Simultaneous First-Price Auctions with Preferences over Combinations: Identification, Estimation and Application*

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Abstract

Motivated by the empirical prevalence of simultaneous bidding across a wide range of auction markets, we develop and estimate a structural model of strategic interaction in simultaneous first-price auctions when objects are heterogeneous and bidders have preferences over combinations. In this model, bidders have stochastic private valuations for each object and stable incremental preferences over combinations, nesting the standard separable model as the special case when incremental preferences over combinations are zero. We establish non-parametric identification under standard exclusion restrictions, providing a basis for both testing on and estimation of preferences over combinations. We then apply our model to data on Michigan Department of Transportation highway procurement auctions, we quantify the magnitude of cost synergies and assess possible efficiency losses arising from simultaneous bidding in this market.

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1 Introduction

Simultaneous bidding in multiple first-price auctions is a commonly occurring but rarely discussed phenomenon in many real-world auction markets. In environments where values over combinations are non-additive in the set of objects won, bidders must account for possible combination wins at the time of bidding. This in turn substantially alters the strategic bidding problem compared to the standard first price auction with ambiguous welfare implications depending on the importance of synergies (either positive[1] or negative[2]) among objects. As a first step toward exploring this issue, we develop a structural model of bidding in simultaneous first-price auctions and study identification and estimation in this framework. We then apply our results to estimate cost synergies arising in Michigan Department of Transportation (MDOT) highway procurement auctions, using the resulting estimates to quantify efficiency losses arising from simultaneous bidding in this application.

To underscore the prevalence of simultaneous bidding in applications, note that most widely studied first-price marketplaces in fact exhibit simultaneous bids. Concrete examples include markets for highway procurement in most US states (Jofret-Bonet and Pesendorfer 2003, Krasnokutskaya 2009, Krasnokutskaya and Seim 2004, Somaini 2013, Li and Zheng 2009, Groeger 2014, many others), snow-clearing in Montreal (Flambard and Perrigne 2006), recycling services in Japan (Kawai 2010), cleaning services in Sweden (Lunander and Lundberg 2012), oil and drilling rights in the US Outer Continental Shelf (Hendricks and Porter 1984, Hendricks, Pinkse and Porter 2003), and to a lesser extent US Forest Service timber harvesting (Lu and Perrigne 2008, Li and Zheng 2012, Li and Zhang 2010, Athey, Levin and Siera 2011, many others). Furthermore, in many of these applications we expect bidders to have non-additive preferences over combinations; due, for instance, to capacity effects in highway procurement (Jofret-Bonet and Pesendorfer 2003), distance between contracts in snow clearing and waste collection, or information spillovers in Outer Continental Shelf drilling (Hendricks and Porter 1984). In such cases strategic simul-

[1] These commonly arise from cost savings in procurement auctions.

[2] These may arise when bidders have resource and capacity constraints or when winning alternative items offered can meet the same need.

taneity may substantially influence both bidder behavior and market performance.

To illustrate the policy questions arising in simultaneous multi-object auctions, note that given a set of L heterogeneous objects for sale, bidders could in general assign either positive or negative synergies to winning multiple objects. Bidder i's preference structure could thus in principle be as complex as a complete 2^L-dimensional set of signals describing the valuations i assigns to each of the 2^L possible subsets of objects. Meanwhile, the simultaneous first-price mechanism allows bidders to submit (at most) L individual bids on the L objects being sold. Consequently, the simultaneous first-price auction format is necessarily inefficient – the "message space" (standalone bids) is insufficiently rich to allow bidders to express their true preferences. Allowing combinatorial bids might help to alleviate this "message space" problem, but need not produce an efficient allocation (see e.g. Cantillon and Pesendorfer 2006, Crampton at al. 2006) and could impose substantial practical costs on both bidders and the seller (the "winner determination problem") . Hence in evaluating the relative merit of the simultaneous first-price format it is first necessary to assess the empirical magnitude of revenue and efficiency losses due to simultaneous bidding. Very little is presently known about these questions, due in part to a near-total lack of methods for analyzing preferences over combinations in simultaneous auctions.

Motivated by this gap in the literature, we develop an empirical model of bidding in simultaneous first-price auctions when objects are heterogeneous and bidders have non-additive preferences over combinations, to our knowledge the first such in the literature. This model turns on a novel decomposition of bidder preferences, which we outline briefly here. We represent the total value i assigns to a given combination as the sum of two components: the sum of the *standalone valuations* i assigns to winning each object in the combination individually, plus a combination-specific *complementarity* (either positive or negative) capturing the change in value i assigns to winning objects in combination.[3] We interpret standalone valuations as private information drawn independently across bidders conditional on observables, but require incremental preferences over combinations to be *stable* in the sense that

[3]Note that this decomposition is without loss of generality; the key identifying restriction is the structure we impose on complementarities.

complementarities are functions of observables.[4] We find this framework natural in a variety of procurement contexts – when, for instance, non-additivity in preferences can be represented as realizations of a utility shock realized after a multiple win. Furthermore – and crucially – our framework collapses immediately to the standard separable model when complementarities are zero, supporting formal testing of this hypothesis. We apply this model to data on Michigan Department of Transportation (MDOT) highway procurement auctions and evaluate the efficiency losses due to simultaneous bidding in this market. In so doing, we make three main contributions to the literature on structural analysis of auction markets.

First, we propose a structural model of simultaneous first-price auctions permitting identification of non-additive preferences over combinations. Identification in this framework rests on two key assumptions. First, as noted above, we assume that complementarities are stable functions of observables. Second, we assume that the marginal distributions of i's standalone valuations are invariant either to the characteristics of i's rivals or to the characteristics of other objects on which i is bidding. We show that optimal behavior in this environment yields an inverse bidding system non-parametrically identified up to the unknown function describing complementarities, collapsing to the standard inverse bidding function of Guerre, Perrigne and Vuong (2000) when complementarities are zero. Building on this inverse bidding system, we translate the exclusion restrictions outlined above into a system of identifying restrictions, with excludable variation in competition yielding non-parametric identification and excludable variation in characteristics of other objects yielding semiparametric identification of model primitives. These results provide a formal basis for structural analysis of simultaneous first-price auctions with non-additive preferences over combinations, to our knowledge the first such in the literature.

Second, building on our identification argument, we develop a three-step procedure yielding empirical estimates of primitives in our structural model. First, in Step 1, we estimate the multi-variate joint distribution of bids as a function of bidder-

[4] Note that this structure does not restrict dependence between i's standalone valuations for different objects in the market. We view this flexibility as critical, as in practice we expect i's standalone valuations to be positively correlated.

and auction-level characteristics. Due to the high-dimensional nature of this estimation problem, we follow several prior studies (e.g. Cantillon and Pesendorfer 2006 and Athey, Levin and Siera 2011) by employing a parametric approximation to the observed bid density in implementation of this step. Next, in Step 2, we parametrize preferences over combinations as a function of bidder- and combination-specific covariates[5] and estimate parameters in this function by minimization of a simulated analog to our semiparametric identification criterion which reduces to a quadratic minimization problem. Finally, in Step 3, we map estimates derived in Step 2 through the inverse bidding system derived in Step 1 to obtain estimates of the distribution of private costs rationalizing observed bidding behavior.

Finally, we apply the framework developed above to analyze simultaneous bidding in Michigan Department of Transportation (MDOT) highway procurement markets. We view this market as prototypical of our target application: large numbers of projects are auctioned simultaneously (an average of 45 per letting round in our 2005-2015 sample period), more than half of bidders bid on at least two projects simultaneously (with an average of 2.7 bids per round across all bidders in the sample), and combination and contingent bidding are explicitly forbidden. Within this marketplace, we show that factors such as size of other projects, number of bidders in other auctions, and the relative distance between projects have substantial reduced-form impacts on i's bid in auction l. This finding is expected when bidders have non-trivial preferences over combinations, but difficult to rationalize within either the standard separable Independent Private Values model or typical extensions of it (e.g. affiliated values, unobserved heterogeneity, and endogenous entry). We then apply the three-step estimation algorithm described above to recover structural estimates of primitives, with results suggesting that winning multiple auctions together leads to cost savings for moderately sized and / or homogeneous projects but cost increases for large and / or heterogeneous projects: roughly 18 percent cost savings for a two-auction combination at the 95th (best) percentile in our sample, transitioning

[5] In our application, combination-specific covariates might include the sum of engineer's estimates across projects in a combination, distance between projects in a combination, and indicators for whether projects in a combination are of the same type, among others.

to roughly 4 percent cost increases for a two-auction combination at the 5th (worst) percentile. Finally, we use our estimation results to analyze the implications of the simultaneous first-price design in the MDOT marketplace. Specifically, we assess the degree of inefficiency in MDOT highway procurement auctions, analyze the extent of the exposure problem,[6] and explore potential gains from switching to a widely used alternative mechanism: the combinatorial clock-proxy auction of Ausubel, Crampton and Milgrom (2006).

Related literature While we are not aware of a structural exploration of bidding in simultaneous first-price auctions, there is a growing empirical literature on *multi-unit auctions*. Many studies in this literature analyze markets for homogenous, divisible goods like electricity and treasury bills; see e.g. Fevrier, Preget, and Visser (2004); Chapman, McAdams, and Paarsch (2007); Kastl (2011); Hortacsu and Puller (2008); Hortacsu and McAdams (2010) and Hortacsu (2011); Wolak (2007); and Reguant (2014). More closely related to our paper are Cantillon and Pesendorfer (2006), Fox and Bajari (2013), and Kim, Oliveres and Weintraub (2014). Fox and Bajari (2013) estimate the deterministic component of bidder valuations in FCC simultaneous ascending spectrum auctions without package bidding. They exploit the assumption that the allocation of licenses is pairwise stable in matches and use the maximum score estimator for matching game to estimate the valuation function. Cantillon and Pesendorfer (2006) analyze combinatorial first-price sealed-bid auctions for London bus routes when bidders can submit package bids. While superficially similar to the first-price setting we study, allowing bidders to submit combinatorial bids substantially alters analysis of the bidding problem; intuitively, the "message space" (bids over combinations) is no longer sparse relative to the type space (preferences over combinations), leading to a fundamentally different and simpler identification problem. More recently, Kim, Oliveres and Weintraub (2014) extend the methodology of Cantillon and Pesendorfer (2006) to large-scale combinatorial auctions used in procurement of Chilean school meals.

[6]The exposure problem in auctions of multiple items involves the risk of bidders winning unwanted items, i.e. winning items at prices above bidders' values for them.

Paralleling these structural studies, there is a small reduced-form literature seeking to quantify the role of preferences over combinations in multi-object auctions: Ausubel, Cramton, McAfee and McMillan (1997) and Moreton and Spiller (1998) measure synergy effects in FCC spectrum auctions; Lunander and Lundberg (2012) show that firms inflate their standalone bids in combinatorial first-price auctions relative to first-price auctions but without significant differences in the procurer's cost of internal cleaning services in Sweden.

From a more theoretical perspective, there exist a few studies analyzing strategic interaction in stylized models involving simultaneous first-price auctions; see for example Szentes and Rosenthal (1996) and Ghosh (2012). Gentry, Komarova, Schiraldi and Shin (2015) study existence and proprieties of equilibrium in a setting closely paralleling that studied here. There is also a substantial literature analyzing properties of various combinatorial auction mechanisms: Ausbel and Milgrom (2002), Ausbel and Cramton (2004), Cramton (1998, 2002, 2006), Krishna and Rosenthal (1996), Klemperer (2008, 2010), Milgrom (2000a, 2000b), and Rosenthal and Wang (1996), to mention just a few. Detailed surveys of this literature are given in de Vreis and Vorha (2003), and Cramton et al. (2006).[7]

2 A model of simultaneous first-price auctions

This section introduces the model and highlights its key features which are then used to build our identification strategy.

A set of $\mathcal{N} = \{1, ..., N\}$ risk-neutral bidders compete for (subsets of) a set $\mathcal{L} = \{1, ..., L\}$ of objects allocated via separate but simultaneous first-price auctions. Each bidder i is endowed with latent preferences over combinations described by a $2^L \times 1$

[7]There is also a growing theoretical literature on simultaneous first-price auctions in computer science; see Feldman et al. 2012, and Syrgkanis 2012 among others. This literature focuses primarily on deriving bounds on the "Bayesian price of anarchy," or fractional efficiency loss, in simultaneous first-price auction markets. Positive results in this literature are largely restricted to settings with negative complementarities, and even in these settings bounds tend to be wide (e.g. Feldman et al. (2012) show that Bayesian Nash equilibrium captures at least half of total social surplus).

vector of *combinatorial valuations* Y_i. Combinatorial valuations Y_i for bidder i are drawn from a joint distribution $F_{Y,i}$ satisfying the following properties:

Assumption 1 (Independent Private Values)**.** *Each bidder i draws latent combinatorial valuations Y_i from an absolutely continuous c.d.f. $F_{Y,i}$ with support on a compact, convex set $\mathcal{Y}_i \subset R^{2^L}$, with $F_{Y,i}$ common knowledge, and value draws independent across bidders: $Y_i \perp Y_j$ for all i, j. $Y_i^0 = 0$.*

While the distribution $F_{Y,i}$ is common knowledge to all participants, realizations of the latent combinatorial valuation vector Y_i are *ex ante* unknown to i and must be discovered through costly entry. As our main focus here is analysis of bid-stage behavior, we abstract somewhat from entry decisions.[8] Informally, however, we interpret observed participation patterns as arising from a broader entry and bidding game along the lines of those considered by Li and Zheng (2014) and Groeger (2014). In this broader game, bidders first simultaneously choose subsets of auctions in which to enter, with entry decisions informed by private entry costs drawn in a first stage. Play then proceeds to the bidding stage, in which entrants learn their valuations for all combinations for which they are bidding, observe the entry decisions of potential rivals, and submit binding standalone bids in each auction they have entered. Finally, the auctioneer assigns allocations and payments according to usual first-price auction rules: every object $l \in \mathcal{L}$ receiving at least one bid is allocated to a high bidder in auction l, with bidders paying their bids for each object they receive.

We proceed to consider bidding behavior taking entry decisions as given. Let an *entry set* $\mathcal{E}_i \subset \mathcal{L}$ for bidder i denote the set of auctions in which bidder i has entered, and an *entry structure* $\mathcal{E} = (\mathcal{E}_1, ..., \mathcal{E}_N)$ describe entry sets for all bidders $i = \{1, ..., N\}$. We aim to analyze the bidding game which arises following realization of a particular entry structure $\mathcal{E}$, taking this entry structure as given. Toward this end, we introduce the following notation and definitions.

Actions, types, and strategies Let $\mathcal{B}_\ell \subset \mathbb{R}^+$ denote the set of feasible bids in auction $\ell = 1, ..., L$; without loss of generality, we take this to be a compact set.

[8] See Appendix D for a detailed description of this broader entry game.

Let L_i denote the number of auctions in which i participates under $\mathcal{E}$. A *bid* $b_i^{\mathcal{E}}$ for player i in subgame $\mathcal{E}$ is an $L_i \times 1$ vector such that $b_{i\ell}^{\mathcal{E}} \in \mathcal{B}_\ell$ for all $\ell \in \mathcal{E}_i$, while a *type* $Y_i^{\mathcal{E}}$ is a $2^{L_i} \times 1$ vector whose elements describe the combinatorial valuations assigned by i to each combination which he could win at $\mathcal{E}_i$ (i.e. to each subset of $\mathcal{L}$ contained in $\mathcal{E}_i$). Let $\mathcal{B}_i^{\mathcal{E}} = \times_{\ell \in \mathcal{E}_i} \mathcal{B}_\ell$ and $\mathcal{Y}_i^{\mathcal{E}} \subset \mathcal{Y}_i$ denote i's bid and type spaces in subgame $\mathcal{E}$, and let $F_{Y,i}^{\mathcal{E}}$ denote the distribution of $Y_i^{\mathcal{E}}$ on $\mathcal{Y}_i^{\mathcal{E}}$; note that $F_{Y,i}^{\mathcal{E}}$ is derived from i's *ex ante* type distribution $F_{Y,i}$ by marginalizing out elements corresponding to combinations which are not subsets of $\mathcal{E}_i$. In what follows, we omit superscripts $\mathcal{E}$ wherever feasible; unless otherwise noted, all objects defined below are defined relative to a given subgame $\mathcal{E}$.

Standalone valuations and complementarities Define an *outcome* for bidder i as an $1 \times L_i$ vector such that for each $l \in \{1, ..., L_i\}$ the element $\omega_{il} = 1$ if the lth element of $\mathcal{E}_i$ is allocated to i and $\omega_{il} = 0$ otherwise. Similarly, let the *outcome matrix* Ω_i for bidder i be the $2^{L_i} \times L_i$ matrix whose rows describe all outcomes possible for i: e.g. if $L_i = 2$, then

$$\Omega_i^T = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Let i's *standalone valuation* for object l, denoted V_{il}, be the valuation i assigns to the outcome "i wins object l alone": $V_{il} \equiv Y_i^{\{l\}}$. Similarly, let i's *standalone valuation vector*, denoted V_i, be the $L_i \times 1$ vector whose elements describe i's standalone valuations for the L_i objects in set $\mathcal{E}_i$. We can now define the *complementarity vector* K_i for bidder i as the difference between i's combinatorial valuation vector Y_i and the sum of i's standalone valuations for each combination in $\mathcal{E}_i$:

$$K_i \equiv Y_i - \Omega_i V_i.$$

Intuitively, $\Omega_i V_i$ represents the additive portion of bidder i's preferences over combinations, while K_i represents the non-additive portion. In particular, $K_i = 0$ if and only if i's preferences are additively separable in the set of objects won.

Marginal and combination win probabilities Let $P_i(b;\sigma_{-i})$ be the $2^L \times 1$ vector describing the probability distribution over combinations arising when i submits bid b_i facing rival strategies σ_{-i}, such that each element of $P_i(b_i;\sigma_{-i})$ describes the probability that i wins the combination associated with the corresponding element of Y_i. Let $\Gamma_i(b_i;\sigma_{-i})$ be the $L_i \times 1$ vector whose lth element $\Gamma_{il}(b_i;\sigma_{-i})$ is the marginal probability that bidder i submits bid vector $b_i \in \mathcal{B}_i$ against rivals who bid according to strategies σ_{-i}. Observe that $\Gamma_i(b_i;\sigma_{-i})$ is related to $P_(b_i;\sigma_{-i})$ by

$$\Gamma_i(b_i;\sigma_{-i}) = \Omega_i^T P_i(b_i;\sigma_{-i}).$$

Note that if ties occur with probability zero at b_{il}, then $\Gamma_{il}(b_i;\sigma_{-i})$ is simply the c.d.f. of the maximum rival bid in auction l, evaluated at b_{il}.

Bidder payoffs Now consider bidder i with type realization $y_i \in \mathcal{Y}_i^{\mathcal{E}_i}$ competing against rivals who bid according to strategy profile σ_{-i}. Let v_i and k_i be the standalone valuation and complementarity vectors corresponding to type realization y_i respectively. Recalling that a bidder winning combination S submits payment $\omega_i^S b_i$, we can then write bidder i's interim payoff function as follows:

$$\begin{aligned}\pi_i(b_i;y_i,\sigma_{-i}) &= (y_i - \Omega_i b_i)^T P_i(b_i;\sigma_{-i}) \\ &= (\Omega v_i - \Omega_i b_i)^T P_i(b_i;\sigma_{-i}) + k_i^T P_i(b_i;\sigma_{-i}) \\ &= (v_i - b_i)^T \Gamma_i(b_i;\sigma_{-i}) + k_i^T P_i(b_i;\sigma_{-i}), \end{aligned} \tag{1}$$

Note that if i's preferences over combinations are additive, then $k_i = 0$ and (1) reduces to the standard separable form

$$\pi_i(b_i;v_i,\sigma_{-i}) = \sum_{l=1}^{L} (v_{il} - b_{il})\Gamma_{il}(b_{il};\sigma_{-i}).$$

In this case, standard first-price theory applied auction by auction will characterize an equilibrium of the overall simultaneous first price auction game.

Bidding equilibrium We apply results in Jackson, Simon, Swinkels and Zame (2002) to the existence of an equilibrium in the "communication extension" of the simultaneous first-price auction game in which bidders send "cheap talk" signals use by the auctioneer to break ties. If ties occur with probability zero, this in turn corresponds to a full equilibrium in distributional strategies.

Let $\mathcal{B}^{\mathcal{E}} = \mathcal{B}_1^{\mathcal{E}} \times \cdots \times \mathcal{B}_N^{\mathcal{E}}$ denote the overall bid space generated by entry structure $\mathcal{E}$. Let the *outcome correspondence* $\Theta^{\mathcal{E}} : \mathcal{B}^{\mathcal{E}} \rightrightarrows \Delta^{N_1} \times \cdots \times \Delta^{N_L}$ be such that for every bid profile $b \in \mathcal{B}^{\mathcal{E}}$, $\Theta^{\mathcal{E}}(b)$ is the set of all probabalistic allocation rules such that every object l is allocated to a high bidder in auction l with probability one. Following Jackson, Simon, Swinkels and Zame (2002), we define the *communication extension* of the simultaneous first-price auction game as the game that results when the auctioneer allows each bidder i to submit *both* a bid $b_i \in \mathcal{B}_i^{\mathcal{E}}$ and a signal $t_i \in \mathcal{T}_i^{\mathcal{E}} \equiv \mathcal{Y}_i^{\mathcal{E}}$ indicating his private type, where signals $t_1, ..., t_N$ may be used to resolve ties but are otherwise irrelevant for allocations and payoffs. Then applying Theorem 1 in Jackson, Simon, Swinkels and Zame (2002), we obtain:

Proposition 1. *For every entry structure $\mathcal{E}$, there exists an equilibrium with endogenous tiebreaking in the communication extension of the simultaneous first-price auction game induced by $\mathcal{E}$: that is, a profile of distributional strategies $\sigma^{\mathcal{E}} = (\sigma_1^{\mathcal{E}}, ..., \sigma_N^{\mathcal{E}})$ and a tiebreaking rule $\theta^{\mathcal{E}} : \mathcal{T}^{\mathcal{E}} \times \mathcal{B}^{\mathcal{E}} \to \Delta^{N_1} \times \cdots \times \Delta^{N_L}$ selected from $\Theta^{\mathcal{E}}$ such that bidding according to $\sigma^{\mathcal{E}}$ and truthfully communicating types is a Bayesian Nash equilibrium of the communication extension under tiebreaking rule $\theta^{\mathcal{E}}$.*

At least two comments on Proposition 1 are worth mentioning here. First, as noted above, if $K_i = 0$ with probability one for all i, then auctions are separable and classical theory applies. In this case any strategy profile in which bidders play Bayesian Nash equilibria auction by auction will be a Bayesian Nash equilibrium of the overall bidding game, and this strategy profile paired with *any* tiebreaking rule will be an equilibrium with endogenous tiebreaking as defined in Proposition 1. In this sense Proposition 1 formally embeds the (very well understood) classical model within a much more general (but far more challenging) framework permitting arbitrary complementarities.

Second, one could alternatively interpret the bid space as *discrete*, in which case existence of pure strategy Bayesian Nash equilibria in every bidding subgame follows immediately from results in Milgrom and Weber (1985). In our view virtually all real-world markets are inherently discrete on some level, either explicitly due to minimum bid increments or implicitly due to minimum currency units. In this sense we see the continuous case as a primarily theoretical (rather than practical) concern. As is conventional in the literature, however, we here choose to stylize discrete bid spaces as approximately continuous.

3 Empirical framework: identification

Suppose the econometrician has access to a "typical" simultaneous first-price auction sample, interpreted as a sample of T auction rounds drawn from some stable underlying data generating process. In each round, the auctioneer offers L_t objects for auction to N_t bidders active in the marketplace (though as above not all bidders need be active in all auctions). Bidders then simultaneously submit sealed bids on the set of auctions in which they are active, with the set of bidders active in each auction common knowledge to a ll participants.[9] For each round t, we assume the econometrician observes data on all bidders i and auctions l present in the market. Asymptotic statements should be interpreted as applying when $T \to \infty$.

For each bidder i present at time t, let $\mathcal{E}_{it}$ be the set of auctions in which i bids at time t, with b_{it} the corresponding vector of i's bids and N_{it} the set of bidders faced by bidder i in $\mathcal{E}_{it}$. For each round t, the econometrician observes $\mathcal{E}_{it}$, b_{it}, and a vector of bidder-specific characteristics Z_{it} for all bidders i active in the round; to simplify notation, we will adopt the convention that Z_{it} includes $\mathcal{E}_{it}$. For future reference, let L_{it} denote the cardinality of $\mathcal{E}_{it}$, and define $Z_t^i \equiv (Z_{1t}, ..., Z_{N_{it},t})$.

Meanwhile, on the auction side, we partition the econometrician's information into two sets of covariates. First, for each object l auctioned at time t, the econo-

[9] While we do not model entry formally here, our analysis can be readily extended to incorporate endogenous participation along the lines of Levin and Smith (1994) and Athey, Levin and Siera (2011)). We outline this extension in more detail below.

metrician observes a vector of covariates X_{lt}; for future reference, we define $X_t^i \equiv (X_{1t}, ..., X_{L_{it},t})$. The econometrician may also observe a vector of W_t^i taken to affect combinatorial valuations but not standalone valuations for auctions in the set $\mathcal{E}_{it}$; we formalize this restriction in Assumption 5 below. In a highway procurement context, X_{lt} would include factors like project size, project location, and type of work in project l, whereas W_t^i could include distance between projects, interaction between project sizes, and other factors assumed irrelevant for V_{il} after conditioning on Z_t^i and X_{lt}. Notice that while the sets Z_t^i, X_t^i and W_t^i are bidder i specific, to simplify the notation we are going to drop the superscript i. As above, we do not explicitly model the entry stage but rather focus on bidding behavior taking entry realizations as given.[10]

3.1 Identifying assumptions

Even cursory analysis of the simultaneous first-price problem suggests a major empirical obstacle: whereas in general the model could involve up to $2^{L_{it}} - 1$ unknown combinatorial valuations for a given bidder, the data generating process yields only L_{it} observed bids corresponding to these unobservables. To obtain a viable empirical model for simultaneous first-price auctions, it is therefore imperative to specialize the model before taking it to data. Obviously, whether a given specialization is plausible will depend crucially on the problem at hand, and no one assumption is likely to be suitable for all applications. Since our main interest here is procurement, however, we here propose two restrictions on primitives which we find natural in many procurement contexts. As we go on to show, these turn out to be sufficient for non-parametric identification of primitives given data of the form above. We thereby provide a formal basis for empirical analysis of simultaneous bidding in a wide range of applications of practical and policy interest.

Assumption 2 (Stochastic V_i, stable K_i). *For all i and t, $K_{it} = \kappa_i(Z_t, W_t, X_t)$, with*

[10] While estimating the entry cost distribution is conceptually standard in this framework, it requires solving a combinatorial optimization problem which is computational very demanding and beyond the scope of the current paper.

V_{it} is distributed according to joint c.d.f. $F_i(\cdot|Z_t, W_t, X_t)$.

Assumption 2 says that complementarities are stable functions of bidder, auction, and combination specific observables. This assumption is motivated by our interpretation of K_i as a pure combination effect; i.e. an incremental cost or benefit derived from winning two objects together. We find this structure reasonable for applications such as procurement contracting, where bidders are obligated to perform all projects won.[11] Note that $\kappa_i(\cdot)$ can also be interpreted as an expectation over a combination-specific utility shock realized after a multiple win. Also, and importantly, Assumption 2 formally nests the hypothesis of additively separable preferences: $\kappa_i(Z_t, W_t, X_t) = 0$.

In addition to the restriction on primitives captured in Assumptions 2, we maintain two regularlity conditions on the underlying auction process:

Assumption 3. *In each letting t, observed bids are generated by play of an equilibrium with endogenous tiebreaking in the JSSZ communication extension to the simultaneous first-price auction game. Furthermore, for any t, t' such that $(Z_t, W_t, X_t) = (Z_{t'}, W_{t'}, X_{t'})$, the equilibrium played at t is the same as the equilibrium played at t'.*

Assumption 4. *For each $(Z_t, W_t, X_t) \in \mathcal{Z} \times \mathcal{W} \times \mathcal{X}$ and each bidder i active in letting t, the joint distribution of bids submitted by i is absolutely continuous.*

The assumption that a single equilibrium is played is widely invoked in the literature; see, e.g., Somaini (2014) and references therein. Assumption 4 is a regularity condition on the observed distribution of bids which we expect to hold in any equilibrium such that bidders do not bid atoms. Note that when $K_i = 0$ any combination of strategies which would represent an equilibrium auction-by-auction will also be an equilibrium in the bidding game, and under standard regularity conditions (e.g. Assumption 1) strategies in any such equilibrium will satisfy Assumption 4. Hence to test the null hypothesis of additively separable preferences ($K_i = 0$), one need not

[11] In Appendix A, we extend our identification results to the case where complementarities are affine functions of standalone valuation.

maintain Assumption 4 as a separate hypothesis.[12]

The model primitives to recover are the distribution of standalone valuations $F_i(\cdot|Z_{lt}, X_t, W_t)$ and the complementarity function $\kappa_i(Z_t, W_t, X_t)$ for each bidder i. Let $G_i(\cdot|Z_t, W_t, X_t)$ be the c.d.f. of the joint distribution of the $L_{it} \times 1$ bid vector b_i submitted by bidder i at observables (Z_t, W_t, X_t); note that under Assumption 3, $G_i(\cdot|Z_t, W_t, X_t)$ is identified directly from observables for all i and t. Consistent with Assumption 1, we permit arbitrary correlation between elements of V_i, but assume vectors V_i, V_j are drawn independently across bidders. Independence of V_i, V_j implies independence of b_i, b_j, so knowledge of $G_1, ..., G_{N_t}$ is sufficient to characterize the joint distribution of all bids submitted by all bidders at time t.

Inverse Bid Function Let $P_{-i}(\cdot|Z_t, W_t, X_t) : \mathcal{B}_i \rightarrow \Delta^{2^{L}_{it}}$ be the probability distribution over outcomes facing bidder i at observables (Z_t, W_t, X_t) taking rival strategies as given, and $\Gamma_{-i}(\cdot|Z_t, W_t, X_t) \equiv \Omega^T P_{-i}(\cdot|Z_t, W_t, X_t)$ be the $L_{it} \times 1$ vector of marginal win probabilities corresponding to $P_{-i}(\cdot|Z_t, W_t, X_t)$. Note that identification of $G_1, ..., G_{N_t}$ implies identification of P_{-i}, Γ_{-i} for all i and (Z_t, W_t, X_t). Given any realization v_i of V_i and any vector of complementarities K_i, we can therefore write the problem facing bidder i at observables (Z_t, W_t, X_t) in terms of directly identified objects as follows:

$$\max_{b \in \mathcal{B}_i} \{(v_i - b) \cdot \Gamma_{-i}(b|Z_t, W_t, X_t) + P_{-i}(b|W_t, Z_t, X_t)^T K_i\}.$$

Temporarily suppose that i's objective is differentiable at $b^* \in \text{int}(\mathcal{B}_i)$; we show below that under Assumption 4 this holds almost surely with respect to the measure on $\mathcal{B}_i$ induced by G_i. Then by hypothesis of equilibrium play, b^* must satisfy necessary

[12] As it is difficult to rule out ties *a priori* in the fully general simultaneous model, we also extend our partial identification results (see Appendix C) to accommodate potential mass points in equilibrium bids. We show that in this case primitives are partially identified, with Monte Carlo analysis suggesting that identified sets are tight in practice. While we do not believe ties are important in the application we consider — they are never observed in the data — this analysis helps to underscore robustness of our results.

first-order conditions for an interior optimum:

$$\nabla_b \Gamma_{-i}(b^*|Z_t, W_t, X_t)(v_i - b^*) = \Gamma_{-i}(b^*|Z_t, W_t, X_t) - \nabla_b P_{-i}(b^*|W_t, Z_t, X_t)^T K_i. \quad (2)$$

Let $\mathcal{K}_{it}$ denote the following $(2^{L_{it}} - L_{it} - 1)$-dimensional subspace of $\mathbb{R}^{2^{L_{it}}}$:

$$\mathcal{K}_{it} = \{k \in \mathbb{R}^{2^{L_{it}}} \, : \, k_1 = k_2 = \ldots = k_{L_{it}+1} = 0\}.$$

That is, $\mathcal{K}_{it}$ contains $2^{L_{it}}$-dimensional vectors whose first $L_{it} + 1$ components are equal to zero. These zero components correspond to the cases of bidder i winning at most one object ($\omega = (0, \ldots, 0)$ or $\omega'\omega = 1$).

Taking $K_i \in \mathcal{K}_{it}$ as given, the first order condition (2) generates for each $b^* \in \text{int}(\mathcal{B}_i)$ an $L_{it} \times 1$ system of equations in the $L_{it} \times 1$ vector of unknown standalone valuations v_i. We now establish that this system may be inverted for v_i at almost every b_i submitted by i. Recall that $\Gamma_{-i}(b|Z_t, W_t, X_t)$ is an $L_{it} \times 1$ vector whose lth element describes the probability that bid vector b wins auction l, which under Assumption 4 is simply the probability that the maximum rival bid in auction l is below b_l. Hence $\nabla_b \Gamma_{-i}(b|Z_t, W_t, X_t)$ will be a diagonal matrix with (l, l)th element given by the p.d.f. of the maximum rival bid in auction l. In equilibrium this p.d.f. must be positive at (almost) every $b^* \in \text{int}(\mathcal{B}_i)$, and again invoking Assumption 4 this will be (almost) every bid submitted. We therefore conclude:

Proposition 2 (Inverse Bidding Function)**.** *Let K be any vector in $\mathcal{K}_{it}$, (Z_t, W_t, X_t) be any realization in $\mathcal{Z} \times \mathcal{W} \times \mathcal{X}$, and maintain Assumptions 1-4. Then for almost every b_i drawn from $G_i(\cdot|Z_t, W_t, X_t)$, there exists a unique vector $\tilde{v} \in \mathbb{R}^{L_{it}}$ satisfying the first-order system (2) at b_i given $(K; Z_t, W_t, X_t)$. This $\tilde{v}$ can be expressed in terms of b_i via the inverse bidding function*

$$\tilde{v} = \xi_i(b_i|K; Z_t, W_t, X_t),$$

where $\xi_i(\cdot|\cdot;Z,W,X) : \mathcal{B}_{it} \times \mathcal{K}_{it} \to \mathbb{R}^{L_{it}}$ *is defined by*

$$\xi_i(b|K;Z_t,W_t,X_t) \equiv b + [\nabla_b \Gamma_{-i}(b|Z_t,W_t,X_t)]^{-1} \\ \times \left[\Gamma_{-i}(b|Z_t,W_t,X_t) - \nabla_b P_{-i}(b|Z_t,W_t,X_t)^T K\right], \qquad (3)$$

and the right-hand expression is identified up to K*.*

$\xi_i(b_i|K;Z_t,X_t,W_t)$ describes the unique vector of *candidate* standalone valuations at which b_i could be a best response under the hypothesis $K = \kappa_i(Z_t,W_t,X_t)$. If in fact $K = \kappa_i(Z_t,W_t,X_t)$, then the first-order system (2) describes the true equilibrium bidding relationship and hence we must have $v_i = \xi_i(b_i|K;Z_t,W_t)$ almost surely. Otherwise, $\xi_i(b_i|K;Z_t,X_t,W_t)$ represents the unique *candidate* for v_i at which b_i satisfies first order necessary conditions for a best response.[13] Note that at $K = 0$ $\xi_i(\cdot)$ reduces to the standard inverse bidding function of Guerre, Perrigne and Vuong (2000) defined auction-by-auction.

Now observe that Proposition 2 implies a unique identified candidate $\tilde{F}_i(\cdot|K;Z_t,W_t,X_t)$ for the unknown c.d.f. $F_i(\cdot|Z_t,W_t,X_t)$:

$$\tilde{F}_i(v|K;Z_{lt},W_t,X_t) = \int_{\mathbf{B}_i} 1[\xi_i(B_i|K;Z_t,W_t,X_t) \le v]\, G_i(dB_i|Z_t,W_t,X_t), \qquad (4)$$

where by construction

$$F_i(\cdot|Z_t,W_t,X_t) = \tilde{F}_i(v|\kappa(Z_t,W_t,X_t);Z_t,W_t,X_t). \qquad (5)$$

Identification of the model thus reduces to identification of κ_i, since given κ_i the distribution of standalone valuation is non-parametrically identified through (5). We therefore turn to consider restrictions yielding identification of κ_i, both non-parametrically through Z_{-i} and semi-parametrically through W.

[13] Obviously, imposing *sufficient* conditions for b_i to be a best response – by, for instance, requiring second-order conditions to hold at $\xi(b_i|K;Z_t,X_t,W_t)$ – can only improve identification.

Non-parametric identification of κ_i based on variation in Z_{-i} As a basis for non-parametric identification of κ_i, we consider the following assumption:

Assumption 5. *$F_i(\cdot|Z_t, W_t, X_t) = F_i(\cdot|Z_{it}, X_t)$ and $\kappa_i(Z_t, W_t, X_t) = \kappa_i(Z_{it}, W_t, X_t)$.*

Assumption 5 imposes two exclusion restrictions: own primitives F_i, κ_i are invariant to competition Z_{-it}, and standalone valuations V_i are invariant to combination characteristics W_t given Z_{it}, X_t. The former is widely invoked in the non-parametric auction literature (e.g. Haile, Hong and Shum (2003), Guerre, Perrigne and Vuong (2009), Somaini (2014)), while the latter formalizes the exclusion restriction underlying the definition of W_t. Note that although our emphasis here is on identification at the bidding stage, one can formally justify the first of these within a two-stage entry and bidding model like that we sketch above: i.e. in which auction-level entry decisions depend on idiosyncratic auction- or combination-specific entry costs, with realizations of V_i discovered after entry.[14]

To understand how variation in Z_{-it} identifies $\kappa_i(\cdot)$, consider a simple two-auction example. Holding (Z_{it}, W_t, X_t) fixed, define $\kappa_0 \equiv \kappa_i(Z_{it}, W_t, X_t)$ as above. Starting from some initial competition structure Z_{-it}, let Z'_{-it} be the competition structure derived from Z_{it} by adding, for example, one additional bidder to Auction 2. Then the *marginal* probability that i wins Auction 1 will be similar at Z_{-it} and Z'_{-it}, but the probability of the *combination* outcome "i wins both 1 and 2" will differ. Furthermore, under Assumption 5, this change in combination win probabilities is the *only* way changing Z_{-i} matters for i's strategy in Auction 1. Therefore to the extent that moving from competition structure Z_{-it} to competition structure Z'_{-i} matters for i's behavior in Auction 1, it can be only through κ_0; if moving from Z_{-it} to Z'_{-it} has no effect, then we must have $\kappa_0 = 0$. The number of feasible "experiments" is limited only by the support of Z_{-i}, with each experiment inducing a continuum of non-linear equations in the finite vector κ_0. Under weak regularity conditions this system will have the unique (overdetermined) solution $\kappa_0 = \kappa_i(Z_{it}, W_t, X_t)$. Iteration of the argument then yields identification of $\kappa_i(\cdot)$ for any (Z_{it}, W_t, X_t).

[14]See Appendix D for further details.

We now formalize this intuition, dropping the subscript t when feasible. By linearity of $\xi_i(B_i|K; Z, W, X)$ in K, we have for any (Z, W, X) and any $K \in \mathcal{K}_{it}$:

$$E_{B_i}[\xi_i(B_i|K; Z, W, X)|Z, W, X] = \Upsilon_i(Z, W, X) - \Psi_i(Z, W, X) \cdot K, \tag{6}$$

where $\Upsilon_i(Z, W, X)$ is an identified $L_{it} \times 1$ vector defined by

$$\Upsilon_i(Z, W, X) = \int_{\mathcal{B}_i} \left(B_i + \nabla_b \Gamma_{-i}(B_i|Z, W, X)^{-1} \Gamma_{-i}(B_i|Z, W, X)\right) G_i(dB_i|Z, W, X)$$

and $\Psi_i(Z, W, X)$ is an identified $L_{it} \times 2^{L_{it}}$ matrix defined by

$$\Psi_i(Z, W, X) = \int_{\mathcal{B}_i} \nabla_b \Gamma_{-i}(B_i|Z, W, X)^{-1} \nabla_b P_{-i}(B_i|Z, W, X)^T G_i(dB_i|Z, W, X).$$

Furthermore, by equation (5) and invariance of $F_i(\cdot|Z_i, X)$ in Z_{-i} we must have for any Z_{-i}, Z'_{-i}:

$$E_{B_i}[\xi(B_i|\kappa_0; Z_i, Z_{-i}, W, X)|Z, W, X] = E_{B_i}[\xi(B_i|\kappa_0; Z_i, Z'_{-i}, W, X)|Z', W, X]. \tag{7}$$

Substituting (6) into (7), we thereby obtain an $L_i \times 1$ system of linear restrictions in the $2^{L_i} \times 1$ vector $\kappa_0 = \kappa(Z_i, W, X)$:

$$(\Upsilon_i(Z, W, X) - \Upsilon_i(Z', W, X)) - (\Psi_i(Z, W, X) - \Psi_i(Z', W, X)) \cdot \kappa_0 = 0. \tag{8}$$

For a single Z_{-i}, Z'_{-i} pair, this system will typically be rank-deficient and thus will not uniquely determine κ_0. But the underlying equality restriction must hold for *every* $Z_{-i}, Z'_{-i} \in \mathcal{Z}_{-i}$. Pooling these restrictions, we therefore conclude:

Proposition 3. *For any $(Z_i, W, X) \in \mathcal{Z}_i \times \mathcal{W} \times \mathcal{X}$, suppose there exist vectors $Z_{-i,0}, Z_{-i,1}, ..., Z_{-i,J}$ in the support of $Z_{-i}|Z_i, W, X$ such that the $JL_i \times 2^{L_i}$ matrix*

$$\mathbf{M}_\Psi \equiv \begin{bmatrix} \Psi_i(Z_i, Z_{-i,1}, W, X) - \Psi_i(Z_i, Z_{-i,0}, W, X) \\ \vdots \\ \Psi_i(Z_i, Z_{-i,J}, W, X) - \Psi_i(Z_i, Z_{-i,0}, W, X) \end{bmatrix}$$

has full column rank when projected onto $\mathcal{K}_{it}$. Then $\kappa_i(Z_i, W, X)$ is identified.

Recall that the expectations criterion (7) exploits only equality of first moments of $F_i(\cdot|Z_i, X_l)$ across Z_{-i}, whereas the underlying invariance restriction implied by Assumption 5 requires equality (and, under Assumption 1, finiteness) of all moments. The system of equations in Proposition 3 merely provides a simple and directly verifiable sufficient condition guaranteeing that the underlying system of functional identities has a unique solution. Note further that variation in, e.g., number of rivals in other auctions will produce exactly the kind of variation in Ψ_i needed for full column rank of $\mathbf{M}_\Psi$: intuitively, changes in combination win probabilities relevant for cross-auction bidding only through κ_0. We thus view full column rank of $\mathbf{M}_\Psi$ as a weak regularity condition guaranteeing identification of model primitives.

Semi-Parametric identification of κ_i based on variation in W In the previous paragraph, non-parametric identification is achieved under the restriction that primitives are invariant to rival characteristics. This assumption may be violated in cases where richer strategic interaction among players leads to stand alone valuation to be function of rival characteristics; if, for instance, sub-contractors are bidders in procurement auctions. In such environments, identification of complementarities can still be achieved without necessarily relying on variation in Z_{-i} once we add the following assumption:

Assumption 6. *$\kappa_i(Z_{it}, W_t, X_t) = \tilde{\mathbf{C}}_i(Z_{it}, W_t, X_t, \theta_{0i})$, where $\tilde{\mathbf{C}}_i(Z_{it}, W_t, X_t, \theta_{0i})$ is a known transformation of $(Z_{it}, W_t, X_t, \theta_{0i})$, and $\theta_{0i} \in \Theta_i \subset \mathbb{R}^{p_i}$.*

We can then can replace Assumption 5 with the weaker assumption:

Assumption 7. *$F_i(\cdot|Z_t, W_t, X_t) = F_i(\cdot|Z_t, X_t)$.*

Notice that Assumption 6 is quite natural in applications, as we will typically wish to impose some parametric structure on $\kappa_i(Z_i, W, X)$. For simplicity, we consider the case when $\kappa_i(Z_i, W, X)$ is linear in parameters – that is, $\kappa_i(Z_i, W, X) = \mathbf{C}_i(Z_i, W, X)\theta_{0i}$. Equation (7) reduces to the linear-in-parameters form

$$(\Upsilon_i - \Upsilon_i') - (\Psi_i \mathbf{C}_i - \Psi_i' \mathbf{C}_i') \cdot \theta_{0i} = 0,$$

where $\Upsilon_i, \Psi_i, \mathbf{C}_i$ are identified functions of (Z, W, X) and $\Upsilon'_i, \Psi'_i, \mathbf{C}'_i$ are identified functions of (Z, W', X). Thus Given an appropriate collection of J "experiments" in which we vary W, $\{(Z_j, W_j, X_j), (Z_j, W'_j, X_j)\}_{j=1}^J$, we can thus express θ_{0i} as the solution to the following $\mathcal{L}^2$-minimization problem

$$\min_{\theta \in \Theta_i} \sum_{j=1}^{J} \left(\Upsilon_{ij} - \Upsilon'_{ij} - (\Psi_{ij}\mathbf{C}_{ij} - \Psi'_{ij}\mathbf{C}'_{ij}) \cdot \theta \right)^T \left(\Upsilon_{ij} - \Upsilon'_{ij} - (\Psi_{ij}\mathbf{C}_{ij} - \Psi'_{ij}\mathbf{C}'_{ij}) \cdot \theta \right), \tag{9}$$

with identification of θ_{0i} implied by a standard rank condition on the differences $(\Psi_{ij}\mathbf{C}_{ij} - \Psi'_{ij}\mathbf{C}'_{ij})$ across j. Note that given $\{\Upsilon_{ij}, \Upsilon'_{ij}, \Psi_{ij}, \Psi'_{ij}\}_{j=1}^J$, the problem of finding θ_{0i} reduces to intercept-free least squares of differences $(\Upsilon_{ij} - \Upsilon'_{ij})$ on differences $(\Psi_{ij}\mathbf{C}_{ij} - \Psi'_{ij}\mathbf{C}'_{ij})$ across j.

Notice that if we deem appropriate for the application studied, we can combine both set of identifying restrictions: the ones induced by the variation in Z_{-i} and those induced by the variation in W.

4 Application: Michigan Highway Procurement

We now turn to consider the marketplace for Michigan Department of Transportation (MDOT) highway construction and maintenance contracts. As common in similar procurement contexts, MDOT allocates contracts for a wide range of highway construction and maintenance services via low-price sealed-bid auctions. More than half (56 percent) of bidders submit bids on multiple contracts within any given "letting date". Bids are submitted to MDOT auction by auction, with combination and contingent bidding explicitly forbidden by MDOT auction rules. Bidders may amend bids up to the letting date, but once announced letting results are legally binding, with winning bidders held liable for failure to complete contracts won (though they may subcontract up to 60 percent of contract work). The MDOT marketplace thus closely parallels our simultaneous first-price structure, with factors like capacity constraints and / or economies of scale and scope due, for example, to project location, and project type inducing potential non-additivity in project payoffs.

4.1 Data and descriptive statistics

MDOT provides detailed records on contracts auctioned, bids received, and letting outcomes on its letting website (http://www.michigan.gov/mdot). Building on these records, we observe data on (almost) all contracts auctioned by MDOT over the sample period January 2005 to March 2014.[15] Our sample includes a total of 8224 auctions, where for each auction the following information is observed: project description, project location, pre-qualification requirements, the internal MDOT engineer's estimate of the total cost of the project, and the list of participating firms and their bids. Based on information in project descriptions, we apply a naive Bayes algorithm to classify projects into five project types: bridge work, major construction, paving (primarily hot-mix asphalt), safety (e.g. signing and signals), and miscellaneous, leading to a final distribution of projects across types summarized in Table 1. As evident from Table 1, roughly 80 percent of contracts are for road and bridge construction and maintenance broadly defined, with the remainder split between safety and other miscellaneous construction.

The data contains information on a total of 859 unique bidders active in the MDOT marketplace over our sample period, which we subclassify by size and scope of activity as follows. We define "regular" bidders to be those who have submitted more than 100 bids in the sample period. This yields a total of 36 regular bidders in the sample, with all remaining bidders classified as "fringe". For the subsample of bidders who have submitted more than 50 bids, we also collect data on number and location of plants by firm. This data is derived from a variety of sources: OneSource North America Business Browser, Dun and Bradstreet, Hoover's, Yellowpages.com and firms' websites. Based on this information, we further classify bidders as "large" or "small" by number of plants in Michigan, with "large" regular bidders defined as those with at least 5 plants. We thus obtain a final classification of 8 large regular bidders, 28 small regular bidders, and 823 fringe bidders (of which 4 large bidders) in the MDOT marketplace.

[15]For a small number of contracts MDOT records are incomplete. We have originally collected data from October 2002. In the estimation, we have discarded the first few years (from October 2002 to December 2004) so to construct the backlog variable.

Table 1: Summary of Projects by Type

Contract Type	Frequency
Bridge	13.33
Major Construction	9.64
Paving	56.33
Safety	12.25
Miscellaneous	8.45

Summary statistics Tables 2 and 3 summarize several key measures of market structure and bidder behavior. Table 2 surveys the auction side of the marketplace. The first key feature emerging from this table is, not surprisingly, the large number of contracts auctioned simultaneously in the market: a mean of 45 per letting date, with a maximum of 133 on a single letting date (note that smaller "supplements" lettings are occasionally held two or three weeks after the main letting in a given month). On average about five bids are received per contract, which is small relative to the average number of bidders (approximately 84) active in any period. For each contract, MDOT prepares an internal "Engineer's Estimate" of expected procurement cost released to bidders before bidding; as evident from the dispersion in this measure, projects in the marketplace vary substantially in size and complexity. The statistic "Money Left on the Table" measures the percent difference between lowest and second-lowest bids; on average this is 7.4 percent or roughly \$112,000 per contract, suggesting the presence of substantial uncertainty in the marketplace.

Table 3 re-frames the auction-level participation variables in Table 2 to provide a clearer picture of bidder behavior in the MDOT marketplace. Again, the key pattern emerging from Table 3 is the prevalence of simultaneous bidding in MDOT auctions, with the average bidder competing in roughly 2.7 auctions per round and large and regular bidders competing in substantially more. The variable "backlog" provides a bidder-specific measure of capacity utilization. As usual in the literature, we define backlog for bidder i at date t as the sum of work remaining among projects l won by i up to t, where work remaining on project l at date t is defined as total project size (measured by the engineer's estimate) times the proportion of scheduled project

Table 2: Auction Level Summary Statistics

	Mean	St. Dev.	Min	Max
Auctions per Round	45.19	35.67	1	133
Total Bids per Round	228.1	180.9	1	669
Distinct Bidders per Round	83.97	57.06	1	207
Number of Bidders per Auction	5.048	3.186	1	28
Large Regular Bidders per Auction	0.398	0.672	0	3
Regular Bidders per Auction	1.500	1.362	0	7
Fringe Bidders per Auction	3.149	2.926	0	23
Engineer's Estimate (in thousands)	1,514	4,689	4.412	165,313
Project Duration (in days)	175.8	205.1	2	1,838
Money Left on the Table	0.0744	0.0966	0	3.016

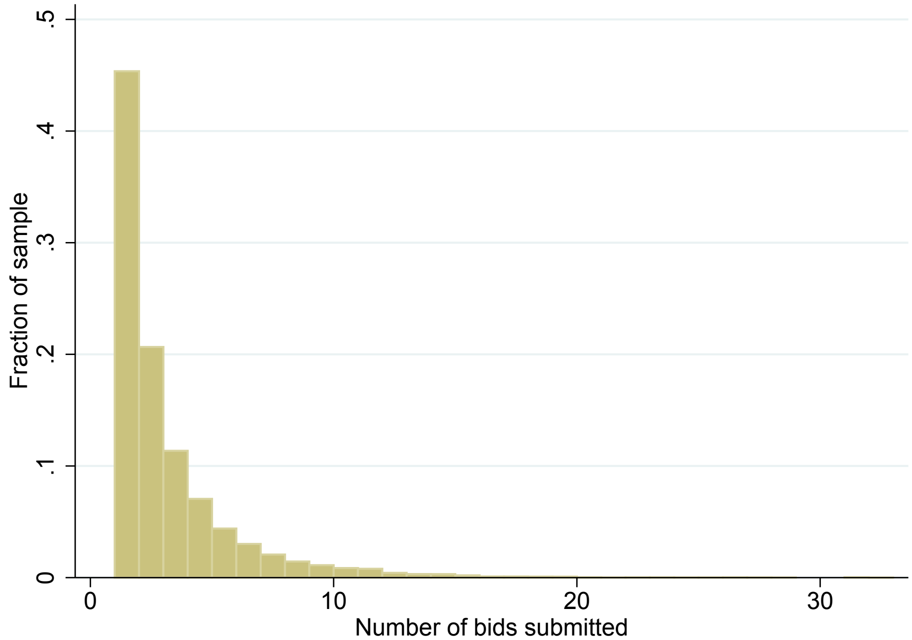
Table 3: Bidder Level Summary Statistics

	Mean	St. Dev.	Min	Max
Bids by Round	2.716	2.785	1	33
Bids by Round if Large	6.65	6.27	1.000	33.000
Bids by Round if Regular	5.96	4.58	1.00	33.00
Backlog (in millions)	5.792	19.01	0	275.5

days remaining at date t. Note that number of bids submitted in any given auction is small relative to the number of bidders in the marketplace, with even large bidders competing in less than fifteen percent of total auctions on average.

As a graphical perspective on the scope of simultaneous bidding in the MDOT marketplace, Figure 1 plots the distribution of the number of bids by round submitted by all bidders in the sample. As evident from Figure 1, more than 55 percent of bidders in our sample submit multiple bids in the same round. Despite this, it is relatively uncommon for a typical bidder to compete in a large number of auctions; almost 90 percent of bidders in our sample bid in 6 or fewer auctions and only 2 percent bid in more than 10. Not surprisingly, the outliers are almost exclusively large regular bidders.

Figure 1: Distribution of Simultaneous Bids Submitted



4.2 Reduced-form regressions

To document the potential implications of simultaneous bidding on bidder behavior and auction outcomes, we next explore a series of reduced-form regressions. The unit of analysis in these regressions is a bidder-auction-round combination, with the dependent variable log of bid submitted by bidder i in auction l in letting t. We regress log bids on a vector of regressors intended to capture effects of own-auction and cross-auction characteristics on i's bid in auction l at time t.

Regression specification As usual, we control for a number of auction-level characteristics which we expect to be key direct determinants of i's bid in auction l: the size of auction l, proxied by the MDOT engineer's estimate of expected project cost, the level of competition i faces in auction l, and the distance between project l and

i's base of operations.[16] To explore potential cross-auction interaction in the MDOT marketplace, we seek a set of covariates relevant for bidding in auction l only through κ: i.e. factors shifting *combination* payoffs but irrelevant for standalone valuations after conditioning on characteristics of auction l.

To control for cross-auction competition which may shift combination win probabilities, we consider the number of rivals across all auctions played by bidder i. The effects of cross-auction competition on i's bids in auction l are theoretically ambiguous, depending both on the sign of κ and on strategic responses by rival bidders. A priori, however, if objects are substitutes, we expect greater competition in auction k to increase marginal returns to winning auction l.

To capture the presence of potential budget constraints or dis-economies of scale, we consider the (log of) the engineer's estimate across all auctions in which i is competing and we also consider the amount of overlapping time there is among projects from the starting day to the completion day of each product assuming a uniform workload over time. Insofar as marginal costs are increasing in capacity utilization, we expect the coefficients on these variables to be positive.

In principle, complementarities arising between similar projects may differ from those arising between different projects. To account for this possibility, we consider a concentration index for projects of the same type in which bidder i participates. A negative sign is interpreted as a relative complementarity between similar projects.

Finally, as an additional proxy for potential economy of scale/substitutabilities among projects, we compute the (log of) distance between the current project and the other projects in which i bids normalized by the total distance between each of these projects and the closest plant owned by bidder i. The further away projects are from each other, the less economies of scale there are (keeping everything else constant). We expect this variable to have a positive sign.

Regression results Table 4 reports OLS estimates for our baseline regression specifications: logs bids by bidder, round, and auction on the own- and cross-auction

[16] We construct for each bidder-project pair the minimum straight-line distance (in miles) between any of i's plants and the centroid of the county in which project l is located.

characteristics defined above. All regression specifications include a full set of bidder type, project type, and letting date indicators, with standard errors clustered at the bidder-round level to allow for correlation within bidder i's bids.

Estimated effects of own-auction characteristics correspond closely both to our priors and to findings elsewhere in the literature. As expected, bids are increasing almost one for one in project size, with the coefficient on log engineer's estimate exceeding 0.97 in all specifications. Similarly, the negative coefficient on number of rivals suggests that competition increases bidder aggressiveness, with one additional competitor associated with a 5 percent decrease in average bids. Finally, the coefficient on log distance to project suggests that a one percent increase in i's distance from the project leads to about a 2 percent increase in i's bid on average.

More importantly, estimated cross-auction effects are also highly significant, with magnitudes stable across specifications and signs broadly consistent with our prior expectations. In particular, the positive coefficient on log of engineer's estimates across auctions suggests that competing in larger auctions leads to a substantial decrease in aggressiveness by bidder i in auction l, with the negative coefficient on same-type projects suggesting that this effect is ameliorated slightly when the two projects are of the same type. Similarly, the coefficient on total number of rivals in auctions partecipated by i suggests that facing more competition across auction leads bidder i to bid more aggressively in auction l. Finally, the positive sign on log distance among projects indicates that increasing distance to other projects reduces the synergies among them, which corroborates the hypothesis that simultaneous bidding induces strategic spillovers.

4.3 Structural estimation

Building on the identification results, we now turn to consider structural estimation of the complementarity vector $\kappa(\cdot)$. In principle, the results in Section 3.1 support fully non-parametric estimation. In practice, of course, the dimensionality of the problem renders this infeasible. We therefore implement estimation of $\kappa(\cdot)$ in two steps. First, following Athey, Levin and Siera (2011) and Cantillon and Pesendorfer

Table 4: OLS Estimates of Cross-Auction Effects

$y = ln(bid)$	1	2
Log engineer's estimate	0.971***	0.9764***
	(0.0011)	(0.0011)
Log number of rivals	-0.0502***	-0.0402***
	(0.0032)	(0.003)
Log distance to project	0.0213**	0.0136***
	(0.0011)	(0.001)
Log days to the project start	0.0039***	0.0039***
	(0.0009)	(0.0009)
Standardize backlog	0.0029**	0.0033***
	(0.001)	(0.0011)
Big bidder	-	0.0026
	-	(0.0044)
Log number of big rivals faced	0.0048***	0.0049
	(0.0024)	(0.0022)
Regular Bidder	-	-0.0042
	-	(0.0026)
Log number of regular rivals faced	0.0257***	0.0304***
	(0.0031)	(0.0028)
Multiple bids dummy	-0.1011***	-0.185***
	(0.0228)	(0.022)
Log sum engineer's estimate across played auctions	0.0064***	0.012***
	(0.0016)	(0.0015)
Log sum number of rivals across played auctions	-0.0152***	-0.0114***
	(0.0025)	(0.0023)
Log distance across played projects	0.0029***	0.0041***
	(0.0013)	(0.0013)
Fraction overlapping time across projects	0.0172***	0.0133***
	(0.0037)	(0.004)
Same-type-auctions concentration index	-0.0107***	-0.0273***
	(0.0051)	(0.0053)
Constant	0.5643***	0.4329***
	(0.0187)	(0.0188)
Year FE, Month FE, Auction type FE	YES	YES
Bidder type FE	NO	YES
Bidder ID FE	YES	NO
R-squared	98.16	97.93

Unit of analysis is bidder-auction-round, with standard errors clustered by bidder within each round. There are 40624 observations. Variables *log of engineer's estimate*, *log of number of rivals in the auction* and *log of distance to the county centroid* measure size, strength of competition, and distance to project l respectively. Remaining variables proxy for cross-auction characteristics: total number of rivals, sum engineer's estimate, distance to auctions scaled by distance to project l in which i is competing and number of overlapping days among projects scaled by the total number of days to completion.

(2006) among others, we estimate a parametric approximation to the equilibrium distribution G_i of bids submitted by each bidder i appearing in bidder i's problem. Second, we translate these estimates through the first-order condition (2) to obtain a minimum-distance criterion paralleling Equation (7).

Specification for G_i Building on our reduced-form analysis, we model i's bid in auction l as depending on the following observables: i's type, characteristics X_{ilt} influencing i's standalone valuation for contract l, characteristics W_{ilt} relevant for i's preferences over combinations involving auction l, competition in auction l, and competition in other auctions in which i bids. In particular, for bidder i facing market structure (Z_t, W_t, X_t), we estimate the following first-step model:

$$\ln(b_{it}) \sim \text{MVN}(\cdot|\mu(Z_t, W_t, X_t), \Sigma(Z_t, W_t, X_t)).$$

As typical in applications, we take $\mu(\cdot)$ to be a linear function of observables:

$$\mu_{ilt} = \beta D^{\mu}_{ilt},$$

where D^{μ}_{ilt} is a subset of (Z_t, W_t, X_t) which includes the following elements: log engineer's estimate for project l, log number of rivals in auction l, log sum of engineer's estimates across other projects in $\mathcal{E}_{it}$, log number of rivals in $\mathcal{E}_{it}$, an indicator for submitting multiple bids, and a constant term.

We specify the variance terms $\sigma^2_l(Z_t, W_t, X_t)$ of $\Sigma(Z_t, W_t, X_t)$ as

$$\sigma^2_l(Z_t, W_t, X_t) = \exp(\alpha D^{\sigma}_{ilt}),$$

and the correlation terms $\rho_{kl}(Z_t, W_t, X_t)$ as

$$\rho_{kl}(Z_t, W_t, X_t) = \frac{\exp(\gamma D^{\rho_{kl}}_{it} - 1)}{\exp(\gamma D^{\rho_{kl}}_{it} + 1)}$$

where D^{σ}_{ilt} and $D^{\rho_{kl}}_{it}$ are known transformations of (Z_t, W_t, X_t). In our baseline specification, D^{σ}_{ilt} includes log engineer's estimate in auction l, log number of rivals in auction l, log sum of engineer's estimate across auctions in $\mathcal{E}_{it}$, and log number of rivals across auctions in $\mathcal{E}_{it}$. Meanwhile, $D_{it}^{\rho_{kl}}$ includes the product of log engineer's estimates for pair kl, the product of number of rivals in pair kl, and indicators for projects of the same type and projects in the same county.

Specification for κ We adopt the following simple linear specification for $\kappa(\cdot)$:

$$\kappa^{\omega}(Z, W, X) = \theta_0^1 + \theta_0^2 \cdot Z_i + \theta_0^3 \cdot W, \tag{10}$$

Estimation algorithm Having specified κ to be linear in parameters θ_0, we implement estimation of θ_0 based on $\mathcal{L}^2$-type criterion (9) derived in Section 3.1. If all Γ_j, Γ'_j, Ψ_j, Ψ'_j, $\mathbf{C}_j$, $\mathbf{C}'_j$ in the criterion (9) were known or their consistent estimators were available, we could immediately obtain an estimate $\hat{\theta}$ for θ_0 by least squares based on (9). In implementing this procedure, however, we must first resolve several practical issues. We address these as follows.

First consider the set of counterfactuals $\{(Z_{i,j}, Z_{-i,j}, W_j, X_j), (Z_{i,j}, Z'_{-i,j}, W'_j, X_j)\}_{j=1}^{J}$ at which to evaluate the criterion (9). In principle, any choice of such that "regressors" $\Psi_j \mathbf{C}_j - \Psi'_j \mathbf{C}'_j$ in (9) satisfy a standard rank condition will be sufficient to construct an estimator for θ_0. In practice, however, we expect selections approximating the empirical distribution of (Z, W, X) to improve estimation performance. For the set of baseline points $\{(Z_j, W_j, X_j)\}_{j=1}^{J}$ in the criterion (9), we fix bidder i and construct a counterfactual pair (Z'_j, W'_j, X'_j) as follows. First, we randomly draw one auction, $q_i \in \mathcal{E}_i$. For this auction, we then draw counterfactual realizations of the number of rivals, type, distance among project, time overlap and log engineer's estimate from their empirical distributions among projects of the same type as Auction q_i, holding all other characteristics fixed. We therefore consider M_j counterfactual points $\left(Z_j^{(m_j)}, W_j^{(m_j)}, X_j^{(m_j)}\right)$, $m_j = 1, \ldots, M_j$, that have the properties described above and repeat for all bidders i. The simplest case is when $M_j = 1$, $j = 1, \ldots, J$. Specifically shifting from (Z_j, W_j, X_j) to $\left(Z_j^{(m_j)}, W_j^{(m_j)}, X_j^{(m_j)}\right)$, we make sure that $X_{l_i,j} = X_{l_i,j}^{m_j}$ for all $l_i \neq q_i \in \mathcal{E}_i$ but $X_{q_i,j}$ and $X_{q_i,j}^{m_j}$ can be different. This guarantees that we move W_{lt} for all auctions $l_i \neq q_i$ without changing X_{lt} (as described in

Section 3.1).

We then simulate (Υ_j, Ψ_j) and $(\Upsilon_j^{(m_j)}, \Psi_j^{(m_j)})$ for each $m_j = 1, \ldots, J$. For a given realization (Z, W, X), we accomplish this in three steps. First, we draw a size-R random sample of bid vectors $\{b_i^r\}_{r=1}^R$ from the joint distribution $\hat{G}_i(\cdot|Z, W, X)$ implied by our first-step estimates for G_i.[17] Next, for each realization b_i^r of B_i, we compute corresponding realizations for $\Gamma_i(b_i^r|Z, W, X)$, $\nabla\Gamma_i(b_i^r|Z, W, X)$, and $\nabla P_i(b_i^r|Z, W, X)$ based on our first-step estimates for $\hat{G}_1(\cdot|Z, W, X)$, ..., $\hat{G}_{N_t}(\cdot|Z, W, X)$ (approximating gradients with finite differences). Finally, we approximate Ψ and Υ by averaging appropriate products of these functions across draws $\{b_i^r\}_{r=1}^R$:

$$\hat{\Upsilon}_i = \frac{1}{R}\sum_{r=1}^{R} b_i^r + \nabla\Gamma_i(b_i^r|Z, W, X)^{-1}\,\Gamma_i(b_i^r|Z, W, X);$$

$$\hat{\Psi}_i = \frac{1}{R}\sum_{r=1}^{R} \nabla\Gamma_i(b_i^r|Z, W, X)^{-1}\nabla P_i(b_i^r|Z, W, X)^T.$$

We ultimately obtain the simulated $\mathcal{L}^2$-type criterion

$$\sum_i \sum_{l_i \neq q_i} \sum_{j=1}^{J} \sum_{m_j=1}^{M_j} \left(\hat{\Upsilon}_{i,l_i,j} - \hat{\Upsilon}_{i,l_i,j}^{(m_j)} - \left(\hat{\Psi}_{i,l_i,j}\mathbf{C}_{ij} - \hat{\Psi}_{i,l_i,j}^{(m_j)}\mathbf{C}_{ij}^{(m_j)}\right)\cdot\theta\right)^2.$$

While Section 3 emphasizes first moments as sufficient for identification, in the actual estimation, we also match the empirical quantiles of the empirical distributions of stand-alone valuations. In principle and in practice, matching also on quantiles conveys further information about the shape of the distribution, thereby improving precision of the estimates. This in turns results in solving a fixed point algorithm as the empirical quantiles are function of the parameter estimates. We solve this by iteration. We start from the parameters obtained using only the criterion above, and we then turn to a richer criterion where we seek to minimize the difference in the empirical quantiles.[18] The empirical quantiles are based on the initial estimates, which in turns yields a new set of estimates and new quantiles. The procedure is

[17] In practice we set R to 500, with larger samples having very little effect on results.

[18] In the estimation we match the 25^{th}, 50^{th} and 75^{th} percentiles

then repeated until convergence. In each iteration, the criterion to minimize reduces to a linear least-squares estimator (OLS), which we implement via the robust regression method to deal with a skewed distribution and outliers as well as non-constant variance in the errors. Standard error are boot-strapped.

4.4 Estimation results

This subsection reports results from applying the structural estimation procedure in Section 4.3 to the sample of bidders competing in two MDOT auctions simultaneously. We first report results from our first-step estimation of bid distributions G_i for all bidders, then discuss estimates of $\kappa(\cdot)$ for the two-bidder sample derived from these through the algorithm outlined above.

Estimates of G_i Table 5 reports results from first-step maximum likelihood estimation of i's bid distribution G_i based on the log-normal approximation described in Section 4.3.

The first panel of Table 5 reports estimates $\hat{\beta}$ for parameters β affecting mean parameters $\mu(\cdot)$. Not surprisingly, these are qualitatively similar to those in our reduced-form specifications 4. In the next two panels of Table 5, we present estimates for parameters in the variance-covariance matrix $\Sigma(Z, W, X)$. Variance parameters (Panel 2) suggest that bidders facing more competition and competing in larger auctions submit less dispersed bids; while we have no strong priors on these effects, the direction seems natural. More interestingly, covariance parameters suggest several broad patterns in bidding across auctions. First, not surprisingly, bidder i bids relatively more similarly in similar auctions: i.e. in the same county and of the same type. Second, competing in larger projects tends to decrease correlation in i's bids. In other words, bidders competing in two large projects tend to compete in one relatively more aggressively than the other. We interpret this as consistent with the presence of increasing costs to multiple wins. Finally, stronger competition within the pair tends to decrease correlation in bids. Since more competition obliges bidder i to compete more aggressively, and thereby decrease mark-ups conditional on winning, we again interpret this as consistent with substitution between projects.

Estimates of κ Building on the first-step estimates in Table 5, we now apply the two-step algorithm outlined in Section 4.3 to obtain estimates of the structural parameters θ_0 appearing in $\kappa(\cdot)$. For the moment in constructing our $\mathcal{L}^2$-type criterion (9), we focus on bidders competing in two auctions. As $\kappa(\cdot)$ depends primarily on characteristics such as total size, overlap, and distance between projects in the final combination, the estimates scale naturally to any other combination sizes with a significant reduction in the estimation time.

Table 6 reports estimates $\hat{\theta}$ derived from this procedure. Results suggest that $\kappa(W)$ is characterized by a positive intercept, with projects of similar types having larger intercepts on average. Big and regular bidders have on average higher synergies. As sum of engineer's estimates (in thousands) and/or the backlog (in thousands) increase, however, projects become more and more substitute, with larger time overlaps, greater heterogeneity, and greater distance among projects amplifying these effects. At the median two-auction combination in our sample, these point estimates imply that a joint win would generate cost savings equal to approximately 1.8 percent of combination size, which is relatively small. However, this point estimate masks substantial heterogeneity in the data: a joint win implies cost savings equal to approximately 18 percent of combination size at the 95th (best) quantile of combinations, transitioning to a cost *increase* of approximately 4 percent at the 5th (worst) quantile.

Taken together, these numbers highlight both the potential importance of combinatorial preferences and the fact that these may differ qualitatively across both auctions and bidders. While declining complementarities between larger projects is natural and consistent with prior findings in the literature (e.g. Jofre-Bonet and Pesendorder (2003)), the changing sign of $\kappa(\cdot)$ is both novel and of considerable economic interest. We view this pattern as consistent with an underlying U-shaped cost curve, with average completion costs falling until firm resources are fully employed and rising substantially thereafter.

Table 5: First-Step MLE Estimates of G_i

Mean μ_l	$\hat{\beta}$	MLE SEs	95% CI	
Constant	0.3872	0.0155	0.3568	0.4176
Log engineer's estimate	0.9808	0.0009	0.979	0.9826
Log rivals in auction	-0.0426	0.0027	-0.0479	-0.0373
Multiple bids dummy	-0.1439	0.0205	-0.1841	-0.1037
Log sum engineer's (across l)	0.0083	0.0014	0.0056	0.011
Log sum rivals (across l)	-0.0061	0.0021	-0.0102	-0.002
log of days to the start	0.0032	0.0008	0.0016	0.0048
Standardize backlog	0.0036	0.001	0.0016	0.0056
Same-type-auctions index	-0.0224	0.005	-0.0322	-0.0126
Fraction overlapping time	0.0177	0.0035	0.0108	0.0246
Log number of big rivals faced	0.0057	0.0022	0.0014	0.01
Log number of regular rivals faced	0.024	0.0024	0.0193	0.0287
Big bidder	0.01	0.0044	0.0014	0.0186
Regular bidder	-0.0051	0.0025	-0.01	-0.0002
Log distance to project	0.0152	0.0009	0.0134	0.017
Log distance across played projects	0.0056	0.0012	0.0032	0.008
Bidder Type FE	YES	-	-	-
Auction Type FE	YES	-	-	-
Year FE	YES	-	-	-
Month FE	YES	-	-	-

Variance σ_l	$\hat{\alpha}$	MLE SEs	95% CI	
Constant	0.1046	0.0742	-0.0408	0.25
Multiple bids dummy	-0.2034	0.019	-0.2406	-0.1662
Log engineer's estimate	-0.2651	0.0054	-0.2757	-0.2545

Covariance ρ_{kl}	$\hat{\gamma}$	MLE SEs	95% CI	
Constant	0.2067	0.023	0.1616	0.2518
Same county projects	0.2206	0.0275	0.1667	0.2745
Same type projects	0.1257	0.0188	0.0889	0.1625
Fraction overlapping time	-0.0288	0.0203	-0.0686	0.011

Table 6: Estimates of θ_0, Two-Auction Subsample

	$\hat{\theta}$	SE
Constant	57.4474	8.9459
Same-type-auctions index	60.6405	10.1404
Fraction overlapping time across projects	-42.723	5.4036
Distance across played projects in KM	-1.9347	0.7926
Big Bidder	74.7563	24.8928
Regular Bidder	64.6683	9.8214
Sum engineer's estimate + Backlog in '000	-0.0062	0.0012
Bidder Type FE	YES	-

Units are in thousands of dollars, positive κ means lower cost.

5 Counterfactuals

While the simultaneous first-price auction is clearly inefficient when bidders have combinatorial preferences, little is known empirically about the magnitude of these inefficiencies in practice. Furthermore, little is known (either theoretically or empirically) about the revenue properties of the simultaneous first-price auction relative to other feasible multi-object mechanisms such as the Vickery-Clarke-Groves (VCG) mechanism, the combinatorial proxy auction (Ausubel and Milgrom 2002), or the clock-proxy auction (Ausubel, Crampton and Milgrom 2006). Given that implementation of such combinatorial mechanisms involves substantial practical costs (even solving the allocation problem once is NP-hard), determining the magnitude of their potential revenue and efficiency effects is crucial in evaluating whether policymakers might want to switch. If efficiency gains or small and / or revenue effects are ambiguous, an optimal policymaker may prefer the simplicity and transparency of the simultaneous first-price auction to better-performant but more complex combinatorial mechanisms. Conversely, if large efficiency and / or revenue gains are feasible, incurring greater combinatorial implementation costs may be worthwhile.

In this section, we compare revenue and efficiency outcomes of the simultaneous low-bid first-price auction with those of two other mechanisms: a descending combinatorial VCG mechanism and a descending combinatorial proxy auction a la

Ausubel and Milgrom (2002). As is well known, the VCG mechanism induces truthful reporting (hence efficiency) as an equilibrium, but is vulnerable to collusion and – particularly in settings with complementarities – can exhibit very poor revenue performance. The Ausbel-Milgrom proxy auction is widely seen to mitigate the potential revenue disadvantages of the VCG auction, while still achieving efficiency so long as bidders report their true preferences to the proxy agent.

Descending proxy auction Adapted to our procurement setting, the descending proxy auction operates as follows. First, each bidder i reports to its proxy agent a $(2^{L_i} - 1) \times 1$ vector describing costs of completion for each possible combination of the L_i products on which i has undertaken cost discovery. Second, proxies compete on behalf of bidders in a virtual descending package auction, bidding according to the following rule: in each bidding round, submit the allowable package bid that, if accepted, would maximize the bidder's profit given its reported costs. After each bidding round, a provisional winning allocation is determined by minimizing procurer costs over existing bids, and bidding proceeds to the next round. If no new bids are submitted in a round, the auction ends.

Consistent with most prior work on proxy auctions, we restrict attention to the case where bidders truthfully report costs. This guarantees that the final allocation is efficient and in the core of the corresponding exchange game. Note, however, that it is uncertain whether truthful reporting is an equilibrium in general.[19] Insofar as false reports distort final allocations, our results may overstate gains from the proxy auction. Nevertheless, we see truthful revelation as a useful and practical benchmark for comparison with the simultaneous first-price auction.

Computation of final outcomes in the Ausubel-Milgrom proxy auction is known to be extremely challenging, requiring one to solve a NP-hard winner determination problem for every bidding round. Since the proxy auction obtains (approximate) efficiency only with a small bid increment, and the number of bidding rounds required for convergence increases substantially as the bid increment decreases, naive application of the Ausubel-Milgrom algorithm can be extremely costly computationally.

[19] See related discussion in, e.g., Ausubel and Milgrom (2002).

We therefore focus instead on two variants of the Ausubel-Milgrom auction identified by Sandholm (2006) as having good computational properties: the *safe-start proxy auction*, in which starting bids for each bidder are determined by the VCG payment rule, and the *increment scaling proxy auction*, in which the bid increment automatically scales down as the auction proceeds. In both variants we target a final-iteration bid increment of \$1000, which is quite small as bids are typically in hundreds of thousands to millions of dollars. These algorithms need not generate the same revenue as the naive proxy auction, but retain its desirable efficiency and revenue properties. See Sandholm (2006) for detailed discussion of these algorithms.

Counterfactual implementation In implementing our counterfactuals, we first restrict attention to the sub-sample of auctions such that all bidders in each auction participate in no more than six auctions. Given the combinatorial nature of the problem this dramatically simplifies the computational burden without changing substantially the representativeness of the bidder-types in the sample as 91.84% of bidders submit 6 or fewer bids. We partion this sub-sample into 432 "virtual lettings" defined such that each bidder in letting t bids only in letting t. Of these virtual lettings, there are 7 with more than 30 bidders; as our counterfactuals require repeatedly solving for optimal allocations and the time needed to do so grows more than exponentially in the number of auctions and bidders, we drop these 7 virtual lettings from the sample. We thus end up with a final counterfactual sample of 425 virtual lettings, of which 154 (our primary interest) involve at least two auctions.

Given this sample, we implement our counterfactual comparisons as follows. First, for each bidder i and letting t in the counterfactual sample, we draw a sample of bids $\{b_i^{rt}\}_{r=1}^{R}$ from the corresponding bid distribution $\hat{G}_i(\cdot)$ estimated in Step 1 of our structural analysis. Second, for each bid vector b_i^{rt} drawn for each bidder i, we recover the corresponding standalone valuation vector v_i^{rt} implied by the inverse bid function (2), taking as given the estimates $\hat{\kappa}_i(\cdot)$ for $\kappa_i(\cdot)$ obtained in Step 2 of our structural analysis. For each letting t in the counterfactual sample and each replication $r \in \{1, ..., R\}$, we then proceed in three steps.

First, we simulate the allocation a_{FPA}^{rt} and procurement cost C_{FPA}^{rt} arising under

Table 7: Counterfactual simulation results, multi-auction lettings

(a) Average project completion costs (total per letting)

Expected project completion costs, simultaneous FPA	$2,641,101$
Expected project completion costs, combinatorial VCG	$2,491,567$

(b) Average payments by MDOT to bidders (total per letting)

Expected MDOT payments, simultaneous FPA	$3,404,234$
Expected MDOT payments, combinatorial VCG	$3,405,602$
Expected MDOT payments, incremental scaling proxy	$3,392,307$

Averages based on $R = 200$ replications of simulation procedure in text.

the simulatenous first-price auction format given bid realizations $\{b_i^{rt}\}_{i=1}^N$ for each bidder in the sample; i.e. awarding each auction to the bidder submitting the lowest standalone bid. Then, taking estimated complementarities $\{\hat{\kappa}_i(\cdot)\}_{i=1}^N$ and estimated valuations $\{v_i^{rt}\}_{i=1}^N$ as given, we simulate total social costs of project complection S_{FPA}^{rt} corresponding to allocation a_{FPA}^{rt}.

Second, we simulate the allocation a_{VCG}^{rt} and social cost S_{VCG}^{rt} induced by the dominant strategy equilibrium of the VCG mechanism given valuation draws $\{v_i^{rt}\}_{i=1}^N$ and estimated complementarities $\{\hat{\kappa}_i(\cdot)\}_{i=1}^N$. We then compute the corresponding total procurement cost C_{VCG}^{rt} by summing VCG payments for each bidder.

Third, assuming truthful reporting of types by bidders, we simulate proxy auction procurement costs C_{PROXY}^{rt} based on the safe-start and incremental scaling algorithms described above. In both variants, we target a final iteration bid increment of \$1000, which is quite small relative to typical bids. While in principle efficiency in proxy auctions obtains only when the bid increment approaches zero, in practice we find that our \$1000 bid increment captures virtually all social gains – departures from VCG allocations were rare and efficiency losses in the event of departures were small, for an overall increase in social costs on the order of a thousandth of one percent. We thus do not report separate efficiency measures for our proxy implementations.

Counterfactual results Table 7 summarizes results of this counterfactual comparison based on $R = 200$ simulation replications, focusing on the subsample of virtual lettings involving at least two auctions. Reported results are averages of counterfactual quantities across both simulation draws r and lettings t, with the latter taken only across multi-auction lettings. For purposes of these simulations, we set MDOT's effective reserve price for each project equal to twice the MDOT engineer's estimate of costs; other plausible values lead to very similar results.

Two striking patterns emerge from Table 7. First, as expected, the simultaneous first-price mechanism involves nontrivial efficiency losses, generating expected social costs of 2.64 million per letting versus 2.49 million per letting for the (socially efficient) combinatorial VCG mechanism. Yet in percentage terms gains from the VCG mechanism are relatively small: roughly 5.7 percent social cost savings relative to total completion costs under the simultaeous first-price mechanism.

Second, and even more striking, expected payments by MDOT to bidders are extremely similar across all three mechanisms considered. Payments under the incremental scaling proxy are marginally lower and payments under the VCG mechanism are marginally higher than simulated payments under the simultaneous first-price auction. But differences in both cases are extremely small, amounting to less than one percent of total MDOT payments under the baseline first-price mechanism. We emphasize that this is not a prediction of the theory; with different parameters, one can easily obtain substantial differences in revenue.

Taken together, we view these findings as strong suggestive evidence that the simultaneous first-price mechanism in fact performs remarkably well in the MDOT procurement setting. If MDOT's objective is to minimize its expected payments, it can (for all practical purposes) do no better by switching to a combinatorial mechanism. If its objective is to minimize social costs, moderately larger gains are possible, but these are still quite modest relative to total social expenditure.

Note that this analysis is only partial in that we effectively hold entry behavior fixed across mechanisms. This is entirely for computational reasons; even if we were to estimate distributions of entry costs, solving for equilibrium entry responses would be an immense computational challenge. We instead discuss briefly how we might expect results to change. By construction, social savings not captured by MDOT must accrue as profit to bidders, and in equilibrium this should translate into greater entry. This in turn might generate slightly larger revenue effects than we estimate here. In contrast, since since new entrants are by definition marginal, we expect true efficiency gains to be similar (probably slightly smaller) than those we report above.

6 Conclusion

Motivated by an institutional framework common in procurement applications, we develop and estimate a structural model of bidding in simultaneous first-price auctions, to our knowledge the first such in the literature. Non-parametric and semi-parametric identification of the model is achieved under standard exclusion restrictions. Finally, we apply this framework to data on Michigan Department of Transportation highway construction and maintenance auctions. While for the median bidder in our estimation sample estimated complementarities are approximately zero, this masks substantial heterogeneity in the sample. Our estimates suggest that winning a two-auction combination generates cost effects ranging from roughly 4 percent cost increases (relative to combination size) at the 5th percentile to roughly 18 percent cost savings (relative to combination size) at the 95th percentile, with combination costs increasing in joint size of, scheduling overlap between, and distance between projects in the combination. Building on these observations, we compare performance of the simultaneous first-price mechanism with performance of two truly combinatorial alternatives: the Vickery-Clarke-Groves mechanism, and a descending proxy auction a la Ausubel and Milgrom (2002). Despite the presence of substantial complementarities (both positive and negative) in the data, we find that these alternative mechanisms generate relatively modest gains: roughly 5.7 percent savings in social costs of project completion, with very little change in MDOT's expected costs. We view this as strong suggestive evidence that simultaneous first-price auctions can perform relatively well even in environments with economically important complementarities. While more research on this is needed, this observation may partially rationalize the widespread popularity of simultaneous first-price auctions in practice.

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Appendix A: Complementarities depending on V

In this appendix, we explore prospects for generalizing our non-parametric identification results to the case where complementarities are additively separable and/or affine functions of standalone valuations. Such a case could arise if, for instance, winning two auctions together increases i's valuation for one or both objects by a fixed percentage.

Notation and definitions Let e_l denote the L-dimensional lth unit vector. We say complementarities are *additively separable in* v if for each ω that contains at least two non-zero components (that is, $\omega^T\omega \geq 2$), the complementarity function is a function of the vector of standalone valuations $v = (v_1, v_2, \ldots, v_L)^T$ such that

$$K^{\omega}(v) = \sum_{l:\,\omega^T e_l = 1} \phi_l(v_l) + \bar{K}^{\omega} \tag{11}$$

for some functions ϕ_l, $l = 1, \ldots, L$. If each function $\phi_l(\cdot)$ is linear in its argument v_l, then we obtain the special case of *complementarities affine in* v:

$$K^{\omega}(v) = \sum_{l:\,\omega^T e_l = 1} \delta^l v_l + \bar{K}^{\omega}, \qquad \text{if } \omega^T\omega \geq 2. \tag{12}$$

As usual, if ω contains at most one component equal to one (that is, $\omega^T\omega \leq 1$), then we set $K^{\omega}(v) \equiv 0$.

An important special case of (12) is when all δ^l are identical and $\bar{K}^{\omega} = 0$ for any ω. This case describes the situation of a constant relative complementarity – that is, when $K^{\omega}(v)$ is a constant ratio of the additive valuation.

Now assume that complementarities are affine in v, and define an $L \times 1$ vector δ and an $L \times L$ matrix $D(\delta)$ as follows:

$$\delta \equiv (\delta^1, \delta^2, \ldots, \delta^L)^T$$
$$D(\delta) \equiv diag(\delta^1, \delta^2, \ldots, \delta^L).$$

Let A denote the $2^L \times 2^L$ matrix such that its $(2^L - L - 1) \times (2^L - L - 1)$ submatrix $(a_{ij})_{i,j=L+2,\ldots,2^L}$ coincides with the identity matrix of size $2^L - L - 1$, with all the other elements of A are 0. We then have

$$K(v) = A\Omega D(\delta)v + \bar{K},$$

where $\bar{K}$ denotes the $2^L \times 1$ vector of constant components $\bar{K}^\omega$ in the functions of complementarities. Clearly, the first $L+1$ elements in $\bar{K}$ are zero since they correspond to the cases of ω such that $\omega^T\omega \leq 1$.

Non-parametric identification Our next step is to generalize the non-parametric identification result above to the case when complementarities are affine functions of standalone valuations. Namely, for a given subset ω containing at least two elements,

$$K^\omega = K^\omega(v_i, Z_i, W, X) = \sum_{l:\,\omega^T e_l = 1} \delta^l(Z_i, W, X)v_{i,l} + \bar{K}^\omega(Z_i, W, X).$$

When $\omega^T\omega \leq 1$, the complementarity is set to 0. As can be seen, the functional form of complementarities does not depend on Z_{-i}. As we show below, under weak conditions there is enough variation in $Z_{-i} \,|\, Z_i, W, X$ to determine the linear (in $v_{i,l}$) part of complementarities as well as the constant part.

Define the $L_i \times 1$ vector $\delta(Z_i, W, X)$ as

$$\delta(Z_i, W, X) = (\delta^1(Z_i, W, X), \delta^2(Z_i, W, X), \ldots, \delta^{L_i}(Z_i, W, X))^T,$$

and the $D(\delta(Z_i, W, X))$ as the $L_i \times L_i$ matrix

$$D(\delta(Z_i, W, X)) = diag(\delta^1(Z_i, W, X), \delta^2(Z_i, W, X), \ldots, \delta^{L_i}(Z_i, W, X)).$$

Then

$$K(v_i, Z_i, X) = A_i\Omega_i D(\delta(Z_i, W, X))v_i + \bar{K}(Z_i, W, X),$$

where $\bar{K}(Z_i, W, X)$ denotes the $2^{L_i} \times 1$ vector of constant components in the complementarities (obviously, $\bar{K}(Z_i, W, X) \in \mathcal{K}_i$). Matrix A_i denotes the $2^{L_i} \times 2^{L_i}$ matrix such that its $(2^{L_i} - L_i - 1) \times (2^{L_i} - L_i - 1)$ submatrix $(a_{ij})_{i,j=L_i+2,\ldots,2^{L_i}}$ coincides with the identity matrix of size $2^{L_i} - L_i - 1$, and all the other elements of A_i are 0. Clearly, the rank of matrix $A_i\Omega_i$ is equal to L_i.

Using the first-order condition and taking into account the form of $K(v_i, Z_i, W, X)$, obtain

$$\begin{aligned} v_i &= b_i + [\nabla_b \Gamma_{-i}(b_i|Z, W, X)]^{-1}\,\Gamma_{-i}(b_i|Z, W, X) \\ &\quad - [\nabla_b \Gamma_{-i}(b_i|Z, W, X)]^{-1}\,\nabla_b P_{-i}(b_i|Z, W, X)^T \left[A_i\Omega_i D(\delta(Z_i, W, X))v_i + \bar{K}(Z_i, W, X)\right], \end{aligned}$$

where, as before, $Z = (Z_i, Z_{-i})$. Denote

$$\Pi(b_i, \delta, Z, W, X) = I_{L_i} + [\nabla_b \Gamma_{-i}(b_i|Z, W, X)]^{-1} \nabla_b P_{-i}(b_i|Z, W, X)^T A_i \Omega_i D(\delta).$$

For given Z_i, Z_{-i}, W and X, define $\Delta(Z_i, Z_{-i}, W, X)$ as the set of $\delta \in \Re^{L_i}$ such that

$$\Pi(b_i, \delta, Z, W, X) \text{ is non-singular for almost all } b_i.$$

E.g., $\Delta(Z_i, Z_{-i}, W, X) \ni 0$. If $\delta = \delta(Z_i, W, X) \in \Delta(Z_i, Z_{-i}, W, X)$, then

$$\begin{aligned}
v_i &= \Pi(b_i, \delta(Z_i, W, X), Z, W, X)^{-1} b_i \\
&+ \Pi(b_i, \delta(Z_i, W, X), Z, W, X)^{-1} [\nabla_b \Gamma_{-i}(b_i|Z, W, X)]^{-1} \Gamma_{-i}(b_i|Z, W, X) \\
&- \Pi(b_i, \delta(Z_i, W, X), Z, W, X)^{-1} [\nabla_b \Gamma_{-i}(b_i|Z, W, X)]^{-1} \nabla_b P_{-i}(b_i|Z, W, X)^T \bar{K}(Z_i, W, X).
\end{aligned}$$

Assuming that $\delta =\in \Delta(Z_i, Z_{-i}, W, X)$, let us denote

$$\begin{aligned}
\mathcal{D}_1(\delta, Z_i, Z_{-i}, W, X) = & E_{B_i}\left[\Pi(B_i, \delta, Z, W, X)^{-1} B_i \middle| Z, W, X\right] \\
& + E_{B_i}\left[\Pi(B_i, \delta, Z, W, X)^{-1} [\nabla_b \Gamma_{-i}(B_i|Z, W, X)]^{-1} \Gamma_{-i}(B_i|Z, W, X) \middle| Z, W, X\right],
\end{aligned}$$

$$\mathcal{D}_2(\delta, Z_i, Z_{-i}, W, X) = E_{B_i}\left[\Pi(B_i, \delta, Z, W, X)^{-1} [\nabla_b \Gamma_{-i}(B_i|Z, W, X)]^{-1} \nabla_b P_{-i}(B_i|Z, W, X)^T \middle| Z, W, X\right],$$

Keeping Z_i, W, X fixed, let us draw another value Z'_{-i} from the support of $Z_{-i}|Z_i, W, X$, and denote $Z' = (Z_i, Z'_{-i})$. Due to the assumptions made on the distribution of the standalone valuations, $E[V_i|Z, W, X] = E[V_i|Z', W, X]$. Therefore, for $\delta = \delta(Z_i, W, X) \in \Delta(Z_i, Z_{-i}, W, X) \cap \Delta(Z_i, Z'_{-i}, W, X)$,

$$\begin{aligned}
&\mathcal{D}_1(\delta(Z_i, W, X), Z_i, Z'_{-i}, W, X) - \mathcal{D}_1(\delta(Z_i, W, X), Z_i, Z_{-i}, W, X) = \\
&\big(\mathcal{D}_2(\delta(Z_i, W, X), Z_i, Z'_{-i}, W, X) - \mathcal{D}_2(\delta(Z_i, W, X), Z_i, Z_{-i}, W, X)\big) \bar{K}(Z_i, W, X).
\end{aligned}$$

For fixed Z_i, W, X, this system has $2^{L_i} - 1$ unknowns (L_i in $\delta(Z_i, W, X)$ and $2^{L_i} - L_i - 1$ in $\bar{K}(Z_i, W, X)$) and L_i equations. This gives us the following result.

Proposition 4. *Suppose that for $(Z_i, W, X) \in \mathcal{Z}_i \times \mathcal{W} \times \mathcal{X}$, there exist $J + 1 \geq (2^{L_i} - 1)/L_i + 1$ vectors $Z_{-i,0}, Z_{-i,1}, ..., Z_{-i,J}$ in the support of $Z_{-i}|Z_i, W, X$ such that there is a unique $\delta \in \bigcap_{j=0}^{J} \Delta(Z_i, Z_{-i,j}, W, X)$ and a unique $\kappa \in \mathcal{K}_i$ that solve the system of $J \cdot L_i$ equations*

$$\begin{aligned}
&\mathcal{D}_1(\delta, Z_i, Z_{-i,j}, W, X) - \mathcal{D}_1(\delta, Z_i, Z_{-i,0}, W, X) = \\
&(\mathcal{D}_2(\delta, Z_i, Z_{-i,j}, W, X) - \mathcal{D}_2(\delta, Z_i, Z_{-i,0}, W, X))\, \kappa, \quad j = 1, \ldots, J.
\end{aligned} \tag{13}$$

Then the values of $\delta(Z_i, W, X)$ and $\bar{K}(Z_i, W, X)$ are identified, and thus, the complemen-

tarity function is identified for these values of Z_i, W, X.

System (13) is non-linear in δ. However, for each fixed $\delta \in \bigcap_{j=0}^{J} \Delta(Z_i, Z_{-i,j}, W, X)$, this system is linear in κ. Proposition 4 implies that in the case of identification it is not possible to have a situation when for different δ_1 and δ_2, where $\delta_1, \delta_2 \in \bigcap_{j=0}^{J} \Delta(Z_i, Z_{-i,j}, W, X)$, system (13) has solutions $\kappa_1 \in \mathcal{K}_i$ and $\kappa_2 \in \mathcal{K}_i$, respectively. Thus, in this sense the question of identification of $\delta(Z_i, W, X)$ and $\bar{K}(Z_i, W, X)$ comes down to the question of existence of solutions to systems of linear equations: (13) can have a solution κ for one δ only, and for that δ it has to be unique. Using the Kronecker-Capelli theorem, which gives the necessary and sufficient conditions for the existence of a solution to a system of linear equations, and also the necessary and sufficient conditions for the uniqueness of such a solution, we formulate the identification result in the Proposition 5 below.

Before we proceed to Proposition 5, let E_i denote the $2^{L_i} \times (2^{L_i} - L_i - 1)$ matrix such that its $(2^{L_i} - L_i - 1) \times (2^{L_i} - L_i - 1)$ submatrix $(\tilde{e}_{ij})_{i=L_i+2,\ldots,2^{L_i},\, j=1,\ldots,2^{L_i}-l_i-1}$ coincides with the identity matrix of size $2^{L_i} - L_i - 1$, and all its other elements (that is, all the elements in the first $L_i + 1$ rows) are equal to zero. For every $\kappa \in \mathcal{K}_i$ there is a unique $\breve{\kappa} \in \mathbb{R}^{2^{L_i}-L_i-1}$ such that

$$\kappa = E_i \breve{\kappa}.$$

Obviously, this $\breve{\kappa}$ is formed by the last $2^{L_i} - L_i - 1$ values in κ. System (13) can be equivalently written as

$$\begin{aligned} \mathcal{D}_1(\delta, Z_i, Z_{-i,j}, W, X) - \mathcal{D}_1(\delta, Z_i, Z_{-i,0}, W, X) = \\ \left(\left(\mathcal{D}_2(\delta, Z_i, Z_{-i,j}, W, X) - \mathcal{D}_2(\delta, Z_i, Z_{-i,0}, W, X) \right) E_i \right) \ \breve{\kappa}, \quad j = 1, \ldots, J, \end{aligned} \tag{14}$$

with $\breve{\kappa} \in \mathbb{R}^{2^{L_i}-L_i-1}$. For a fixed δ, system (13) is linear in κ, has the $J \cdot L_i \times 2^{L_i}$ matrix of coefficients, and imposes restrictions on the solution κ by requiring that $\kappa \in \mathcal{K}_i$. Its equivalent representation (14) is linear in $\breve{\kappa}$ for a fixed δ, has the $J \cdot L_i \times (2^{L_i} - L_i - 1)$ matrix of coefficients, and does not impose any restrictions on the solution $\breve{\kappa} \in \mathbb{R}^{2^{L_i}-L_i-1}$. This allows us to apply the Kronecker-Capelli theorem to system (14) in a straightforward way.

Proposition 5. *Suppose that for $(Z_i, W, X) \in \mathcal{Z}_i \times \mathcal{W} \times \mathcal{X}$, there exist $J + 1 \geq (2^{L_i} - 1)/L_i + 1$ vectors $Z_{-i,0}, Z_{-i,1}, \ldots, Z_{-i,J}$ in the support of $Z_{-i} | Z_i, W, X$ such that there is a unique $\delta \in \bigcap_{j=0}^{J} \Delta(Z_i, Z_{-i,j}, W, X)$ that satisfies the following two conditions:*

1. *First,*

$$rank\left(\left[\mathbf{M}_1(\delta, Z_i, W, X) \,|\, \mathbf{M}_2(\delta, Z_i, W, X) \right] \right) = rank\left(\mathbf{M}_2(\delta, Z_i, W, X) \right), \tag{15}$$

where $\mathbf{M}_2(\delta, Z_i, W, X)$ *denotes the* $J \cdot L_i \times (2^{L_i} - L_i - 1)$ *matrix*

$$\mathbf{M}_2(\delta, Z_i, W, X) \equiv \begin{bmatrix} (\mathcal{D}_2(\delta, Z_i, Z_{-i,1}, W, X) - \mathcal{D}_2(\delta, Z_i, Z_{-i,0}, W, X))\, E_i \\ \vdots \\ (\mathcal{D}_2(\delta, Z_i, Z_{-i,J}, W, X) - \mathcal{D}_2(\delta, Z_i, Z_{-i,0}, W, X))\, E_i \end{bmatrix},$$

and $\mathbf{M}_1(\delta, Z_i, W, X)$ *denotes the* $J \cdot L_i \times 1$ *vector*

$$\mathbf{M}_1(\delta, Z_i, W, X) \equiv \begin{bmatrix} \mathcal{D}_1(\delta, Z_i, Z_{-i,1}, W, X) - \mathcal{D}_1(\delta, Z_i, Z_{-i,0}, W, X) \\ \vdots \\ \mathcal{D}_1(\delta, Z_i, Z_{-i,J}, W, X) - \mathcal{D}_1(\delta, Z_i, Z_{-i,0}, W, X) \end{bmatrix}.$$

2. *Moreover, this* δ *is such that* $\mathbf{M}_2(\delta, Z_i, W, X)$ *has full column rank:*

$$rank\,(\mathbf{M}_2(\delta, Z_i, W, X)) = 2^{L_i} - L_i - 1. \tag{16}$$

Then the values of $\delta(Z_i, W, X)$ *and* $\bar{K}(Z_i, W, X)$ *are identified, and thus, the complementarity function is identified for these values of* Z_i, W, X.

Condition (15) requires that in system (14), the rank of the matrix of coefficients $\mathbf{M}_2(\delta, Z_i, W, X)$ is equal to the rank of the augmented matrix $[\mathbf{M}_1(\delta, Z_i, W, X) \,|\, \mathbf{M}_2(\delta, Z_i, W, X)]$ for one δ only. The Kronecker-Capelli theorem guarantees then that (14) has a solution $\breve{\kappa}$ for that δ only. Condition (16) then guarantees this $\breve{\kappa}$ is determined uniquely, and, thus, $\kappa = E_i\breve{\kappa}$ is determined uniquely.

Note that all the identification conditions in Proposition 5 are formulated in terms of δ. The closed form for $\delta(Z_i, W, X)$ cannot be found but in practice one can find $\delta(Z_i, W, X)$ and $\bar{K}(Z_i, W, X)$ by solving, e.g., the following optimization problem:

$$\min_{\delta \in \bigcap_{j=0}^{J} \Delta(Z_i, Z_{-i,j}, W, X),\, \breve{\kappa} \in \mathbb{R}^{2^{L_i} - L_i - 1}} Q(\delta, \breve{\kappa}, Z_i, W, X),$$

where

$$Q(\delta, \breve{\kappa}, Z_i, W, X) \equiv (\mathbf{M}_1(\delta, Z_i, W, X) - \mathbf{M}_2(\delta, Z_i, W, X)\breve{\kappa})^T\, (\mathbf{M}_1(\delta, Z_i, W, X) - \mathbf{M}_2(\delta, Z_i, W, X)\breve{\kappa})\,.$$

Appendix B: Proofs

Proof of Proposition 2. The proof of Proposition 2 rests on two key claims. First, the first-order system (2) must be well-defined for almost every b_i submitted by i, i.e. almost everywhere with respect to the measure induced by $G_i(\cdot|Z, W, X)$. Second, at almost every

b_i at which first order conditions hold, the first-order system (2) must be invertible. We establish each claim in turn.

First show that the first order system (2) is well-defined for almost every b_i submitted by i. Recall that we can write bidder i's objective as

$$\pi(v_i, b|K; Z, W, X) = (\Omega v_i + K - \Omega b)^T P_{-i}(b|Z, W, X).$$

where v_i and K are given at the time of maximization. Note that the system (2) necessarily holds at any best respose where $\pi(v_i, \cdot|K; Z, W, X)$ is differentiable and that Assumption 3 implies that each observed b_i is a best response. Hence the system (2) will be well defined for almost every b_i submitted by i if and only if $\pi(v_i, \cdot|K; Z, W, X)$ is differentiable almost everywhere with respect to the measure on B_i induced by $G_i(\cdot|Z, W, X)$. But under Assumption 4 $G_i(\cdot|Z, W, X)$ is absolutely continuous. To establish the claim, it is thus sufficient to prove differentiability of $\pi(v_i, \cdot|K; Z, W, X)$ a.e. with respect to the Lebesgue measure on B_i.

Clearly $(\Omega v_i + K - \Omega b)$ is differentiable in b for any $v_i, K \in R^{L_{it}} \times \mathcal{K}$. Thus differentiability of $\pi(v_i, \cdot|K; Z, W, X)$ at b is equivalent to differentiability of $P_{-i}(\cdot|Z, W, X)$ at b. Let B_{-i} be the $L_i \times 1$ random vector describing maximum rival bids in the set of auctions in which i participates. Again applying Assumption 4 to rule out ties, the probability i wins combination ω at bid b is

$$P^\omega(b|Z, W, X) = \Pr(\{\cap_{\{l:\omega_l=1\}} 0 \leq B_{-i,l} \leq b_{i,l}\} \cap \{\cap_{\{l:\omega_l=0\}} b_{i,l} \leq B_{-i,l} < \infty\}).$$

For each $\omega \in \Omega_i$, let b^ω be the $(\sum \omega) \times 1$ sub-vector of b describing i's bids for objects in ω, B^ω_{-i} be the $(\sum \omega) \times 1$ sub-vector of B_{-i} describing maximum rival bids for objects in ω, and $G^\omega_{-i}(b^\omega|Z, W, X)$ be the equilibrium joint c.d.f. of B^ω_{-i} at (Z, W, X). Applying the formula for a rectangular probability and simplifying, we can then represent $P_{-i}(\cdot|Z, W, X)$ in the form

$$P^\omega_{-i}(b|Z, W, X) = \sum_{\omega' \in \Omega} a^\omega_{\omega'} G^{\omega'}_{-i}(b^{\omega'}|Z, W, X),$$

where each $a^\omega_{\omega'}$ is a known scalar (determined by ω, ω') taking values in $\{-1, 0, 1\}$. But by absolute continuity each c.d.f. $G^\omega_{-i}(\cdot|Z, W, X)$ is differentiable a.e. (Lebesgue) in its support, and interpreted as a function from $\mathcal{B}_i$ to $\mathbb{R}^{L_i}$, each $b^{\omega'}$ is continuously differentiable in b. Thus interpreted as a function from $\mathcal{B}_i$ to $\mathbb{R}$, each $G^{\omega'}_{-i}(b^{\omega'}|Z, W, X)$ is differentiable on a set of full Lebesgue measure in B_{-i}. The set of points in $\mathcal{B}_i$ at which all $G^{\omega'}_{-i}(b^{\omega'}|Z, W, X)$ are differentiable is the intersection of points in $\mathcal{B}_i$ at which each $G^{\omega'}_{-i}(b^{\omega'}|Z, W, X)$ is differentiable, i.e. the intersection of a finite collection of sets of full Lebesgue measure in $\mathcal{B}_i$. But from above differentiability of $G^{\omega'}_{-i}(b|Z, W, X)$ for all ω' implies differentiability of $P^\omega_{-i}(b|Z, W, X)$. Hence $P^\omega_{-i}(\cdot|Z, W, X)$ is differentiable on a set of full Lebesgue measure in $\mathcal{B}_i$. This in turn implies differentiability of $\pi(v_i, \cdot|K; Z, W, X)$ a.e. with respect to the measure on $\mathcal{B}_i$ induced by $G_i(\cdot|Z, W, X)$, as was to be shown.

We next establish that the first-order system (2) must yield a unique solution $\tilde{v}$ for almost every b_i submitted by i. Let $\tilde{B}_i$ be the set of points in $\mathcal{B}_i$ at which $\pi(\cdot,\cdot|K;W,Z,X)$ is differentiable in b; from above, $\tilde{B}_i$ is a subset of full Lebesgue measure in $\mathcal{B}_i$. Choosing any $b \in \tilde{B}_i$ and rearranging (2) yields

$$\nabla_b \Gamma_{-i}(b|Z,W,X)\tilde{v} = \nabla_b \Gamma_{-i}(b|Z,W,X)b + \Gamma_{-i}(b|Z,W,X) - \nabla_b P_{-i}(b|W,Z,X)^T K_i.$$

Hence uniqueness of $\tilde{v}$ is equivalent to invertibility of the $L_i \times L_i$ matrix $\nabla_b \Gamma_{-i}(b|Z,W,X)$. Recall that $\Gamma_{-i}(b|Z,W,X)$ is an $L_i \times 1$ vector whose lth element describes the probability that bid vector b wins auction l. Note that $b \in \tilde{B}_i$ rules out ties at b. Thus for $b \in \tilde{B}_i$ the lth element of $\Gamma_{-i}(b|Z,W,X)$ is the marginal c.d.f. $G^l_{-i}(b|Z,W,X)$ of $B_{-i,l}$, from which it follows that $\nabla_b \Gamma_{-i}(b|Z,W,X)$ is a diagonal matrix whose l,lth element is the marginal p.d.f. $g^l_{-i}(b|Z,W,X)$ of $B_{-i,l}$. Hence $\nabla_b \Gamma_{-i}(b|Z,W,X)$ will be invertible at b if and only if $g^l_{-i}(b|Z,W,X) > 0$ for all l.

But by hypothesis each submitted bid b_i is a best response to rival play at (Z,W,X) for some (v,K). Suppose that there exists an $\epsilon > 0$ such that $g^l_{-i}(\cdot|Z,W,X) = 0$ on $(b_{il} - \epsilon, b_i]$. Then player i could infintesimally reduce b_{il} without affecting either Γ_{-i} or P_{-i}, a profitable deviation for any (v,K). Hence we must have $g^l_{-i}(\cdot|Z,W,X) > 0$ almost everywhere (Lebesgue) in the support of B_i. By Assumption 4, this in turn implies $g^l_{-i}(\cdot|Z,W,X) > 0$ for almost every b_i submitted by i. Since l was arbitrary, we must have $\nabla_b \Gamma_{-i}(b_i|Z,W,X)$ invertible for almost every bid b_i submitted by i. Hence for almost every b_i submitted by i there will exist a unique $\tilde{v}$ satisfying (2) at b_i, given by

$$\tilde{v} = b_i + \nabla_b \Gamma_{-i}(b_i|Z,W,X)^{-1}\Gamma_{-i}(b_i|Z,W,X) \\ + \nabla_b \Gamma_{-i}(b_i|Z,W,X)^{-1}\nabla_b P_{-i}(b_i|W,Z,X)^T K.$$

The RHS of this expression is identified up to K, establishing the claim.

□

Appendix C: Partial identification with general G_i

The point identification result for the complementarity function $\kappa_i(Z_i,W,X)$ and the conditional distribution of $V_i|Z_i,W,X$ relied on the equations in the first order conditions obtained from bidder's optimization of the payoff function. To derive those equations we employed the absolute continuity of bid distribution functions G_i. That, in particular, eliminated the possibility of bidders playing atoms in the equilibrium. In this appendix, we want to illustrate an approach to the identification question when no continuity restrictions are imposed on G_i. Our identification method is based on using inequalities for bidder's best responses and employing the exclusion restrictions in Assumption 5 to obtain bounds on the complementarity function and the distributions of standalone valuations. Throughout our analysis here we continue to impose Assumptions 1-5.

Let us fix $(Z_i, W, X) \in \mathcal{Z}_i \times \mathcal{W} \times \mathcal{X}$. As before, $F_i(\cdot|Z_i, W, X)$ denotes the L-variate distribution of the vector of standalone valuations V_i of the bidder conditional on Z_i, W, X. As $\mathcal{F}_{ac}(\mathbb{R}^p)$ we denote the set of all absolutely continuous cumulative distribution functions on $\mathbb{R}^p$.

Observable are the distribution functions $G_i(b|Z, W, X)$, where $Z = (Z_i, Z_{-i})$, of the bids in the equilibrium played by the bidders for any $Z_{-i} \in \mathcal{Z}_{-i}|Z, W, X$, $i = 1, \ldots, N$.

The identification set is defined as the set

$$\mathcal{H}_i(Z_i, W, X) = \{(F_i(\cdot|Z_i, W, X), \kappa_i(Z_i, W, X)) : \text{ Conditions } (C1), (C2), (C3) \text{ satisfied}\}.$$

When writing the identification set in this form, we already make us of the exclusion restrictions in Assumption 5. The conditions in the definition of $\mathcal{H}_i(Z_i, W, X)$ are the following:

(C1) $F_i(\cdot|Z_i, W, X) \in \mathcal{F}_{ac}(\mathbb{R}^{L_i})$, and the support of $F_i(\cdot|Z_i, W, X)$ is a compact, convex set in $\mathbb{R}^{L_i}$ (consistent with Assumption 1).

(C2) $\kappa_i(Z_i, W, X) \in \mathcal{K}_i$.

(C3) For each $Z_{-i} \in \mathcal{Z}_{-i}|Z_i, W, X$, bidder's behavior is consistent with the maximization of the payoff function

$$\pi(v_i, b; Z, W, X) = v_i^T \Gamma_{-i}(b|Z, W, X) - b^T \Gamma_{-i}(b|Z, W, X) + P_{-i}(b|Z, W, X)^T \kappa_i(Z_i, W, X)$$

with respect to $b \in \mathcal{B}_i$. That is, for each $Z_{-i} \in \mathcal{Z}_{-i}|Z_i, X, W$, every bidder i's bid vector b_i observed in the equilibrium satisfies the inequality

$$\begin{aligned} v_i^T \Gamma_{-i}(b_i|Z_{-i}) - b_i^T \Gamma_{-i}(b_i|Z_{-i}) + P_{-i}(b_i|Z_{-i})^T \kappa(Z_i, W, X) \geq \\ v_i^T \Gamma_{-i}(b|Z_{-i}) - b^T \Gamma_{-i}(b|Z_{-i}) + P_{-i}(b|Z_{-i})^T \kappa(Z_i, W, X) \\ \forall\, b \in \mathcal{B}_i, \end{aligned} \quad (17)$$

where for notational simplicity we wrote $\Gamma_{-i}(\cdot|Z_{-i})$ and $P_{-i}(b_i|Z_{-i})$ instead of $\Gamma_{-i}(b_i|Z, W, X)$ and $P_{-i}(b_i|Z, W, X)$ respectively, thus omitting fixed (Z_i, W, X) from the notation.

Let $\mathcal{H}_{i,\kappa}(Z_i, W, X)$ stand for the projection of the identification set onto the second component – on the set of $\kappa_i(Z_i, W, X)$. Let $\mathcal{H}_{i,F}(Z_i, W, X)$ stand for the projection of the identification set onto the first component – on the set of $F_i(\cdot|Z_i, W, X)$; and let $\mathcal{H}_{i,F_l}(Z_i, W, X)$ stand for the projection of $\mathcal{H}_{i,F}(Z_i, W, X)$ onto the marginal distribution of V_{il} conditional on Z_i, W, X – that is, on the set of univariate distribution functions $F_{il}(\cdot|Z_i, W, X)$. Even though it is very difficult to obtain a closed form description of the sets $\mathcal{H}_{i,F}(Z_i, W, X)$ and $\mathcal{H}_{i,\kappa}(Z_i, W, X)$, it is possible to give closed form characterizations of their supersets. These supersets are given in Proposition 6 below.

First, we introduce some notations. Let $\Delta^+_{\epsilon,l}[f(u)]$ and $\Delta^-_{\epsilon,l}[f(u)]$ denote differences in the values of $f(\cdot)$ associated with adding ϵ and $-\epsilon$ to the $l$the component of u respectively:

$$\Delta^+_{\epsilon,l}[f(u)] = f(u + \epsilon e_l) - f(u),$$
$$\Delta^-_{\epsilon,l}[f(u)] = f(u - \epsilon e_l) - f(u),$$

where e_l denotes the L_i-dimensional lth unit vector.

Suppose there exist known scalars $\underline{v} \geq 0$ and $\bar{v} < \infty$ such that $\underline{v} \leq v_{il} \leq \bar{v}$ for any l; note that these could be strictly outside the support of V_{il}. For each $b_i \in \mathcal{B}_i$, define $I^-_{\epsilon,l}(b_i|Z_{-i})$, $I^+_{\epsilon,l}(b_i|Z_{-i})$ as follows:

$$I^-_{\epsilon,l}(b_i|Z_{-i}) = \left\{ \underline{v} \text{ if } \Delta^-_{\epsilon,l}[\Gamma_{-i}(b_i|Z_{-i})] = 0,\ \frac{\Delta^-_{\epsilon,l}[b_i^T \Gamma_{-i}(b_i|Z_{-i})]}{\Delta^-_{\epsilon,l}[\Gamma_{-i,l}(b_i|Z_{-i})]} \text{ else} \right\};$$
$$I^+_{\epsilon,l}(b_i|Z_{-i}) = \left\{ \bar{v} \text{ if } \Delta^+_{\epsilon,l}[\Gamma_{-i}(b_i|Z_{-i})] = 0,\ \frac{\Delta^+_{\epsilon,l}[b_i^T \Gamma_{-i}(b_i|Z_{-i})]}{\Delta^+_{\epsilon,l}[\Gamma_{-i,l}(b_i|Z_{-i})]} \text{ else} \right\}.$$

Also, for each $b_i \in \mathcal{B}_i$, define the following $S^-_{\epsilon,l}(b_i|Z_{-i})$ and $S^+_{\epsilon,l}(b_i|Z_{-i})$:

$$S^-_{\epsilon,l}(b_i|Z_{-i}) = \left\{ 0 \text{ if } \Delta^-_{\epsilon,l}[\Gamma_{-i}(b_i|Z_{-i})] = 0,\ \frac{\Delta^-_{\epsilon,l}[P_{-i}(b_i|Z_{-i})]}{\Delta^-_{\epsilon,l}[\Gamma_{-i,l}(b_i|Z_{-i})]} \text{ else} \right\};$$
$$S^+_{\epsilon,l}(b_i|Z_{-i}) = \left\{ 0 \text{ if } \Delta^+_{\epsilon,l}[\Gamma_{-i}(b_i|Z_{-i})] = 0,\ \frac{\Delta^+_{\epsilon,l}[P_{-i}(b_i|Z_{-i})]}{\Delta^+_{\epsilon,l}[\Gamma_{-i,l}(b_i|Z_{-i})]} \text{ else} \right\}.$$

For any $K \in \mathcal{K}_i$, let $\tilde{F}^-_{il}(\cdot|K; Z_{-i})$ denote the c.d.f. of

$$\sup_{\epsilon>0} \left(I^-_{\epsilon,l}(b_i|Z_{-i}) - S^-_{\epsilon,l}(b_i|Z_{-i})^T K \right),$$

and let $\tilde{F}^+_{il}(\cdot|K; Z_{-i})$ denote the c.d.f. of

$$\inf_{\epsilon>0} \left(I^+_{\epsilon,l}(b_i|Z_{-i}) - S^+_{\epsilon,l}(b_i|Z_{-i})^T K \right)$$

.

Hereafter we assume that ties are broken independently across auctions.

Proposition 6. *a) A superset of $\mathcal{H}_{i,\kappa}(Z_i, W, X)$ can be found in the following way:*

$$\mathcal{H}^{(1)}_{i,\kappa}(Z_i, W, X) = \bigcap_{l=1}^{L} \tilde{\mathcal{K}}_{i,l},$$

where $\tilde{\mathcal{K}}_{i,l}$ is defined as

$$\tilde{\mathcal{K}}_{i,l} = \{K \in \mathcal{K}_i \bigm| \tilde{F}^{+}_{il}(\cdot|K; Z_{-i}) \leq \tilde{F}^{-}_{il}(\cdot|K; Z'_{-i}) \quad \forall\, Z_{-i}, Z'_{-i} \in \mathcal{Z}_{-i}|Z_i, W, X\}.$$

b) A superset of $\mathcal{H}_{i,F_l}(Z_i, W, X)$ can be found as the set of univariate functions $F_{il}(\cdot) \in \mathcal{F}_{ac}(\mathbb{R})$ such that for any $\eta \in \mathbb{R}$,

$$F_{il}(\eta) \in \bigcap_{\kappa_0 \in \mathcal{H}^{(1)}_{i,\kappa}(Z_i,W,X)} \bigcap_{Z_{-i}, Z'_{-i} \in \mathcal{Z}_{-i}|Z_i,W,X} [\tilde{F}^{+}_{il}(\eta|\kappa_0; Z_{-i}), \tilde{F}^{-}_{il}(\eta|\kappa_0; Z'_{-i})]\}.$$

Let us denote this superset as $\mathcal{H}^{(1)}_{i,F_l}(Z_i, W, X)$.

c) A superset of $\mathcal{H}_{i,F}(Z_i, W, X)$ can be found as the set of L_i-variate functions $F_i(\cdot) \in \mathcal{F}_{ac}(\mathbb{R}^{L_i})$ such that each lth marginal distribution function generated by $F_i(\cdot)$ belongs to $\mathcal{H}^{(1)}_{i,F_l}(Z_i, W, X)$, $l = 1, \ldots, L_i$.

Moreover, for any $\eta = (\eta_1, \ldots, \eta_{L_i})$,

$$F_i(\eta) \leq \min_{l=1,\ldots,L_i} \inf_{\kappa_0 \in \mathcal{H}^{(1)}_{i,\kappa}(Z_i,W,X)} \inf_{Z_{-i} \in \mathcal{Z}_{-i}|Z_i,W,X} \tilde{F}^{+}_{il}(\eta_l|\kappa_0; Z_{-i}),$$

$$F_i(\eta) \geq \max\left\{\sum_{l=1}^{L_i} \sup_{\kappa_0 \in \mathcal{H}^{(1)}_{i,\kappa}(Z_i,W,X)} \sup_{Z_{-i} \in \mathcal{Z}_{-i}|Z_i,W,X} \tilde{F}^{-}_{il}(\eta_l|\kappa_0; Z_{-i}) - L_i + 1, 0\right\}.$$

Proof. Let $\Delta_\delta[f(u)] = (f(u+\delta) - f(u))$ denote changes in $f(u)$ induced by adding the vector δ to the vector u. For notational simplicity, let $\kappa_i(Z_i, W, X) = \kappa_0$. Then we can equivalently restate (17) as follows:

$$v_i^T \Delta_\delta[\Gamma_{-i}(b_i|Z_{-i})] - \Delta_\delta[b_i^T \Gamma_{-i}(b_i|Z_{-i})] + \Delta_\delta[P_{-i}(b_i|Z_{-i})^T]\kappa_0 \;\leq\; 0 \;\forall\, \delta \;\in\; \mathcal{B}_i - b_i, \quad (18)$$

where the difference $\mathcal{B}_i - b_i$ is understood as the Minkowski difference.

System (18) is linear in v_i and κ_0. The set of solutions of (v_i, κ_0) to this system can be shown to be convex. Its intersection with $[\underline{v}, \overline{v}]^L \times \mathcal{K}_i$ is convex as well. In a special case when $\mathcal{B}_i$ consists of a finite number of points, the set of solutions is a convex (closed) polyhedron in $\Re^{2^L - 1}$.

Any subsystem of a finite number of inequalities from system (18) can easily be resolved to give an upper bound (a lower bound) on the lth component v_{il} in terms of the minimum (maximum) of some linear functions with of κ_0 with known coefficients. That could be used to get bounds on the distribution functions of standalone valuations in terms of κ_0. However, the formulas for the bounds on each component obtained from a finite system of linear inequalities are quite complicated. An easier way to obtain bounds on each v_{il} is to consider only those alternative bids b that differ from b_i in the lth coordinate.

Since the ties are broken independently across auctions at b_i, then a change in the lth component of b_i affects only the lth component of Γ_{-i}. Noting that $\Gamma_{-i,l}$ is monotone in b_{il} and rearranging, we thus must have for any ϵ such that $b_i - \epsilon e_l \in \mathcal{B}_i$ and $b_i + \epsilon e_l \in \mathcal{B}_i$

$$\begin{aligned}\frac{\Delta^-_{\epsilon,l}[b_i^T\Gamma_{-i}(b_i|Z_{-i})]}{\Delta^-_{\epsilon,l}[\Gamma_{-i,l}(b_i|Z_{-i})]} - \frac{\Delta^-_{\epsilon,l}[P_{-i}(b_i|Z_{-i})^T]}{\Delta^-_{\epsilon,l}[\Gamma_{-i,l}(b_i|Z_{-i})]}\kappa_0 \leq v_{il} \\ \leq \frac{\Delta^+_{\epsilon,l}[b_i^T\Gamma_{-i}(b_i|Z_{-i})]}{\Delta^+_{\epsilon,l}[\Gamma_{-i,l}(b_i|Z_{-i})]} - \frac{\Delta^+_{\epsilon,l}[P_{-i}(b_i|Z_{-i})^T]}{\Delta^+_{\epsilon,l}[\Gamma_{-i,l}(b_i|Z_{-i})]}\kappa_0. \quad (19)\end{aligned}$$

If for a given ϵ, we have that $b_i - \epsilon e_l \notin \mathcal{B}_i$ or $b_i + \epsilon e_l \notin \mathcal{B}_i$, then at least one of $\Delta^-_{\epsilon,l}[\Gamma_{-i,l}(b_i|Z_{-i})]$ and $\Delta^+_{\epsilon,l}[\Gamma_{-i,l}(b_i|Z_{-i})]$ is equal to 0, and then in order to bound v_{il} we can use our prior knowledge that $v_{il} \in [\underline{v}, \overline{v}]$.

Using our notations above, we can say that for any b_i in the support of G_i:

$$I^-_{\epsilon,l}(b_i|Z_{-i}) - S^-_{\epsilon,l}(b_i|Z_{-i})^T\kappa_0 \leq v_{il} \leq I^+_{\epsilon,l}(b_i|Z_{-i}) - S^+_{\epsilon,l}(b_i|Z_{-i})^T\kappa_0,$$

and hence,

$$\sup_{\epsilon>0}\left(I^-_{\epsilon,l}(b_i|Z_{-i}) - S^-_{\epsilon,l}(b_i|Z_{-i})^T\kappa_0\right) \leq v_{il} \leq \inf_{\epsilon>0}\left(I^+_{\epsilon,l}(b_i|Z_{-i}) - S^+_{\epsilon,l}(b_i|Z_{-i})^T\kappa_0\right). \quad (20)$$

Then by the inequalities in (20), we must have for any $Z_{-i} \in \mathcal{Z}_{-i}|Z_i, W, X$:

$$\tilde{F}^+_{il}(\cdot|\kappa_0; Z_{-i}) \leq F_{il}(\cdot) \leq \tilde{F}^-_{il}(\cdot|\kappa_0; Z_{-i}). \quad (21)$$

Pooling information across Z_{-i}, it follows that we can have $K = \kappa_0$ only if

$$\tilde{F}^+_{il}(\cdot|K; Z_{-i}) \leq \tilde{F}^-_{il}(\cdot|K; Z'_{-i}) \quad \forall\, Z_{-i}, Z'_{-i} \in \mathcal{Z}_{-i}|Z_i, W, X. \quad (22)$$

This gives part a) of this theorem. Part b) immediately follows from (21). Part c) follows from (21) and the well known result on sharp Frechet-Hoeffding bounds for joint distributions. □

The next proposition provides an expectations version of the partial identification argument. Even though the supersets it gives are larger than those in Proposition 6, computationally they are easier to obtain. Before formulating this proposition, let us define

$L_i \times 1$ vectors $\Psi_\epsilon^-(Z_{-i})$, $\Psi_\epsilon^+(Z_{-i})$ and $L_i \times 2^{L_i}$ matrices $\chi_\epsilon^-(Z_{-i})$, $\chi_\epsilon^+(Z_{-i})$ as follows:

$$\begin{aligned}
\Psi_\epsilon^-(Z_{-i}) &\equiv \left[E[I_{\epsilon,l}^-(B_i|Z_{-i})|Z_{-i}]\right]_{l=1}^{L_i} \\
\Psi_\epsilon^+(Z_{-i}) &\equiv \left[E[I_{\epsilon,l}^+(B_i|Z_{-i})|Z_{-i}]\right]_{l=1}^{L_i} \\
\chi_\epsilon^-(Z_{-i}) &\equiv \left[E[S_{\epsilon,l}^-(B_i|Z_{-i})|Z_{-i}]^T\right]_{l=1}^{L_i} \\
\chi_\epsilon^+(Z_{-i}) &\equiv \left[E[S_{\epsilon,l}^+(B_i|Z_{-i})|Z_{-i}]^T\right]_{l=1}^{L_i}.
\end{aligned}$$

Proposition 7. *A superset of $\mathcal{H}_{i,\kappa}(Z_i, W, X)$ can be found in the following way:*

$$\mathcal{H}_{i,\kappa}^{(2)}(Z_i, W, X) = \bigcap_{\epsilon>0} \hat{\mathcal{K}}_i^\epsilon,$$

where $\hat{\mathcal{K}}_i^\epsilon$ is defined as

$$\begin{aligned}
\hat{\mathcal{K}}_i^\epsilon \equiv \Big\{K \in \mathcal{K}_i \;\Big|\; & \left(\Psi_\epsilon^-(Z_{-i}) - \Psi_\epsilon^+(Z'_{-i})\right) \\
& - \left(\chi_\epsilon^-(Z_{-i}) - \chi_\epsilon^+(Z'_{-i})\right) K \leq 0 \text{ for all } Z_{-i}, Z'_{-i} \in \mathcal{Z}_{-i}|Z_i, W, X\Big\}.
\end{aligned}$$

Results analogous to parts b) and c) in Proposition 7 hold as well with $\mathcal{H}_{i,\kappa}^{(2)}(Z_i, W, X)$ replacing $\mathcal{H}_{i,\kappa}^{(1)}(Z_i, W, X)$.

Proof. Taking expectations of (20) across b_{il}, we obtain

$$\begin{aligned}
E[I_{\epsilon,l}^-(B_i|Z_{-i})|Z_{-i}] - E[S_{\epsilon,l}^-(B_i|Z_{-i})|Z_{-i}]\kappa_0 &\leq E[V_{il}|Z_{-i}] \\
\equiv E[V_{il}] &\leq E[I_{\epsilon,l}^+(B_i|Z_{-i})|Z_{-i}] - E[S_{\epsilon,l}^+(B_i|Z_{-i})|Z_{-i}]\kappa_0. \quad (23)
\end{aligned}$$

Then pooling restrictions of the form (23) across Z_{-i}, Z'_{-i} and l, we obtain

$$\Psi_\epsilon^-(Z_{-i}) - \chi_\epsilon^-(Z_{-i})\kappa_0 \leq \Psi_\epsilon^+(Z'_{-i}) - \chi_\epsilon^+(Z'_{-i})\kappa_0 \quad \forall\; Z_{-i}, Z'_{-i} \in \mathcal{Z}_{-i}|Z_i, W, X. \quad (24)$$

□

Set $\mathcal{H}_{i,\kappa}^{(2)}(Z_i, W, X)$ is larger than $\mathcal{H}_{i,\kappa}^{(1)}(Z_i, W, X)$ because first-order stochastic dominance implies inequalities for expectations.

Note two features of $\mathcal{H}_{i,\kappa}^{(2)}(Z_i, W, X)$. First, it can be represented as the intersection of a set of half-spaces in $\mathcal{K}_i$, where half-spaces are bounded by hyperplanes involving slope vectors $(\chi_{\epsilon,l}^-(Z_{-i}) - \chi_{\epsilon,l}^+(Z'_{-i}))$ and intercepts $(\Psi_{\epsilon,l}^-(Z_{-i}) - \Psi_{\epsilon,l}^+(Z'_{-i}))$, and the intersection is taken over collections of $(Z_{-i}, Z'_{-i}, \epsilon, l)$.

Second, if G_i is absolutely continuous, then $\mathcal{H}_{i,\kappa}^{(2)}(Z_i, W, X)$ is a singleton, and as we show below, the analysis of $\mathcal{H}_{i,\kappa}^{(2)}(Z_i, W, X)$ essentially becomes our identification strategy in the case of point identification. Indeed, bidder i's objective function is differentiable at almost every observed b_i. Hence as $\epsilon \to 0$ we will have for all l

$$\lim_{\epsilon \to 0} \frac{\Delta^-_{\epsilon,l} b_i \Gamma_{-i}(b_i|Z_{-i})}{\Delta^-_{\epsilon,l} \Gamma_{-i,l}(b_i|Z_{-i})} = \lim_{\epsilon \to 0} \frac{\Delta^-_{\epsilon,l} b_i \Gamma_{-i}(b_i|Z_{-i})/\epsilon}{\Delta^-_{\epsilon,l} \Gamma_{-i,l}(b_i|Z_{-i})/\epsilon} = \frac{\partial (b_i \Gamma_{-i}(b_i|Z_{-i}))/\partial b_{il}}{d\Gamma_{-i,l}(b_i|Z_{-i}))/db_{il}},$$

and therefore $\Psi^-_\epsilon(\cdot) \to \Psi(\cdot)$. Analogously, it is straightforward to show that $\Psi^+_\epsilon(\cdot) \to \Psi(\cdot)$, $\chi^-_\epsilon \to \chi(\cdot)$, and $\chi^+_\epsilon \to \chi(\cdot)$. Hence the restriction (24) implies

$$\Psi(Z_{-i}) - \chi(Z_{-i})\kappa_0 \leq \Psi(Z'_{-i}) - \chi(Z'_{-i})\kappa_0 \quad \forall\ Z_{-i}, Z'_{-i} \in \mathcal{Z}_{-i}|Z_i, W, X.$$

Noting that Z_{-i}, Z'_{-i} are interchangeable, we thus have for any $Z_{-i}, Z'_{-i} \in \mathcal{Z}_{-i}|Z_i, W, X$:

$$\begin{aligned} \Psi(Z_{-i}) - \chi(Z_{-i})\kappa_0 &\leq \Psi(Z'_{-i}) - \chi(Z'_{-i})\kappa_0 \\ \Psi(Z'_{-i}) - \chi(Z'_{-i})\kappa_0 &\leq \Psi(Z_{-i}) - \chi(Z_{-i})\kappa_0, \end{aligned}$$

or equivalently

$$\Psi(Z_{-i}) - \chi(Z_{-i})\kappa_0 = \Psi(Z'_{-i}) - \chi(Z'_{-i})\kappa_0 \quad \forall\ Z_{-i}, Z'_{-i} \in \mathcal{Z}_{-i}|Z_i, W, X.$$

But this is exactly the identification restriction invoked in Proposition 3 in the current paper. Thus we can strictly generalize our existing identification results (which depend on differentiability a.e.) to partial identification for arbitrary G_i.

Appendix D: Entry

In this Appendix, we formally embed the bidding model we describe above within a two-stage entry-plus-bidding model paralleling those considered by Li and Zhang (2015) and Groeger (2014) among others. This discovery process proceeds as follows.

At the beginning of the game, each bidder i is endowed by nature with a combinatorial valuation vector Y_i drawn by nature from $F_{Y,i}$. However, realizations of Y_i are *ex ante unknown* to i and can be discovered by i only through costly entry. Specifically, at the beginning of Stage 1, each bidder i observes a $2^L \times 1$ vector of private *combinatorial entry costs* C_i, with element C_i^S of C_i describing the total cost i must incur to enter auctions for the set of objects $S \in \mathcal{S}$. This cost vector C_i is drawn from a joint distribution satisfying the following properties:

Assumption 8 (Private Entry Costs)**.** *For each bidder i, C_i is drawn independently of combinatorial preferences Y_i from cost distribution $F_{C,i}$ with support on a compact, convex set $\mathcal{C}_i \subset R^{2^L}$, with C_i private information, $F_{C,i}$ common knowledge, and cost draws independent across bidders: $C_i \perp C_j$ for all i, j.*

Having observed C_i, bidder i chooses a set of auctions $S \in \mathcal{S}$ to enter, pays the corresponding entry cost C_i^S, and proceeds to Stage 2. Next, at the beginning of Stage 2, Bidder i observes the realizations of her combinatorial valuations $Y_i^{S'}$ for all combinations feasible at $\mathcal{E}_i$; that is, for each $S' \in \mathcal{S}$ such that $S' \subset S$. Lastly, bidder i submits a single bid b_{il} for each object l in the entry set S. Conditional on any entry realization $\mathcal{E}$, the bidding subgame proceeds exactly as described in the main text.

Following Milgrom and Weber (1985), define a *distributional entry strategy* for player i as a measure ξ_i over $\mathcal{C}_i \times \mathcal{S}$ whose marginal over $\mathcal{C}_i$ is $F_{C,i}$, with $\xi = (\xi_1, ..., \xi_N)$ a profile of distributional entry strategies. Recall that for each entry structure $\mathcal{E}$ potentially resulting from Stage 1, there exists at least one equilibrium with endogenous tiebreaking in the corresponding Stage 2 bidding subgame; furthermore, since entry costs are sunk at the time of bidding, these equilibria do not depend on the particular Stage 1 entry strategies giving rise to $\mathcal{E}$. For each possible entry structure $\mathcal{E}$, fix a single post-entry equilibrium with endogenous tiebreaking, and let $\Pi(\mathcal{E}) = (\Pi_1(\mathcal{E}), ..., \Pi_N(\mathcal{E}))$ denote expected *ex ante* profits for each bidder in this equilibrium. We seek to characterize an equilibrium in distributional strategies taking these continuation payoffs as given.

Toward this end, consider player i with combinatorial entry cost vector C_i choosing a set of auctions $\mathcal{E}_i \subset \mathcal{L}$ to enter in Stage 1. Let

$$\Xi(S, \xi_{-i}) = E[\Pi_i(S, \mathcal{E}_{-i})|\xi_{-i}]$$

be i's expected net profit from entering auction combination $S \in \mathcal{S}$ given rival entry strategies ξ_{-i} (where the expectation is taken over rival entry sets $\mathcal{E}_{-i}$). We can then write bidder i's Stage 1 problem as:

$$\max_{S \in \mathcal{S}} \Xi(S, \xi_{-i}) - C_i^S.$$

The Stage 1 action set for each bidder is the finite set $\mathcal{S}$, and bidders' private entry costs are independent. Hence by Proposition 1 of Milgrom and Weber (1985) there exists an equilibrium in distributional strategies in the Stage 1 entry game with continuation payoffs described by $\Pi(\cdot)$. Furthermore, by construction, these continuation payoffs arise from play of equilibria with endogenous tiebreaking in every post-entry bidding subgame. Hence there exists an equilibrium with endogenous tiebreaking in the simultaneous first price auction game with costly bidder entry.

Note that the only property of the entry cost distributions $F_{C,1}, ..., F_{C,N}$ we have required thus far is independence − in particular, an equilibrium in distributional strategies exists even when C_i is deterministic. If to this we add the restriction that $F_{C,i}$ is atomless on $\mathcal{C}_i$ for each i, then Proposition 4 of Milgrom and Weber (1985) implies existence of a equilibrium with endogenous tiebreaking in which bidders play pure entry strategies.

Now consider the set of cost vectors C_i at which bidder i chooses to enter set $S \in \mathcal{S}$;

denote this set $\mathcal{C}_i^S$. As usual, this set is simply the affine cone

$$\mathcal{C}_i^S = \{C_i \in \mathbb{R}^{2^L} : C_i^S - C_i^{S'} \leq \Xi(S, \xi_{-i}) - \Xi(S', \xi_{-i}) \, \forall S' \in \mathcal{S}\},$$

which is nonempty for each S. If C_i has full support on $\mathbb{R}^{2^L}$, bidder i will thus enter each auction set $S \in \mathcal{S}$ with positive probability. In other words, equilibrium behavior will involve *variation in participation* by bidder i which is *effectively exogenous* from the perspective of rival bidders. This is precisely the form of variation we exploit in our identification argument.