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"Convergence rate of Euler-Maruyama scheme for McKean-Vlasov SDEs with density-dependent drift"

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CONVERGENCE RATE OF EULER-MARUYAMA SCHEME FOR MCKEAN-VLASOV SDES WITH DENSITY-DEPENDENT DRIFT

ANH-DUNG LE AND STÉPHANE VILLENEUVE

Abstract. In this paper, we study weak well-posedness of a McKean-Vlasov stochastic differential equations (SDEs) whose drift is density-dependent and whose diffusion is constant. The existence part is due to Hölder stability estimates of the associated Euler-Maruyama scheme. The uniqueness part is due to that of the associated Fokker-Planck equation. We also obtain convergence rate in weighted L^1 norm for the Euler-Maruyama scheme.

Keywords: McKean-Vlasov SDEs, density-dependent SDEs, Euler-Maruyama scheme

2020 MSC: 60B10, 60H10

Contents

1. Introduction

We fix $p \in [1, \infty)$ and let $\mathcal{P}_p(\mathbb{R}^d)$ be the space of Borel probability measures on $\mathbb{R}^d$ with finite p-th moment. We endow $\mathcal{P}_p(\mathbb{R}^d)$ with Wasserstein metric W_p. We fix $T \in (0, \infty)$ and let $[0, T]$ be the interval $[0, T]$. We consider a measurable function

$$b : [0, T] \times \mathbb{R}^d \times \mathbb{R}_+ \times \mathcal{P}_p(\mathbb{R}^d) \to \mathbb{R}^d.$$

Let $(B_t, t \geq 0)$ be a d-dimensional Brownian motion and $\mathbb{F} := (\mathcal{F}_t, t \geq 0)$ be an admissible filtration on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. We assume that $(\Omega, \mathcal{A}, \mathbb{F}, \mathbb{P})$ satisfies the usual conditions. In this paper, we consider the SDE

$$\mathrm{d}X_t = b(t, X_t, \ell_t(X_t), \mu_t)\,\mathrm{d}t + \sqrt{2}\,\mathrm{d}B_t, \quad t \in [0, T], \tag{1.1}$$

where the marginal distribution of X_0 is ν, that of X_t is μ_t, and the marginal density of X_t is ℓ_t. The term "McKean-Vlasov" is due to the presence of μ_t while the term "density-dependent" is due to that of $\ell_t(x)$.

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To show the existence of a weak solution, we consider an Euler-Maruyama scheme $X^n := (X^n_t, t \in [0,T])$ constructed as follows. Let $n \in \mathbb{N}$ and $\varepsilon_n := T/n$. Let $t_k := k\varepsilon_n$ for $k \in [\![0,n]\!] := \{0,1,\ldots,n\}$. Let $\tau^n_t := t_k$ if $t \in [t_k, t_{k+1})$ for some $k \in [\![0,n-1]\!]$. For every $t \in [0,T]$, let

$$X^n_t := X_0 + \int_0^t b(s, X^n_{\tau^n_s}, \ell^n_{\tau^n_s}(X^n_{\tau^n_s}), \mu^n_{\tau^n_s}) 1_{(\varepsilon_n, T]}(s) \,\mathrm{d}s + \sqrt{2} B_t, \tag{1.2}$$

where ℓ^n_s, μ^n_s are the density and distribution of X^n_s respectively.

We consider the following set of assumptions:

(A) There exist constants $\alpha \in (0,1), p \in [1,\infty), C > 0$ such that for all $t \in [0,T]; x, y \in \mathbb{R}^d; r, r' \in \mathbb{R}_+$ and $\varrho, \varrho' \in \mathcal{P}_p(\mathbb{R}^d)$:

- (A1) $|b(t,x,r,\varrho)| \le C$.
- (A2) $\nu \in \mathcal{P}_{p+\alpha}(\mathbb{R}^d)$ has a density $\ell_\nu \in C^\alpha_b(\mathbb{R}^d)$.
- (A3) $|b(t,x,r,\varrho) - b(t,x,r',\rho')| \le C\{|r-r'| + W_p(\varrho,\varrho')\}$.

We require $\nu \in \mathcal{P}_{p+\alpha}(\mathbb{R}^d)$ and not just $\nu \in \mathcal{P}_p(\mathbb{R}^d)$. By Markov's inequality, this gives us a direct control (4.2) of the tail of p-moment. We gather parameters in **(A)**:

$$\Theta_1 := (d, T, \alpha, p, C).$$

First, we prove the following estimates about Hölder regularity:

Theorem 1.1. *Let **(A)** hold. There exist constants $c_1 > 0$ (depending on Θ_1) and $c_2 > 0$ (depending on Θ_1, ν) such that for all $0 \le s < t \le T$:*

$$\sup_n \sup_{t\in[0,T]} \|\ell^n_t\|_{C^\alpha_b} \le c_1 \|\ell_\nu\|_{C^\alpha_b}, \tag{1.3}$$

$$\sup_n \|\ell^n_t - \ell^n_s\|_\infty \le c_2 (t-s)^{\frac{\alpha}{2}}, \tag{1.4}$$

$$\sup_n W_p(\mu^n_t, \mu^n_s) \le c_1 (t-s)^{\frac{1}{2}}, \tag{1.5}$$

$$\sup_n \int_{\mathbb{R}^d} (1+|x|^p) |\ell^n_t(x) - \ell^n_s(x)| \,\mathrm{d}x \le c_2 (t-s)^{\frac{\alpha}{2}} s^{-\frac{\alpha}{2}}. \tag{1.6}$$

If we assume, in addition, that $\int_{\mathbb{R}^d} (1+|x|^p)\sqrt{\ell_\nu(x)} \,\mathrm{d}x \le C$, then

$$\sup_n \int_{\mathbb{R}^d} (1+|x|^p) |\ell^n_t(x) - \ell^n_s(x)| \,\mathrm{d}x \le c_2 (t-s)^{\frac{\alpha}{4}}. \tag{1.7}$$

Estimate (1.6) will be used in the proof of Theorem 1.3, but it explodes as $s \downarrow 0$. As in [SH24], we use time cutoff $1_{(\varepsilon_n, T]}$ in (1.2) to tackle this issue. As a direct application of Theorem 1.1, we obtain

Theorem 1.2. *Let **(A)** hold. Then (1.1) has a weak solution whose marginal density satisfies the estimates in Theorem 1.1.*

The main result of the paper is the following rate of convergence:

Theorem 1.3. *Let **(A)** hold and $p = 1$. There exists a constant $c > 0$ (depending on Θ_1, ν) such that for any $n \in \mathbb{N}$:*

$$\sup_{t\in[0,T]} \int_{\mathbb{R}^d} (1+|x|) |\ell^n_t(x) - \ell_t(x)| \,\mathrm{d}x \le c n^{-\frac{\alpha}{2}}.$$

Next we review works related to (1.1) which is a special case of the SDE

$$\mathrm{d}X_t = b(t, X_t, \ell_t(X_t), \mu_t)\,\mathrm{d}t + \sigma(t, X_t, \ell_t(X_t), \mu_t)\,\mathrm{d}B_t, \quad t \in [0, T], \tag{1.8}$$

for some measurable functions

$$\begin{aligned} b &: [0, T] \times \mathbb{R}^d \times \mathbb{R}_+ \times \mathcal{P}_p(\mathbb{R}^d) \to \mathbb{R}^d, \\ \sigma &: [0, T] \times \mathbb{R}^d \times \mathbb{R}_+ \times \mathcal{P}_p(\mathbb{R}^d) \to \mathbb{R}^d \otimes \mathbb{R}^d. \end{aligned}$$

Equation (1.8) is usually studied in two separate cases:

(1) The first case is where $b(t, x, r, \mu) = b(t, x, \mu)$ and $\sigma(t, x, r, \mu) = \sigma(t, x, \mu)$, i.e., there is only distribution-dependence. Then (1.8) is commonly called (classical) McKean-Vlasov SDEs and appears as the limit of various interacting particle systems in biology (see e.g. [NPT10; MT14; Deg18]), mean-filed games (see e.g. [CD+18; Car19]), and data science (see e.g. [MMN18; SS20; RV22]). This form of (1.8) has been extensively studied in various aspects, including well-posedness (see e.g. [MV20; HW19; RZ21]), derivative formulas (see e.g. [HW21; WC23]), long-time behavior (see e.g. [LM22; RW21; Ren23]), and propagation of chaos (see e.g. [Lac23; Szn91; Ern22]).

(2) The second case is where $b(t, x, r, \mu) = b(t, x, r)$ and $\sigma(t, x, r, \mu) = \sigma(t, x, r)$, i.e., there is only density-dependence. Examples of this form are Burgers' equations [CP83; GK83; OK85; Szn86] and (generalized) porous media equations [Ino91; Ben+96; BRR10; BRR11; BR12; BCR13]. Recent works on this form include [BR18; BJ19; BR20; BR21a; BR21b; BR23b; BR23a]. If σ is density-dependent, then the map $r \mapsto r(\sigma\sigma^\top)(t, x, r)$ is usually assumed to satisfy some monotonic condition. For Euler-Maruyama scheme in the special case of constant-diffusion, we refer to [HRZ21; Hao23; WH23; SH24].

There appears to be a shortage of results about SDEs that contain both distribution- and density-dependence. In this direction, we have found only four papers [Wan23; HRZ24; Zha22; Le24]. This scarcity is the motivation for our paper. To our best knowledge, the current paper is the first to study Euler-Maruyama scheme of an SDE that contains both distribution- and density-dependent. While [HRZ21; Hao23; WH23; SH24] obtained convergence rate of marginal density in L^1 norm, we obtain that in (stronger) weighted L^1 norm where the weight is $x \mapsto 1 + |x|$.

The paper is organized as follows. In Section 2, we recall definition of Wasserstein space and heat kernel estimates; prove some crude estimates about the scheme; and establish Duhamel representation of marginal density. We prove our results in Section 3, Section 4, and Section 5, respectively.

We recall two notions of a solution of (1.1):

Definition 1.4.

(1) A *strong solution* to (1.1) is a continuous $\mathbb{R}^d$-valued $\mathbb{F}$-adapted process $(X_t, t \in [0, T])$ on $(\Omega, \mathcal{A}, \mathbb{P})$ such that for each $t \in [0, T]$: the distribution of X_t is $\mu_t \in \mathcal{P}_p(\mathbb{R}^d)$ and admits a density ℓ_t; and

$$X_t = X_0 + \int_0^t b(s, X_s, \ell_s(X_s), \mu_s)\,\mathrm{d}s + B_t \quad \mathbb{P}\text{-a.s.} \tag{1.9}$$

(2) A *weak solution* to (1.1) is a continuous $\mathbb{R}^d$-valued process $(X_t, t \in [0, T])$ on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$ where there exist some d-dimensional Brownian motion $(B_t, t \geq 0)$ and some admissible filtration $\mathbb{F} := (\mathcal{F}_t, t \geq 0)$ such that $(X_t, t \in [0, T])$ is $\mathbb{F}$-adapted, its marginal distribution admits a density, and (1.9) holds.

(3) Equation (1.1) has *strong uniqueness* if any two strong solutions on a given probability space $(\Omega, \mathcal{A}, \mathbb{P})$ for a given Brownian motion and a given admissible filtration coincide

$\mathbb{P}$-a.s. on the path space $C([0,T];\mathbb{R}^d)$. Equation (1.1) has *weak uniqueness* if any weak solution induces the same distribution on $C([0,T];\mathbb{R}^d)$.

Throughout the paper, we use the following conventions:

(1) We denote by ∇, ∇^2, Δ the gradient, the Hessian and the Laplacian with respect to (w.r.t) the spatial variable. We denote by ∂_t (or ∂_s) the derivative w.r.t time. For $\alpha \in (0,1]$ and a function $f : \mathbb{R}^d \to \mathbb{R}$, let $[f]_\alpha := \sup_{x \neq y} \frac{|f(x)-f(y)|}{|x-y|^\alpha}$ be the α-Hölder constant of f. For $\alpha \in (0,1)$, let $C_b^\alpha(\mathbb{R}^d)$ be the Hölder space of functions $f : \mathbb{R}^d \to \mathbb{R}$ with the norm $\|f\|_{C_b^\alpha} := \|f\|_\infty + [f]_\alpha$.

(2) Let $L^0(\mathbb{R}^d)$ be the space of real-valued measurable functions on $\mathbb{R}^d$. Let $L^0_+(\mathbb{R}^d)$ be the subset of $L^0(\mathbb{R}^d)$ that consists of non-negative functions. Let $L^0_b(\mathbb{R}^d)$ be the subset of $L^0(\mathbb{R}^d)$ that consists of bounded functions. We denote by I_d the identity matrix in $\mathbb{R}^d \otimes \mathbb{R}^d$. We denote by $\lceil r \rceil$ the smallest integer greater than or equal to $r \in \mathbb{R}$.

(3) For brevity, we write ∞ for $+\infty$. Let $\mathbb{N}$ be the set of strictly positive natural numbers. Let $\mathbb{R}_+ := \{x \in \mathbb{R} : x \geq 0\}$. For natural numbers m, n with $m \leq n$, we denote $[\![m, n]\!] := \{m, m+1, \ldots, n\}$. We denote by $\mathcal{B}(\mathbb{R}^d)$ the Borel σ-algebra on $\mathbb{R}^d$. Let $\mathcal{P}(\mathbb{R}^d)$ be the space of Borel probability measures on $\mathbb{R}^d$. We denote by $M_p(\varrho)$ the p-th moment of $\varrho \in \mathcal{P}(\mathbb{R}^d)$, i.e.,
$$M_p(\varrho) := \int_{\mathbb{R}^d} |x|^p \, d\varrho(x).$$

2. Preliminaries

2.1. **Wasserstein space** $(\mathcal{P}_p(\mathbb{R}^d), W_p)$**.** Let $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$. The set of *transport plans* (or *couplings*) between μ and ν is defined by
$$\Gamma(\mu,\nu) := \{\varrho \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d) : \mu = \pi^1_\sharp \varrho \text{ and } \nu = \pi^2_\sharp \varrho\}.$$

Above, π^i is the projection of $\mathbb{R}^d \times \mathbb{R}^d$ onto its i-th coordinate, and $\pi^i_\sharp \varrho \in \mathcal{P}(\mathbb{R}^d)$ is the push-forward measure of ϱ through π^i , i.e., $\{\pi^i_\sharp \varrho\}(A) = \varrho(\{\pi^i\}^{-1}(A))$ for all $A \in \mathcal{B}(\mathbb{R}^d)$. For $\mu, \nu \in \mathcal{P}_p(\mathbb{R}^d)$, we define
$$W_p(\mu,\nu) := \inf_{\varrho \in \Gamma(\mu,\nu)} \left\{ \int_{\mathbb{R}^d} |x-y|^p \, d\varrho(x,y) \right\}^{1/p}.$$

By [Vil09, Theorem 6.18], $(\mathcal{P}_p(\mathbb{R}^d), W_p)$ is a Polish space. For more information about Wasserstein space, we refer to [Vil03; Vil09; AG13; San15; FG21; ABS21; Mag23].

2.2. **The scheme is well-defined.** First, we recall the freezing lemma. For the sake of completeness, we include its proof in the Appendix.

Lemma 2.1. *Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and $\mathcal{D}, \mathcal{G}$ independent sub-σ-algebras of $\mathcal{A}$. Let $(E, \mathcal{E})$ be a measurable space and $Y : \Omega \to E$ measurable w.r.t $\mathcal{D}$. Assume that $\varphi : E \times \Omega \to \mathbb{R}^d$ is measurable w.r.t $\mathcal{E} \otimes \mathcal{G}$ and that $\varphi(Y, \cdot)$ is integrable.*

(1) Let $N := \{x \in E : \varphi(x, \cdot) \text{ not integrable}\}$. Let μ be the distribution of Y on $(E, \mathcal{E})$. Then $N \in \mathcal{E}$ and $\mu(N) = 0$.

(2) We define $\Phi : E \to \mathbb{R}^d$ by $\Phi(y) := 0$ for $y \in N$ and $\Phi(y) := \mathbb{E}[\varphi(y, \cdot)]$ for $y \in E \setminus N$. Then Φ is measurable and $\mathbb{E}[\varphi(Y, \cdot)|\mathcal{D}] = \Phi(Y)$.

We define $b^n : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ by

$$b^n(t,x) := b(t, x, \ell^n_{\tau^n_t}(x), \mu^n_{\tau^n_t}) 1_{(\varepsilon_n, T]}(t). \tag{2.1}$$

Then X^n satisfies the SDE

$$\mathrm{d}X^n_t = b^n(t, X^n_{\tau^n_t})\,\mathrm{d}t + \sqrt{2}\,\mathrm{d}B_t. \tag{2.2}$$

Second, we verify that (1.2) is well-defined.

Lemma 2.2. *Let **(A)** hold. Then each X^n_t in (1.2) has distribution $\mu^n_t \in \mathcal{P}_p(\mathbb{R}^d)$ and admits a density.*

Proof. We fix $t \in (t_k, t_{k+1}]$ for some $k \in [\![0, n-1]\!]$. By induction argument, we assume WLOG that the statement holds for t_k. We have

$$X^n_t = X^n_{t_k} + \int_{t_k}^t b^n(s, X^n_{t_k})\,\mathrm{d}s + \sqrt{2}(B_t - B_{t_k}).$$

It follows from $M_p(\mu^n_{t_k}) + \|b^n\|_\infty < \infty$ that $M_p(\mu^n_t) < \infty$. It remains to prove that X^n_t admits a density. Let Σ^1_t be the σ-algebra generated by $(B_s - B_r, t_k \le r < s \le t)$. Let Σ^2_t be the σ-algebra generated by $X^n_{t_k}$. We define $F_t : \mathbb{R}^d \times \Omega \to \mathbb{R}^d$ by

$$F_t(x, \cdot) := x + \int_{t_k}^t b^n(s,x)\,\mathrm{d}s + \sqrt{2}(B_t - B_{t_k}).$$

Because b^n is bounded, F_t is well-defined and measurable w.r.t $\mathcal{B}(\mathbb{R}^d) \otimes \Sigma^1_t$. First, $X^n_t = F_t(X^n_{t_k}, \cdot)$. Second, $F_t(x,\cdot)$ has a non-degenerate normal distribution. We fix a Lebesgue-null set $A \in \mathcal{B}(\mathbb{R}^d)$. We need to prove $\mathbb{P}[X^n_t \in A] = 0$. We define $\bar{F}_t : \mathbb{R}^d \to \mathbb{R}^d$ by $\bar{F}_t(x) := \mathbb{P}[F_t(x,\cdot) \in A]$. Then $\bar{F}_t = 0$. Because $\mathbb{F}$ is admissible, Σ^1_t is independent of Σ^2_t. We have

$$\begin{aligned}
\mathbb{P}[X^n_t \in A] &= \mathbb{P}[F_t(X^n_{t_k}, \cdot) \in A] \\
&= \mathbb{E}[\mathbb{P}[F_t(X^n_{t_k}, \cdot) \in A | \Sigma^2_t]] \\
&= \mathbb{E}[\bar{F}_t(X^n_{t_k})] \quad \text{by Lemma 2.1} \\
&= 0.
\end{aligned}$$

This completes the proof. ■

2.3. **Heat kernel estimates.** We consider the standard Gaussian heat kernel defined for all $t > 0$ and $x \in \mathbb{R}^d$ by

$$p_t(x) := \frac{1}{(4\pi t)^{\frac{d}{2}}} \exp\left(-\frac{|x|^2}{4t}\right).$$

Then

$$p_t(x) \le 2^{\frac{d}{2}} p_{2t}(x), \tag{2.3}$$

$$p_t(x+y) \le 2^{\frac{d}{2}} \exp\left(\frac{|y|^2}{4t}\right) p_{2t}(x). \tag{2.4}$$

The following estimates are classical. See e.g. [HRZ21, Lemma 2.1] for their proofs.

Lemma 2.3. *(1) There exists a constant $c > 0$ (depending on d) such that for all $t > 0$ and $x, y \in \mathbb{R}^d$:*

$$|\nabla p_t(x)| \le c t^{-\frac{1}{2}} p_t(x). \tag{2.5}$$

(2) For $\alpha \in (0,1)$, there exists a constant $c > 0$ (depending on d, T, α) such that for all $0 < t \le T; x, y \in \mathbb{R}^d$ and $i \in \{0,1\}$:

$$|\nabla^i p_t(x) - \nabla^i p_t(y)| \le c|x-y|^\alpha t^{-\frac{i+\alpha}{2}} \{p_{4t}(x) + p_{4t}(y)\}. \tag{2.6}$$

(3) For $\alpha \in (0,1)$, there exists a constant $c > 0$ (depending on d, T, α) such that for all $0 \le s < t \le T; x \in \mathbb{R}^d$ and $i \in \{0,1\}$:

$$|\nabla^i p_t(x) - \nabla^i p_s(x)| \le c|t-s|^{\frac{\alpha}{2}} \{t^{-\frac{i+\alpha}{2}} p_{2t}(x) + s^{-\frac{i+\alpha}{2}} p_{2s}(x)\}. \tag{2.7}$$

The associated semigroup $(P_t, t > 0)$ is defined for all $x \in \mathbb{R}^d$ and $f \in L^0_+(\mathbb{R}^d) \cup L^0_b(\mathbb{R}^d)$ by

$$P_t f(x) := \int_{\mathbb{R}^d} p_t(x-y) f(y) \, \mathrm{d}y.$$

Finally, we prove some crude estimates.

Lemma 2.4. *Let (A) hold. Let $t \in (t_k, t_{k+1}]$ for some $k \in [\![0, n-1]\!]$. There exists a constant $c \ge 1$ (depending on Θ_1) such that for every $\varphi \in L^0_+(\mathbb{R}^d \times \mathbb{R}^d) \cup L^0_b(\mathbb{R}^d \times \mathbb{R}^d)$:*

$$\mathbb{E}[\varphi(X^n_{t_k}, X^n_t)] = \int_{\mathbb{R}^d \times \mathbb{R}^d} \varphi(x,y) \ell^n_{t_k}(x) p_{t-t_k}\left(x - y + \int_{t_k}^t b^n(s,x) \, \mathrm{d}s\right) \mathrm{d}x \, \mathrm{d}y \tag{2.8}$$

$$\le c \int_{\mathbb{R}^d \times \mathbb{R}^d} \varphi(x,y) \ell^n_{t_k}(x) p_{2(t-t_k)}(x-y) \, \mathrm{d}x \, \mathrm{d}y, \tag{2.9}$$

$$\|\ell^n_t\|_\infty \le c \|\ell^n_{t_k}\|_\infty. \tag{2.10}$$

Proof. Let Σ^1_t, Σ^2_t and F_t be defined as in the proof of Lemma 2.2. We have

$$\begin{aligned}\mathbb{E}[\varphi(X^n_{t_k}, X^n_t)] &= \mathbb{E}[\varphi(X^n_{t_k}, F_t(X^n_{t_k}, \cdot))] \\ &= \mathbb{E}[\mathbb{E}[\varphi(X^n_{t_k}, F_t(X^n_{t_k}, \cdot)) | \Sigma^2_t]]. \end{aligned} \tag{2.11}$$

We define $\hat{F}_t : \mathbb{R}^d \to \mathbb{R}^d$ by $\hat{F}_t(x) := \mathbb{E}[\varphi(x, F_t(x, \cdot))]$. Notice that $F_t(x, \cdot)$ has a normal distribution with mean $x + \int_{t_k}^t b^n(s,x) \, \mathrm{d}s$ and covariance matrix $2tI_d$. Consequently, the density of $F_t(x, \cdot)$ is $y \mapsto p_{t-t_k}(x - y + \int_{t_k}^t b^n(s,x) \, \mathrm{d}s)$. There exists a constant $c_1 \ge 1$ (depending on Θ_1) such that

$$\hat{F}_t(x) = \int_{\mathbb{R}^d} \varphi(x,y) p_{t-t_k}\left(x - y + \int_{t_k}^t b^n(s,x) \, \mathrm{d}s\right) \mathrm{d}y \tag{2.12}$$

$$\le c_1 \int_{\mathbb{R}^d} \varphi(x,y) p_{2(t-t_k)}(x-y) \, \mathrm{d}y \quad \text{by (2.4) and (A1)}. \tag{2.13}$$

WLOG, we assume $\varphi \in L^0_b(\mathbb{R}^d \times \mathbb{R}^d)$. Then

$$\mathbb{E}[\varphi(X^n_{t_k}, X^n_t)] = \mathbb{E}[\hat{F}_t(X^n_{t_k})] \quad \text{by (2.11) and Lemma 2.1}$$

$$= \int_{\mathbb{R}^d} \ell^n_{t_k}(x) \hat{F}_t(x) \, \mathrm{d}x \tag{2.14}$$

$$\le c_1 \int_{\mathbb{R}^d \times \mathbb{R}^d} \varphi(x,y) \ell^n_{t_k}(x) p_{2(t-t_k)}(x-y) \, \mathrm{d}x \, \mathrm{d}y \quad \text{by (2.13)}. \tag{2.15}$$

Thus (2.9) follows. Clearly, (2.12) and (2.14) imply (2.8). The case $\varphi \in L^0_+(\mathbb{R}^d \times \mathbb{R}^d)$ follows from a truncated argument and monotone convergence theorem. For every $f \in L^0_+(\mathbb{R}^d)$,

$$\begin{aligned}\mathbb{E}[f(X^n_t)] &\le c_1 \int_{\mathbb{R}^d \times \mathbb{R}^d} f(y) \ell^n_{t_k}(x) p_{2(t-t_k)}(x-y) \, \mathrm{d}x \, \mathrm{d}y \quad \text{by (2.15)} \\ &\le c_1 \|\ell^n_{t_k}\|_\infty \int_{\mathbb{R}^d \times \mathbb{R}^d} f(y) p_{2(t-t_k)}(x-y) \, \mathrm{d}x \, \mathrm{d}y\end{aligned}$$

$$= c_1 \|\ell^n_{t_k}\|_\infty \|f\|_{L^1}.$$

By duality, $\|\ell^n_t\|_\infty \le c_1 \|\ell^n_{t_k}\|_\infty$. This completes the proof. ∎

2.4. **Duhamel representation.**

Lemma 2.5. *Let (A) hold. It holds for every $x \in \mathbb{R}^d$ that*

$$\ell^n_t(x) = P_t \ell_\nu(x) + \int_0^t \mathbb{E}[\langle b^n(s, X^n_{\tau^n_s}), \nabla p_{t-s}(X^n_s - x)\rangle] \,\mathrm{d}s, \tag{2.16}$$

$$\ell_t(x) = P_t \ell_\nu(x) + \int_0^t \mathbb{E}[\langle b(s, X_s, \ell_s(X_s), \mu_s), \nabla p_{t-s}(X_s - x)\rangle] \,\mathrm{d}s. \tag{2.17}$$

Proof. Let's prove (2.16). We fix $t \in (0, T]$ and $f \in C^\infty_c(\mathbb{R}^d)$. We define

$$g : [0, t] \times \mathbb{R}^d \to \mathbb{R},\ (s, y) \mapsto (P_{t-s} f)(y).$$

Then $(\partial_s + \Delta) g = 0$. We have

$$\begin{aligned} \mathrm{d}g(s, X^n_s) &= \langle b^n(s, X^n_{\tau^n_s}), \nabla g(s, X^n_s)\rangle \,\mathrm{d}s \qquad \text{by Itô's lemma} \\ &\quad + (\partial_s + \Delta) g(s, X^n_s) \,\mathrm{d}s + \mathrm{d}M_s \\ &= \langle b^n(s, X^n_{\tau^n_s}), \nabla g(s, X^n_s)\rangle \,\mathrm{d}s + \mathrm{d}M_s. \end{aligned}$$

Above, $M_0 = 0$ and $\mathrm{d}M_s = \{\nabla g(s, X^n_s)\}^\top \,\mathrm{d}B_s$. Then

$$f(X^n_t) = g(0, X_0) + \int_0^t \langle b^n(s, X^n_{\tau^n_s}), \mu^n_{\tau^n_s}), \nabla g(s, X^n_s)\rangle \,\mathrm{d}s + M_t.$$

To apply Fubini's theorem, we next verify $\sup_{s\in[0,T]} \|\nabla g(s, \cdot)\|_\infty < \infty$. It suffices to prove that the Lipschitz constant of $g(s, \cdot)$ is bounded uniformly in time. We consider the Gaussian process governed by the SDE

$$\mathrm{d}Y^x_{s,t} = \sqrt{2}\,\mathrm{d}B_t, \quad t \in [s, T], Y^x_{s,s} = x.$$

We have

$$\begin{aligned} |g(s, x) - g(s, y)| &= |\mathbb{E}[f(Y^x_{s,t})] - \mathbb{E}[f(Y^y_{s,t})]| \\ &\le \|\nabla f\|_\infty \mathbb{E}[|Y^x_{s,t} - Y^y_{s,t}|] \\ &\le c_1 \|\nabla f\|_\infty |x - y|. \end{aligned}$$

Above, the constant $c_1 > 0$ is given by [HW22, Theorem 1.1(2)]. By Fubini's theorem,

$$\begin{aligned} \int_{\mathbb{R}^d} f(x) \ell^n_t(x) \,\mathrm{d}x &= \mathbb{E}[g(0, X_0)] + \int_0^t \mathbb{E}[\langle b^n(s, X^n_{\tau^n_s}), \nabla g(s, X^n_s))\rangle] \,\mathrm{d}s \\ &=: I_1 + I_2. \end{aligned}$$

By Leibniz integral rule,

$$\nabla g(s, y) = \nabla_y \int_{\mathbb{R}^d} p_{t-s}(y - x) f(x) \,\mathrm{d}x = \int_{\mathbb{R}^d} \nabla p_{t-s}(y - x) f(x) \,\mathrm{d}x.$$

We have

$$\begin{aligned} I_1 &= \int_{\mathbb{R}^d} \ell_\nu(y) \{P_t f\}(y) \,\mathrm{d}y \\ &= \int_{\mathbb{R}^d} f(x) \left(\int_{\mathbb{R}^d} p_t(y - x) \ell_\nu(y) \,\mathrm{d}y \right) \mathrm{d}x \\ &= \int_{\mathbb{R}^d} f(x) \{P_t \ell_\nu\}(x) \,\mathrm{d}x, \end{aligned}$$

$$\begin{aligned} I_2 &= \int_0^t \mathbb{E}[\langle b^n(s, X^n_{\tau^n_s}), \int_{\mathbb{R}^d} f(x) \nabla p_{t-s}(X^n_s - x) \, \mathrm{d}x \rangle] \, \mathrm{d}s \\ &= \int_{\mathbb{R}^d} f(x) \int_0^t \mathbb{E}[\langle b^n(s, X^n_{\tau^n_s}), \nabla p_{t-s}(X^n_s - x) \rangle] \, \mathrm{d}s \, \mathrm{d}x. \end{aligned}$$

Then (2.16) follows. The proof of (2.17) is a straightforward modification of above reasoning. ■

For brevity, we adopt the following notation in the remaining of the paper:

(1) We write $M_1 \lesssim M_2$ if there exists a constant $c > 0$ (depending on Θ_1) such that $M_1 \le c M_2$.

(2) We write $M_1 \preccurlyeq M_2$ if there exists a constant $c > 0$ (depending on Θ_1, ν) such that $M_1 \le c M_2$.

3. Stability estimates of the scheme

This section is devoted to the proof of Theorem 1.1. We utilize techniques from [Wan23; HRZ21]. By (1.2),

$$X^n_t - X^n_s = \int_s^t b^n(r, X^n_{\tau^n_r}) \, \mathrm{d}r + \sqrt{2}(B_t - B_s).$$

Then

$$\mathbb{E}[|X^n_t - X^n_s|^p] \lesssim (t-s)^p + \mathbb{E}[|B_{t-s}|^p] \lesssim (t-s)^{\frac{p}{2}}.$$

Thus (1.5) follows.

3.1. **Bound supremum norm of marginal density.** By Lemma 2.5 and (A1), there exists a constant $c_1 > 0$ (depending on Θ_1) such that

$$\begin{aligned} \ell^n_t(x) &\le \|\ell_\nu\|_\infty + c_1 \int_0^t \mathbb{E}[|\nabla p_{t-s}(X^n_s - x)|] \, \mathrm{d}s \\ &=: \|\ell_\nu\|_\infty + c_1 \int_0^t E_s \, \mathrm{d}s. \end{aligned} \tag{3.1}$$

Step 1: We are going to prove the existence of $m \in [\![1, n]\!]$ such that $\frac{m}{n}$ depends only on Θ_1 and that $\|\ell^n_{t_k}\|_\infty \le 2\|\ell_\nu\|_\infty$ for every $k \in [\![0, m]\!]$. The idea is to pick m such that the following induction argument is valid. We fix $k \in [\![1, m]\!]$ and assume that $\|\ell^n_{t_i}\|_\infty \le 2\|\ell_\nu\|_\infty$ for every $i \in [\![0, k-1]\!]$. Let $s \in (t_i, t_{i+1})$ for some $i \in [\![0, k-1]\!]$. We have

$$\begin{aligned} E_s &\lesssim \int_{\mathbb{R}^d \times \mathbb{R}^d} \ell^n_{t_i}(y) p_{2(s-t_i)}(y-z) |\nabla p_{t_k - s}(x-z)| \, \mathrm{d}y \, \mathrm{d}z \quad \text{by (2.9)} \\ &\lesssim \frac{1}{\sqrt{t_k - s}} \int_{\mathbb{R}^d \times \mathbb{R}^d} \ell^n_{t_i}(y) p_{2(s-t_i)}(y-z) p_{t_k - s}(x-z) \, \mathrm{d}y \, \mathrm{d}z \quad \text{by (2.5)} \\ &\lesssim \frac{1}{\sqrt{t_k - s}} \int_{\mathbb{R}^d \times \mathbb{R}^d} \ell^n_{t_i}(y) p_{2(s-t_i)}(y-z) p_{2(t_k - s)}(x-z) \, \mathrm{d}z \, \mathrm{d}y \quad \text{by (2.3)} \\ &= \frac{1}{\sqrt{t_k - s}} \int_{\mathbb{R}^d} \ell^n_{t_i}(y) p_{2(t_k - t_i)}(x-y) \, \mathrm{d}y \quad \text{by Chapman–Kolmogorov equation} \\ &\le \frac{2\|\ell_\nu\|_\infty}{\sqrt{t_k - s}} \quad \text{by inductive hypothesis.} \end{aligned} \tag{3.2}$$

By (3.1) and (3.2), there exists a constant $c_2 > 0$ (depending on Θ_1) such that

$$\begin{aligned} \ell^n_{t_k}(x) &\le \|\ell_\nu\|_\infty \left[1 + c_1 c_2 \int_0^{t_k} \frac{\mathrm{d}s}{\sqrt{t_k - s}} \right] \\ &= \|\ell_\nu\|_\infty (1 + 2 c_1 c_2 \sqrt{t_k}). \end{aligned}$$

It suffices to choose m such that

$$c_1 c_2 \sqrt{t_k} \le \frac{1}{2}. \tag{3.3}$$

Clearly, there exists a constant $c_3 \in (0,1)$ (depending on Θ_1) such that if $\frac{m}{n} \le c_3$ then (3.3) holds.

Step 2: Repeating **Step 1** at most $\lceil 1/c_3 \rceil$ times, we have $\|\ell^n_{t_k}\|_\infty \le 2^{\lceil 1/c_3 \rceil}\|\ell_\nu\|_\infty$ for every $k \in [\![0, n]\!]$.

Step 3: By (2.10), there exists a constant $c_4 \ge 1$ (depending on Θ_1) such that if $t \in (t_k, t_{k+1})$ then $\|\ell^n_t\|_\infty \le c_4 \|\ell^n_{t_k}\|_\infty$. Thus

$$\sup_{t\in[0,T]} \|\ell^n_t\|_\infty \le c_4 2^{\lceil 1/c_3 \rceil}\|\ell_\nu\|_\infty. \tag{3.4}$$

3.2. **Hölder continuity in space.** We fix $x, x' \in \mathbb{R}^d$. By Lemma 2.5 and (A1),

$$\begin{aligned} |\ell^n_t(x) - \ell^n_t(x')| &\lesssim |P_t\ell_\nu(x) - P_t\ell_\nu(x')| \\ &\qquad + \int_0^t \mathbb{E}[|\nabla p_{t-s}(X^n_s - x) - \nabla p_{t-s}(X^n_s - x')|]\,\mathrm{d}s \\ &=: I_1 + \int_0^t I^n_2(s)\,\mathrm{d}s. \end{aligned} \tag{3.5}$$

We consider the Gaussian process governed by the SDE

$$\mathrm{d}Y^x_t = \sqrt{2}\,\mathrm{d}B_t, \quad t \in [0,T], Y^x_0 = x.$$

By [HW22, Theorem 1.1(2)],

$$\mathbb{E}[\sup_{t\in[0,T]} |Y^x_t - Y^{x'}_t|] \lesssim |x - x'|. \tag{3.6}$$

On the other hand,

$$\begin{aligned} I_1 &= |\mathbb{E}[\ell_\nu(Y^x_t)] - \mathbb{E}[\ell_\nu(Y^{x'}_t)]| \\ &\le [\ell_\nu]_\alpha \mathbb{E}[|Y^x_t - Y^{x'}_t|^\alpha] \\ &\le [\ell_\nu]_\alpha (\mathbb{E}[|Y^x_t - Y^{x'}_t|])^\alpha \quad \text{by Jensen's inequality} \\ &\lesssim [\ell_\nu]_\alpha |x - x'|^\alpha \quad \text{by (3.6)}. \end{aligned} \tag{3.7}$$

Next we bound $I^n_2(s)$. We have

$$\begin{aligned} I^n_2(s) &\lesssim s^{-\frac{1+\alpha}{2}} |x - x'|^\alpha \mathbb{E}[p_{4(t-s)}(X^n_s - x) + p_{4(t-s)}(X^n_s - x')] \quad \text{by (2.6)} \\ &= s^{-\frac{1+\alpha}{2}} |x - x'|^\alpha \int_{\mathbb{R}^d} \ell^n_s(y)\{p_{4(t-s)}(y - x) + p_{4(t-s)}(y - x')\}\,\mathrm{d}y \\ &\lesssim \|\ell_\nu\|_\infty s^{-\frac{1+\alpha}{2}} |x - x'|^\alpha \quad \text{by (3.4)}. \end{aligned} \tag{3.8}$$

By (3.5), (3.7) and (3.8),

$$\begin{aligned} \frac{|\ell^n_t(x) - \ell^n_t(x')|}{|x - x'|^\alpha} &\lesssim [\ell_\nu]_\alpha + \|\ell_\nu\|_\infty \int_0^t s^{-\frac{1+\alpha}{2}}\,\mathrm{d}s \\ &\lesssim \|\ell_\nu\|_{C^\alpha_b} \quad \text{because } \alpha \in (0,1). \end{aligned} \tag{3.9}$$

Then (3.4) and (3.9) imply (1.3).

3.3. **Hölder continuity in time.** We fix $x \in \mathbb{R}^d$ and $0 \le s < t \le T$.

Step 1: we will prove (1.4). By Lemma 2.5,

$$\begin{aligned} \ell_t^n(x) &= P_t \ell_\nu(x) + \int_0^t \mathbb{E}[\langle b^n(r, X^n_{\tau^n_r}), \nabla p_{t-r}(X_r^n - x)\rangle] \,\mathrm{d}r, \\ \ell_s^n(x) &= P_s \ell_\nu(x) + \int_0^s \mathbb{E}[\langle b^n(r, X^n_{\tau^n_r}), \nabla p_{s-r}(X_r^n - x)\rangle] \,\mathrm{d}r. \end{aligned} \tag{3.10}$$

By (3.10) and (A1),

$$\begin{aligned} |\ell_t^n(x) - \ell_s^n(x)| &\lesssim |P_t \ell_\nu(x) - P_s \ell_\nu(x)| \\ &\qquad + \int_0^s \mathbb{E}[|\nabla p_{t-r}(X_r^n - x) - \nabla p_{s-r}(X_r^n - x)|] \,\mathrm{d}r \\ &\qquad + \int_s^t \mathbb{E}[|\nabla p_{t-r}(X_r^n - x)|] \,\mathrm{d}r \\ &=: I_1(x) + I_2^n(x) + I_3^n(x). \end{aligned} \tag{3.11}$$

It holds for every $i \in \{0, 1\}$ that

$$\begin{aligned} &\nabla^i p_{t-r}(y-x) - \nabla^i p_{s-r}(y-x) \\ &= \nabla_y^i \int_{\mathbb{R}^d} p_{s-r}(y-z) p_{t-s}(z-x) \,\mathrm{d}z - \nabla^i p_{s-r}(y-x) \\ &= \nabla_y^i \int_{\mathbb{R}^d} p_{s-r}(y-z) p_{t-s}(z-x) \,\mathrm{d}z - \nabla_y^i \int_{\mathbb{R}^d} p_{s-r}(y-x) p_{t-s}(z-x) \,\mathrm{d}z \\ &= \int_{\mathbb{R}^d} \{\nabla^i p_{s-r}(y-z) - \nabla^i p_{s-r}(y-x)\} p_{t-s}(z-x) \,\mathrm{d}z. \end{aligned} \tag{3.12}$$

First,

$$\begin{aligned} I_1(x) &= \left| \int_{\mathbb{R}^d} \ell_\nu(y) \{p_t(y-x) - p_s(y-x)\} \,\mathrm{d}y \right| \\ &= \left| \int_{\mathbb{R}^d} \left[\int_{\mathbb{R}^d} \ell_\nu(y) \{p_s(y-z) - p_s(y-x)\} \,\mathrm{d}y \right] p_{t-s}(z-x) \,\mathrm{d}z \right| \quad \text{by (3.12)} \\ &\le \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} \ell_\nu(y) \{p_s(y-z) - p_s(y-x)\} \,\mathrm{d}y \right| p_{t-s}(z-x) \,\mathrm{d}z \\ &= \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} p_s(y) \{\ell_\nu(y+z) - \ell_\nu(y+x)\} \,\mathrm{d}y \right| p_{t-s}(z-x) \,\mathrm{d}z \\ &\preccurlyeq \int_{\mathbb{R}^d} |z-x|^\alpha p_{t-s}(z-x) \,\mathrm{d}z \quad \text{because } \ell_\nu \in C_b^\alpha(\mathbb{R}^d) \\ &\lesssim (t-s)^{\frac{\alpha}{2}}. \end{aligned} \tag{3.13}$$

Second,

$$\begin{aligned} I_2^n(x) &= \int_0^s \mathbb{E}\left[\left| \int_{\mathbb{R}^d} \{\nabla p_{s-r}(X_r^n - z) - \nabla p_{s-r}(X_r^n - x)\} p_{t-s}(z-x) \,\mathrm{d}z \right| \right] \mathrm{d}r \quad \text{by (3.12)} \\ &\le \int_0^s \int_{\mathbb{R}^d} \mathbb{E}[|\nabla p_{s-r}(X_r^n - z) - \nabla p_{s-r}(X_r^n - x)|] p_{t-s}(z-x) \,\mathrm{d}z \,\mathrm{d}r \\ &\lesssim \int_0^s (s-r)^{-\frac{1+\alpha}{2}} \int_{\mathbb{R}^d} |z-x|^\alpha \mathbb{E}[p_{4(s-r)}(X_r^n - z)] p_{t-s}(z-x) \,\mathrm{d}z \,\mathrm{d}r \qquad \text{by (2.6)} \\ &\qquad + \int_0^s (s-r)^{-\frac{1+\alpha}{2}} \int_{\mathbb{R}^d} |z-x|^\alpha \mathbb{E}[p_{4(s-r)}(X_r^n - x)] p_{t-s}(z-x) \,\mathrm{d}z \,\mathrm{d}r \\ &=: I_{21}^n(x) + I_{22}^n(x). \end{aligned}$$

We have

$$
\begin{aligned}
I_{21}^n(x) &= \int_0^s (s-r)^{-\frac{1+\alpha}{2}} \int_{\mathbb{R}^d\times\mathbb{R}^d} |z-x|^\alpha \ell_s^n(y) p_{4(s-r)}(y-z) p_{t-s}(z-x)\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}r \\
&\preccurlyeq \int_0^s (s-r)^{-\frac{1+\alpha}{2}} \int_{\mathbb{R}^d\times\mathbb{R}^d} |z-x|^\alpha p_{4(s-r)}(y-z) p_{t-s}(z-x)\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}r \quad \text{by (3.4)} \\
&\lesssim \int_0^s (s-r)^{-\frac{1+\alpha}{2}} \int_{\mathbb{R}^d\times\mathbb{R}^d} |z-x|^\alpha p_{4(s-r)}(y-z) p_{4(t-s)}(z-x)\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}r \quad \text{by (2.3)} \\
&= \int_0^s (s-r)^{-\frac{1+\alpha}{2}} \int_{\mathbb{R}^d} |z-x|^\alpha p_{4(t-s)}(z-x)\,\mathrm{d}z\,\mathrm{d}r \\
&\lesssim (t-s)^{\frac{\alpha}{2}} \int_0^s (s-r)^{-\frac{1+\alpha}{2}}\,\mathrm{d}r. \qquad (3.14)
\end{aligned}
$$

Similarly,

$$
I_{22}^n(x) \preccurlyeq (t-s)^{\frac{\alpha}{2}} \int_0^s (s-r)^{-\frac{1+\alpha}{2}}\,\mathrm{d}r. \qquad (3.15)
$$

Third,

$$
\begin{aligned}
I_3^n(x) &= \int_s^t \int_{\mathbb{R}^d} \ell_r^n(y) |\nabla p_{t-r}(y-x)|\,\mathrm{d}y\,\mathrm{d}r \\
&\lesssim \int_s^t \frac{1}{\sqrt{t-r}} \int_{\mathbb{R}^d} \ell_r^n(y) p_{t-r}(y-x)\,\mathrm{d}y\,\mathrm{d}r \quad \text{by (2.5)} \qquad (3.16) \\
&\preccurlyeq \int_s^t \frac{1}{\sqrt{t-r}} \int_{\mathbb{R}^d} p_{t-r}(y-x)\,\mathrm{d}y\,\mathrm{d}r \quad \text{by (3.4)} \\
&= \int_s^t \frac{\mathrm{d}r}{\sqrt{t-r}} \lesssim \sqrt{t-s} \lesssim (t-s)^{\frac{\alpha}{2}}. \qquad (3.17)
\end{aligned}
$$

By (3.13), (3.14), (3.15) and (3.17),

$$
\begin{aligned}
|\ell_t^n(x) - \ell_s^n(x)| &\preccurlyeq (t-s)^{\frac{\alpha}{2}} \left[1 + \sup_{s\in[0,T]} \int_0^s (s-r)^{-\frac{1+\alpha}{2}}\,\mathrm{d}r \right] \\
&\lesssim (t-s)^{\frac{\alpha}{2}} \quad \text{because } \alpha \in (0,1).
\end{aligned}
$$

This implies (1.4).

Step 2: we will prove (1.6). We have

$$
\begin{aligned}
I_1(x) &\le \int_{\mathbb{R}^d} \ell_\nu(y) |p_t(y-x) - p_s(y-x)|\,\mathrm{d}y \\
&\lesssim |t-s|^{\frac{\alpha}{2}} \int_{\mathbb{R}^d} \ell_\nu(y) \{ t^{-\frac{\alpha}{2}} p_{2t}(y-x) + s^{-\frac{\alpha}{2}} p_{2s}(y-x) \}\,\mathrm{d}y \quad \text{by (2.7)} \\
&\lesssim (t-s)^{\frac{\alpha}{2}} s^{-\frac{\alpha}{2}} \int_{\mathbb{R}^d} \ell_\nu(y) \{ p_{2t}(y-x) + p_{2s}(y-x) \}\,\mathrm{d}y.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\int_{\mathbb{R}^d} (1+|x|^p) I_1(x)\,\mathrm{d}x \\
&\lesssim (t-s)^{\frac{\alpha}{2}} s^{-\frac{\alpha}{2}} \int_{\mathbb{R}^d} \ell_\nu(y) \left[\int_{\mathbb{R}^d} (1+|x|^p) \{ p_{2t}(y-x) + p_{2s}(y-x) \}\,\mathrm{d}x \right] \mathrm{d}y \\
&\lesssim (t-s)^{\frac{\alpha}{2}} s^{-\frac{\alpha}{2}} \int_{\mathbb{R}^d} (1+|y|^p) \ell_\nu(y)\,\mathrm{d}y \\
&\preccurlyeq (t-s)^{\frac{\alpha}{2}} s^{-\frac{\alpha}{2}}. \qquad (3.18)
\end{aligned}
$$

We have

$$\begin{aligned}
I_2^n(x) \lesssim |t-s|^{\frac{\alpha}{2}} \int_0^s & \mathbb{E}[(t-r)^{-\frac{1+\alpha}{2}} p_{2(t-r)}(X_r^n - x) \qquad \text{by (2.7)} \\
& + (s-r)^{-\frac{1+\alpha}{2}} p_{2(s-r)}(X_r^n - x)] \, \mathrm{d}r \\
\lesssim |t-s|^{\frac{\alpha}{2}} & \int_0^s (s-r)^{-\frac{1+\alpha}{2}} \mathbb{E}[p_{2(t-r)}(X_r^n - x)] \, \mathrm{d}r.
\end{aligned}$$

Then

$$\begin{aligned}
& \int_{\mathbb{R}^d} (1+|x|^p) I_2^n(x) \, \mathrm{d}x \\
& \lesssim |t-s|^{\frac{\alpha}{2}} \int_0^s (s-r)^{-\frac{1+\alpha}{2}} \mathbb{E}\left[\int_{\mathbb{R}^d} (1+|x|^p) p_{2(t-r)}(X_r^n - x) \, \mathrm{d}x \right] \mathrm{d}r \\
& \lesssim |t-s|^{\frac{\alpha}{2}} \int_0^s (s-r)^{-\frac{1+\alpha}{2}} \mathbb{E}[1 + |X_r^n|^p] \, \mathrm{d}r \\
& \preccurlyeq |t-s|^{\frac{\alpha}{2}} \int_0^s (s-r)^{-\frac{1+\alpha}{2}} \, \mathrm{d}r \quad \text{by (1.5)} \\
& \lesssim |t-s|^{\frac{\alpha}{2}} \quad \text{because } \alpha \in (0,1).
\end{aligned} \tag{3.19}$$

We have

$$\begin{aligned}
& \int_{\mathbb{R}^d} (1+|x|^p) I_3^n(x) \, \mathrm{d}x \\
& \lesssim \int_s^t \frac{1}{\sqrt{t-r}} \int_{\mathbb{R}^d} \ell_r^n(y) \left[\int_{\mathbb{R}^d} (1+|x|^p) p_{t-r}(y-x) \, \mathrm{d}x \right] \mathrm{d}y \, \mathrm{d}r \quad \text{by (3.16)} \\
& \lesssim \int_s^t \frac{1}{\sqrt{t-r}} \left[\int_{\mathbb{R}^d} (1+|y|^p) \ell_r^n(y) \, \mathrm{d}y \right] \mathrm{d}r \\
& \preccurlyeq \int_s^t \frac{\mathrm{d}r}{\sqrt{t-r}} \quad \text{by (1.5)} \\
& \lesssim \sqrt{t-s} \lesssim (t-s)^{\frac{\alpha}{2}}.
\end{aligned} \tag{3.20}$$

We have (3.11) together with (3.18), (3.19) and (3.20) implies (1.6).

Step 3: we will prove (1.7). We assume, in addition, that $\int_{\mathbb{R}^d} (1+|x|^p) \sqrt{\ell_\nu(x)} \, \mathrm{d}x \le C$ holds. Let Y be a random variable whose density is p_1. Then

$$\begin{aligned}
I_1(x) & = |\mathbb{E}[\ell_\nu(x+\sqrt{t}Y) - \ell_\nu(x+\sqrt{s}Y)]| \\
& \le \mathbb{E}[\{|\ell_\nu(x+\sqrt{t}Y) - \ell_\nu(x+\sqrt{s}Y)|^2\}^{\frac{1}{2}}] \\
& \preccurlyeq \mathbb{E}[|(\sqrt{t}-\sqrt{s})Y|^{\frac{\alpha}{2}} \times |\ell_\nu(x+\sqrt{t}Y) - \ell_\nu(x+\sqrt{s}Y)|^{\frac{1}{2}}] \quad \text{because } \ell_\nu \in C_b^\alpha(\mathbb{R}^d) \\
& \lesssim (t-s)^{\frac{\alpha}{4}} \mathbb{E}[|Y|^{\frac{\alpha}{2}} \{|\ell_\nu(x+\sqrt{t}Y)|^{\frac{1}{2}} + |\ell_\nu(x+\sqrt{s}Y)|^{\frac{1}{2}}\}].
\end{aligned}$$

Thus

$$\begin{aligned}
& \int_{\mathbb{R}^d} (1+|x|^p) I_1(x) \, \mathrm{d}x \\
& \preccurlyeq (t-s)^{\frac{\alpha}{4}} \mathbb{E}[|Y|^{\frac{\alpha}{2}} \int_{\mathbb{R}^d} (1+|x|^p) \{|\ell_\nu(x+\sqrt{t}Y)|^{\frac{1}{2}} + |\ell_\nu(x+\sqrt{s}Y)|^{\frac{1}{2}}\} \, \mathrm{d}x] \\
& \lesssim (t-s)^{\frac{\alpha}{4}} \mathbb{E}[|Y|^{\frac{\alpha}{2}} \int_{\mathbb{R}^d} (1+|x|^p + |Y|^p) \sqrt{\ell_\nu(x)} \, \mathrm{d}x] \\
& \preccurlyeq (t-s)^{\frac{\alpha}{4}} \mathbb{E}[|Y|^{\frac{\alpha}{2}} (1+|Y|^p)] \\
& \le (t-s)^{\frac{\alpha}{4}}.
\end{aligned} \tag{3.21}$$

We have (3.11) together with (3.21), (3.19) and (3.20) implies (1.7). This completes the proof.

4. Existence of a weak solution

This section is dedicated to the proof of Theorem 1.2.

4.1. Convergence of marginal densities. By (1.2) and (A1),

$$\sup_{n\in\mathbb{N}} \sup_{t\in[0,T]} M_{p+\alpha}(\mu^n_t) \lesssim 1 + M_{p+\alpha}(\nu). \tag{4.1}$$

We denote $B^c_R := \mathbb{R}^d \setminus B(0,R)$ for every $R>0$. Then we have a uniform control of tail moment:

$$\begin{aligned} &\sup_{n\in\mathbb{N}} \sup_{t\in[0,T]} \int_{B^c_R} |x|^p \, \mathrm{d}\mu^n_t(x) \\ &\le \frac{1}{R} \sup_{n\in\mathbb{N}} \sup_{t\in[0,T]} \int_{\mathbb{R}^d} |x|^{p+\alpha} \, \mathrm{d}\mu^n_t(x) \quad \text{by Markov's inequality} \\ &\lesssim \frac{1+M_{p+\alpha}(\nu)}{R} \quad \text{by (4.1)} \\ &\preccurlyeq \frac{1}{R} \quad \text{by (A2).} \end{aligned} \tag{4.2}$$

Similarly,

$$\sup_{n\in\mathbb{N}} \sup_{t\in[0,T]} \mu^n_t(B^c_R) \preccurlyeq \frac{1}{R}. \tag{4.3}$$

By Theorem 1.1 and Arzelà–Ascoli theorem, there exist a sub-sequence (also denoted by (ℓ^n) for simplicity) and a continuous function $\ell : [0,T]\times\mathbb{R}^d \to \mathbb{R}_+$ such that

$$\lim_n \sup_{t\in[0,T]} \sup_{x\in B(0,R)} |\ell^n_t(x) - \ell_t(x)| = 0 \quad \text{for} \quad R>0. \tag{4.4}$$

Above, $\ell_t := \ell(t,\cdot)$. Clearly, $\ell_0 = \ell_\nu$ and

$$\begin{aligned} \sup_{t\in[0,T]} \|\ell_t\|_{C^\alpha_b} &\lesssim \|\ell_\nu\|_{C^\alpha_b}, \\ \|\ell_t - \ell_s\|_\infty &\preccurlyeq |t-s|^{\frac{\alpha}{2}} \quad \text{for} \quad s,t\in[0,T]. \end{aligned}$$

Using above estimates together with the same reasoning as in [Le24, Section 4.2], we get that ℓ_t is a density whose induced distribution $\mu_t \in \mathcal{P}_p(\mathbb{R}^d)$; and that

$$\limsup_n \sup_{t\in[0,T]} W_p(\mu^n_t, \mu_t) = 0, \tag{4.5}$$

$$W_p(\mu_t,\mu_s) \lesssim |t-s|^{\frac{1}{2}}, \tag{4.6}$$

$$\begin{aligned} \sup_{t\in[0,T]} \int_{B^c_R} |x|^p \, \mathrm{d}\mu_t(x) &\preccurlyeq \frac{1}{R}, \\ \sup_{t\in[0,T]} \mu_t(B^c_R) &\preccurlyeq \frac{1}{R} \quad \text{for} \quad R>0. \end{aligned} \tag{4.7}$$

4.2. Existence of a weak solution. We fix $(f,g)\in C^\infty_c(0,T)\times C^\infty_c(\mathbb{R}^d)$. We apply Itô's lemma to (2.2) and the map $(t,x)\mapsto f(t)g(x)$. Then

$$\begin{aligned} \mathbb{E}\Big[\int_0^t f'(s) g(X^n_s)\,\mathrm{d}s\Big] = &-\mathbb{E}\Big[\int_0^t \langle f(s)\nabla g(X^n_s), b^n(s, X^n_{\tau^n_s})\rangle \,\mathrm{d}s\Big] \\ &- \mathbb{E}\Big[\int_0^t f(s)\Delta g(X^n_s)\,\mathrm{d}s\Big]. \end{aligned} \tag{4.8}$$

Let

$$
\begin{aligned}
I_1^n(s) &:= \mathbb{E}[g(X_s^n)], \\
I_3^n(s) &:= \mathbb{E}[\Delta g(X_s^n)], \\
I_2^n(s) &:= \mathbb{E}[\langle \nabla g(X_s^n), b^n(s, X_{\tau_s^n}^n)\rangle] \\
&= \int_{\mathbb{R}^d \times \mathbb{R}^d} \langle \nabla g(y), b^n(s,x)\rangle \ell_{\tau_s^n}^n(x) \qquad \text{by (2.8)} \\
&\qquad \times p_{s-\tau_s^n}\left(x - y + \int_{\tau_s^n}^s b_r^n(x)\, \mathrm{d}r\right) \mathrm{d}x\, \mathrm{d}y.
\end{aligned} \tag{4.9}
$$

By (4.8) and Fubini's theorem,

$$
\int_0^t f'(s) I_1^n(s)\, \mathrm{d}s = -\int_0^t f(s) I_2^n(s)\, \mathrm{d}s - \int_0^t f(s) I_3^n(s)\, \mathrm{d}s. \tag{4.10}
$$

By (4.5),

$$
\begin{aligned}
\lim_n I_1^n(s) &= \int_{\mathbb{R}^d} g(x)\ell(s,x)\, \mathrm{d}x, \\
\lim_n I_3^n(s) &= \int_{\mathbb{R}^d} \Delta g(x)\ell(s,x)\, \mathrm{d}x.
\end{aligned} \tag{4.11}
$$

Next we consider $I_2^n(s)$. It holds for n large enough that $\operatorname{supp} f \subset (\varepsilon_n, T)$. WLOG, we assume $s \in (\varepsilon_n, T)$ and thus $b^n(s,x) = b(s, x, \ell_{\tau_s^n}^n(x), \mu_{\tau_s^n}^n)$. For every $x \in \mathbb{R}^d$, we define $\nu^{n,x} \in \mathcal{P}(\mathbb{R}^d)$, $h^n(x) \in \mathbb{R}$ and $I_2(s) \in \mathbb{R}$ by

$$
\mathrm{d}\nu^{n,x}(y) := p_{s-\tau_s^n}\left(x - y + \int_{\tau_s^n}^s b^n(r,x)\, \mathrm{d}r\right) \mathrm{d}y, \tag{4.12}
$$

$$
h^n(x) := \int_{\mathbb{R}^d} \langle \nabla g(y), b(s, x, \ell_{\tau_s^n}^n(x), \mu_{\tau_s^n}^n)\rangle\, \mathrm{d}\nu^{n,x}(y), \tag{4.13}
$$

$$
I_2(s) := \int_{\mathbb{R}^d} \ell_s(x) \langle \nabla g(x), b(s, x, \ell_s(x), \mu_s)\rangle\, \mathrm{d}x. \tag{4.14}
$$

Then

$$
\begin{aligned}
\int_{\mathbb{R}^d} |x-y|\, \mathrm{d}\nu^{n,x}(y) &= \int_{\mathbb{R}^d} |y| p_{s-\tau_s^n}\left(y + \int_{\tau_s^n}^s b^n(r,x)\, \mathrm{d}r\right) \mathrm{d}y, \\
&\lesssim \sqrt{s - \tau_s^n} \quad \text{by (A1) and (2.4)},
\end{aligned} \tag{4.15}
$$

$$
I_2^n(s) = \int_{\mathbb{R}^d} \ell_{\tau_s^n}^n(x) h^n(x)\, \mathrm{d}x \quad \text{by (4.9), (4.12) and (4.13)}. \tag{4.16}
$$

WLOG, we assume $\|f\|_\infty + \|\nabla g\|_\infty + \|\nabla^2 g\|_\infty \le 1$. By (4.14) and (4.16),

$$
\begin{aligned}
|I_2^n(s) - I_2(s)| \lesssim& \int_{\mathbb{R}^d} |\ell_{\tau_s^n}^n(x) - \ell_s(x)| \times |\langle \nabla g(x), b(s, x, \ell_s(x), \mu_s)\rangle|\, \mathrm{d}x \\
&+ \int_{\mathbb{R}^d} \ell_{\tau_s^n}^n(x) |h^n(x) - \langle \nabla g(x), b(s, x, \ell_s(x), \mu_s)\rangle|\, \mathrm{d}x.
\end{aligned} \tag{4.17}
$$

First,

$$
\begin{aligned}
&|h^n(x) - \langle \nabla g(x), b(s, x, \ell_s(x), \mu_s)\rangle| \\
\lesssim& \int_{\mathbb{R}^d} |\nabla g(y) - \nabla g(x)|\, \mathrm{d}\nu^{n,x}(y) \qquad \text{by (4.13)} \\
&+ |b(s, x, \ell_{\tau_s^n}^n(x), \mu_{\tau_s^n}^n) - b(s, x, \ell_s(x), \mu_s)| \\
\lesssim& \int_{\mathbb{R}^d} |x-y|\, \mathrm{d}\nu^{n,x}(y) + |\ell_{\tau_s^n}^n(x) - \ell_s(x)| + W_p(\mu_{\tau_s^n}^n, \mu_s) \quad \text{by (A3)}
\end{aligned}
$$

$$\lesssim \sqrt{s-\tau_s^n} + |\ell^n_{\tau_s^n}(x) - \ell_s(x)| + W_p(\mu^n_{\tau_s^n}, \mu_s) \quad \text{by (4.15)}. \tag{4.18}$$

Let $S := \operatorname{supp} g$. Then S is compact. By (4.17) and (4.18),

$$|I_2^n(s) - I_2(s)| \lesssim \int_S |\ell^n_{\tau_s^n}(x) - \ell_s(x)| \, \mathrm{d}x + \sqrt{s-\tau_s^n} \tag{4.19}$$
$$+ \int_{\mathbb{R}^d} \ell^n_{\tau_s^n}(x) |\ell^n_{\tau_s^n}(x) - \ell_s(x)| \, \mathrm{d}x + W_p(\mu^n_{\tau_s^n}, \mu_s).$$

By (1.4) and (4.4), it holds for every $R>0$ that

$$\lim_n \sup_{x \in B(0,R)} |\ell^n_{\tau_s^n}(x) - \ell_s(x)| = 0. \tag{4.20}$$

By (1.5) and (4.5),

$$\lim_n W_p(\mu^n_{\tau_s^n}, \mu_s) = 0. \tag{4.21}$$

We have

$$\begin{aligned}
&\limsup_n |I_2^n(s) - I_2(s)| \\
&\lesssim \limsup_n \int_{\mathbb{R}^d} \ell^n_{\tau_s^n}(x) |\ell^n_{\tau_s^n}(x) - \ell_s(x)| \, \mathrm{d}x \quad \text{by (4.19), (4.20) and (4.21)} \\
&\preccurlyeq \limsup_n \int_{\mathbb{R}^d} |\ell^n_{\tau_s^n}(x) - \ell_s(x)| \, \mathrm{d}x \quad \text{by (1.3)} \\
&\le \limsup_n \int_{B(0,R)} |\ell^n_{\tau_s^n}(x) - \ell_s(x)| \, \mathrm{d}x + \limsup_n \int_{B_R^c} |\ell^n_{\tau_s^n}(x) - \ell_s(x)| \, \mathrm{d}x \\
&=: \limsup_n I_{41}^n(s,R) + \limsup_n I_{42}^n(s,R) \quad \text{for every} \quad R>0.
\end{aligned}$$

By (4.20), $\limsup_n I_{41}^n(s,R) = 0$. By (4.3) and (4.7), $\limsup_n I_{42}^n(s,R) \preccurlyeq \frac{1}{R}$. Then $\limsup_n |I_2^n(s) - I_2(s)| \preccurlyeq \frac{1}{R}$ and thus $\lim_n I_2^n(s) = I_2(s)$. This together with (4.10), (4.11) and dominated convergence theorem implies

$$\int_0^t \int_{\mathbb{R}^d} f'(s) g(x) \ell(s,x) \, \mathrm{d}x \, \mathrm{d}s = - \int_0^t \int_{\mathbb{R}^d} f(s) \langle \nabla g(x), b(s,x,\ell(s,x),\mu_s) \ell(s,x) \rangle \, \mathrm{d}x \, \mathrm{d}s$$
$$- \int_0^t \int_{\mathbb{R}^d} f(s) \Delta g(x) \ell(s,x) \, \mathrm{d}x \, \mathrm{d}s.$$

Hence ℓ satisfies the following Fokker-Planck PDE in distributional sense

$$\partial_t \ell(t,x) = - \sum_{i=1}^d \partial_{x_i} \{ b^i(t,x,\ell(t,x),\mu_t) \ell(t,x) \} + \Delta \ell(t,x).$$

Clearly,

(1) $(t,x) \mapsto b(t,x,\ell(t,x),\mu_t)$ is measurable with

$$\int_{[0,T]} \int_{\mathbb{R}^d} |b(t,x,\ell(t,x),\mu_t)| \, \mathrm{d}\mu_t(x) \, \mathrm{d}t < \infty.$$

(2) $[0,T] \to \mathcal{P}_p(\mathbb{R}^d)$, $t \mapsto \mu_t$ is continuous by (4.6).

We apply superposition principle as in [BR20, Section 2] and get that (1.1) has a weak solution whose marginal distribution is exactly $(\mu_t, t \in [0,T])$. This completes the proof.

5. Rate of convergence

This section is dedicated to the proof of Theorem 1.3.

5.1. **Decomposition of error.** For every $(s,z)\in[0,T]\times\mathbb{R}^d$, we define

$$
\begin{aligned}
I_1^n(s,z) &:= \mathbb{E}[\langle b^n(s,X^n_{\tau^n_s}), \nabla p_{t-s}(X^n_s - z) - \nabla p_{t-s}(X^n_{\tau^n_s} - z)\rangle],\\
I_2^n(s,z) &:= \mathbb{E}[\langle b^n(s,X^n_{\tau^n_s}), \nabla p_{t-s}(X^n_{\tau^n_s} - z)\rangle] - \mathbb{E}[\langle b^n(s,X^n_s), \nabla p_{t-s}(X^n_s - z)\rangle],\\
I_3^n(s,z) &:= 1_{(\varepsilon_n,T]}(s)\mathbb{E}[\langle b(s,X^n_s,\ell^n_{\tau^n_s}(X^n_s),\mu^n_{\tau^n_s}) - b(s,X^n_s,\ell^n_s(X^n_s),\mu^n_s), \nabla p_{t-s}(X^n_s - z)\rangle],\\
I_4^n(s,z) &:= 1_{[0,\varepsilon_n]}(s)\mathbb{E}[\langle b(s,X^n_s,\ell^n_s(X^n_s),\mu^n_s), \nabla p_{t-s}(X^n_s - z)\rangle].
\end{aligned}
$$

We also define the density-like function $\hat{\ell}^n_t:\mathbb{R}^d\to\mathbb{R}$ by

$$
\hat{\ell}^n_t(z) := P_t\ell_\nu(z) + \int_0^t \mathbb{E}[\langle b(s,X^n_s,\ell^n_s(X^n_s),\mu^n_s), \nabla p_{t-s}(X^n_s - z)\rangle]\,\mathrm{d}s.
$$

By Lemma 2.5,

$$
\ell^n_t(z) - \hat{\ell}^n_t(z) = \int_0^t \{I_1^n(s,z) + I_2^n(s,z) + I_3^n(s,z) - I_4^n(s,z)\}\,\mathrm{d}s.
$$

We define $f,\hat{f}:[0,T]\to\mathbb{R}_+$ by

$$
\begin{aligned}
f(t) &:= \int_{\mathbb{R}^d}(1+|z|^p)|\ell^n_t(z) - \ell_t(z)|\,\mathrm{d}z,\\
\hat{f}(t) &:= \int_{\mathbb{R}^d}(1+|z|^p)|\hat{\ell}^n_t(z) - \ell_t(z)|\,\mathrm{d}z.
\end{aligned}
$$

By (1.5) and (4.6), f is bounded. Clearly,

$$
f(t) \le \hat{f}(t) + \int_0^t\int_{\mathbb{R}^d}(1+|z|^p)\{|I_1^n(s,z)| + |I_2^n(s,z)| + |I_3^n(s,z)| + |I_4^n(s,z)|\}\,\mathrm{d}z\,\mathrm{d}s. \tag{5.1}
$$

We have

$$
\begin{aligned}
|I_1^n(s,z)| &\lesssim \mathbb{E}[|\nabla p_{t-s}(X^n_s - z) - \nabla p_{t-s}(X^n_{\tau^n_s} - z)|] \quad \text{by (A1)}\\
&\lesssim \frac{1}{\sqrt{t-s}}\mathbb{E}[|X^n_s - X^n_{\tau^n_s}|^\alpha\{p_{4(t-s)}(X^n_s - z) + p_{4(t-s)}(X^n_{\tau^n_s} - z)\}] \quad \text{by (2.6)}\\
&\lesssim \frac{1}{\sqrt{t-s}}\int_{(\mathbb{R}^d)^2}|x-y|^\alpha\{p_{4(t-s)}(y-z) + p_{4(t-s)}(x-z)\} \quad \text{by (2.9)}\\
&\qquad \times \ell^n_{\tau^n_s}(x)p_{2(s-\tau^n_s)}(x-y)\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}
$$

There exists a constant $\kappa>0$ (depending on Θ_1) such that

$$
\begin{aligned}
\int_{\mathbb{R}^d}(1+|z|^p)\{p_{4(t-s)}(y-z) + p_{4(t-s)}(x-z)\}\,\mathrm{d}z &\lesssim 1+|x|^p+|y|^p,\\
|x-y|^\alpha p_{2(s-\tau^n_s)}(x-y) &\lesssim (s-\tau^n_s)^{\frac{\alpha}{2}}p_{\kappa(s-\tau^n_s)}(x-y).
\end{aligned}
$$

Then

$$
\begin{aligned}
&\int_{\mathbb{R}^d}(1+|z|^p)|I_1^n(s,z)|\,\mathrm{d}z\\
&\lesssim \frac{(s-\tau^n_s)^{\frac{\alpha}{2}}}{\sqrt{t-s}}\int_{(\mathbb{R}^d)^2}(1+|x|^p+|y|^p)\ell^n_{\tau^n_s}(x)p_{\kappa(s-\tau^n_s)}(x-y)\,\mathrm{d}x\,\mathrm{d}y\\
&\lesssim \frac{(s-\tau^n_s)^{\frac{\alpha}{2}}}{\sqrt{t-s}}\int_{\mathbb{R}^d}(1+|x|^p)\ell^n_{\tau^n_s}(x)\,\mathrm{d}x\\
&\preccurlyeq \frac{(s-\tau^n_s)^{\frac{\alpha}{2}}}{\sqrt{t-s}} \quad \text{by (1.5)}.
\end{aligned} \tag{5.2}
$$

We have

$$\begin{aligned}|I_2^n(s,z)| &\le \int_{\mathbb{R}^d} |\langle b^n(s,x), \nabla p_{t-s}(x-z)\rangle| \times |\ell^n_{\tau^n_s}(x) - \ell^n_s(x)| \,\mathrm{d}x \\ &\lesssim \frac{1_{(\varepsilon_n,T]}(s)}{\sqrt{t-s}} \int_{\mathbb{R}^d} p_{t-s}(x-z)|\ell^n_{\tau^n_s}(x) - \ell^n_s(x)| \,\mathrm{d}x \quad \text{by (2.1), (2.5) and (A1).}\end{aligned}$$

Then

$$\begin{aligned}&\int_{\mathbb{R}^d} (1+|z|^p)|I_2^n(s,z)| \,\mathrm{d}z \\ &\lesssim \frac{1_{(\varepsilon_n,T]}(s)}{\sqrt{t-s}} \int_{\mathbb{R}^d} \left[\int_{\mathbb{R}^d} (1+|z|^p) p_{t-s}(x-z) \,\mathrm{d}z \right] |\ell^n_{\tau^n_s}(x) - \ell^n_s(x)| \,\mathrm{d}x \\ &\lesssim \frac{1_{(\varepsilon_n,T]}(s)}{\sqrt{t-s}} \int_{\mathbb{R}^d} (1+|x|^p)|\ell^n_{\tau^n_s}(x) - \ell^n_s(x)| \,\mathrm{d}x \\ &\preccurlyeq \frac{(s-\tau^n_s)^{\frac{\alpha}{2}} 1_{(\varepsilon_n,T]}(s)}{|\tau^n_s|^{\frac{\alpha}{2}} \sqrt{t-s}} \quad \text{by (1.6)} \\ &\lesssim \frac{(s-\tau^n_s)^{\frac{\alpha}{2}} 1_{(\varepsilon_n,T]}(s)}{(s-\varepsilon_n)^{\frac{\alpha}{2}} \sqrt{t-s}}. \end{aligned} \tag{5.3}$$

We have

$$\begin{aligned}|I_3^n(s,z)| &\lesssim \mathbb{E}[|\nabla p_{t-s}(X^n_s - z)|]\{\|\ell^n_{\tau^n_s} - \ell^n_s\|_\infty + W_p(\mu^n_{\tau^n_s}, \mu^n_s)\} \quad \text{by (A3)} \\ &\lesssim \frac{\|\ell^n_{\tau^n_s} - \ell^n_s\|_\infty + W_p(\mu^n_{\tau^n_s}, \mu^n_s)}{\sqrt{t-s}} \mathbb{E}[p_{t-s}(X^n_s - z)] \quad \text{by (2.5)} \\ &\preccurlyeq \frac{(s-\tau^n_s)^{\frac{\alpha}{2}}}{\sqrt{t-s}} \mathbb{E}[p_{t-s}(X^n_s - z)] \quad \text{by Theorem 1.1.}\end{aligned}$$

Then

$$\begin{aligned}\int_{\mathbb{R}^d} (1+|z|^p)|I_3^n(s,z)| \,\mathrm{d}z &\preccurlyeq \frac{(s-\tau^n_s)^{\frac{\alpha}{2}}}{\sqrt{t-s}} \mathbb{E}\Big[\int_{\mathbb{R}^d} (1+|z|^p) p_{t-s}(X^n_s - z) \,\mathrm{d}z \Big] \\ &\lesssim \frac{(s-\tau^n_s)^{\frac{\alpha}{2}}}{\sqrt{t-s}} \mathbb{E}[1+|X^n_s|^p] \\ &\preccurlyeq \frac{(s-\tau^n_s)^{\frac{\alpha}{2}}}{\sqrt{t-s}} \quad \text{by (1.5).}\end{aligned} \tag{5.4}$$

By (A1) and (2.5),

$$|I_4^n(s,z)| \lesssim \frac{1_{[0,\varepsilon_n]}(s)}{\sqrt{t-s}} \mathbb{E}[p_{t-s}(X^n_s - z)].$$

Then

$$\begin{aligned}\int_{\mathbb{R}^d} (1+|z|^p)|I_4^n(s,z)| \,\mathrm{d}z &\lesssim \frac{1_{[0,\varepsilon_n]}(s)}{\sqrt{t-s}} \mathbb{E}\Big[\int_{\mathbb{R}^d} (1+|z|^p) p_{t-s}(X^n_s - z) \,\mathrm{d}z \Big] \\ &\lesssim \frac{1_{[0,\varepsilon_n]}(s)}{\sqrt{t-s}} \mathbb{E}[1+|X^n_s|^p] \\ &\preccurlyeq \frac{1_{[0,\varepsilon_n]}(s)}{\sqrt{t-s}} \quad \text{by (1.5).}\end{aligned} \tag{5.5}$$

5.2. **Bound weighted L^1 norm.** As in [Le24, Section 5.1], we have

$$\begin{aligned}\hat{f}(t) &\lesssim (1 + \|\ell_\nu\|_\infty + M_p(\nu)) \int_0^t (T-s)^{-\frac{1}{2}} \{f(s) + |f(s)|^{\frac{1}{p}}\} \,\mathrm{d}s \\ &\preccurlyeq \int_0^t (T-s)^{-\frac{1}{2}} \{f(s) + |f(s)|^{\frac{1}{p}}\} \,\mathrm{d}s. \end{aligned} \tag{5.6}$$

By (5.1), (5.2), (5.3), (5.4), (5.5) and (5.6),

$$\begin{aligned} f(t) &\preccurlyeq (s-\tau_s^n)^{\frac{\alpha}{2}} \int_0^t \left\{ \frac{1}{\sqrt{t-s}} + \frac{1_{[0,\varepsilon_n]}(s)}{(s-\tau_s^n)^{\frac{\alpha}{2}}\sqrt{t-s}} + \frac{1_{(\varepsilon_n,T]}(s)}{(s-\varepsilon_n)^{\frac{\alpha}{2}}\sqrt{t-s}} \right\} \mathrm{d}s \\ &\quad + \int_0^t (T-s)^{-\frac{1}{2}} \{f(s) + |f(s)|^{\frac{1}{p}}\} \,\mathrm{d}s \\ &\lesssim (s-\tau_s^n)^{\frac{\alpha}{2}} + \int_0^t (T-s)^{-\frac{1}{2}} \{f(s) + |f(s)|^{\frac{1}{p}}\} \,\mathrm{d}s. \end{aligned}$$

Because $p = 1$ and $s - \tau_s^n \le \frac{1}{n}$,

$$f(t) \preccurlyeq n^{-\frac{\alpha}{2}} + \int_0^t (T-s)^{-\frac{1}{2}} f(s) \,\mathrm{d}s.$$

By Gronwall's lemma,

$$\sup_{t\in[0,T]} f(t) \preccurlyeq n^{-\frac{\alpha}{2}}.$$

This completes the proof.

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Appendix

Proof of Lemma 2.1. WLOG, we assume $d = 1$ and φ is non-negative. The claims that $N \in \mathcal{E}$ and that Φ is measurable follows from Tonelli theorem, measurability and non-negativity of φ.

(1) Let ν be the joint distribution of $(Y, \mathrm{id}) : (\Omega, \mathcal{A}) \to (E \times \Omega, \mathcal{E} \otimes \mathcal{G})$. Here id the identity map on Ω. Let $\mathbb{P}_\mathcal{G}$ be the restriction of $\mathbb{P}$ to $(\Omega, \mathcal{G})$. We have $\nu = \mu \otimes \mathbb{P}_\mathcal{G}$ because it holds for $A \in \mathcal{E}$ and $B \in \mathcal{G}$ that

$$\begin{aligned}\nu(A \times B) &= \mathbb{P}[(Y, \mathrm{id}) \in A \times B] \\ &= \mathbb{P}[\{Y \in A\} \cap \{\mathrm{id} \in B\}] \\ &= \mathbb{P}[Y \in A] \times \mathbb{P}[\mathrm{id} \in B] \quad \text{by independence of } \mathcal{D}, \mathcal{G} \\ &= \mu(Y \in A) \times \mathbb{P}_\mathcal{G}[B].\end{aligned}$$

Then

$$\begin{aligned}\mathbb{E}[|\varphi(Y, \cdot)|] &= \int_{E\times\Omega} |\varphi(y, \omega)| \,\mathrm{d}\nu(y, \omega) \quad \text{because } \varphi \text{ is measurable w.r.t } \mathcal{E} \otimes \mathcal{G} \\ &= \int_E \left(\int_\Omega |\varphi(y, \omega)| \,\mathrm{d}\mathbb{P}_\mathcal{G}(\omega) \right) \mathrm{d}\mu(y) \quad \text{by Tonelli theorem} \\ &= \int_E \mathbb{E}[|\varphi(y, \cdot)|] \,\mathrm{d}\mu(y). \end{aligned} \tag{5.7}$$

By integrability of $\varphi(Y, \cdot)$, it holds for μ-a.e. $y \in E$ that $\mathbb{E}[|\varphi(y, \cdot)|] < \infty$. Thus $\mu(N) = 0$.

(2) It remains to prove $\mathbb{E}[\varphi(Y, \cdot)|\mathcal{D}] = \Phi(Y)$. First, we verify that $\Phi(Y)$ is integrable. Indeed,

$$\begin{aligned}\mathbb{E}[|\Phi(Y)|] &= \int_E |\Phi(y)| \,\mathrm{d}\mu(y) \\ &= \int_E |\mathbb{E}[\varphi(y, \cdot)]| \,\mathrm{d}\mu(y) \\ &\le \int_E \mathbb{E}[|\varphi(y, \cdot)|] \,\mathrm{d}\mu(y) \\ &= \mathbb{E}[|\varphi(Y, \cdot)|] \quad \text{by (5.7)} \\ &< \infty.\end{aligned}$$

Let $D \in \mathcal{D}$. We need to prove $\mathbb{E}[\varphi(Y, \cdot)1_D] = \mathbb{E}[\Phi(Y)1_D]$. WLOG, we assume $\mathbb{P}[D] > 0$. We define a probability measure $\widetilde{\mathbb{P}}$ on $(\Omega, \mathcal{A})$ by $\widetilde{\mathbb{P}}[A] := \frac{\mathbb{P}[A\cap D]}{\mathbb{P}[D]}$ for $A \in \mathcal{A}$. Let $\widetilde{\mathbb{E}}$ be the expectation w.r.t $(\Omega, \mathcal{A}, \widetilde{\mathbb{P}})$. Approximating with simple functions, we get $\mathbb{E}[Z1_D] = \mathbb{P}[D] \times \widetilde{\mathbb{E}}[Z]$ for $Z \in L^1(\Omega, \mathcal{A}, \widetilde{\mathbb{P}})$. So it suffices to prove $\widetilde{\mathbb{E}}[\varphi(Y, \cdot)] = \widetilde{\mathbb{E}}[\Phi(Y)]$. Let $\widetilde{\mu}$ be the distribution of Y on $(E, \mathcal{E})$ under $\widetilde{\mathbb{P}}$. Let $\widetilde{\nu}$ be the joint distribution of

$(Y, \mathrm{id}) : (\Omega, \mathcal{A}) \to (E \times \Omega, \mathcal{E} \otimes \mathcal{G})$ under $\widetilde{\mathbb{P}}$. Let $\widetilde{\mathbb{P}}_{\mathcal{G}}$ be the restriction of $\widetilde{\mathbb{P}}$ to $(\Omega, \mathcal{G})$. Let's prove that $\widetilde{\nu} = \widetilde{\mu} \otimes \widetilde{\mathbb{P}}_{\mathcal{G}}$. Indeed, it holds for $A \in \mathcal{E}$ and $B \in \mathcal{G}$ that

$$\begin{aligned}
\widetilde{\nu}(A \times B) &= \widetilde{\mathbb{P}}[(Y, \mathrm{id}) \in A \times B] \\
&= \widetilde{\mathbb{P}}[\{Y \in A\} \cap \{\mathrm{id} \in B\}] \\
&= \frac{\mathbb{P}[(\{Y \in A\} \cap D) \cap \{\mathrm{id} \in B\}]}{\mathbb{P}[D]} \\
&= \frac{\mathbb{P}[\{Y \in A\} \cap D] \times \mathbb{P}[\{\mathrm{id} \in B\} \cap D]}{(\mathbb{P}[D])^2} \quad \text{by independence of } \mathcal{D}, \mathcal{G} \text{ under } \mathbb{P} \\
&= \widetilde{\mathbb{P}}[Y \in A] \times \widetilde{\mathbb{P}}[\mathrm{id} \in B] \\
&= \widetilde{\mu}(Y \in A) \times \widetilde{\mathbb{P}}_{\mathcal{G}}[B].
\end{aligned}$$

Then

$$\begin{aligned}
\widetilde{\mathbb{E}}[\varphi(Y, \cdot)] &= \int_{E \times \Omega} \varphi(y, \omega) \, \mathrm{d}\widetilde{\nu}(y, \omega) \quad \text{because } \varphi \text{ is measurable w.r.t } \mathcal{E} \otimes \mathcal{G} \\
&= \int_E \left(\int_\Omega \varphi(y, \omega) \, \mathrm{d}\widetilde{\mathbb{P}}_{\mathcal{G}}(\omega) \right) \mathrm{d}\widetilde{\mu}(y) \quad \text{by Fubini's theorem} \\
&= \int_E \widetilde{\mathbb{E}}[\varphi(y, \cdot)] \, \mathrm{d}\widetilde{\mu}(y), \\
\widetilde{\mathbb{E}}[\varphi(y, \cdot)] &= \frac{\mathbb{E}[\varphi(y, \cdot) 1_D]}{\mathbb{P}[D]} \\
&= \frac{\mathbb{E}[\varphi(y, \cdot)] \times \mathbb{E}[1_D]}{\mathbb{P}[D]} \quad \text{by independence of } \mathcal{D}, \mathcal{G} \text{ under } \mathbb{P} \\
&= \mathbb{E}[\varphi(y, \cdot)] = \Phi(y).
\end{aligned}$$

Then $\widetilde{\mathbb{E}}[\varphi(Y, \cdot)] = \int_E \Phi(y) \, \mathrm{d}\widetilde{\mu}(y) = \widetilde{\mathbb{E}}[\Phi(Y)]$. This completes the proof.

■

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