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Abstract

In this paper, we define additive dyadic social interactions games (ADG), in which each player cares not only about the selected action, but also about interactions with other players, especially those who choose the same action. This class of games includes alliance formation games, network games, and discrete choice problems with network externalities. While it is known that games in the ADG class admit a pure strategy Nash equilibrium that is a maximizer of the game's potential, the potential approach does not always apply if all coalitional deviations are allowed. In this paper, we systematically identify both restrictions on possible coalitional deviations to preserve the potential function and types of equilibria that ensures their existence for various subclasses of ADG or its subclasses, extending the approach taken by Le Breton et al. (2021). An additional avenue of our results is the examination of games with *outside option* which expand the set of alternatives. Although this modification complicates the analysis significantly, positive results are obtained with different class of restrictions.

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1 Introduction

The impact of social interactions is evident in a wide range of individual decisions. These include choosing where to live (Schelling, 1969), deciding whether to join a club (Dixit, 2003), quitting smoking (Harris and Lopez-Valcárcel, 2004), engaging in criminal activities (Glaeser et al., 1996), choosing a side in an international conflict (Altfeld and Bueno de Mesquita, 1979; Axelrod and Bennett, 1993), and deciding whether to acquire a non-native language (Brock et al., 2025). More generally, social interactions arise in many discrete choice problems with network externalities (see Brock and Durlauf, 2001, 2002; Soetevent and Kooreman, 2007), as well as in a variety of network games, including the quadratic-utility continuous-action games of Ballester et al. (2006) and Bramoullé and Kranton (2007). Collective action problems have been studied by Schelling (1978), who considered a setting in which an individual's decision to participate in a protest depends on the fraction of the population already engaged in the protest.

In games with social interactions, players care not only about their own chosen actions but also about externalities generated by actions of others. Many of such a class of games have multiple Nash equilibria.[1] This naturally motivates refinements of Nash equilibrium that allow for coordinated deviations by groups of players. However, unrestricted coordination among players may fail to produce an equilibrium allocation because of the complex pattern of social interactions across different groups. In this paper, we study *Additive Dyadic Games* (*ADG*), which formally defined below, and analyze equilibrium concepts that are coalitionally stable. Our objective is to identify restrictions on coalitional deviations and subclasses of *ADG* that guarantee the existence of each equilibrium concept. We also investigate the properties of these equilibria across different subclasses of *ADG*. In particular, we ask the following questions. Do our equilibrium concepts satisfy Pareto efficiency? What is the relationship between strong Nash equilibrium and our equilibrium concepts? Does the availability of an outside option affect the equilibrium?

To address these questions, we consider a society with n individuals and a set of m feasible actions. When an individual i selects an action k, her payoff is given by the sum of n terms:

$$v_k^i + \sum_{j \neq i} v_{kl}^{ij}, \tag{1}$$

where the first term v_k^i represents the intrinsic payoff derived by i from selecting action k, and v_{kl}^{ij} is the benefit to player i from her interaction with j when j selects action l. We further impose a standard *symmetry* assumption: for each pair of players i and j, and two actions k and l, the benefit that i choosing k derived from j choosing l, is equal to the dyadic interaction payoff of j choosing k derives from i when she selects l. Note that if the payoff contained only the first term in (1), every individual would simply choose an action that maximizes her intrinsic payoff. However, in general, the

[1]For example, Dower et al. (2026) identified 86 pure-strategy Nash equilibria in a social interaction game with 46 players and two feasible actions.

presence of $n-1$ other terms that represent the benefit to player i derived from her social interactions with other players could alter player i's action choice. To reflect the additive and dyadic nature of our payoff structure that relies on pairwise interactions, we shall call the class of these social interactions games *Additive Dyadic Games* and refer to it as *ADG*.

The class of *ADG* was introduced by Le Breton and Weber (2011) to unify the examination of various games with social interactions explored in the literature. They show that *ADG* has a *potential function* on the set of action profiles whose local maximizers are Nash equilibria of the game (Monderer and Shapley 1996).[2] Thus, there is a potential-maximizing Nash equilibrium in *ADG* generally.

A possibility of coalitional deviation in *ADG* has been studied by Le Breton et al. (2021) who examined a restricted class of *gradual* coalitional deviations. and defined a *landscape equilibrium* as immune to all *gradual* coalitional deviations. They proved the existence of landscape equilibrium for a subclass of *ADG*, which is, in our terminology, *neutral* and *basic* *ADG*, denoted by *NBADG*.[3] They showed that the potential function of *ADG* is preserved for gradual and that the (weighted-) potential-maximizing strategy profile is a strongly Pareto efficient in *NBADG*. Dower et al. (2026) considered a special case of Le Breton et al. (2021) with two actions, and showed that the potential-maximizing equilibrium is a strong Nash equilibrium, since a landscape equilibrium in a two action game is a strong Nash equilibrium.

In this paper, we systematically identify restrictions on possible coalitional deviations to preserve the potential function, and the subclasses *ADG* that admit equilibria under these deviations. As our Example 2 indicates, the potential maximization is inconsistent with the existence of a strong Nash equilibrium (Aumann 1959) with no restriction on coalitional deviations (Banerjee et al. 2001). Our response is two-fold. First, we offer a new notion of *weak landscape equilibrium*, which is weaker than landscape equilibrium, and prove its existence for a subclass of *uniform* games in *ADG*, denoted by *UADG* (Theorem 2). Second, we demonstrate the existence of landscape equilibrium for the class of *basic* (*uniform* and *local*) *ADG*, denoted by *BADG* (Theorem 3), which contains the original class *NBADG* considered by Le Breton et al. (2021). As is shown below, Theorem 3 is particularly useful since it allows to apply the result in discrete choice models with network externalities, and it paves the way for exploration of econometric side of the models.[4] Following Binmore, et al. (1989), we also introduce an outside option to subclasses of *ADG*, and investigate how the results would be modified by this change. It turns out that many results are upset in the presence of an outside option—in general, landscape equilibrium may not exist. However, we obtain positive results with different class of restrictions.

[2] The study of potential games goes back to Rosenthal (1973) who considered a congestion game and showed that there exists a Nash equilibrium in pure strategies.

[3] When the intrinsic payoff v_k^i is action-independent, it is said *neutral*, and the dyadic social interaction v_{kl}^{ij} is said (i) *local* if it is in effect only within the same action, and (ii) *uniform* if its value is action-independent. If (i) and (ii) are simultaneously satisfied, v_{kl}^{ij} is said *basic*.

[4] That could be done by defining the intrinsic component v_k^i as the sum of a deterministic component and a random component.

This paper is organized as follows. In Section 2, we provide a review of the relevant literature. In Section 3, the existence of Nash equilibrium within the *ADG* class is discussed. In Section 4, we introduce two versions of landscape equilibrium (weak landscape equilibrium and landscape equilibrium. We demonstrate that the distinction between the sets of weak landscape, landscape and strong equilibria is not vacuous, and, in general, they do not coincide (Examples 2 and 3). In Section 5, we show that weak landscape equilibrium exists when propensity matrices within the same actions are uniform (Theorem 2), but this *uniformity* only cannot guarantee the existence of landscape equilibrium (Example 4). In Section 6, we focus on *basic* additive social interactions games and extend the result of Le Breton et al. (2021) to prove the existence of a landscape equilibrium within this class of games (Theorem 3). We also show that there may not be strongly Pareto landscape equilibrium without *neutrality* (Example 6). In Section 7, even with *neutrality*, we show that a potential-maximizing landscape equilibrium is not necessarily a strong Nash equilibrium even if the set of latter is nonempty (Example 7), and that there could be two Pareto-ranked landscape equilibria (Example 8). In Section 8 we consider a class of basic games *BADG* that allow for *an outside option.* Since those games are no longer uniform, theorem 2 does not apply her, and, indeed, we show that for this class of games, the existence of a landscape equilibrium is no longer guaranteed (Example 10), and a potential-maximizing action profile may not even be a weak landscape equilibrium (Example 11). However, we demonstrate that there exists a strong Nash equilibrium if there are two feasible alternatives ($m = 2$) and dyadic social interactions are non-negative (Theorem 4), and that there exists a landscape equilibrium when dyadic social externality is non-positive (Theorem 5). In Section 9 we address the question of when a game admits an *ADG* payoff representation. This question is a difficult challenge and we offer a partial answer (Proposition 2), whereas Example 12 illustrates the difficulties in assuring the symmetry in this setting. Finally, Example 14 demonstrates that a monotone transformation of a game in the *NBADG* class, may alter the set of potential maximizers. Section 10 concludes the paper with some remarks, including an application to a class of matching problems with peer interactions.

2 The Literature Review

We start with the papers that belong to the three main examples of the *ADG* class.

First, the *ADG* class contains the family of network games examined in Ballester et al. (2006). They assume that a player's payoff function is quadratic:

$$u^i(x^1, ..., x^n) = a^i x^i + \frac{1}{2}b(x^i)^2 + \sum_{j\neq i} p^{ij} x^i x^j,$$

where $x^i \in \mathbb{R}_+$ is player i's effort level, and a^i, b, and p^{ij} are parameters. Although the strength of social interaction p^{ij} is assumed to be *symmetric*, the sign of p^{ij} can be positive or negative. Now, letting actions k and l represent the levels of effort

$e_k, e_l \in \mathbb{R}_+$ exerted by players i and j choosing actions k and l, respectively, we can treat their network game as an *ADG* with payoff function (1) with:

$$\begin{aligned} v_k^i &= a^i e_k + \frac{1}{2} b e_k^2, \\ v_{kl}^{ij} &= p^{ij} e_k e_l. \end{aligned}$$

Thus, the fact that their game belongs to the class of *ADG* implies that the existence of a Nash equilibrium of the game does not rely on the continuous single-dimensional action space. Ballester et al. (2006) show that if an interior Nash equilibrium of this game exists, it is unique and is described by a set of linear equations.[5] Bramoullé and Kranton (2007) analyzed a similar network games, and Bramoullé et al. (2014) show that there is a potential function for this game and examine various applications of social interactions games, including R&D competition and societal crime patterns.

Second, another subclass of *ADG* is represented by a discrete choice problem, where social interactions are generated by the impact of players choosing the same alternative (see Brock and Durlauf, 2001, 2002, and Soetevent and Kooreman, 2007), who study a two-stage game on networks that assigns links between players. In their setting, action k corresponds to an action (alternative) each player can choose, and the dyadic social interaction v_{kl}^{ij} is *basic*, which is *local* and *uniform* with anonymous propensity $p^{ij} = 1$ for all i, j, thus social interactions among players depend on n_k, which is the number of players choosing action k. Additive dyadic social interactions games have been used for empirical research of social interactions under the common term v^{ij} in (1).[6] In Brock and Durlauf (2001, 2002), players are assumed to make their action choice based on their expected memberships of actions, $(n_k)_{k=1}^m$, computed from the distributions of player-specific random terms. Brock and Durlauf (2001, 2002) and Blume et al. (2015) analyze the equilibrium of the game with rational expectations. In contrast, Soetevent and Kooreman (2007) assume that player-specific random terms are common knowledge and use Nash equilibrium as a solution concept when $m = 2$. They prove existence of a Nash equilibrium, provide bounds on the number of equilibria, and explore the econometric side of the model, offering an empirical illustration.

Third, note that various facets of international alliances such as bloc formation in a conflict (Altfeld and Bueno de Mesquita, 1979) or voting behavior in the UN Assembly (Russett, 1966) constitute potential applications of the *ADG* framework. In fact, Axelrod and Bennett (1993), and Bennett (2000) modeled an international alliance game as a *basic ADG*, with players choosing one of the two existing blocs, and examined the stability of nations' alliance structure. The key ingredient in Axelrod and Bennett's landscape theory of aggregation is the *propensity matrix* $(p^{ij})_{i,j=1,\ldots,n}$

[5] Under further assumptions on the parameters, Ballester et al. (2006) explore the properties of equilibrium as a function of the matrix $\left(p^{ij}\right)_{i,j=1,\ldots,n}$.

[6] In the one-player setting with $n = 1$, our model corresponds to the celebrated random utility model used extensively in the standard econometrics of discrete choice models (McFadden, 1981, Manski, 2000). In that respect, the *ADG* class can just be viewed as an n-player extension of the random utility model with Nash equilibrium and its refinements replacing utility maximization.

which describes the sign and magnitude of the dyadic social externality between each pair of players i and j. The sign of a matrix entry again indicates the nature of the externality (negative for foes and positive for allies) and is symmetric, while the absolute value represents the intensity of this negative or positive interaction.
They apply their model to the empirical study of the alignment of 17 European nations with the Allies and Axis during World War II. To do so, Axelrod and Bennett (1993) presented a method for the empirical identification of the propensity matrix. Following the Axelrod-Bennett contribution, Florian and Galam (2000) generalized the landscape theory to more than two blocs and extended their analysis to the break-up of Yugoslavia more than thirty years ago. Kijima (2001) applied this extended framework to analyze alliance formation in the civil aviation industry.

It is worth pointing out that the notion of *dyadic affinities* (the terminology coined by Kuperman et al., 2006), which is central in the *ADG* framework, is a key feature of academic research in international politics. As we alluded to earlier, Bueno de Mesquita (1975, 1981) was the first to transform the notion of structural affinity into an empirical measure via the similarity of alliance portfolios. Signorino and Ritter (1999) offer an alternative way of deriving the propensity matrix. Kuperman et al. (2006) discuss the properties and limitations of these two measures of structural affinity and offer an alternative one.

Some papers examine equilibrium stability and the robustness of equilibria as a refinement concept of Nash equilibrium in potential games. Ui (2001) provides support for such an equilibrium selection by showing its robustness in the sense of Kajii and Morris (1997). Carbonell-Nicolau and McLean (2014) show that the set of potential maximizers contains a Kohlberg and Mertens (1986) stable set of pure-strategy Nash equilibria. Blume (1993) shows that the potential maximizers correspond to stochastically stable states under log-linear dynamics such as the logit choice rule. Bramoullé et al. (2014) demonstrate the stability of the potential-maximizing Nash equilibrium.

Newton and Sercombe (2020) consider coalitional deviations as well as unilateral deviations in analyzing the diffusion of an innovation on a network using a two-action symmetric coordination game with a potential function. They compare the graph-theoretic properties under which potential-maximizing strategy profiles are evolutionarily stable with unilateral (stochastic) and coalitional (coordinated) deviations. Le Breton et al. (2021) extend the analysis of Axelrod and Bennett (1993) by allowing for single-group landscape coalitional deviations to show that the potential-maximizing Nash equilibrium is immune to *landscape* deviations and is strongly Pareto efficient. Dower et al. (2026) analyze the two paths (Western and Eastern) of the post-war development strategies chosen by African countries and estimate a propensity matrix, similarly to Axelrod and Bennett (1993). They show the existence of a strong Nash equilibrium and compute a potential-maximizing strong Nash equilibrium of the game between African countries.

3 Potential Functions and Nash Equilibria in the Additive Dyadic Games

Here, we introduce a class of additive dyadic social interactions games. There are a finite set of players $N = \{1, 2, ..., n\}$, and a finite set of feasible actions $M = \{1, 2, ..., m\}$, where each player in N chooses an action s^i from M. The resulting strategy profile $\mathbf{s} = (s^i)_{i \in N}$ partitions players over the set of actions $\mathbf{G}(\mathbf{s}) = (G_\ell(\mathbf{s}))_{\ell \in M}$, where $G_\ell(\mathbf{s}) \equiv \{j \in N : s^j = \ell\}$ is the set of players who chose the same action ℓ.

Let $p_{k\ell}^{ij} \in \mathbb{R}$ denote player i's dyadic propensity from player j when player i chooses action k and player j chooses action ℓ, and let $\sigma^j > 0$ be an influence parameter—it indicates the impact of player j on others.

The propensities described here are represented by the following matrices:

$$P_{kk} = \begin{pmatrix} p_{kk}^{11} & \cdots & p_{kk}^{1i} & \cdots & p_{kk}^{1j} & \cdots & p_{kk}^{1n} \\ \vdots & \ddots & \vdots & & \vdots & & \vdots \\ p_{kk}^{i1} & \cdots & p_{kk}^{ii} & \cdots & \underline{p_{kk}^{ij}} & \cdots & p_{kk}^{in} \\ \vdots & & \vdots & \ddots & \vdots & & \vdots \\ p_{kk}^{j1} & \cdots & \underline{p_{kk}^{ji}} & \cdots & p_{kk}^{jj} & \cdots & p_{kk}^{jn} \\ \vdots & & \vdots & & \vdots & \ddots & \vdots \\ p_{kk}^{n1} & \cdots & p_{kk}^{ni} & \cdots & p_{kk}^{nj} & \cdots & p_{kk}^{nn} \end{pmatrix}$$

and

$$P_{k\ell} = \begin{pmatrix} p_{k\ell}^{11} & & & \\ & \ddots & \boxed{p_{k\ell}^{ji}} & \\ & \mathbf{p}_{k\ell}^{ij} & \ddots & \\ & & & p_{k\ell}^{nn} \end{pmatrix}; \ \& \ \ P_{\ell k} = \begin{pmatrix} p_{\ell k}^{11} & & & \\ & \ddots & \mathbf{p}_{\ell k}^{ji} & \\ & \boxed{p_{\ell k}^{ij}} & \ddots & \\ & & & p_{\ell k}^{nn} \end{pmatrix}$$

Here, P_{kk} is a symmetric matrix for any $k \in M$, but across action propensity matrices $P_{k\ell}$ and $P_{\ell k}$ are not necessarily symmetric (meaningless diagonal terms satisfy $p_{k\ell}^{ii} = p_{\ell k}^{ii} = 0$).

Then the payoff specification (1) for player i who chooses an action $k \in M$ under strategy profile $\mathbf{s}$ yields

$$u_k^i(\mathbf{G}) = v_k^i + \sum_{\ell \in M} \sum_{j \in G_\ell} p_{k\ell}^{ij} \sigma^j = v_k^i + \sum_{j \in G_k} p_{kk}^{ij} \sigma^j + \sum_{\ell \in M \setminus \{k\}} \sum_{j \in G_\ell} p_{k\ell}^{ij} \sigma^j,$$

where $v_k^i \in \mathbb{R}$ is, to recall, an intrinsic benefit derived by player i from choosing action k.

Note that if we multiply the right-hand side of the last expression by σ^i, then a new game that is strategically equivalent to the original one.

We further impose the straightforward normalization conditions: (i) $p_{kk}^{ii} = 0$ for all $i \in N$ and all $k \in M$, and (ii) $p_{k\ell}^{ii} = 0$ for all $i \in N$ and all $k, \ell \in M$ with $k \neq \ell$.

Note the symmetry condition $p^{ij}_{k\ell} = p^{ji}_{\ell k}$ for all $i, j \in N$ and all $k, \ell \in M$, simply means that propensities are symmetric as the dyadic interactions are symmetric.

A game that belongs to this class is called an **additive dyadic game**, and the set of additive dyadic games is denoted by ADG.

We now present a short proof of a variant of the Le Breton and Weber (2011) result, which showed that every game in the ADG class admits a Nash equilibrium in pure strategies.[7] The proof underlines the importance of the potential functions technique (Monderer and Shapley, 1996)) whose local maximizers are Nash equilibria of the game. For this end, we define the following function, which will be shown to be a weighted potential of the game $\Gamma \in ADG$.

$$\mathbb{F}(\mathbf{s}) \equiv \sum_{\ell \in M} \sum_{j \in G_\ell(\mathbf{s})} \sigma^j v^j_\ell + \frac{1}{2} \sum_{\ell \in M} \sum_{j \in G_\ell(\mathbf{s})} \sum_{\ell' \in M} \sum_{j' \in G_{\ell'}(\mathbf{s})} \sigma^j p^{jj'}_{\ell\ell'} \sigma^{j'}. \tag{2}$$

To see that $\mathbb{F}(\mathbf{s})$ is a weighted potential, we let $\mathbf{s} = (s^j)_{j \in N}$ with $s^j = k$, and consider a potential of a reduced game without player i and strategy profile $\mathbf{s}^{-i}$. Then, we have

$$\mathbb{F}(\mathbf{s}) - \mathbb{F}(\mathbf{s}^{-i}) = \sigma^i v^i_k + \sum_{\ell \in M} \sum_{j \in G_\ell(\mathbf{s})} \sigma^i p^{ij}_{k\ell} \sigma^j,$$

since there are two cases $k = \ell$ and $k = \ell'$ with $p^{ij}_{k\ell} = p^{ji}_{\ell k}$ (and $p^{ii}_{kk} = 0$). Let $\tilde{\mathbf{s}} = (\tilde{s}^i, \mathbf{s}^{-i})$ with $\tilde{s}^i = h$. Similarly we have

$$\mathbb{F}(\tilde{\mathbf{s}}) - \mathbb{F}(\mathbf{s}^{-i}) = \sigma^i v^i_h + \sum_{\ell \in M} \sum_{j \in G_\ell(\tilde{\mathbf{s}})} \sigma^i p^{ij}_{h\ell} \sigma^j.$$

This implies

$$\mathbb{F}(\tilde{\mathbf{s}}) - \mathbb{F}(\mathbf{s}) = \sigma^i v^i_h + \sum_{\ell \in M} \sum_{j \in G_\ell(\tilde{\mathbf{s}})} \sigma^i p^{ij}_{h\ell} \sigma^j - \sigma^i v^i_k - \sum_{\ell \in M} \sum_{j \in G_\ell(\mathbf{s})} \sigma^i p^{ij}_{k\ell} \sigma^j.$$

If any player $i \in N$ switches her strategy from $s^i = k$ to $\tilde{s}^i = h$ unilaterally, then we have

$$\Delta u^i = u^i(\tilde{\mathbf{s}}) - u^i(\mathbf{s}) = v^i_h + \sum_{\ell \in M} \sum_{j \in G_\ell(\tilde{\mathbf{s}})} p^{ij}_{h\ell} \sigma^j - v^i_k - \sum_{\ell \in M} \sum_{j \in G_\ell(\mathbf{s})} p^{ij}_{k\ell} \sigma^j,$$

and we conclude

$$\mathbb{F}(\tilde{\mathbf{s}}) - \mathbb{F}(\mathbf{s}) = \sigma^i \Delta u^i.$$

Therefore, $\mathbb{F}(\mathbf{s})$ is a weighted potential of the game Γ. In contrast with the weighted Benthamite social welfare function, the second term of $\mathbb{F}(\mathbf{s})$ has the coefficient $\frac{1}{2}$. That is, when player i switches from k to h to join player j, the i's net welfare gain from the relation with a player j is the same as welfare gain derived by player j from

[7] Since in our paper we refer only to pure strategies Nash equilibria, without any confusion, we will use the term of Nash equilibrium rather than Nash equilibrium in pure strategies.

the relation to i. The following is a variation of the result by Le Breton and Weber (2011).[8]

Theorem 1. Every game $\Gamma \in ADG$ admits a Nash equilibrium. In particular, a weighted-potential-maximizing strategy profile is a Nash equilibrium.

Proof. Let $\bar{\mathbf{s}} \in \arg\max_{\mathbf{s}} \mathbb{F}(\mathbf{s})$. Suppose that $\bar{\mathbf{s}}$ is not a pure strategy Nash equilibrium. Then, there is player $i \in N$ and strategies $k \neq h$ such that $s^i = k$ and $\tilde{s}^i = h$ with $\Delta u^i = u^i(\tilde{s}^i, \bar{\mathbf{s}}^{-i}) - u^i(\bar{\mathbf{s}}) > 0$. However, this implies $\mathbb{F}(\tilde{s}^i, \bar{\mathbf{s}}^{-i}) > \mathbb{F}(\bar{\mathbf{s}})$, a contradiction.$\square$

Note that Theorem 1 covers the finite versions of the existence results in Ballester et al. (2006) and Bramoullé et al. (2014). Moreover, it is more general than the equilibrium result in Axelrod and Bennett (1993). While Brock and Durlauf (2001, 2002) use a different equilibrium notion, Soetevent and Kooreman (2007) Nash equilibrium existence result in that setting is also covered by Theorem 1.

It is important to point out that the relationship between the set of potential maximizers and the set of equilibria breaks down if we allow for coalitional deviations rather than individual switches of action. It is shown by the following example:

Example 1. Let $N = \{1, 2, 3, 4\}$ and $M = \{a, b\}$ with $v^i_\ell = 0$ for all $i \in N$ and all $\ell \in M$. Let game $\Gamma \in ADG$ be local: i.e., off-diagonal propensity matrices are $P_{ab} = P_{ba} = \mathbf{0}$. We set $\sigma^i = 1$ for all $i \in N$. Let

$$P_{aa} = \begin{pmatrix} 0 & 1.5 & 0 & 0 \\ 1.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.5 \\ 0 & 0 & 1.5 & 0 \end{pmatrix} \quad P_{bb} = \begin{pmatrix} 0 & 3 & 0 & -1 \\ 3 & 0 & -1 & 0 \\ 0 & -1 & 0 & 3 \\ -1 & 0 & 3 & 0 \end{pmatrix}$$

In this case, potential-maximizing allocations are $\mathbf{s} = (a, a, b, b)$ (or $\mathbf{s}' = (b, b, a, a)$). Consider a coalitional deviation by $\{1, 2\}$ moving to action b. With this deviation, players 1 and 2 improve their payoffs from 1.5 to 2 each, but the value of potential goes down from $\frac{1}{2}\{(1.5 + 1.5) + (3 + 3)\} = 4.5$ to $\frac{1}{2}\{2 + 2 + 2 + 2\} = 4$.$\blacksquare$

Thus, in order to guarantee the existence of equilibria immune against coalitional deviations, we need to limit the scope of permissible deviations and/or restrict the subset of the ADG games under consideration. It is done in the following section.

[8] When $\sigma^i = 1$ for all $i \in N$, Le Breton and Weber (2011) proved the existence of Nash equilibrium, allowing for nonlinear population externalities. Chakrabarti et al. (2025) extend their result to prove the existence of a Nash equilibrium under a more general structure. However, their results do not hold if influence parameters σ^is are heterogeneous.

4 Coalitional Deviations and Subclasses of the ADG

In this section, we first identify several subclasses of ADG, We then define various equilibrium concepts that allow for coalitional deviations.

Definition 1.

1. A game $\Gamma \in ADG$ is **local** if $P_{k\ell} = (\mathbf{0})_{n\times n}$ for all $k, \ell \in M$ with $k \neq \ell$: i.e., social interactions take place only within groups of players who chose the same action. The set of local games is denoted by $LADG$.

2. A game $\Gamma \in ADG$ is **uniform** if diagonal propensity matrices are the same for all actions: i.e., $P_{kk} = P$ for all $k \in M$. The set of local games is denoted by $UADG$.

3. A game $\Gamma \in ADG$ is **externally uniform** if off-diagonal propensity matrices are the same for all heterogeneous action pairs: i.e., $P_{k\ell} = Q$ for all $k, \ell \in M$ with $k \neq \ell$. The set of local games is denoted by $EUADG$.

4. A game Γ is **basic** if it is *local* and *uniform*. The set of *basic* games is denoted by $BADG$.[9]

5. A game Γ is **neutral** if it is *basic*, and, moreover, $v^i_k = 0$ for all $i \in N$ and all $k \in M$. The set of neutral games is denoted by $NBADG$.

Naturally, we have the following straightforward inclusions

$$NBADG \subset BADG \subset UADG \subset ADG.$$

We introduce some solution concepts for our games. First, a strategy profile is a **Nash equilibrium** if and only if for all $i \in N$, $u^i(\mathbf{s}) \geq u^i(\tilde{s}^i, \mathbf{s}^{-i})$ holds for all $\tilde{s}^i \in M$. While Nash equilibrium is based on individual players' unilateral deviations is the most fundamental solution concept, we are interested in exploring the framework that allows for coalitional deviations.

For every two strategy profiles $\mathbf{s}$ and $\tilde{\mathbf{s}}$, denote by $S(\mathbf{s}, \tilde{\mathbf{s}})$ the set of players who choose different strategies at $\mathbf{s}$ and $\tilde{\mathbf{s}}$, i.e.,

$$S(\mathbf{s}, \tilde{\mathbf{s}}) \equiv \left\{ i \in N : \tilde{s}^i \neq s^i \right\}.$$

A coalitional deviation from a strategy profile $\mathbf{s}$ via $\tilde{\mathbf{s}}$ is **strictly improving** if $u^i(\tilde{\mathbf{s}}) > u^i(\mathbf{s})$ for all $i \in S(\mathbf{s}, \tilde{\mathbf{s}})$. Without placing any restrictions on the set of coalitional deviations, we obtain the notion of strong Nash equilibrium (Aumann 1959).

[9] Note that *uniform* and *externally uniform* ADG class is equivalent to $BADG$ class. Assume uniformity and external uniformity together. Translating the propensity matrices by subtracting Q, and redefining all propensity matrices $P_{k\ell}$ by $P_{k\ell} - Q$ for all $k, \ell \in M$, the newly created ADG becomes *local*, thus it belongs to the $BADG$ class.

Definition 2. A strategy profile $\mathbf{s}$ is a **strong Nash equilibrium** if there is no strictly improving coalitional deviation $S(\mathbf{s}, \tilde{\mathbf{s}})$. The set of all strong Nash equilibria is denoted by SNE.

While being an attractive solution concept, a strong Nash equilibrium may often fail to exist for $BADG$ games, too.[10]

Example 2. Let $N = \{1, 2, 3\}$ and $M = \{a, b, c, d\}$. Suppose that game $\Gamma \in BADG$ has the following propensity matrix,

$$P = \begin{pmatrix} 0 & 3 & 1 \\ 3 & 0 & -2 \\ 1 & -2 & 0 \end{pmatrix}$$

and intrinsic preferences over actions,

$k \in M$	a	b	c	d
v_k^1	0	-10	-3.5	-10
v_k^2	-10	0	-0.5	-10
v_k^3	-10	-10	0	-0.5

We set $\sigma^i = 1$ for all $i \in N$. Players 1, 2, and 3 will choose their actions from $\{a, c\}$, $\{b, c\}$, and $\{c, d\}$, respectively. Player 1 may choose c only when both players 2 and 3 choose c. The following profiles are candidates for a strong Nash equilibrium.

	u^1	u^2	u^3	potential
(a, b, c)	0	0	0	0
(c, c, c)	0.5	0.5	-1	-2
(c, c, d)	-0.5	2.5	-0.5	-1.5
(a, c, d)	0	-0.5	-0.5	-1
(a, b, d)	0	0	-0.5	-0.5

Notice that there is a unique Nash equilibrium $\mathbf{s} = (a, b, c)$. However, it is not a strong Nash equilibrium, since players 1 and 2 will jointly move to c. Thus, a strong Nash equilibrium does not exist. The reason is as follows. At strategy profile $\mathbf{s}$, players 1 and 2 are choosing different actions, but they both shift to c. This coordinated action generates benefits which are internal for players 1 and 2, where no outsider benefits from this move. Thus, the change in the value of our potential function caused by this coalitional deviation does not corresponds to the joint benefit of the coalition. Indeed, this profitable coalitional deviation reduces the value of our potential.■

Note that the above example shows that if *a coalitional deviation that involves coordinated moves by two players choosing different actions in the original strategy (action) profile, a potential function approach does not work, threatening the existence*

[10] While Banerjee et al. (2001) show that SNE may fail to exist even within the $NBADG$ class. Our simple Example 2 clearly illustrates the type of coalitional deviations that rule out the existence of SNE.

of an equilibrium. Thus, we will need to limit the scope of coalitional deviations. Let $\mathbf{s}$ and $\tilde{\mathbf{s}}$ be two different profiles of actions, and k and h be two different actions. Let

$$S_{kh}(\mathbf{s}, \tilde{\mathbf{s}}) \equiv \left\{ i \in S(\mathbf{s}, \tilde{\mathbf{s}}) : s^i = k \text{ and } \tilde{s}^i = h \right\}$$

be the set of players who switched their strategy from k to h at $\mathbf{s}$. Clearly, the collection of (not necessarily nonempty) sets $\{S_{kh}(\mathbf{s}, \tilde{\mathbf{s}})\}_{k,h \in M, k \neq h}$ is a partition of $S(\mathbf{s}, \tilde{\mathbf{s}})$, which is the set of all players whose chose different strategies at $\mathbf{s}$ and $\tilde{\mathbf{s}}$.

We consider the following two types of coalitional deviations:

Definition 3. Two types of coalitional deviations with their associated equilibrium concepts are defined as:

1. We say that a coalitional deviation $S(\mathbf{s}, \tilde{\mathbf{s}}) \equiv \cup_{k \neq h} S_{kh}(\mathbf{s}, \tilde{\mathbf{s}})$ is of **a single-group** if there is at a single (k, h) combination with $k \neq h$ and $|S_{kh}(\mathbf{s}, \tilde{\mathbf{s}})| > 0$. This condition means that if there are players who switch their action, they are all changing actions from k to h. A strategy profile $\mathbf{s}$ is a **weak landscape equilibrium** if it is immune to all single-group coalitional deviations. The set of those equilibria is denoted by WLE.

2. (Le Breton et al., 2021) We say that a coalitional deviation $S(\mathbf{s}, \tilde{\mathbf{s}}) \equiv \cup_{k \neq h} S_{kh}(\mathbf{s}, \tilde{\mathbf{s}})$ is **gradual** if for any $h \in M$, there is at most one single k with $|S_{kh}(\mathbf{s}, \tilde{\mathbf{s}})| > 0$. That is, if there are players who choose newly adopted action h under $\tilde{\mathbf{s}}$, all such players must have been choosing the same action k under $\mathbf{s}$. A strategy profile $\mathbf{s}$ is a **landscape equilibrium** if $\mathbf{s}$ is immune to all landscape deviations. The set of those equilibria is denoted by LE.

The *gradual* coalitional deviation and landscape equilibrium were first defined by Le Breton et al. (2021). The requirements imposed by coalitional deviations in definition 3 can be illustrated on a directed graph over the set of actions M. The changes of actions by the coalition members of deviation $S(\mathbf{s}, \tilde{\mathbf{s}})$ are summarized by a directed graph on action set M, $(M, \mathcal{G}(\mathbf{s}, \tilde{\mathbf{s}}))$, such that $kh \in \mathcal{G}(\mathbf{s}, \tilde{\mathbf{s}})$ if and only if $S_{kh}(\mathbf{s}, \tilde{\mathbf{s}}) \neq \emptyset$. Thus, the single-group deviation forms a single directed edge $\{kh\} = \mathcal{G}(\mathbf{s}, \tilde{\mathbf{s}})$. Since gradual coalitional deviation $S(\mathbf{s}, \tilde{\mathbf{s}}) \equiv \cup_{k \neq h} S_{kh}(\mathbf{s}, \tilde{\mathbf{s}})$ requires that all action h is pointed by at most one action, each component of $\mathcal{G}(\mathbf{s}, \tilde{\mathbf{s}})$ is a cycle, an out-tree, or a cycle with out-trees stemming out of it.[11] It is important to note that the in these gradual deviations, the dyadic payoffs derived among partners remain unchanged if ADG is *uniform* and *local*. The class of gradual coalitional deviations covers much wider class of directed graphs, thus, yields allows for a wider range of coalitional deviations than that of the former two types. It is easy to see that the unique NE of Example 2 is also WLE, and LE, while SNE does not exist.

The deviations of the former type are allowed only for a group of players who all choose the same actions before and after a deviation. The deviations of the latter

[11] A path $\{(k_1, k_2), (k_2, k_3), ..., (k_{q-1}, k_q)\}$ can form a trivial cycle by including (k_q, k_1) with $S_{k_q k_1}(\mathbf{s}, \tilde{\mathbf{s}}) = \emptyset$.

type are less restrictive and require that only those members of the deviating group who choose the same action at $\tilde{\mathbf{s}}$, also share the same action at $\mathbf{s}$. Naturally, we have the following inclusion relationships:

$$SNE \subset LE \subset WLE \subset NE,$$

where NE denotes the set of pure strategies Nash equilibria. Example 2 shows that the sets SNE and LE do not always coincide. Example 3 demonstrates that, in general, LE, WLE, and NE are different from each other even under $BADG$ case.

Example 3. Let $N = \{1,2,3,4,5\}$ and $M = \{a,b,c\}$. Suppose that in game $\Gamma \in BADG$ with $\sigma^i = 1$ for all $i \in N$, players' payoff information is summarized by the following table and propensity matrix P:

k	a	b	c
v_k^1	0	1.5	-10
v_k^2	0	1.5	-10
v_k^3	0	-10	0.5
v_k^4	-10	0	-10
v_k^5	0.5	-10	0

$$P = \begin{pmatrix} 0 & 1 & 0 & -1 & -1 \\ 1 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 \\ -1 & -1 & 0 & 0 & 0 \\ -1 & -1 & -1 & 0 & 0 \end{pmatrix}$$

The relevant strategy profiles and their payoff vectors are:

i	1	2	3	4	5		potential
$\mathbf{s}^1 = (a,a,a,b,c)$	1	1	0	0	0	NE	1
$\mathbf{s}^2 = (b,a,a,b,c)$	0.5	0	0	-1	0	no	0.5
$\mathbf{s}^3 = (a,a,c,b,a)$	0	0	0.5	0	-1.5	no	-1
$\mathbf{s}^4 = (b,b,a,b,c)$	1.5	1.5	0	-2	0	WLE	2
$\mathbf{s}^5 = (b,b,c,b,a)$	1.5	1.5	0.5	-2	0.5	$LE = SNE$	3
$\mathbf{s}^6 = (b,a,a,b,a)$	0.5	-1	-0.5	-1	-0.5	no	-3

Note that players 1 and 2 are symmetric. In this game, there are three Nash equilibria, $\mathbf{s}^1$, $\mathbf{s}^4$, and $\mathbf{s}^5$. From Nash equilibrium $\mathbf{s}^1$, players 1 and 2 can move together to b through a single-group coalitional deviation, generating $\mathbf{s}^4$. Although player 4 is negatively affected by the deviation by players 1 and 2, the new strategy profile $\mathbf{s}^4$ is immune to any further single-group coalitional deviations. Thus, $\mathbf{s}^4$ is a weak landscape equilibrium. However, from $\mathbf{s}^4$, a gradual coalitional deviation composed by players 3 and 5 swapping their actions generates $\mathbf{s}^5$, and $\mathbf{s}^5$ is a landscape equilibrium. Note that $\mathbf{s}^5$ can be brought directly from $\mathbf{s}^1$ by a three-group gradual coalitional deviation involving players 1, 2, 3, and 5 as well. As we saw earlier, the values of the potential of the game for $\mathbf{s}^1$, $\mathbf{s}^4$, and $\mathbf{s}^5$ are 1, 2, and 3, respectively. Thus, gradual coalitional deviations allow the value of the potential to raise without being stuck at a local maximum potential Nash equilibrium. Note that $\mathbf{s}^5$ is also a strong Nash equilibrium.■

5 Existence of a Weak Landscape Equilibrium in the $UADG$ Class

Example 1 shows that potential maximizers are not necessarily weak landscape equilibria. The ADG game in Example 1 is local but not uniform. Theorem 2 states that any game in the $UADG$ class admits a weak landscape equilibrium.

Theorem 2. Every game $\Gamma \in UADG$ admits a weak landscape equilibrium. In particular, a weighted-potential-maximizing strategy profile is a weak landscape equilibrium.

Proof. Suppose that $\bar{\mathbf{s}} \in \arg\max_{\mathbf{s}} \mathbb{F}(\mathbf{s})$, and suppose that a coalition $\emptyset \neq I \subseteq G_k(\bar{\mathbf{s}})$ is a profitable deviation by switching their actions from k to h together given $\bar{\mathbf{s}}_{-I}$ intact. Let $\tilde{\mathbf{s}} \equiv (\bar{\mathbf{s}}_{-I}, (h)_{i\in I})$. Consider a potential of a reduced game without players in coalition I and strategy profile $\mathbf{s}^{-i}$. Then, we have

$$\mathbb{F}(\bar{\mathbf{s}})-\mathbb{F}(\bar{\mathbf{s}}^{-I}) = \sum_{i\in I}\sigma^i v_k^i+\sum_{i\in I}\sum_{\ell\in M\setminus\{k\}}\sum_{j\in G_\ell(\bar{\mathbf{s}})\setminus I}\sigma^i p_{k\ell}^{ij}\sigma^j+\sum_{i\in I}\sum_{j\in G_k(\bar{\mathbf{s}})\setminus I}\sigma^i p_{kk}^{ij}\sigma^j+\sum_{i\in I}\sum_{j\in I}\sigma^i p_{kk}^{ij}\sigma^j,$$

since there are two cases $k=\ell$ and $k=\ell'$ with $p_{k\ell}^{ij}=p_{\ell k}^{ji}$ (and $p_{kk}^{ii}=0$). Let $\tilde{\mathbf{s}}=(\tilde{s}^i,\bar{\mathbf{s}}^{-i})$ with $\tilde{s}^i=h$. Similarly we have

$$\mathbb{F}(\tilde{\mathbf{s}})-\mathbb{F}(\bar{\mathbf{s}}^{-I}) = \sigma^i v_h^i+\sum_{i\in I}\sum_{\ell\in M\setminus\{h\}}\sum_{j\in G_\ell(\bar{\mathbf{s}})}\sigma^i p_{h\ell}^{ij}\sigma^j+\sum_{i\in I}\sum_{j\in G_h(\bar{\mathbf{s}})}\sigma^i p_{hh}^{ij}\sigma^j+\sum_{i\in I}\sum_{j\in I}\sigma^i p_{hh}^{ij}\sigma^j.$$

Since uniformity implies $p_{kk}^{ij}=p_{hh}^{ij}=p^{ij}$ for all $i,j\in N$, this implies

$$\begin{aligned}\mathbb{F}(\tilde{\mathbf{s}})-\mathbb{F}(\bar{\mathbf{s}}) &= \sum_{i\in I}\sigma^i v_h^i+\sum_{i\in I}\sum_{\ell\in M\setminus\{h\}}\sum_{j\in G_\ell(\bar{\mathbf{s}})}\sigma^i p_{h\ell}^{ij}\sigma^j+\sum_{i\in I}\sum_{j\in G_h(\bar{\mathbf{s}})}\sigma^i p^{ij}\sigma^j\\ &\quad -\sum_{i\in I}\sigma^i v_k^i-\sum_{i\in I}\sum_{\ell\in M\setminus\{k\}}\sum_{j\in G_\ell(\bar{\mathbf{s}})}\sigma^i p_{k\ell}^{ij}\sigma^j-\sum_{i\in I}\sum_{j\in G_k(\bar{\mathbf{s}})\setminus I}\sigma^i p^{ij}\sigma^j.\end{aligned}$$

Since all player $i\in I$ is better off by joining coalitional deviation I, we have

$$\begin{aligned}u^i(\tilde{\mathbf{s}})-u^i(\bar{\mathbf{s}}) &= v_h^i+\sum_{\ell\in M\setminus\{h\}}\sum_{j\in G_\ell(\bar{\mathbf{s}})}p_{h\ell}^{ij}\sigma^j+\sum_{j\in G_h(\bar{\mathbf{s}})}p^{ij}\sigma^j-v_k^i-\sum_{\ell\in M\setminus\{k\}}\sum_{j\in G_\ell(\bar{\mathbf{s}})}p_{k\ell}^{ij}\sigma^j-\sum_{j\in G_k(\bar{\mathbf{s}})\setminus I}p^{ij}\sigma^j\\ &> 0,\end{aligned}$$

since uniformity implies $p_{kk}^{ij}=p_{hh}^{ij}=p^{ij}$ for all $i,j\in N$. Thus, we can conclude

$$\begin{aligned}\sum_{i\in I}\sigma^i\left(u^i(\tilde{\mathbf{s}})-u^i(\bar{\mathbf{s}})\right) &= \sum_{i\in I}\sigma^i v_h^i+\sum_{i\in I}\sum_{\ell\in M\setminus\{h\}}\sum_{j\in G_\ell(\bar{\mathbf{s}})}\sigma^i p_{h\ell}^{ij}\sigma^j+\sum_{i\in I}\sum_{j\in G_h(\bar{\mathbf{s}})}\sigma^i p^{ij}\sigma^j\\ &\quad -\sum_{i\in I}\sigma^i v_k^i-\sum_{i\in I}\sum_{\ell\in M\setminus\{k\}}\sum_{j\in G_\ell(\bar{\mathbf{s}})}\sigma^i p_{k\ell}^{ij}\sigma^j-\sum_{i\in I}\sum_{j\in G_k(\bar{\mathbf{s}})\setminus I}\sigma^i p^{ij}\sigma^j\\ &= \mathbb{F}(\tilde{\mathbf{s}})-\mathbb{F}(\bar{\mathbf{s}})>0.\end{aligned}$$

This is a contradiction. Hence, under uniformity, there is no profitable type 1 deviation from $\bar{\mathbf{s}}$, and $\bar{\mathbf{s}}$ is a weak landscape equilibrium.$\square$

The following example shows that uniformity is necessary for the existence of weak landscape equilibrium.

Example 4. Let $N = \{1, 2\}$ and $M = \{a, b\}$. Suppose that in game $\Gamma \in LADG$ with $\sigma^i = 1$ for all $i \in N$, players' payoff information is summarized by the following table

k	a	b
v_k^1	2	-3
v_k^2	2	-3

and the propensity matrices are

$$P_{aa} = \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix} \quad P_{bb} = \begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix} \quad P_{ab} = P_{ba} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

violating uniformity. The unique Nash equilibrium is $\mathbf{s} = (a, a)$ with payoff vector $(0, 0)$, but players 1 and 2 together can form a profitable single-group coalitional deviation by moving from a to b, generating payoff vector $(1, 1)$. However, from $\mathbf{s}' = (b, b)$, either player wants to go back to a unilaterally. Thus, there is no weak landscape equilibrium without uniformity even if ADG is local.$\blacksquare$

We may consider an intermediate equilibrium concept between WLE and LE. Let a coalitional deviation $S(\mathbf{s}, \tilde{\mathbf{s}}) \equiv \cup_{k \neq h} S_{kh}(\mathbf{s}, \tilde{\mathbf{s}})$ be of **a splitting-single-group** if there is at a single $k \in M$ with $|S_{kh}(\mathbf{s}, \tilde{\mathbf{s}})| > 0$, and $|S_{\ell h}(\mathbf{s}, \tilde{\mathbf{s}})| = 0$ for all $\ell \neq k$. This condition allows that a coalitional deviation from k can split to different destinations. A strategy profile $\mathbf{s}$ is a **intermediate landscape equilibrium** if it is immune to all splitting-single-group coalitional deviations. Note that while the sets WLE and ILE coincide if ADG is *local*. it is not necessarily true if the game is not *local* even if it is *uniform*. More precisely, the game does not admit ILE, although WLE does exist.

Example 5. Let $N = \{1, 2\}$ and $M = \{a, b, c\}$. Suppose that in game $\Gamma \in UADG$ with $\sigma^i = 1$ for all $i \in N$, players' payoff information is summarized by the following table

k	a	b	c
v_k^1	2	-3	-10
v_k^2	2	-10	-3

and the propensity matrices are

$$P_{aa} = P_{bb} = \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix} \quad P_{bc} = \begin{pmatrix} 0 & 4 \\ 0 & 0 \end{pmatrix} \quad P_{cb} = \begin{pmatrix} 0 & 0 \\ 4 & 0 \end{pmatrix},$$

and all other $P_{k\ell} = \mathbf{0}$. The unique Nash equilibrium is $\mathbf{s} = (a, a)$ with payoff vector $(0, 0)$, but players 1 and 2 together can form a profitable single-group coalitional deviation by moving from a to b and c, respectively, generating payoff vector $(1, 1)$. However, from $\mathbf{s}' = (b, c)$, either player wants to go back to a unilaterally. Thus, although $\mathbf{s}$ is a weak landscape equilibrium, it is not immune to a splitting coalitional deviation, and intermediate landscape equilibrium does not exist.■

Examples 4 and 5 may motivate to impose *uniformity* and *external uniformity* together on ADG class. However, as footnote 9 indicates, with uniformity and external uniformity together, ADG class becomes *basic* (*local* and *uniform*). Thus, it suffices to focus on the class of $BADG$ to deal with *uniformity* and *external uniformity* together.

6 Existence of Landscape Equilibria in the $BADG$ Class

Le Breton et al. (2021) proved the existence of landscape equilibrium in $NBADG$ class by assuming $v_k^i = 0$ for all $i \in N$ and $k \in M$. In this section, we slightly generalize their result by expanding the class, dropping neutrality:

$$\mathbb{F}(\mathbf{s}) \equiv \sum_{\ell \in M} \sum_{j \in G_\ell(\mathbf{s})} \sigma^j v_\ell^j + \frac{1}{2} \sum_{j \in G_\ell(\mathbf{s})} \sum_{j' \in G_{\ell'}(\mathbf{s})} \sigma^j p^{jj'} \sigma^{j'}.$$

In order for $\mathbb{F}(\mathbf{s})$ to be a potential function for coalitional deviations, we need to restrict the set of admissible coalitional deviations. The following lemma proves the desired result for a coalitional deviation that forms a single cycle.

Lemma 1. Suppose that game $\Gamma \in BADG$. Consider a coalitional deviation $S(\mathbf{s}, \tilde{\mathbf{s}})$ which forms a single cycle $\{k_1, k_2, ..., k_q, k_{q+1}\}$ with $k_{q+1} = k_1$ in its graph representation $(M, \mathcal{G}(\mathbf{s}, \tilde{\mathbf{s}}))$. Then, we have $\mathbb{F}(\mathbf{s})$ as a weighted potential function for such a class of coalitional deviations $S(\mathbf{s}, \tilde{\mathbf{s}})$.

Proof. Suppose that a coalitional deviation $S(\mathbf{s}, \tilde{\mathbf{s}})$ which forms a single $\{k_1, ..., k_q, ..., k_Q, k_{Q+1}\}$ with $k_{Q+1} = k_1$. Then, $\{k_1, ..., k_q, ..., k_Q, k_{Q+1}\} = S(\mathbf{s}, \tilde{\mathbf{s}})$, and for all $k_q \in S(\mathbf{s}, \tilde{\mathbf{s}})$, $G_q(\tilde{\mathbf{s}}) = G_q(\mathbf{s}) + S_{q-1,q}(\mathbf{s}, \tilde{\mathbf{s}}) - S_{q,q+1}(\mathbf{s}, \tilde{\mathbf{s}})$, where G_q and $S_{q,q+1}$ are abbreviations of G_{k_q} and $S_{k_q k_{q+1}}$. Then, we have

$$\begin{aligned}
\mathbb{F}(\tilde{\mathbf{s}}) - \mathbb{F}(\mathbf{s}) \quad = \quad & \sum_{q=1}^{Q} \sum_{i \in S_{q-1,q}(\mathbf{s}, \tilde{\mathbf{s}})} \sigma^i v_k^i - \sum_{q=1}^{Q} \sum_{S_{q,q+1}(\mathbf{s}, \tilde{\mathbf{s}})} \sigma^i v_k^i \\
& + \sum_{q=1}^{Q} \sum_{i \in S_{q-1,q}(\mathbf{s}, \tilde{\mathbf{s}})} \sum_{j \in G_k(\mathbf{s}) \setminus S_{q,q+1}(\mathbf{s}, \tilde{\mathbf{s}})} \sigma^i p^{ij} \sigma^j \\
& - \sum_{q=1}^{Q} \sum_{i \in S_{q,q+1}(\mathbf{s}, \tilde{\mathbf{s}})} \sum_{j \in G_k(\mathbf{s}) \setminus S_{q,q+1}(\mathbf{s}, \tilde{\mathbf{s}})} \sigma^i p^{ij} \sigma^j,
\end{aligned}$$

since the benefits or losses the group remaining at action k_q $(G_q(\mathbf{s})\backslash S_{q,q+1}(\mathbf{s},\tilde{\mathbf{s}}))$ receives is exactly the same as the benefits or losses the relevant departing group $S_{q,q+1}(\mathbf{s},\tilde{\mathbf{s}})$ and arriving group $S_{q-1,q}(\mathbf{s},\tilde{\mathbf{s}})$ obtaining from action k_q. Rewriting this, we obtain

$$\begin{aligned}\mathbb{F}(\tilde{\mathbf{s}})-\mathbb{F}(\mathbf{s}) &= \sum_{q=1}^{Q}\sum_{i\in S_{q-1,q}(\mathbf{s},\tilde{\mathbf{s}})}\sigma^i u^i(\tilde{\mathbf{s}})-\sum_{q=1}^{Q}\sum_{S_{q,q+1}(\mathbf{s},\tilde{\mathbf{s}})}\sigma^i u^i(\mathbf{s})\\ &= \sum_{q=1}^{Q}\sum_{i\in S_{q-1,q}(\mathbf{s},\tilde{\mathbf{s}})}\sigma^i\left(u^i(\tilde{\mathbf{s}})-u^i(\mathbf{s})\right).\end{aligned}$$

Thus, if all $i\in S(\mathbf{s},\tilde{\mathbf{s}})$ is improving by joining the coalition $S(\mathbf{s},\tilde{\mathbf{s}})$, then $\mathbb{F}(\tilde{\mathbf{s}})>\mathbb{F}(\mathbf{s})$ must hold.■

Note that the statement of Lemma 1 holds for coalitional deviation $S(\mathbf{s},\tilde{\mathbf{s}})$ which forms a path $\{k_1,k_2,...,k_q\}$ in its graph representation $(M,\mathcal{G}(\mathbf{s},\tilde{\mathbf{s}}))$ (since it can be seen as a trivial cycle), by simply setting $S_{Q,Q+1}(\mathbf{s},\tilde{\mathbf{s}})(=S_{Q,1}(\mathbf{s},\tilde{\mathbf{s}}))=\emptyset$.

The next lemma follows from Lemma 1.

Lemma 2. Suppose that game $\Gamma\in BADG$. Consider a profitable gradual coalitional deviation $S(\mathbf{s},\tilde{\mathbf{s}})$. Then, there is a subset $T\subseteq S(\mathbf{s},\tilde{\mathbf{s}})$ such that $T=S(\mathbf{s},(\tilde{\mathbf{s}}_T,\mathbf{s}_{-T}))$ is a profitable coalitional deviation that forms a cycle (or a path).

Proof. Since $S(\mathbf{s},\tilde{\mathbf{s}})$ is a gradual coalitional deviation, any component of $\mathcal{G}(\mathbf{s},\tilde{\mathbf{s}})$ is composed of a cycle, an out-tree, or a combination of cycles and out-trees in a directed graph. Suppose that there is a terminating node h: i.e., $S_{kh}(\mathbf{s},\tilde{\mathbf{s}})\neq\emptyset$, but $S_{h\ell}(\mathbf{s},\tilde{\mathbf{s}})=\emptyset$ for all $\ell\neq h$. Then, $S_{kh}(\mathbf{s},\tilde{\mathbf{s}})$ itself is a profitable coalitional deviation since Γ is local. Thus, $\{k,h\}$ forms a trivial cycle (a path). Suppose that there is no terminal node. Then, $\mathcal{G}(\mathbf{s},\tilde{\mathbf{s}})$ is composed of a cycle, or non-overlapping cycles. In the latter case, pick a cycle. Such a cycle forms a profitable coalitional deviation, since Γ is local. We have completed the proof.■

Lemmas 1 and 2 yield a generalization of Le Breton et al. (2021). This generalization opens up applications for discrete choice models with network externalities and paves the way for exploration of econometric side of the model.

Theorem 3. Every game $\Gamma\in BADG$ admits a landscape equilibrium. In particular, a weighted-potential-maximizing strategy profile is a landscape equilibrium.

Proof of Theorem 3. Let $\bar{\mathbf{s}}\in\arg\max_{\mathbf{s}}\mathbb{F}(\mathbf{s})$. Suppose that $\bar{\mathbf{s}}$ is not immune to a landscape coalitional deviation $S(\bar{\mathbf{s}},\tilde{\mathbf{s}})\neq\emptyset$. By Lemma 2, there is a profitable coalitional deviation that forms a cycle or a path. By Lemma 1, there is a strategy profile $\tilde{\mathbf{s}}$ with $\mathbb{F}(\tilde{\mathbf{s}})>\mathbb{F}(\bar{\mathbf{s}})$, a contradiction.□

Note that Theorem 3 generalizes Dower et al. (2026) who considered the games in the $NBADS$ class with two actions $|M|=2$ that allow various types of coalitional deviations. When $|M|=2$, landscape equilibrium is equivalent to strong Nash equilibrium, since directed graph $(m, \mathcal{G}(\mathbf{s}, \tilde{\mathbf{s}}))$ of any coalitional deviation $S(\mathbf{s}, \tilde{\mathbf{s}})$ is a single-group deviation or is to form a cycle.[12] As Theorem 3, we can dispense neutrality in showing the existence of strong Nash equilibrium with two actions.

Corollary 1. For every game in $BADG$ with two feasible actions, a weighted-potential-maximizing strategy profile is a strong Nash equilibrium.

7 Landscape Equilibria in the $NBADG$ Class

Here, we investigate the role of the neutrality assumption. Consider a σ^i-weighted Benthamite social welfare:

$$\begin{aligned}\mathbb{W}(\mathbf{s}) &\equiv \sum_{k\in M}\left[\sum_{i\in G_k(\mathbf{s})}\sigma^i\left(v_k^i+\sum_{j\in G_k(\mathbf{s})}p^{ij}\sigma^j\right)\right]\\ &= \sum_{k\in M}\left[\sum_{i\in G_k(\mathbf{s})}\sigma_k^i v_k^i+\sum_{i\in G_k(\mathbf{s})}\sum_{j\in G_k(\mathbf{s})}\sigma^i p^{ij}\sigma^j\right].\end{aligned}$$

Under neutrality, we have $v_k^i = 0$ for all $k \in M$ and $i \in N$, and $\mathbb{F}(\mathbf{s})$ becomes:

$$\mathbb{W}(\mathbf{s}) = \sum_{k\in M}\left[\sum_{i\in G_k(\mathbf{s})}\sum_{j\in G_k(\mathbf{s})}\sigma^i p^{ij}\sigma^j\right].$$

The potential $\mathbb{F}$ in this case becomes:

$$\mathbb{F}(\mathbf{s}) \equiv \frac{1}{2}\sum_{k\in M}\left[\sum_{i\in N}\sum_{j\in N}\sigma_k^i p^{ij}\sigma_k^j\right] = \frac{1}{2}\mathbb{W}(\mathbf{s}).$$

Thus, we have $\mathbb{F}(\mathbf{s}) = \frac{1}{2}\mathbb{W}(\mathbf{s})$, we have the result by Le Breton et al. (2021).

Proposition 1. For every game $\Gamma \in NBADG$, a strategy profile $\mathbf{s}$ that maximizes potential function $\mathbb{F}(\mathbf{s})$ is a strongly Pareto optimal landscape equilibrium.

Although Proposition 1 shows that the potential-maximizing strategy profile is not only a landscape equilibrium but also is a strongly Pareto efficient one under neutrality, the next example shows that a potential-maximizing strategy profile may not be Pareto-efficient when neutrality is dropped. In fact, the constructed game does not admit a strongly Pareto efficient Nash equilibrium.

[12] Dower et al. (2026) computed a potential-maximizing strong Nash equilibrium of the game. Although this class of problems is known NP hard, they could use the fact that the potential maximization for a $NBADG$ game is equivalent to the celebrated MAXCUT problem in combinatorial optimization (Goemans and Williamson, 1995) in the case where $p_{ij} \leq 0$ for all i and j.

Example 6. Suppose that game $\Gamma \in BADG$ has four players $N = \{1, 2, 3, 4\}$ and three actions $M = \{a, b, c\}$ with the following payoff information:

v_k^i	1	2	3	4
a	2.6	2.6	-10	-10
b	-10	-10	2.6	2.6
c	0	0	0	0

$$P = \begin{pmatrix} 0 & -1 & 1.5 & 1.5 \\ -1 & 0 & 1.5 & 1.5 \\ 1.5 & 1.5 & 0 & -1 \\ 1.5 & 1.5 & -1 & 0 \end{pmatrix}$$

We set $\sigma^i = 1$ for all $i \in N$. In this example, there are four relevant strategy profiles: $\mathbf{s}^1$, $\mathbf{s}^2$, $\mathbf{s}^3$, and $\mathbf{s}^4$:

$\mathbf{s}$	(u^1, u^2, u^3, u^4)	$\mathbb{F}(\mathbf{s})$
$\mathbf{s}^1 = (a, a, b, b)$	$(1.6, 1.6, 1.6, 1.6)$	8.4
$\mathbf{s}^2 = (c, c, c, c)$	$(2, 2, 2, 2)$	4
$\mathbf{s}^3 = (a, c, c, c)$	$(2.6, 3, 0.5, 0.5)$	4.6
$\mathbf{s}^4 = (a, c, b, c)$	$(2.6, 1.5, 2.6, 1.5)$	6.7

The potential-maximizing strategy profile is $\mathbf{s}^1$, achieving $u(\mathbf{s}^1) = (1.6, 1.6, 1.6, 1.6)$ and $\mathbb{F}(\mathbf{s}^1) = 8.4$. This is a landscape equilibrium, Pareto dominated by $\mathbf{s}^2$, which is not even a Nash equilibrium. In fact, $\mathbf{s}^1$ is unique Nash equilibrium of this game.$\blacksquare$

Now, we will focus on the class of $NBADG$. The next example shows that the Benthamite-welfare-maximizing landscape equilibrium is not necessarily a strong Nash equilibrium, even if the game belongs to $NBADG$ and admits a strong Nash equilibrium. This example is a *dichotomous*, where $p^{ij} \in \{-1, 1\}$, and there are only (equally desirable) allies and (equally undesirable) foes for all players. This negative result is quite robust.

Example 7. Suppose that game $\Gamma \in NBADG$ has twelve players and five feasible actions. The game is dichotomous and we set $\sigma^i = 1$ for all $i \in N$. Let propensity matrix P be:

$$P = \begin{pmatrix}
\mathbf{0} & 1 & 1 & \mathbf{1} & -1 & -1 & \mathbf{1} & -1 & -1 & \mathbf{1} & -1 & -1 \\
1 & 0 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
1 & 1 & 0 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
\mathbf{1} & -1 & -1 & \mathbf{0} & 1 & 1 & \mathbf{1} & -1 & -1 & \mathbf{1} & -1 & -1 \\
-1 & -1 & -1 & 1 & 0 & 1 & -1 & -1 & -1 & -1 & -1 & -1 \\
-1 & -1 & -1 & 1 & 1 & 0 & -1 & -1 & -1 & -1 & -1 & -1 \\
\mathbf{1} & -1 & -1 & \mathbf{1} & -1 & -1 & \mathbf{0} & 1 & 1 & \mathbf{1} & -1 & -1 \\
-1 & -1 & -1 & -1 & -1 & -1 & 1 & 0 & 1 & -1 & -1 & -1 \\
-1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 0 & -1 & -1 & -1 \\
\mathbf{1} & -1 & -1 & \mathbf{1} & -1 & -1 & \mathbf{1} & -1 & -1 & \mathbf{0} & 1 & 1 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 0 & 1 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 0
\end{pmatrix}$$

The Benthamite social welfare is maximized under $\mathbf{s}$ which generates three groups $G(\mathbf{s}) = (\{1, 2, 3\}, \{4, 5, 6\}, \{7, 8, 9\}, \{10, 11, 12\})$ with its payoff vector $(2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2)$, and this partition is a potential-maximizing landscape equilibrium ($\mathbb{F}(\mathbf{s}) = 12$). However, players $\{1, 4, 7, 10\}$ can deviate from $\mathbf{s}$ to create $\mathbf{s}'$ which generates $G(\mathbf{s}') = (\{2, 3\}, \{5, 6\}, \{8, 9\}, \{11, 12\}, \{1, 4, 7, 10\})$ with its payoff vector $(3, 1, 1, 3, 1, 1, 3, 1, 1, 3, 1, 1)$. Thus, $\mathbf{s}$ is not a strong Nash equilibrium. In contrast, $\mathbf{s}'$ is not Benthamite-welfare-maximizing ($\mathbb{F}(\mathbf{s}') = 10$), but it is a strong Nash equilibrium.■

The following example shows that there may also be a strictly Pareto-dominated landscape equilibrium, even though Le Breton et al. (2021) show that a potential-maximizing partition is always a strongly Pareto-efficient landscape equilibrium under neutrality.

Example 8. Suppose that game $\Gamma \in NBADG$ has six players, and six actions with the following propensity matrix P:

$$P = \begin{pmatrix} 0 & 1 & 1 & 1.5 & -1 & -1 \\ 1 & 0 & 0 & -1 & -1 & -1 \\ 1 & 0 & 0 & -1 & -1 & -1 \\ 1.5 & -1 & -1 & 0 & 1 & 1 \\ -1 & -1 & -1 & 1 & 0 & 0 \\ -1 & -1 & -1 & 1 & 0 & 0 \end{pmatrix}$$

We set $\sigma^i = 1$ for all $i \in N$. The Benthamite social welfare is maximized under $\mathbf{s}$ which generates two groups $G(\mathbf{s}) = (\{1, 2, 3\}, \{4, 5, 6\})$ with its payoff vector $(2, 1, 1, 2, 1, 1)$, and this partition is a landscape equilibrium. Actually, $\mathbf{s}$ is a strong Nash equilibrium. However, $G(\mathbf{s}') = (\{1, 4\}, \{2\}, \{3\}, \{5\}, \{6\})$ with its payoff vector $(1.5, 0, 0, 1.5, 0, 0)$ is also a landscape equilibrium.■

Finally, we show that there may be multiple strong Nash equilibria, while only one of them is potential-maximizing. In other words, there may be a strong Nash equilibrium that is not potential-maximizing even under neutrality.

Example 9. Suppose that game $\Gamma \in NBADG$ with six players, and six actions in a neutral basic game with propensity matrix P:

$$P = \begin{pmatrix} 0 & 2.5 & 1 & -10 & 1 & 2 \\ 2.5 & 0 & 2 & 1 & -10 & -10 \\ 1 & 2 & 0 & 2 & -10 & -10 \\ -10 & 1 & 2 & 0 & 2.5 & 1 \\ 1 & -10 & -10 & 2.5 & 0 & 2 \\ 2 & -10 & -10 & 1 & 2 & 0 \end{pmatrix}$$

We set $\sigma^i = 1$ for all $i \in N$. There are two strong Nash equilibrium group structures under $\mathbf{s}$ and $\tilde{\mathbf{s}}$, which generate $G(\mathbf{s}) = (\{1, 2, 3\}, \{4, 5, 6\})$ with its payoff vector $(3.5, 4.5, 3, 3.5, 4.5, 3)$, and $G(\tilde{\mathbf{s}}) = (\{2, 3, 4\}, \{5, 6, 1\})$ with its payoff vector $(3, 3, 4, 3, 3, 4)$, respectively. The former is a potential-maximizer, but the latter is not.■

8 Landscape Equilibria in Local ADG Class with an Outside Option

In this section we consider the possibility of having an outside option: players can choose their action from an augmented action set $\bar{M} \equiv M \cup \{0\}$.[13] We regard an outside option 0 as a special action with a $n \times n$ propensity matrix $P_{00} \equiv (\mathbf{0})_{i,j \in N}$, all of whose elements are zero. *We naturally set* $v_0^i = 0$ *for all* $i \in N$. We call a local ADG game with an outside option M**-uniform** if $P_{kk} = P$ for all $k \in M$, and $P_{00} = (\mathbf{0})_{i,j \in N}$. Note that such a game does not satisfy *uniformity* and Theorem 2 is not applicable in this domain. Moreover, the following simple example demonstrates a possibility of nonexistence of a weak landscape equilibrium in presence of an outside option.

Example 10. Suppose that game $\Gamma \in LADG$ with three players $N = \{1, 2, 3\}$ and a single action $M = \{a\}$ and an outside option 0 with M*-uniformity*. The propensity matrix is given by

$$P = P_{aa} = \begin{pmatrix} 0 & -1 & 2 \\ -1 & 0 & 3 \\ 2 & 3 & 0 \end{pmatrix},$$

and $v_a^1 = -0.5$, $v_a^2 = v_a^3 = -2.5$. We set $\sigma^i = 1$ for all $i \in N$. The payoff of each player i is determined by

$$u_k^i(\mathbf{s}) = v_k^i + \sum_{j \in N} p^{ij}$$

for all $i \in N$ and $k \in M$, while $u_0^i = 0$ for all $i \in N$ irrespective of who are choosing 0. There are five relevant strategy profiles:

$\mathbf{s}$	(u^1, u^2, u^3)
$\mathbf{s}^1 = (0, 0, 0)$	$(0, 0, 0)$
$\mathbf{s}^2 = (0, a, a)$	$(0, 0.5, 0.5)$
$\mathbf{s}^3 = (a, a, a)$	$(0.5, -0.5, 2.5)$
$\mathbf{s}^4 = (a, 0, a)$	$(1.5, 0, -0.5)$
$\mathbf{s}^5 = (a, 0, 0)$	$(-0.5, 0, 0)$

As is clear from the table, $\mathbf{s}^1$ is a unique Nash equilibrium. But it is not immune to a joint deviation by players 2 and 3. Thus, there is no weak landscape equilibrium in this game.■

Actually, it is hard to obtain any positive result for any equilibrium concept based on coalitional deviations. The main reason for this is that even if we restrict the class of games by imposing *neutrality*, a potential function does not perform well with an outside option. The next example shows that even with a single feasible

[13] Binmore, et al. (1989) analyzed how bargaining outcomes could be affected in the presence of an outside option.

action ($|M| = 1$), the potential-maximizing strategy profile may not even be a weak landscape equilibrium with an outside option.

Example 11. Suppose that local game $\Gamma \in NLADG$ with four players $N = \{1, 2, 3, 4\}$, a single action $M = \{a\}$ and an outside option 0 with *M-uniformity.* The propensity matrix is given by

$$P = \begin{pmatrix} 0 & 2.1 & -1 & -1 \\ 2.1 & 0 & -1 & -1 \\ -1 & -1 & 0 & 2.1 \\ -1 & -1 & 2.1 & 0 \end{pmatrix}.$$

We set $\sigma^i = 1$ for all $i \in N$. Clearly, weak landscape equilibrium is unique $\mathbf{s}^* = (a, a, a, a)$ with $u^i(\mathbf{s}^*) = 0.1$, but the potential is not maximized at $\mathbf{s}^*$. Action profiles $\mathbf{s}^1 = (a, a, 0, 0)$ and $\mathbf{s}^2 = (0, 0, a, a)$ achieve the maximum potential $\mathbb{F}(\mathbf{s}^1) = \mathbb{F}(\mathbf{s}^2) = 2.1$, while the unique weak landscape equilibrium $\mathbf{s}^*$ has $\mathbb{F}(\mathbf{s}^*) = 0.2$. Notice that $\mathbf{s}^*$ is a strong Nash equilibrium, too.$\blacksquare$

Example 11 shows that (1) some further restrictions on the class of $NLADG$ games are needed to get existence of equilibrium and (2) that techniques other than potential maximization will have to be used. As far as we know, the question of existence of a SNE, a LE, or a WLE in the presence of an outside option is open. However, if we consider the case with *positive externalities* ($p^{ij} \geq 0$ holds for any $i, j \in N$), we can obtain a positive result. The logic of the proof uses neither our functional specification nor a potential function. We will show the existence of strong Nash equilibrium in the case of $|M| = 2$.

Theorem 4. Suppose that $p^{ij} \geq 0$ holds for any $i, j \in N$ in a local and *M-uniform* $LADG$ with an outside option. If $|M| = 2$, there is a strong Nash equilibrium.

Proof. In this case, even if $p^{ij} = 0$, there is no negative impact from other players joining an existing group. Thus, this case can be considered as a (weakly) positive externalities case. In what follows, it is more convenient to use group structures $\mathbf{G} \equiv (G_a, G_b, G_0)$ instead of $\mathbf{s}$, where $G_k \subseteq N$ denotes the set of players at action $k \in \{a, b, 0\}$. We consider the following algorithm.

1. Let $\mathbf{G}^0 \equiv (G_a^0, G_b^0, G_0^0) = (\emptyset, N, \emptyset)$.
2. Suppose that $\mathbf{G}^{t'} \equiv (G_a^{t'}, G_b^{t'}, G_0^{t'})$ has been defined for all $t' = 0, ..., t$. Let $\mathbf{G}^{t+1} \equiv (G_a^{t+1}, G_b^{t+1}, G_0^{t+1})$ with $G_b^{t+1} \equiv \{i \in G_b^t : u_b^i(G_b^t) > 0\}$ and $G_0^{t+1} \equiv \{i \in G_0^t : u^i(G_b^t) \leq 0\} \cup G_0^t$.

Due to positive externalities, $G_b^{t+1} \subseteq G_b^t$ for all $t = 0, 1, ...$, and there is T such that $\mathbf{G}^t = \mathbf{G}^T$ for all $t \geq T$ due to finiteness of N. Let $\bar{\mathbf{G}} = \mathbf{G}^T$. If $\bar{\mathbf{G}}$ is a strong Nash equilibrium, we are done. Thus, suppose that $\bar{\mathbf{G}}$ is not a strong Nash equilibrium. Then, a subgroup of $S \subseteq \bar{G}_0 \cup \bar{G}_b$ wants to join a together. Let $S_b = S \cap \bar{G}_b$ and $S_0 = S \cap \bar{G}_0$. Let S move to a. That is, there is $\mathbf{G}' = (G'_a, G'_b, G'_0)$ where $G'_a = \bar{G}_a \cup S_b \cup S_0$,

$G_b' = \bar{G}_b \backslash S_b$, and $G_0' = \bar{G}_0 \backslash S_0$ such that $u^i(\mathbf{G}') > u^i(\bar{\mathbf{G}})$ for all $i \in S$. Let $\bar{\mathbf{G}}' = (\bar{G}_a', \bar{G}_b', \bar{G}_0')$ with $\bar{G}_b' \equiv \{i \in G_b' : u_b^i(G_b') > 0\}$ and $\bar{G}_0' \equiv \left\{i \in \bar{G}_b : u_b^i(\bar{G}_b) \le 0\right\} \cup \bar{G}_0$. If $\bar{\mathbf{G}}'$ is a strong Nash equilibrium, we are done. Otherwise, there are $S' \subset \bar{G}_b' \cup \bar{G}_0'$ and $\mathbf{G}'' = (\bar{G}_a'', \bar{G}_b'', \bar{G}_0'')$, where $G_a'' = \bar{G}_a' \cup S_b' \cup S_0'$, $G_b'' = \bar{G}_b' \backslash S_b'$, and $G_0'' = \bar{G}_0' \backslash S_0'$ such that $u^i(\mathbf{G}'') > u^i(\bar{\mathbf{G}}')$ for all $i \in S'$. Let $\bar{\mathbf{G}}'' = (\bar{G}_a'', \bar{G}_b'', \bar{G}_0'')$ with $\bar{G}_b'' \equiv \{i \in G_b'' : u_b^i(G_b'') > 0\}$ and $\bar{G}_0'' \equiv \left\{i \in \bar{G}_0' : u_b^i(G_b'') \le 0\right\} \cup \bar{G}_0'$. In this procedure, $\bar{G}_a \subseteq \bar{G}_a' \subseteq \bar{G}_a'' \subseteq ...$, and the members' payoff u_a^i does not decline. In contrast, $\bar{G}_b \supseteq \bar{G}_b' \supseteq \bar{G}_b'' \supseteq ...$, and the members' payoff u_b^i does not rise. If we continue the process until there is no further deviations, we arrive at a strong Nash equilibrium. Similarly, we can launch our algorithm with $\mathbf{G}^0 \equiv (G_a^0, G_b^0, G_0^0) = (N, \emptyset, \emptyset)$, instead. The same procedure converges to a strong Nash equilibrium.$\square$

Example 12. Suppose that game $\Gamma \in LADG$ with three players $N = \{1, 2, 3\}$ and action set $M = \{a, b, c\}$ and an outside option 0 with *M-uniformity*. The propensity matrix is given by

$$P = \begin{pmatrix} 0 & 3 & 3 \\ 3 & 0 & 3 \\ 3 & 3 & 0 \end{pmatrix},$$

and $v_a^1 = v_b^2 = v_c^3 = -2$, $v_b^1 = v_c^2 = v_a^3 = -1$, and $v_c^1 = v_a^2 = v_b^3 = -10$. We set $\sigma^i = 1$ for all $i \in N$. There are three landscape equilibria $\mathbf{s}^1 = (a, 0, a)$, $\mathbf{s}^2 = (b, b, 0)$, and $\mathbf{s}^3 = (0, c, c)$. However, none of them is a strong Nash equilibrium. For example, $\mathbf{s}^1$ is not immune to a deviation by players 2 and 3 jointly switching to action b.$\blacksquare$

Adding a set of players as an argument, let $\mathbb{F}(\mathbf{s}, N)$ be a weighted-potential function for player set N on an augmented action set $M \cup \{0\}$:

$$\mathbb{F}(\mathbf{s}, N) \equiv \frac{1}{2} \sum_{k \in M} \sum_{i \in G_k(\mathbf{s})} \sum_{j \in G_k(\mathbf{s})} \sigma^i p^{ij} \sigma^j + \frac{1}{2} \sum_{i \in G_0(\mathbf{s})} 0.$$

We have a counterpart of Proposition 2 with *negative externalities* ($p^{ij} \le 0$ holds for all $i, j \in N$). The logic of the proof is again based on potential function approach again.

Theorem 5. Suppose that $p^{ij} \le 0$ holds for all $i, j \in N$ in a *local* and *M-uniform* *LADG* with an outside option. The weighted-potential maximizing action profile $\mathbf{s}^*$ is a landscape equilibrium.

Proof. We want to show that $\mathbf{s}^* \in \arg\max_{\mathbf{s}} \mathbb{F}(\mathbf{s}, N)$ is a landscape equilibrium. This means that $\mathbf{s}^*$ is immune to any profitable gradual deviations.

First of all, notice that since $\mathbb{F}(\mathbf{s}, N)$ is a weighted-potential function, $\mathbf{s}^*$ is a Nash equilibrium by Theorem 1. This means that no player with $s_i^* \in M$ do not want to move to the outside option 0. Since the outside option payoff is normalized to zero, $u^i(\mathbf{s}^*) \ge 0$ must hold for all players with $s_i^* \in M$. This implies that no player with $s_i^* \in M$ would switch to outside option 0 unilaterally or jointly.

Now, to the contrary, suppose that there is a profitable gradual deviation $S(\mathbf{s}^*, \tilde{\mathbf{s}}) \equiv \{S_{kh}(\mathbf{s}^*, \tilde{\mathbf{s}})\}_{k,h \in M \cup \{0\}, k \neq h}$ from $\mathbf{s}^*$. From the above argument, $S_{k0}(\mathbf{s}^*, \tilde{\mathbf{s}}) = \emptyset$ holds for all $k \in M$. If $S_{0k}(\mathbf{s}^*, \tilde{\mathbf{s}}) = \emptyset$ holds for all $k \in M$, $G_0(\mathbf{s}^*) = G_0(\tilde{\mathbf{s}})$ and $\cup_{k\in M} G_k(\mathbf{s}^*) = \cup_{k\in M} G_k(\tilde{\mathbf{s}})$ must hold. Then, if we maximize potential $\mathbb{F}(\mathbf{s}, N\backslash G_0(\mathbf{s}^*))$ on M by excluding $G_0(\mathbf{s}^*)$ from N, we should have the same maximizing action profile, $(s^{i*})_{i\in N\backslash G_0(\mathbf{s}^*)}$ for $N\backslash G_0(\mathbf{s}^*)$. Then, $S(\mathbf{s}^*, \tilde{\mathbf{s}})$ cannot improve upon $\mathbf{s}^*$ by Theorem 3 for a reduced set of players $N\backslash G_0(\mathbf{s}^*)$ without an outside option. This means that there is at least one $k_1 \in M$ with $S_{0k_1}(\mathbf{s}^*, \tilde{\mathbf{s}}) \neq \emptyset$.

Let $(M\cup\{0\}, \mathcal{G}(\mathbf{s}, \tilde{\mathbf{s}}))$ be a directed graph formed by $S(\mathbf{s}^*, \tilde{\mathbf{s}}) \equiv \{S_{kh}(\mathbf{s}^*, \tilde{\mathbf{s}})\}_{k,h\in M\cup\{0\},k\neq h}$. Since $S(\mathbf{s}^*, \tilde{\mathbf{s}})$ is a gradual coalitional deviation and $S_{\ell 0}(\mathbf{s}^*, \tilde{\mathbf{s}}) = \emptyset$ holds for all $\ell \in M$, no component of the graph $(M \cup \{0\}, \mathcal{G}(\mathbf{s}, \tilde{\mathbf{s}}))$ starting from 0 can form a cycle, thus such a component must be a path $\{0, k_1, ..., k_m\}$ with $S_{k_m\ell}(\mathbf{s}^*, \tilde{\mathbf{s}}) = \emptyset$ for all $\ell \in M \cup \{0\}$. Focus on the last edge of the path (k_{m-1}, k_m). We have $S_{k_{m-1}k_m}(\mathbf{s}^*, \tilde{\mathbf{s}}) \neq \emptyset$ with $S_{k_m\ell}(\mathbf{s}^*, \tilde{\mathbf{s}}) = \emptyset$ for all $\ell \in M \cup \{0\}$. Due to negative externality, no player $i \in S_{k_{m-1}k_m}(\mathbf{s}^*, \tilde{\mathbf{s}})$ is better off by switching from k_{m-1} to k_m unilaterally. This contradicts with the fact that $\mathbf{s}^*$ is a Nash equilibrium. Hence, we conclude that there is no profitable gradual coalitional deviation from $\mathbf{s}^*$, and $\mathbf{s}^*$ is a landscape equilibrium.□

Remark. Theorem 4 implies that under positive externality, Proposition 2.2 of Konishi et al. (1997a) still holds even with an outside option if $|M| = 2$. It is an open question whether or not M-uniform $LADG$ admits a landscape equilibrium if $|M| \geq 3$. For negative externality, Theorem 5 shows how Proposition 3.3 in Konishi et al. (1997b) needs to be modified. Konishi et al. (1997b) proved that there is a strong Nash equilibrium under negative externality and anonymous congestion without assuming ADG structure.[14] Our Theorem 5 holds only for ADG, but we allow for heterogeneous negative externalities and the equilibrium concept is landscape equilibrium. It is an open question whether ADG admit a strong Nash equilibrium under the conditions of Proposition 3.[15]

9 Re-scaling Payoffs

We have demonstrated that any game $BADG$ class is a weighted potential game. If the propensity matrix is not symmetric, could it be the case that the game remains the $BADG$ class once the payoffs of each player i are transformed by a monotone affine transformation f^i? In the two player case, a simple sign symmetry condition is sufficient. to provide an affirmative answer,

Proposition 2. Suppose that $|N| = 2$ with payoffs: $\tilde{u}^i(\mathbf{s}) = \tilde{v}^i_k + w^{ij}$ if $s^i = s^j = k$, and $\tilde{u}^i(\mathbf{s}) = \tilde{v}^i_k$ if $s^i = k \neq s^j$ for all $k \in M$, and all $i, j \in N$ with $i \neq j$. If either

[14] In the context of ADG, anonymous congestion means that $p_{ij} = p_{ij'}$ for all $i \in N$ and all $j, j' \in N\backslash\{i\}$.

[15] See also Holzman and Law-Yone (1997), Milchtaich (1996), Quint and Shubik (1994) for further results on the existence of Nash and strong Nash equilibria in group formation games with *anonymity* and either positive or negative externalities.

$w^{12} \times w^{21} > 0$ or $w^{12} = w^{21} = 0$, then there is an affine transformations of the payoff functions such that the new game is in the $BADG$ class.

Proof. If $w^{12} = w^{21} = 0$, let $u^i(\mathbf{s}) = \tilde{u}^i(\mathbf{s})$, and we are done. If $w^{12} \times w^{21} > 0$, let $u^i(\mathbf{s}) \equiv \frac{1}{|w^{ij}|}\tilde{u}^i(\mathbf{s}) = \frac{\tilde{v}^i_k}{|w^{ij}|} + \frac{w^{ij}}{|w^{ij}|}$. Let $p^{ij} = \frac{w^{ij}}{|w^{ij}|}$. Then, p^{ij} is either 1 or -1. Since $w^{12} \times w^{21} > 0$, $p^{ij} = p^{ji}$ holds, which completes the proof.$\square$

However, in the game with more than two players, the following example shows that the situation is more complicated,

Example 13. Suppose that game Γ has three players $|N| = 3$ with neutral quasi-linear utility function $\tilde{u}^i = \sum_{j \in G_k} w^{ij}$, where w^{ij} denotes player i's benefit from player j in the same group G_k. We set $\sigma^i = 1$ for all $i \in N$. Now, suppose further that $0 < w^{13} < w^{12}$, $0 < w^{21} < w^{23}$, and $0 < w^{32} < w^{31}$. Then, there is no symmetric propensity matrix P. If these three players' preferences are respected, by symmetry, we have $0 < p^{13} < p^{12} = p^{21} < p^{23} = p^{32} < p^{31} = p^{13}$. It is impossible to satisfy these inequalities, similarly to the nonexistence of stable matching in a roommate problem. This example illustrates the necessity of a common ranking over all pairs of players consistent with players' preferences (see the *common-ranking property* in Banerjee et al. 2001).$\blacksquare$

Example 13 refers to preferences over single partner players, but there may be multiple partners in a group in our additive social interactions game. Finding a common ranking over all subsets of players is harder than the issue described above. A single ranking over all subsets of players is not necessarily consistent, even if we could find a single ranking over all pairs of players. This issue becomes more complicated as the number of players increases. It is hard to identify an additively separable preference profile that has a weighted potential function $\mathbb{F}(\mathbf{s})$.

Profiles that maximize a weighted potential of a game of the $BADG$ class occupy a central place in this paper. If a weighted potential game admits several weighted potentials, then as stated by Monderer and Shapley (1996), the potential maximizers remain the same. In contrast to the affine case, if the new game with transformed payoff functions is an ordinal potential game, the set of potential maximizers in the transformed game may be different if some of the f_i are not affine monotone transformations.

Example 14. Suppose that game $\Gamma \in NBADG$ has six players $N = \{1, 2, 3, 4, 5, 6\}$ and six neutral actions $M = \{a, b, c, d, e, f\}$ with a propensity matrix:

$$P = \begin{pmatrix} 0 & -10 & 1.8 & -10 & 0.6 & 2 \\ -10 & 0 & 2 & 0.6 & -10 & 0.4 \\ 1.8 & 2 & 0 & -10 & 0.4 & -10 \\ -10 & 0.6 & -10 & 0 & 2 & 1.8 \\ 0.6 & -10 & 0.4 & 2 & 0 & -10 \\ 2 & 0.4 & -10 & 1.8 & -10 & 0 \end{pmatrix}$$

We set $\sigma^i = 1$ for all $i \in N$. Under this P, each agent's preferences over relevant coalitions are:

$$
\begin{array}{rcl}
\boxed{\{1,3,5\}} & \succ & _1\{\mathbf{1},\mathbf{6}\} \succ_1 \{1,3\} \succ_1 \{1,5\} \succ_1 \{1\} \\
\{\mathbf{2},\mathbf{3}\} & \succ & _2\boxed{\{2,4,6\}} \succ_2 \{2,4\} \succ_2 \{2,6\} \succ_2 \{2\} \\
\boxed{\{1,3,5\}} & \succ & _3\{\mathbf{2},\mathbf{3}\} \succ_3 \{1,3\} \succ_3 \{3,5\} \succ_3 \{3\} \\
\boxed{\{2,4,6\}} & \succ & _4\{\mathbf{4},\mathbf{5}\} \succ_4 \{4,6\} \succ_4 \{2,4\} \succ_4 \{4\} \\
\{\mathbf{4},\mathbf{5}\} & \succ & _5\boxed{\{1,3,5\}} \succ_5 \{1,5\} \succ_5 \{3,5\} \succ_5 \{5\} \\
\boxed{\{2,4,6\}} & \succ & _6\{\mathbf{1},\mathbf{6}\} \succ_6 \{4,6\} \succ_6 \{2,6\} \succ_6 \{6\}
\end{array}
$$

Thus, in this game, there are two (equivalent-class) strong Nash equilibria (thus, landscape equilibria): $\mathbf{G} \equiv \{\{1,3,5\},\{2,4,6\}\}$ and $\mathbf{G}' \equiv \{\{1,6\},\{2,3\},\{4,5\}\}$. The propensity matrix P yields the following values of potential under $\mathbf{G}$ and $\mathbf{G}'$ are:

$$\mathbb{F}(\mathbf{G}) = \frac{1}{2}\mathbb{W}(\mathbf{G}) = \frac{1}{2}\{2.4+1+2.2+2.4+1+2.2\} = 5.6,$$

and

$$\mathbb{F}(\mathbf{G}') = \frac{1}{2}\mathbb{W}(\mathbf{G}') = \frac{1}{2}\{2+2+2+2+2+2\} = 6.$$

Thus, $\mathbf{G}'$ is the potential-maximizing strong Nash equilibrium. Now, we re-scale propensity matrix P slightly.

$$\hat{P} = \begin{pmatrix} 0 & -10 & 1.8 & -10 & 0.6 & \mathbf{1.85} \\ -10 & 0 & \mathbf{1.85} & 0.6 & -10 & 0.4 \\ 1.8 & \mathbf{1.85} & 0 & -10 & 0.4 & -10 \\ -10 & 0.6 & -10 & 0 & \mathbf{1.85} & 1.8 \\ 0.6 & -10 & 0.4 & \mathbf{1.85} & 0 & -10 \\ \mathbf{1.85} & 0.4 & -10 & 1.8 & -10 & 0 \end{pmatrix}.$$

This modification does not alter agents' preference orderings. However, we have:

$$\mathbb{F}_{\hat{P}}(\mathbf{G}) = \frac{1}{2}\{2.4+1+2.2+2.4+1+2.2\} = 5.6,$$

and

$$\mathbb{F}_{\hat{P}}(\mathbf{G}') = \frac{1}{2}\{1.85+1.85+1.85+1.85+1.85+1.85\} = 5.55.$$

This means that $\mathbf{G}$ is the weighted potential-maximizing strong Nash equilibrium under $\hat{P}$. Thus, potential-maximizing equilibrium selection is sensitive to the cardinal form of the payoff function even if players' ordinal preferences over coalitions remain the same. It is worth noting that while both functions are ordinal potentials, the second is not a weighted potential, which explains the emergence of different potential-maximizers in these two games. ■

10 Concluding Remarks

In this paper, we examine non-cooperative games with additive dyadic social interactions and focus on strategy profiles that are immune to various classes of group deviations. We show that there exists a landscape equilibrium in *basic* (*local* and *uniform*) *ADG*, and further investigate the properties of potential-maximizing landscape equilibria in various settings. In particular, we show that a potential-maximizing landscape equilibrium may not be a strong Nash equilibrium even if there is a strong Nash equilibrium in *neutral* and *basic ADG*. We also indicate that the introduction of an outside option makes our examination more complex. Since some of our results do not hold any longer, we need to impose constraints on the value of social interaction externalities to rescue the existence of equilibrium based on coalitional deviations.

We conclude this paper by illustrating how our potential function approach could be applied to a class of matching problems. Since we have been considering coalitional deviations, it is natural to explore applications of our approach to matching problems with a quota for each action. As long as a potential function is well-defined for admissible coalitional deviations, a profitable swapping of players' actions will increase the potential of the matching problem. The existence of quotas in fact limits feasible coalitional deviations. For example, consider a student assignment problem for sports activities, each having a capacity limit, and where students have dyadic preferences over their peers. We could then apply a potential-improving cyclical swapping of students, respecting quotas, to find a *landscape stable assignment* from an arbitrary initial assignment. Afacan et al. (2025) consider a seat assignment problem in public transportation with passengers possessing symmetric preferences for those sitting next to them. We can apply our approach to this problem as well.[16]

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