k)\right)',\mathbb{D}^{\mathbb{Q}}_{k,i-1}(u_0,\ldots,u_k)\right]'\right).\end{aligned}$$

Since the conditional Laplace transform of f_t given $\underline{f_{t-1}}$ under the physical measure $\phi_{t-1}(u)$ is the same function as the risk-neutral one $\phi^{\mathbb{Q}}_{t-1}(u)$, but plugging in the physical parameters instead of the risk-neutral ones, we easily obtain:

$$\begin{aligned}&\varphi_{t-1}(u_0,\ldots,u_k)=\mathbb{E}\left[\exp\left(\sum_{i=0}^{k}u_i'f_{t+i}\right)\Big|\underline{f_{t-1}}\right]\\ =:\ &\exp\left(\mathbb{A}_k(u_0,\ldots,u_k)+\mathbb{B}'_k(u_0,\ldots,u_k)X_{t-1}+X'_{t-1}\mathbb{C}_k(u_0,\ldots,u_k)\,X_{t-1}+\mathbb{D}_k(u_0,\ldots,u_k)\mathfrak{r}_{t-1}\right)\end{aligned}\tag{37}$$

where:

$$\begin{aligned}
\mathbb{A}_k(u_0,\ldots,u_k) &:= \mathbb{A}_{k,k}(u_0,\ldots,u_k)\\
\mathbb{B}_k(u_0,\ldots,u_k) &:= \mathbb{B}_{k,k}(u_0,\ldots,u_k)\\
\mathbb{C}_k(u_0,\ldots,u_k) &:= \mathbb{C}_{k,k}(u_0,\ldots,u_k)\\
\mathbb{D}_k(u_0,\ldots,u_k) &:= \mathbb{D}_{k,k}(u_0,\ldots,u_k),
\end{aligned}$$

with initial conditions $\mathbb{A}_{k,1}(u_0,\ldots,u_k)=\mathbb{A}(u_k)$, $\mathbb{B}_{k,1}(u_0,\ldots,u_k)=\mathbb{B}(u_k)$, $\mathbb{C}_{k,1}(u_0,\ldots,u_k)=\mathbb{C}(u_k)$ and $\mathbb{D}_{k,1}(u_0,\ldots,u_k)=\mathbb{D}(u_k)$, and $\forall i\in\{2,\ldots,k\}$,

$$\begin{aligned}
\mathbb{A}_{k,i}(u_0,\ldots,u_k) &= \mathbb{A}_{k,i-1}(u_0,\ldots,u_k)\\
&+ \mathbb{A}\left(u_{k-i+1}+\left[\mathbb{B}'_{k,i-1}(u_0,\ldots,u_k),\mathrm{Vec}\left(\mathbb{C}_{k,i-1}(u_0,\ldots,u_k)\right)',\mathbb{D}_{k,i-1}(u_0,\ldots,u_k)\right]'\right)\\
\mathbb{B}_{k,i}(u_0,\ldots,u_k) &= \mathbb{B}\left(u_{k-i+1}+\left[\mathbb{B}'_{k,i-1}(u_0,\ldots,u_k),\mathrm{Vec}\left(\mathbb{C}_{k,i-1}(u_0,\ldots,u_k)\right)',\mathbb{D}_{k,i-1}(u_0,\ldots,u_k)\right]'\right)\\
\mathbb{C}_{k,i}(u_0,\ldots,u_k) &= \mathbb{C}\left(u_{k-i+1}+\left[\mathbb{B}'_{k,i-1}(u_0,\ldots,u_k),\mathrm{Vec}\left(\mathbb{C}_{k,i-1}(u_0,\ldots,u_k)\right)',\mathbb{D}_{k,i-1}(u_0,\ldots,u_k)\right]'\right)\\
\mathbb{D}_{k,i}(u_0,\ldots,u_k) &= \mathbb{C}\left(u_{k-i+1}+\left[\mathbb{B}'_{k,i-1}(u_0,\ldots,u_k),\mathrm{Vec}\left(\mathbb{C}_{k,i-1}(u_0,\ldots,u_k)\right)',\mathbb{D}_{k,i-1}(u_0,\ldots,u_k)\right]'\right).
\end{aligned}$$

Using the properties of Monfort et al. (2016), the probabilities to stay at zero for k periods are given by the following limit:

$$\begin{aligned}
\mathbb{P}\left(r_{t+1:t+k}=\delta_0\middle|\underline{f}_t\right) &= \mathbb{P}\left(\mathfrak{r}_{t+1:t+k}=0\middle|\underline{f}_t\right)=\lim_{v\to-\infty}\varphi_t(u_k,\cdots,u_k), \quad u_k=(0,\cdots,0,v)\\
\mathbb{Q}\left(r_{t+1:t+k}=\delta_0\middle|\underline{f}_t\right) &= \mathbb{Q}\left(\mathfrak{r}_{t+1:t+k}=0\middle|\underline{f}_t\right)=\lim_{v\to-\infty}\varphi_t^{\mathbb{Q}}(u_k,\cdots,u_k), \quad u_k=(0,\cdots,0,v).
\end{aligned}$$

Using a continuity argument, we flip the limit and the exponential such that:

$$\begin{aligned}
\mathbb{P}\left(r_{t+1:t+k}=\delta_0\middle|\underline{f}_t\right) &= \exp\Bigg(\lim_{v\to-\infty}\mathbb{A}_k(u_k,\ldots,u_k)+\lim_{v\to-\infty}\mathbb{B}'_k(u_k,\ldots,u_k)X_{t-1}\\
&+ X'_{t-1}\left[\lim_{v\to-\infty}\mathbb{C}_k(u_k,\ldots,u_k)\right]X_{t-1}+\lim_{v\to-\infty}\mathbb{D}_k(u_k,\ldots,u_k)\,\mathfrak{r}_{t-1}\Bigg)\\
\mathbb{Q}\left(r_{t+1:t+k}=\delta_0\middle|\underline{f}_t\right) &= \exp\Bigg(\lim_{v\to-\infty}\mathbb{A}^{\mathbb{Q}}_k(u_k,\ldots,u_k)+\lim_{v\to-\infty}\mathbb{B}^{\mathbb{Q}'}_k(u_k,\ldots,u_k)X_{t-1}\\
&+ X'_{t-1}\left[\lim_{v\to-\infty}\mathbb{C}^{\mathbb{Q}}_k(u_k,\ldots,u_k)\right]X_{t-1}+\lim_{v\to-\infty}\mathbb{D}^{\mathbb{Q}}_k(u_k,\ldots,u_k)\,\mathfrak{r}_{t-1}\Bigg),
\end{aligned}$$

for $u_k = (0, \cdots, 0, v)'$. We obtain:

$$\begin{aligned}
\mathcal{D}_{0,k} &= \lim_{v \to -\infty} \mathbb{A}_k(u_k, \ldots, u_k) \\
\mathcal{D}_k &= \lim_{v \to -\infty} \left[\mathbb{B}_k'(u_k, \ldots, u_k), \operatorname{Vec}\left(\mathbb{C}_k(u_k, \ldots, u_k)\right)', \mathbb{D}_k(u_k, \ldots, u_k)\right]' \\
\mathcal{D}_{0,k}^{\mathbb{Q}} &= \lim_{v \to -\infty} \mathbb{A}^{\mathbb{Q}}(u_k, \ldots, u_k) \\
\mathcal{D}_k &= \lim_{v \to -\infty} \left[\mathbb{B}_k^{\mathbb{Q}'}(u_k, \ldots, u_k), \operatorname{Vec}\left(\mathbb{C}_k^{\mathbb{Q}}(u_k, \ldots, u_k)\right)', \mathbb{D}_k^{\mathbb{Q}}(u_k, \ldots, u_k)\right]' .
\end{aligned}$$

A.6 Primary Dealer Survey data

The primary dealer surveys (PDS) are publicly available from January 2011 on. They are conducted by the New York Fed to inform the FOMC members of primary dealer's expectation about the economy, monetary policy and financial markets developments. They are conducted on a regular basis, prior to the FOMC meetings (8 per year) in January, March, April, June, July, September, October and December.[31] The questions and statistics collected have evolved to adapt to the economic environment, which makes it difficult to create homogeneous time-series on the probability to stay at the zero lower bound for a year.

We construct the conditional probabilities of staying at zero for a year using the question: *Of the possible outcomes below, please indicate the percent chance you attach to the timing of the first federal funds target rate increase.* (question #2 of each survey). The answer takes the form of a table associating the average of all participant answers per horizon. Table 1 provides two examples.

As can be seen on Table 1, the horizons of the question can be for next quarter or next semester. For all time periods where the horizons are quarterly or below, we aggregate the answers to get semi-annual horizons for homogeneity. We then compute the probabilities as follow. Let $\mathcal{M}_t = \{1, \ldots, 12\}$ be the number of the current date-t month, $\mathcal{Y}_t$ the number of date-t year, and $\mathcal{H}_t = 1 + \mathbb{1}\{\mathcal{M}_t \in \{7, \ldots, 12\}\}$ the indicator of the semester. Let $p_t(\mathcal{H}_t, \mathcal{Y}_t)$ be the answer given in the survey. Our probabilities are given by:

$$\begin{aligned}
&\left[1 - p_t\left(\mathcal{H}_t, \mathcal{Y}_t\right)\right] \times \left[1 - p_t\left(\mathcal{H}_t + 1, \mathcal{Y}_t\right)\right] \times \left[1 - p_t\left(\mathcal{H}_t, \mathcal{Y}_t + 1\right) \tfrac{\mathcal{M}_t - 1}{12}\right] \qquad \text{if} \quad \mathcal{H}_t = 1 \\
&\left[1 - p_t\left(\mathcal{H}_t, \mathcal{Y}_t\right)\right] \times \left[1 - p_t\left(\mathcal{H}_t - 1, \mathcal{Y}_t + 1\right)\right] \times \left[1 - p_t\left(\mathcal{H}_t, \mathcal{Y}_t + 1\right) \tfrac{\mathcal{M}_t - 1}{12}\right] \quad \text{if} \quad \mathcal{H}_t = 2 .
\end{aligned}$$

The previous formula assumes that inside the last semester considered, the timing of the first increase is uniformly distributed. For example, for the two panels of Table 1, we obtain

[31] See the survey results on https://www.newyorkfed.org/markets/primarydealer_survey_questions.html.

Table 1: Examples of primary dealer survey answers

Panel(a): January 2011

	2011				2012				$\geqslant$ 2013
	Q1	Q2	Q3	Q4	Q1	Q2	Q3	Q4	Q1
Average	0%	1%	2%	11%	14%	13%	16%	17%	25%

Panel(b): March 2013

	2013		2014		2015		2016		2017	
	H1	H2	H1	H2	H1	H2	H1	H2	H1	$\geqslant$H2
Average	0%	1%	5%	10%	23%	27%	18%	8%	4%	4%

the probabilities:

$$\begin{aligned} 2011-01 &\rightarrow [1-(0+0.01)]\times[1-(0.02+0.11)] \simeq 0.86 \\ 2013-03 &\rightarrow (1-0)\times(1-0.01)\times\left(1-\frac{3-1}{12}0.05\right) \simeq 0.965\,. \end{aligned}$$

Last, in order to avoid the fitted series to be too volatile, we fill out the missing data with the last available data point (step function) and impose that the measurement errors standard deviation is equal to 15% of the obtained series standard deviation.

B Technical Appendix

B.1 Including longer-period price variations in M_t

The class of models we consider is models where the inflation rate between $t-k$ and t, denoted by $\pi_{t-k,t}$, is directly included as the first macroeconomic variable, that is the first component of M_t. For notation simplicity let us also assume that there is no other macroeconomic variable, that is $M_t = \pi_{t-k,t}$. By definition we have:

$$\pi_{t-k,t} = \sum_{i=1}^{k} \pi_{t-k+i-1,t-k+i} \iff \pi_{t-1,t} = \pi_{t-k,t} - \pi_{t-k-1,t-1} + \pi_{t-k-1,t-k}\,.$$

Hence, using the VAR(1) dynamics of X_t (see Equation (1)):

$$\pi_{t-1,t} = \mu_\pi + (\Phi_\pi - 1)\pi_{t-k-1,t-1} + \Phi_{\pi_Z} Z_{t-1} + \Sigma_\pi^{1/2}\varepsilon_{\pi,t} + \pi_{t-k-1,t-k}\,.$$

Denoting by:

$$\widetilde{X}_t = \left[\pi_{t-k,t},\, Z_t',\, \pi_{t-1,t}, \ldots,\, \pi_{t-k,t-k+1}\right]'\,,$$

the vector of size $n+k$, we can form a new Gaussian VAR(1) dynamic system with $\widetilde{X}_t$ as:

$$\widetilde{X}_t = \left(\begin{array}{c} \mu_\pi \\ \hline \mu_Z \\ \hline \mu_\pi \\ \hline 0 \\ \vdots \\ 0 \end{array}\right) + \left(\begin{array}{cc|ccccc} \Phi_\pi & \Phi_{\pi,Z} & 0 & & \cdots & & 0 \\ \Phi_{Z,\pi} & \Phi_Z & 0 & & \cdots & & 0 \\ \hline \Phi_\pi - 1 & \Phi_{\pi,Z} & 0 & \cdots & 0 & & 1 \\ \hline 0 & 0 & 1 & 0 & \cdots & & 0 \\ \vdots & \vdots & 0 & \ddots & & & \vdots \\ 0 & 0 & 0 & 0 & & 1 & 0 \end{array}\right) \widetilde{X}_{t-1} + \left(\begin{array}{cc} \Sigma_\pi^{1/2} & 0 \\ 0 & I_{n_Z} \\ \hline \Sigma_\pi^{1/2} & 0 \\ \hline 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{array}\right) \varepsilon_t\,.$$

It is immediate to see that the short-rate dynamics can also be transformed in terms of $\widetilde{X}_t$, that is:

$$\mathfrak{r}_t | (\underline{\mathfrak{r}_{t-1}}, \underline{\widetilde{X}_t}) \sim \gamma_0 \left(\theta_0 + \widetilde{\theta}'\widetilde{X}_t + \widetilde{X}_t'\widetilde{\Theta}\widetilde{X}_t + \beta\mathfrak{r}_{t-1},\, \varsigma\right)\,,$$

where

$$\widetilde{\theta} = [\theta',\, 0, \cdots 0]' \quad \text{and} \quad \widetilde{\Theta} = \left(\begin{array}{cc} \Theta & 0 \\ 0 & 0 \end{array}\right)\,.$$

Risk-neutral dynamics and conditional Laplace transforms under any measure can be easily derived for the vector $\widetilde{f}_t = [\widetilde{X}_t', \text{Vec}(\widetilde{X}_t\widetilde{X}_t')', \mathfrak{r}_t]'$. The pricing of nominal bonds and inflation-indexed bonds hence follow exactly the same pattern as presented in the main text.

B.2 Identification constraints

In this section, we prove that the constraints imposed for the estimation are sufficient to identify the physical parameters and that the latent factors cannot be rotated. Let us consider an alternative vector of factors $\widetilde{X}_t$ such that:

$$\begin{aligned}
\widetilde{X}_t &= q + QX_t = q + Q(\mu + \Phi X_{t-1} + \Sigma^{1/2}\varepsilon_t) \\
&= q + Q\mu + Q^{-1}q + Q\Phi Q^{-1}\widetilde{X}_t + Q\Sigma^{1/2}\varepsilon_t \qquad (38) \\
&=: \mu^* + \Phi^*\widetilde{X}_{t-1} + \Sigma^{*^{1/2}}\varepsilon_t\,. \qquad (39)
\end{aligned}$$

Let us partition q and Q such that:

$$q = \begin{pmatrix} q_M \\ q_Z \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} Q_M & Q_{M,Z} \\ Q_{Z,M} & Q_Z \end{pmatrix}.$$

We show that the conditions $\mu = \mu^*$, $\Phi = \Phi^*$ and $\Sigma = \Sigma^*$ are sufficient to obtain $q = 0$ and $Q = I_n$. First, since the macroeconomic variables are observed, we have $\widetilde{M}_t = M_t$. This implies:

$$\begin{cases} q_M = 0 \\ Q_M = I_{n_M} \\ Q_{M,Z} = 0 \end{cases}$$

Second, since $\mu_Z = 0$, we have:

$$q_Z + Q_{Z,M}\mu_M + Q_Z^{-1}q_Z = 0 \Longleftrightarrow q_Z = -(I + Q_Z^{-1})^{-1}Q_{Z,M}\mu_M\,.$$

Then, we consider the condition $\Sigma = \Sigma^*$:

$$\begin{aligned}
0 &= Q_{Z,M}\Sigma_M \\
I &= Q_{Z,M}\Sigma_M Q_{Z,M}' + Q_Z Q_Z'
\end{aligned}$$

which, by invertibility of Σ_M translates into:

$$Q_{Z,M} = 0 \quad \text{and} \quad Q_Z Q_Z' = I\,.$$

In particular, these conditions imply that $q_Z = 0$ thus $q = 0$, and Q_Z is orthogonal. We just need to prove now that $Q_Z = I_{n_Z}$. Let us consider the condition $\Phi = \Phi^*$.

$$\Phi = Q\Phi Q^{-1} \iff \Phi = \begin{pmatrix} \Phi_M & \Phi_{M,Z} Q_Z^{-1} \\ Q_Z \Phi_{Z,M} & Q_Z \Phi_Z Q_Z^{-1} \end{pmatrix}.$$

which can be rewritten as:

$$\begin{cases} \Phi_{M,Z} = \Phi_{M,Z} Q_Z' \\ \Phi_{Z,M} = Q_Z \Phi_{Z,M} \\ \Phi_Z = Q_Z \Phi_Z Q_Z' \end{cases}$$

which is only possible if $Q_Z = I$.

B.3 Campbell-Shiller regression coefficients

In the following, we focus on a 12-month holding period. In the nominal world, the one-year excess returns of holding a nominal bond of maturity h are given by:

$$\frac{1}{12}\log\left(\frac{B(t+12, h-12)}{B(t,h)}\right) - R(t,12)\,.$$

In the real world, the nominal return of this one-year holding-period strategy must be corrected from the realized inflation rate and compared to the real rates $R_a^*(t,12)$:

$$\begin{aligned} & \frac{1}{12}\log\left(\frac{B(t+12, h-12)}{B(t,h)}\right) - \frac{1}{12}\Pi_{t+12} - R_a^*(t,12) \\ = & \left[\underbrace{\frac{1}{12}\log\left(\frac{B(t+12, h-12)}{B(t,h)}\right) - R(t,12)}_{Nominal\ excess\ returns}\right] + \left[\underbrace{R(t,12) - R_a^*(t,12) - \frac{1}{12}\Pi_{t+12}}_{Breakeven\ -\ Inflation}\right]. \end{aligned}$$

The real excess returns of nominal bonds are the sum of the nominal excess returns and the spread between the so-called breakeven inflation rate ($R(t,12) - R_a^*(t,12)$) and the realized inflation during the holding period. This last term would be close to the inflation risk premium would the inflation forecasting errors be small. Therefore, real excess returns of

nominal bonds include information about both the evolution of nominal term premia and inflation risk premia separately.

For the excess returns of TIPS, I denote by $B_t^*(t+12, h-12)$ the price at $t+12$ of the TIPS issued at time t of maturity h.

$$B_t^*(t+12, h-12) = \mathbb{E}\left[m_{t+12,t+h}\frac{CPI_{t+h}}{CPI_t}\Big|\underline{f_{t+12}}\right], \tag{40}$$

where the principal is adjusted by the reference price-index variation between the inception and the maturity date (t and $t+h$). Rearranging formula (40), this price can be expressed with the price of a newly issued TIPS at date $t+12$.

$$B_t^*(t+12, h-12) = \mathbb{E}\left[m_{t+12,t+h}\frac{CPI_{t+h}}{CPI_{t+12}}\Big|\underline{f_{t+12}}\right]\frac{CPI_{t+12}}{CPI_t} = B^*(t+12, h-12)\exp(\Pi_{t+12}). \tag{41}$$

Therefore, the real and nominal excess returns of holding TIPS for k-holding periods are respectively given by:

$$\begin{aligned} & \frac{1}{12}\log\left(\frac{B^*(t+12,h-12)}{B^*(t,h)}\exp(\Pi_{t+12})\right) - \frac{1}{12}\Pi_{t+12} - R_a^*(t,12) \\ = \quad & \frac{1}{12}\log\left(\frac{B^*(t+12,h-12)}{B^*(t,h)}\right) - R_a^*(t,12) \end{aligned}$$

and,

$$\begin{aligned} & \frac{1}{12}\log\left(\frac{B^*(t+12,h-12)}{B^*(t,h)}\exp(\Pi_{t+12})\right) - R(t,12) \\ = \quad & \underbrace{\left[\frac{1}{12}\log\left(\frac{B^*(t+12,h-12)}{B^*(t,h)}\right) - R_a^*(t,12)\right]}_{Real\ excess\ returns} - \underbrace{\left[R(t,12) - R_a^*(t,12) - \frac{1}{12}\Pi_{t+12}\right]}_{Breakeven\ -\ Inflation}. \end{aligned}$$

Similarly to nominal bonds, TIPS excess returns involve only real term premia in real terms, and both real term premia and inflation risk premia in nominal terms.

We turn now to the model-implied slopes for Campbell-Shiller regressions (28) to (31). Due to the similarities of the different specifications, we only present the computations for (28). Let us consider the general linear regression $Y = \omega + \phi X + \epsilon$. The optimal ϕ is given by:

$$\phi = \frac{\mathbb{C}ov(Y, X)}{\mathbb{V}(X)}.$$

Replacing Y and X by the Campbell-Shiller specification variables, we obtain:

$$\phi_h = \frac{h-12}{12} \times \frac{\mathbb{C}ov\left[R(t+12,h-12) - R(t,h), R(t,h) - R(t,12)\right]}{\mathbb{V}\left[R(t,h) - R(t,12)\right]}.$$

Using the affine interest rate formulas, we obtain:

$$\begin{aligned}
\phi_h &= \frac{h-12}{12} \times \frac{\mathbb{C}ov\left[\mathcal{B}'_{h-12} f_{t+12} - \mathcal{B}'_h f_t, \mathcal{B}'_h f_t - \mathcal{B}'_{12} f_t\right]}{\mathbb{V}\left[\mathcal{B}'_h f_t - \mathcal{B}'_{12} f_t\right]} \\
&= \frac{h-12}{12} \times \frac{\mathbb{C}ov\left[\left(\mathcal{B}'_{h-12}\Psi^{12} - \mathcal{B}'_h\right) f_t, \left(\mathcal{B}'_h - \mathcal{B}'_{12}\right) f_t\right]}{\mathbb{V}\left[\left(\mathcal{B}'_h - \mathcal{B}'_{12}\right) f_t\right]} \\
&= \frac{h-12}{12} \times \frac{\left(\mathcal{B}'_{h-12}\Psi^{12} - \mathcal{B}'_h\right) \mathrm{Vec}^{-1}\left(\left(I_{(n+n^2+1)^2} - (\Psi \otimes \Psi)\right)^{-1} (\Omega_0 + \Omega\mathbb{E}(f_t))\right) (\mathcal{B}_h - \mathcal{B}_{12})}{\left(\mathcal{B}'_h - \mathcal{B}'_{12}\right) \mathrm{Vec}^{-1}\left(\left(I_{(n+n^2+1)^2} - (\Psi \otimes \Psi)\right)^{-1} (\Omega_0 + \Omega\mathbb{E}(f_t))\right) \left(\mathcal{B}'_h - \mathcal{B}'_{12}\right)},
\end{aligned}$$

where $\mathbb{E}(f_t) = (I_{n+n^2+1} - \Psi)^{-1}\Psi_0$. The proofs for the other regressions are of similar fashion, since all dependent and independent variables of all regressions can be expressed as affine functions of the process f_t.

C Tables and figures

Table 2: Descriptive statistics

	Nominal rates (1990-2015)						
	1-month	1-year	2-year	3-year	5-year	7-year	10-year
mean	2.883	3.375	3.642	3.893	4.336	4.699	5.103
sd	2.228	2.382	2.351	2.272	2.098	1.952	1.800
$\rho(1)$	0.981	0.986	0.985	0.984	0.982	0.980	0.979
	Inflation	Real rates (2004-2015)					
	y-o-y	1-year	2-year	3-year	5-year	7-year	10-year
mean	2.607	-0.071	-0.100	-0.034	0.234	0.532	0.909
mean (excl. crisis)		-0.221	-0.206	-0.111	0.174	0.473	0.855
sd	1.237	1.592	1.423	1.312	1.153	1.069	0.954
sd (excl. crisis)		1.431	1.362	1.286	1.146	1.055	0.936
$\rho(1)$	0.942	0.938	0.963	0.964	0.969	0.962	0.956

Notes: All units are annualized percentage points. 'mean' are sample averages, 'sd' are sample standard deviations, and '$\rho(1)$' are autocorrelation of order 1. The 'excl. crisis' rows present descriptive statistics calculated on the TIPS data excluding the period from September 2008 to February 2009.

Table 3: Parameter estimates: X_t dynamics

	estimates	std.		estimates	std.
μ_Π	2.3948***	(0.5892)	$\mu^{\mathbb{Q}}_\Pi$	9.7565***	(2.2447)
μ_{Z_1}	0	–	$\mu^{\mathbb{Q}}_{Z_1}$	2.5501***	(0.7398)
μ_{Z_2}	0	–	$\mu^{\mathbb{Q}}_{Z_2}$	0	–
μ_{Z_3}	0	–	$\mu^{\mathbb{Q}}_{Z_3}$	0	–
Φ_Π	0.9006***	(0.0107)	$\Phi^{\mathbb{Q}}_\Pi$	0.7894***	(0.0109)
$\Phi_{Z_1,\Pi}$	0	–	$\Phi^{\mathbb{Q}}_{Z_1,\Pi}$	0	–
$\Phi_{Z_2,\Pi}$	0.2579***	(0.0292)	$\Phi^{\mathbb{Q}}_{Z_2,\Pi}$	0.0482***	(0.0037)
$\Phi_{Z_3,\Pi}$	0.1042***	(0.0287)	$\Phi^{\mathbb{Q}}_{Z_3,\Pi}$	0.0378***	(0.0049)
Φ_{Π,Z_1}	0	–	$\Phi^{\mathbb{Q}}_{\Pi,Z_1}$	0	–
Φ_{Z_1}	0.9816***	(0.0025)	$\Phi^{\mathbb{Q}}_{Z_1}$	0.9574***	(0.0008)
Φ_{Z_2,Z_1}	0	–	$\Phi^{\mathbb{Q}}_{Z_2,Z_1}$	0	–
Φ_{Z_3,Z_1}	0	–	$\Phi^{\mathbb{Q}}_{Z_3,Z_1}$	0	–
Φ_{Π,Z_2}	-0.0109***	(0.0027)	$\Phi^{\mathbb{Q}}_{\Pi,Z_2}$	-0.0795***	(0.008)
Φ_{Z_1,Z_2}	0	–	$\Phi^{\mathbb{Q}}_{Z_1,Z_2}$	0	–
Φ_{Z_2}	0.9739***	(0.00008)	$\Phi^{\mathbb{Q}}_{Z_2}$	0.9961***	(0.0005)
Φ_{Z_3,Z_2}	-0.0097***	(0.003)	$\Phi^{\mathbb{Q}}_{Z_3,Z_2}$	0	–
Φ_{Π,Z_3}	-0.0204***	(0.0026)	$\Phi^{\mathbb{Q}}_{\Pi,Z_3}$	-0.0781***	(0.0079)
Φ_{Z_1,Z_3}	0	–	$\Phi^{\mathbb{Q}}_{Z_1,Z_3}$	-0.0267***	(0.0021)
Φ_{Z_2,Z_3}	0	–	$\Phi^{\mathbb{Q}}_{Z_2,Z_3}$	0	–
Φ_{Z_3}	0.9998***	(0.000001)	$\Phi^{\mathbb{Q}}_{Z_3}$	0.9984***	(0.0003)
Σ_Π	0.1357***	(0.006)	$\Sigma^{\mathbb{Q}}_\Pi$	0.552709***	(0.07786)
$\Sigma_{Z_1,\Pi}$	0	–	$\Sigma^{\mathbb{Q}}_{Z_1,\Pi}$	0	–
$\Sigma_{Z_2,\Pi}$	0	–	$\Sigma^{\mathbb{Q}}_{Z_2,\Pi}$	0	–
$\Sigma_{Z_3,\Pi}$	0	–	$\Sigma^{\mathbb{Q}}_{Z_3,\Pi}$	0	–
Σ_{Z_1}	1	–	$\Sigma^{\mathbb{Q}}_{Z_1}$	1	–
Σ_{Z_2,Z_1}	0	–	$\Sigma^{\mathbb{Q}}_{Z_2,Z_1}$	0	–
Σ_{Z_3,Z_1}	0	–	$\Sigma^{\mathbb{Q}}_{Z_3,Z_1}$	0	–
Σ_{Z_2}	1	–	$\Sigma^{\mathbb{Q}}_{Z_2}$	1	–
Σ_{Z_3,Z_2}	0	–	$\Sigma^{\mathbb{Q}}_{Z_3,Z_2}$	0	–
Σ_{Z_3}	1	–	$\Sigma^{\mathbb{Q}}_{Z_3}$	1	–

Notes: Standard deviations are in parentheses and are calculated using the outer-product Hessian approximation. The '–' sign indicates that the parameter has been calibrated hence does not possess any standard deviation. Significance level: * <0.1, ** <0.05, *** <0.01.

Table 4: Parameter estimates: short-rate and the prices of risk

	r_t dynamics				
	estimates	std.		estimates	std.
Θ_{Π}	0.013784***	(0.001804)	$\Theta^{\mathbb{Q}}_{\Pi}$	0.013784***	(0.001804)
$\Theta_{Z_1,\Pi}$	0.010387***	(0.000837)	$\Theta^{\mathbb{Q}}_{Z_1,\Pi}$	0.010387***	(0.000837)
$\Theta_{Z_2,\Pi}$	0.005234***	(0.000707)	$\Theta^{\mathbb{Q}}_{Z_2,\Pi}$	0.005234***	(0.000707)
$\Theta_{Z_3,\Pi}$	-0.000278*	(0.000163)	$\Theta^{\mathbb{Q}}_{Z_3,\Pi}$	-0.000278*	(0.000163)
Θ_{Z_1}	0.007828***	(0.000594)	$\Theta^{\mathbb{Q}}_{Z_1}$	0.007828***	(0.000594)
Θ_{Z_2,Z_1}	0.003944***	(0.000411)	$\Theta^{\mathbb{Q}}_{Z_2,Z_1}$	0.003944***	(0.000411)
Θ_{Z_3,Z_1}	-0.000209*	(0.000115)	$\Theta^{\mathbb{Q}}_{Z_3,Z_1}$	-0.000209*	(0.000115)
Θ_{Z_2}	0.001988***	(0.000388)	$\Theta^{\mathbb{Q}}_{Z_2}$	0.001988***	(0.000388)
Θ_{Z_3,Z_2}	-0.000106*	(0.00006)	$\Theta^{\mathbb{Q}}_{Z_3,Z_2}$	-0.000106*	(0.00006)
Θ_{Z_3}	0.000006	(0.000006)	$\Theta^{\mathbb{Q}}_{Z_3}$	0.000006	(0.000006)
β	0.8541***	(0.0045)	$\beta^{\mathbb{Q}}$	0.8541***	(0.0045)
ς	1	–	$\varsigma^{\mathbb{Q}}$	1	–
$\delta_0 \cdot 1200$	0.1574***	(0.0056)	$\delta \cdot 1200$	0.2053***	(0.014)
	Prices of risk and measurement errors standard deviations				
	estimates	std.		estimates	std.
$\lambda_{0,\Pi}$	0	–	λ_{0,Z_2}	0	–
λ_{0,Z_1}	2.5501***	(0.7398)	λ_{0,Z_3}	0	–
λ_{Π}	-5.2097***	(0.2816)	λ_{Π,Z_2}	-0.0635***	(0.0136)
$\lambda_{Z_1,\Pi}$	0	–	λ_{Z_1,Z_2}	0	–
$\lambda_{Z_2,\Pi}$	-0.2098***	(0.0277)	λ_{Z_2}	0.0222***	(0.0005)
$\lambda_{Z_3,\Pi}$	-0.0664**	(0.0261)	λ_{Z_3,Z_2}	0.0097***	(0.003)
λ_{Π,Z_1}	0	–	λ_{Π,Z_3}	0.0088**	(0.0045)
λ_{Z_1}	-0.0243***	(0.0024)	λ_{Z_1,Z_3}	-0.0267***	(0.0021)
λ_{Z_2,Z_1}	0	–	λ_{Z_2,Z_3}	0	–
λ_{Z_3,Z_1}	0	–	λ_{Z_3}	-0.0014***	(0.0003)
Λ_r	0	–	$\Lambda_{X,\Pi}$	2.7808***	(0.1474)
σ_R	0.0534***	(0.0006)	σ^*_R	0.1277***	(0.0027)
$\sigma^{(12)}_{\Pi}$	0.509	–	$\sigma^{(120)}_{\Pi}$	0.389	–
$\sigma^{(3)}_{S_R}$	0.231	–	$\sigma^{(12)}_{S_R}$	0.422	–
σ_{ZLB}	0.0522	–			

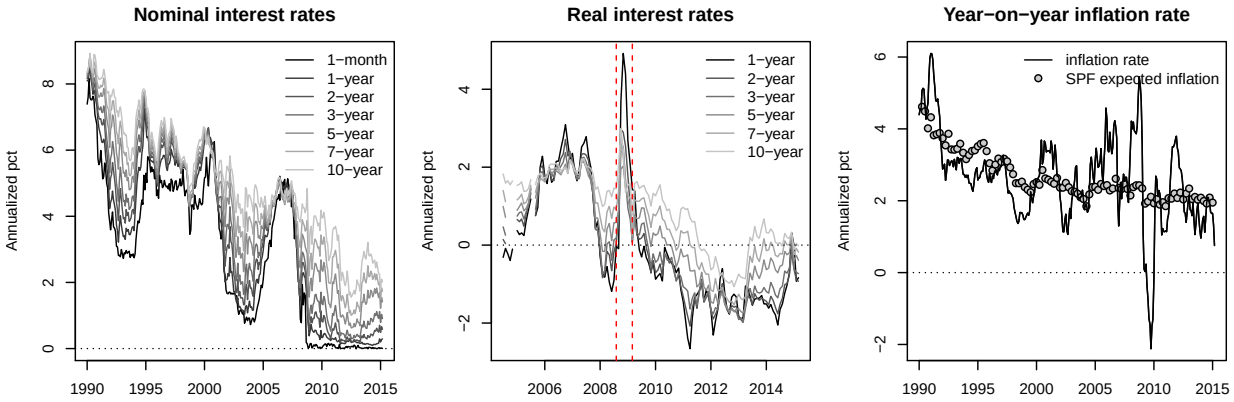
Notes: Standard deviations are in parentheses and are calculated using the outer-product Hessian approximation. The '–' sign indicates that the parameter has been calibrated hence does not possess any standard deviation. Significance level: * <0.1, ** <0.05, *** <0.01.

Table 5: Model fit and characteristics

Maturities (months)	1	12	24	36	60	84	120
Nominal rates RMSE (bps)	6.1	8.3	7.61	8.62	9.33	8.00	8.17
Real rates RMSE (bps)	-	16.4	8.37	10.19	15.15	13.32	9.62
Probabilities (in %)	$\mathbb{P}(r_t = 0) = 24.31$			$\mathbb{P}(r_t = 0 \mid r_{t-1} = 0) = 84.35$			

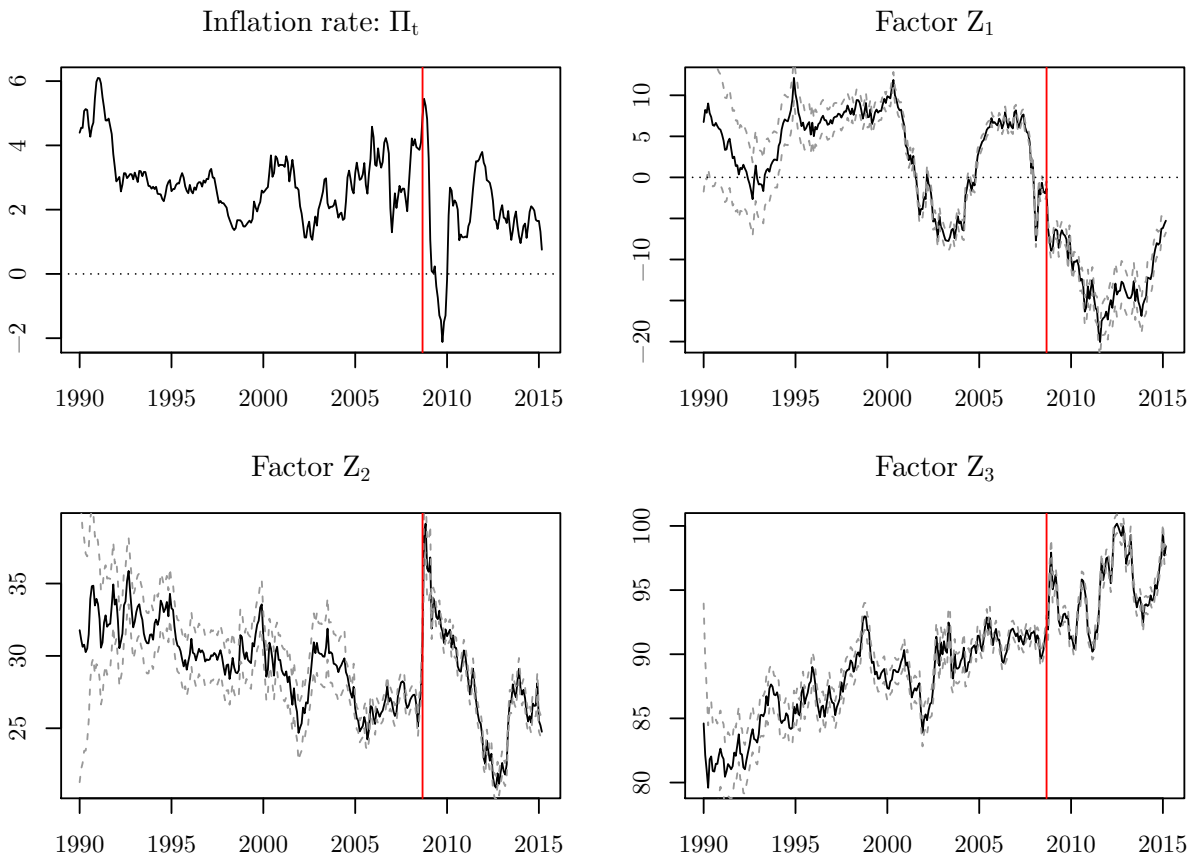
Note: Probabilities are calculated with simulated paths of length 1,000,000.

Figure 1: Nominal and real term structures and inflation data



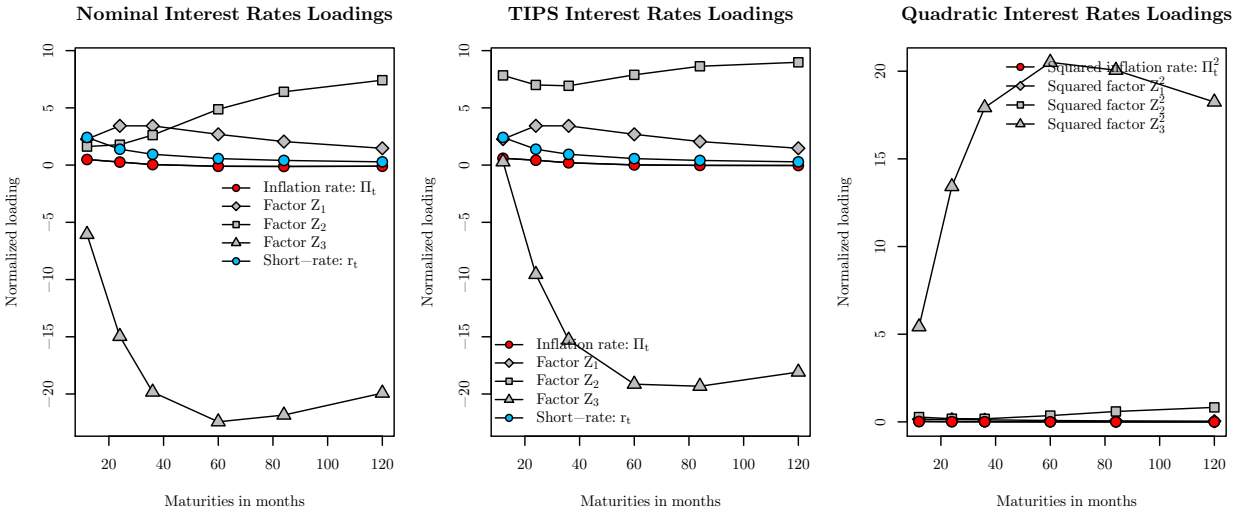
Notes: The left plot presents the time-series of the nominal term structure of interest rates from January 1990 to March 2015. Maturities range from 1 month to 10 years. The middle plot presents the term structure of real rates built as the difference between the nominal zero-coupon interest rates and the inflation swap rates of the same maturity. Observations start in July 2004 and run to March 2015. The vertical red dashed lines indicate the beginning and end of a reduced market liquidity period, that we treat as missing data in the estimation. The right plot presents the realized year-on-year inflation lagged of 3 months (black solid line). The dots superimpose the expected average inflation rate over the next year as measured by the survey of professional forecasters.

Figure 2: Filtered factors



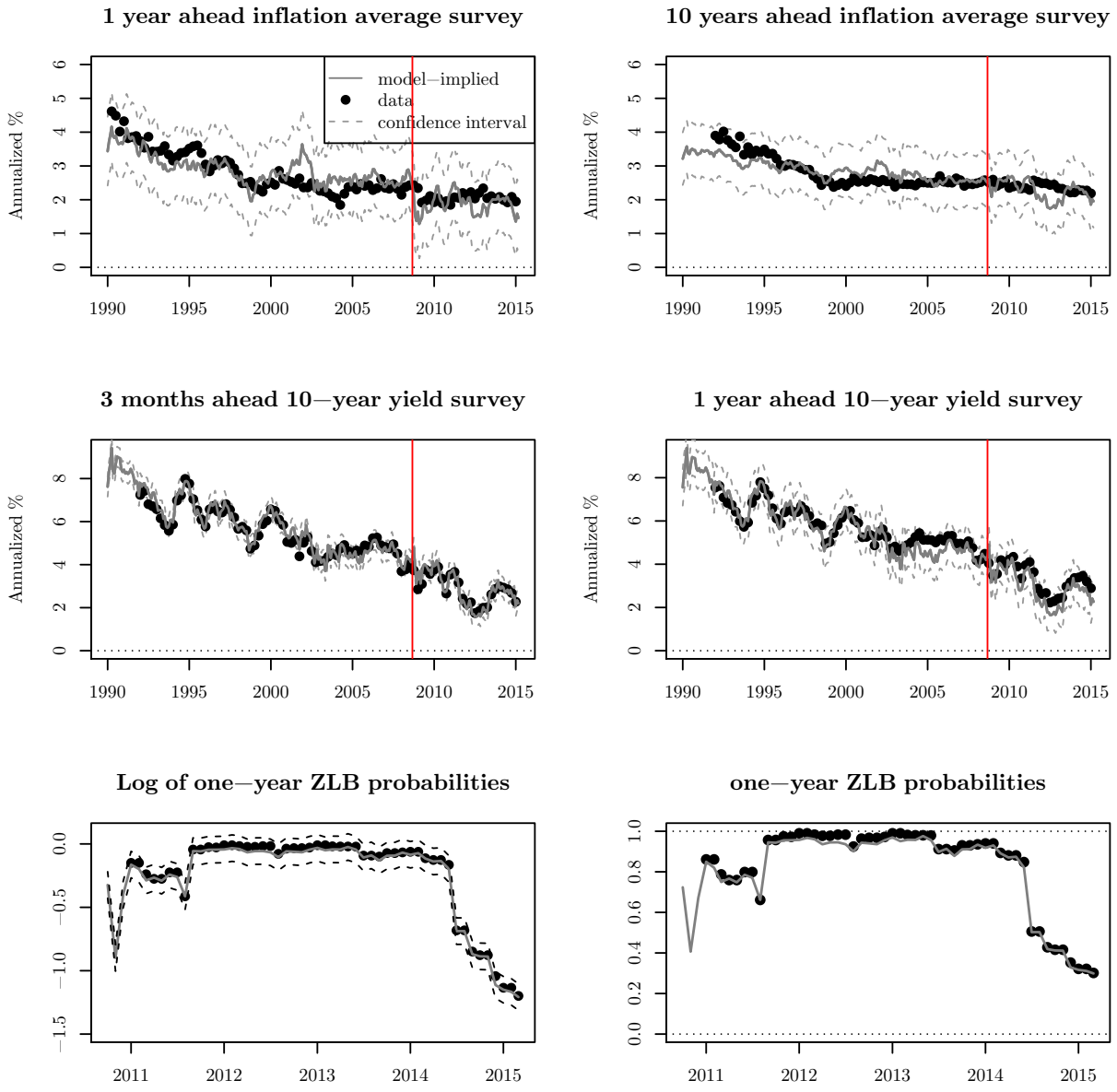
Notes: The first factor is the observed year-on-year inflation rate. It hence possesses no filtering standard deviations. The other 3 factors are estimated using the quadratic Kalman filter and 95% confidence bounds are plotted with dashed grey lines. The red vertical line delimits the beginning of the zero lower bound period.

Figure 3: Factors loadings



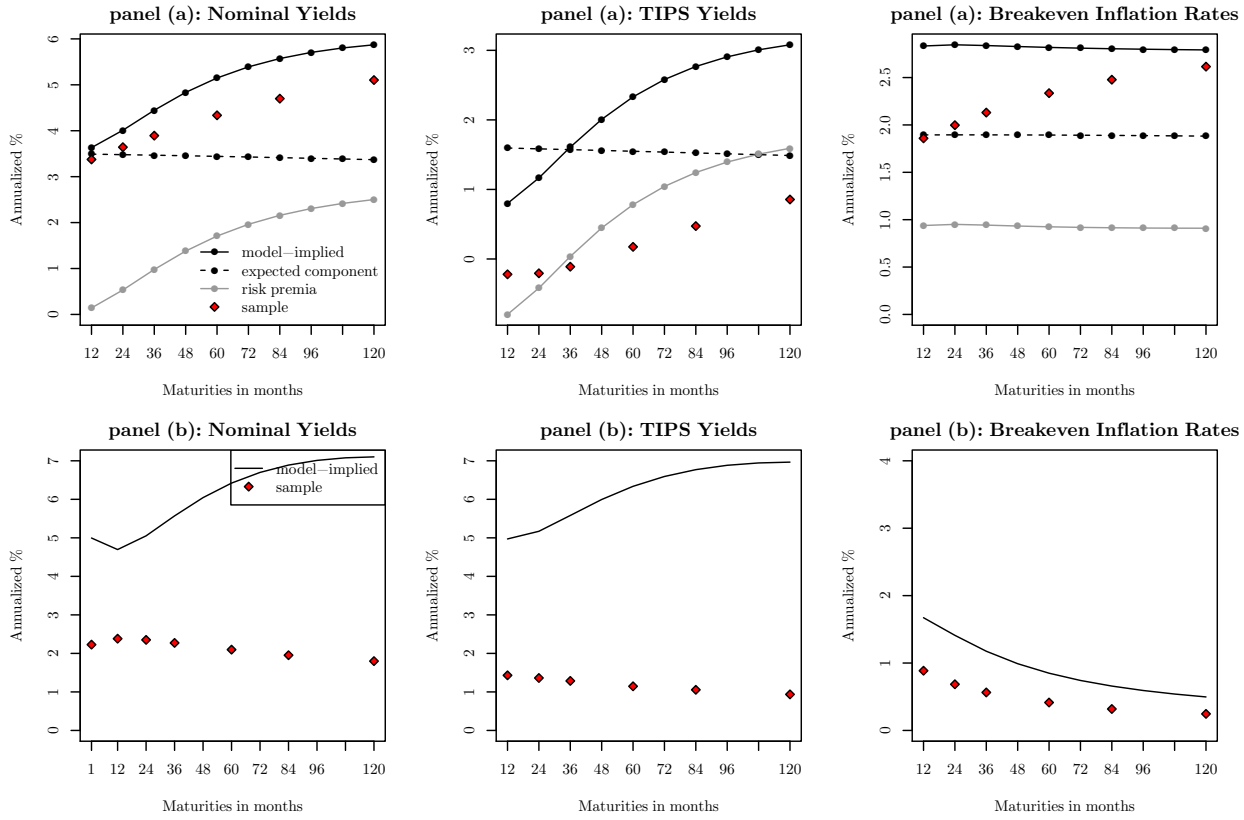
Notes: From left to right, this plot gathers the linear loadings of the nominal interest rates, of the real rates, and the quadratic loadings (which are the same for both yield curves) with respect to maturity. These loadings are normalized by the in-sample standard deviation of the corresponding filtered factor to be comparable with each other.

Figure 4: Fitted series of survey data



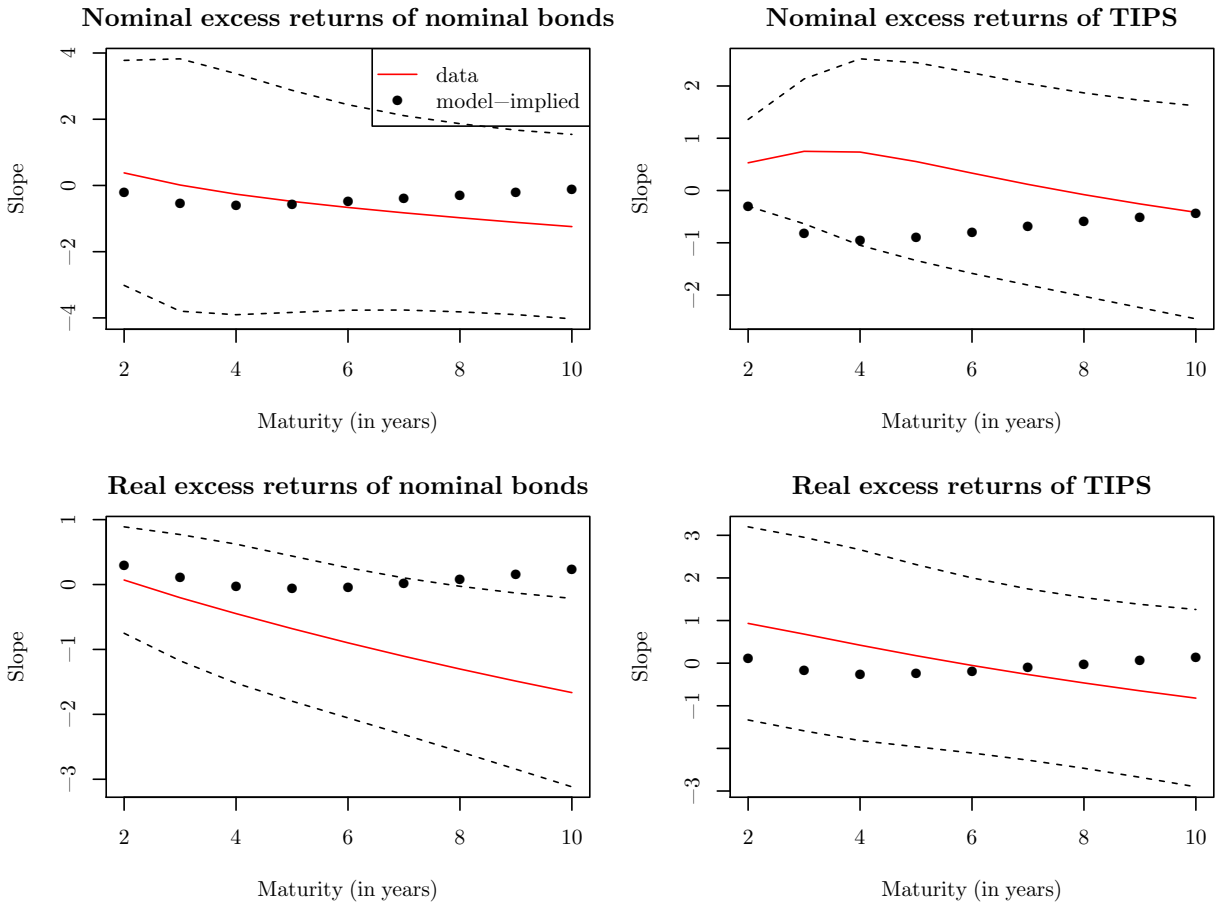
Notes: The black dots correspond to observed forecast data. The grey solid lines correspond to the model-implied forecasted values. Top graphs correspond respectively to the one-year ahead and 10-year ahead inflation average surveys. Medium graphs correspond respectively to the three-months ahead and one-year ahead 10-year yield survey. Units are in annualized percentage points. Bottom graphs correspond respectively to the fitted natural logarithm of ZLB probabilities, and of the exponential of the latter. Confidence intervals computed using the measurement errors standard deviations are plotted in grey dashed lines. The red vertical line delimits the beginning of the zero lower bound period.

Figure 5: Marginal term structures



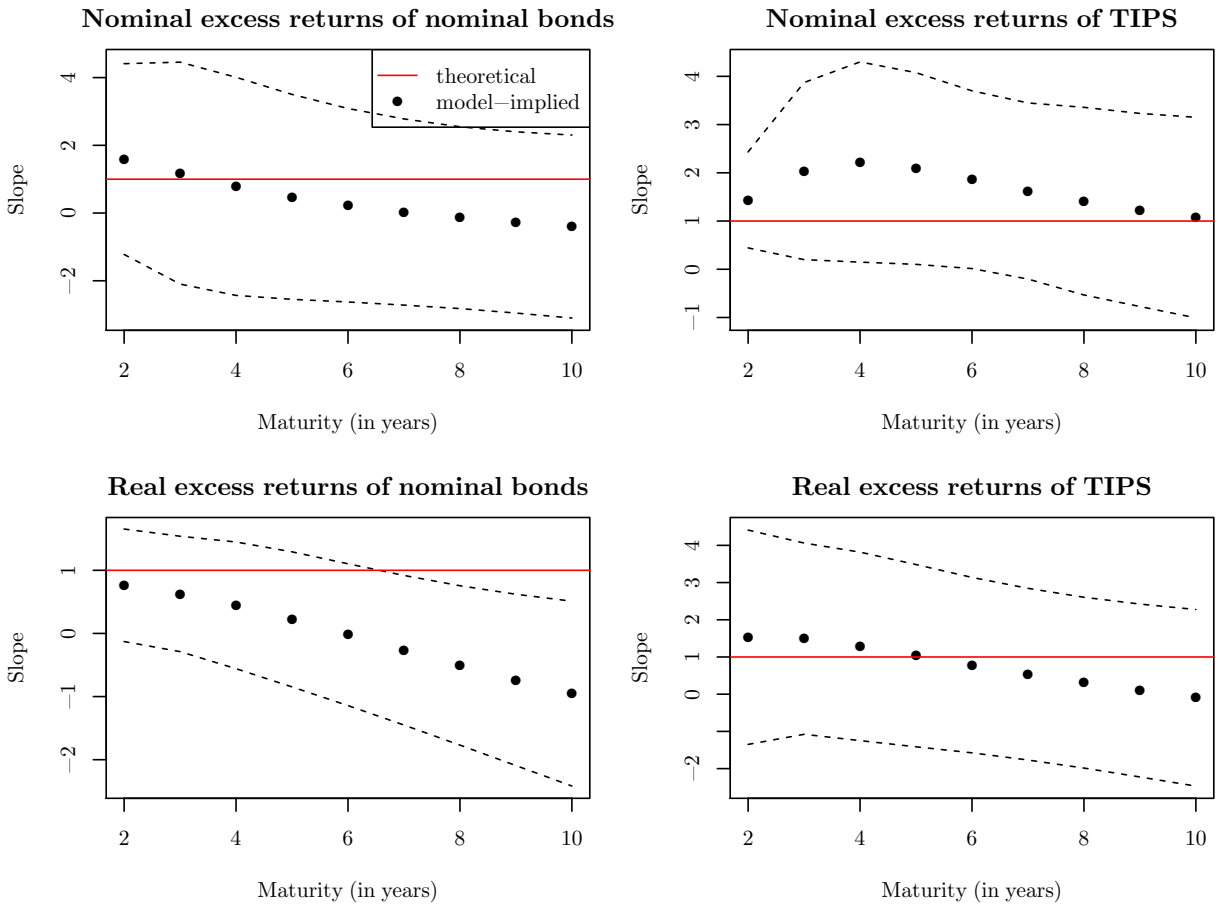
Notes: Panel (a) presents the marginal mean term structure of nominal yields, TIPS yields, and breakeven inflation rates (resp. left, middle and right plots). For each graph of panel (a), we present the data average (red diamonds), the model-implied counterpart (black solid line) and its decomposition into expected component (grey solid line) and risk premium (black dashed line). Panel(b) presents the model-implied marginal term structure of standard deviations of nominal yields, TIPS yields, and breakeven inflation rates (resp. black solid lines of left, middle and right plots) along with data counterparts (red diamonds). All model-implied quantities are obtained with the closed-form marginal first two moments of the transition equation (14).

Figure 6: Campbell-Shiller regression slopes: LPY-I



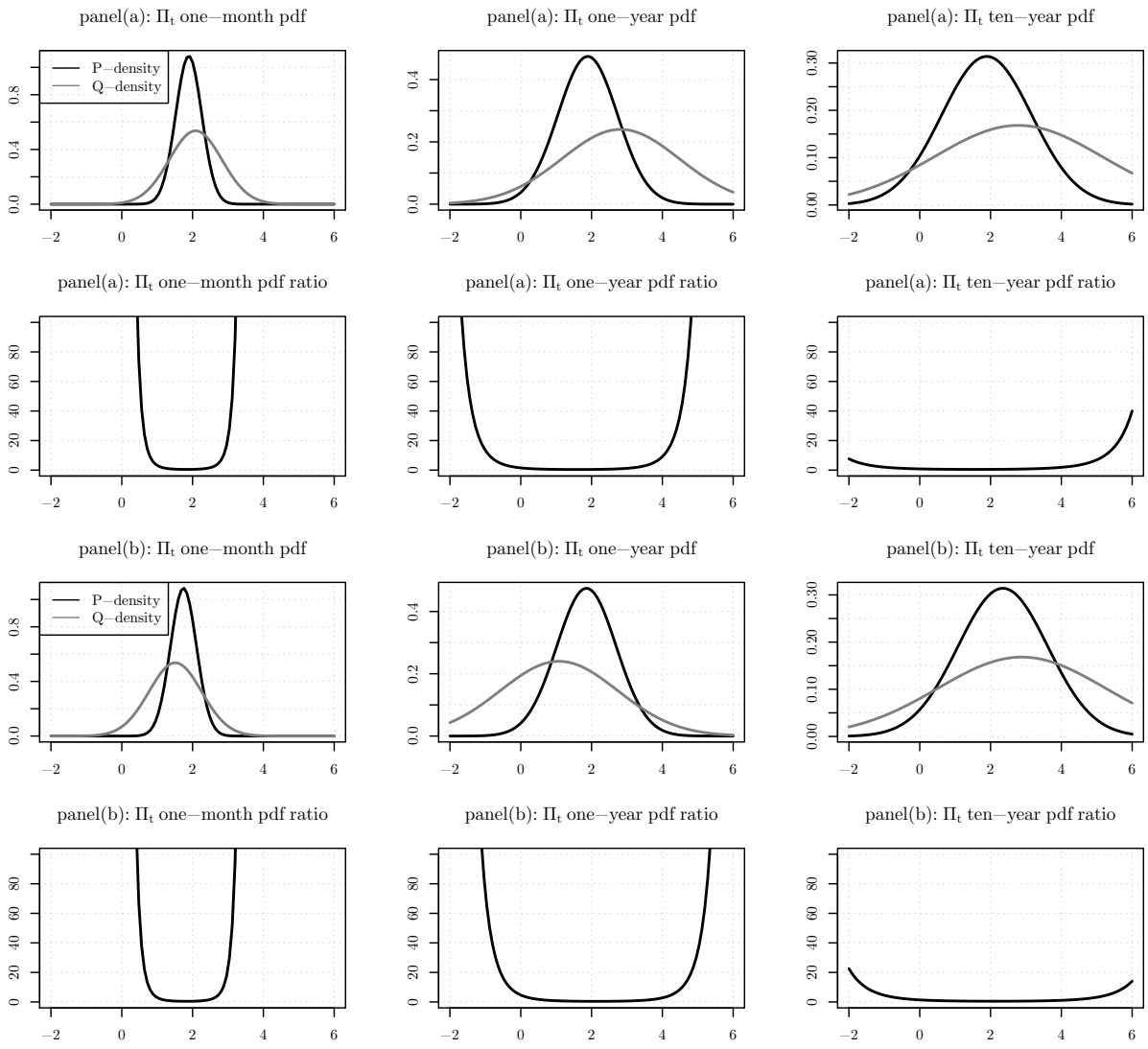
Notes: From left to right, from top to bottom, the graphs present the slopes of Campbell and Shiller regressions for (28), (29), (30) and (31) with a 12-months holding period. The red solid line gathers the slope estimates obtained with filtered yields data from January 1990 to August 2014. 95% Confidence intervals are computed using Newey-West robust estimators with automatically selected lag and are indicated with the dashed lines. Model-implied estimates are indicated with the black dots and computed with the yields and inflation expectation and variance formulas.

Figure 7: Campbell-Shiller regression slopes: LPY-II



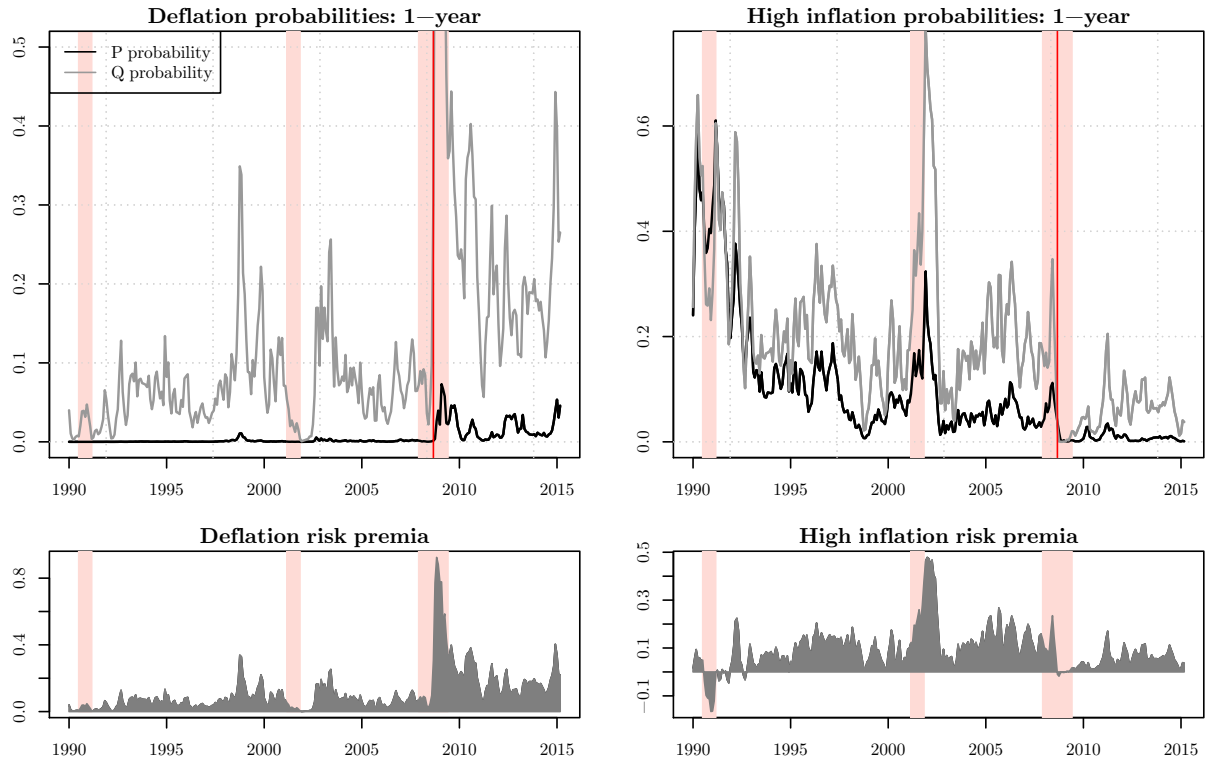
Notes: From left to right, from top to bottom, the graphs present the slopes of Campbell and Shiller regressions for (28), (29), (30) and (31) when regressors are adjusted by the corresponding model-implied expected excess return series. The red solid line represents the theoretical values of the regression, namely one for all maturities. Model-implied estimates are indicated with the black dots and computed performing the Campbell and Shiller regressions where the dependent variable is adjusted by the model-implied expected excess returns. 95% Confidence intervals are computed using Newey-West robust estimators with automatically selected lag and are indicated with the dashed lines.

Figure 8: Inflation conditional densities and density ratios



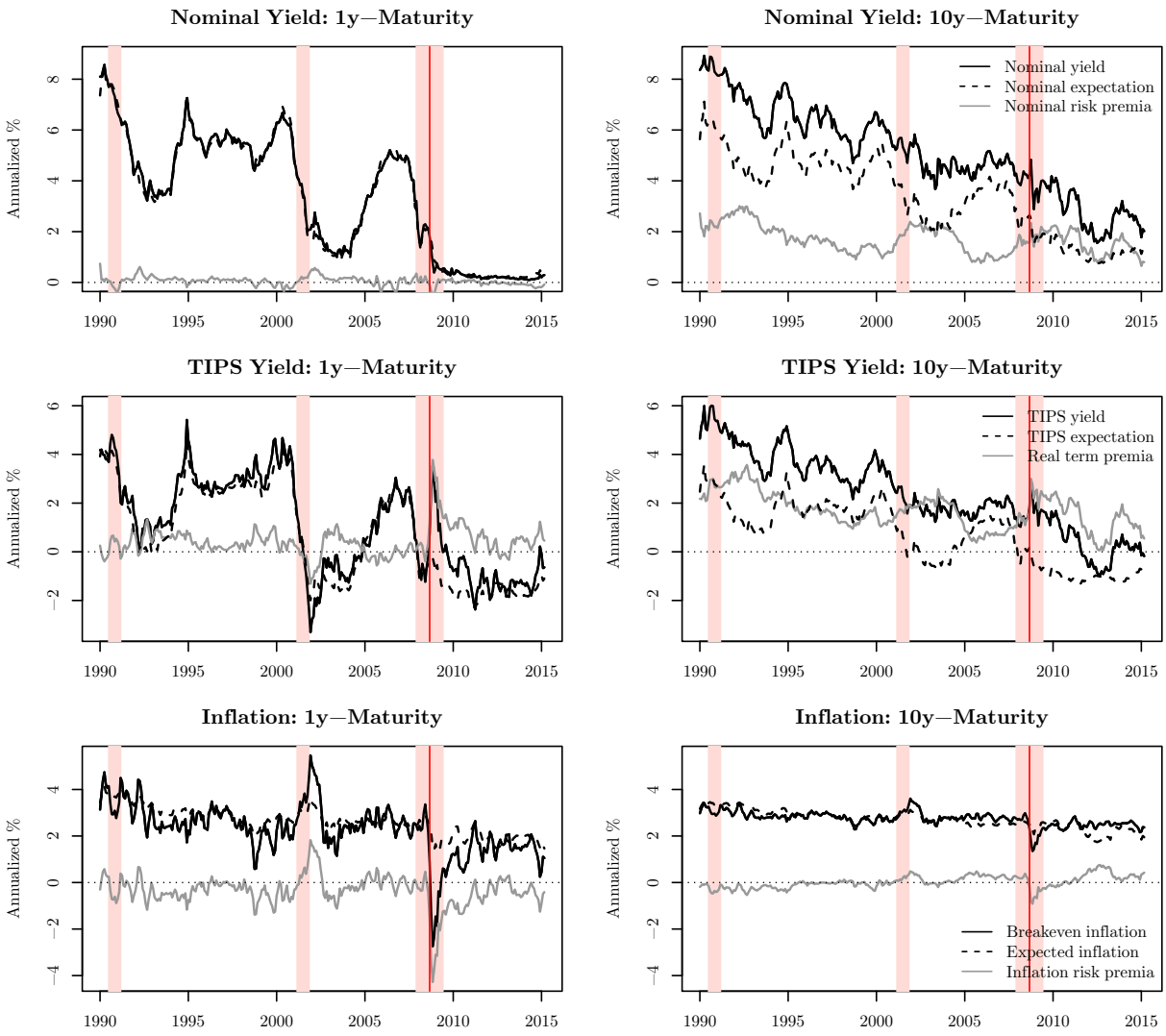
Notes: Panel (a) and (b) present the conditional inflation densities from the starting point $X_t = \mathbb{E}(X_t)$ and $X_t = \bar{X}_{zlb}$ respectively, the model-implied mean of the factors and the empirical mean measured during the ZLB period. For both panels, the first row presents the physical and risk-neutral conditional Gaussian pdfs (black and grey lines respectively) and the second row presents the $\mathbb{Q}/\mathbb{P}$-ratio. The three columns represent different horizons: one-month (left), one-year (middle) and ten-year (right). x-axis units are in percentage points.

Figure 9: Deflation and high inflation risk



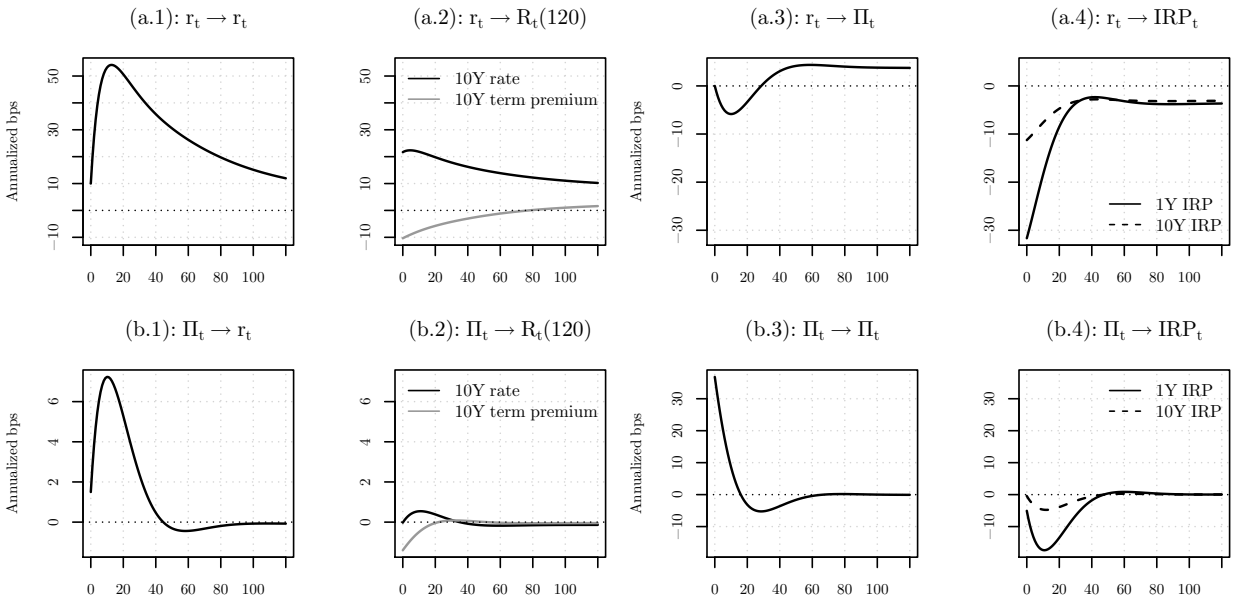
Notes: The first column presents the one-year ahead physical and risk-neutral conditional probabilities that the year-on-year inflation rate goes negative (resp. black and grey solid lines) and calculates the associated risk premium (bottom graph). The second column presents the one-year ahead physical and risk-neutral conditional probabilities that the year-on-year inflation rate goes above 4% (resp. black and grey solid lines) and calculates the associated risk premium (bottom graph). The red vertical line delimits the beginning of the zero lower bound period. Pink shaded areas are NBER recession periods.

Figure 10: Decomposition of interest rates



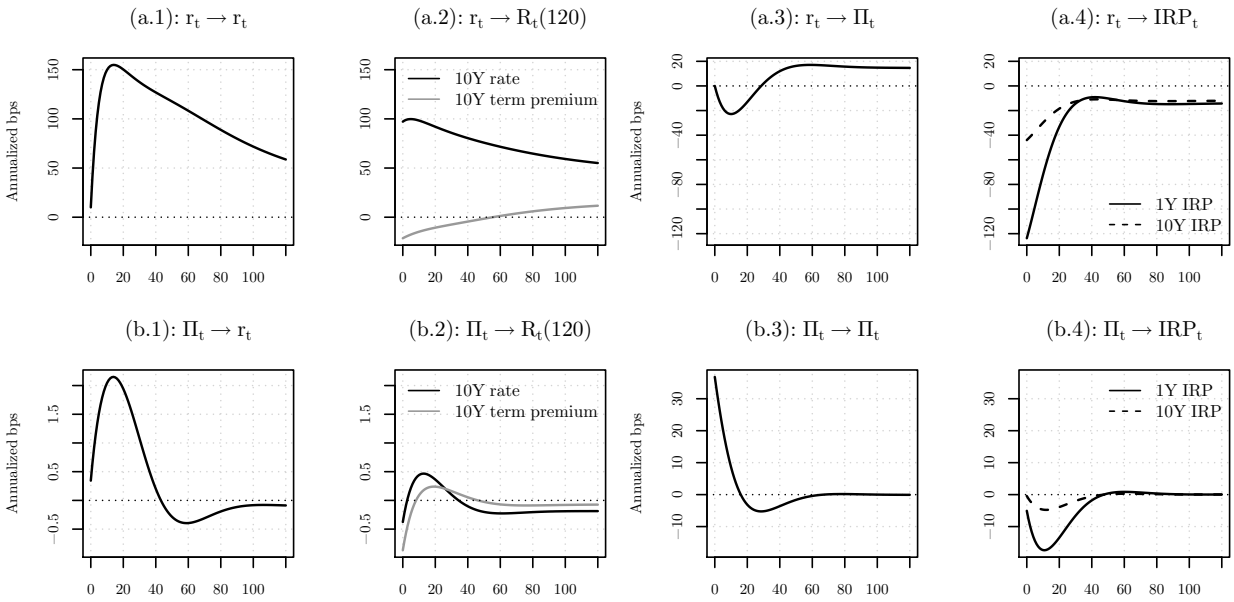
Notes: First column presents results for the 1-year maturity yields, whereas second column presents results for the 10-year maturity yields. The first row presents to the observed nominal yield (black solid line), the nominal term premia (grey solid line), and the expected component (black dashed line). The second row presents to the filtered TIPS yield (black solid line), the real term premia (grey solid line) and the expected real rate (black dashed line). The last row presents the filtered inflation breakeven rate (black solid line), the inflation risk premia (grey solid line) and the inflation expectation (black dashed line). Units are in annualized percentage points. The red vertical line delimits the beginning of the zero lower bound period. Pink shaded areas are NBER recession periods.

Figure 11: Impulse-response functions in the steady-state



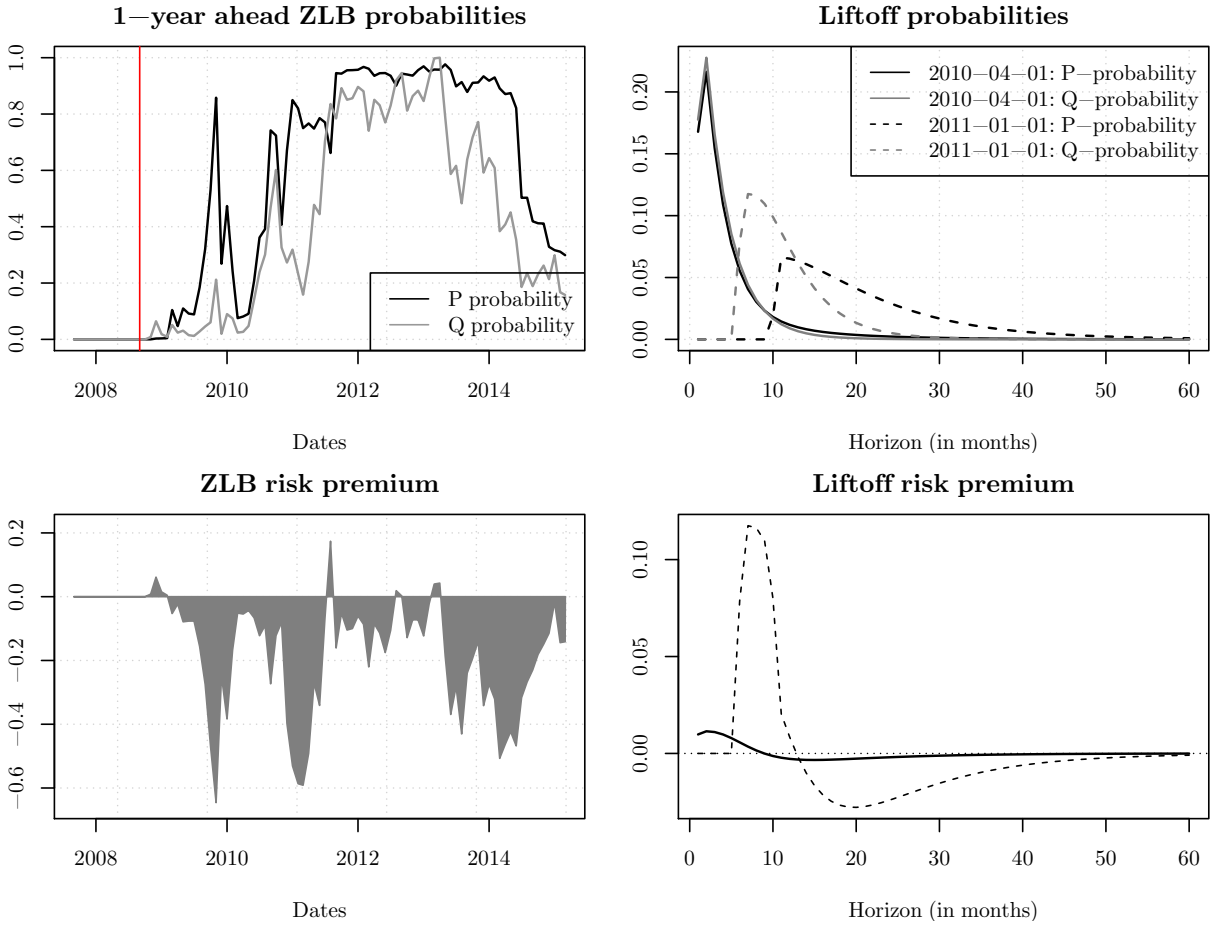
Notes: These graphs present the effect of an upward monetary policy shock (panel (a)) and of an upward inflation shock (panel (b)) conditionally on being at the steady state (see Section 3.8 for the detailed procedure). Column 1 to 4 respectively present the effects of the shocks on the short-term nominal interest rate, the long-term nominal rate, the inflation rate, and the inflation risk premium. Units are in annualized basis points.

Figure 12: Impulse-response functions at the zero lower bound



*Notes:*These graphs present the effect of an upward monetary policy shock (panel (a)) and of an upward inflation shock (panel (b)) conditionally on being at the zero lower bound (see Section 3.8 for the detailed procedure). Column 1 to 4 respectively present the effects of the shocks on the short-term nominal interest rate, the long-term nominal rate, the inflation rate, and the inflation risk premium. Units are in annualized basis points.

Figure 13: Physical and risk-neutral liftoff probabilities



Notes: The two columns present respectively the 1-year ahead zero lower bound probabilities under the physical and the risk-neutral measure and the physical and risk-neutral liftoff probabilities at two chosen dates. $\mathbb{P}$- and $\mathbb{Q}$-probabilities are respectively represented with a black and a grey solid line. The second row presents the difference between the risk-neutral and physical probabilities. All quantities are computed before applying corrections on the factors. The red vertical bar delimits the beginning of the zero lower bound period.