

Optimally Stubborn*

Anna Sanktjohanser[†]

PRELIMINARY AND INCOMPLETE

March 12, 2017

Abstract

I consider a bargaining game with two types of players – rational and stubborn. Rational players choose demands at each point in time. Stubborn players are restricted to choose a bargaining strategy from a proper subset of strategies available to rational players. In the simplest case, stubborn players are restricted to choose from the set of “insistent” strategies that always make the same demand and never accept anything less. However, their initial choice of demand is unrestricted. I characterize the equilibria in this game, showing how the flexibility of the stubborn type changes equilibrium predictions.

1 Introduction

The literature on reputation builds on the idea that some players may be limited in their choice of strategies. The results in this literature rely on the specific choice of behavioral type. With the “right” behavioral (Stackelberg) type present, we can derive sharp predictions in term of lower (or upper) bounds on payoffs (Fudenberg and Levine, 1989). However, there is no explicit description of equilibrium behavior as these bounds are derived using strategies available to the rational player, rather than the optimal one. With one major exception (Abreu and Gul (2000) and follow-up papers), this literature does not characterize the equilibrium. The literature takes as given that such “behavioral” types exist, and does not address the question why particular

*I would like to thank Péter Esö, Johannes Hörner, Margaret Meyer, Sujoy Mukerji, Rajiv Sethi, William Zame, as well as seminar participants at the University of Oxford for helpful discussions and comments.

[†]Department of Economics, University of Oxford, Oxford, OX1 3UQ, UK,
anna.sanktjohanser@economics.ox.ac.uk.

behavioral types occur. The aim of this paper is to endogenize these types by giving them some choice over the form of their “stubbornness.” I do so in the framework of bargaining, as modeled by Abreu and Gul (AG hereafter) and Myerson (1991).

I consider a bargaining game with two types of players – rational and stubborn. The game ends when a player adjusts his demand so as to make it compatible with his opponent’s. Rational players choose demands at each point in time. The key feature of my model is that unlike in the literature, stubborn players can choose their initial demand freely, but cannot adjust their demand.

My paper has two main results. First, I derive strong behavioral predictions: as the probability of stubbornness goes to 0, the set of equilibrium offers converges to at most a three-point set, that is, players are indifferent over at most three offers. Moreover, in the limit, at most two offers are made with positive probability. Second, even as the probability of stubbornness goes to 0, delay (and hence, inefficiency) may not disappear. Rather, there is a Folk theorem like payoff multiplicity. Any feasible payoff can arise as an equilibrium payoff when the probability that a player is stubborn is sufficiently small (at least when no refinement is applied).

Both results stand in contrast with the results when stubborn players cannot choose their initial demand freely. First, when there is an exogenous distribution of offers of the stubborn type as in AG, rational players will mimic any demand sufficiently high. Secondly, in AG, when the probability of a player being stubborn is small, there is no delay (and hence, inefficiency) if the “right” stubborn type is present. My paper shows that the right stubborn type may not be present when given choice over his initial demand. However, in the set of symmetric equilibria with at most three offers, there is a unique equilibrium satisfying refinements such as D1 or passive beliefs. In this equilibrium, there is no delay and inefficiency – players receive a payoff of $1/2$ when they are equally patient as they do with exogenous stubborn types in AG.

My model builds on the framework by Myerson (1991) and AG. They consider a bargaining environment, where there is a small probability of a player being behavioral. Behavioral types in AG have no choice over their actions, and the distribution of behavioral types is exogenously given. They derive weak behavioral predictions: any demand above some threshold is mimicked by the rational type. If the “right” behavioral type is present, AG derive strong predictions in terms of the payoffs when the probability of a player being stubborn is sufficiently small. The right behavioral type is the type which makes a demand proportional to a player’s patience. If this type is present, then for a sufficiently small probability of a player being stubborn, there is no delay or inefficiency. A rational player receives a payoff proportional to his patience. I show that the right behavioral type may not be present once behavioral players choose their initial demand freely. More generally, my paper is related to the literature on reputation (Fudenberg

and Levine, 1989 and 1992; Abreu, Pearce and Stacchetti, 2015; Fanning, 2016a and 2016b) and bargaining (Nash, 1953; Abreu and Pearce, 2015). The most closely related papers are Abreu and Sethi (2003), Atakan and Ekmekci (2014), Kambe (1999) and Wolitzky (2012), who all build on Abreu and Gul (2000). Abreu and Sethi (2003) endogenize behavioral types using an evolutionary stability approach. In contrast, in my model stubborn types pick their initial demand so as to maximize their payoff. They show that if a behavioral type is present in an evolutionary equilibrium, the complementary demand must also be present – this is not true in my model. Similar to my model, they find that inefficient delays may occur in equilibrium. Atakan and Ekmekci (2014) endogenize behavioral types in a two sided search market. The matching market serves as an endogenous outside option. Unlike in my model, stubborn types cannot choose their initial demand, but they can exit the current trade when they are sure they face a stubborn player. Given the differences in modeling, it is difficult to compare their results to mine. In Kambe (1999) and Wolitzky (2012), players do not know at the demand stage whether they are behavioral or not. Rather, a player becomes “committed” with some exogenous probability after demands have been chosen. In my model, a player knows his type (behavioral or rational) at the time of choosing his demand. Unlike in my model, the lower bound on the payoff of a rational player in Wolitzky (2012) is non-zero. Kambe (1999) focusses on one offer equilibria, and insofar the results are similar to mine in this special case.

The structure of this paper is as follows. I first describe the model in Section 2. Section 3 analyzes the benchmark case with an exogenous distribution of stubborn types as in AG. Section 4 discusses necessary conditions for equilibrium existence with endogenous stubborn types. The main results, which focus on mixing equilibria, are presented in Section 5. Section 6 briefly discusses existence of equilibria with a separating offer by the stubborn type. Section 7 concludes.

2 Model

The model and the notation (mostly) follows AG. Time is continuous, and the horizon is infinite. Two players decide on how to split a unit surplus. At time 0, players i and j simultaneously announce demands, α_i and α_j . If $\alpha_i + \alpha_j \leq 1$, the demands are said to be *compatible*. In this case, the game ends. If $\alpha_i + \alpha_j > 1$, the demands are *incompatible*. In this case, a concession game starts. The game ends when one player concedes. Concession means agreeing to the opponent’s demand.

Each player i is rational with probability $1 - z$, and stubborn with probability z , where $z \in (0, 1)$. Before the game starts, each player privately learns whether he is stubborn or

rational. A rational player $i = 1, 2$ can make any demand $\alpha_i \in [0, 1]$ at time 0, and concede to his opponent at any point in time. Stubborn player i can choose his initial demand $\alpha_i \in [0, 1]$, but cannot concede to his opponent. Note that this is unlike in AG, where a stubborn player cannot choose his initial demand.¹

A strategy for a stubborn player, i , σ_i^S , is defined by a probability distribution s_i on $[0, 1]$. A strategy for a rational player i , σ_i^R , is defined by a probability distribution r_i on $[0, 1]$, and a collection of cumulative distributions F_{α_i, α_j}^i on $\mathcal{R}_+ \cup \{\infty\}$, for all $\alpha_i + \alpha_j > 1$. $F_{\alpha_i, \alpha_j}^i(t)$ is the probability of player i conceding to player j by time t (inclusive). Therefore,

$$\lim_{t \rightarrow \infty} F_{\alpha_i, \alpha_j}^i(t) \leq 1 - \pi_i(\alpha_i),$$

where

$$\pi_i(\alpha_i) = \frac{zs_i(\alpha_i)}{zs_i(\alpha_i) + (1-z)r_i(\alpha_i)} \quad (1)$$

is the posterior probability that player i is stubborn immediately after it is observed that i demands α_i at time zero given σ_i^R and σ_i^S . Note that $F_{\alpha_i, \alpha_j}^i(0)$ may be positive. It is the probability that i immediately concedes to j .

Player i 's discount rate is $\rho > 0$, for $i = 1, 2$. The continuous-time bargaining problem is denoted $B = \{z, \rho\}$. If $\alpha_i + \alpha_j \leq 1$ at $t = 0$, the demands are compatible and player i receives α_i and $1 - \alpha_j$ with probability $1/2$. Suppose $\bar{\alpha} = (\alpha_i, \alpha_j)$ is observed at time 0, with $\alpha_i + \alpha_j > 1$. Then player i 's expected payoff from conceding at time t , given strategy profile $\bar{\sigma} = (\sigma_i, \sigma_j)$, where $\sigma_i = (\sigma_i^R, \sigma_i^S)$, is:

$$U_i(t, \sigma^j \mid \bar{\alpha}) = \alpha_i \int_{y < t} e^{-\rho y} dF_{\bar{\alpha}}^j(y) + \frac{\alpha_i + 1 - \alpha_j}{2} (F_{\bar{\alpha}}^j(t) - F_{\bar{\alpha}}^j(t^-)) e^{-\rho t} \\ + (1 - \alpha_j) (1 - F_{\bar{\alpha}}^j(t)) e^{-\rho t}, \quad (2)$$

where $F_{\bar{\alpha}}^j(t^-) = \lim_{y \uparrow t} F_{\bar{\alpha}}^j(y)$. Hence, player i receives the discounted value of his demand α_i if player j concedes to i before i concedes to j . If players concede simultaneously, then player i receives his own demand and the complement of player j 's demand with equal probability. Player i receives the discounted value of the complement of player j 's demand, $1 - \alpha_j$, if player i concedes first. Player i 's expected payoff from never conceding is:

¹In AG, there are $N + 1$ types of players: one rational type and N stubborn types. A stubborn player of type α^i in AG always demands α^i , accepts any offer of at least α^i , and rejects all smaller offers. They assume an exogenously given finite set of stubborn types: $C = \{\alpha^1, \alpha^2, \dots, \alpha^N\}$.

$$U_i(\infty, \sigma^j \mid \bar{\alpha}) = \alpha_i \int_{y \in [0, \infty)} e^{-\rho y} dF_{\bar{\alpha}}^j(y). \quad (3)$$

This is a stubborn player's payoff from facing a demand which is incompatible with his own demand. Since F_{α_i, α_j}^i describes the concession behavior of a player, unconditional on his type, a rational player i 's concession behavior is described by:

$$\frac{1}{1 - \pi_i(\alpha_i)} F_{\alpha_i, \alpha_j}^i.$$

Therefore, a rational player i 's expected utility from a mixed action F^i conditional on $\bar{\alpha} = (\alpha_i, \alpha_j)$ being observed at time 0 is:

$$U_i(\bar{\sigma} \mid \bar{\alpha}) = \frac{1}{1 - \pi_i(\alpha_i)} \int_{y \in [0, \infty)} U_i(y, \sigma_j \mid \bar{\alpha}) dF_{\bar{\alpha}}^i(y). \quad (4)$$

A rational player i 's expected utility from the strategy profile $\bar{\sigma}$ is:

$$\begin{aligned} U_i(\bar{\sigma}) = & \sum_{\alpha_i} r_i(\alpha_i) \left(\sum_{\alpha_j \leq 1 - \alpha_i} \frac{\alpha_i + 1 - \alpha_j}{2} ((1 - z)r_j(\alpha_j) + zs_j(\alpha_j)) \right) \\ & + \sum_{\alpha_i} r_i(\alpha_i) \left(\sum_{\alpha_j > 1 - \alpha_i} U_i(\bar{\sigma} \mid \alpha_i, \alpha_j) ((1 - z)r_j(\alpha_j) + zs_j(\alpha_j)) \right). \end{aligned} \quad (5)$$

The first term is the payoff a rational player receives from demanding α_i when $\alpha_i + \alpha_j \leq 1$. The second term is the payoff from demanding α_i when being faced with an incompatible demand.

Leaving aside the technical issues of defining a revision of offers in continuous time, revising one's offer reveals rationality. As we know from Myerson (1991), revealing rationality when the opponent is stubborn with positive probability leads to immediate concession (i.e., revising one's demand so as to make it compatible with the opponent's). Hence, I can restrict attention to the reduced form game, where offers are made at time 0 once and for all, without loss.

For the analysis in $B = \{z, \rho\}$, I use the solution concept of Perfect Bayesian equilibrium (PBE). As usual, a PBE is a profile of strategies $\sigma^* = (\sigma_1^*, \sigma_2^*)$, and the probability of facing a stubborn type such that the strategy maximizes a player's expected utility (given beliefs), and the beliefs are formed according to Bayes' rule, where possible (see Fudenberg and Tirole, 1991 for a formal definition). From now on, equilibrium refers to PBE.

3 Benchmark

In this section, I recall the unique equilibrium outcome when stubborn players have no choice over their initial demand, as modeled by Abreu and Gul (2000).² This serves as a benchmark for the analysis that follows.

There is an exogenously given set of stubborn types $C = \{\alpha^1, \alpha^2, \dots, \alpha^N\}$, where $\alpha^n < \alpha^{n+1}$ and $\alpha^N < 1$. A stubborn player of type α^i always demands α^i , accepts any offer of at least α^i , and rejects all smaller offers.

I denote the probability that stubborn player i is of type α^n by $s_i(\alpha^n)$. Hence, s_i is a probability distribution on C . The continuous-time bargaining problem is denoted $B^{AG} = \{(C, z, s_i, \rho)_{i=1}^2\}$. Proposition 1 (AG) establishes existence and uniqueness of the equilibrium outcomes with a given distribution of stubborn types.

Proposition 1 (AG, Proposition 2) *For any bargaining game B^{AG} , a Perfect Bayesian equilibrium exists. Furthermore, all equilibria yield the same distribution over outcomes.*

The unique equilibrium outcome in this game can be characterized by the two choices a rational player makes: (1) when to concede, and (2) whom to mimic. In the equilibrium, after the initial choice of demands, (i) at most one player concedes with positive probability immediately; (ii) players concede at a constant rate that makes the opponent indifferent between waiting and conceding; (iii) there is a finite time, call it T_0 , by which the posterior probability of stubbornness reaches 1 simultaneously for both players and concessions by the rational type stop. Moreover, any demand above some threshold is mimicked with positive probability.

I illustrate the mimicking behavior of the rational type in Figure 1. The figure shows the posterior probability of stubbornness in an equilibrium.³ We can see that the lower three demands are not mimicked by the rational type, i.e., $\pi(\alpha | \alpha \leq \frac{1}{3}) = 1$. On the other hand, any demand of $\frac{2}{5}$ or higher is mimicked with positive probability, i.e., $\pi(\alpha | \alpha \geq \frac{2}{5}) < 1$. The U-shaped structure of the posterior probability above the threshold is driven by the concept of strength, as defined and discussed below.

Let me be more precise regarding the rate of concession and the stopping time of the rational type. After time 0, player i demanding α_i concedes to player j demanding α_j at a rate:

$$\lambda_i^{\alpha_i, \alpha_j} = \frac{\rho(1 - \alpha_i)}{\alpha_i + \alpha_j - 1}.$$

²In AG, players make offers sequentially rather than simultaneously (as in my model).

³In particular, I choose a PBE with seven stubborn types $C = \{\frac{1}{15}, \frac{1}{10}, \frac{1}{3}, \frac{2}{5}, \frac{3}{5}, \frac{2}{3}, \frac{9}{10}\}$, and $z = \frac{1}{3}$.

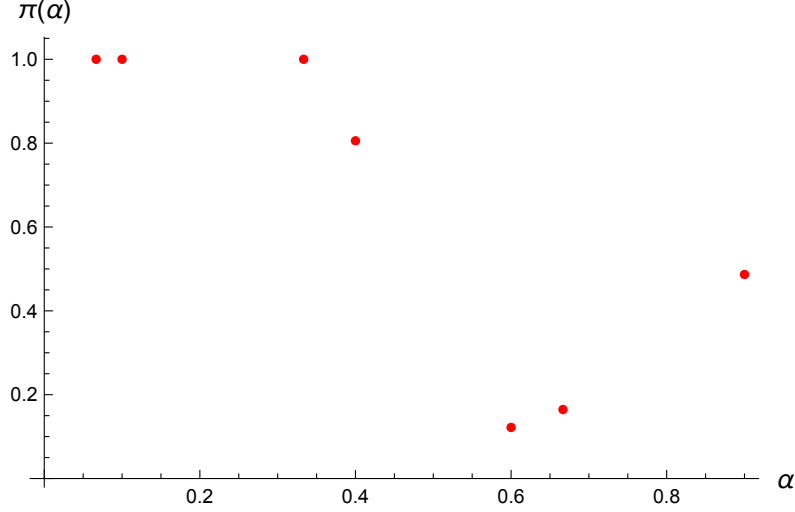


Figure 1: Probability of stubbornness conditional on the demand α , $\pi(\alpha)$, in a PBE with exogenous stubborn types (see body for specification of parameters)

Requirement (iii) pins down the identity of the player who concedes at time 0 as well as the probability with which this happens.⁴ Let $T_i^{\alpha_i, \alpha_j}$ denote the time at which player i is stubborn with probability 1 conditional on not conceding with positive probability at time 0. Then the time T_0 is given by:

$$T_0 = \min\{T_1^{\alpha_1, \alpha_2}, T_2^{\alpha_2, \alpha_1}\},$$

where

$$T_i^{\alpha_i, \alpha_j} = -\frac{1}{\lambda_i^{\alpha_i, \alpha_j}} \log \pi_i(\alpha_i)$$

for $i = 1, 2$. Player i is *stronger* than player j if and only if:

$$T_i^{\alpha_i, \alpha_j} < T_j^{\alpha_j, \alpha_i}.$$

In other words, a player is stronger the sooner the time at which he is known to be stubborn. The weaker player i has to concede with sufficient probability at time zero, so that conditional on not conceding, and given the concession rates, his probability of stubbornness reaches 1 at the same time as player j . The strength of player i relative to player j depends on (i) how likely i is thought to be stubborn conditional on his demand, (ii) how high i 's demand is. Clearly, the more likely a player is thought to be stubborn, the more willing the opponent is to give up. The

⁴For an intuition for (iii), see AG page 10.

higher a player's demand, the more willing his opponent is to wait. This is because conditional on giving up, a player gets less the higher his opponent's demand. Hence, the lower the demand a player makes, the stronger he is because it makes his opponent more willing to give up. To capture this intuition, it is useful to introduce the concept of "offer-adjusted reputation." Let $\mu_i(\alpha_i)$ denote the offer-adjusted reputation of player i when demanding α_i :

$$\mu_i(\alpha_i) = \pi_i(\alpha_i)^{\frac{1}{1-\alpha_i}}. \quad (6)$$

Player i is weaker than player j , (and hence, i concedes to j at time 0 with positive probability) if and only if i has a lower offer-adjusted reputation: $\mu_i(\alpha_i) < \mu_j(\alpha_j)$. In particular, the probability of immediate concession of player i is given by:

$$F_{\alpha_i, \alpha_j}^i(0) = \max \left\{ 1 - \left(\frac{\mu_i(\alpha_i)}{\mu_j(\alpha_j)} \right)^{1-\alpha_i}, 0 \right\}.$$

The derivation follows AG.

Let me return to the U-shape of the posterior probability of stubbornness in Figure 1. Suppose player j demands α_j with probability 1 and is thought to be stubborn with probability $\pi(\alpha_j)$. Then, fixing the probability of player i being stubborn, the preferences of a rational player i are single-peaked in his own demand α_i : He trades off the probability with which his opponent concedes at time 0, with how high his payoff is conditional on his opponent conceding (see Section 5.2 for a more detailed discussion of preferences). This implies that in equilibrium, the conditional probability of stubbornness must be single-bottomed in α_i as Figure 1 shows.

To convey the message of AG, I allow discount rates to differ in the next proposition. Let $v_i = \rho_i + \rho_j$ and let $\underline{v}_i := \max\{\alpha \in C \cup \{0\} \mid \alpha < v_i\}$.

Proposition 2 (AG, Corollary in Section 5) *Let $B_n^{AG} = \{(C, z_n, s_i, \rho_i)_{i=1}^2\}$ be a sequence of continuous-time bargaining games such that $\lim_{n \rightarrow \infty} z_n = 0$. Let ϵ be the mesh of C .⁵ Then for n sufficiently large, the equilibrium payoff of agent i is at least $\frac{\rho_i}{\rho_i + \rho_j} - \epsilon$, and hence, the inefficiency due to delay is at most 2ϵ .*

Proposition 2 says that as the probability of a player being stubborn goes to 0, delay and inefficiency disappear provided the "right" behavioral type is present. The right type is the type making a demand proportional to a player's patience. In the symmetric discounting case, the right type then is a type demanding 1/2. A rational player in AG gets a payoff proportional to his patience.

⁵I.e., $\max_k (\alpha_{k+1} - \alpha_k)$, ordering the offers from smallest to largest.

As I will explain, the results differ markedly once stubborn players can choose their initial demand freely—I derive strong predictions in terms of behavior, but delay (and hence, inefficiency) may not disappear even in the limit, at least when no refinement is applied.

4 Necessary conditions for equilibrium existence

This section is divided into two parts. In the first part, I show that there can be at most one separating offer. By separating offer, I mean an offer which is exclusively made by only one type, rational or stubborn. I argue that if it exists, it is made by the stubborn type. The nature of such “hybrid equilibria” is postponed to Section 6. In the second part of this section, I focus on symmetric PBE in which all offers are made with positive probability by both types. I provide necessary conditions for such equilibria to exist.

4.1 (Non-)Existence of separating offers

In this subsection, I use the following refinement of PBE:

Assumption 1 *When there are two offers revealing rationality, each player receives as payoff the limit of the (unique) equilibrium payoff in the game, where each player is believed to be stubborn with probability z .*

There are two reasons to focus on such equilibria. The first is technical: it ensures continuity of the equilibrium payoff correspondence as $z \rightarrow 0$. The second is that such a payoff is an equilibrium payoff in the game with complete information.

Lemma 1 *There can be at most one offer which is only made by the stubborn type.*

Proof (Sketch). Suppose there were two demands which are not mimicked by the rational type. Then both of these demands would be conceded to immediately with probability of almost 1 as $z \rightarrow 0$. In that case, the higher demand would be strictly preferred by the stubborn type. Hence, there can be at most one such demand. By virtue of a rational player’s payoff being strictly higher than a rational player’s payoff, a rational player would not want to mimic this demand. ■

Lemma 2 *There can be at most one offer which is only made by the rational type.*

Proof (Sketch). In the following, I refer to an offer which is made by both types as a *hybrid* offer. There can be no hybrid offer which is higher than a rational offer. A rational player would

receive a strictly higher payoff regardless of the offer faced: when being faced with a compatible offer, then the higher offer (i.e., the hybrid) would receive a higher payoff. When being faced with an incompatible offer, the opponent would never concede to the rational offer at time 0; However, the rational offer would immediately concede to the highest hybrid offer. Hence, there cannot be a hybrid offer which is higher than a rational offer. Note that this implies that there is no offer which is compatible with the rational offer. As a result, if there are multiple rational offers, then all rational offers have to be higher than the highest hybrid offer. Suppose there were several rational offers. Then they would receive the same payoff from facing any incompatible hybrid offer. However, the payoff when being faced with another rational offer is strictly higher for the lowest rational offer by Assumption 1. ■

Lemma 3 *There exists $\bar{z} > 0$ such that for any $z < \bar{z}$, there exists no symmetric equilibrium with $\text{supp } r = \{\alpha_h, \alpha_r\}$ and $\text{supp } s = \{\alpha_h\}$.*

Proof. See appendix. ■

From the previous lemma, it is clear that the rational offer α_r must be the higher offer. However, if it is the higher offer, then the rational type strictly prefers to demand it over the lower offer α_h .

I will discuss existence of equilibria with an offer only made by the stubborn type in Section 6.

4.2 Necessary conditions for existence of mixed PBE

For the remaining part of this section, and the next section I will focus exclusively on symmetric equilibria, where all offers are made by both types. Hence, I will drop subscript i . In this subsection, I provide necessary conditions for existence of symmetric equilibria, where all offers are made by both types. The analysis of the model is considerably simplified by the preliminary proposition below. The proposition imposes structure on the demand configurations that can occur in equilibrium.

Proposition 3 *Fix any set of demands C . In any symmetric equilibrium with $\text{supp } r = \text{supp } s = C$, the following holds:*

1. *the offer-adjusted reputation, $\mu(\alpha)$, is decreasing in $\alpha \in C$;*
2. *the lowest and the highest demand in C played with positive probability (by some type) are incompatible (unless there is only one offer);*

3. *the set (or, equivalently, number) of compatible equilibrium demands is strictly decreasing in the demand; i.e., if $\alpha < \alpha'$, with $\alpha, \alpha' \in C$, then there exists $\alpha'' \in C$ such that $\alpha + \alpha'' \leq 1 < \alpha' + \alpha''$.*

Proof. See Appendix. ■

Proposition 3 is trivial if $|C| = 1$. Hence, from now on, suppose $|C| \geq 2$. The proof of Proposition 3 builds on the following insight. The difference between a rational and a stubborn player demanding α is the payoff when faced with an incompatible demand α' . The rational player has the option value of being able to concede. Once a player is certain that the opponent is stubborn a player would strictly prefer to concede. However, a stubborn player is unable to do so. Hence, when faced with an incompatible demand, the payoff for a rational player is strictly higher than the payoff for a stubborn player. Therefore, unless every demand made with positive probability is compatible with every other demand, the equilibrium payoff for a rational player must be strictly higher than the payoff for a stubborn player. Suppose every demand is compatible with every other demand made with positive probability, i.e., all demands have to lie below $1/2$. This cannot be an equilibrium since a rational player would then strictly prefer to deviate to a demand above $1/2$. Hence, the payoff a rational player receives is strictly higher than the payoff a stubborn player receives.

Sketch of proof of Proposition 3

1. $F_{\alpha_i, \alpha_j}^i(0)$ is increasing in $\mu_j(\alpha_j)$. If $\mu_j(\alpha_j)$ was increasing in α_j , a player would always benefit from increasing his demand α_j . This is inconsistent with a player being indifferent between demands.
2. Suppose the lowest demand was compatible with the highest demand. Then the payoff from making the lowest demand is the same for a rational and a stubborn player. But if it is the same for the lowest demand, it must be the same for every other demand made with positive probability. By the above argument, this cannot be.
3. Consider two consecutive demands that a rational player is indifferent between. This implies that the increased gain from the higher offer is precisely offset by the decrease in the option value of concession. The stubborn player does not have this option and hence, strictly prefers the higher demand.

5 Existence

This section presents the main results of the paper. In the first part, I show that there is a unique symmetric equilibrium with one offer. I then argue that the real challenge is to create indifference over multiple demands because the two types of players have different payoff functions. I show that the two types of players can indeed be made indifferent over the same set of demands. However, in the limit, as the probability of stubbornness goes to 0, there are at most three offers over which players are indifferent. Moreover, at most two offers are made strictly positive probability. I then turn to the question of inefficiency, showing that even in the limit, delay may not disappear. As a result, there is a Folk theorem like payoff multiplicity (at least when no refinement is applied). I then turn to equilibrium selection with refinements, in particular passive beliefs and D1.

5.1 Existence with one offer

Lemma 4 below establishes that there is a unique symmetric equilibrium with one offer. In this equilibrium, there is no delay (and hence, no inefficiency).

Lemma 4 *There is a unique symmetric equilibrium, where players make only one demand. In this equilibrium, players demand $1/2$.*

Proof. Suppose players choose a demand $\alpha < 1/2$. Then both types of players have an incentive to deviate to $1 - \alpha$. Suppose instead players choose a demand $\alpha > 1/2$. The expected payoff for a rational player in this candidate equilibrium is $1 - \alpha$. The expected payoff for a stubborn player from demanding α is $(1 - \alpha)(1 - z^{\frac{\alpha}{1-\alpha}})$. However, a stubborn player could receive $1 - \alpha$, by demanding $1 - \alpha$. If players demand $1/2$, then $\alpha = 1 - \alpha$, and hence, there is no such deviation. Suppose $\alpha = 1/2$. Then if any deviation is believed to come from a rational type, neither player type wants to deviate. ■

Hence, when focussing on symmetric equilibria with one demand, I derive very strong predictions in terms of payoffs and behavior. Independent of the prior probability of a player being stubborn, there is no inefficiency. Subsection 5.4 shows that this does not generalize to equilibria with more than one demand. In fact, there is delay in any equilibrium with more than one demand, even when the probability of stubbornness goes to 0. The reader might wonder if it is reasonable to require players to put probability 1 on any deviation coming from the rational type – I defer the discussion on off-equilibrium path beliefs to Section 5.5, where refinements are introduced. Given this belief however, it is clear that deterring deviations is straightforward.

Hence, the real question is whether indifference over multiple demands is possible. This is what I turn to next.

5.2 Existence with two offers

In this subsection, I show that equilibria, where the two types of players are mixing over multiple (two) demands exist. After stating the result formally, I provide intuition by discussing the preferences of the two types.

Proposition 4 (a) *Fix a sequence $z^n \rightarrow 0$. Fix a corresponding convergent sequence of equilibria $(\alpha^n, \beta^n, r^n, s^n)$. Then there exist $a \in (0, 1/2]$ and $b \in (1 - a, 1]$ such that*

$$\lim_{n \rightarrow \infty} \alpha^n = a, \quad \lim_{n \rightarrow \infty} \beta^n = b. \quad (7)$$

Moreover, in any such sequence,

$$\lim_{n \rightarrow \infty} \begin{pmatrix} r_n(\alpha^n) \\ r^n(\beta^n) \end{pmatrix} = \begin{cases} \frac{2(a+b-1)}{2b-1}, \\ \frac{1-2a}{2b-1}, \end{cases} \quad \text{and} \quad \lim_{n \rightarrow \infty} \begin{pmatrix} s^n(\alpha^n) \\ s^n(\beta^n) \end{pmatrix} = \begin{cases} \frac{1-b}{2-a-b}, \\ \frac{1-a}{2-a-b}. \end{cases} \quad (8)$$

(b) *For any $a \in (0, 1/2]$ and $b \in (1 - a, 1]$, there exists a sequence $z^n \rightarrow 0$ and a convergent sequence of corresponding equilibria $(\alpha^n, \beta^n, r^n, s^n)$ satisfying (7) and (8).*

Proof. See Appendix. ■

Proposition 4 says that a stubborn player and a rational player can be indifferent between the same two demands, despite the distinct preferences in the reduced game, given their strategic differences. Note that even in the limit, as the probability of stubbornness goes to 0, both types put strictly positive probability on both demands. This may be surprising to the reader – it implies that unlike in standard models of signaling, preferences in my (reduced-form) model do not satisfy the single-crossing property.

Figure 2 shows the 3D-payoff profile of a rational and stubborn player i respectively as a function of his own demand α and $\pi(\alpha)$ when faced with an opponent j who places positive probability on two demands, 3/10 and 8/10, for a given pair of conditional probabilities of stubbornness for player j .⁶ In particular, it shows the equilibrium payoff of a rational (stubborn) player i when the opponent j mixes over 3/10 and 8/10, and I take α_i and $\pi(\alpha)$ as given (not necessarily optimal). For both types of player, the payoff is increasing in the probability of being

⁶In particular, I use the equilibrium probabilities of stubbornness conditional on the demands 3/10 and 8/10, respectively.

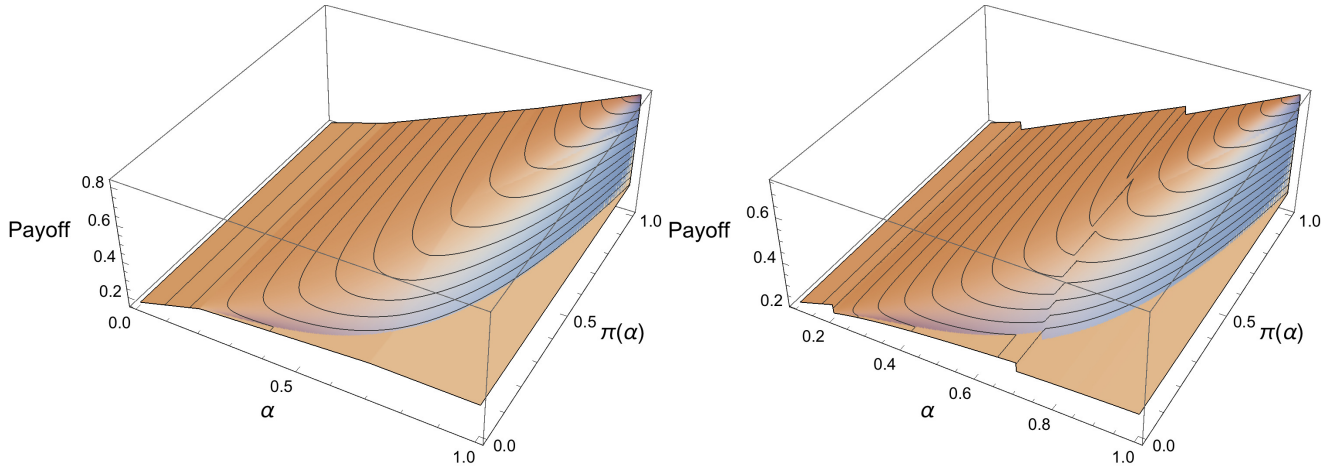


Figure 2: 3D-Payoff profile for a rational type (left) and a stubborn type (right) for a fixed set of demands and posterior probabilities of the opponent

thought to be stubborn, $\pi(\alpha)$. This is not surprising: the higher the probability that a player is thought to be stubborn, the more likely an opponent is to give up immediately at time 0. The payoff is non-monotonic in the demand α . Conditional on the opponent conceding, there is a benefit to making a higher demand. However, the probability of immediate concession is decreasing in the demand.

Consider an equilibrium with $z = 1/100$, $\alpha = 3/10$, and $\beta = 8/10$. Figure 4 shows the indifference correspondences of a rational and stubborn type respectively in this equilibrium (rational type in red, stubborn type in black). We can see that the indifference correspondences cross at $3/10$ and $8/10$. Figure 3 shows a cross-section of the 3D-payoff profile of player i as a function of α_i and π_i . In particular, I take the cross-section through $(3/10, \pi(3/10))$ and $(8/10, \pi(8/10))$, where $\pi(3/10)$ and $\pi(8/10)$ are the equilibrium probabilities of stubbornness. We can see that there is a discontinuity in the payoff of the stubborn type at $\alpha_i = 2/10$ and $\alpha_i = 7/10$, as α_i becomes incompatible with $8/10$ and $3/10$ respectively.

The difference between a rational type and a stubborn type demanding α is the payoff when faced with an incompatible demand α' . The rational type has the option value of concession – if his opponent is known to be stubborn, the rational type can (and strictly prefers to) concede, while the stubborn type cannot. Suppose a player is faced with a demand of α' . Then a rational and stubborn type alike receives $1 - \alpha'$ from demanding $1 - \alpha'$. A rational type receives at least $1 - \alpha'$ from demanding $1 - \alpha' + \epsilon$ for any $\epsilon > 0$ – he can always concede to this opponent and receive $1 - \alpha'$ immediately.

On the other hand, for $\epsilon > 0$ small enough, a stubborn type receives strictly less than $1 - \alpha'$

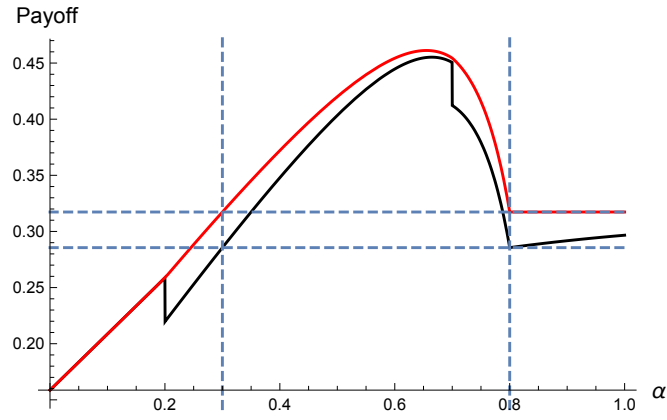


Figure 3: Cross-sections of the 3D-payoff profile for rational (red) and stubborn (black) type (see body for specification of z)

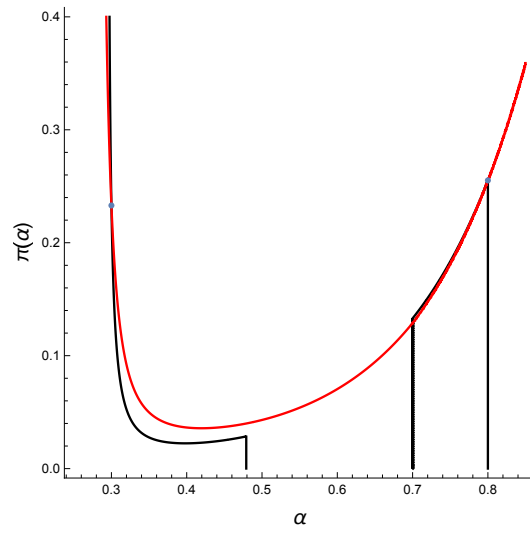


Figure 4: Indifference correspondences for rational (red) and stubborn (black) type (see body for specification of parameters)

from demanding $1 - \alpha' + \epsilon$. When making a compatible demand, such as $1 - \alpha'$, a stubborn type pays no cost of being faced with a (possibly) stubborn opponent. When demanding $1 - \alpha' + \epsilon$, his demand is incompatible with his opponent's. There is a positive probability that the player will be faced with a stubborn opponent, in which case he receives a payoff of 0. As $\epsilon \rightarrow 0$, he receives at most $1 - \alpha' + \epsilon \rightarrow 1 - \alpha'$ when facing a rational opponent. However, with probability $\pi_{\alpha'}$ he faces a stubborn opponent. This implies that there is a downward jump in the payoff of a stubborn player i 's when demanding $1 - \alpha' + \epsilon$ rather than $1 - \alpha'$. More generally, fixing a set of possible demands by the opponent, the payoff function of a stubborn player i from demanding α_i is discontinuous in α_i . Each demand made by the opponent with positive probability implies a discontinuity in the payoff of a stubborn player from demanding α_i as we vary α_i . The size of the downward jump in the payoff at $1 - \alpha'$ is determined by the probability that the demand α' is made by a stubborn opponent; i.e., the downward jump in the payoff is larger the more likely the stubborn type is faced with a stubborn opponent. This implies that when the (unconditional) probability of a player being stubborn is small, the size of the jumps goes to 0. The cost of facing an incompatible demand from a stubborn opponent implies that the indifference correspondence of the stubborn type can cross the indifference curve of the rational type multiple times.

More generally, fixing the opponent's demand and probability of stubbornness, a rational player i 's payoff is single-peaked in α_i . In other words, there is a unique best reply for a rational player to a given demand of the opponent. The same is not true for a stubborn player – fixing the opponent's demand and probability of stubbornness, a stubborn player's payoff has three local peaks as we vary α_i (either a demand is compatible with the opponent's, it is incompatible but has a higher offer-adjusted reputation, or it is incompatible and has a lower offer-adjusted reputation).

The smaller the (unconditional) probability of a player being stubborn, the smaller the payoff jumps of the stubborn type are. The smaller the jump, the closer together any two demands between which both types of players are indifferent (one demand weakly below the demand at which the jump occurs and one above the jump). However, Proposition 3 imposes significant structure on the demand configurations that can occur in equilibrium. In the case of three demands, the lower two demands need to be compatible. Hence, as $z \rightarrow 0$, it becomes “more difficult” to make players indifferent between different demands. This is what I turn to in the next subsection.

5.3 Existence with three or more offers

The proposition below states that players can be made indifferent over more than two offers. However, the demand configurations over which players can be indifferent have a very specific structure. Define

$$\begin{aligned} s_a &= \begin{cases} \frac{1-c}{2-a-c} & \text{if } a > 1 - \frac{c}{4} - \sqrt{c(8-7c)}, \\ 0 & \text{if } a < 1 - \frac{c}{4} - \sqrt{c(8-7c)}, \end{cases} \\ s_b &= \begin{cases} 0 & \text{if } a > 1 - \frac{c}{4} - \sqrt{c(8-7c)}, \\ 1 & \text{if } a < 1 - \frac{c}{4} - \sqrt{c(8-7c)}, \end{cases} \\ s_c &= \begin{cases} \frac{1-a}{2-a-c} & \text{if } a > 1 - \frac{c}{4} - \sqrt{c(8-7c)}, \\ 0 & \text{if } a < 1 - \frac{c}{4} - \sqrt{c(8-7c)}. \end{cases} \end{aligned}$$

Proposition 5 (a) Fix any set of three demands $C = \{a, b, c\}$, with $1/2 < b < 1 - a < c \leq 1$.

Then there exists $\bar{z} > 0$, such that for any $z < \bar{z}$, there exists no symmetric equilibrium with support C .

(b) Fix a sequence $z^n \rightarrow 0$, and a corresponding convergent sequence of equilibria $(\alpha^n, \beta^n, \gamma^n, r^n, s^n)$.

Then there exist $a \in (0, 1/2]$ and $c \in (1 - a, 1]$ such that

$$\lim_{n \rightarrow \infty} (\alpha^n, \beta^n, \gamma^n) = (a, 1 - a, c). \quad (9)$$

Moreover, in any such sequence,

$$\lim_{n \rightarrow \infty} r^n = (0, 1, 0), \quad \lim_{n \rightarrow \infty} s^n = (s_a, s_b, s_c). \quad (10)$$

(c) For any $a \in (0, 1/2]$ and $c \in (1 - a, 1]$, there exists a sequence $z^n \rightarrow 0$ and a corresponding convergent sequence of equilibria $(\alpha^n, \beta^n, \gamma^n, r^n, s^n)$ satisfying (9) and (10).

Proof. See Appendix. ■

Note that Proposition 5 says in the limit, players are faced with a demand of $1 - \alpha$ with probability 1.

Conjecture 1 (a) Fix any set of demands C , where $|C| > 3$. Then there exists $\bar{z} > 0$, such that for any $z < \bar{z}$, there exists no equilibrium with support C .

(b) Fix a sequence $z^n \rightarrow 0$, and fix a corresponding convergent sequence of equilibria $(\alpha_1^n, \dots, \alpha_K^n, r^n, s^n)$, where $K > 3$. Then there exists $a \in (0, 1/2]$ and $c \in (1 - a, 1]$ such that

$$\lim_{n \rightarrow \infty} (\alpha_1^n, \dots, \alpha_{k-1}^n, \alpha_k^n, \dots, \alpha_N^n) = (a, \dots, a, 1 - a, \dots, 1 - a, c), \quad (11)$$

where $\alpha_k = \lceil K/2 \rceil$.

(c) For any $a \in (0, 1/2]$ and $c \in (1 - a, 1]$, there exists a sequence $z^n \rightarrow 0$ and a corresponding convergent sequence of equilibria $(\alpha_1^n, \dots, \alpha_K^n, r^n, s^n)$ satisfying (11).

Proof. To be completed. ■

The intuition for Proposition 5 and the heuristic argument for Conjecture 1 is as follows. I order demands from lowest to highest, denoting the lowest demand by α_1 , and the highest demand by α_N . Throughout, suppose that there are no offers made exclusively by one type.

Step 1: If there are more than three offers, then α_1 and α_2 have to be virtually identical. If there are exactly three offers, then α_1 and α_2 have to be exactly compatible (i.e., add up to 1). Recall that the lowest demand is compatible with all but the highest demand; and that the second lowest demand is compatible with all but the highest and second highest demand – this follows from Proposition 3. Hence, the payoff from demanding the second lowest offer α_2 is strictly higher than the payoff from demanding the lowest offer α_1 except when faced with the highest demand α_N . Hence, for a player to be indifferent between α_1 and α_2 , then either (1) α_N has to concede to α_1 much more likely than to α_2 so as to offset the payoff difference, or (2) α_1 and α_2 are nearly identical.

Suppose (1). Recall that the probability of immediate concession is pinned down by the probability of being thought to be stubborn adjusted for the level of the demand (i.e., by the offer adjusted reputation). Fixing demands, the more likely a player is thought to be stubborn, the more likely he will be conceded to immediately. Fixing the probability of being thought to be stubborn, the higher a player's demand the less likely he is conceded to immediately.

If α_N is to concede to α_1 much more likely than α_N concedes to α_2 , (a) α_N must be thought to be stubborn much more likely than α_2 or (b) α_2 is nearly identical to α_N and similarly likely to be thought to be stubborn.

Suppose (a). If z is close to 0, then this implies that the rational type is unlikely to demand α_N . However, this implies that for a rational player to be indifferent between α_1 and α_2 , then either (i) the two offers α_1 and α_2 are nearly identical so as to minimize the payoff difference from facing any offer $\alpha_i \neq \alpha_N$. Note that if α_1 and α_2 are almost identical, then α_{N-1} is nearly identical to $1 - \alpha_1$; or (ii) players are faced with a demand of α_{N-1} almost certainly; and the

payoff from facing a demand of α_{N-1} is nearly identical for α_1 and α_2 . When making a demand of α_1 , the payoff from facing an α_{N-1} is α_1 and $1 - \alpha_{N-1}$ with equal probability. When making a demand of α_2 , the payoff is a weighted sum of α_2 and $1 - \alpha_{N-1}$. Recall that

$$\alpha_1 \leq 1 - \alpha_{N-1} \leq \alpha_2.$$

Hence, for the payoff from the two demands α_1 and α_2 to be “the same,” α_{N-1} must be equal to $1 - \alpha_1$. Secondly, it requires that α_{N-1} concedes to α_2 with low probability (or α_2 is virtually identical to α_1). But for α_{N-1} to concede to α_2 with low probability, it must be that α_2 is close to α_{N-1} .⁷ Suppose (b). Then it must be that $\alpha_2 = \alpha_{N-1}$.

Hence, if there are more than three offers, it must be that α_1 and α_2 are almost identical, and hence, α_{N-1} must be equal to $1 - \alpha_1$. Moreover, if there are exactly three offers, then these offers must be close to $(\alpha_1, 1 - \alpha_1, \alpha_N)$.

Step 2: If there are more than three demands, then all offers below $1/2$ are clustered around α_1 , and all offers above $1/2$ except the very highest, are clustered around $1 - \alpha_1$. Suppose, α_2 is almost identical to α_1 ; and hence, α_{N-1} is almost identical to $1 - \alpha_1$.

The rational payoff from demanding α_N is higher than the payoff from demanding α_1 except when faced with a demand of α_N . Recall, α_N is incompatible with all offers; and α_1 is compatible with all but the highest offer. It is then clear that α_N gets a strictly higher payoff than α_1 from any demand lower than or equal to α_{N-2} . Recall that α_{N-1} is virtually identical to $1 - \alpha_1$, which implies that the payoff from demanding α_1 and α_N is virtually identical when faced with a demand of α_{N-1} .

For a player to be indifferent between α_N and α_1 , then either (1) the highest offer has to be played with sufficient probability so as to offset the payoff difference; or (2) offers where α_N and α_1 receive a different payoff are played with probability close to 0. Suppose (1). If α_{N-1} is virtually identical to α_1 , then α_N and α_{N-1} receive the same payoff from all demands by their opponent except when faced with a demand of α_N . If α_N concedes with positive probability to α_{N-1} , then α_{N-1} is preferable to α_N . Hence, for a player to be indifferent between the highest two offers, it must be that α_N hardly concedes to α_{N-1} . But this is inconsistent with α_N being played with high probability. Suppose (2). When faced with a demand of less than $1/2$, the payoff from demanding α_N (strictly more than $1/2$) is strictly higher than the payoff from demanding α_1 (at most $1/2$). Hence, any demand less than $1/2$ is played with probability close to 0. Moreover, any demand above $1/2$ is either close to $1 - \alpha_1$ or is played with probability close to 0. If demands above $1/2$ are close to $1 - \alpha_1$, this implies that all demands below $1/2$ have to be close to α_1 .

⁷It cannot be that α_{N-1} is thought to be much more likely to be crazy than α_2 if α_{N-1} is players are faced with a demand of α_{N-1} almost certainly.

5.4 Inefficiency and payoffs in the limit

Proposition 4 states that the necessary conditions for equilibrium existence in Proposition 3 are sufficient if players put strictly positive probability on two demands only. For such an equilibrium to exist, the lower demand α must be (weakly) less than $1/2$, and α and β must add up to strictly more than 1. In such an equilibrium, when the probability of stubbornness is small, the higher demand β immediately concedes to the lower demand α with probability close to 1. When both players choose the higher demand, they engage in a war of attrition with an expected payoff of $1 - \beta$. Therefore, even in the limit, delay (and hence, inefficiency) may not disappear. It is clear, that when α is close to 0 (and hence, β), a rational player's expected equilibrium payoff is close to 0. If on the other hand both demands are close to $1/2$, a rational player's expected payoff is close to $1/2$ (when players are equally patient). By adjusting α and β one can generate in this fashion any payoff between 0 and $1/2$. Corollary 2 formalizes this insight.

Corollary 2 *Fix any $v \in (0, 1/2]$. Then there exists \bar{z} such that for any $z < \bar{z}$, a symmetric equilibrium exists such that the equilibrium payoff for a rational agent is v .*

Proof. This follows immediately from Proposition 4. Fix any equilibrium characterized in Proposition 4. Denote the payoff of a rational player in this equilibrium by v_r . Then

$$\lim_{z \rightarrow 0} v_r = \frac{2(a + b - 1)}{2b - 1}(1 - a) + \frac{1 - 2a}{2b - 1}(1 - b).$$

Fix any $\epsilon > 0$ and set $a < \epsilon$. Then $b > 1 - \epsilon$. The result immediately follows. ■

Hence, unlike with an exogenously given distribution of stubborn types, there is a Folk theorem like payoff-multiplicity when stubborn types can choose their initial demand freely. This is induced by the delay to agreement. For delay to disappear in the limit with exogenous stubborn types, AG require the “right” stubborn type to be present. In the symmetric discounting case, this would be the type demanding $1/2$. Corollary 2 shows that when the stubborn type is given choice over his initial demand, the “right” stubborn type may not be present. When he is not, delay (and hence, inefficiency) do not disappear even when the probability of a player being stubborn is infinitely small. It is natural to ask if I can derive stronger predictions regarding payoffs (and inefficiency) when using refinements.

5.5 Refinements

5.5.1 Passive beliefs

Denote the belief about a player being stubborn conditional on the out-of-equilibrium demand d by π_d . Suppose a player puts probability z on his opponent being stubborn if the opponent

chooses an out-of-equilibrium demand d , i.e., $\pi_d = z$. An equilibrium is a passive belief equilibrium if there exists no profitable deviation d conditional on $\pi_d = z$.

Lemma 5 *There is a unique passive belief equilibrium. In this equilibrium, a player makes a demand proportional to his patience.*

Proof. See appendix. ■

When players are equally patient as assumed throughout this paper, this implies that players will demand $1/2$. In equilibria with more than one offer, at least one offer must have a posterior probability of stubbornness less than z . This implies that the rational type would benefit from deviating to an offer within $\epsilon > 0$ of this offer. Note that Lemma 5 makes no restriction on the type of equilibrium considered, i.e., it includes asymmetric equilibria.

5.5.2 Divinity

Loosely speaking, the refinement D1 attaches probability 1 to the type with the strongest incentive to deviate to a given demand. More formally, denote the set of types by $\Theta = \{R, S\}$, where R stands for rational and S for stubborn. Let $u_i^*(\theta)$ be the equilibrium payoff of type $\theta \in \{R, S\}$. Define $D(\theta, S, d)$ to be the set of mixed-strategy best responses (MBR) F_2 to demand d and beliefs concentrated on S that make type θ strictly prefer d to his equilibrium strategy,

$$D(\theta, S, d) = \cup_{\mu: \mu(S|d)=1} \{F_2 \in MBR(\mu, d) \text{ s.t. } u_1^*(\theta) < u_1(d, F_1, \theta)\},$$

and let $D^0(\theta, S, d)$ be the set of mixed best responses that make type θ exactly indifferent. A type θ is deleted for demand d under criterion D1 if there is a θ' such that

$$\{D(\theta, \Theta, d) \cup D^0(\theta, \Theta, d)\} \subset D(\theta', \Theta, d).$$

In other words, if the set of best responses (and associated beliefs about a player being stubborn conditional on d) for which a rational player benefits from deviating to d is strictly smaller than the set of best responses for which a stubborn player benefits from deviating to d , then D1 puts probability 0 on the deviation coming from a rational player. Note that D1 is not defined for dynamic games to beyond signaling games. However, first, note that, given the realized demands and associated beliefs, I can compute the expected payoff from the continuation game. Hence, I can associate to my game a corresponding game which ends once offers are chosen. This is the game on which I apply D1.

Lemma 6 *The unique symmetric one-offer equilibrium satisfies D1.*

Proof. The payoff for both types of player in the symmetric one-offer equilibrium is $\alpha = 1/2$. If a player was to deviate, he would deviate to $d > 1/2$. Denote the belief that the demand d is made by a stubborn player by s_d . The payoff from d for a stubborn player is:

$$v_s^d = (1 - z^{2d}) \left(d \left(1 - z s_d^{-\frac{1}{2(1-d)}} \right) + \frac{1}{2} z s_d^{-\frac{1}{2(1-d)}} \right).$$

A rational player's payoff from demanding d is:

$$v_r^d = d \left(1 - z s_d^{-\frac{1}{2(1-d)}} \right) + \frac{1}{2} z s_d^{-\frac{1}{2(1-d)}}.$$

It follows immediately that the threshold belief (and the resulting action by the opponent) at which the stubborn player prefers to deviate is strictly above the threshold belief at which the rational player prefers to deviate. Hence, D1 puts probability 1 on a deviation coming from a rational player. As a result, there is no profitable deviation from $\alpha = 1/2$ for either type of player. ■

Note that, unlike with passive beliefs, my claim is only about symmetric equilibria. Any asymmetric one offer equilibrium (i.e., where player i demands α_i and j demands $\alpha_j = 1 - \alpha_i$) satisfies D1.

Lemma 7 *There is a unique equilibrium satisfying D1, in the set of symmetric equilibria with at most two offers.*

Proof. See appendix. ■

Conjecture 3 *There is a unique equilibrium satisfying D1 in the set of symmetric equilibria with at most three offers.*

Proof. To be completed. ■

Let me illustrate D1 in a simple two demand equilibrium. Consider an equilibrium with $(\alpha, \beta, z) = (\frac{1}{3}, \frac{3}{4}, \frac{1}{100})$. It is straightforward to derive the threshold MBR $D^0(\theta, S, d)$ for $\theta = R, S$ which makes a player of type θ indifferent between α (or equivalently β) and d . Figure 5 shows the ratio of the “threshold beliefs” at which a player is indifferent between the equilibrium demand and deviating to a demand $d \in (\alpha, 1 - \alpha]$. We can see that the ratio is consistently above 1. Hence, the set of MBR and associated beliefs for which a rational player benefits from deviating to d is a proper subset of the set of MBRs and associated beliefs for a stubborn player for any $d \in (\alpha, 1 - \alpha]$. But if a player was thought to be stubborn with probability 1 if

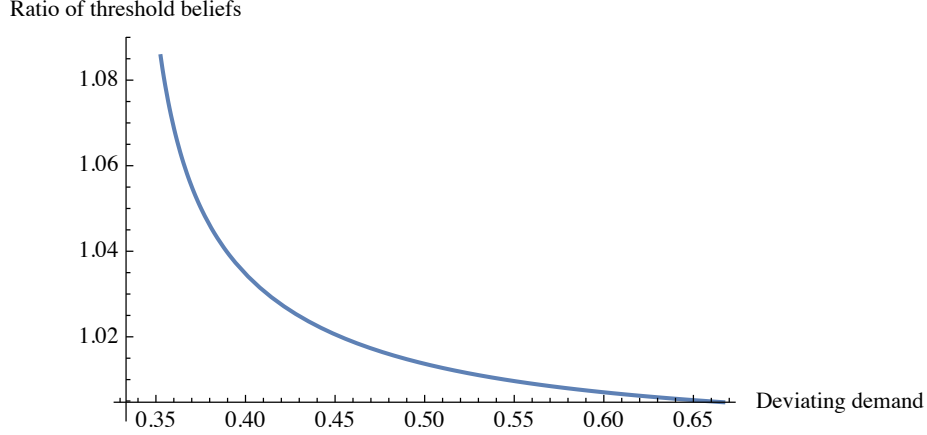


Figure 5: Ratio of the threshold belief of a rational player/threshold belief of a stubborn player

he makes any demand in $(a, 1 - a]$, this is a profitable deviation. Hence, the equilibrium with $(\alpha, \beta, z) = (\frac{1}{3}, \frac{3}{4}, \frac{1}{100})$ does not satisfy $D1$.⁸

The difference between a rational and a stubborn player is the payoff when faced with an incompatible demand. The rational player has the option value of concession – if his opponent is known to be stubborn, the rational player can (and strictly prefers to) concede, while the stubborn player cannot. In fact, the stubborn player’s payoff is 0 when faced with an incompatible demand from a stubborn opponent. Hence, the stubborn player is at a disadvantage when being faced with an incompatible demand relative to the rational player. When faced with a demand of α , the payoff of a stubborn player as a function of his own demand jumps down at $1 - \alpha$. This makes a deviation $d \in (a, 1 - a]$ “more attractive” for the stubborn type. This suggests that $D1$ (or passive beliefs) may substitute for exogenous behavioral types as in AG.

6 Existence with a separating offer

In Section 4, I show that there can be at most one separating offer, and that this separating offer has to be made by the stubborn type. In this section, I show that such equilibria do indeed exist.

Lemma 8 *(a) Fix any $\alpha > 1/2$. There exists $\bar{z} > 0$, such that for all $z < \bar{z}$, there exists a symmetric equilibrium with $\text{supp } r = \{\alpha\}$ and $\text{supp } s = \{1 - \alpha, \alpha\}$.*

⁸My choice of demands as well as the probability of a player being stubborn is without loss. Fixing $d = 1/2$, I have plotted the ratio of the threshold beliefs for any α, β . It is clear that the ratio is consistently above 1, and hence, $D1$ rules out any such equilibrium.

- (b) Fix any α_h, α_s with $\alpha_h + \alpha_s \neq 1$. There exists no symmetric equilibrium with $\text{supp } r = \{\alpha_h\}$ and $\text{supp } s = \{\alpha_h, \alpha_s\}$.

While I have not established existence for such hybrid equilibria with more than two offers, I conjecture that one can always add a separating offer for the stubborn type.

7 Conclusion

This paper shows that the predictions of the reputation literature are reliant on the assumption of exogenous stubborn types. Once the stubborn type is given choice over his initial demand, delay (and hence, inefficiency) may not disappear even when the probability of stubbornness goes to 0. Unlike in the literature, I am able to derive strong behavioral predictions in terms of the demand configurations that can occur in equilibrium.

A natural extension to the current paper would be to broaden the set of strategies to available to the stubborn type. For instance, it may be natural to introduce an exit option for the stubborn type, when known to be faced with a stubborn opponent. This may help to understand better the tradeoff between the predictions of the reputation literature and the flexibility given to behavioral types. While this paper focusses on endogenizing behavioral types in a bargaining setting, it would also be interesting to consider this question in other settings, for instance in a repeated games framework.

References

- [1] Abreu, D., and F. Gul (2000): “Bargaining and Reputation,” *Econometrica*, 68, 85–117.
- [2] Abreu, D., and D. Pearce (2015): “A Dynamic Reinterpretation of Nash Bargaining with Endogenous Threats,” *Econometrica*, 83, 1641–1655.
- [3] Abreu, D., D. Pearce and E. Stacchetti (2015): “One-sided uncertainty and delay in reputational bargaining,” *Theoretical Economics*, 10, 719–773.
- [4] Abreu, D., and R. Sethi (2003): “Evolutionary Stability in a Reputational Model of Bargaining,” *Games and Economic Behavior*, 44, 195–216.
- [5] Atakan, A., and M. Ekmekci (2014): “Bargaining and Reputation in Search Markets,” *Review of Economic Studies*, 81, 1–29.

- [6] Fanning, J. (2016a): “Reputational Bargaining and Deadlines,” *Econometrica*, 84, 1131–1179.
- [7] Fanning, J. (2016b): “No compromise: uncertain costs in reputational bargaining,” *Working paper, Brown University*.
- [8] Fudenberg, D., and D. Levine (1989): “Reputation and Equilibrium Selection in Games with a Patient Player,” *Econometrica*, 57, 759–778.
- [9] Fudenberg, D., and D. Levine (1992): “Maintaining a Reputation when Strategies are Imperfectly Observed,” *Review of Economic Studies*, 59, 561–579.
- [10] Fudenberg, D., and J. Tirole (1991): *Game Theory*. Cambridge, MA: The MIT Press.
- [11] Kambe, S. (1999): “Bargaining With Imperfect Commitment,” *Games and Economic Behavior*, 28, 217–237.
- [12] Myerson, R. (1991): *Game Theory: Analysis of Conflict*. Cambridge, MA: Harvard University Press.
- [13] Nash, J. (1953): “Two-Person Cooperative Games,” *Econometrica*, 21, 128–140.
- [14] Wolitzky, A. (2012): “Reputational Bargaining with Minimal Knowledge of Rationality,” *Econometrica*, 80, 2047–2087.

Appendix

Proof of Lemma 3. Suppose there was one offer which is only made by the rational type, call it α_r , and one offer which was made by both types, call it α_h . Denote the probability with which the rational type demands α_h by r_h .

There exists no such PBE if $\alpha_h > 1/2$, and $\alpha_h + \alpha_r > 1$. In this case, the payoff of a rational player from playing α_h is strictly higher than the payoff from playing α_r :

$$((1 - z)r_h + z)(1 - \alpha_h) + (1 - z)(1 - r_h)(1 - \alpha_r) < ((1 - z)r_h + z)(1 - \alpha_h) + (1 - z)(1 - r_h)\alpha_h.$$

Suppose $\alpha_r > 1/2$, and $\alpha_h < 1/2$, with $\alpha_r + \alpha_h > 1$.

Then the payoff of a rational player demanding α_r is:

$$v_r^{\alpha_r} = ((1 - z)r_h + z)(1 - \alpha_h) + (1 - z)(1 - r_h)(1 - \alpha_r).$$

The payoff from demanding α_h is the same for a rational and stubborn type:⁹

$$v^{\alpha_h} = ((1 - z)r_h + z) 1/2 + (1 - z)(1 - r_h)\alpha_h.$$

Hence,

$$r_h^* = 1 - \frac{1 - 2\alpha_h}{(1 - 2\alpha_r)(1 - z)}.$$

Note $r_h^* < 0$ if

$$z < \frac{2(\alpha_h + \alpha_r - 1)}{2\alpha_r - 1}.$$

Hence, for small z there exists no symmetric PBE with $\alpha_r > 1/2$, and $\alpha_h < 1/2$, with $\alpha_r + \alpha_h > 1$. Moreover, under assumption 1, a rational player has an incentive to deviate to $1 - \alpha_h$:

$$v_r^{1-\alpha_h} = 1 - \alpha_h.$$

Suppose $\alpha_r > 1/2$, and $\alpha_h < 1/2$, with $\alpha_r + \alpha_h < 1$. Then the payoff of a rational player is:

$$v_r^{\alpha_r} = ((1 - z)r_h + z) (1/2(1 - \alpha_h) + 1/2\alpha_r) + (1 - z)(1 - r_h)(1 - \alpha_r)$$

$$v_r^{\alpha_r} = ((1 - z)r_h + z) 1/2 + (1 - z)(1 - r_h) (\alpha_h 1/2 + (1 - \alpha_r)1/2).$$

Hence, the payoff from α_r is strictly higher than from α_h . ■

Proof of Proposition 3. For the proof that follows it is useful to introduce some notation.

Define:

$$2^{\mathcal{W}} = \{\mu_2(\alpha_2) < \mu_1(\alpha_1)\},$$

and

$$2^{\mathcal{S}} = \{\mu_2(\alpha_2) \geq \mu_1(\alpha_1)\}.$$

Moreover, I denote by q_i^j the probability of player i making a demand α^j , i.e.,

$$q_i^j = z_i s_i(\alpha^j) + (1 - z_i) r_i(\alpha^j).$$

Note that for existence of equilibrium, it is necessary that $\sum_{\alpha^i} q_1^i(\alpha^i) = 1$, and

$$\sum_{\alpha^i} \pi_1(\alpha^i) q_1^i = z.$$

⁹Clearly, a stubborn type has no incentive to deviate (he receives the same payoff as a rational player).

I can write the payoff of a rational player 2 demanding α_2 in equilibrium as:

$$\begin{aligned} V_2^{R*}(\alpha_2) &= \sum_{\alpha_i \leq 1-\alpha_2} q_1^i \frac{1-\alpha_i+\alpha_2}{2} + \sum_{\alpha_i \text{ s.t. } 2^{\mathcal{W}}} q_1^i (1-\alpha_i) + \sum_{\alpha_i \text{ s.t. } 2^{\mathcal{S}}} q_1^i (\alpha_2 F_1^{\alpha_i, \alpha_2}(0) + (1-\alpha_i)(1-F_1^{\alpha_i, \alpha_2}(0))) \\ &= \sum_{\alpha_i \leq 1-\alpha_2} q_1^i \frac{1-\alpha_i+\alpha_2}{2} + \sum_{\alpha^i \text{ s.t. } 2^{\mathcal{W}}} q_1^i (1-\alpha^i) + \sum_{\alpha^i \text{ s.t. } 2^{\mathcal{S}}} q_1^i \left(\alpha_2 + (1-\alpha^i - \alpha_2) \left(\frac{\mu_1(\alpha^i)}{\mu_2(\alpha_2)} \right)^{1-\alpha^i} \right). \end{aligned}$$

Similarly, I can write the payoff of a stubborn player 2 demanding α_2 in equilibrium as:

$$\begin{aligned} V_2^{S*}(\alpha_2) &= \sum_{\alpha_i \leq 1-\alpha_2} q_1^i \frac{1-\alpha_i+\alpha_2}{2} + \sum_{\alpha_i \text{ s.t. } 2^{\mathcal{W}}} q_1^i \alpha_2 \int_0^{T_1^{\alpha_2 \alpha_i}} e^{-\rho s} dF_1^{\alpha_i \alpha_2}(s) \\ &\quad + \sum_{\alpha_i \text{ s.t. } 2^{\mathcal{S}}} q_1^i \alpha_2 \left(F_1^{\alpha_i, \alpha_2}(0) + \int_0^{T_2^{\alpha_2 \alpha_i}} e^{-\rho s} dF_1^{\alpha_i \alpha_2}(s) \right). \end{aligned}$$

This requires the evaluation of two integrals:

$$\begin{aligned} \int_0^{T_1^{\alpha_i \alpha_2}} e^{-\rho s} dF_1^{\alpha_i \alpha_2}(s) &= \left[-\frac{\lambda_1^{\alpha_i \alpha_2}}{\lambda_1^{\alpha_i \alpha_2} + \rho} \exp(-s(r + \lambda_1^{\alpha_i \alpha_2})) \right]_0^{T_1^{\alpha_i \alpha_2}} \\ &= \frac{\lambda_1^{\alpha_i \alpha_2}}{\lambda_1^{\alpha_i \alpha_2} + \rho} (1 - \exp(-T_1^{\alpha_i \alpha_2}(\rho + \lambda_1^{\alpha_i \alpha_2}))) \\ &= \frac{\lambda_1^{\alpha_i \alpha_2}}{\lambda_1^{\alpha_i \alpha_2} + \rho} \left(1 - \exp\left(\frac{1}{\lambda_1^{\alpha_i \alpha_2}} \log \pi_1(\alpha_i)(\rho + \lambda_1^{\alpha_i \alpha_2})\right) \right) \\ &= \frac{1-\alpha_i}{\alpha_2} (1 - \mu_1(\alpha_i)^{\alpha_2}). \end{aligned}$$

$$\begin{aligned} \int_0^{T_2^{\alpha_2 \alpha_i}} e^{-\rho s} dF_1^{\alpha_i \alpha_2}(s) &= [\dots] = \frac{1-\alpha_i}{\alpha_2} \pi_1(\alpha_i) \pi_2(\alpha_2)^{-\frac{1-\alpha_i}{1-\alpha_2}} \left(1 - \pi_2(\alpha_2)^{\frac{\alpha_2}{1-\alpha_2}} \right) \\ &= \frac{1-\alpha_i}{\alpha_2} \left(\frac{\mu_1(\alpha_i)}{\mu_2(\alpha_2)} \right)^{1-\alpha_i} (1 - \mu_2(\alpha_2)^{\alpha_2}). \end{aligned}$$

I can write $V_2^{S*}(\alpha_2)$ as:

$$\begin{aligned} V_2^{S*}(\alpha_2) &= V_2^{R*}(\alpha_2) - \sum_{\alpha^i \text{ s.t. } 2^{\mathcal{W}}} q_1^i (1-\alpha_i) \mu_1(\alpha^i)^{\alpha_2} - \sum_{\alpha^i \text{ s.t. } 2^{\mathcal{S}}} q_1^i (1-\alpha_i) \left(\frac{\mu_1(\alpha_i)}{\mu_2(\alpha_2)} \right)^{1-\alpha^i} \mu_2(\alpha_2)^{\alpha_2} \\ &= V_2^{R*}(\alpha_2) - \sum_{\alpha^i > 1-\alpha_2} q_1^i (1-\alpha^i) \mu_1(\alpha_i)^{\alpha_2} \max \left\{ 1, \left(\frac{\mu_2(\alpha_2)}{\mu_1(\alpha^i)} \right)^{\alpha^i + \alpha_2 - 1} \right\} \end{aligned}$$

(1) Suppose the offer-adjusted reputation $\mu_2(\alpha_2)$ was not decreasing in α_2 . Suppose further that an increase in α_2 does not change the sets $\alpha^i < 1 - \alpha_2$, 2^W , and 2^S in V_2^{R*} . By inspection, V_2^{R*} is increasing in α_2 : if α_i is such that 2^S , then

$$1 - \left(\frac{\mu_1(\alpha_i)}{\mu_2(\alpha_2)} \right)^{1-\alpha_i} > 0.$$

Moreover, V_2^{R*} is increasing in $\mu_2(\alpha_2)$: $1 - \alpha_i - \alpha_2 < 0$ by virtue of the demands being incompatible. And hence, since $\frac{\mu_1(\alpha_i)}{\mu_2(\alpha_2)}$ is decreasing in $\mu_2(\alpha_2)$, V_2^{R*} is increasing in $\mu_2(\alpha_2)$. This implies that provided an increase in α_2 does not change the sets, $\mu_2(\alpha_2)$ has to be strictly decreasing in α_2 for player 2 to be indifferent between demands.

Now suppose that the offer-adjusted reputation $\mu_2(\alpha_2)$ is not decreasing in α_2 ; and an increase in α_2 does change the sets from $\alpha^i < 1 - \alpha_2$ to 2^W , or from 2^W to 2^S , or both. I show that any such increase in α_2 increases a rational player's payoff. Note that

$$\frac{1 - \alpha_i + \alpha_2}{2} < (1 - \alpha_i)$$

for any $\alpha_2 < 1 - \alpha_i$ (where α_2 is the “lower” and hence, compatible demand). Hence, an increase in α_2 which causes a shift from the set $\alpha_2 < 1 - \alpha_i$ to 2^W increases player 2's payoff. Similarly,

$$(1 - \alpha_i) < \left(\alpha_2 + (1 - \alpha_i - \alpha_2) \left(\frac{\mu_1(\alpha_i)}{\mu_2(\alpha_2)} \right)^{1-\alpha_i} \right),$$

for any $\alpha_2 > 1 - \alpha_i$. Hence, if the offer-adjusted reputation is not decreasing, then an increase in α_2 strictly increases player 2's payoff. Hence, for a player to be indifferent between demands it must be that $\mu_2(\alpha_2)$ is decreasing in α_2 .

(3) Suppose the set of compatible demands was constant between α^n and α^{n+1} . Then I could write the payoff differences for a stubborn player as:

$$\begin{aligned} (V_2^{S*}(\alpha_n) - V_2^{R*}(\alpha_n)) - (V_2^{S*}(\alpha_{n+1}) - V_2^{R*}(\alpha_{n+1})) = & - \sum_{\alpha_i > 1 - \alpha_n} q^i (1 - \alpha_i) \cdot \\ & \left(\mu_1(\alpha_i)^{\alpha_n} \max \left\{ 1, \left(\frac{\mu_2(\alpha_n)}{\mu_1(\alpha_i)} \right)^{\alpha_i + \alpha_n - 1} \right\} - \mu_1(\alpha_i)^{\alpha_{n+1}} \max \left\{ 1, \left(\frac{\mu_2(\alpha_{n+1})}{\mu_1(\alpha_i)} \right)^{\alpha_i + \alpha_{n+1} - 1} \right\} \right). \end{aligned}$$

I show that this difference is always negative. There are three cases to consider: (i) the maximum is 1 in both max operators, (ii) the maximum is 1 for neither case, (iii) 1 is the maximum when α^{n+1} , but not when α^n . Suppose (i):

$$\max \left\{ 1, \left(\frac{\mu_2(\alpha_n)}{\mu_1(\alpha_i)} \right)^{\alpha_i + \alpha_n - 1} \right\} = 1$$

and

$$\max \left\{ 1, \left(\frac{\mu_2(\alpha_{n+1})}{\mu_1(\alpha_i)} \right)^{\alpha_i + \alpha_{n+1} - 1} \right\} = 1.$$

Clearly,

$$\mu_1(\alpha_i)^{\alpha_n} > \mu_1(\alpha_i)^{\alpha_{n+1}}.$$

Therefore, the payoff difference is negative in this case.

Suppose instead (ii):

$$\max \left\{ 1, \left(\frac{\mu_2(\alpha_n)}{\mu_1(\alpha_i)} \right)^{\alpha_i + \alpha_n - 1} \right\} = \left(\frac{\mu_2(\alpha_n)}{\mu_1(\alpha_i)} \right)^{\alpha_i + \alpha_n - 1}$$

and

$$\max \left\{ 1, \left(\frac{\mu_2(\alpha_{n+1})}{\mu_1(\alpha_i)} \right)^{\alpha_i + \alpha_{n+1} - 1} \right\} = \left(\frac{\mu_2(\alpha_{n+1})}{\mu_1(\alpha_i)} \right)^{\alpha_i + \alpha_{n+1} - 1}.$$

Note that for any such α_i , $\mu(\alpha_n) > \mu(\alpha_{n+1})$, and $0 < \alpha_n + \alpha_i - 1 < \alpha_{n+1} + \alpha_i - 1 < 1$. Therefore:

$$\mu(\alpha_i)^{1-\alpha_i} (\mu(\alpha_n)^{\alpha_n + \alpha_i - 1} - \mu(\alpha_{n+1})^{\alpha_{n+1} + \alpha_i - 1}) > 0.$$

Therefore, the payoff difference is negative in this case.

Finally suppose (iii):

$$\max \left\{ 1, \left(\frac{\mu_2(\alpha_n)}{\mu_1(\alpha_i)} \right)^{\alpha_i + \alpha_n - 1} \right\} = \left(\frac{\mu_2(\alpha_n)}{\mu_1(\alpha_i)} \right)^{\alpha_i + \alpha_n - 1}$$

and

$$\max \left\{ 1, \left(\frac{\mu_2(\alpha_{n+1})}{\mu_1(\alpha_i)} \right)^{\alpha_i + \alpha_{n+1} - 1} \right\} = 1.$$

Note that this implies $\mu(\alpha_n) > \mu(\alpha_i)$. Moreover, by assumption

$$0 < \alpha_i + \alpha_n - 1 < \alpha_i + \alpha_{n+1} - 1 < 1.$$

Therefore,

$$\mu(\alpha_i)^{1-\alpha_i} \mu(\alpha_n)^{\alpha_i + \alpha_n - 1} - \mu(\alpha_k)^{\alpha_{n+1}} > 0.$$

Hence, for a stubborn player to be indifferent the set of compatible demands must be decreasing in α . ■

Proof of Proposition 4. I can write the payoff for a rational player from making demands α and β wpp (and only those) as

$$\begin{aligned}
V_\alpha^R &= (zs(\alpha) + (1-z)r(\alpha)) \frac{1}{2} \\
&\quad + (zs(\beta) + (1-z)r(\beta)) \left(\alpha + (1-\beta-\alpha) \frac{zs(\beta)}{zs(\beta) + (1-z)r(\beta)} \left(\frac{zs(\alpha)}{zs(\alpha) + (1-z)r(\alpha)} \right)^{-\frac{1-\beta}{1-\alpha}} \right) \\
V_\beta^R &= (zs(\alpha) + (1-z)r(\alpha)) (1-\alpha) + (zs(\beta) + (1-z)r(\beta)) (1-\beta)
\end{aligned} \tag{12}$$

Moreover, I can write the payoff differences $V_\alpha^R - V_\alpha^S$ and $V_\beta^R - V_\beta^S$ as

$$\begin{aligned}
V_\alpha^R - V_\alpha^S &= -(zs(\beta) + (1-z)r(\beta)) (1-\beta) \frac{zs(\beta)}{zs(\beta) + (1-z)r(\beta)} \left(\frac{zs(\alpha)}{zs(\alpha) + (1-z)r(\alpha)} \right)^{\frac{\alpha+\beta-1}{1-\alpha}} \\
V_\beta^R - V_\beta^S &= -(zs(\alpha) + (1-z)r(\alpha)) (1-\alpha) \left(\frac{zs(\alpha)}{zs(\alpha) + (1-z)r(\alpha)} \right)^{\frac{\beta}{1-\alpha}} \\
&\quad - (zs(\beta) + (1-z)r(\beta)) (1-\beta) \frac{zs(\beta)}{zs(\beta) + (1-z)r(\beta)} \left(\frac{zs(\alpha)}{zs(\alpha) + (1-z)r(\alpha)} \right)^{\frac{\beta}{1-\beta}}
\end{aligned} \tag{13}$$

An equilibrium requires

$$\begin{aligned}
V_\alpha^R - V_\beta^R &= 0, \\
V_\alpha^R - V_\alpha^S - (V_\beta^R - V_\beta^S) &= 0, \\
r(\alpha) + r(\beta) &= 1, \\
s(\alpha) + s(\beta) &= 1.
\end{aligned} \tag{14}$$

I replace

$$\frac{zs(\alpha)}{zs(\alpha) + (1-z)r(\alpha)}$$

by π_α , and

$$\frac{zs(\beta)}{zs(\beta) + (1-z)r(\beta)}$$

by π_β . Moreover, I replace $s(\alpha)/\pi_\alpha$ by z_α . Similarly, for $s(\beta)/\pi_\beta$. Then I get:

$$\begin{aligned}
V_\alpha^R - V_\beta^R &= z \left((\beta + \alpha - 1) z_\beta - \frac{(1 - 2\alpha)z_\alpha}{2} - (\beta + \alpha - 1) \pi_\alpha^{-\frac{1-\beta}{1-\alpha}} \pi_\beta z_\beta \right) \\
(V_\alpha^R - V_\alpha^S) - (V_\beta^R - V_\beta^S) &= z \left((1 - \beta) \pi_\beta^{\frac{\beta}{1-\beta}} z_\beta + \pi_\alpha^{\frac{\alpha+\beta-1}{1-\alpha}} ((1 - \alpha) \pi_\alpha z_\alpha - (1 - \beta) \pi_\beta z_\beta) \right) \\
1 &= z_\alpha \pi_\alpha + z_\beta \pi_\beta \\
\frac{1 - z}{z} &= z_\alpha (1 - \pi_\alpha) + z_\beta (1 - \pi_\beta)
\end{aligned} \tag{15}$$

Note that the payoff differences are linear in z . Hence, I divide by z . Moreover, I replace z_β by $\frac{1 - \pi_\alpha z_\alpha}{\pi_\beta}$. I denote the new expressions, A^R and A^S :

$$\begin{aligned}
A^R &= (\beta + \alpha - 1) \frac{1 - \pi_\alpha z_\alpha}{\pi_\beta} - \frac{(1 - 2\alpha)z_\alpha}{2} - (\beta + \alpha - 1) \pi_\alpha^{-\frac{1-\beta}{1-\alpha}} (1 - \pi_\alpha z_\alpha) \\
A^S &= (1 - \beta) \pi_\beta^{\frac{\beta}{1-\beta}} \frac{1 - \pi_\alpha z_\alpha}{\pi_\beta} + \pi_\alpha^{\frac{\alpha+\beta-1}{1-\alpha}} ((1 - \alpha) \pi_\alpha z_\alpha - (1 - \beta) (1 - \pi_\alpha z_\alpha)) \\
\frac{1 - z}{z} &= z_\alpha (1 - \pi_\alpha) + \frac{1 - \pi_\alpha z_\alpha}{\pi_\beta} (1 - \pi_\beta).
\end{aligned} \tag{16}$$

For the constraint to be satisfied as $z \rightarrow 0$, it is necessary that $z_\alpha \rightarrow \infty$, or $\pi_\beta \rightarrow 0$. If $z_\alpha \rightarrow \infty$, then for a solution to the system to exist, it is necessary that $\pi_\beta \rightarrow 0$ (from A^R). If $\pi_\beta \rightarrow 0$, then for a solution to exist, it is necessary that either $\pi_\alpha \rightarrow 0$, or $\pi_\alpha z_\alpha \rightarrow \frac{1-\beta}{2-\alpha-\beta}$. Hence, if a solution exists, then

$$\pi_\alpha \rightarrow 0 \text{ and } \pi_\beta \rightarrow 0,$$

or

$$\pi_\alpha z_\alpha \rightarrow \frac{1 - \beta}{2 - \alpha - \beta} \text{ and } \pi_\beta \rightarrow 0.$$

I can solve $A^R = 0$ for z_α .

$$z_\alpha = \frac{(\alpha + \beta - 1) \left(\pi_\alpha^{-\frac{1-\beta}{1-\alpha}} - \pi_\beta^{-1} \right)}{-1/2 + \alpha + (\alpha + \beta - 1) \left(\pi_\alpha^{\frac{\beta-\alpha}{1-\alpha}} - \pi_\alpha \pi_\beta^{-1} \right)} \tag{17}$$

I replace z_α in A^S . I denote the new expression B^S :

$$B^S = (1 - \beta) \left(1 - \frac{\pi_\alpha \left((\alpha + \beta - 1) \left(\pi_\alpha^{-\frac{1-\beta}{1-\alpha}} - \pi_\beta^{-1} \right) \right)}{-1/2 + \alpha + (\alpha + \beta - 1) \left(\pi_\alpha^{-\frac{\beta-\alpha}{1-\alpha}} - \pi_\alpha \pi_\beta^{-1} \right)} \right) \pi_\beta^{-2 + \frac{1}{1-\beta}}$$

$$+ \pi_\alpha^{-1 + \frac{\beta}{1-\alpha}} \left(-1 + \beta + \frac{2(2 - \alpha - \beta)(\alpha + \beta - 1) \pi_\alpha \left(\pi_\alpha^{-\frac{\beta}{1-\alpha}} - \pi_\alpha^{-\frac{1}{1-\alpha}} \pi_\beta \right)}{-2(\alpha + \beta - 1) \pi_\alpha^{-\frac{\alpha}{1-\alpha}} \pi_\beta + \pi_\alpha^{-\frac{\beta}{1-\alpha}} (2(\alpha + \beta - 1) \pi_\alpha + \pi_\beta - 2\alpha \pi_\beta)} \right)$$

I can bring the two fractions to a common denominator, multiply B^S by this common denominator, and then simplify. I denote the new expression C^S .

$$C^S = 2(1 - \beta) - 2\alpha(2 - \alpha - \beta) + (1 + 2\alpha(-1 + \beta) - \beta) \pi_\beta \pi_\alpha^{-1}$$

$$- (1 - 2\alpha)(1 - \beta) \pi_\alpha^{-\frac{\beta}{1-\alpha}} \pi_\beta^{\frac{\beta}{1-\beta}} + 2(1 - \alpha)(\alpha + \beta - 1) \pi_\alpha^{-\frac{1-\beta}{1-\alpha}} \pi_\beta$$

For a solution to exist, it is necessary that one of the three terms involving π_α and π_β in C^S goes to a constant, and the other two terms go to 0.

Case 1 $\pi_\alpha/\pi_\beta \rightarrow K$, where K is some constant to be determined. Then

$$\pi_\alpha^{-\frac{\beta}{1-\alpha}} \pi_\beta^{\frac{\beta}{1-\beta}} = (\pi_\alpha/\pi_\beta)^{-\frac{\beta}{1-\alpha}} \pi_\beta^{-\frac{\beta}{1-\alpha}} \pi_\beta^{\frac{\beta}{1-\beta}} \rightarrow K^{-\frac{\beta}{1-\alpha}} \pi_\beta^{\frac{\beta(\beta-\alpha)}{(1-\alpha)(1-\beta)}}$$

Therefore, if $\pi_\alpha/\pi_\beta \rightarrow K$, then $\pi_\alpha^{-\frac{\beta}{1-\alpha}} \pi_\beta^{\frac{\beta}{1-\beta}} \rightarrow 0$. Moreover,

$$\pi_\alpha^{-\frac{1-\beta}{1-\alpha}} \pi_\beta = (\pi_\alpha/\pi_\beta)^{-\frac{1-\beta}{1-\alpha}} \pi_\beta^{-\frac{1-\beta}{1-\alpha}} \pi_\beta \rightarrow K^{-\frac{1-\beta}{1-\alpha}} \pi_\beta^{\frac{\beta-\alpha}{1-\alpha}}.$$

Hence, if $\pi_\alpha/\pi_\beta \rightarrow K$, then $\pi_\alpha^{-\frac{1-\beta}{1-\alpha}} \pi_\beta \rightarrow 0$. Hence, I can solve for the ratio $\pi_\alpha/\pi_\beta \rightarrow K$ in the limit:

$$\pi_\alpha/\pi_\beta \rightarrow \frac{(1 - 2\alpha)(1 - \beta)}{2(1 - \alpha)(\alpha + \beta - 1)},$$

or equivalently,

$$K = \frac{(1 - 2\alpha)(1 - \beta)}{2(1 - \alpha)(\alpha + \beta - 1)}.$$

I can therefore also derive an expression for z_α (equation (6)) in the limit:

$$\begin{aligned}
z_\alpha &= \frac{(\alpha + \beta - 1) (\pi_\alpha / \pi_\beta)^{-\frac{1-\beta}{1-\alpha}} \pi_\beta^{-\frac{1-\beta}{1-\alpha}} - (\alpha + \beta - 1) \pi_\beta^{-1}}{-1/2 + \alpha + (\alpha + \beta - 1) (\pi_\alpha / \pi_\beta)^{\frac{\beta-\alpha}{1-\alpha}} \pi_\beta^{\frac{\beta-\alpha}{1-\alpha}} - (\alpha + \beta - 1) (\pi_\alpha / \pi_\beta)}, \\
z_\alpha \rightarrow z_\alpha^* &= \frac{(\alpha + \beta - 1) K^{-\frac{1-\beta}{1-\alpha}} \pi_\beta^{-\frac{1-\beta}{1-\alpha}} - (\alpha + \beta - 1) \pi_\beta^{-1}}{-1/2 + \alpha + (\alpha + \beta - 1) K^{\frac{\beta-\alpha}{1-\alpha}} \pi_\beta^{\frac{\beta-\alpha}{1-\alpha}} - (\alpha + \beta - 1) K}.
\end{aligned} \tag{18}$$

By using the constraint in (5) I can solve for π_β as a function of z only:

$$\begin{aligned}
\frac{1-z}{z} &= z_\alpha (1 - \pi_\alpha) + \frac{1 - \pi_\beta}{\pi_\beta} (1 - \pi_\alpha z_\alpha) \\
&= z_\alpha (1 - (\pi_\alpha / \pi_\beta) \pi_\beta) + \frac{1 - \pi_\beta}{\pi_\beta} (1 - (\pi_\alpha / \pi_\beta) \pi_\beta z_\alpha) \\
\frac{1-z}{z} &\approx z_\alpha^* (1 - K \pi_\beta) + \frac{1 - \pi_\beta}{\pi_\beta} (1 - K \pi_\beta z_\alpha^*) \\
&\approx \pi_\beta^{-1} - 1
\end{aligned} \tag{19}$$

Since $s(\alpha) = z_\alpha / \pi_\alpha$:

$$\begin{aligned}
s(\alpha) &\rightarrow \frac{(\alpha + \beta - 1) K^{-\frac{1-\beta}{1-\alpha}} \pi_\beta^{\frac{\beta-\alpha}{1-\alpha}} - (\alpha + \beta - 1)}{-1/2 + \alpha + (\alpha + \beta - 1) K^{\frac{\beta-\alpha}{1-\alpha}} \pi_\beta^{\frac{\beta-\alpha}{1-\alpha}} - (\alpha + \beta - 1) K} K \\
&\approx \frac{(\alpha + \beta - 1) K^{-\frac{1-\beta}{1-\alpha}} z^{\frac{\beta-\alpha}{1-\alpha}} - (\alpha + \beta - 1)}{-1/2 + \alpha + (\alpha + \beta - 1) K^{\frac{\beta-\alpha}{1-\alpha}} z^{\frac{\beta-\alpha}{1-\alpha}} - (\alpha + \beta - 1) K} K.
\end{aligned} \tag{20}$$

Hence,

$$\lim_{z \rightarrow 0} s(\alpha) = \frac{1 - \beta}{2 - \alpha - \beta}.$$

Recall

$$\pi_\alpha = \frac{zs(\alpha)}{zs(\alpha) + (1 - z)r(\alpha)}.$$

Moreover,

$$\pi_\alpha / \pi_\beta \rightarrow K.$$

Hence, I can solve for $\lim_{z \rightarrow 0} r(\alpha)$ in the same fashion:

$$\lim_{z \rightarrow 0} r(\alpha) = \frac{2(\alpha + \beta - 1)}{2\beta - 1}.$$

Case 2 $\pi_\alpha/\pi_\beta^{\frac{1-\alpha}{1-\beta}} \rightarrow K$, where as before K is some constant to be determined. However,

$$\pi_\beta/\pi_\alpha = \pi_\beta \left(\pi_\alpha/\pi_\beta^{\frac{1-\alpha}{1-\beta}} \right)^{-1} \pi_\beta^{\frac{1-\alpha}{1-\beta}} \rightarrow K^{-1} \pi^{-\frac{\beta-\alpha}{1-\alpha}}.$$

Hence, $\pi_\alpha/\pi_\beta^{\frac{1-\alpha}{1-\beta}} \not\rightarrow K$.

Therefore, there is a unique candidate solution. Note that $r(\alpha)$ and $s(\alpha)$ are well-defined for any α, β such that $\alpha < 1/2$ and $\alpha + \beta > 1$. The second part of the proposition follows by the implicit function theorem. A and B below are the payoff differences between two offers a and b for a rational and stubborn type respectively.

$$A = (s_a - 1)z(a + b - 1) \left(\frac{s_a z}{r_a(-z) + r_a + s_a z} \right)^{\frac{1-b}{a-1}} + \frac{1}{2}(2a + (2b - 1)z(r_a - s_a) - 2br_a + 2b + r_a - 2)$$

$$B = (b-1)(s_a-1)z \left(\frac{(s_a-1)z}{r_a(-z) + r_a + s_a z - 1} \right)^{\frac{1}{1-b}-2} - z(s_a(a+b-2)-b+1) \left(\frac{s_a z}{r_a(-z) + r_a + s_a z} \right)^{-\frac{b}{a-1}-1}$$

Claim 4 *The system*

$$\begin{aligned} A &= 0 \\ B &= 0 \end{aligned} \tag{21}$$

can be solved locally around $z = 0$, with $s_a \in (0, 1)$, $r_a \in (0, 1)$.

Replace s_a by $\frac{r_a x(z-1)}{xz-1}$ in A and B . Then I can solve $A = 0$ for r_a as a function of x and z only:

$$r_a = \frac{2(a+b-1)(xz-1) \left(x^{\frac{1}{a-1}} z^{\frac{a}{a-1}} - x^{\frac{b}{a-1}} z^{\frac{b}{a-1}} \right)}{(z-1) \left(2(a+b-1)x^{\frac{a}{a-1}} z^{\frac{a}{a-1}} + (1-2b)x^{\frac{b}{a-1}} z^{\frac{b}{a-1}} \right)}.$$

Replacing r_a in B and simplifying, I get:

$$B1 = x^{\frac{b-2}{a-1}} z^{-\frac{ab+a+b}{(a-1)(b-1)}} (b_1 + b_2 b_3), \tag{22}$$

where

$$\begin{aligned}
b_1 = & - (1 - 2b)^2(b - 1)x^{\frac{b+2}{a-1}}z^{\frac{(a-1)b+a+2b^2}{(a-1)(b-1)}} \\
& + 2(2b - 1)(a + b - 1)(a + 2b - 3)x^{\frac{a+b+1}{a-1}}z^{\frac{(a-1)b+a+2b^2}{(a-1)(b-1)}} \\
& - 4(a + b - 2)(a + b - 1)^2x^{\frac{2a+b}{a-1}}z^{\frac{(a-1)b+a+2b^2}{(a-1)(b-1)}} \\
& + 4(a - 1)(a + b - 1)^2x^{\frac{3}{a-1}+2}z^{\frac{b(2a+b)}{(a-1)(b-1)}} \\
& - 2(a - 1)(2b - 1)(a + b - 1)x^{\frac{a+2}{a-1}}z^{\frac{b(2a+b)}{(a-1)(b-1)}} \\
b_2 = & \left(\frac{x^{\frac{b}{a-1}}(2x(a + b - 1) - 2b + 1)z^{\frac{b}{a-1}}}{(2a - 1)x^{\frac{b}{a-1}}z^{\frac{b}{a-1}} - 2(a + b - 1)\left(x^{\frac{1}{a-1}} - x^{\frac{a}{a-1}}\right)z^{\frac{a}{a-1}}} \right)^{\frac{1}{1-b}} \\
b_3 = & - 8(b - 1)(a + b - 1)^2x^{\frac{4}{a-1}+2}z^{\frac{2ab-a+b^2+b}{(a-1)(b-1)}} \\
& + 4(b - 1)(a + b - 1)^2x^{\frac{4}{a-1}+3}z^{\frac{2ab-a+b^2+b}{(a-1)(b-1)}} \\
& + 4(b - 1)(a + b - 1)^2x^{\frac{a+3}{a-1}}z^{\frac{2ab-a+b^2+b}{(a-1)(b-1)}} \\
& - 4(2a - 1)(b - 1)(a + b - 1)x^{\frac{a+b+2}{a-1}}z^{\frac{b(a+2b)}{(a-1)(b-1)}} \\
& + 4(2a - 1)(b - 1)(a + b - 1)x^{\frac{2a+b+1}{a-1}}z^{\frac{b(a+2b)}{(a-1)(b-1)}} \\
& + (1 - 2a)^2(b - 1)x^{\frac{a+2b+1}{a-1}}z^{\frac{a+b(3b-1)}{(a-1)(b-1)}}
\end{aligned} \tag{23}$$

I replace x by

$$u + \frac{(1 - b)(2b - 1)}{2(2 - a - b)(a + b - 1)}$$

and z by $s^{\frac{1-a}{b-a}}$ in B1. Moreover, multiply B1 by

$$-\frac{(a + b - 2)s^{-\frac{2b}{a-b}}\left(\frac{(b-1)(2b-1)}{2(a+b-2)(a+b-1)} + u\right)^{\frac{2-b}{a-1}}}{2u(a + b - 2)(a + b - 1) - 2ab + a + 2b - 1}.$$

Call the new function $B2(s, u)$. I then take partial derivatives of $B2(s, u)$ with respect to s and u ; and evaluate

$$-\frac{\partial B2(s, u)/\partial u}{\partial B2(s, u)/\partial s}$$

at $(s, u) = (0, 0)$. I get

$$\frac{\Delta s}{\Delta u} \Big|_{(s,u)=(0,0)} = -\frac{\partial B2(s, u)/\partial u}{\partial B2(s, u)/\partial s} \Big|_{(s,u)=(0,0)} = \frac{(2 - a - b)}{1 - a} \left(\frac{2(2 - a - b)(a + b - 1)}{(1 - b)(2b - 1)} \right)^{\frac{b-a}{1-a}},$$

which is clearly finite. Hence, the system

$$\begin{aligned} A &= 0 \\ B &= 0 \end{aligned} \tag{24}$$

can be solved locally around $z = 0$, with $r_a \in (0, 1)$, and $s_a \in (0, 1)$.

■

Proof of Proposition 5. In a similar fashion to the two offer case, replace $\frac{s_i}{zs_i + (1-z)r_i}$ by x_i for $i = a, \beta$ in the payoff differences for rational and stubborn players respectively. Moreover, replace $\frac{zs_\gamma}{zs_\gamma + (1-z)r_\gamma}$ by $(zx_\gamma)^{\frac{1-\gamma}{1-\beta}}$. Denote by A_1 and A_2 the payoff differences for a rational player between a and β and between β and c respectively. Further, denote by A_3 and A_4 the respective payoff differences for a stubborn player.

$$\begin{aligned} A_1 &= s_\gamma z (-1 + \beta + \gamma) (x_\beta z)^{-\frac{1-\gamma}{1-\beta}} - (1 - a - \beta) \frac{s_\beta}{2x_\beta} - (\beta - a) \frac{s_\alpha}{2x_\alpha} - s_\gamma z (a + \gamma - 1) (x_\alpha z)^{-\frac{1-\gamma}{1-\alpha}} \\ &\quad - s_\gamma z (\beta - \alpha) (x_\gamma z)^{-\frac{1-\gamma}{1-\beta}} \end{aligned} \tag{25}$$

$$A_2 = (\beta + \gamma - 1) s_\gamma z \left((x_\gamma z)^{-\frac{1-\gamma}{1-\beta}} - (x_\beta z)^{-\frac{1-\gamma}{1-\beta}} \right) - (1 - \alpha - \beta) \frac{s_\alpha}{2x_\alpha} \tag{26}$$

$$A_3 = (1 - \beta) (x_\beta z)^{\frac{\beta}{1-\beta}} \frac{s_\beta}{x_\beta} - (1 - c) s_\gamma \left((x_\alpha z)^{\frac{c}{1-\alpha}} x_\alpha^{-1} - (x_\beta z)^{\frac{\gamma}{1-\beta}} x_\beta^{-1} \right) \tag{27}$$

$$\begin{aligned} A_4 &= (1 - \alpha) (x_\alpha z)^{\frac{c}{1-\alpha}} \frac{s_\alpha}{x_\alpha} - (1 - c) s_\gamma \left((x_\beta z)^{\frac{c}{1-\beta}} x_\beta^{-1} - z (zx_\gamma)^{\frac{2\gamma-1}{1-\beta}} \right) - (1 - \beta) \frac{s_\beta}{x_\beta} \left((x_\beta z)^{\frac{\beta}{1-\beta}} - (x_\beta z)^{\frac{\gamma}{1-\beta}} \right) \end{aligned} \tag{28}$$

and the constraint on x_i :

$$\frac{zs_\gamma \left(1 - (x_\gamma z)^{-\frac{1-\gamma}{1-\beta}} \right)}{z - 1} + \frac{s_\alpha - s_\alpha x_\alpha z}{x_\alpha - x_\alpha z} + \frac{s_\beta - s_\beta x_\beta z}{x_\beta - x_\beta z} = 1 \tag{29}$$

Hence, an equilibrium requires

$$A_1 = 0, A_2 = 0, A_3 = 0, A_4 = 0,$$

and equation (29) to be satisfied. First, note that the system is linear in s_i . Therefore, I can normalize $s_\gamma = 1$. Second, replace x_i by π_i/z . Third, replace s_i by $z_i\pi_i$. Note that $\pi_i \in [0, 1]$,

and $z_i \in [0, \infty)$. Conversely, for every $z_i \in [0, \infty)$ and $\pi_i \in [0, 1]$, there exists, $x_i \in [?, 1/z]$ and $s_i \in [0, 1]$.

With these replacements in A_1 through to A_4 , I can then define:

$$B_1 = z(-1 + \beta + \gamma)(\pi_\beta)^{-\frac{1-\gamma}{1-\beta}} - (1 - \alpha - \beta) \frac{z_\beta z}{2} - (\beta - a) \frac{z_\alpha z}{2} - z(a + \gamma - 1)(\pi_\alpha)^{-\frac{1-\gamma}{1-\alpha}} - z(\beta - \alpha)(\pi_\gamma)^{-\frac{1-\gamma}{1-\beta}} \quad (30)$$

$$B_2 = (\beta + \gamma - 1)z \left((\pi_\gamma)^{-\frac{1-\gamma}{1-\beta}} - (\pi_\beta)^{-\frac{1-\gamma}{1-\beta}} \right) - (1 - \alpha - \beta) \frac{z_\alpha z}{2} \quad (31)$$

$$B_3 = (1 - \beta)(\pi_\beta)^{\frac{\beta}{1-\beta}} z_\beta z - (1 - \gamma) \left((\pi_\alpha)^{\frac{\gamma}{1-\alpha}} z \pi_\alpha^{-1} - (\pi_\beta)^{\frac{\gamma}{1-\beta}} z \pi_\beta^{-1} \right) \quad (32)$$

$$B_4 = (1 - \alpha)(\pi_\alpha)^{\frac{\gamma}{1-\alpha}} z_\alpha z - (1 - c) \left((\pi_\beta)^{\frac{\gamma}{1-\beta}} z \pi_\beta^{-1} - z(\pi_\gamma)^{\frac{2\gamma-1}{1-\beta}} \right) - (1 - \beta) z_\beta z \left((\pi_\beta)^{\frac{\beta}{1-\beta}} - (\pi_\beta)^{\frac{\gamma}{1-\beta}} \right) \quad (33)$$

and the constraint on x_i :

$$\frac{z \left(1 - (\pi_\gamma)^{-\frac{1-\gamma}{1-\beta}} \right)}{z - 1} + z z_\alpha \frac{1 - \pi_\alpha}{1 - z} + z z_\beta \frac{1 - \pi_\beta}{1 - z} = K_0, \quad (34)$$

where K_0 is a positive constant.

$$\left(\pi_\gamma^{-\frac{1-\gamma}{1-\beta}} - 1 \right) + z_a(1 - \pi_\alpha) + z_\beta(1 - \pi_\beta) = \frac{1 - z}{z} K_0 \quad (35)$$

Note that the system B_1 through to B_4 is linear in z , and hence I can divide by z . Define:

$$C_1 = (\beta + \gamma - 1) \pi_\beta^{-\frac{1-\gamma}{1-\beta}} - (\alpha + \gamma - 1) \pi_\alpha^{-\frac{1-\gamma}{1-\alpha}} - (\beta - \alpha) \pi_\gamma^{-\frac{1-\gamma}{1-\beta}} - (1 - \alpha - \beta) \frac{z_\beta}{2} - (\beta - a) \frac{z_\alpha}{2} \quad (36)$$

$$C_2 = (\beta + \gamma - 1) \left(\pi_\gamma^{-\frac{1-\gamma}{1-\beta}} - \pi_\beta^{-\frac{1-\gamma}{1-\beta}} \right) - (1 - \alpha - \beta) \frac{z_\alpha}{2} \quad (37)$$

$$C_3 = (1 - \beta) \pi_\beta^{\frac{\beta}{1-\beta}} z_\beta - (1 - \gamma) \left(\pi_\alpha^{\frac{\gamma}{1-\alpha}-1} - \pi_\beta^{\frac{\gamma}{1-\beta}-1} \right) \quad (38)$$

$$C_4 = (1 - \alpha) \pi_\alpha^{\frac{\gamma}{1-\alpha}} z_\alpha - (1 - \gamma) \left(\pi_\beta^{\frac{\gamma}{1-\beta}-1} - \pi_\gamma^{\frac{2\gamma-1}{1-\beta}} \right) - (1 - \beta) z_\beta \left(\pi_\beta^{\frac{\beta}{1-\beta}} - \pi_\beta^{\frac{\gamma}{1-\beta}} \right) \quad (39)$$

Hence, an equilibrium is a vector $(\pi, z) \in [0, 1]^3 \times [0, 1]^2$ such that

$$C_1 = 0, \quad C_2 = 0, \quad C_3 = 0, \quad C_4 = 0,$$

and equation (35) is satisfied. Note that:

$$C_2 = 0 \text{ and } z_\alpha > 0 \rightarrow \pi_\gamma \leq \pi_\beta,$$

$$C_3 = 0 \text{ and } z_\beta > 0 \rightarrow \pi_\beta \leq \pi_\alpha.$$

In short, if a solution $(\pi_\alpha, \pi_\beta, \pi_\gamma, z_\alpha, z_\beta)$ exists,

$$\pi_\alpha \geq \pi_\beta \geq \pi_\gamma.$$

Note that C_2 is linear in z_α , and C_3 is linear in z_β . Hence, I can solve for z_α and z_β :¹⁰

$$z_\alpha = \frac{2(\beta + \gamma - 1) \left(\pi_\gamma^{-\frac{1-\gamma}{1-\beta}} - \pi_\beta^{-\frac{1-\gamma}{1-\beta}} \right)}{1 - \alpha - \beta},$$

$$z_\beta = \frac{(1 - \gamma) \left(\pi_\beta^{-\frac{\beta}{1-\beta}} \pi_\alpha^{\frac{a+\gamma-1}{1-\alpha}} - \pi_\beta^{-\frac{1-\gamma}{1-\beta}} \right)}{1 - \beta}.$$

I can then plug z_α and z_β into C_1 and C_4 . I denote them D_1 and D_4 :

$$D_1 = \frac{(\alpha^2(1 - \gamma) - 2\alpha(1 - \beta)(2\beta + \gamma - 1) + (1 - \beta)(-1 + \beta + \gamma + \beta\gamma)) \pi_\beta^{-\frac{1-\gamma}{1-\beta}}}{2(1 - \beta)(1 - \alpha - \beta)} \quad (40)$$

$$- (\alpha + \gamma - 1) \pi_\alpha^{-\frac{1-\gamma}{1-\alpha}} - \frac{(1 - \alpha - \beta)(1 - \gamma) \pi_\alpha^{\frac{\gamma}{1-\alpha}-1} \pi_\beta^{-\frac{\beta}{1-\beta}}}{2(1 - \beta)} - \frac{(\beta - \alpha)(\gamma - \alpha) \pi_\gamma^{-\frac{1-\gamma}{1-\beta}}}{1 - \alpha - \beta}$$

$$D_4 = \frac{2(1 - \alpha)(\beta + \gamma - 1) \pi_\alpha^{\frac{\gamma}{1-\alpha}} \left(\pi_\gamma^{-\frac{1-\gamma}{1-\beta}} - \pi_\beta^{-\frac{1-\gamma}{1-\beta}} \right)}{1 - \alpha - \beta} - (1 - \gamma) \pi_\alpha^{\frac{\gamma}{1-\alpha}-1} \left(1 - \pi_\beta^{\frac{\gamma-\beta}{1-\beta}} \right) - (1 - \gamma) \left(\pi_\beta^{\frac{2\gamma-1}{1-\beta}} - \pi_\gamma^{\frac{2\gamma-1}{1-\beta}} \right) \quad (41)$$

Hence, an equilibrium requires $D_1 = 0$, $D_4 = 0$, and equation (35). Note that the first term in D_1 is positive for any a, β, γ .

¹⁰Note that C_2 is independent of z_β ; and C_3 is independent of z_α . Moreover, $1 - \alpha - \beta > 0$ and $\beta < 1$. Hence, a solution is well-defined.

Claim 5 When $z \rightarrow 0$, $\pi_i \rightarrow 0$ for $i = \alpha, \beta, \gamma$.

Proof. Consider equation (35). To satisfy this equation as $z \rightarrow 0$, and hence $Z \rightarrow \infty$, it is necessary that either (i) $z_\alpha \rightarrow \infty$, (ii) $z_\beta \rightarrow \infty$, or (iii) $\pi_\gamma \rightarrow 0$. Suppose (i) $z_\alpha \rightarrow \infty$. Then, from (36) it follows that $\pi_\beta \rightarrow 0$, and from (37) it follows that $\pi_\gamma \rightarrow 0$. Suppose (ii) $z_\beta \rightarrow \infty$. Then, from (38) it follows that $\pi_\beta \rightarrow 0$, and from (37) it follows that $\pi_\gamma \rightarrow 0$. Suppose finally (iii) $\pi_\gamma \rightarrow 0$. Then, from (36) it follows that $\pi_\beta \rightarrow 0$. Hence, it must be that $y_i \rightarrow 0$ for $i = \beta, \gamma$.

From (41), it follows that $\pi_\alpha \rightarrow 0$ – otherwise, the second term in D_4 cannot be 0 in the limit (the other two terms go to 0). Hence, we know that

$$y_i \rightarrow 0 \text{ for } i = \alpha, \beta, \gamma.$$

■

Claim 6 Fixing the demands α , β , and γ , a solution does not exist, i.e., there is no π_α and π_β satisfying $E_1^{**} = 0$.

Consider D_4 . Note that as $z \rightarrow 0$, $\pi_\alpha^{\frac{\gamma}{1-\alpha}-1}$ is infinitely bigger than $\pi_\beta^{\frac{\gamma-\beta}{1-\beta}}$, $\pi_\beta^{\frac{2\gamma-1}{1-\beta}}$ and $\pi_\gamma^{\frac{2\gamma-1}{1-\beta}}$. The exponent on π_α is smaller than on π_β or π_γ , and π_α is bigger than π_β and π_γ for any z . Eliminating those terms that are negligible gives us E_4 :

$$E_4 = \frac{2(1-\alpha)(\beta+\gamma-1)\pi_\alpha^{\frac{\gamma}{1-\alpha}-1} \left(\pi_\gamma^{-\frac{1-\gamma}{1-\beta}} - \pi_\beta^{-\frac{1-\gamma}{1-\beta}} \right)}{1-\alpha-\beta} - (1-\gamma)\pi_\alpha^{\frac{\gamma}{1-\alpha}-1} \quad (42)$$

With E_4 , I can solve for π_γ .

$$\pi_\gamma^{-\frac{1-\gamma}{1-\beta}} \rightarrow \frac{(1-\alpha-\beta)(1-\gamma)}{2(1-\alpha)(\beta+\gamma-1)\pi_\alpha} + \pi_\beta^{\frac{\gamma-1}{1-\beta}}.$$

I can plug this into D_1 . I denote the new expression E_1 :

$$\begin{aligned} E_1 = & \frac{\beta+\gamma-1-\beta\gamma+\alpha(1-2\beta+\gamma)}{2(1-\beta)} \pi_\beta^{-\frac{1-\gamma}{1-\beta}} - (\alpha+\gamma-1)\pi_\alpha^{-\frac{1-\gamma}{1-\alpha}} \\ & - \frac{(1-\alpha-\beta)(1-\gamma)\pi_\alpha^{\frac{\gamma}{1-\alpha}-1} \pi_\beta^{-\frac{\beta}{1-\beta}}}{2(1-\beta)} - \frac{(\beta-\alpha)(\gamma-\alpha)(1-\gamma)\pi_\alpha^{-1}}{2(1-\alpha)(\beta+\gamma-1)} \end{aligned} \quad (43)$$

I can multiply this by $\pi_\beta^{\frac{1-\gamma}{1-\beta}}$:

$$E_1^* = \pi_\beta^{\frac{1-\gamma}{1-\beta}} E_1 = \frac{\beta + \gamma - 1 - \beta\gamma + \alpha(1 - 2\beta + \gamma)}{2(1 - \beta)} - \frac{(\beta - \alpha)(\gamma - \alpha)(1 - \gamma)\pi_\alpha^{-1}\pi_\beta^{\frac{1-\gamma}{1-\beta}}}{2(1 - \alpha)(\beta + \gamma - 1)} \\ - (\alpha + \gamma - 1)\pi_\alpha^{-\frac{1-\gamma}{1-\alpha}}\pi_\beta^{\frac{1-\gamma}{1-\beta}} - \frac{(1 - \alpha - \beta)(1 - \gamma)\pi_\alpha^{\frac{\gamma}{1-\alpha}-1}\pi_\beta^{-\frac{\beta+\gamma-1}{1-\beta}}}{2(1 - \beta)} \quad (44)$$

Note that the second and third term have the same power on π_β , but $(1 - \gamma)/(1 - \alpha) < 1$, and hence π_α^{-1} is infinitely bigger in the limit than $\pi_\alpha^{-\frac{1-\gamma}{1-\alpha}}$. Hence, there are two remaining terms involving π_α and π_β . Define E_1^{**} :

$$E_1^{**} = \frac{\beta + \gamma - 1 - \beta\gamma + \alpha(1 - 2\beta + \gamma)}{2(1 - \beta)} - \frac{(\beta - \alpha)(\gamma - \alpha)(1 - \gamma)\pi_\alpha^{-1}\pi_\beta^{\frac{1-\gamma}{1-\beta}}}{2(1 - \alpha)(\beta + \gamma - 1)} \\ - \frac{(1 - \alpha - \beta)(1 - \gamma)\pi_\alpha^{\frac{\gamma}{1-\alpha}-1}\pi_\beta^{-\frac{\beta+\gamma-1}{1-\beta}}}{2(1 - \beta)} \quad (45)$$

There are three cases to consider:

Case 1: $\pi_\alpha^{-1}\pi_\beta^{\frac{1-\gamma}{1-\beta}} \rightarrow K_1$, and $\pi_\alpha^{\frac{\gamma+\alpha-1}{1-\alpha}}\pi_\beta^{-\frac{\beta+\gamma-1}{1-\beta}} \rightarrow K_2$,

Case 2: $\pi_\alpha^{-1}\pi_\beta^{\frac{1-\gamma}{1-\beta}} \rightarrow K_3$, and $\pi_\alpha^{\frac{\gamma+\alpha-1}{1-\alpha}}\pi_\beta^{-\frac{\beta+\gamma-1}{1-\beta}} \rightarrow 0$,

Case 3: $\pi_\alpha^{\frac{\gamma+\alpha-1}{1-\alpha}}\pi_\beta^{-\frac{\beta+\gamma-1}{1-\beta}} \rightarrow K_4$, and $\pi_\alpha^{-1}\pi_\beta^{\frac{1-\gamma}{1-\beta}} \rightarrow 0$,

where K_1 through to K_4 are constants.

Case 1 If $\pi_\alpha^{-1}\pi_\beta^{\frac{1-\gamma}{1-\beta}} \rightarrow K_1$, and $\pi_\alpha^{\frac{\gamma+\alpha-1}{1-\alpha}}\pi_\beta^{-\frac{\beta+\gamma-1}{1-\beta}} \rightarrow K_2$, then

$$\pi_\alpha \propto \pi_\beta^{\frac{1-\gamma}{1-\beta}}, \text{ and}$$

$$\pi_\alpha \propto \pi_\beta^{\frac{\beta+\gamma-1}{1-\beta} \frac{1-\alpha}{\alpha+\gamma-1}}.$$

However,

$$\frac{1-\gamma}{1-\beta} \neq \frac{\beta+\gamma-1}{1-\beta} \frac{1-\alpha}{\alpha+\gamma-1},$$

since $1 - \gamma < 1 - \alpha$, and $\frac{\beta+\gamma-1}{\alpha+\gamma-1} > 1$. Hence, the two terms in E_1^{**} involving π_α and π_β cannot both go to a constant.

Case 2 If $\pi_\alpha^{-1} \pi_\beta^{\frac{1-\gamma}{1-\beta}} \rightarrow K_3$, then

$$\pi_\alpha \propto \pi_\beta^{\frac{1-\gamma}{1-\beta}}.$$

Since

$$\frac{(\alpha + \gamma - 1)}{1 - \alpha} \frac{1 - \gamma}{1 - \beta} - \frac{\beta + \gamma - 1}{1 - \beta} < 0,$$

it follows that

$$\pi_\alpha^{\frac{\gamma+\alpha-1}{1-\alpha}} \pi_\beta^{-\frac{\beta+\gamma-1}{1-\beta}} \rightarrow \infty.$$

Hence, if $\pi_\alpha^{-1} \pi_\beta^{\frac{1-\gamma}{1-\beta}} \rightarrow K_3$,

$$\pi_\alpha^{\frac{\gamma+\alpha-1}{1-\alpha}} \pi_\beta^{-\frac{\beta+\gamma-1}{1-\beta}} \not\rightarrow 0.$$

Case 3 If $\pi_\alpha^{\frac{\gamma+\alpha-1}{1-\alpha}} \pi_\beta^{-\frac{\beta+\gamma-1}{1-\beta}} \rightarrow K_4$, then

$$\pi_\alpha \propto \pi_\beta^{\frac{(\beta+\gamma-1)}{1-\beta} \frac{1-\alpha}{(\alpha+\gamma-1)}}.$$

Since

$$-\frac{1 - \alpha}{\gamma + \alpha - 1} \frac{(\beta + \gamma - 1)}{1 - \beta} + 1 - \gamma < 0,$$

it follows that

$$\pi_\alpha^{-1} \pi_\beta^{\frac{1-\gamma}{1-\beta}} \rightarrow \infty.$$

Hence, if $\pi_\alpha^{\frac{\gamma+\alpha-1}{1-\alpha}} \pi_\beta^{-\frac{\beta+\gamma-1}{1-\beta}} \rightarrow K_4$, then

$$\pi_\alpha^{-1} \pi_\beta^{\frac{1-\gamma}{1-\beta}} \not\rightarrow 0.$$

Hence, fixing α , β and γ , there is no π_α and π_β satisfying $E_1^{**} = 0$. Hence, fix any three demands, where $\beta \neq 1 - \alpha$, then there exists $\bar{z} > 0$ such that for any $z < \bar{z}$ no equilibrium exists, where players put strictly positive probability on all three demands.

Claim 7 When $\beta = 1 - \alpha$, a solution to the the system of equations

$$C_1 = 0, C_2, C_3 = 0, C_4 = 0,$$

and equation (35) exists.

Take C_1 to C_4 and simplify by the fact that $\beta = 1 - \alpha$. Denote the new expressions F_1 to F_4 .

$$F_1 = (\gamma - \alpha) \pi_\beta^{-\frac{1-\gamma}{\alpha}} - (\alpha + \gamma - 1) \pi_\alpha^{-\frac{1-\gamma}{1-\alpha}} - (1 - 2\alpha) \pi_\gamma^{-\frac{1-\gamma}{\alpha}} - (1 - 2\alpha) \frac{z_\alpha}{2} \quad (46)$$

$$F_2 = (\gamma - \alpha) \left(\pi_\gamma^{-\frac{1-\gamma}{\alpha}} - \pi_\beta^{-\frac{1-\gamma}{\alpha}} \right) \quad (47)$$

$$F_3 = \alpha \pi_\beta^{\frac{1-\alpha}{\alpha}} z_\beta - (1 - \gamma) \left(\pi_\alpha^{\frac{\gamma}{1-\alpha}-1} - \pi_\beta^{\frac{\gamma}{\alpha}-1} \right) \quad (48)$$

$$F_4 = (1 - \alpha) \pi_\alpha^{\frac{\gamma}{1-\alpha}} z_\alpha - (1 - \gamma) \left(\pi_\beta^{\frac{\gamma}{\alpha}-1} - \pi_\gamma^{\frac{2\gamma-1}{\alpha}} \right) - \alpha z_\beta \left(\pi_\beta^{\frac{1-\alpha}{\alpha}} - \pi_\beta^{\frac{\gamma}{\alpha}} \right) \quad (49)$$

Note that for a solution to exist, $\pi_\gamma = \pi_\beta$ (from $F_2 = 0$). I can solve for z_α and z_β by setting $F_1 = 0$ and $F_3 = 0$:

$$\begin{aligned} z_\alpha &= \frac{2 \left((\alpha + \gamma - 1) \pi_\alpha^{-\frac{1-\gamma}{1-\alpha}} + (\alpha - \gamma) \pi_\beta^{-\frac{1-\gamma}{\alpha}} + (1 - 2\alpha) \pi_\gamma^{-\frac{1-\gamma}{\alpha}} \right)}{2\alpha - 1} \\ z_\beta &= \frac{(1 - \gamma) \pi_\alpha^{\frac{\alpha+\gamma-1}{1-\alpha}} \left(\pi_\beta - \pi_\alpha^{-\frac{\alpha+\gamma-1}{1-\alpha}} \pi_\beta^{\frac{\gamma}{\alpha}} \right)}{\alpha} \end{aligned} \quad (50)$$

I replace z_α and z_β in F_4 . Moreover, I replace π_γ by π_β ; and multiply F_4 by $\pi_\alpha^{-\frac{\gamma+\alpha-1}{1-\alpha}}$. I denote the new expression G_4 :

$$G_4 = \frac{2(1 - \alpha)(\alpha + \gamma - 1) \pi_\alpha^{\frac{\gamma-\alpha}{1-\alpha}}}{1 - 2\alpha} \left(\pi_\alpha \pi_\beta^{-\frac{1-\gamma}{\alpha}} - \pi_\alpha^{\frac{\gamma-\alpha}{1-\alpha}} \right) - (1 - \gamma) \left(1 - \pi_\beta^{\frac{\alpha+\gamma-1}{\alpha}} \right) \quad (51)$$

Note that $\pi_\alpha^{\frac{\gamma-\alpha}{1-\alpha}} \rightarrow 0$. Similarly, $\pi_\beta^{\frac{\alpha+\gamma-1}{\alpha}} \rightarrow 0$. Hence, for $G_4 = 0$ as $z \rightarrow 0$, I require

$$\frac{2(1 - \alpha)(\alpha + \gamma - 1)}{1 - 2\alpha} \pi_\alpha \pi_\beta^{-\frac{1-\gamma}{\alpha}} - (1 - \gamma) \rightarrow 0.$$

Hence,

$$\pi_\alpha / \pi_\beta^{\frac{1-\gamma}{\alpha}} \rightarrow \frac{2(1 - \alpha)(\alpha + \gamma - 1)}{1 - 2\alpha}.$$

Define

$$K = \frac{2(1 - \alpha)(\alpha + \gamma - 1)}{1 - 2\alpha}.$$

Recall that $\pi_\gamma = \pi_\beta$. Therefore, I can write all variables as a function of the demands, π_β and z :

$$\begin{aligned}
s(\alpha) &\approx \frac{2(\alpha + \gamma - 1) \left(K^{\frac{\gamma-\alpha}{1-\alpha}} \pi_\beta^{\frac{(\gamma-\alpha)(1-\gamma)}{(1-\alpha)\alpha}} - K \right)}{2\alpha - 1} \\
s(\beta) &\approx \frac{(1 - \gamma) \left(\pi_\beta^{\frac{2(1-\alpha)^2 + (-2+\alpha)\gamma + \gamma^2}{(-1+\alpha)\alpha}} - K^{\frac{\gamma+\alpha-1}{1-\alpha}} \pi_\beta^{\frac{\alpha+\gamma-1}{\alpha}} \right)}{\alpha} \\
s(\gamma) &= 1 \\
r(\alpha) &\approx \frac{2(\alpha + \gamma - 1) \left(K^{\frac{\gamma-\alpha}{1-\alpha}} \pi_\beta^{\frac{(\gamma-\alpha)(1-\gamma)}{(1-\alpha)\alpha}} - K \right)}{2\alpha - 1} \frac{(-1 + K \pi_\beta^{\frac{1-\gamma}{\alpha}}) z}{K \pi_\beta^{\frac{1-\gamma}{\alpha}} (-1 + z)} \\
r(\beta) &\approx \frac{(1 - \gamma) \left(\pi_\beta^{\frac{2(1-\alpha)^2 + (-2+\alpha)\gamma + \gamma^2}{(-1+\alpha)\alpha}} - K^{\frac{\gamma+\alpha-1}{1-\alpha}} \pi_\beta^{\frac{\alpha+\gamma-1}{\alpha}} \right)}{\alpha} \frac{(-1 + \pi_\beta) z}{\pi_\beta (-1 + z)} \\
r(\gamma) &\approx \frac{z (\pi_\beta^{-\frac{1-\gamma}{\alpha}} - 1)}{1 - z} \\
r(\alpha) + r(\beta) + r(\gamma) &= s(\alpha) + s(\beta) + s(\gamma).
\end{aligned} \tag{52}$$

The constraint on the probabilities allows me to solve for π_β as a function of z only. It is then straightforward algebra to show that if

$$\begin{aligned}
\alpha &< \frac{1}{4} \left(4 - \gamma - \sqrt{(8 - 7\gamma)\gamma} \right), \\
\pi_\beta &\approx z.
\end{aligned}$$

If

$$\begin{aligned}
\alpha &> \frac{1}{4} \left(4 - \gamma - \sqrt{(8 - 7\gamma)\gamma} \right), \\
\pi_\beta &\approx \left(\frac{(1 - \alpha)(1 - \gamma)}{\alpha(2 - \alpha - \gamma)} z \right)^{\frac{\alpha - \alpha^2}{2 - 3\alpha + \alpha^2 - 2\gamma + \alpha\gamma + \gamma^2}}.
\end{aligned}$$

Case 1 $\pi_\beta \approx z$; $\alpha < \frac{1}{4} \left(4 - \gamma - \sqrt{(8 - 7\gamma)\gamma} \right)$. Then I can solve for the probabilities first as a function of z only, and then take limits with respect to z :

$$\begin{aligned}
\lim_{z \rightarrow 0} s(\alpha) &= \frac{1 - c}{1 - a}, & \lim_{z \rightarrow 0} s(\beta) &= \infty, & \lim_{z \rightarrow 0} s(\gamma) &= 1, \\
\lim_{z \rightarrow 0} r(\alpha) &= \frac{2(a + c - 1)}{1 - 2a}, & \lim_{z \rightarrow 0} r(\beta) &= \infty, & \lim_{z \rightarrow 0} r(\gamma) &= 0.
\end{aligned} \tag{53}$$

I denote the scaled probabilities with the subscript “s”. Then:

$$\begin{aligned}\lim_{z \rightarrow 0} s^s(\alpha) &= 0, & \lim_{z \rightarrow 0} s^s(\beta) &= 1, & \lim_{z \rightarrow 0} s^s(\gamma) &= 0, \\ \lim_{z \rightarrow 0} r^s(\alpha) &= 0, & \lim_{z \rightarrow 0} r^s(\beta) &= 1, & \lim_{z \rightarrow 0} r^s(\gamma) &= 0.\end{aligned}\tag{54}$$

Case 2 $\pi_\beta \approx \left(\frac{(1-\alpha)(1-\gamma)}{\alpha(2-\alpha-\gamma)} z \right)^{\frac{\alpha-\alpha^2}{2-3\alpha+\alpha^2-2\gamma+\alpha\gamma+\gamma^2}}; \alpha > \frac{1}{4} \left(4 - \gamma - \sqrt{(8-7\gamma)\gamma} \right)$. Then in the same fashion, I derive:

$$\begin{aligned}\lim_{z \rightarrow 0} s(\alpha) &= \frac{1-c}{1-a}, & \lim_{z \rightarrow 0} s(\beta) &= 0, & \lim_{z \rightarrow 0} s(\gamma) &= 1, \\ \lim_{z \rightarrow 0} r(\alpha) &= 0, & \lim_{z \rightarrow 0} r(\beta) &= \frac{2-a-c}{1-a}, & \lim_{z \rightarrow 0} r(\gamma) &= 0.\end{aligned}\tag{55}$$

Hence, the correctly scaled probabilities are:

$$\begin{aligned}\lim_{z \rightarrow 0} s^s(\alpha) &= \frac{1-c}{2-a-c}, & \lim_{z \rightarrow 0} s^s(\beta) &= 0, & \lim_{z \rightarrow 0} s^s(\gamma) &= \frac{1-a}{2-a-c}, \\ \lim_{z \rightarrow 0} r^s(\alpha) &= 0, & \lim_{z \rightarrow 0} r^s(\beta) &= 1, & \lim_{z \rightarrow 0} r^s(\gamma) &= 0.\end{aligned}\tag{56}$$

When $\alpha = \frac{1}{4} \left(4 - \gamma - \sqrt{(8-7\gamma)\gamma} \right)$, then $\lim_{z \rightarrow 0} s(\beta) = \frac{1-c}{a}$. Hence, in this case all three probabilities of the stubborn player are strictly interior (everything else unchanged). The third part of the proposition follows by the implicit function theorem:

The payoff differences between the offers a , b , and c for a rational and stubborn player

respectively can be written as:

$$\begin{aligned}
A_1 &= -\frac{1}{2}(-a+b+1)(r_\alpha(-z) + r_\alpha + s_\alpha z) + \frac{1}{2}(a-b+1)(r_\beta(-z) + r_\beta + s_\beta z) \\
&\quad + s_\gamma z(-a-c+1) \left(\frac{s_\alpha z}{r_\alpha(-z) + r_\alpha + s_\alpha z} \right)^{\frac{1-c}{a-1}} + a(r_\gamma(-z) + r_\gamma + s_\gamma z) \\
&\quad + s_\gamma z(b+c-1) \left(\frac{s_\beta z}{r_\beta(-z) + r_\beta + s_\beta z} \right)^{\frac{1-c}{b-1}} \\
&\quad - (b-1)(r_\beta(z-1) - s_\beta z) + b(r_\gamma(z-1) - s_\gamma z) + \frac{1}{2}(r_\alpha(-z) + r_\alpha + s_\alpha z) \\
A_2 &= \frac{1}{2}z \left(s_\alpha(a+b-1) + 2s_\gamma(b+c-1) \left(\left(\frac{s_\beta z}{r_\beta(-z) + r_\beta + s_\beta z} \right)^{\frac{c-1}{b-1}} - 1 \right) \left(\frac{s_\beta z}{r_\beta(-z) + r_\beta + s_\beta z} \right)^{\frac{1-c}{b-1}} \right) \\
&\quad - \frac{1}{2}r_\alpha(z-1)(a+b-1) - r_\gamma(z-1)(b+c-1) \\
A_3 &= (c-1)s_\gamma z \left(\frac{s_\alpha z}{r_\alpha(-z) + r_\alpha + s_\alpha z} \right)^{-\frac{a+c-1}{a-1}} - (c-1)s_\gamma z \left(\frac{s_\beta z}{r_\beta(-z) + r_\beta + s_\beta z} \right)^{-\frac{b+c-1}{b-1}} \\
&\quad - (b-1)s_\beta z \left(\frac{s_\beta z}{r_\beta(-z) + r_\beta + s_\beta z} \right)^{\frac{1-2b}{b-1}} \\
A_4 &= (a-1)(r_\alpha(z-1) - s_\alpha z) \left(\frac{s_\alpha z}{r_\alpha(-z) + r_\alpha + s_\alpha z} \right)^{\frac{c}{1-a}} + (c-1)s_\gamma z \left(\frac{s_\beta z}{r_\beta(-z) + r_\beta + s_\beta z} \right)^{-\frac{b+c-1}{b-1}} \\
&\quad - (b-1)s_\beta z \left(\frac{s_\beta z}{r_\beta(-z) + r_\beta + s_\beta z} \right)^{\frac{c}{1-b}-1} - (b-1)(r_\beta(z-1) - s_\beta z) \left(\frac{s_\beta z}{r_\beta(-z) + r_\beta + s_\beta z} \right)^{\frac{b}{1-b}} \\
&\quad - (c-1)s_\gamma z \left(\frac{s_\gamma z}{r_\gamma(-z) + r_\gamma + s_\gamma z} \right)^{\frac{1-2c}{c-1}}
\end{aligned} \tag{57}$$

I can replace $\frac{s_i}{zs_i+(1-z)r_i}$ by u_i for $i = a, b, c$. Moreover, note that from the first part of the proof $b = 1 - a$. Moreover, I replace $r_c = 1 - r_a - r_b$. Then I can solve $A_1 = 0$ for r_a as a function of r_b , and u_i . Moreover, I can solve $A_2 = 0$ for u_c as a function of u_b only. I can then plug the expressions for r_a and u_b into A_3 and A_4 . I can then solve $A_3 = 0$ for r_b as a function of u_i . I

plug this into A_4 . I replace u_i by y_i/z for $i = a, b$. Then I get:

$$\begin{aligned}
A_4 &= \frac{n_{4.0}n_{4.1}}{d_4}, \text{ where} \\
n_{4.0} &= a(z-1)\pi_\alpha^{-\frac{c}{a-1}} \left(\pi_\beta^{\frac{1-c}{a}}\right)^{-\frac{2c}{c-1}} \\
n_{4.1} &= 2(a-1)(a+c-1)\pi_\alpha^{\frac{a+1}{a-1}}\pi_\beta^{\frac{1}{a}} \left(\pi_\beta^{\frac{1-c}{a}}\right)^{\frac{2c}{c-1}} - (2a-1)(c-1)\pi_\alpha^{\frac{c+1}{a-1}} \left(\pi_\beta^{\frac{1-c}{a}}\right)^{\frac{2c}{c-1}} \left(\pi_\beta^{\frac{1}{a}} - \pi_\beta^{\frac{a+c}{a}}\right) \\
&\quad - 2(a-1)(a+c-1)\pi_\alpha^{\frac{a+c}{a-1}}\pi_\beta^{\frac{c}{a}} \left(\pi_\beta^{\frac{1-c}{a}}\right)^{\frac{2c}{c-1}} + (2a-1)(c-1)\pi_\alpha^{\frac{a+2c}{a-1}} \left(\pi_\beta^{\frac{1}{a}} \left(\pi_\beta^{\frac{1-c}{a}}\right)^{\frac{1}{c-1}} - \pi_\beta^{\frac{2c}{a}} \left(\pi_\beta^{\frac{1-c}{a}}\right)^{\frac{2c}{c-1}}\right) \\
d_4 &= -2a(a+c-1)\pi_\alpha^{\frac{2a}{a-1}}\pi_\beta^{\frac{1}{a}} \\
&\quad + 2a(a+c-1)\pi_\alpha^{\frac{a+1}{a-1}}\pi_\beta^{\frac{1}{a}} + \pi_\alpha^{\frac{a+c}{a-1}} \left(-((2a-1)(c-1)\pi_\beta + a+c-1)\pi_\beta^{\frac{c}{a}} - a(2a-1)\pi_\beta^{\frac{1}{a}}\right) \\
&\quad + 2a(a+c-1)\pi_\alpha^{\frac{2a+c-1}{a-1}}\pi_\beta^{\frac{c}{a}} + (2a-1)(c-1)(\pi_\beta-1)\pi_\beta\pi_\alpha^{\frac{1}{a-1}}
\end{aligned} \tag{58}$$

The remaining equation is the constraint:

$$r_a + r_b + r_c = s_a + s_b + s_c.$$

Doing the same change of variables as with the payoff differences, I get:

$$\begin{aligned}
C_1 &= \frac{c_1(-1+z)}{c_2 z} - 1, \text{ where} \\
c_1 &= 2a(a+c-1)\pi_\alpha^{\frac{2a+1}{a-1}}\pi_\beta^{\frac{1}{a}} + (2a-1)\pi_\alpha^{\frac{a+c+1}{a-1}} \left((c-1)\pi_\beta^{\frac{a+c}{a}} + a\pi_\beta^{\frac{1}{a}}\right) \\
&\quad - 2a(a+c-1)\pi_\alpha^{\frac{2a+c}{a-1}}\pi_\beta^{\frac{c}{a}} + (1-2a)(c-1)\pi_\beta^2\pi_\alpha^{\frac{2}{a-1}} \\
c_2 &= \pi_\alpha^{\frac{a+c+1}{a-1}} \left(2a^2\pi_\beta^{\frac{1}{a}} - (c-1)(\pi_\beta-1)\pi_\beta^{\frac{c}{a}} + a \left(\pi_\beta^{\frac{c}{a}} + 2(c-1)\pi_\beta^{\frac{a+c}{a}} - \pi_\beta^{\frac{1}{a}}\right)\right) \\
&\quad - 2a(a+c-1)\pi_\alpha^{\frac{a+2}{a-1}}\pi_\beta^{\frac{1}{a}} + 2a(a+c-1)\pi_\alpha^{\frac{2a+1}{a-1}}\pi_\beta^{\frac{1}{a}} - 2a(a+c-1)\pi_\alpha^{\frac{2a+c}{a-1}}\pi_\beta^{\frac{c}{a}} \\
&\quad + (1-2a)(c-1)(\pi_\beta-1)\pi_\beta\pi_\alpha^{\frac{2}{a-1}}
\end{aligned} \tag{59}$$

Hence, we can express the system as $A_4 = 0$, and $C_1 = 0$.

Claim 8 *The system*

$$\begin{aligned}
C_1 &= 0 \\
A_4 &= 0
\end{aligned} \tag{60}$$

can be solved locally around $z = 0$, with $s_i \in (0, 1)$, $r_i \in (0, 1)$ for $i \in \{a, 1-a, c\}$.

Case 1:

$$\frac{(a-2)c + 2(a-1)^2 + c^2}{(a-1)a} < 0.$$

In this case, $\pi_\beta \approx z$. I replace π_α by

$$x_\alpha^{\frac{1-c}{a}}$$

in A_4 and C_1 . Then all variables in A_4 and C_1 go to 0 at the same rate. I denote the numerator of these expressions (with the replacement of π_α) by D_1 and D_2 :

$$\begin{aligned} D_1 &= a(z-1)x_\alpha^{-\frac{a(c-2)+2c^2+c-1}{(a-1)a}} \pi_\beta^{-\frac{2c}{a}} d_1, \text{ where} \\ d_1 &= -2(a-1)(a+c-1)\pi_\beta^{\frac{1}{a}} x_\alpha^{\frac{-ac+a-2c^2+c}{a-a^2}} \\ &\quad + 2(a-1)(a+c-1)x_\alpha^{\frac{a(c-1)+c^2+c-1}{(a-1)a}} \pi_\beta^{\frac{c}{a}} + (2a-1)(c-1)x_\alpha^{\frac{2a(c-1)+c^2}{(a-1)a}} \left(\pi_\beta^{\frac{1}{a}} - \pi_\beta^{\frac{a+c}{a}} \right) \\ D_2 &= -(2a-1)x_\alpha^{\frac{a-c^2}{(a-1)a}} \left((c-1)\pi_\beta^{\frac{a+c}{a}} + a\pi_\beta^{\frac{1}{a}} \right) \\ &\quad + 2a(z-1)(a+c-1)\pi_\beta^{\frac{1}{a}} x_\alpha^{-\frac{a(c-2)+c}{(a-1)a}} - 2az(a+c-1)x_\alpha^{-\frac{(a+1)(c-1)}{(a-1)a}} (x_\alpha \pi_\beta)^{\frac{1}{a}} \\ &\quad - z(a+c-1)x_\alpha^{-\frac{(c-1)(a+c)}{(a-1)a}} (x_\alpha \pi_\beta)^{\frac{c}{a}} + 2az(a+c-1)\pi_\beta^{\frac{1}{a}} x_\alpha^{\frac{a-2c+1}{(a-1)a}} \\ &\quad + 2a(a+c-1)x_\alpha^{\frac{1-\frac{(c-1)(a+c)}{a-1}}{a}} \pi_\beta^{\frac{c}{a}} + (2a-1)(c-1)\pi_\beta(\pi_\beta - z)x_\alpha^{\frac{(a-2)c+1}{(a-1)a}} \end{aligned} \quad (61)$$

For D_1 to be 0, we need $d_1 = 0$, and hence, I can ignore the remaining terms in D_1 . Further, I multiply d_1 by

$$x_\alpha^{\frac{1+a^2-2ac-c^2}{a^2-a}};$$

similarly, I multiply D_2 by

$$\pi_\beta^{-1+\frac{1-2c+ac}{a-a^2}}.$$

Recall that as $z \rightarrow 0$, $\pi_\beta \rightarrow 0$ and $x_\alpha \rightarrow 0$. Hence, I delete all but the biggest term in d_1 ; similarly in D_1 . Call the new expressions E_1 and E_2 , respectively:

$$\begin{aligned} E_1 &= x_\alpha^{\frac{a^2-2ac-c^2+1}{(a-1)a}} \left(2(a-1)(a+c-1)x_\alpha^{\frac{a(c-1)+c^2+c-1}{(a-1)a}} \pi_\beta^{\frac{c}{a}} + (2a-1)(c-1)\pi_\beta^{\frac{1}{a}} x_\alpha^{\frac{2a(c-1)+c^2}{(a-1)a}} \right) \\ E_2 &= (2a-1)(c-1)(\pi_\beta - z)x_\alpha^{\frac{(a-2)c+1}{(a-1)a}} \pi_\beta^{\frac{ac-2c+1}{a-a^2}} \end{aligned} \quad (62)$$

I can then take derivatives with respect to π_β , x_α and z respectively. I then evaluate those derivatives at $z = 0$, $x_\alpha = m\pi_\beta$, where

$$m = 2^{-\frac{a}{1-c}} \left(\frac{(1-2a)(1-c)}{(1-a)(a+c-1)} \right)^{\frac{a}{1-c}}.$$

Then I can write the derivatives as:

$$\begin{aligned}
\left. \frac{dE_1}{dx_\alpha} \right|_{z=0, x_\alpha=m\pi_\beta} &= \frac{(a-1) \left((2a-1)(c-1)m^{-1/a} + 2(a-c)(a+c-1)m^{-\frac{c}{a}} \right)}{a} \\
\left. \frac{dE_1}{d\pi_\beta} \right|_{z=0, x_\alpha=m\pi_\beta} &= \frac{m^{\frac{a^2-3ac+2a-2c^2+2}{a-a^2}} \left(2(a-1)c(a+c-1)m^{\frac{a+c}{(a-1)a}} + (2a-1)(c-1)m^{\frac{ac+1}{(a-1)a}} \right)}{a} \\
\left. \frac{dE_2}{dx_\alpha} \right|_{z=0, x_\alpha=m\pi_\beta} &= \frac{(2a-1)(c-1)((a-2)c+1)m^{\frac{(a-2)c+1}{(a-1)a}-1}}{(a-1)a} \\
\left. \frac{dE_2}{d\pi_\beta} \right|_{z=0, x_\alpha=m\pi_\beta} &= \frac{(2a-1)(c-1)(a^2-a(c+1)+2c-1)m^{\frac{(a-2)c+1}{(a-1)a}}}{(a-1)a}
\end{aligned} \tag{63}$$

Denote the Jacobian

$$J_1 = \left[\begin{array}{cc} \left. \frac{dE_1}{dx_\alpha} \right|_{z=0, x_\alpha=m\pi_\beta} & \left. \frac{dE_1}{d\pi_\beta} \right|_{z=0, x_\alpha=m\pi_\beta} \\ \left. \frac{dE_2}{dx_\alpha} \right|_{z=0, x_\alpha=m\pi_\beta} & \left. \frac{dE_2}{d\pi_\beta} \right|_{z=0, x_\alpha=m\pi_\beta} \end{array} \right].$$

Similarly, I evaluate

$$\begin{aligned}
\left. \frac{dE_1}{dz} \right|_{z=0, x_\alpha=m\pi_\beta} &= 0 \\
\left. \frac{dE_2}{dz} \right|_{z=0, x_\alpha=m\pi_\beta} &= (1-2a)(c-1) \left(2^{\frac{a}{c-1}} \left(-\frac{(a-1)(a+c-1)}{(2a-1)(c-1)} \right)^{\frac{a}{c-1}} \right)^{\frac{(a-2)c+1}{(a-1)a}}
\end{aligned} \tag{64}$$

Denote

$$Z_1 = \left[\begin{array}{c} \left. \frac{dE_2}{dz} \right|_{z=0, x_\alpha=m\pi_\beta} \end{array} \right].$$

Finally,

$$\left[\begin{array}{c} \left. \frac{dx_\alpha}{dz} \right|_{z=0, x_\alpha=m\pi_\beta} \\ \left. \frac{d\pi_\beta}{dz} \right|_{z=0, x_\alpha=m\pi_\beta} \end{array} \right] = -J_1^{-1} \times Z_1.$$

It can be verified that

$$\begin{aligned}
\left. \frac{dx_\alpha}{dz} \right|_{z=0, x_\alpha=m\pi_\beta} &> 0 \\
\left. \frac{d\pi_\beta}{dz} \right|_{z=0, x_\alpha=m\pi_\beta} &> 0.
\end{aligned} \tag{65}$$

Hence, if

$$\frac{(a-2)c + 2(a-1)^2 + c^2}{(a-1)a} < 0,$$

then $A_4 = 0$ and $C_2 = 0$ can be solved locally around $z = 0$.

Case 2:

$$\frac{(a-2)c + 2(a-1)^2 + c^2}{(a-1)a} > 0.$$

In this case, $z \approx \frac{a^2 + ac - 3a + c^2 - 2c + 2}{a - a^2} \frac{\pi_\beta}{(1-a)(1-c)}$. I replace π_β by

$$x_\beta^{\frac{a-a^2}{a^2+ac-3a+c^2-2c+2}}.$$

Similarly, I replace π_α by

$$x_\alpha^{\frac{(a-1)(c-1)}{a^2+a(c-3)+(c-2)c+2}}.$$

Note that now all variables go to 0 at the same rate. Denote the numerator of C_1 and D_2 by D_1 and D_2 :

$$\begin{aligned} D_1 &= -a(z-1)x_\alpha^{-\frac{1}{a^2+a(c-3)+(c-2)c+2}} x_\beta^{\frac{(a-1)a}{a^2+a(c-3)+(c-2)c+2}} d_1, \text{ where} \\ d_1 &= -(2a-1)(c-1)x_\alpha^{\frac{c^2}{a^2+a(c-3)+(c-2)c+2}} \left(x_\beta^{\frac{(a-1)c}{a^2+a(c-3)+(c-2)c+2}} - x_\beta^{\frac{(a-1)(a+2c-1)}{a^2+a(c-3)+(c-2)c+2}} \right) \\ &\quad - 2(a-1)(a+c-1)x_\alpha^{\frac{a(c-1)+c}{a^2+a(c-3)+(c-2)c+2}} x_\beta^{\frac{(a-1)(a+2c-1)}{a^2+a(c-3)+(c-2)c+2}} \\ &\quad + 2(a-1)(a+c-1)x_\alpha^{\frac{a(c-1)+c(c-1)+1}{a^2+a(c-3)+(c-2)c+2}} x_\beta^{\frac{(a-1)(a+c)}{a^2+a(c-3)+(c-2)c+2}} \\ D_2 &= x_\alpha^{-\frac{1}{a^2+a(c-3)+(c-2)c+2}} x_\beta^{-\frac{3a}{a^2+a(c-3)+(c-2)c+2}} d_2, \text{ where} \\ d_2 &= 2a(a+c-1)x_\alpha^{\frac{a(-a+c+1)}{a^2+a(c-3)+(c-2)c+2}+1} x_\beta^{\frac{2a^2-ac+a+c}{a^2+a(c-3)+(c-2)c+2}} \\ &\quad + 2az(a+c-1)x_\alpha^{\frac{a(c-1)+c}{a^2+a(c-3)+(c-2)c+2}} x_\beta^{\frac{2a^2+1}{a^2+a(c-3)+(c-2)c+2}} \\ &\quad - x_\alpha^{\frac{a(c-1)+c(c-1)+1}{a^2+a(c-3)+(c-2)c+2}} x_\beta^{\frac{a(a-2c)}{a^2+a(c-3)+(c-2)c+2}} d_3 \\ &\quad + (2a-1)(c-1)x_\alpha^{\frac{c}{a^2+a(c-3)+(c-2)c+2}} \left(x_\beta^{\frac{3a}{a^2+a(c-3)+(c-2)c+2}} - zx_\beta^{\frac{a(a+2)}{a^2+a(c-3)+(c-2)c+2}} \right) \\ &\quad - 2a(a+c-1)x_\alpha^{\frac{2a(c-1)+1}{a^2+a(c-3)+(c-2)c+2}} x_\beta^{\frac{2a^2+1}{a^2+a(c-3)+(c-2)c+2}}, \text{ and} \\ d_3 &= \left(z(a+c-1)x_\beta^{\frac{(a+1)(a+c)}{a^2+a(c-3)+(c-2)c+2}} + a(2a-1)x_\beta^{\frac{a^2+2ac+1}{a^2+a(c-3)+(c-2)c+2}} + (2a-1)(c-1)x_\beta^{\frac{a(c+2)+c}{a^2+a(c-3)+(c-2)c+2}} \right) \end{aligned} \tag{66}$$

I multiply D_1 by

$$x_\alpha^{\frac{2-a(a+c)}{a^2+a(c-3)+(c-2)c+2}},$$

and D_2 by

$$x_\alpha^{\frac{-a^2+a-c+1}{a^2+a(c-3)+(c-2)c+2}}.$$

Then I do first order Taylor expansions (i.e., keeping the largest term in both D_1 and D_2). I call the new expressions E_1 and E_2 :

$$\begin{aligned} E_1 &= 2(a-1)a(z-1)(a+c-1)x_\alpha^{\frac{(c-1)(a+c)}{a^2+a(c-3)+(c-2)c+2} + \frac{2-a(a+c)}{a^2+a(c-3)+(c-2)c+2}} x_\beta^{\frac{(a-1)(2a+c)}{a^2+a(c-3)+(c-2)c+2}} \\ E_2 &= (1-2a)(c-1)zx_\alpha^{\frac{-a^2+a-c+1}{a^2+a(c-3)+(c-2)c+2} + \frac{c-1}{a^2+a(c-3)+(c-2)c+2}} x_\beta^{\frac{(a-1)a}{a^2+a(c-3)+(c-2)c+2}} \end{aligned} \quad (67)$$

I can then take derivatives with respect to x_β , x_α and z respectively. I then evaluate those derivatives at $z=0$, $x_\alpha = nx_\beta$, where

$$n = 2^{-\frac{a-1}{c-1} - \frac{c}{a-1} - 1} \left(\frac{-2a(c-1) + c-1}{(a-1)(a+c-1)} \right)^{\frac{a-1}{c-1} + \frac{c}{a-1} + 1}.$$

Then I can write the derivatives as:

$$\begin{aligned} \left. \frac{dE_1}{dx_\alpha} \right|_{z=0, x_\alpha=nx_\beta} &= \frac{2(a-1)a(a+c-1)(a^2+a-c^2+c-2)n^{-\frac{(a-1)(2a+c)}{a^2+a(c-3)+c^2-2c+2}}}{a^2+a(c-3)+c^2-2c+2} \\ \left. \frac{dE_1}{dx_\beta} \right|_{z=0, x_\alpha=nx_\beta} &= -\frac{2(a-1)^2a(a+c-1)(2a+c)n^{-\frac{a^2+a-c^2+c-2}{a^2+a(c-3)+c^2-2c+2}}}{a^2+a(c-3)+c^2-2c+2} \\ \left. \frac{dE_2}{dx_\alpha} \right|_{z=0, x_\alpha=nx_\beta} &= -\frac{a^2(2a-1)(a+c-2)n^{-\frac{(a-1)a}{a^2+a(c-3)+c^2-2c+2}-1}}{a^2+a(c-3)+c^2-2c+2} \\ \left. \frac{dE_2}{dx_\beta} \right|_{z=0, x_\alpha=nx_\beta} &= \frac{a^2(2a-1)(a+c-2)n^{-\frac{(a-1)a}{a^2+a(c-3)+c^2-2c+2}}}{a^2+a(c-3)+c^2-2c+2} \end{aligned} \quad (68)$$

Denote the Jacobian

$$J_2 = \left[\begin{array}{cc} \frac{dE_1}{dx_\alpha} & \frac{dE_1}{d\pi_\beta} \\ \frac{dE_2}{dx_\alpha} & \frac{dE_2}{d\pi_\beta} \end{array} \right] \bigg|_{z=0, x_\alpha=nx_\beta}.$$

Similarly, I evaluate

$$\begin{aligned} \left. \frac{dE_1}{dz} \right|_{z=0, x_\alpha=nx_\beta} &= 0 \\ \left. \frac{dE_2}{dz} \right|_{z=0, x_\alpha=nx_\beta} &= (1-2a)(c-1)2^{-\frac{a}{1-c}} \left(\frac{-2a(c-1) + c-1}{(a-1)(a+c-1)} \right)^{\frac{a}{1-c}} \end{aligned} \quad (69)$$

Denote

$$Z_2 = \left[\begin{array}{c} \frac{dE_1}{dz} \\ \frac{dE_2}{dz} \end{array} \right] \bigg|_{z=0, x_\alpha = nx_\beta}.$$

Finally,

$$\left[\begin{array}{c} \frac{dx_\alpha}{dz} \\ \frac{dx_\beta}{dz} \end{array} \right] \bigg|_{z=0, x_\alpha = nx_\beta} = -J_2^{-1} \times Z_2.$$

$$\begin{aligned} \frac{dx_\alpha}{dz} &= - \frac{(a-1)(c-1)2^{-\frac{a^2+a(c-3)+c^2-2c+2}{(a-1)(c-1)}}(2a+c) \left(\frac{-2a(c-1)+c-1}{(a-1)(a+c-1)} \right)^{\frac{a^2+a(c-3)+c^2-2c+2}{(a-1)(c-1)}}}{a^2(a+c-2)} > 0 \\ \frac{dx_\beta}{dz} &= - \frac{(c-1)(a^2+a-c^2+c-2)}{a^2(a+c-2)} > 0. \end{aligned} \quad (70)$$

Hence, if

$$\frac{(a-2)c + 2(a-1)^2 + c^2}{(a-1)a} > 0,$$

the system $C_1 = 0$, $A_4 = 0$, can be solved locally around $z = 0$. ■

Proof of Lemma 5. In PBE with more than one offer, at least one offer must have a posterior probability of stubbornness less than z . This implies that the rational type would benefit from deviating to an offer within $\epsilon > 0$ of this offer – he would be conceded to with higher probability, and hence, receive a higher payoff.

Consider one-offer PBE with asymmetric discounting. With passive beliefs, the payoff for a rational player i from a deviation to d when his opponent demands α_j is:

$$V_{Ri}^d = d \left(1 - z z^{-\frac{\rho_i}{\rho_j} \frac{1-\alpha_j}{1-d}} \right) + (1 - \alpha_j) z z^{-\frac{\rho_i}{\rho_j} \frac{1-\alpha_j}{1-d}} \quad (71)$$

The first step is to show that $(\alpha_1, \alpha_2) = \left(\frac{\rho_2}{\rho_1 + \rho_2}, \frac{\rho_1}{\rho_1 + \rho_2} \right)$ is a passive belief equilibrium. I then show that for any other candidate equilibrium, there is a profitable deviation for the rational player.

Step 1 If $(\alpha_1, \alpha_2) = \left(\frac{\rho_2}{\rho_1 + \rho_2}, \frac{\rho_1}{\rho_1 + \rho_2} \right)$, then the payoff to a rational player and a stubborn player is the same. Hence, if there is a profitable deviation, it must be a profitable deviation for the rational player. Clearly, a rational player i would not want to deviate to a demand below $\frac{\rho_j}{\rho_1 + \rho_2}$. Hence, consider a deviation to $d > \frac{\rho_j}{\rho_1 + \rho_2}$. His payoff from this deviation is:

$$V_{R2i}^d = d \left(1 - z z^{-\frac{\rho_i}{\rho_1 + \rho_2} \frac{1}{1-d}} \right) + \frac{\rho_j}{\rho_1 + \rho_2} z z^{-\frac{\rho_i}{\rho_1 + \rho_2} \frac{1}{1-d}} \quad (72)$$

Clearly, $V_{Ri}^d < \frac{\rho_j}{\rho_1 + \rho_2}$ if and only if:

$$-\frac{\rho_i}{\rho_1 + \rho_2} \frac{1}{1-d} + 1 > 0.$$

Since $d > \frac{\rho_j}{\rho_1 + \rho_2}$, there is no profitable deviation for $i = 1, 2$.

Step 2 Note that there cannot be an asymmetric one-offer equilibrium with $\alpha_1 + \alpha_2 > 1$. A stubborn player would have an incentive to deviate to a demand which makes his demand just compatible with his opponent. Hence, consider a candidate equilibrium with $\alpha_i = 1 - \alpha_j < \frac{\rho_j}{\rho_1 + \rho_2}$. I claim that a rational player then has an incentive to deviate to $\frac{\rho_j}{\rho_1 + \rho_2}$. A rational player's payoff from deviating to $d = \frac{\rho_j}{\rho_1 + \rho_2}$ is:

$$\begin{aligned} V_{Ri}^d &= d \left(1 - z z^{-\frac{\rho_i}{\rho_j} \frac{(1-\alpha_j)}{(1-d)}} \right) + (1 - \alpha_j) z z^{-\frac{\rho_i}{\rho_j} \frac{(1-\alpha_j)}{(1-d)}} \\ &= \frac{\rho_j}{\rho_1 + \rho_2} \left(1 - z z^{-\frac{\rho_2 + \rho_1}{\rho_j} (1-\alpha_j)} \right) - (1 - \alpha_j) z z^{-\frac{\rho_2 + \rho_1}{\rho_j} (1-\alpha_j)}. \end{aligned} \quad (73)$$

Clearly, $V_{Ri}^d > 1 - \alpha_j = \alpha_i$ if and only if

$$1 - \frac{\rho_2 + \rho_1}{\rho_1} (1 - \alpha_1) > 0.$$

Since $1 - \alpha_j < \frac{\rho_j}{\rho_1 + \rho_2}$, $V_{Ri}^d > 1 - \alpha_j$. Hence, unless $\alpha_i = \frac{\rho_j}{\rho_1 + \rho_2}$, there always is a profitable deviation for the rational player i . ■

Proof of Lemma 7. Suppose a rational player i deviates to $d \in (\alpha, 1 - \alpha)$. Then the rational type's payoff is:

$$v_{ri}^d = (z s_\alpha + (1 - z) r_\alpha) \left(\frac{d + 1 - \alpha}{2} \right) + (z s_\beta + (1 - z) r_\beta) \left(d F_j^{\beta, d}(0) + (1 - \beta)(1 - F_j^{\beta, d}(0)) \right),$$

where

$$F_j^{\beta, d}(0) = 1 - \frac{z s_\beta}{z s_\beta + (1 - z) r_\beta} (z s_d)^{-\frac{1-\beta}{1-d}}$$

and $z s_d$ is the belief that j puts on i being stubborn conditional demanding d . I can then solve for the value of s_d which makes the rational type indifferent between his equilibrium demands and the deviating demand d , call it s_d^r :

$$s_d^r = \frac{1}{z} \left(\frac{2(-1 + \beta + d)(-1 + s_\alpha)z}{2 - 2\beta - 2d - (1 + \alpha)r_\alpha + 2\beta r_\alpha + d r_\alpha + (1 + \alpha - 2\beta - d)(r_\alpha - s_\alpha)z} \right)^{\frac{1-d}{1-\beta}}.$$

In the following I show that at s_d^r the stubborn type strictly prefers to deviate to d . The payoff of a stubborn type from a deviation $d \in (\alpha, 1 - \alpha)$ is:

$$v_{si}^d = (zs_\alpha + (1 - z)r_\alpha) \left(\frac{d + 1 - \alpha}{2} \right) + (zs_\beta + (1 - z)r_\beta) \left(1 - \left(\frac{zs_\beta}{zs_\beta + (1 - z)r_\beta} \right)^{\frac{d}{1 - \beta}} \right) \left(dF_j^{\beta, d}(0) + (1 - \beta)(1 - F_j^{\beta, d}(0)) \right). \quad (74)$$

The payoff from the equilibrium demand β is: I first replace s_d in $v_{si}^d - v_{si}^\beta$ by s_d^r . Moreover, I replace $\frac{zs_\beta}{zs_\beta + (1 - z)r_\beta}$ by π_β , and $\frac{zs_\alpha}{zs_\alpha + (1 - z)r_\alpha}$ by π_α . I can then write the payoff difference as:

I show that $v_{si}^d|_{s_d^r} - v_{si}^\beta > 0$. [To be completed]. ■

Proof of Lemma 8. (a) Denote the probability with which the stubborn type demands α by s_h . The payoff of a rational player from demanding α is:

$$v_r^\alpha = (1 - z + zs_h)(1 - \alpha) + z(1 - s_h)\alpha.$$

Suppose an opponent puts probability 1 on any deviation coming from a rational type. A rational player has no strict incentive to play any offer above α – he will receive the same payoff from the deviation as in the candidate equilibrium. From any deviating demand less than α , he receives a strictly lower payoff: when being faced with a demand of α , he receives at most $1 - \alpha$; yet, when being faced with a demand of $1 - \alpha$ his payoff is strictly less than α . Hence, a rational player has no incentive to deviate.

The payoff of a stubborn type from demanding α , and $1 - \alpha$ respectively is:

$$v_s^\alpha = (1 - z + zs_h)(1 - \alpha) \left(1 - z^{\frac{\alpha}{1 - \alpha}} \right) + z(1 - s_h)\alpha,$$

$$v_s^{1 - \alpha} = (1 - z + zs_h)(1 - \alpha) + z(1 - s_h)1/2.$$

I can solve this for the mixing probability s_h^* :

$$s_h^* = \frac{-2(\alpha - 1)z^{\frac{\alpha}{1 - \alpha}} + 2(\alpha - 1)z^{\frac{1}{1 - \alpha}} - 2\alpha z + z}{z \left(-2\alpha + 2(\alpha - 1)z^{\frac{\alpha}{1 - \alpha}} + 1 \right)},$$

where $s_h^* \in [0, 1]$ for any $\alpha > 1/2$, as $z \rightarrow 0$. (b) Suppose there was one offer which is only made by the stubborn type, call it α_s , and one offer which was made by both types, call it α_h . The offer made by the stubborn type only, α_s , cannot be above $1/2$, otherwise, there exists $\bar{z} > 0$ such

that for any $z < \bar{z}$, $v_s > 1/2$. But $v_r > v_s$, and $v_r \leq 1/2$. The hybrid offer α_h cannot be below $1/2$, otherwise there is an incentive for both types to deviate to $1 - \alpha_h$. Hence, $\alpha_s \leq 1/2 \leq \alpha_h$.

Suppose now $\alpha_s + \alpha_h > 1$. Then the payoff of a stubborn type from demanding α_h is:

$$v_s^{\alpha_h} = (1 - z + zs_h)(1 - z^{\frac{\alpha_h}{1-\alpha_h}})(1 - \alpha_h).$$

But a stubborn type could receive $1 - \alpha_h$ by demanding $1 - \alpha_h$. Hence, if a hybrid equilibrium exists, $\alpha_s \leq 1/2 \leq \alpha_h \leq 1 - \alpha_s$. The payoff of a rational player in such a candidate equilibrium is:

$$v_r^{\alpha_h} = (1 - z + zs_h)(1 - \alpha_h) + z(1 - s_h)((1 - \alpha_s)1/2 + \alpha_h 1/2).$$

Recall that $\alpha_2 \leq 1/2$. If $\alpha_s < 1 - \alpha_h$, then a rational player benefits from deviating to $1 - \alpha_h$:

$$v_r^{1-\alpha_h} = (1 - z + zs_h)(1 - \alpha_h) + z(1 - s_h)(1 - \alpha_s).$$

Hence, if a hybrid equilibrium exists, then $\alpha_h = 1 - \alpha_s = \alpha$. ■