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"Creative Destruction and Energy Transition"

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Abstract

The energy transition is often described as a 'wall of investment' issue. But not only renewable energy expansion must cope with the carbon free energy needs resulting from the Paris agreement but most of the present day fossil energy production capital will have to be abandoned by the middle of the century. We consider a partial equilibrium model where energy is produced from a CO2 polluting source and a carbon free source. Crude energy transformation into energy services requires specific capital infrastructures, accumulated through a costly investment process submitted to adjustment costs. Fossil fuels burning generates CO2 emissions and the total cumulated emissions is caped by a global atmospheric CO2 concentration stabilisation objective. We describe the socially optimal policy in this context. Prior to the phase when the society is actually constrained by the climate cap, we show that the pattern of evolution of the production capacities of the fossil fuel industry exhibits a three regimes structure: fist an ascending phase, followed by a stabilisation phase when the industry stops investing and its production capacity peaks before falling when approaching the time at which the climate constraint becomes binding. During the two first phases, the energy transition occurs in a context of increasing energy supply while it happens in a context of increasing energy scarcity during the last phase, capacity accumulation in the renewable energy industry being lower than the scraping of capacities in the fossil fuels industry. The non-monotonicity of the energy price path induces the possibility of waves of investment in the clean renewable energy sector. If fossil energy gets a larger share in the fuel mix at the beginning of the climate regulation, fossil energy production will peak at a larger level. We show that the regulator must adapt to this situation not by preventing a larger peak but by making it happen earlier, thus building initially a form of carbon 'saving' that will be progressively consumed afterwards before the climate constrained episode.

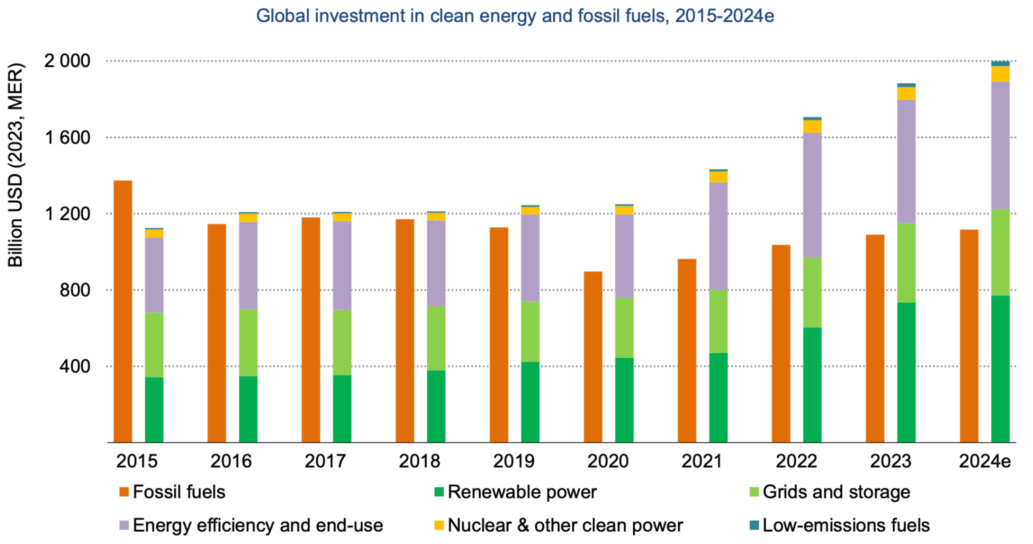
Keywords: Climate change; Energy transition; Capacity constraints, Adjustment costs.

JEL classifications: Q32, Q35, Q42, Q54.

1 Introduction

That the fossil energy industry goes on investing in production capacities either directly or indirectly through exploration and development of new reserves is a commonplace observation. How to understand that more than ten years after the Paris agreement, fossil fuels are not only maintaining their share in the world fuel mix but continue to expand even controlling for population increase? Several explanations have been proposed to explain this apparent paradox. Some have stressed the weight of subsidies to fossil energies, either at the consumer or the producer stage, and strongly advocated for their removal. Others highlight the lack of carbon pricing, or its presently too small level, for a correct monitoring of investments in dirty or green energy sources. Some cynically mention the weak credibility of climate policies, heavily dependent on political or electoral agendas in the main world economies.

Recent trends offer a more optimistic view. While still significant, the investment rate in fossil energy generation slows down while investments in clean renewable energy production accelerate (see Figure 1).[1]



IEA. CC BY 4.0

Note: Other clean power = fossil fuel power with CCUS, hydrogen, ammonia, and large-scale heat pumps. Low-emissions fuels = modern bioenergy, low-emissions H_2 based fuels, and CCUS associated with fossil fuels and also includes direct air capture. 2024e = estimated values for 2024.

Figure 1: Recent trends in energy production investments.

The dispatching by world regions of these investments shows clearly that today China, Europe and the US are leading the transition while other countries in Africa, the Middle-East or Latin America still rely excessively on fossil fuels. As for many other environmental issues, the problem becomes a North-South divide one, or a poor against rich countries problem. The need for huge investments in the transition is one of the most critical issue that governments try to solve by inflating the green fund over the successive COP rounds and now through the Just Energy Transition initiative.

However the continuous expansion of the fossil energy industry could be perfectly

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[1] Chart from "World energy investment 2024", IEA.

rational even without considering the permanent growth of the world energy demand, especially in the South. In view of the relative competitiveness of fossil and renewable energy and their respective investment costs, the cost effective mix of investments has no reason to exclude *a priori* fresh investments in fossil energy even under a climate constraint.

The deployment of a clean renewable energy conversion system able to cope with the energy demand while mitigating climate change is often described as a ’wall of investment’ issue. But in parallel, the need to build new capacities has to be complemented with the dismantling of most of the fossil energy production capacity. Energy transition is a massive ’creative-destruction’ problem. The capitalisation of the fossil energy industry amounts to 4 trillion US $. To this sum must be added all the capital stock of downstream firms transforming fossil energy into energy services: carmakers, steel producers, cementeries and chemical industries. To fulfil the net zero objective by the middle of the century, most of this productive capital has to be dismantled, left idle or reallocated to non carbon emitting uses, like electric vehicles production. After the Industrial Revolution, the energy industry has progressively piled up all technologically accessible energy sources, a move powered by the continuous expansion of the energy demand (see Figure 2). The figure shows clearly that the energy history has never been one of substitution from old exploited energy sources to new ones thanks to technical progress, but a progressive mobilisation of more and more energy sources, especially fossil fuels. This pattern is no more sustainable because of climate change. For the first time in its history the energy industry must proceed simultaneously to the dismantling of its fossil fuels based production component and to the building of clean energy production capacities, that is an actual substitution between energy sources.

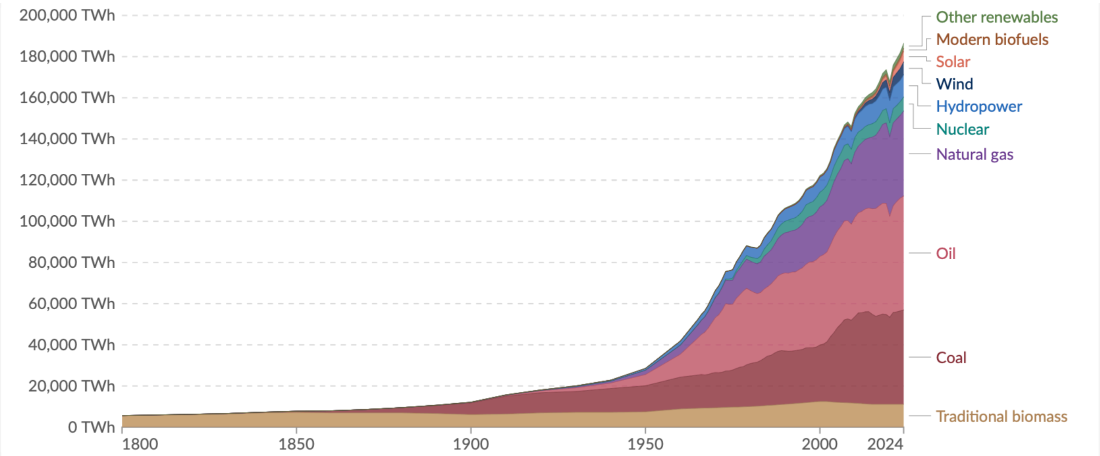

Figure 2: The history of energy supply by sources.

The theme of ’creative destruction’ is very popular in the innovation literature under the heading of ’Schumpeterian growth’ or ’disruptive’ innovations. It is also widely studied in the trade and global change studies dealing with the offshoring of industrial activities from formerly industrialised countries to emerging countries. In the energy transition field, with the exception of the directed technical change studies applied to climate change, the issues of ’creation’ and ’destruction’ have been studied in isolation. There exists a long-standing tradition of studies about the inertia of the energy system and the lock-in of industries in fossil fuel energy based production and transformation technologies. Over the last twenty years a significant set of studies emerged on the deployment of renewable energy conversion capacities in the context of climate change. The objective of this paper is to merge these two lines of research into a comprehensive dynamic analysis. In particular we want to assess what should be an optimal strategy of carbon pricing to simultaneously smoothen the destruction cost of existing capacities in the fossil energy sector and enhance the creation of capacities in the renewable energy sector. Say otherwise, our objective is to better characterise the inter-temporal trade-offs between the costs of the destruction of dirty energy capacities and the benefits of the creation of non-carbon emitting ones.

Many studies of capital inertia in the fossil energy based industries have relied of putty-clay formulations of the capital structure while studies of the renewable energy sector are mostly putty-putty. To harmonize the presentation and avoid intricacies we adopt the following simplifications. We assume that useful energy services can be produced either from fossil crude energy or renewable, carbon-free, crude energy. Production of energy services is submitted to capacity constraints in the two energy conversion industries. These capacities can be increased through investments submitted to adjustment costs and we adopt a convex investment cost formulation to take account of these costs.

We want to describe explicitly the dismantling decisions of the fossil energy conversion industry. To this aim we assume that keeping in operational status the equipments requires paying a maintenance cost. Equipments not benefiting from maintenance at any moment are definitively out-of-order while properly maintained equipments can run forever. Thus we transform the dismantling decision problem of the industry into a problem of optimal maintenance policy of its park of installations. Thanks to this simplifying assumption we avoid the complexities of putty-clay formulations or of vintage capital models.

The burning of fossil fuels releases carbon emissions in the atmosphere. The global society wants to stabilise the resulting carbon stock at a concentration limiting the impacts of climate change, implying to maintain the pollution stock below some threshold, or carbon cap. We want to put the analysis inside the perspective of the so-called 'net zero target' (NZT). Thus we assume that some fraction of the carbon stock self-depletes allowing for some consumption of fossil fuels compatible with the carbon stock regeneration when the society is constrained by the cap. Assuming that society is not initially constrained by the cap, a three episodes climate history results. First society is not constrained by the cap but emissions accumulate until the cap becomes binding. Next society has to face the cap constraint and limit its emissions to the level compatible with the stabilisation of the carbon cycle. Last because of the energy transition, the emissions from the consumption of fossil fuels falls below what is allowed by the NZT and society escapes definitively the cap constraint.

The dynamic problem we want to study is potentially very complex, explaining why the related literature mostly relies on numerical simulations. At least the time evolutions of four state variables must be described: the available stock of fossil fuels, the carbon stock and the two productive capital stocks of the fossil energy conversion sector and the renewable energy conversion sector. To simplify slightly the analysis we assume that the fossil resource is abundant, more precisely we put assumptions on the relative competitiveness of the two energy sources ensuring that the use of fossil energy will be phased-out in finite time. Thus the cumulated amount of fossil fuels consumption will be bounded from above and we shall assume an upper bound lower than the available reserves, meaning that the fossil energy industry leaves permanently underground some reserves. We hence focus on a three state variables problem, involving the carbon stock dynamics together with the two capital stocks dynamics.

Inside this setting, we show the following. The investment dynamics of the fossil energy conversion industry obeys what we call our 'central scenario' before the economy is actually constrained by the cap. This scenario is composed of a first phase during which the industry invests in more capacities, followed by a stabilisation phase when the industry no more invests and maintains its previously accumulated production capacity before a third phase when the industry scraps capacities in order to satisfy the NZT at the time the carbon cap constraint begins to bind. The investment rate should constantly decline during the first phase and the scraping process should accelerate during the third phase. Even if the initial capacities of the fossil energy industry are lower than the NZT constraint, it is generally optimal for the industry to overshoot the NZT capacity and one should not expect that the industry starts immediately to reduce its capacities when the climate policy is announced.

In parallel the renewable industry constantly build up capacities but it is possible that its investment policy be made of waves of investments with decelerating and accelerating episodes. In vintage capital models such waves are frequently observed but in the present context they result from the carbon pricing policy which must follow before the binding cap regime an exponential trend *à la* Hotelling. More precisely we show that if the energy price must decrease before the scraping phase of fossil energy conversion capacities, it must increase during this phase, the fossil energy industry having to scrap capacities at a higher rate the renewable energy industry builds its own capacities. This non monotonous pattern of the energy price explains the possibility of investment waves in the renewable energy industry.

We show the existence of decreasing relationships between the size of the carbon stock and the size of the capital stock of the fossil energy industry, one at the time the industry decides to stop investing, what we call the *stabilisation border* in the carbon stock-fossil energy capital stock space, and another one at the time the industry decides to scrap its capacities, what we call the *scraping border*. This suggests that if the fossil energy industry decides to cap its capacities at a larger peak, it must do so at a lower level of the carbon stock, that is earlier. Then a larger capacity peak implying more carbon emissions during the stabilisation phase, the carbon stock size will rise faster. Despite this acceleration of the accumulation of carbon in the atmosphere, the industry starts to scrap its capacities at a lower carbon stock level. From the regulator point-of-view this means that she has to adapt its carbon pricing policy to the industry decision by making the capacity peak happen earlier and be left also earlier, building up initially a form of carbon 'savings', to be consumed over the stabilisation and the scraping phase. We show that this is actually the case in our model.

During the carbon cap constrained phase, the optimal carbon pricing policy depends on what has happened prior to this phase. If the fossil energy industry has begun to dismantle its capacities, then the value of the fossil energy capital is nil and the industry has no reason to keep it. What justifies the existence of a climate constrained phase is the insufficient accumulation of renewable energy conversion capacities. We show the existence of a unique renewable energy production capacity above which fossil energy is no more competitive and the fossil energy industry should scrap its capacities even without a climate constraint. The length of the carbon cap constrained phase is thus the time needed for the renewable energy industry to accumulate this capacity.

In the meantime the carbon regulation policy should incentivise the fossil energy industry to maintain its capacities and not scraping them, even if they have a nil value. This is achieved by progressively reducing the carbon price down to zero, at the time the renewable energy is able to completely outcompete fossil energy. If the fossil energy industry is very constrained by the carbon cap, it would have been willing to accumulate more capacities when the carbon cap becomes binding. Thus the value of the fossil energy conversion capital is still positive at the beginning of the carbon constrained phase. In this case we show that the carbon price trajectory is indeterminate, the only constraint for the regulator being to maintain the profitability of investment in the fossil energy sector below its minimal cost.

The next section 2 presents the model. In section 3 we lay down the optimality problem faced by society and makes some preliminary comments on its solution. We next solve the problem recursively. We describe the optimal policy when society is no more constrained by the climate in section 4. We do the same job for the carbon cap constrained phase in section 5. We split the study of the time phase before the binding carbon cap regime in two. We first present in section 6 the possible optimal scenarios in the fossil energy industry and the renewable energy industry respectively. Next we perform a phase space analysis to determine the optimal scenarios depending on the initial state of the carbon stock and the the two capital stocks of the energy sector in section 7. Because of the complexity of the exercise we browse through two benchmarks before presenting the general analysis: a first where there is no capital accumulation in the renewable energy sector, in order to focus of the interplay between the carbon accumulation dynamics and the capital accumulation dynamics in the polluting energy sector, and a second where there is no climate constraint, focusing on the interplay between the capital accumulation dynamics in the fossil energy sector and in the renewable energy sector. The last section 8 concludes.

Related literature

The present work borrows from several branches of the literature. The first is the long tradition of production economics and econometric studies of the inertia of the capital structure of energy systems (Pindyck and Rothenberg, 1983, Atkeson and Kehoe, 1999). These models adopt a putty-clay formulation of the capital structure to account for this inertia. Hawkins-Pierot and Wagner, (2024), extend this literature by incorporating productivity differentials inside a reflection on the management of climate change. We depart from these approaches by adopting a simpler view of the capital structure. We assume that the dismantling costs *per se* are negligible, however we introduce sunk costs of investments, implying that once installed the industry has to keep for some time their equipments to recover at least the investment sunk cost, introducing some inertia in their capital structure.

Coulomb *et al.*, (2019), and Rozenberg *et al.*, (2020), are recent contributions raising the same issue as in the present paper. We depart from their analysis in several ways. First instead of a macroeconomic growth framework we adopt a partial equilibrium approach to get more focused results. Both contributions adopt a rather strange way of modelling the dismantlement process of fossil energy production capital. They assume that the investments in the energy industry are irreversible but that the capital is progressively worn out through natural decay, implying that when the fossil energy industry becomes strongly constrained by the climate policy it stops investing, keeps its excess capacities (because of the investment irreversibility assumption) and waits for the passage of time to make progressively decline the capital stock. It is well known (Arrow, 1968) that imposing the irreversibility of the investment process results in a capital accumulation path made of phases of active investments separated by no capital accumulation phases. This complicated pattern mixes with the dynamics of carbon accumulation in the aforementioned contributions, generating complex dynamics that can be dealt with only thanks to numerical simulations (and very hard to interpret economically).

In our setting equipments are run at full capacity, identifying the whole capital stock to the technically efficient productive capacity so that the capacity price, equivalently the opportunity cost of the capacity constraint, can become nil only when the industry scraps capacities. This approach we think gives a more insightful view of the dynamics of the capacity prices. Instead of assuming an exogenous depreciation rate, hard to justify if the capital is not used in production, we endogenies the capital depreciation rate through a two variables decision problem and not just one: an investment decision and a critical maintenance decision, allowing the effective capital stock to be increasing or decreasing at any moment.

Last the aforementioned works account explicitly for the scarcity of fossil fuels, complicating even more the analysis. There exists today a large body of literature on 'stranded assets' (see below) concluding that a complete burning of the oil, gas and coal existing reserves is incompatible with the ability of the carbon cycle to absorb the resulting CO_2 emissions without triggering catastrophic climate change. Thus the physical depletion of fossil fuels reserves issue can be discarded, which we assume by considering abundant fossil fuels resources in order to simplify somewhat the analysis.

The implications of climate change for the future of the fossil fuels energy industry have attracted a lot of attention under the heading of 'stranded assets'. Primarily the literature was centred on the need to bound permanently underground a significant fraction of the proven reserves of fossil fuels to preserve the climate of the planet (Unruh, 2000, Seto *et al.* 2016). Then the scope of the studies expanded to many aspects of 'carbon lock-in' in relation to asset reserves (Van der Ploeg and Rezai, 2017, Van der Ploeg, 2020), to the implications for the capital dynamics of the industry either broadly (van Sluisveld *et al.*, 2015, Fowlie *et al.*, 2016, Curtin *et al.*, 2019, Baldwin *et al* ., 2020), either for specific sectors (Jonhson *et al.*, 2015) or countries (Kumar Saha and Carter, 2022). We share with this literature their concern for the productive capital dynamics in the energy sector and the idea that mitigating climate change implies to put some limits on the consumption of fossil fuels resources. In our model these limits are not physical constraints on the cumulated consumption of the resources but take the form of limits on the maximum amount of fossil energy conversion capacities.

These limits arise from our modelling approach of the climate problem. We depart from a climate damages framework and instead adopt a carbon budget, or carbon 'ceiling', view of climate regulation, in line with the stabilisation of atmospheric carbon concentration objective of the Paris agreement. Initiated by the seminal contribution of Chakravorty *et al.* (2004), the carbon budget framework is increasingly used in recent climate economics papers (Eichner *et al.*, 2023, Kollenbach, 2023, Moreaux *et al.*, 2024, Amigues and Lafforgue, 2025).

Even if we do not consider explicitly technical progress, our paper shares some common features with the now well developed body of literature about innovation and climate change. In the macroeconomic models of directed technical change, the assumption usually made of linear knowledge accumulation functions gives a Ricardian flavour to their framework, inducing a progressive specialisation of the economy in the research direction in which it has initially a comparative advantage. It has been shown that this problem is more general and that directed technical change models should largely exhibit path dependency. The issue has been explored in the context of natural resources scarcity by Hassler *et al.* (2021), and in the context of climate policy by Lennox and Witajewski-Baltvilks, (2017), Acemoglu *et al*, (2012), and more recently in Acemoglu *et al.*, (2023). Our approach shares with these models a bi-sectoral structure of energy use and provision, and because we assume that fossil fuels are abundant, the sectors differ only in their contribution to CO_2 pollution and cost characteristics, as they assume. The capital accumulation trajectories we describe are also partially path dependent like the innovation trajectories in these models, but our specialisation trajectories differ, being typically non time monotonous in our framework.

Modelling the rise of renewable energy production as a time-to-build problem is today frequently adopted in the literature. Initiated by early contributions of Wirl (1991), Wirl and Withagen, (2000), Fisher *et al.* (2004), this view is developed in Amigues *et al.* (2015) in the context of natural resources extraction and used in the context of climate change in many recent contributions (Kollenbach, 2017, Schubert and Pommeret, 2022). The Kollenbach model is a two capital stocks model like ours but one of them is a general productive capital stock not responsible for carbon pollution, the other one being the renewable energy sector capital stock. The contribution of Schubert and Pommeret uses one capital stock of renewable energy and electricity storage capacities, being mainly focused on the issues of variability and intermittency of renewable energy provision. To our knowledge, our contribution is the first one which attempts to study analytically the joint dynamics of carbon accumulation in the atmosphere and capital accumulation in a two-sector model under a climate constraint.

2 The model

Useful energy services (U.E) can be obtained through energy conversion technologies applied to two possible crude energy sources: a polluting fossil resource and a carbon-free renewable energy source (e.g. wind and solar energy).

The fossil energy sector

The economy produces useful energy services (U.E) from fossil fuels (fossil crude energy) through a Leontief production process:

$$q_x = \min\{K_x, \bar{s}_x.v_x, \bar{r}_x x\} \tag{2.1}$$

where q_x denotes the production of U.E from fossil fuels, K_x is the installed production capacity of U.E, v_x is a vector of variables inputs used to produce U.E from fossil fuels, $\bar{s}_x$ a vector of technical coefficients and $\bar{s}_x.v_x$ their scalar product. The production of an amount q_x of U.E requires x units of fossil fuels with a technical coefficient $\bar{r}_x$. Thus producing one unit of U.E requires $s_x = 1/\bar{s}_x$ units of the variable input v_x. Assuming a constant given vector price p_{vx} of the variable inputs, the variable production cost is given by $p_{vx}.v_x = p_{vx}.s_x q_x$ (because $q_x = \bar{s}_x.v_x$ from technical efficiency). Denoting by $a_x = p_{vx}.s_x$, the variable unitary cost is a_x. Let $r_x = 1/\bar{r}_x$. Assuming a constant extraction cost e_x per unit of processed fossil fuel, to the variable input cost $a_x q_x$ must be added the fossil fuel processing cost $r_x e_x q_x = b_x q_x$. Let $c_x \equiv a_x + b_x$ be the average and marginal cost of U.E production from fossil and $c_x q_x$ is the total production cost of U.E.

In addition, technical efficiency requires that $q_x = K_x$. The fossil UE production is constrained by the available production capacity, an energy conversion capital stock that can be increased over time through investment. But it can be also possible that some part of the existing capital becomes useless depending on the economic conditions and should be scraped accordingly.

Increasing the production capacity K_x requires the purchase of new equipments. Denoting by $K_x(t)$ the installed capacity at time t and $k_x(t)$ the investment rate in capacity building, $K_x(t)$ evolves over time according to:

$$\dot{K}_x(t) = k_x(t) - \delta_x(t)K_x(t) \tag{2.2}$$

where $\delta_x(t)$ denotes the (endogenous) scrapping rate of existing equipments at time t. To keep in operating conditions the installed equipments $K_x(t)$ requires a maintenance effort $m_x K_x(t)$. Assume that properly maintained, the equipment stays for any time amount in productive capacity but maintenance being not applied at some time $\bar{t}$, the equipment will be definitively out of order after $\bar{t}$ and cannot be repaired. Therefore, the scrapping rate $\delta_x(t)$ is the proportion of equipments not benefiting from maintenance at time t. For the sake of simplicity assume that scraping does not bring specific dismantlement costs (an assumption not suited for nuclear plants), equivalently that when the industry cannot cover the maintenance cost, it simply stops using the equipments and leaves them idle.

The purchase of new equipments brings a cost $N_x(k_x)$. The function $N_x(.)$ is twice continuously differentiable, strictly increasing: $n_x(k_x) \equiv dN_x(k_x)/dk_x > 0$, strictly convex: $n_x'(k_x) = d^2N_x(k_x)/dk_x^2 > 0$, $N_x(0) = 0$ and $n_x(0^+) = \underline{n}_x > 0$. This last assumption stresses that no equipment supply can start from scratch and requires to pay a sunk cost. The function $N_x(.)$ can also be interpreted as the supply curve of the equipment providing industry, assumed strictly increasing in k_x with a positive intercept at $k_x = 0^+$.

To avoid awkward investment strategies where the industry would prefer to buy permanently fresh equipments rather than maintaining them, assume that $m_x < \underline{n}_x$, so that whatever the economic conditions of equipment supply, it is better to pay the maintenance cost than purchasing constantly fresh capacity units and let idle the already installed equipment pieces.

Burning fossil fuels to produce U.E generates polluting emissions proportional to the consumption of fossil fuels, ζx, where ζ is the polluting content of fossil fuels thus an emission rate $\zeta r_x q_x$ in terms of U.E production from a fossil energy source. This formulation shows that the parameter r_x can be interpreted as an inefficiency measure of the fossil energy conversion technique. Through technical progress, the industry could make decrease this parameter, allowing for less CO_2 emissions per fossil U.E production unit.

Last assume that the fossil fuels are abundant, more exactly that they are available initially in an amount larger than the total cumulated consumption of the resource along the optimal path. We give later precise conditions for this to be the case.

The renewable energy sector

The economy produces U.E from the solar source in amount q_y according to a Leontief production process:

$$q_y = \min\{K_y, \bar{s}_y v_y, \bar{r}_y y\} \tag{2.3}$$

where q_y denotes the production of U.E from solar energy, K_y is the installed production capacity of U.E, v_y is a vector of variables inputs used to produce U.E from solar energy, $\bar{s}_y$ a vector of technical coefficients and $\bar{s}_y.v_y$ their scalar product. The production of an amount q_y of U.E requires y units of solar energy with a technical conversion coefficient $\bar{r}_y$. Our specification of the technology implies that producing one unit of U.E requires $s_y = 1/\bar{s}_y$ units of the variable input v_y. Assuming a constant given vector price p_{vy} of the variable inputs, the variable production cost is given by $p_{vy}.v_y = p_{vy}.s_y q_y$ (because $q_y = \bar{s}_y.v_y$ from technical efficiency). Denoting by $a_y = p_{vy}.s_y$, the variable unitary cost is a_y.

Converting solar energy into U.E faces access to space constraints. To formalise this idea, accessing solar energy, either directly (solar panels farms), of indirectly (windmills, biofuels), brings a cost $b_y(K_y)$ depending on the installed production capacity K_y, an increasing and strictly convex function of K_y. In addition, $b_y(0^+) = \underline{b}_y > 0$, accessing the first unit of space for solar energy conversion has a strictly positive cost. Therefore the total production cost of U.E from solar energy is $a_y q_y + b_y(K_y)$ with $q_y \leq K_y$.

As for U.E produced from fossil fuels, the production capacity of solar energy conversion can be increased over time through the purchase of new equipments. Denoting by $k_y(t)$ the investment rate at time t, the capacity dynamics are given by:

$$\dot{K}_y(t) = k_y(t) - \delta_y(t) K_y(t) \tag{2.4}$$

We make similar assumptions as for the production of U.E from fossil fuels. Keeping the installed capacity in order requires a maintenance effort $m_y K_y(t)$. Defaulting on maintenance puts the capacity out of order and $\delta_y(t)$ is thus the proportion of equipments not benefiting from maintenance at time t. As for the fossil energy sector, purchasing new equipments brings a cost $N_y(k_y)$, an increasing and convex function of k_y, and $n_y(0^+) =$

$dN_y(0^+)/dk_y = \underline{n}_y > 0$. Furthermore $m_y < \underline{n}_y$ to avoid awkward investment strategies in the solar energy sector.

Energy surplus and the green energy regime

At the final users stage, U.E produced from any source is perfectly substitutable and assumed not storable so that $q(t) = q_x(t) + q_y(t)$, where $q(t)$ denotes the aggregate consumption rate of U.E at time t. Consuming U.E in amount q generates a gross social surplus $u(q)$ where $u(q)$ is a continuous and twice differentiable function, strictly increasing, $u'(q) > 0$, strictly concave, $u''(q) < 0$, and satisfying the first Inada condition: $\lim_{q \downarrow 0} u'(q) = +\infty$, energy services are deemed essential for the society.

If the solar energy conversion sector would be the sole provider of U.E and the industry in a mature state of full deployment of capacities, the consumption rate of energy services should remain constant at a level $\tilde{q}$ solution of:

$$u'(q) = a_y + b_y'(q) + m_y + \rho \underline{n}_y$$

Under our concavity assumption on $u(.)$ and convexity assumption on $b(.)$, the solution is unique. The r.h.s of the above equation is the full marginal cost of UE provision, the sum of the direct marginal cost $a_y + b_y'(q)$ (since $q = K_y$), the maintenance marginal cost and the minimum rental cost on the last capacity unit installed to attain the mature capacity. The production of the UE amount $\tilde{q}$ requires an installed production capacity at least equal to $\tilde{K} = \tilde{q}$. Let $\tilde{p} = u'(\tilde{q})$ be the corresponding shadow price of U.E, in short the long run 'energy price'. We call *long run green energy regime* a situation where all energy services would be produced from carbon free energy and there would be no more accumulation of capacities by the industry.

We want to focus on the case of an economy converging to the long run green energy regime, that is where renewable energy would finally be able to outcompete fossil energy. For this to be the case it is necessary and sufficient to assume:

Assumption A. 1. *The long run energy price in a green energy regime is lower than the O&M cost of fossil energy provision:* $\tilde{p} < c_x + m_x$

We are going to show that under A.1, the transition to the green energy regime happens in finite time. For the fossil resource to be abundant it is thus sufficient that its initial availability be larger than the cumulated consumption of the resource before the complete transition to the green energy regime.

Carbon pollution

Let $Z(t)$ denote the CO_2 concentration in the atmosphere at time t, we call equivalently the 'carbon stock' by a slight abuse of language. The carbon stock self-depletes at a constant rate α. Thus in terms of UE from fossil energy, the carbon stock evolves according to:

$$\dot{Z}(t) = \zeta r_x q_x(t) - \alpha Z(t) \tag{2.5}$$

To avoid triggering catastrophic climate damage, the society decides to keep the CO_2 concentration below some security level $\bar{Z}$, that is stick to the constraint $\bar{Z} - Z(t) \geq 0$. Assume $Z^0 = Z(0) < \bar{Z}$ to give content to the problem.

3 The society optimality problem

The society has to compute UE production plans from the two crude energy sources, $\{q_x(t), q_y(t)\}$, thus a CO_2 polluting emission policy, $\zeta r_x q_x(t)$, a capital maintenance policy, equivalently scrapping plans, $\{\delta_x(t), \delta_y(t)\}$, and an investment policy in both capacities, $\{k_x(t), k_y(t)\}$, maximising the discounted cumulated sum of instantaneous net surpluses. Let $\rho > 0$ denote the social discount rate. The society solves the problem:

$$\begin{aligned}
\max \int_0^\infty & [u(q_x(t) + q_y(t)) - c_x q_x(t) - a_y q_y(t) - b_y(K_y(t)) - m_x K_x(t) \\
& \quad -m_y K_y(t) - N_x(k_x(t)) - N_y(k_y(t))] \, e^{-\rho t} dt \\
s.t. \quad & \dot{K}_x(t) = k_x(t) - \delta_x(t) K_x(t) \\
& \dot{K}_y(t) = k_y(t) - \delta_y(t) K_y(t) \\
& \dot{Z}(t) = \zeta r_x q_x(t) - \alpha Z(t) \\
& K_x(t) - q_x(t) \geq 0; K_y(t) - q_y(t) \geq 0 \\
& \bar{Z} - Z(t) \geq 0 \\
& q_x(t) \geq 0, q_y(t) \geq 0, k_x(t) \geq 0, k_y(t) \geq 0, \delta_x(t) \geq 0, \delta_y(t) \geq 0 \\
& K_x(t) \geq 0; K_y(t) \geq 0
\end{aligned}$$

with the vector of initial endowments, $K_x(0) = K_x^0$, $K_y(0) = K_y^0$, $Z(0) = Z^0$.

Let $\lambda_x(t)$ and $\lambda_y(t)$ denote respectively the co-state variables associated to the time evolutions of the state variables $K_x(t)$ and $K_y(t)$, respectively. λ_x and λ_y have the usual economic interpretation of being the shadow marginal values, or capital 'prices', of the specific productive capital stocks in the fossil and solar energies conversion sectors. Let $-\lambda_Z(t)$ denote the co-state variable associated to the dynamics of the carbon stock, $Z(t)$. Being the shadow cost of carbon pollution, $-\lambda_Z(t) \leq 0$, allowing to interpret $\lambda_Z(t) \geq 0$ as the carbon 'price', equivalently the carbon tax that should be levied on the production of U.E from the fossil fuels source, at any time t. The current value Hamiltonian of this optimal control problem reads (dropping time dependency):

$$\begin{aligned}
\mathcal{H} = & \, u(q_x + q_y) - c_x q_x - a_y q_y - b_y(K_y) - m_x K_x - m_y K_y - N_x(k_x) - N_y(k_y) \\
& + \lambda_x [k_x - \delta_x K_x] + \lambda_y [k_y - \delta_y K_y] - \lambda_Z [\zeta r_x q_x - \alpha Z]
\end{aligned}$$

Let $\eta_x(t)$ and $\eta_y(t)$ denote the Lagrange multipliers associated to the capacity constraints, $K_x(t) \geq q_x(t)$ and $K_y(t) \geq q_y(t)$ respectively. Similarly let $\nu_Z(t)$ the multiplier associated to the carbon constraint $\bar{Z} \geq Z(t)$, $\nu_{Kx}(t)$ and $\nu_{Ky}(t)$ the multipliers associated to the positivity constraints on $K_x(t)$ and $K_y(t)$ respectively. Last let $\gamma_h(t)$ be the Lagrange multipliers associated to the positivity constraints on $h \in \{q_x, q_y, k_x, k_y, \delta_x, \delta_y\}$. The current value Lagrangian of this problem reads:

$$\begin{aligned}
\mathcal{L} = & \, \mathcal{H} + \eta_x(K_x - q_x) + \eta_y(K_y - q_y) + \nu_Z(\bar{Z} - Z) + \nu_{Kx} K_x + \nu_{Ky} K_y \\
& + \gamma_{qx} q_x + \gamma_{qy} q_y + \gamma_{kx} k_x + \gamma_{ky} k_y + \gamma_{\delta x} \delta_x + \gamma_{\delta y} \delta_y
\end{aligned}$$

A first set of almost necessary optimality conditions is:

$$u'(q_x + q_y) = c_x + \zeta r_x \lambda_Z + \eta_x - \gamma_{qx} \tag{3.6}$$

$$u'(q_x + q_y) = a_y + \eta_y - \gamma_{qy} \tag{3.7}$$

$$\lambda_x = n_x(k_x) - \gamma_{kx} \tag{3.8}$$

$$\lambda_y = n_y(k_y) - \gamma_{ky} \tag{3.9}$$

$$\lambda_x K_x = \gamma_{\delta x} \tag{3.10}$$

$$\lambda_y K_y = \gamma_{\delta y}. \tag{3.11}$$

When time differentiable the motion of the co-state variables obey:

$$\dot{\lambda}_x = (\rho + \delta_x)\lambda_x - \eta_x + m_x - \nu_{Kx} \tag{3.12}$$

$$\dot{\lambda}_y = (\rho + \delta_y)\lambda_y - \eta_y + m_y + b'_y(K_y) - \nu_{Ky} \tag{3.13}$$

$$\dot{\lambda}_Z = (\rho + \alpha)\lambda_Z - \nu_Z \tag{3.14}$$

3.1 Preliminary considerations

When the economy is constrained by the climate ceiling $\bar{Z}$, the production of UE from fossils must stay constant at the level $\bar{q}_x = \alpha\bar{Z}/(\zeta r_x)$, corresponding to a production capacity at least equal to $\bar{K}_x = \bar{q}_x$. Under A.1, the transition to renewable energy should be complete in the long run, implying the existence of a time $\bar{t}_Z$, after which the capacity $\bar{K}_x$ should be progressively dismantled and the economy should not be anymore constrained by the climate ceiling.

We conclude that the optimal path exhibits a three phases structure. During a first time interval, $[0, \underline{t}_Z)$, because $Z^0 < \bar{Z}$, the economy is not yet constrained by the climate. The polluting emissions, $\zeta r_x q_x(t)$, are larger than the natural self-depletion and the carbon stock, $Z(t)$, increases up to the cap $\bar{Z}$, reached at time $\underline{t}_Z$, $Z(\underline{t}_Z) = \bar{Z}$. The economy is constrained by the carbon ceiling, $Z(t) = \bar{Z}$, during a second time interval, $[\underline{t}_Z, \bar{t}_Z)$. There is no more accumulation of capacity in the fossil fuels energy sector, $K_x(t) = \bar{K}_x$, $t \in [\underline{t}_Z, \bar{t}_Z)$. The renewable energy conversion sector goes on accumulating capacities and $q(t) = \bar{q}_x + q_y(t)$ increases, driving down the energy price $p(t) = u'(\bar{q}_x + q_y(t))$. The consequence is a progressive fall of the profitability of the fossil energy conversion sector down to a point when its productive capacity becomes in excess at time $\bar{t}_Z$. After this time, the fossil energy conversion sector scraps progressively its capacity allowing for an escape from the climate constraint. During the third phase, $[\bar{t}_Z, \infty)$, the production capacity of the fossil energy conversion sector is driven down to 0 while the production capacity of the renewable energy conversion sector increases up to its long run value, $\tilde{K}_y$. We call respectively *pre-ceiling* phase, *ceiling* phase, and *post-ceiling* phase the three time intervals, $[0, \underline{t}_Z)$, $[\underline{t}_Z, \bar{t}_Z)$ and $[\bar{t}_Z, \infty)$.

Over the pre-ceiling phase, $\nu_Z = 0$, by complementary slackness and $\lambda_Z(t)$ increases exponentially at the rate $(\rho + \alpha)$. After $\bar{t}_Z$, the economy is no more submitted to the carbon cap constraint and $\lambda_Z(t) = 0$, $t \geq \bar{t}_Z$. The condition (3.10) and the complementary slackness condition $\gamma_{\delta x}\delta_x = 0$ imply that $\lambda_x(t) = 0$ when $\delta_x(t) > 0$. Hence $\lambda_x(t) = 0$, $t \geq \bar{t}_Z$, over the post-ceiling phase. We now browse recursively through the different phases.

4 The post-ceiling phase

On one hand $\lambda_Z(t) = 0$ after $\bar{t}_Z$. On the other hand, $\lambda_x(t) = 0$, thus λ_x is a nil constant, hence $\dot{\lambda}_x(t) = 0$, $t > \bar{t}_Z$, implying through (3.12) that $\eta_x(t) = m_x$ when $K_x > 0$, since $\nu_{Kx} = 0$ by complementary slackness when $K_x > 0$. Hence (3.6) reads:

$$u'(K_x(t) + K_y(t)) = c_x + m_x \equiv \bar{p}_x \tag{4.1}$$

whenever $q_x(t) = K_x(t) > 0$.

Under our assumption of a 100 % renewable energy long run regime, the long run capacity of the renewable energy conversion sector, $\tilde{K}_y$, is determined through:

$$u'(\tilde{K}_y) = a_y + b'_y(\tilde{K}_y) + m_y + \rho\underline{n}_y \tag{4.2}$$

Since $\tilde{p} < c_x + m_x = \bar{p}_x$ under A.1 and $p(t)$ is a time continuous function along any optimum, the post-ceiling phase is composed of two successive sub-phases: a first sub-phase, $[\bar{t}_Z, \bar{t}_x)$, during which the fossil energy production capacity is progressively dismantled until the complete elimination of fossil energy in the fuel mix, so that $K_x(\bar{t}_x) = 0$, and a second sub-phase, $[\bar{t}_x, \infty)$, of infinite duration, during which the renewable energy sector is the sole energy provider and continues to expand its production capacity up to $\tilde{K}_y$. Let $K_y^x = K_y(\bar{t}_x)$ denote the renewable energy production capacity at the final closure of the fossil energy sector. This production capacity is the unique solution of $u'(K_y) = \bar{p}_x = c_x + m_x$. We now describe the optimal policy over the two sub-phases recursively.

4.1 The ultimate sub-phase, $[\bar{t}_x, \infty)$

Using (4.1) and (3.7), an equivalent expression of (3.13) reads:

$$\dot{\lambda}_y(t) = \rho\lambda_y(t) - \Big[u'(K_y(t)) - a_y - b'_y(K_y(t)) - m_y\Big] \tag{4.3}$$

On the other hand, $n_y(k_y)$ being a strictly increasing function of k_y, its inverse $n_y^{-1}(.)$ is well defined and (3.9) can be written as:

$$\dot{K}_y(t) = k_y(t) = n_y^{-1}(\lambda_y(t)) \equiv k_y(\lambda_y(t)) \tag{4.4}$$

introducing the expression of the inverse function $k_y(\lambda_y)$ by a slight abuse of notation. Since $n_y(.)$ is an increasing function, $k_y(.)$ is also an increasing function of λ_y.

The optimal time evolution of $\{K_y(t), \lambda_y(t)\}$ for $t \geq \bar{t}_x$, is the solution of the autonomous differential system (4.4)-(4.3). In the phase plane (K_y, λ_y), the isocline $\dot{\lambda}_y = 0$ is a curve of equation:

$$\lambda_y = \frac{1}{\rho}\Big[u'(K_y) - a_y - b'_y(K_y) - m_y\Big]$$

a decreasing function of K_y since $u''(.) < 0$ and $b_y''(.) > 0$. $\lambda_y(t)$ is time increasing in the phase plane above the isocline $\dot{\lambda}_y = 0$ and time decreasing below it. Observe that if at some time $\bar{t}$, $\dot{\lambda}_y(\bar{t}) > 0$ then:

$$\rho\lambda_y(\bar{t}) > u'(K_y(\bar{t})) - a_y - b_y'(K_y(\bar{t})) - m_y$$

For $t > \bar{t}$, the l.h.s of the previous inequality would increase since $\dot{\lambda}_y(t) > 0$ while the r.h.s. would decrease since $K_y(t)$ increases. Thus the inequality would remain valid after $\bar{t}$.

By construction the isocline $\dot{\lambda}_y = 0$ cuts the horizontal $\lambda_y = \underline{n}_y$ at $K = \tilde{K}_y$ in the phase plane, (K_y, λ_y). Thus if there exists a time $\bar{t}$ such that $\dot{\lambda}_y(\bar{t}) > 0$, $\lambda_y(t)$ should increase after $\bar{t}$ above the isocline $\dot{\lambda}_y = 0$ and converge toward a value of λ_y strictly larger than $\underline{n}_y$, a contradiction. We conclude that $\dot{\lambda}_y(t) < 0$ throughout the sub-phase, $[\bar{t}_x, \infty)$.

With the particular long run solution, $(\tilde{K}_y, \underline{n}_y)$, the differential system (4.4)-(4.3) admits a unique general solution. Let $\lambda_y^* = \hat{\lambda}_y(K_y^*)$ denote the implicit relationship between K_y^* and λ_y^* along the optimal path. Since $\dot{K}_y^* > 0$ and $\dot{\lambda}_y^* < 0$, we conclude that $d\hat{\lambda}_y(K_y)/dK_y = \dot{\lambda}_y/\dot{K}_y < 0$. The relationship between the investment value, λ_y, and the installed capacity, K_y, takes the form of a decreasing curve in the phase plane. This implies the existence of a unique value of λ_y, we denote λ_y^x, defined by $\lambda_y^x = \hat{\lambda}_y(K_y^x)$, the value of an investment in extra renewable energy production capacities at the beginning of the sub-phase, $[\bar{t}_x, \infty)$, $\lambda_y(\bar{t}_x) = \lambda_y^x$.

4.2 The first sub-phase $[\bar{t}_Z, \bar{t}_x)$

Using (4.1) and (3.7), an equivalent expression of (3.13) reads:

$$\dot{\lambda}_y(t) = \rho\lambda_y(t) - \Big[\bar{p}_x - a_y - b_y'(K_y(t)) - m_y\Big] \tag{4.5}$$

The optimal dynamics of $(K_y(t), \lambda_y(t)$ during the phase, $[\bar{t}_Z, \bar{t}_x)$ is the general solution of the differential autonomous system (4.4)-(4.5). With the particular solution at time $\bar{t}_x$, (K_y^x, λ_y^x), the system admits a unique general solution $\{K_y^*(t), \lambda_y^*(t)\}$. Let $\lambda_y^* = \hat{\lambda}_y(K_y^*)$ denote the implicit relationship between K_y and λ_y along the optimal path during the post-ceiling phase.

The argument developed for the second sub-phase still applies to the first sub-phase. If there exists some time $\bar{t}$, $\bar{t}_Z < \bar{t} < \bar{t}_x$, such that $\dot{\lambda}_y(\bar{t}) > 0$, then $\dot{\lambda}_y(t) > 0$. Hence $\dot{\lambda}_y(\bar{t}_x^-) > 0$ which yields a contradiction since $\dot{\lambda}_y(\bar{t}_x^+) < 0$, as shown before, and $\lambda_y(t)$ must be continuously time differentiable at time $\bar{t}_x$. We conclude that $\dot{\lambda}_y(t) < 0$ during the first post-ceiling sub-phase. Then $\dot{K}_y^* > 0$ and $\dot{\lambda}_y^* < 0$ imply that $d\hat{\lambda}_y/dK_y = \dot{\lambda}_y^*/\dot{K}_y^* < 0$.

At the beginning time of the first post-ceiling sub-phase, $\bar{p}_x = u'(\bar{K}_x + K_y(\bar{t}_Z))$ determines $\bar{K}_y$, the optimal capacity of the renewable energy conversion sector at that time. $\hat{\lambda}_y(K_y)$ being a strictly decreasing function of K_y, to $\bar{K}_y$ is associated a unique value of λ_y at time $\bar{t}_Z$ we denote $\bar{\lambda}_y$, $\lambda_y^*(\bar{t}_Z) = \hat{\lambda}_y(\bar{K}_y) = \bar{\lambda}_y$.

Because $\bar{p}_x = u'(K_x(t) + K_y(t))$, the aggregate capital stock is held constant during the first post-ceiling sub-phase. Hence $K_y(t) \to K_y^x$ when $K_x(t) \to 0$, implies that $K_y^x =$

$K_x(t)+K_y(t)$. There is no expansion of the energy sector at the aggregate level during the first sub-phase. The increase of the renewable energy conversion capacity just compensates the progressive scraping of the fossil energy conversion capacities. This gives an expression of $\bar{K}_y$ as $\bar{K}_y = K^x_y - \bar{K}_x$.

The differential system being time autonomous, it takes a unique given time length T_x for the system to move from its initial position, $(\bar{K}_y, \bar{\lambda}_y)$, to its final position, (K^x_y, λ^x_y). This unique time length determines $\bar{t}_x$ for a given $\bar{t}_Z$, through $\bar{t}_x = \bar{t}_Z + T_x$.

We get through time differentiation $\dot{K}_x(t) = -k_y(t) = -k_y(\lambda^*_y(t))$. Thus the optimal scraping rate of capacities in the fossil energy conversion sector is given by $\delta^*_x(t) = k_y(\lambda^*_y(t))/K_x(t)$. Since $K_x(t) = K^x_y - K_y(t)$, an equivalent expression of δ_x as a function of K_y reads:

$$\delta^*_x(t) = \frac{k_y(\hat{\lambda}_y(K^*_y(t))}{K^x_y - K^*_y(t)}$$

The dynamics of the scraping rate is indeterminate because $k_y(t)$ decreases while $K_y(t)$ increases.

The following Proposition 1 summaries the main qualitative properties of the optimal path during the post-ceiling phase:

Proposition P. 1. *The post-ceiling phase is a sequence of two successive sub-phases. During the first sub-phase, $[\bar{t}_Z, \bar{t}_x)$:*

1. *The energy price remains constant at the level $\bar{p}_x = c_x + m_x$. Thus the aggregate consumption rate of U.E stays constant at the level $\bar{q}$ solution of $u'(q) = \bar{p}_x$.*
2. *The fossil energy conversion rate, $q_x(t)$, declines at the same rate the renewable energy conversion rate, $q_y(t)$, increases. The conversion capacity of fossil energy, $K_x(t)$, falls down to 0 from the initial level, $\bar{K}_x$, while the conversion capacity of renewable energy increases from the level $K^x_y - \bar{K}_x$ up to K^x_y, the unique renewable energy conversion capacity solution of $u'(K^x_y) = \bar{p}_x$.*
3. *The investment rate in renewable energy conversion capacities, $k_y(t)$, falls during the phase down to $k^x_y = k_y(\lambda^x_y)$ at time $\bar{t}_x$.*
4. *The value of one additional renewable energy conversion capacity unit, $\lambda_y(t)$, falls from the level $\bar{\lambda}_y$ down to λ^x_y at time $\bar{t}_x$.*
5. *The time length of the first sub-phase, T_x, is uniquely determined by the initial vector $(K^x_y - \bar{K}_x, \bar{\lambda}_y)$ and the final vector (K^x_y, λ^x_y), so that $\bar{t}_x = \bar{t}_Z + T_x$ is determined for any given $\bar{t}_Z$.*

During the second sub-phase of infinite duration, $[\bar{t}_x, \infty)$:

1. *The energy price decreases from the level $\bar{p}_x = c_x + m_x$ down to the level $\tilde{p} = u'(\tilde{K}_y)$, where $\tilde{K}_y$ is the long run renewable energy conversion capacity, the unique solution of $u'(K_y) = a_y + b'_y(K_y) + m_y + \rho\underline{n}_y$.*

2. *The renewable energy conversion rate, $q_y(t)$ increases. The conversion capacity of renewable energy increases from the level K_y^x up to $\tilde{K}_y$, the long run renewable energy conversion capacity.*
3. *The investment rate in renewable energy conversion capacities, $k_y(t)$, falls asymptotically during the phase down to 0.*
4. *The value of one additional renewable energy conversion capacity unit, $\lambda_y(t)$, falls from the level λ_y^x asymptotically down to $\underline{n}_y$.*

The main takeaway for the next steps of the study are: 1) There exists a unique renewable energy conversion capacity to be installed, $\bar{K}_y$, when the economy escapes from the climate constraint; 2) This capacity is determined independently of the history of the energy transition before $\bar{t}_Z$; 3) Associated to $\bar{K}_y$, there exists a unique value of the renewable energy capital price, $\bar{\lambda}_y$, at time $\bar{t}_Z$, a decreasing function of $\bar{K}_x$.

Once the renewable energy sector has accumulated the capacity $\bar{K}_y$, the energy price is still larger than $\tilde{p}_x$ thus the sector should continue to expand its production capacity. This is compatible with the survival of the fossil energy sector only if this sector scraps an equal amount to the accumulation of capacities in the renewable energy sector in order to keep the energy price at its floor level $\bar{p}_x$ allowing the fossil energy sector to operate, that is able to cover the variable unit cost and the maintenance cost.

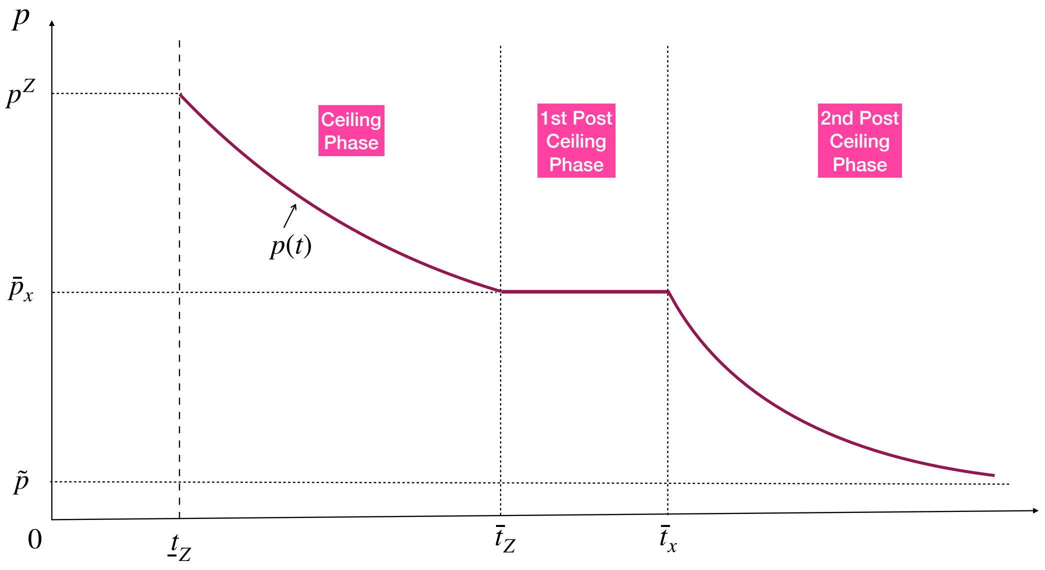

Figure 3: Time evolution of the energy price after $\underline{t}_Z$.

Observe that the energy price dynamics is very different from the textbook ceiling model. In this model the energy price is capped at a constant level during the ceiling phase and next rises up to the price at which the renewable energy sector is able to supply all the energy demand. We are going to show that the renewable energy sector expands permanently its capacities during the ceiling phase, driving down the energy price. Thus the energy price trajectory after $\underline{t}_Z$, that is when the economy begins to face the climate constraint is made of a first phase, $[\underline{t}_Z, \bar{t}_Z)$, of declining energy prices, followed by a phase, $[\bar{t}_Z, \bar{t}_x)$, of energy price stabilisation at the level $\bar{p}_x$, and ended by a declining price trend, converging asymptotically down to the level $\tilde{p}$. The pattern of evolution of the energy price is shown on the Figure 3.

The interim phase, $[\bar{t}_Z, \bar{t}_x)$, thus corresponds to a structural adjustment phase when the fossil energy sector declines irreversibly while the renewable energy sector takes the lead progressively. During this phase, the energy price is stabilised.

4.3 The scraping border

In the phase plane, (K_x, K_x), the time evolution of $K_x(t)$ and $K_y(t)$ during the scraping phase, $[\bar{t}_Z, \bar{t}_x)$ defines a decreasing line of equation $K_x = K_y^x - K_y$, intersecting the horizontal axis at $K_y = K_y^x$ and cutting the $K_x = \bar{K}_x$ horizontal at $K_y = \bar{K}_y$. For different levels of $\bar{K}_x$, and thus of the carbon ceiling constraint, $\bar{Z}$, the line defines a critical border in the plane, (K_x, K_y), we call the *scraping border* (see Figure 4).

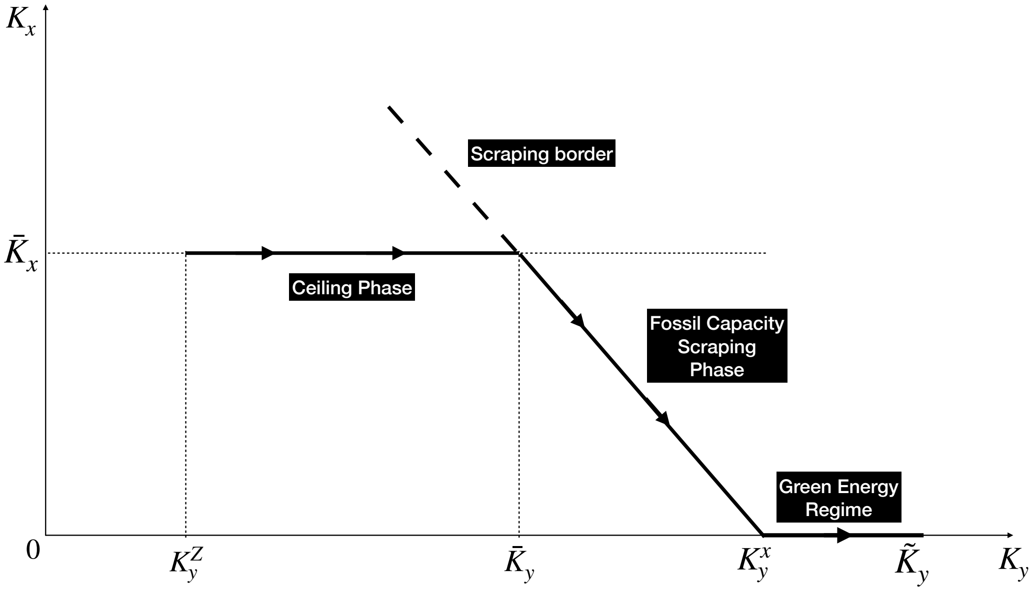

Figure 4: The scraping border in the (K_y, K_x) plane.

5 The ceiling phase

The study of the post-ceiling phase has shown that at the end of the phase, $\lambda_x(\bar{t}_Z) = 0$, $\lambda_y(\bar{t}_Z) = \bar{\lambda}_y$, $\lambda_Z(\bar{t}_Z) = 0$ while $K_x(\bar{t}_Z) = \bar{K}_x$ and $K_y(\bar{t}_Z) = \bar{K}_y = K_y^x - \bar{K}_x$. In addition the energy price is equal to $\bar{p}_x$, $p(\bar{t}_Z) = \bar{p}_x$. From the study of the phase diagram we can remark that λ_y is time continuous and time differentiable at time $\bar{t}_Z$, and since $\dot{\lambda}_y(\bar{t}_Z^+) < 0$, we can infer that $\dot{\lambda}_y(\bar{t}_Z^-) = \dot{\lambda}_y(\bar{t}_Z^+) < 0$.

Two possibilities must be considered. Either the fossil energy sector has begun to scrap its capacities before the ceiling phase. This should be the case if the initial capacity, $K_x^0 = K_x(0)$ is larger than $\bar{K}_x$. Either the fossil energy sector has constantly built up capacities before the ceiling phase and would have been willing to continue accumulating capacities if not constrained by the carbon ceiling. In the first case the value of fossil energy conversion capacity units should be nil while it should be still positive in the second case.

Observe that the dynamics of capital accumulation in the renewable energy sector does not depend on the profitability conditions in the fossil energy sector. For a given ceiling constraint, and thus a given $\bar{K}_x$, the energy price $p = u'(\bar{K}_x + K_y)$, is a function of K_y alone. It is then easily checked that the dynamical system determining the investment policy in the renewable energy sector depends only on K_y. The optimal investment policy in renewable energy conversion capacities is independent from the value of fossil energy conversion capacity units during the ceiling phase, either nil or still positive.

All these considerations suggest the following study plan. First we are going to describe the optimal investment policy in the renewable energy sector. Second we study the capital value dynamics in the fossil energy sector, either nil or positive, and deduce the implications of these dynamics on the temporal evolution of the carbon price (e.g. the carbon tax) during the ceiling phase.

5.1 Investment policy in the renewable energy sector

Let $\beta_y(t)$ denote the net operational margin (that is net of the unit maintenance cost) on one installed capacity unit converting crude renewable energy into U.E. at time t.

$$\beta_y(t) \equiv p(t) - a_y - b_y'(K_y(t)) - m_y$$

The dynamics of λ_y given by (3.13) reads equivalently $\dot{\lambda}_y = \rho\lambda_y - \beta_y$.

The same argument as used in the study of the post-ceiling phase applies. First observe that because $p(t) = u'(\bar{K}_x + K_y(t))$:

$$\dot{\beta}_y(t) = \Big[u''(t) - b_y''(t)\Big]\, k_y(t) < 0 \quad \text{if} \quad k_y(t) > 0$$

Thus $\beta_y(t)$ is a strictly decreasing time function if $k_y(t) > 0$. If there exists some time instant $\bar{t}$, $\bar{t} \in [\underline{t}_Z, \bar{t}_Z)$ such that $\dot{\lambda}_y(\bar{t}) > 0$, then $\rho\lambda_y(\bar{t}) > \beta_y(\bar{t})$, and the l.h.s being time increasing while the r.h.s is time decreasing implies that the inequality is preserved. Thus $\lambda_y(t)$ should keep increasing through time if it is increasing at some time $\bar{t}$ during the ceiling phase. However $\dot{\lambda}_y(\bar{t}_Z) < 0$ yields a contradiction, thus $\lambda_y(t)$ must be time decreasing during the whole ceiling phase.

To the decrease of $\lambda_y(t)$ is associated a fall of the investment rate in renewable energy conversion capacities, $k_y(t)$ decreases. On the other hand $K_y(t)$ increases, although at a declining rate, implying that $q(t) = \bar{K}_x + K_y(t)$ increases and $p(t) = u'(q(t))$ decreases. Independently of the profitability conditions in the fossil energy conversion sector, the energy price should decrease because of the expansion of energy supply resulting from capacity investments in the renewable energy sector.

During the ceiling phase the optimal trajectory $\{K_y(t), \lambda_y(t)\}$ is a general solution of the autonomous differential system:

$$
\begin{aligned}
\dot{K}_y(t) &= k_y(\lambda_y(t)) \\
\dot{\lambda}_y(t) &= \rho\lambda_y(t) - \beta_y(K_y(t))
\end{aligned} \tag{5.1}
$$

With the particular solution, $(\bar{K}_y, \bar{\lambda}_y)$, this system admits a unique general solution. Observe that at time $\underline{t}_Z$, $\dot{\lambda}_y(\underline{t}_Z) < 0$ implies that $\lambda_y(\underline{t}_Z) < \beta_y(\underline{t}_Z)/\rho$.

To a given capacity at the beginning of the ceiling phase, $K_y^Z = K(\underline{t}_Z)$, is associated a time duration of the ceiling phase, $T^c \equiv \bar{t}_Z - \underline{t}_Z$. The differential system being time autonomous and $K_y(\bar{t}_Z) = \bar{K}_y$, being a parametric constant, it is clear that T^c is a decreasing function of K_y^Z. Thus to a given pair, $(K_y^Z, \underline{t}_Z)$, is associated a unique value of $\bar{t}_Z$, the end time of the ceiling phase. The end time increases linearly with $\underline{t}_Z$: $\partial\bar{t}_Z/\partial\underline{t}_Z = 1$ and decreases with K_y^Z: $\partial\bar{t}_Z/\partial K_y^Z < 0$. From the study of the post-ceiling phases, we deduce that to a pair $(\underline{t}_Z, K_y^Z)$ is associated a unique value of $\bar{t}_x$, the phasing out time of fossil energy. It is immediate that $\partial\bar{t}_x/\partial\underline{t}_Z = 1$ and $\partial\bar{t}_x/\partial K_y^Z = 0$. Because the critical renewable energy conversion capacity, $\bar{K}_y$, triggering the exit from the carbon ceiling regime, is parametrically determined, the time length of the first post-ceiling phase is independent from K_y^Z.

To sum up, because $\lambda_y(t)$ should decrease during the ceiling phase, $\lambda_y(\underline{t}_Z) \equiv \lambda_y^Z > \bar{\lambda}_y > \underline{n}_y$ and the renewable energy sector should also invest at least during some time interval $[\underline{t}_Z - \Delta, \underline{t}_Z)$, $\Delta > 0$, before the ceiling phase. The energy price constantly falls during the phase and $q_x(t) = \bar{K}_x$ stays constant while $q_y(t) = K_y(t)$ increases, inducing a rise of $q(t)$, the aggregate U.E. consumption rate. The investment rate in renewable energy conversion capacities into U.E falls after $\underline{t}_Z$, thus the growth of $K_y(t)$ slows down together with the growth of $q(t)$ during the ceiling phase. Remember that over the time interval $[\bar{t}_Z, \bar{t}_x)$, the aggregate capital stock stays constant, so that $q(t)$ is constant after $\bar{t}_Z$ until the complete scraping of fossil energy conversion units.

We now turn to the fossil energy sector and distinguish two scenarios. In the scenario $\#c1$, where 'c' stands for 'ceiling', the fossil energy industry has begun to scrap its capacities before $\underline{t}_Z$, while in the alternative scenario $\#c2$, the industry has constantly built up capacities until $\underline{t}_Z$.

5.2 The Scenario $\#c1$

In this scenario, it should be the case that $\lambda_x(\underline{t}_Z) = 0$. And since there can be no more investments in fossil energy conversion capacities building after $\underline{t}_Z$, $\lambda_x(t)$ should remain nil during the whole ceiling phase. Let $\beta_x(t)$ denote the net operational margin (net of the maintenance costs) on one installed capacity unit in fossil energy conversion at time t:

$$
\beta_x(t) \equiv p(t) - c_x - m_x - \zeta r_x \lambda_Z(t)
$$

An equivalent writing of (3.12) reads: $\dot{\lambda}_x(t) = (\rho + \delta_x(t))\lambda_x(t) - \beta_x(t)$. Thus we conclude from $\lambda_x(t) = 0$, and hence $\dot{\lambda}_x(t) = 0$, for $t \in [\underline{t}_Z, \bar{t}_Z)$, that $\beta_x(t) = 0$, during the whole ceiling phase.

Since $\gamma_{\delta x} = 0$ through (3.10) when $\lambda_x = 0$, it could be the case through complementary slackness that δ_x be > 0. However the energy sector has to keep fossil energy production capacities waiting for a sufficient accumulation of renewable energy capacities to be able to exit the ceiling phase. Thus $\delta_x(t)$ should be nil during the ceiling phase, the fossil energy sector conserving the capacity $\bar{K}_x$ until the renewable energy capacity has risen up to the level $\bar{K}_y$.

Because $\beta_x(t) = 0$, $t \in [\underline{t}_Z, \bar{t}_Z)$, we get:

$$p(t) = c_x + m_x + \zeta r_x \lambda_Z$$

Let $\pi_x(t) \equiv p(t) - c_x - m_x = p(t) - \bar{p}_x$ denote the operational margin before tax. During the ceiling phase, $\pi_x(t)$ decreases down to 0 at time $\bar{t}_Z$. The carbon tax is set at the level: $\lambda_Z(t) = \pi_x / \zeta r_x$. The regulator keeps $\beta_x(t)$ at a nil level by collecting the operational margin before tax through the carbon tax levy. The dynamics of $\pi_x(t)$ is the same as the energy price dynamics. Time differentiating:

$$\dot{\lambda}_Z(t) = \frac{\dot{p}(t)}{\zeta r_x} < 0$$

The carbon price constantly decreases down to 0 at time $\underline{t}_Z$.

Taking into account the optimal accumulation trajectory of capacities in the renewable energy sector, the optimal carbon tax, $\lambda_Z^*(t)$, is given by:

$$\lambda_Z^*(t) = \frac{1}{\zeta r_x} \left[u'(\bar{K}_x + K_y^*(t)) - c_x - m_x \right]$$

which determines $\lambda_Z^*(t)$ once the optimal capital accumulation policy in the renewable energy conversion sector, $\{k_y^*(t), K_y^*(t)\}$, has been determined. Time differentiating, we obtain also the following expression of $\nu_Z^*(t)$, the optimal value of the opportunity cost of the climate constraint:

$$\begin{aligned} \nu_Z^*(t) &= (\rho + \alpha)\lambda_Z^*(t) - \dot{\lambda}_Z^*(t) = \frac{1}{\zeta r_x} \left[(\rho + \alpha)(p(t) - \bar{p}_x) - \dot{p}(t) \right] \\ &= \frac{1}{\zeta r_x} \left[(\rho + \alpha) \left(u'(\bar{K}_x + K_y^*(t)) - c_x - m_x \right) - u''(\bar{K}_x + K_y^*(t)) k_y^*(t) \right] \end{aligned} \tag{5.2}$$

In the textbook carbon ceiling model, the energy price must be constant during the ceiling phase. This implies that not only the carbon price should decrease during the phase but also the opportunity cost of the climate constraint $\nu_Z(t)$. The issue is more tricky in the present model. Time differentiating (5.2), we obtain:

$$\dot{\nu}_Z(t) = \frac{1}{\zeta r_x} \left[(\rho + \alpha)\dot{p}(t) - \ddot{p}(t) \right]$$

We know already that $\dot{p}(t) < 0$ but the sign of $\ddot{p}(t)$ is indeterminate. Twice differentiating $p(t)$ we obtain:

$$\dot{p}(t) = u''(t) k_y(t) \Longrightarrow \ddot{p}(t) = u'''(t) k_y^2(t) + u''(t) \dot{k}_y(t)$$

Because $\dot{k}_y(t) < 0$, as shown before, the second term of the sum is > 0, since $u''(t) < 0$. But the sign of the first term depends on the sign of the third derivative of the gross surplus function. A sufficient condition for $\ddot{p}(t)$ to be > 0 is that $u'''(t) \geq 0$. Remark that this assumption is trivially satisfied by a linear-quadratic $u(.)$ function (a linear demand where u'' is a constant), or a constant elasticity function (an iso-elastic demand function of the form $p = q^{-\epsilon}$, with $\epsilon > 0$, where $u''' > 0$). We thus conclude that with a linear of iso-elastic energy demand function, $\dot{\nu}_Z(t) < 0$ because $\dot{p}(t) < 0$ and $\ddot{p}(t) > 0$.

Because $\zeta r_x \lambda_Z(t) = p(t) - \bar{p}_x$, the carbon price spans the demand curve over time. This is one of the reason for which at the second order, $\ddot{p}(t)$ can increase or decrease depending on the variations of the slope of the demand curve (the third order derivative). On the other side, the speed of move along the curve depends on the speed of capital accumulation in the renewable energy sector, that is of the rate of change of $k_y(t)$. We have shown that $k_y(t)$ decreases, but this decrease is itself not constant so that the acceleration or deceleration of the move along the demand curve also influences the dynamics of $\dot{p}(t)$ and $\ddot{p}(t)$. In turn the dynamics of ν_Z is affected and $\nu_Z(t)$ can be non monotonously evolving over time. The opportunity cost of the climate constraint in this model is equivalently the opportunity cost of the insufficient accumulation of renewable energy conversion capacities with respect to an escape of the carbon ceiling regime.

That the fossil energy sector does not continue to scrap capacities at the time $\underline{t}_Z$, thus eliminating the ceiling phase, requires that $K_y^Z = K_y(\underline{t}_Z)$ be lower than $\bar{K}_y$. Even if the fossil energy sector scraps capacities before the beginning of the ceiling phase, in order to adjust to the net zero constraint, it must stop doing so when the ceiling phase begins. Thus what explains the existence of a non degenerate ceiling phase in this scenario is an insufficient renewable energy conversion capital building before $\underline{t}_Z$, the time at which the carbon stock level reaches the carbon cap. The economy must wait under the ceiling constraint until the renewable energy sector has built up enough capacities to provide a possibility to exit the ceiling and resume the capacity scraping process in the fossil energy conversion sector.

In the textbook ceiling model, what justifies the existence of a blockaded time episode at the ceiling is the need to wait until the fossil crude energy reserves be sufficiently depleted to allow escaping the ceiling phase when the unconstrained extraction rate would fall below the net zero cap. In the present model, the fossil crude energy reserves have been assumed abundant. This is now the scarcity of renewable energy production capacities which explains the existence of a ceiling phase in a time to build context, that is under convex investment costs. Observe that with linear investment costs, like in a conventional macro growth model, the investment rate would be bounded from above by the consumption versus savings arbitrage equation so that a ceiling phase should also happen, the economy not willing to invest too much in renewable energy conversion capital building. Such a slowing down effect is of course magnified with convex building costs.

5.3 The Scenario $\#c2$

It is also possible that a combination of high building costs in the two sectors and a severe climate constraint result in a situation where not only the renewable energy sector has not built up enough capacities to avoid a climate constrained episode but the fossil energy sector has just accumulated the net zero capacity at the time the ceiling regime begins. This scenario requires of course that $K_x^0 < \bar{K}_x$. Although not very empirically relevant, this possibility recalls that the existence of a ceiling regime is not the consequence of an excess production capacity in the fossil energy sector with respect to the net zero target but from an insufficient accumulation of capacities in the renewable energy sector, as outlined before.

Because it is no more possible to invest in capacity building in the fossil energy sector when the climate constraint binds, the sector must design an investment policy during the pre-ceiling phase such that the value of an extra piece of capital be just equal to $\underline{n}_x$ at time $\underline{t}_Z$. This is a consequence of the time continuity of $\lambda_x(t)$ at time $\underline{t}_Z$. Therefore, $\lambda_x(t)$ is no more nil during the ceiling phase and should vary over time, together with $\beta_x(t)$. Thus we need to perform a specific study of the dynamics of $\lambda_x(t)$ and $\beta_x(t)$ during the phase.

Through (3.8), positive investment in the fossil energy sector before $\underline{t}_Z$ implies that $\lambda_x(t) > \underline{n}_x$ over a time interval $[\underline{t}_Z - \Delta, \underline{t}_Z)$, $\Delta > 0$ and that $\lambda_x(\underline{t}_Z) = \underline{n}_x$. We thus conclude that $\dot{\lambda}_x(\underline{t}) < 0$ over some time interval $[\underline{t}_Z - \Delta', \underline{t}_Z)$, $\Delta' \leq \Delta$. The function $\dot{\lambda}_x(t)$ being time continuous, we also conclude that $\dot{\lambda}_x(\underline{t}_Z^-) = \dot{\lambda}_x(\underline{t}_Z^+) = \dot{\lambda}_x(\underline{t}_Z) < 0$. Last we have , $\lambda_x(\bar{t}_Z) = 0$.

The difficulty now is that the carbon pricing policy is indeterminate. However this policy is submitted to some constraints, so it is only partially indeterminate. Through (3.6):

$$\beta_x(t) = p(t) - c_x - m_x - \zeta r_x \lambda_Z(t) = p(t) - \bar{p}_x - \zeta r_x \lambda_Z(t) \equiv \pi_x(t) - \zeta r_x \lambda_Z(t)$$

The time evolution of $\pi_x(t)$ is given by the optimal investment policy of the renewable energy sector and is thus given to the regulator. Remember that we have shown that $\pi_x(t)$ constantly decreases down to 0 at time $\bar{t}_Z$.

This is not the only constraint that influences the carbon pricing policy of the regulator. He (she) must also take into account that the dynamics of $\lambda_x(t)$ is given by $\dot{\lambda}_x(t) = \rho\lambda_x(t) - \beta_x(t) = \rho\lambda_x(t) - \pi_x(t) + \zeta r_x \lambda_Z(t)$.

If for example, the regulator decides to set $\lambda_Z(t) = \pi_x(t)/\zeta r_x$ over some non degenerate time interval, $[\underline{t}_Z, \bar{t})$, $\bar{t} > \underline{t}_Z$, then $\beta_x(t) = 0$ during the interval implies that $\dot{\lambda}_x(t) = \rho\lambda_x(t) > 0$ and $\lambda_x(t)$ would grow over $\underline{n}_x$, inducing investments in the fossil energy sector which are prevented by the carbon ceiling.

Thus, rather counterintuitively, this is by lowering the carbon price below $\pi_x(t)/\zeta r_x$, that the regulator can prevent investments in fossil energy generation. It is a consequence of the fact that the regulator cannot incentivise the capital accumulation in the renewable energy sector through its carbon pricing policy during the ceiling phase. Once the carbon ceiling has been chosen, the renewable energy sector implements an investment policy completely independent from the carbon pricing policy of the regulator. Of course by strengthening the ceiling constraint, that is by reducing the net zero target, the energy price trajectory would be shifted upward, inducing an acceleration of the investment pace of the renewable energy sector and thus of the speed of the energy transition.

The message is completely different from the textbook ceiling model. In this model, carbon pricing is constantly needed during the ceiling regime to prevent the extraction rate to grow over the net zero target. Here the only policy constraint for the regulator is to keep the value of investments in the fossil energy sector below the sunk cost level, $\underline{n}_x$, which can be achieved in a variety of ways.

Have a look to a parallel thought experiment where the regulator would set the carbon price at the level $\pi_x(t)/\zeta r_x$ over a non degenerate time interval, $[\bar{t}, \bar{t}_Z)$, that is just before the end of the ceiling phase. Then if $\bar{n}_x > \lambda_x(\bar{t}) > 0$, $\lambda_x(t)$ should increase exponentially after $\bar{t}$, which is incompatible with $\lambda_x(\bar{t}_Z) = 0$. Thus either the regulator maintains $\lambda_x(t)$ above 0 during the ceiling phase, making it falling down to 0 only at the end of the phase, either it drives down $\lambda_x(t)$ to 0 strictly before $\bar{t}_Z$, and next applies the pricing policy $\lambda_Z(t) = \pi_x(t)/\zeta r_x$ in order to keep $\lambda_x(t)$ equal to 0 until the end of the ceiling regime.

Suppose that it decides to keep $\lambda_x(t) > 0$ before $\bar{t}_Z$. We face different possibilities. We have already excluded the possibility that $\beta_x(t)$ be nil over a first time interval, $[\underline{t}_Z, t_\beta)$ and next > 0 after t_β. The reverse possibility can be also excluded. Assume that there exists a critical time, t_β, $\underline{t}_Z < t_\beta < \bar{t}_Z$, such that $\beta_x(t) > 0$ if $t < t_\beta$ and $\beta_x(t) = 0$, if $t \geq t_\beta$. Because $\dot{\lambda}_x(t) = \rho\lambda_x(t) > 0$, when $t \geq t_\beta$, $\lambda_x(t)$ could not fall down to 0 at time $\bar{t}_Z$, yielding a contradiction.

It could also be possible that there exists at least one time interval, $[\underline{t}_\beta, \bar{t}_\beta]$, $\underline{t}_Z < \underline{t}_\beta < \bar{t}_\beta < \bar{t}_Z$, during which $\beta_x(t) = 0$, and $\beta_x(t) > 0$ outside the time interval. We now show how to exclude this possibility. Integrating (3.12) over $[t, \bar{t}_Z)$ using $\lambda_x(\bar{t}_Z) = 0$ yields:

$$\lambda_x(t) = \int_{\underline{t}}^{\bar{t}_Z} \beta_x(\tau) e^{-\rho(\tau - \underline{t}_Z)} d\tau$$

Integrating by parts and using $\beta_x(\bar{t}_Z) = 0$, we obtain:

$$\begin{aligned}
\lambda_x(t) &= -\beta_x(\tau) \frac{e^{\rho(\tau - t)}}{\rho} \Bigg|_t^{\bar{t}_Z} + \frac{1}{\rho} \int_t^{\bar{t}_Z} \dot{\beta}_x(\tau) e^{-\rho(\tau - t)} dt \\
&= \frac{1}{\rho} \left[\beta_x(t) + \int_t^{\bar{t}_Z} \dot{\beta}_x(\tau) e^{-\rho(\tau - t)} dt \right]
\end{aligned}$$

Thus using $\dot{\lambda}_x(t) = \rho\lambda_x(t) - \beta_x(t)$:

$$\dot{\lambda}_x(t) e^{-\rho t} = \int_t^{\bar{t}_Z} \dot{\beta}_x(\tau) e^{-\rho\tau} d\tau$$

We know already that $\dot{\lambda}_x(\underline{t}_Z) < 0$ and $\dot{\lambda}_x(\bar{t}_Z) = 0$. Assume that there exists a time interval $[t_0, t_1] \subset [\underline{t}_Z, \bar{t}_Z)$, such that $\dot{\lambda}_x(t) > 0$, $t \in [t_0, t_1]$. The function $\lambda_x(t)$ being time differentiable, $\dot{\lambda}_x(t_0) = \dot{\lambda}_x(t_1) = 0$, so that:

$$\begin{aligned}
&\dot{\lambda}_x(t_1) e^{-\rho t_1} = 0 = \int_{t_1}^{\bar{t}_Z} \dot{\beta}_x(\tau) e^{-\rho\tau} d\tau \\
&\dot{\lambda}_x(t_0) e^{-\rho t_0} = 0 = \int_{t_0}^{\bar{t}_Z} \dot{\beta}_x(\tau) e^{-\rho\tau} d\tau = \\
&\int_{t_0}^{t_1} \dot{\beta}_x(\tau) e^{-\rho\tau} d\tau + \int_{t_1}^{\bar{t}_Z} \dot{\beta}_x(\tau) e^{-\rho\tau} d\tau = \int_{t_0}^{t_1} \dot{\beta}_x(\tau) e^{-\rho\tau} d\tau
\end{aligned}$$

But for $t_0 < t < t_1$:

$$\dot{\lambda}_x(t)e^{-\rho t} = \int_t^{\bar{t}_Z} \dot{\beta}_x(\tau)e^{-\rho\tau}d\tau = \int_t^{t_1} \dot{\beta}_x(\tau)e^{-\rho\tau}d\tau > 0$$

implies that:

$$\int_{t_0}^{t} \dot{\beta}_x(\tau)e^{-\rho\tau}d\tau < 0$$

should hold true for any $t < t_1$ which yields a contradiction. Hence $\dot{\lambda}_x(t)$ cannot change sign during the ceiling phase, implying that $\dot{\lambda}_x(t) < 0$.

Since $\beta_x(t) = 0$ during the time interval $[\underline{t}_\beta, \bar{t}_\beta]$ implies that $\dot{\lambda}_x(t) = \rho\lambda_x(t) > 0$ inside the interval, we get a contradiction. Thus $\beta_x(t) > 0$ during the time interval $[\underline{t}_Z, \bar{t}_Z)$, provided that $\lambda_x(\underline{t}_Z) = \underline{n}_x > 0$ and thus $\beta_x(\underline{t}_Z) > 0$.

The same argument as presented before then shows that $\dot{\beta}_x(t) < 0$. We conclude that both $\lambda_x(t)$ and $\beta_x(t)$ are time decreasing over the ceiling phase. Assume now that the regulator decides to drive down $\lambda_x(t)$ to 0 during a first time interval, $[\underline{t}_Z, t_\lambda)$ and next keep $\lambda_x(t)$ equal to 0 by setting $\lambda_Z(t) = \pi_x(t)/\zeta r_x$ during the second time interval, $[t_\lambda, \bar{t}_Z)$. The same reasoning as developed before for the time interval, $[\underline{t}_Z, \bar{t}_Z)$, still applies to the time interval, $[\underline{t}_Z, t_\lambda)$ (observe that this reasoning does not make use of the condition $\lambda_Z(\bar{t}_Z) = 0$, which defines in the price space the end of the ceiling regime). Thus we can conclude that during the time interval, $[\underline{t}_Z, t_\lambda)$, the regulator can implement any carbon pricing policy able to induce a simultaneous decrease of $\lambda_x(t)$ and $\beta_x(t)$ down to 0 at time t_λ.

Hence two main carbon pricing policies during the ceiling phase emerge : either maintain permanently $\zeta r_x \lambda_Z < \pi_x(t)$ to keep $\beta_x(t) > 0$ and at the same time larger than $\rho\lambda_x(t)$ to induce a parallel decrease of $\beta_x(t)$ and $\lambda_x(t)$ down to 0 at time $\bar{t}_Z$; either set a carbon price lower than $\pi_x(t)/\zeta r_x$ able to drive down both $\beta_x(t)$ and $\lambda_x(t)$ to 0 strictly before the end of the ceiling phase and next apply the carbon pricing policy $\zeta r_x \lambda_Z(t) = \pi_x(t)$ until $\bar{t}_Z$. In this last case $\lambda_Z(t)$ should decrease during this second time interval.

Whatever the policy chosen, because both $\pi_x(t)$ and $\beta_x(t)$ decrease during the ceiling phase, the dynamics of $\lambda_Z(t)$ is ambiguous, implying that the regulator can implement its policy with a non time monotonous carbon pricing strategy.

The following Proposition 2 sums up the main properties of the optimal paths during the ceiling phase.

Proposition P. 2. *During the ceiling phase, $[\underline{t}_Z, \bar{t}_Z)$:*

1. *The production capacity of the fossil energy sector stays constant at the level $\bar{K}_x$.*
2. *If the fossil energy sector accumulates capacities until $\underline{t}_Z$, the opportunity cost of the capacity constraint in this sector, $\eta_x(t)$, decreases progressively down to m_x. If the industry stops accumulating capacities strictly before $\underline{t}_Z$, $\eta_x(t) = m_x$, throughout the phase.*
3. *If the fossil energy sector accumulates capacities until $\underline{t}_Z$, the shadow value of fossil energy conversion equipments, $\lambda_x(t)$, falls from the level $\underline{n}_x$ down to 0. If the industry stops accumulating capacities strictly before $\underline{t}_Z$, $\lambda_x(t) = 0$ throughout the phase.*

4. *The production capacity of the renewable energy sector expands up to* $\bar{K}_y$.

5. *The opportunity cost of the capacity constraint of the renewable energy sector,* $\eta_y(t)$*, falls.*

6. *The shadow value of renewable energy conversion equipments,* $\lambda_y(t)$*, decreases.*

7. *The energy price falls and the energy supply increases, thanks to the expansion of the renewable energy sector. Renewable energy takes an increasing share in the fuel mix.*

8. *If the fossil energy sector accumulates capacities until* $\underline{t}_Z$*, the time evolution of the carbon price is ambiguous, the only constraint for the regulator being to maintain the value of an investment in fossil energy generation,* $\lambda_x(t)$*, below the lower bound over the investment cost,* $\underline{n}_x$*, in order to prevent any capacity accumulation above the net zero constraint. If the fossil energy sector has begun to scrap capacities before* $\underline{t}_Z$*, there exists a unique optimal carbon pricing policy levying the operational margin before tax of the fossil fuels industry. The carbon price constantly decreases down to* 0 *during the ceiling phase.*

6 The pre-ceiling phase

The study of the ceiling phase has shown that for the climate problem to be meaningful, the available renewable energy conversion capacity at the beginning of the phase, $K_y^Z = K_y(\underline{t}_Z)$, must be lower than $\bar{K}_y$, the capacity level allowing for an escape from the climate constraint. We assume it is the case for an interesting study. It has also shown that the value of investing in renewable energy constantly declines during the ceiling regime while staying constantly above $\underline{n}_y$, the investment sunk cost. Hence the value of investing in renewable energy is larger than $\underline{n}_y$ at the beginning of the phase, $\lambda_y(\underline{t}_Z) > \underline{n}_y$. The renewable energy conversion industry should thus also build up capacities during the pre-ceiling phase.

The behaviour of the fossil energy sector is potentially more complex. It can ramp up investments in new capacities until the beginning of the ceiling regime. It can also do the contrary and constantly scrap capacities down to $\bar{K}_x$, the capacity compatible with the net zero target. Last it can combine positive investments episodes with scraping episodes.

We proceed through successive steps. First we show some preliminary results. Next we deduce from these preliminaries the possible investment scenarios in the fossil energy conversion sector and in the renewable energy conversion sector.

Result 1: The renewable energy sector should not scrap capacities before the ceiling phase.

Assume to the contrary the existence of some non degenerate time interval, $[t_0, t_1)$, during which the renewable energy sector scraps capacities. Thus $\lambda_y(t) = \beta_y(t) = 0$ during this phase. Because $\lambda_y(\underline{t}_Z) > \underline{n}_y > 0$, $\lambda_y(t)$ should increase after t_1 and become larger than $\underline{n}_y$, $\lambda_y(t)$ before $\underline{t}_Z$ being a time differentiable and hence time continuous function.

Next it cannot be optimal that the fossil energy sector scraps its capacities at the same time the renewable energy sector scraps its own capacities. This would require that along some non degenerate sub-interval $[\underline{t}, \bar{t}) \subseteq [t_0, t_1)$ both $\beta_x(t) = 0$ and $\beta_y(t) = 0$, $t \in [\underline{t}, \bar{t})$. Thus inside the time interval:

$$\beta_x(t) = p(t) - c_x - m_x - \zeta r_x \lambda_Z(t) = 0$$
$$\beta_y(t) = p(t) - a_y - m_y - b_y'(K_y(t)) = 0$$

implying that:

$$c_x + m_x + \zeta r_x \lambda_Z(t) = a_y + m_y + b_y'(K_y(t)) \quad t \in [\underline{t}, \bar{t})$$

Because the l.h.s. of this equation increases over time since $\lambda_Z(t)$ increases before $\underline{t}_Z$ and the r.h.s. decreases because $K_y(t)$ decreases, the renewable energy sector scraping its capacities and $b_y'' > 0$, the equation can be satisfied only at one instant yielding a contradiction.

Hence during the time interval $[t_0, t_1)$, the fossil energy sector must increase its capacities and by continuity also over some time interval $[t_1, t_1 + \Delta)$, $\Delta > 0$, after t_1. The trajectory of $\lambda_y(t)$ having to be time continuous, there should exist some non degenerate time interval, $[t_1, t_1 + \Delta)$, with $\Delta > 0$, such that $0 \leq \lambda_y(t) < \underline{n}_y$ throughout the time interval. Since $\lambda_y(t) < \underline{n}_y$, the renewable energy sector still does not invest and keeps its capacities at some constant level, say $\hat{K}_y$. Thus $\beta_y(t)$ during the time interval, $[t_1, t_1 + \Delta)$, is given by:

$$\beta_y(t) = p(K_x(t) + \hat{K}_y) - a_y - m_y - b_y'(\hat{K}_y)$$

Because $K_x(t)$ increases, the fossil energy sector investing in new capacities while the renewable energy conversion capacity stays constant, the energy price should decrease during the time interval, $[t_1, t_1 + \Delta)$, so that $\beta_y(t)$ should decrease. $\beta_y(t)$ being a time continuous function, and because $\beta_y(t_1) = 0$, this implies that $\beta_y(t) < 0$ over the time interval, $[t_1, t_1 + \Delta)$. This cannot be optimal, the operational margin being negative, the renewable energy sector could not cover the maintenance cost of its equipment and should scrap them accordingly, a contradiction since it should keep them at a constant level.

Hence if the renewable energy conversion industry scraps its capacities during some time episode there cannot exist an optimal investment plan after this episode able to satisfy the requirement that the renewable energy sector actively invests, although at a declining rate, just before the ceiling phase. We conclude that the renewable energy sector should not scrap capacities before $\underline{t}_Z$.

Result 2: If both sectors should invest just before the ceiling phase, they should both invest throughout the pre-ceiling phase.

Assume that both sectors invest over some non degenerate time interval, $[\underline{t}_Z - \Delta, \underline{t}_Z)$, $\Delta > 0$. We have shown previously that $\lambda_y(\underline{t}_Z) > \underline{n}_y$ and $\dot{\lambda}_y(\underline{t}_Z) < 0$. The fossil energy sector being actively investing just before $\underline{t}_Z$, we have also shown that $\lambda_x(\underline{t}_Z) = \underline{n}_x$ and that $\dot{\lambda}_x(\underline{t}_Z)$ must be ≤ 0 to prevent investments in this sector after $\underline{t}_Z$. During the time interval, $[\underline{t}_Z - \Delta, \underline{t}_Z)$, $\beta_x(t)$ and $\beta_y(t)$ are given by:

$$\beta_x(t) = p(K_x(t) + K_y(t)) - c_x - m_x - \zeta r_x \lambda_Z(t)$$
$$\beta_y(t) = p(K_x(t) + K_y(t)) - a_y - m_y - b'_y(K_y(t))$$

Because $K_x(t)$ and $K_y(t)$ both increase during the interval, the energy price, $p(t)$, must decrease so that $\beta_x(t)$ and $\beta_y(t)$ both decrease, $\lambda_Z(t)$ and $b'_y(K_y(t))$ being time increasing. We can rely on the same argument as presented before to assess that if there exists a time $\bar{t} \in [\underline{t}_Z - \Delta, \underline{t}_Z)$, at which $\lambda_x(t)$ would become larger than $\beta_x(t)/\rho$, $\dot{\lambda}_x(t)$ would turn > 0 and stay strictly positive after $\bar{t}$, which is impossible since $\dot{\lambda}_x(\underline{t}_Z) \leq 0$. The same applies also to $\lambda_y(t)$.

We thus conclude that both $\lambda_x(t)$ and $\lambda_y(t)$ decrease during the time interval, $[\underline{t}_Z - \Delta, \underline{t}_Z)$. Hence $\lambda_x(\underline{t}_Z - \Delta) > \underline{n}_x$ and $\lambda_y(\underline{t}_Z - \Delta) > \underline{n}_y$. If both sectors should invest at $\underline{t}_Z^-$, this must also be the case at $\underline{t}_Z - \Delta$, whatever large Δ. Therefore, simultaneous investment in both sectors just before the ceiling phase implies that the optimal investment scenario during the pre-ceiling phase is made of a unique phase of positive investments in both sectors. This scenario can only be valid if $K_x^0 < \bar{K}_x$ and $K_y^0 < \bar{K}_y$.

Result 3: If the fossil energy sector invests until the ceiling phase, the renewable energy sector must also invest until $\underline{t}_Z$.

This is a corollary of the preceding point. Assume that the fossil energy sector invests throughout the ceiling phase. Then $\beta_y(t)$ should decrease over time even if the renewable energy sector does not invest. In addition we have shown that $\beta_y(\underline{t}_Z)/\rho > \lambda_y(\underline{t}_Z) > \underline{n}_y$. If the renewable energy sector does not invest, $\lambda_y(t) \leq \underline{n}_y$. Thus for this sector to be an active investor at the beginning of the ceiling phase, $\lambda_y(t)$ should increase above $\underline{n}_y$. This is impossible since $\beta_y(t) > \beta_y(\underline{t}_Z)$, $\beta_y(t)$ being time decreasing, implies that $\beta_y(t)/\rho > \beta_y(\underline{t}_Z)/\rho > \underline{n}_y \geq \lambda_y(t)$, so that $\dot{\lambda}_y(t) \leq 0$.

Result 4: If the fossil energy sector scraps capacities just before $\underline{t}_Z$, it either constantly scraps capacities or after a stabilisation phase.

Assume that the fossil energy sector should scrap capacities just before the beginning of the ceiling phase. Over a non degenerate time interval, $[\underline{t}_Z - \Delta, \underline{t}_Z)$, $\Delta > 0$, then $\lambda_x(t) = \beta_x(\underline{t}_Z) = 0$. Thus:

$$\beta_x(t) = 0 \Longrightarrow p(t) = c_x + m_x + \zeta r_x \lambda_Z(t)$$

The energy price must increase over time during the time interval, $[\underline{t}_Z - \Delta, \underline{t}_Z)$, $\lambda_Z(t)$ being time increasing.

Before $\underline{t}_Z - \Delta$, the fossil energy sector could have scraped capacities since $t = 0$. Or after some preliminary investment episode ending at some time t_0, it has stabilised the fossil energy conversion capacity to some level $\hat{K}_x$ during the time interval, $[t_0, \underline{t}_Z - \Delta)$, before entering the scraping phase over $[\underline{t}_Z - \Delta, \underline{t}_Z)$.

That the fossil energy sector should not start to scrap capacities immediately after accumulating them comes from the sunk cost in investment effort. The value of the last piece of accumulated equipment at t_0 must just cover this sunk cost, thus $\lambda_x(t_0) = \underline{n}_x$, while we know that $\lambda_x(\underline{t}_Z - \Delta) = 0$. The trajectory of $\lambda_x(t)$ having to be time continuous, a minimal delay is needed to drive down $\lambda_x(t)$ from the level $\underline{n}_x$ to 0. This minimal delay corresponds to the exploitation time that is needed to recover the cost on the last purchased piece of equipment at time t_0.

Formally, integrating the motion of $\lambda_x(t)$ over the time interval, $[t_0, \underline{t}_Z - \Delta)$, taking account of $\lambda_x(t_0) = \underline{n}_x$ and $\lambda_x(\underline{t}_Z - \Delta) = 0$, we obtain:

$$\underline{n}_x = \int_{t_0}^{\underline{t}_Z - \Delta} \beta_x(t) e^{-\rho(t-t_0)} dt$$

Remembering that $\beta_x(t)$ is the net operational margin on one unit of equipment, the r.h.s. of this equation is the present value of the stream of benefits coming from the exploitation of this unit of equipment between the times t_0 and $\underline{t}_Z - \Delta$. The time length $\underline{t}_Z - \Delta - t_0$ is thus the minimal delay for this present value of benefits to cover the sunk cost, $\underline{n}_x$.

Result 5: A scraping phase cannot precede a stabilisation phase.

This is trivial. If the industry stops scraping its capacities at a level $\hat{K}_x$ at some time t_0, then $\beta_x(t_0) = 0$ and for $t \in [t_0, t_0 + \Delta)$, $\Delta > 0$, $\beta_x(t)$ should be given by:

$$\beta_x(t) = p(t) - c_x - m_x - \zeta r_x \lambda_Z(t)$$

a strictly decreasing time function, since $\lambda_Z(t)$ increases and $p(t)$ either decreases (if the renewable energy sector invests), or stays constant (if the renewable energy sector does not invest, remembering that we have shown already that the renewable energy sector never scraps capacities). Hence $\beta_x(t) < 0$ inside the open time interval, $(t_0, t_0 + \Delta)$, implying that the fossil energy sector cannot cover the O&M marginal cost and thus should scrap capacities rather than maintaining them.

Result 6: The fossil energy industry never resumes investments after a stabilisation phase.

This is also trivial. We have shown that $\beta_x(t) < \underline{n}_x$ throughout the stabilisation phase. Thus by continuity, $\beta_x(t)$ cannot be strictly larger then $\underline{n}_x$ after the end of the stabilisation phase, and thus fresh investments are not justified.

Result 7: The investment, stabilisation and scraping phases of the fossil energy capital stocks are unique if they happen along an optimal path.

This is an immediate consequence of the results 4, 5, 6. For the same reasons as for Result 6, an active investment phase cannot follow a scraping phase of fossil energy conversion capacities. Thus active investment phases and scraping phases must be separated by stabilisation phases. But any stabilisation phase can be followed only by a scraping phase and only preceded by an active investment phase. Hence the stabilisation phase must be unique, implying in turn that the active investment and scraping phases are also unique.

Result 8: During a scraping phase of fossil energy conversion capital, the operational margin on renewable equipments can either decrease or increase and last decrease.

Since during the scraping phase, $\dot{\beta}_y(t) = \zeta r_x \dot{\lambda}_Z(t) - b''_y(K_y(t))k_y(t)$, and $\dot{\lambda}_Z(t) > 0$, $\dot{\beta}_y(t)$ can be either > 0 or < 0. However we have shown that the scraping phase can occur only during a time interval ending at $\underline{t}_Z$, and that $\dot{\lambda}_y(\underline{t}_Z) < 0$. Thus $\beta_y(t)$ eventually decreases at the end of the pre-ceiling phase.

6.1 Possible pre-ceiling investment scenarios in the fossil energy industry

From the previous results we can conclude that there should exist three mains categories of investment scenarios in the fossil energy industry before the beginning of the ceiling phase. To these main families are associated different possible investment scenarios in the renewable energy industry. We first present the scenarios and next establish their properties.

1. **Scenario 1 :** The fossil energy sector scraps its initial capacity down to $\bar{K}_x$. To be possibly optimal, this scenario requires that $K^0_x \geq \bar{K}_x$.
2. **Scenario 2 :** The fossil energy sector makes increase its production capacity up to some $\hat{K}_x > \bar{K}_x$ during a first time interval, $[0, t^m)$. K^0_x can be larger or smaller than $\bar{K}_x$ in the scenario 2. Next the capacity $\hat{K}_x$ is maintained during a second time interval, $[t^m, t^s)$. The fossil energy industry last scraps the capacity $\hat{K}_x - \bar{K}_x$ during the time interval, $[t^s, \underline{t}_Z)$.
3. **Scenario 3 :** The fossil energy sector constantly invests in capacity building before the ceiling phase, $k_x(t) > 0$, $t \leq \underline{t}_Z$. This scenario is only possible if $K^0_x < \bar{K}_x$.

6.1.1 The Scenario 1: Permanent capacity scraping in the fossil fuels energy sector

This is a one phase scenario, $[0, \underline{t}_Z)$, during which the fossil energy sector constantly scraps its capacities, driving the fossil energy conversion capital from its initial level K^0_x down to $\bar{K}_x$ at time $\underline{t}_Z$.

Throughout the phase, $\lambda_x(t) = \beta_x(t) = 0$, thus the energy price is given through (3.6) by $p(t) = c_x + m_x + \zeta r_x \lambda_Z(t)$. The energy price constantly increases in the Scenario 1 in view of (3.14). To the increase of the energy price corresponds a contraction of the energy consumption rate $q(t) = K_x(t) + K_y(t)$, which implies that $K_y(t)$ increases at a lower rate than $K_x(t)$ decreases when the renewable energy sector invests. Whatever the investment policy of the renewable energy sector, the scraping of fossil energy conversion capacities overcompensates the increase in renewable energy conversion capacities. In this Scenario the energy transition from fossil energy to renewable energy occurs in the context of an increasing scarcity of the energy supply.

6.1.2 The Scenario 2: Building and next scraping of capacities in the fossil fuels energy sector

This is a three phases scenarios. During a first phase, $[0, t^m)$, (where 'm' stands for 'maintenance'), the fossil energy sector accumulates capacities above K_x^0 up to a level $\hat{K}_x$. Next it maintains constant this capacity during a second time interval, $[t^m, t^s)$ (where 's' stands for 'scraping'). Last the fossil energy sector scraps its capacities during a third phase, $[t^s, \underline{t}_Z)$, from the level $\hat{K}_x$ down to the level, $\bar{K}_x$.

The initial investment phase.

Whatever the investment policy of the renewable energy sector, $\beta_x(t) = p(t) - c_x - m_x - \zeta r_x \lambda_Z$ decreases during the first phase, $[0, t^m)$, because $p(t)$ decreases, the renewable energy sector never scraping its capacities (c.f. Result 1) and $\lambda_Z(t)$ being time increasing in view of (3.14). Furthermore $\lambda_x(t^m) = \underline{n}_x$ and because the investment rate falls to 0 when $t \to t^m$, $\dot{\lambda}_x(t^m) \leq 0$. The same argument as used already then shows that this implies that $\dot{\lambda}_x(t) < 0$, $t \in [0, t^m)$. Thus the investment rate in fossil energy conversion equipments constantly declines during the first accumulation phase, $[0, t^m)$. The energy price decreasing over time, the overall energy production capacity expands. Hence the energy transition (if any) occurs in the context of an increasing abundance of energy.

The stabilisation phase.

During the second phase, $\lambda_x(t)$ falls from the level $\underline{n}_x$ at time t^m, down to 0 at time t^s, while $\beta_x(t)$ also decreases down to 0 at time t^s. This can be checked through the following argument. Because $\dot{\lambda}_x(t^m) = \rho\lambda_x(t^m) - \beta_x(t^m) \leq 0$ and $\lambda_x(t^m) = \underline{n}_x$, $\beta_x(t^m) \geq \rho\underline{n}_x > 0$.

If the renewable energy sector invests during a time interval $\mathcal{T} \subset [t^m, t^s)$, the energy price decreases for $t \in \mathcal{T}$, while it stays constant if the renewable energy sector does not invest. Thus whatever the investment policy of the renewable energy sector, $\dot{p}(t) \leq 0$. Hence:

$$\beta_x(t) = p(t) - c_x - m_x - \zeta r_x \lambda_Z(t) \implies \dot{\beta}_x(t) = \dot{p}(t) - \zeta r_x \dot{\lambda}_Z(t) < 0$$

Thus $\beta_x(t)$ constantly declines down to 0 at time t^s.

Next integrating over $[t, t^s)$ the motion of $\lambda_x(t)$, taking into account that $\lambda_x(t^s) = 0$ yields;

$$\lambda_x(t)e^{-\rho t} = \int_t^{t^s} \beta_x(\tau)e^{-\rho\tau} d\tau$$

Integrating by parts setting $U = \beta_x$, $dU = \dot{\beta}_x d\tau$, $dV = e^{-\rho\tau}d\tau$, $V = -e^{-\rho\tau}/\rho$, we obtain:

$$\begin{aligned}\lambda_x(t)e^{-\rho t} =& -\frac{1}{\rho}\beta_x(\tau)e^{-\rho\tau}\Big|_t^{t^s} + \frac{1}{\rho}\int_t^{t^s}\dot{\beta}_x(\tau)e^{-\rho\tau}d\tau \\ \Longrightarrow \rho\lambda_x(t)e^{-\rho t} =& -\beta_x(t^s)e^{-\rho t^s} + \beta_x(t)e^{-\rho t} + \int_t^{t^s}\dot{\beta}_x(\tau)e^{-\rho\tau}d\tau\end{aligned}$$

Since $\beta_x(t^s) = 0$ and $\dot{\lambda}_x(t) = \rho\lambda_x(t) - \beta_x(t)$, this is equivalent to:

$$\dot{\lambda}_x(t)e^{-\rho t} = \int_t^{t^s}\dot{\beta}_x(\tau)e^{-\rho\tau}d\tau$$

Hence $\dot{\beta}_x(\tau) < 0$, $\tau \in [t, t^s)$ implies that $\dot{\lambda}_x(t) < 0$, $t \in [t^m, t^s)$. The value of the fossil energy capacities, $\lambda_x(t)$, constantly declines during the stabilisation (or 'maintenance') phase, falling from $\underline{n}_x$ down to 0 over the phase. Since $\lambda_x(t)$ also decreases during the first accumulation phase, $[0, t^m)$, we conclude that in the Scenario 2, the value of an investment in fossil energy conversion capacities, $\lambda_x(t)$, constantly falls from a level higher than $\underline{n}_x$ at time $t = 0$ down to 0 at time t^s.

If the renewable energy sector expands its capacities, the energy price falls and the energy transition occurs in the context of an increased abundance of energy, since the fossil energy sector keeps at a constant level its own capacities. If the renewable energy sector does not invest during the stabilisation phase, the energy system is in a *statu quo* mode with a constant energy price and no energy transition from fossils to renewables.

The last scraping phase

During the third phase, $[t^s, \underline{t}_Z)$, $\lambda_x(t) = \beta_x(t) = 0$. The exact details of the scraping policy of the fossil energy sector depends on the investment policy of the renewable energy sector. Whatever this policy the energy price increases, meaning that the energy supply declines, the move from fossil fuel to renewable energy being a less than one to one substitution process. The energy transition occurs in the context of an increasing energy scarcity.

6.1.3 The Scenario 3: Constant building of capacities in the fossil fuels energy sector

It is a one phase scenario over $[0, \underline{t}_Z)$. The fossil energy sector constantly accumulates capacities from the level K_x^0 up to $\bar{K}_x$. Of course this scenario is only valid when $K_x^0 < \bar{K}_x$.

Whatever the investment policy of the renewable energy sector, $\beta_x(t)$ constantly decreases in this scenario and the already used argument shows that this implies that $\lambda_x(t)$ should decrease down to $\underline{n}_x$ at time $\underline{t}_Z$. Thus the investment rate in fossil energy conversion capacities , after an initial impulse up at time $t = 0$, constantly declines down to 0 in this Scenario. The renewable energy sector never scraping capacities, the energy supply expands, thus the energy price falls, and the aggregate energy production rate increases, the renewable energy sector never scraping capacities under A.1. The energy transition takes place inside an increasing energy abundance context.

6.2 Investment scenarios in the renewable energy conversion sector

The investment scenarios in the renewable energy sector depend strongly of the investment policy adopted by the fossil energy sector. We thus describe these scenarios for each of the investment scenarios of the fossil energy industry.

6.2.1 Renewables expansion in the investment scenario 1 of the fossil energy sector.

In the constant scraping scenario 1, since $p(t)$ increases on the one hand and $b'_y(K_y(t))$ either stays constant or increase, it is possible that $\beta_y(t) = p(t) - a_y - m_y - b'_y(K_y(t))$ be time increasing. Thus it becomes also possible that $\dot{\lambda}_y(t)$ be > 0 even if we know that $\dot{\lambda}_y(t) < 0$ when $t \to \underline{t}_Z$.

It is also possible that the investment in renewable energy capacities be delayed after $t = 0$. Let $t_{ky} \geq 0$, be the time at which the investment process into renewable energy begins. That t_{ky} could be > 0 results from the building costs and the energy price dynamics. On the other hand, if $K_y^0 = 0$, a consequence of cost convexity is that the renewable energy sector could start building capacities without producing renewable energy. Let t_y be the time at which the renewable energy sector begins to produce renewable energy, then it could be the case that $0 \leq t_{ky} < t_y$. Note that in this case, $q_y(t_y) = K_y(t_y) > 0$. For this scenario to be optimal, the fossil energy production capacity must fall from the level, $K_x(t_y^-) = q(t_y)$, down to the level, $K_x(t_y^+) = q(t_y) - K_y(t_y)$, to preserve the time continuity of $q(t)$ and thus of $p(t)$. To sum up, the possibly optimal investment scenarios in the renewable energy sector when the fossil energy industry scraps constantly its capacities before t_Z are:

(i) : The scenario (a): $0 < t_{ky} < t_y$. This scenario can be valid only if $K_y^0 = 0$. The renewable energy industry starts building capacities with a delay ($t_{ky} > 0$) and first does not produce renewable energy until a time $t_y > t_{ky}$. Because of the strict convexity of the investment cost function, the investment process should start smoothly so that $\lambda_y(t_{ky}) = \underline{n}_y$ and $k_y(t_{ky}) = 0$. After t_y, the industry builds up capacities and produces simultaneously renewable energy, $q_y(t) = K_y(t)$, until $\underline{t}_Z$.

(ii) : The scenario (b): $0 = t_{ky} < t_y$. It is also possibly valid only if $K_y^0 = 0$. The only difference with the previous scenario (a) is that the capacities building process starts right from $t = 0$ and $k_y(0^+) > 0$, while $\lambda_y(0^+) > \underline{n}_y$.

(iii) : The scenario (c) : $0 < t_{ky} = t_y$. This scenario can be valid whatever $K_y^0 \geq 0$. The renewable energy industry begins to build up capacities with a delay and begins to produce renewable energy at the same time. $k_y(t_{ky}) = 0$ and $\lambda_y(t_{ky}) = \underline{n}_y$ in this scenario, the investment process starts smoothly.

(iv) : The scenario (d) : $0 = t_{ky} = t_y$. The renewable energy industry starts building capacities and producing energy right from $t = 0$. $k_y(0^+) > 0$, $q_y(0^+) = K_y^0 \geq 0$, $\lambda_y(0^+) > \underline{n}_y$ in this scenario.

Result 9: Only the scenario (d) of immediate start by positive impulse of the capacity building process in the renewable energy sector is compatible with our assumption A.1 about the relative competitiveness of the two energy sources.

Proof

In the scenarios (a), $p(0) < a_y + b'_y(0)$ so that $\eta_y = 0$. The energy price is initially too low to justify producing renewable energy. At time t_{ky}, the renewable energy industry begins to build capacities. The capital price is $\lambda_y(t_{ky}) = \underline{n}_y$ and $k_y(t_{ky}) = 0$, the investment process starts smoothly. Over the time interval, $[t_{ky}, t_y)$, $\eta_y(t) = 0$, so that $\dot{\lambda}_y(t) = \rho\lambda_y(t) + m_y$. Hence $\lambda_y(t)$ constantly increases during the time interval. On the other hand, $p(t) < a_y + b'_y(0)$ so that $q_y(t) = \eta_y(t) = 0$. At time t_y, the energy price reaches the breakeven price of renewables, $p(t_y) = a_y + b'_y(0)$. In the scenario (b), $p(t) < a_y + b'_y(0)$ and $\eta_y(t) = 0$ if $t < t_y$. The only difference with the scenario (a) is thus that $\lambda_y(0^+) > \underline{n}_y$ and $k_y(0^+) > 0$. These two scenarios can be eliminated because to get a 100% renewable energy regime, we have assumed that:

$$\begin{aligned} & a_y + b'_y(0) < a_y + m_y + b'_y(0) < a_y + m_y + b'_y(\tilde{K}_y) + \rho\underline{n}_y \\ & < c_x + m_x = \bar{p}_x < c_x + m_x + \zeta r_x \lambda_Z = p \end{aligned}$$

Hence production of renewable energy at full capacity is always immediately competitive when capacities are built, so that $q_y(t) = K_y(t)$ and $\eta_y(t) > 0$ by complementary slackness. On the other hand, $K_y^0 < \tilde{K}_y$ implies that:

$$a_y + m_y + b'_y(K_y^0) + \rho\underline{n}_y < a_y + m_y + b'_y(\tilde{K}_y) + \rho\underline{n}_y < p(0)$$

The initial energy price is able to cover the marginal cost of renewable energy production and the minimum rental cost of investment, $\rho\underline{n}_y$. Thus the renewable energy industry should begin to accumulate capacities right from $t = 0$ and the scenario (c) can be eliminated. Only the scenario (d) is compatible with the assumption A.1 on the relative costs of fossil and renewable energy.

The investment policies in the Scenario 1 being a particular case of the investment policies in the Scenario 2, we discuss the investment dynamics in renewable energy production in the next subsection devoted to the Scenario 2.

6.2.2 Renewables expansion in the investment scenario 2 of the fossil energy sector.

In the 3 phases Scenario 2, the energy price, $p(t)$, decreases during the first phase, $[0, t^m)$, whatever the investment policy of the renewable energy sector because $K_x(t)$ increases. Thus $\beta_y(t)$ decreases during the phase. During the stabilisation phase, $[t^m, t^s)$, the energy price either stays constant or decreases, thus $\beta_y(t)$ is also a non increasing time function over the time interval. Last, during the scraping phase, $[t^s, \underline{t}_Z)$, $\beta_y(t)$ can be either increasing or decreasing, as was the case in the Scenario 1.

If $\dot{\beta}_y(t) < 0$ during the scraping phase, then $\beta_y(t)$ constantly decreases before the ceiling phase and $\beta_y(\underline{t}_Z) > \rho\underline{n}_y$ implies that $\beta_y(t)/\rho$ stays above $\underline{n}_y$. The argument already used then shows that because $\dot{\beta}_y(\underline{t}_Z) < 0$, $\lambda_y(t)$ must constantly decrease during the pre-ceiling phase. Thus the investment policy of the renewable energy sector is made of an initial push at a positive level, $k_y(0^+) > 0$, followed by a steady decline of the investment rate down to $k_y^Z \equiv k_y(\underline{t}_Z)$.

If $\beta_y(t)$ increases during the scraping phase, two possibilities emerge. Let $\theta \equiv c_x + m_x - a_y - m_y$, under A.1 $\theta > 0$, and the dynamics of $\lambda_y(t)$ during the scraping phase is given by:

$$\dot{\lambda}_y = \rho\lambda_y - (\zeta r_x \lambda_Z + \theta) + b_y'(K_y)$$

The two possibilities are thus either $\zeta r_x \lambda_Z(t^s) + \theta - b_y'(K_y(t^s))$ is larger than $\rho\underline{n}_y$ or is smaller than $\rho\underline{n}_y$.

In the first case $\beta_y(t) > \rho\underline{n}_y$ over the pre-ceiling phase. Through the same argument developed for the Scenario 1, it is possible to show that the renewable energy sector should invest in capacity building throughout the pre-ceiling phase, so that $t_{ky} = t_y = 0$. This is because $\dot{\lambda}_y(t) > 0$ only if $\lambda_y(t) > \beta_y/\rho$. Thus if there exists $t_{ky} > 0$ at which investment in capacity building starts, $\lambda_y(t_{ky}) = \underline{n}_y < \beta_y(t_{ky})/\rho$ would imply that $\dot{\lambda}_y(t_{ky}) \leq 0$, a contradiction since the industry should actively invest after t_{ky}, so that $\lambda_y(t)$, for $t > t_{ky}$, has to be larger than $\underline{n}_y$.

The second possibility can be eliminated by our assumption on the relative competitiveness of the two energy sources:

$$\begin{aligned}
& c_x + m_x > a_y + m_y + b_y'(\tilde{K}_y) + \rho\underline{n}_y \\
\Longrightarrow & 0 > \rho\underline{n}_y - \theta + b_y'(\tilde{K}_y) \\
& > \rho\underline{n}_y - \theta + b_y'(K_y(T^s)) \\
& \text{(because } \tilde{K}_y > K_y(T^s)) \\
& > \rho\underline{n}_y - (\theta + \zeta r_x \lambda_Z(T^s)) + b_y'(K_y(T^s)) \\
\Longrightarrow & (\theta + \zeta r_x \lambda_Z(T^s)) - b_y'(K_y(T^s)) > \rho\underline{n}_y
\end{aligned}$$

However it is worth giving some attention to this case because its suggests that investments in the renewable energy industry could happen over disconnected time episodes if our relative competitiveness assumption A.1 is not satisfied, what we call the pro-fossil competitive edge case. More precisely, it becomes possible that after an initial phase of active investment until some time $\bar{t}_y < t^s$, the renewable energy production sector stops accumulating capacities. At time $\bar{t}_y$, $\lambda_y(\bar{t}_y) = \underline{n}_y$, and $\lambda_y(t)$ is decreasing before $\bar{t}_y$. After time $\bar{t}_y$, $\lambda_y(t)$ falls and stays below $\underline{n}_y$, preventing investments in the renewable energy industry. Because $K_y(t)$ stays constant, $K_y(t^s) = K_y(\bar{t}_y)$. If $\rho\lambda_y(t^s) < \zeta r_x \lambda_Z(t^s) + \theta - b_y'(K_y(\bar{t}_y))$, $\lambda_y(t)$ should permanently fall after t_s preventing investments before $\underline{t}_Z$, a contradiction since the renewable energy industry must invest during the ceiling phase. Thus there exists some time t_λ such that $\rho\lambda_y(t_\lambda) = \beta_y(t_\lambda)$, so that $\dot{\lambda}_y(t_\lambda) = 0$ and $t_\lambda < t^s$. After t_λ, $\lambda_y(t)$ rises again and there will exist some time t_y such that $\lambda_y(t_y) = \underline{n}_y$ while $\dot{\lambda}_y(t_y) > 0$. Thus the investment in the renewable energy sector increases after t_y before having to fall before $\underline{t}_Z$. The Figure 5 illustrates this investment scenario, labeled #1 on the Figure. Of course it also possible that $\lambda_y(0) < \underline{n}_y$, implying the absence of any investments in the renewable energy industry before t_y, the scenario labeled #2 on the Figure.

Return to our main model. Under A.1 the renewable energy industry must begin to

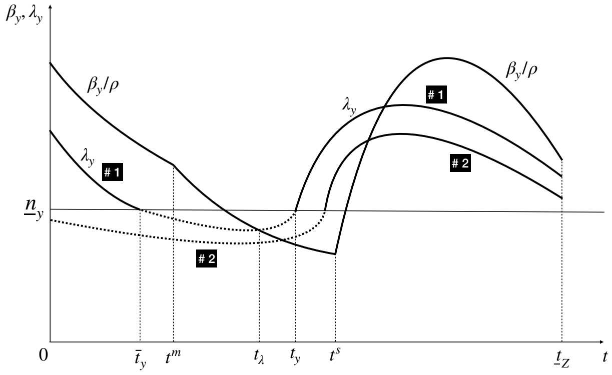

Figure 5: Phases of Investment in renewable energy. The pro-fossil competitive edge case.

invest right from the initial time and keep investing throughout the pre-ceiling phase. Three main categories of investment scenarios in the renewable energy sector appear possible when the investment policy of the fossil energy industry is a scenario 2. They are illustrated on the Figure 6, where the function $\beta_y(t)$ has been pictured as a bell shaped function, which is just one possibility of course.

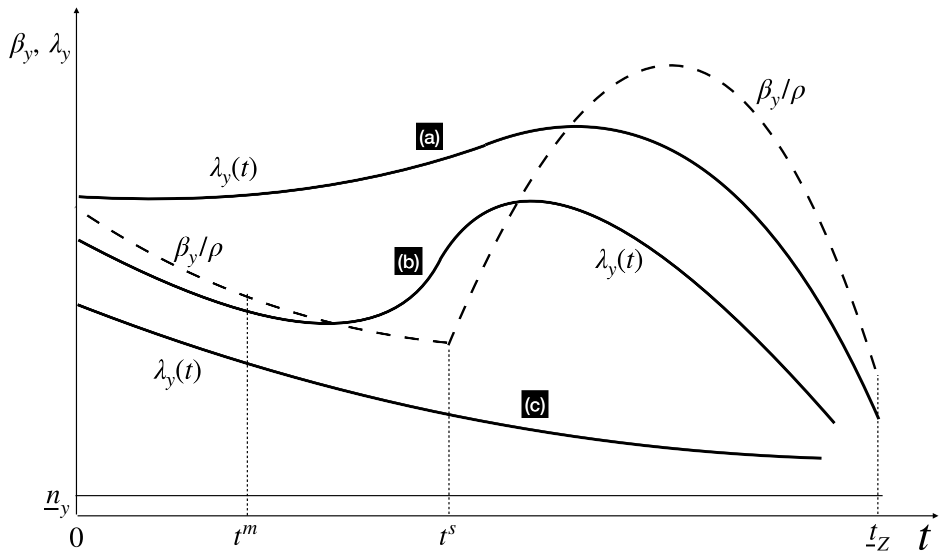

Figure 6: Investment scenarios in renewable energy. The Scenario 2.

The first investment scenario, labelled (a) on the Figure 6 is a two phases path composed of a first time phase during which $\lambda_y(t)$ increases, followed by a second time phase during which $\lambda_y(t)$ decreases. Thus $k_y(t)$ first increases and next decreases in this scenario. Observe that the investment rate peaks during the scraping phase, that is after t^s. The second investment scenario, labelled (b) on the figure, is a three phases path. During a first time phase, $\lambda_y(t)$ decreases. The minimum can be attained either before or after t^m, the beginning time of the stabilisation phase. Then $\lambda_y(t)$ increases during a second time phase and the peak is attained after t^s by construction. Last, during a third time phase, $\lambda_y(t)$ decreases. In this scenario, the investment rate in renewable energy conversion capacities first declines from a strictly positive initial level, next improves and last falls over time. The third investment scenario, labelled (c) on the Figure 6 is a one phase path during which $\lambda_y(t)$, and thus $k_y(t)$, constantly decline until $\underline{t}_Z$.

In the Scenario 2, that the fossil energy industry has to scrap some part of its production capacities in order to adjust to the carbon ceiling induces the possibility of waves of investment in the renewable energy sector. The investment rate, $k_y(t)$, can experience alternating accelerating and decelerating episodes before the ceiling phase. Such investment waves are the consequences of the energy price increase during the scraping process of fossil energy production capacities, an increase itself the consequence of carbon pricing, the carbon price having to rise exponentially at the rate $(\rho + \alpha)$ before the ceiling regime.

Waves of investment are customary in vintage capital models as a result of the internal adjustment process of the demographic structure of the capital stock. They can also appear when investment is irreversible. It is worth remarking that waves can here happen even with the simple rectangular capital mortality table of the present model and even if investment in renewable energy production capacities is *a priori* not irreversible, although we have shown that the renewable energy sector should never scrap its capacities when the fossil energy sector scraps its own capacities. The possibility of investment waves is implied by the particular exponential shape of the carbon price, equivalently the Hotelling like behaviour of this price before the CO_2 stock reaches the cap level, $\bar{Z}$. To conclude, in the Scenario 2, the dynamics of the net operational margin on the renewable energy conversion capital gives rise to three main investment policies in this sector:

1) A two phases policy with first an acceleration of the renewable energy capital building followed by a progressive slow down until $\underline{t}_Z$; 2) A three phases policy with first an investment slow down phase followed by an acceleration phase and concluded with another investment slow down phase ; 3) A one phase policy consisting in a progressive slow down of capital accumulation in the renewable energy conversion sector until $\underline{t}_Z$.

6.2.3 Renewables expansion in the investment Scenario 3 of the fossil energy sector.

The result 3 states that the renewable energy sector should constantly invest when the fossil energy sector actively invests until $\underline{t}_Z$. Thus in the scenario 3, the energy price, $p(t) = u'(K_x(t) + K_y(t))$ constantly declines as the energy supply increases. We thus conclude that $\beta_y(t)$ constantly decreases before $\underline{t}_Z$. We can rely on the argument already invoked to state that $\lambda_y(t)$ should also constantly decline down to $\lambda_y(\underline{t}_Z) > \underline{n}_y$. Hence the investment policy in renewable energy production is made of a unique phase, $[0, \underline{t}_Z)$, of positive investment at a declining rate.

6.2.4 Energy transitions?

In the Scenario 1 where the fossil energy sector dismantles capacities while the renewable energy sector accumulates capacities, the issue of the energy transition is trivial. In the Scenario 2, energy transition happens during the stabilisation and scraping phases but the issue is less clear during the first phase of simultaneous investments by the two sectors. The same ambiguity exists for the Scenario 3.

In the context of the present model the issue of an 'energy transition' at the beginning of the Scenario 2 or in a Scenario 3, that is an of increasing share over time of renewable energy in the fuel mix, appears as an empirical question. Since we have made no assumptions on the relative building costs of the specific capital stocks used by the two energy sectors, nothing can be said on the relative speed of accumulation of capacities and the fuel share dynamics is generally indeterminate. However it is possible to derive some results concerning the values of investing in one or the other type of capital. We take the example of the Scenario 3.

First it is possible to rank the operational margins. Using the expressions of $\beta_x(t)$ and $\beta_y(t)$:

$$\beta_x(t) - \beta_y(t) = -c_x - m_x - \zeta r_x \lambda_Z(t) + a_y + m_y + b'_y(K_y)$$

In order to get a 100 % renewable energy system in the long run, we have assumed that:

$$c_x + m_x > a_y + m_y + b'_y(\tilde{K}_y) + \rho \underline{n}_y$$

where $\tilde{K}_y$ stands as the long run capital stock in the renewable energy conversion sector. Thus:

$$\beta_x - \beta_y < -b'_y(\tilde{K}_y) - \rho \underline{n}_y + b'_y(K_y)$$

Because $K_y < \tilde{K}_y$, and $b''_y(K_y) > 0$, $b'_y(K_y) < b'_y(\tilde{K}_y)$, thus $\beta_x(t) < \beta_y(t)$. Along an investment path where the two sectors accumulate capacities until the ceiling regime, the net operational margin on fossil energy conversion equipments must be lower at all times than the net operational margin on renewable energy conversion equipments.

We can use this property to compare the dual variables values, λ_x and λ_y, that is the respective investment rates in the two sectors. Consider the dual plane, (λ_y, λ_x). There exists in this plane an implicit relationship between λ_x and λ_y, we denote $\lambda_x^*(\lambda_y)$ and $\lambda_x^{*'}(\lambda_y) = \dot{\lambda}_x / \dot{\lambda}_y > 0$, because $\dot{\lambda}_x < 0$ and $\dot{\lambda}_y < 0$ in the Scenario 3. We want to show that $\lambda_x^{*'} < 1$. Assume to the contrary that $\lambda_x^{*'} > 1$, then this would imply:

$$\begin{aligned}
\lambda_x^{*'} = \frac{\dot{\lambda}_x}{\dot{\lambda}_y} = \frac{\rho\lambda_x - \beta_x}{\rho\lambda_y - \beta_y} > 1 \\
\Longrightarrow \rho\lambda_x - \beta_x < \rho\lambda_y - \beta_y \text{ since } \dot{\lambda}_y = \rho\lambda - \beta_y < 0 \\
\Longrightarrow \rho(\lambda_x - \lambda_y) < \beta_x - \beta_y < 0 \Longrightarrow \lambda_x < \lambda_y
\end{aligned}$$

However, if the curve $\lambda_x^*(\lambda_y)$ is of slope larger than one in a zone where $\lambda_x < \lambda_y$, it cannot belong to this zone for larger and larger values of λ_x and λ_y. Thus we get a contradiction and conclude that $\lambda_x^{*'} < 1$.

Thus two main scenarios appear possible in the dual plane. They are illustrated on the Figure 7.

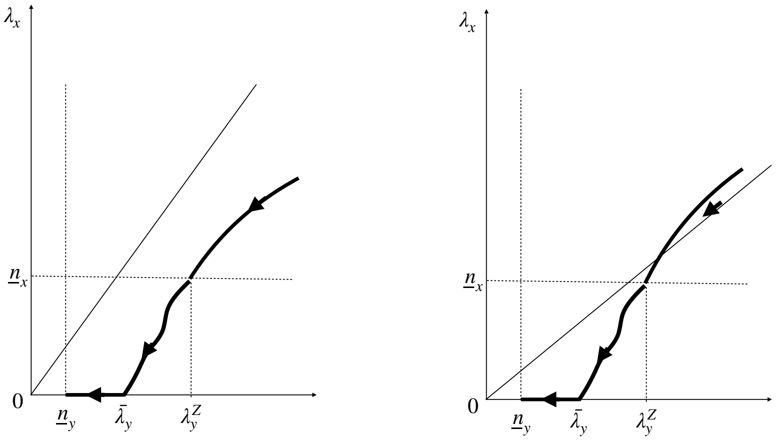

Figure 7: Dynamics in the dual plane (λ_y, λ_x). The Scenario 3.

In the first scenario, the trajectory of $\lambda_x^*(\lambda_y)$ remains below the 45^0 line so that $\lambda_x < \lambda_y$. This implies that $n_x(k_x) < n_y(k_y)$. The investment cost is lower in the fossil energy industry than in the renewable energy industry. If the cost structures are comparable, which seems to be the case empirically when comparing the capacity unit cost for renewable and fossil energy production plants, we can conclude that $k_x < k_y$, the investment rate is larger for renewable than for fossil energy. In the second scenario, it becomes possible that the value of an investment in renewable energy after being smaller than the value of an investment in fossil energy becomes larger when reaching the ceiling regime.

That the accumulation of capacities in renewable energy conversion be faster than in fossil energy conversion does not imply an energy transition from fossil to renewable energy. Denoting by $\sigma \equiv K_y/K$, the renewables share in the fuel mix, it is immediate that:

$$\dot{\sigma} > 0 \Longleftrightarrow \ k_x < \frac{1-\sigma}{\sigma} k_y$$

Starting from initially nil capacities, $\sigma = 0$ implies trivially that σ increases. The same happens when approaching $\underline{t}_Z$, because we know that $k_x \to 0$. In the meantime, $(1-\sigma)/\sigma$ being decreasing over time when σ increases, k_y must not fall too fast with respect to the fall of k_x for the inequality to remain valid.

7 Validity of the scenarios

We present in the Appendix A.1 algorithmic procedures able to compute the closed form solutions of the optimality problem for the three possible investment scenarios in the fossil energy industry. The objective of the present section is to assess in the phase space (K_x, K_y, Z) the domain of validity of the 3 investment/disinvestment policies in fossil energy conversion capital.

In that respect two subspaces are interesting. The first one is the plane (Z, K_x), because the CO_2 emission rate is now ζK_x, establishing a direct link between the accumulation of carbon dioxide in the atmosphere and the capital stock size in the fossil energy industry. Intuitively, it should be expected that the carbon cap constraint put a limit on the potential of accumulation of capital in the fossil energy industry. This limit should take the form of a critical border in the (Z, K_x) plane. Another limit should be the carbon stock size above which the industry should start to scrap its capacities in anticipation of the climate constrained regime. We shall call *stabilisation border* and *scraping border*, the corresponding borders in the (Z, K_x) plane. Second, we want to describe the capital accumulation dynamics in the (K_x, K_y) plane. The focus here is on the issue of 'energy transition', that is on the substitution dynamics away from fossil energy toward renewable energy.

We proceed in successive steps. In the next subsection, we consider the benchmark case of a constant production capacity in the renewable energy sector, thus focusing on the dynamics in the (Z, K_x) plane. The Scenarios 1 and 3 are indeed limit cases of the Scenario 2, where either the two phases $[0, t^m)$ and $[t^m, t^s)$ vanish (in the scenario 1), or either the phases $[t^m, t^s)$ and $[t^s, \underline{t}_Z)$ vanish (in a Scenario 3). Thus we focus on the central Scenario 2 and show how the limit Scenarios 1 or 3 can become optimal for particular initial values of K_x^0 and Z^0. This benchmark setting will prove useful to characterise the borders in the (Z, K_x) plane.

Next we study the climate unconstrained optimal policies, focusing on the (K_x, K_y) plane. Under our relative competitiveness assumptions, renewable energy should ultimately phase-out fossil energy in the long run. Thus some form of energy transition from fossils to renewables should occur even without a climate change concern. We are also more particularly interested in computing the unconstrained optimal path that should peak at one instant at $\bar{Z}$, the carbon concentration mandate. This particular path describes the set of critical initial vectors, (K_x^0, K_y^0), separating the initial vectors for which society would never be constrained by the climate from the initial vectors for which it should actually face the climate constraint. Last with the benchmarks results in hand, we study the general case.

7.1 The benchmark case of a constant K_y

Assume that $K_y(t)$ is fixed at a level $\hat{K}_y$. The dynamic optimisation problem collapses to a two state variables one, (K_x, Z). We focus on a scenario 2 composed of three successive phases. During the first phase, $[0, t^m)$, the industry accumulates fossil energy conversion capacities from the initial level K_x^0, up to a level $\hat{K}_x$ larger than $\bar{K}_x$. The accumulated capacity, $\hat{K}_x$, is next maintained over a second time interval, $[t^m, t^s)$. Because $\hat{K}_x > \bar{K}_x$, the carbon stock grows over time, first from the level Z^0 up to a level we denote $\hat{Z} = Z(t^m)$, at time t^m, the beginning time of the stabilisation phase, and second from the level $\hat{Z}$ up to the level $Z^s = Z(t^s)$ at the end of the phase, with $Z^s < \bar{Z}$, to make sense of the scenario. During the third phase, $[t^s, \underline{t}_Z)$, the industry scraps its capacities, from $\hat{K}_x$ down to $\bar{K}_x$. The carbon stock continues to grow from Z^s up to $\bar{Z}$. We construct the optimal policy recursively.

7.1.1 The scraping regime

We are going to show the existence of a decreasing relationship between $\hat{K}_x$ and Z^s, for a given $\hat{K}_y$, describing what we call the *scraping border* in the (Z, K_x) plane. The intuition is simple: for a given $\hat{K}_x$ to be maintained during the second phase, there should exist a unique level of the carbon stock at the end of the stabilisation phase, Z^s, such that the carbon ceiling level, $\bar{Z}$, and the net zero fossil energy conversion capacity, $\bar{K}_x$, are attained at the same time $\underline{t}_Z$.

Let T^s denote the time length of the phase, $T^s \equiv \underline{t}_Z - t^s$. Make the change of the time variable: $v = t - t^s$, then $v = 0$ at time t^s and $v = T^s$ at time $\underline{t}_Z$. Because, $\lambda_x = \beta_x = 0$ during the scraping phase, (3.6) reads:

$$u'(K_x(v) + \hat{K}_y) = c_x + m_x + \zeta r_x \lambda_Z^s e^{(\rho+\alpha)v} \qquad\qquad v \in [0, T^s)$$

where λ_Z^s is the carbon price level at the beginning of the scraping phase. Differentiating this equation, we obtain:

$$dK_x(v) = \frac{\zeta r_x}{u''(v)} d\lambda_Z^s e^{(\rho+\alpha)v} - d\hat{K}_y \tag{7.1}$$

The carbon stock balance equation over $[0, T^s)$ reads:

$$\bar{Z}e^{\alpha T^s} = Z^s + \zeta r_x \int_0^{T^s} K_x(v) e^{\alpha v} dv$$

The differentiation of this equation yields:

$$\alpha \bar{Z} e^{\alpha T^s} dT^s = dZ^s + \zeta r_x \bar{K}_x e^{\alpha T^s} dT^s + \zeta r_x \int_0^{T^s} dK_x(v) e^{\alpha v} dv$$

Since $\alpha \bar{Z} = \zeta r_x \bar{K}_x$, the dT^s terms cancel and using the expression (7.1) of $dK_x(v)$, we obtain:

$$dZ^s + (\zeta r_x)^2 \left(\int_0^{T^s} \frac{e^{(\rho+2\alpha)v}}{u''(v)} dv \right) d\lambda_Z^s - \zeta r_x \left(\int_0^{T^s} e^{\alpha v} dv \right) d\hat{K}_y = 0$$

Denoting:

$$U_{2\alpha} \equiv - \int_0^{T^s} \frac{e^{(\rho+2\alpha)v}}{u''(v)} dv > 0 \quad \text{and} \quad I_\alpha = \int_0^{T^s} e^{\alpha v} dv > 0$$

the above equation can be written in the more compact form:

$$dZ^s - (\zeta r_x)^2 U_{2\alpha} d\lambda_Z^s - \zeta r_x I_\alpha d\hat{K}_y = 0$$

Evaluating the optimality condition at the initial time and differentiating gets:

$$d\lambda_Z^s = \frac{u''(0)}{\zeta r_x} \left(d\hat{K}_x + d\hat{K}_y \right) \tag{7.2}$$

which defines λ_Z^s as a function of $\hat{K}_x$ parametrised by $\hat{K}_y$, we denote $\lambda_Z^s(\hat{K}_x; \hat{K}_y)$, and:

$$\frac{\partial \lambda_Z^s(\hat{K}_x; \hat{K}_y)}{\partial \hat{K}_x} = \frac{\partial \lambda_Z^s(\hat{K}_x, \hat{K}_y)}{\partial \hat{K}_y} = \frac{u''(0)}{\zeta r_x} < 0$$

The initial carbon price is a decreasing function of both $\hat{K}_x$ and $\hat{K}_y$, that is of $\hat{K} = \hat{K}_x + \hat{K}_y$. Inserting the expression (7.2) of $d\lambda_Z^s$ inside the differentiated carbon balance equation, we obtain:

$$dZ^s - (\zeta r_x)^2 U_{2\alpha} \frac{u''(0)}{\zeta r_x} d\hat{K}_x - \left[(\zeta r_x)^2 U_{2\alpha} \frac{u''(0)}{\zeta r_x} + \zeta r_x I_\alpha\right] d\hat{K}_y = 0$$

Simplifying gets:

$$dZ^s = \zeta r_x U_{2\alpha} u''(0) d\hat{K}_x + \zeta r_x \left[U_{2\alpha} u''(0) + I_\alpha\right] d\hat{K}_y$$

The carbon stock balance equation defines a relationship between Z^s and $\hat{K}_x$, parametrised by $\hat{K}_y$, and such that:

$$\frac{\partial Z^s}{\partial \hat{K}_x} = \zeta r_x U_{2\alpha} u''(0) < 0 \text{ and } \frac{\partial Z^s}{\partial \hat{K}_y} = \zeta r_x \left[U_{2\alpha} u''(0) + I_\alpha\right] (?)$$

In the phase plane, (Z, K_x), the relationship takes the form of a decreasing curve going through the point $(\bar{Z}, \bar{K}_x)$, a point itself located along the null isocline, $\dot{Z} = 0$, which is a line of equation $K_x = (\alpha/\zeta r_x)Z$. The economic interpretation of this result is straightforward. The carbon price should be an increasing function of the pollution stock size, Z. Being at the same time a decreasing function of K_x, there should exist a negative relationship between K_x and Z.

Let $K_x^s(Z; \hat{K}_y)$ denote this relationship. Turn to the sensitivity of this relationship to the level of $\hat{K}_y$, the renewable energy production capacity. Because $u''(0) < 0$, $U_{2\alpha} > 0$ and $I_\alpha > 0$, the sign of $J \equiv u''(0)U_{2\alpha} + I_\alpha$ is ambiguous. If the gross surplus function is quadratic, and thus the energy demand function is linear, $u''(0) = u''(v) = \bar{u}''$, a < 0 constant. Thus:

$$J = -\bar{u}'' \int_0^{T^s} \frac{e^{(\rho+2\alpha)v}}{\bar{u}''} dv + I_\alpha = \int_0^{T_x^s} e^{\alpha v} \left(1 - e^{(\rho+\alpha)v}\right) dv < 0$$

With a linear demand, $J < 0$, thus $\partial Z^s / \partial \hat{K}_y < 0$, an increase of $\hat{K}_y$ makes rotate the curve $K_x^s(Z)$ counterclockwise.

We want to extend the analysis beyond the linear energy demand case. To this aim, time differentiating the optimality condition, we obtain:

$$\begin{cases} k(0)u''(0) = & \zeta r_x(\rho+\alpha)\lambda_Z^s \\ k(v)u''(v) = & \zeta r_x(\rho+\alpha)\lambda_Z^s e^{(\rho+\alpha)v} \end{cases} \Longrightarrow \frac{u''(0)}{u''(v)} = \frac{k(v)}{k(0)} e^{-(\rho+\alpha)v}$$

Hence:

$$\begin{aligned} J &= \int_0^{T^s} \left[-\frac{u''(0)}{u''(v)} e^{(\rho+2\alpha)v} + e^{\alpha v}\right] dv \\ &= \int_0^{T^s} \left[-\frac{k(v)}{k(0)} \frac{e^{(\rho+2\alpha)v}}{e^{(\rho+\alpha)v}} + e^{\alpha v}\right] dv = \int_0^{T^s} e^{\alpha v} \left(1 - \frac{k(v)}{k(0)}\right) dv \end{aligned}$$

Remember that $k(0) < 0$ and $k(v) = \dot{K}_x(v) < 0$. Thus if $k(v) > k(0)$, the scraping process slows down over time while it accelerates in the reverse case and:

$$k(v) > / < k(0) \Longrightarrow \frac{k(v)}{k(0)} < / > 1 \Longrightarrow J > / < 0 \Longrightarrow \frac{\partial Z^s}{\partial \hat{K}_y} > / < 0$$

If the scraping process slows down then the curve $K_x(Z)$ rotates clockwise after an upward shift of $\hat{K}_y$. If the process accelerates, then the curve $K_x^s(Z)$ rotates counterclockwise after an upward shift of $\hat{K}_y$. A set of sufficient conditions for one or the other case is thus given by comparing the curvature of the marginal surplus function (the demand) with the growth rate of λ_Z. Log-differentiating the time differentiated optimality condition yields:

$$k = \zeta r_x \frac{\dot{\lambda}_Z}{u''} \Longrightarrow \frac{\dot{k}}{k} = \frac{\ddot{\lambda}_Z}{\dot{\lambda}_Z} - \frac{\dot{u}''}{u''} = (\rho + \alpha) - \frac{u'''}{u''} k$$

If $u''' < 0$, the r.h.s. is > 0, because $u'' < 0$ and $k < 0$. Thus $\dot{k}/k > 0$, implying that $\dot{k} < 0$, because $k < 0$. Hence $u''' < 0$ implies that $\partial Z^s / \partial \hat{K}_y < 0$. But the indeterminacy remains when $u''' > 0$ or changes sign.

It is anyhow possible to say something even when $u''' > 0$. To show this, take the example of an iso-elastic marginal surplus function, $u'(q) = q^{-\epsilon}$, with $\epsilon > 0$, where it is well known that $u''(q) < 0$ and $u'''(q) > 0$. Remember that we have denoted by $\bar{p}_x$, the sum $c_x + m_x$. Then we get through the optimality condition:

$$q(v)^{-\epsilon} = \bar{p}_x + \zeta r_x \lambda_Z(v) \Longrightarrow q(v) = \left[\bar{p}_x + \zeta r_x \lambda_Z(v)\right]^{-\frac{1}{\epsilon}}$$

Thus:

$$u''(v) = -\epsilon q(v)^{-(1+\epsilon)} = -\epsilon \left[\bar{p}_x + \zeta r_x \lambda_Z(v)\right]^{\frac{1+\epsilon}{\epsilon}}$$

This yields the following expression of J:

$$\begin{aligned} J &= \int_0^{T^s} \left[- \frac{-\epsilon \left[\bar{p}_x + \zeta r_x \lambda_Z^s\right]^{\frac{1+\epsilon}{\epsilon}}}{-\epsilon \left[\bar{p}_x + \zeta r_x \lambda_Z(v)\right]^{\frac{1+\epsilon}{\epsilon}}} e^{(\rho + 2\alpha)v} + e^{\alpha v} \right] dv \\ &= \int_0^{T^s} e^{\alpha v} \left[1 - \left(\frac{\bar{p}_x + \zeta r_x \lambda_Z^s}{\bar{p}_x + \zeta r_x \lambda_Z(v)} \right)^{\frac{1+\epsilon}{\epsilon}} e^{(\rho+\alpha)v} \right] dv \\ &\equiv \int_0^{T^s} e^{\alpha v} \left(1 - \Phi\right) dv \end{aligned}$$

Since $\epsilon > 0$:

$$\Phi > \frac{\bar{p}_x + \zeta r_x \lambda_Z^S}{\bar{p}_x + \zeta r_x \lambda_Z(v)} e^{(\rho+\alpha)v} = \frac{\bar{p}_x e^{(\rho+\alpha)v} + \zeta r_x \lambda_Z(v)}{\bar{p}_x + \zeta r_x \lambda_Z(v)} > 1$$

Thus $J < 0$, implying that $\partial Z^s / \partial \hat{K}_y < 0$.

To determine the time length of the phase, evaluate the optimality condition at time T^s to get:

$$u'(\bar{K}_x + \hat{K}_y) = c_x + m_x + \zeta r_x \lambda_Z^s e^{(\rho+\alpha)T^s}$$

Then differentiating, we obtain:

$$u''(T^s) d\hat{K}_y = \zeta r_x \left[d\lambda_Z^s + (\rho + \alpha) \lambda_Z^s dT^s \right] e^{(\rho+\alpha)T^s}$$

The expression (7.2) of $d\lambda_Z^s$ yields:

$$\begin{aligned} \zeta r_x (\rho + \alpha) \lambda_Z^S dT^s =& u''(T^s) e^{-(\rho+\alpha)T^s} d\hat{K}_y - \zeta r_x \left[\frac{u''(0)}{\zeta r_x} d\hat{K}_x + \frac{u''(0)}{\zeta r_x} d\hat{K}_y \right] \\ =& - u''(0) d\hat{K}_x + \left[u''(T^s) e^{-(\rho+\alpha)T^s} - u''(0) \right] d\hat{K}_y \end{aligned}$$

As expected the time length of the scraping phase is an increasing function of $\hat{K}_x$ (remember that $u''(0) < 0$). The effect of an upward shift of $\hat{K}_y$ is ambiguous. It depends on the comparison between the second order derivative of $u(.)$ at times 0 and T^s weighted by the exponential term, $e^{(\rho+\alpha)T^s}$.

If u'' is constant (the case of a linear marginal surplus function), then the term into brackets is > 0, so that T^s increases with $\hat{K}_y$. Furthermore through our previous computations:

$$\begin{aligned} u''(T^s)e^{-(\rho+\alpha)T^s} - u''(0) &= u''(T^s)e^{-(\rho+\alpha)T^s} - \frac{k(T^{s-})}{k(0)}u''(T^s)e^{-(\rho+\alpha)T^s} \\ &= u''(T^s)e^{-(\rho+\alpha)T^s}\left(1 - \frac{k(T^{s-})}{k(0)}\right) \end{aligned}$$

Thus for $v \leq T^s$, because $u''(T^s) < 0$:

$$k(v) > / < k(0) \Longrightarrow \frac{k(v)}{k(0)} < / > 1 \Longrightarrow \frac{\partial Z^s}{\partial \hat{K}_y} > / < 0 \text{ and } \frac{\partial T^s}{\partial \hat{K}_y} < / > 0$$

As could be expected, if the scraping process slows down, by increasing the initial level of Z^s at the beginning of the phase, the time needed to reach the ceiling, $\bar{Z}$, is reduced. Thus T^s is decreased by a larger $\hat{K}_y$. The opposite happens if the scraping process accelerates, Z^s being reduced and the time needed to attain the ceiling enlarged. Remember that we have shown this to be the case with an iso-elastic demand function. The following proposition summarises our findings.

Proposition P. 3. *1. There exists a decreasing relationship between the maintained fossil energy conversion capital stock at t^s, the beginning time of the scraping phase, and $Z^s = Z(t^s)$, the carbon stock size at the same time.*

2. *The initial carbon price, $\lambda_Z^s = \lambda_Z(t^s)$ is a decreasing function of $\hat{K}_x$, because Z^s is reduced by a larger $\hat{K}_x$, and also a decreasing function of $\hat{K}_y$.*

3. *The time length of the scraping phase, $T^s = \underline{t}_Z - t^s$, increases with $\hat{K}_x$.*

4. *With a linear or iso-elastic energy demand function, the curve $\hat{K}_x(Z^s)$ rotates counterclockwise when $\hat{K}_y$ is increased, equivalently Z^s is decreased by a larger $\hat{K}_y$.*

5. *The length of the scraping phase is increased by a larger $\hat{K}_y$ when the energy demand function is either linear or iso-elastic.*

7.1.2 The stabilisation regime, $[t^m, t^s)$

Sensitivity of the price of carbon to $\hat{K}_x$ and $\hat{K}_y$.

Let $T^m \equiv t^s - t^m$ denote the length of the phase and make the change of the time variable, $v = t - t^m$, so that $v = 0$ when $t = t^m$ and $v = T^m$ when $t = t^s$. Since $K_x(v) = \hat{K}_x$ during the phase, the energy price is constant at the level: $\hat{p} = u'(\hat{K}_x + \hat{K}_y)$. Let $\hat{\lambda}_Z = \lambda_Z(t^m)$ denote the carbon price level at the beginning of the phase, then β_x is defined as:

$$\beta_x(v) = u'(\hat{K}_x + \hat{K}_y) - c_x - m_x - \zeta r_x \hat{\lambda}_Z e^{(\rho+\alpha)v}$$

It is immediate that β_x is a decreasing function of $(\hat{K}_x, \hat{K}_y, \hat{\lambda}_Z)$.

Since $\lambda_x(0) = \underline{n}_x$ and $\lambda_x(T^m) = 0$, the integration of the motion of $\lambda_x(t)$ over $[0, T^m)$ yields:

$$\int_0^{T^m} \left[u'(\hat{K}_x + \hat{K}_y) - c_x - m_x - \zeta r_x \hat{\lambda}_Z e^{(\rho+\alpha)v}\right] e^{-\rho v} dv = \underline{n}_x$$

Differentiating yields:

$$\beta_x(T^m)dT^m + \left(\int_0^{T^m} u''(v)e^{-\rho v}dv\right) d\hat{K}_x + \left(\int_0^{T^m} u''(v)e^{-\rho v}dv\right) d\hat{K}_y \\ - \zeta r_x \left(\int_0^{T^m} e^{\alpha v}dv\right) d\hat{\lambda}_Z = 0$$

Remember that $\beta_x(T^m) = 0$ and denote:

$$U_\rho \equiv -\int_0^{T^m} u''(v)e^{-\rho v}dv > 0 \text{ and } I_\alpha \equiv \int_0^{T^m} e^{\alpha v}dv > 0$$

then the previous equation writes in a more compact form:

$$-U_\rho d\hat{K}_x - U_\rho d\hat{K}_y - \zeta r_x I_\alpha d\hat{\lambda}_Z = 0$$

There exists a minimal delay between the time at which the industry stops accumulating capacities and the time at which it begins to dismantle them. This minimal delay condition defines a relationship between the carbon price level at the beginning of the stabilisation phase, $\hat{\lambda}_Z$, and $\hat{K}_x$, parametrised by $\hat{K}_y$, a relation we denote $\hat{\lambda}_Z(\hat{K}_x; \hat{K}_y)$ and :

$$\frac{\partial \hat{\lambda}_Z(\hat{K}_x, \hat{K}_y)}{\partial \hat{K}_x} = \frac{\partial \hat{\lambda}_Z(\hat{K}_x, \hat{K}_y)}{\partial \hat{K}_y} = -\frac{U_\rho}{\zeta r_x I_\alpha} < 0$$

The negative effect of $\hat{K}_x$ on the initial carbon price, $\hat{\lambda}_Z$, and thus on the carbon price trajectory during the stabilisation phase, $\lambda_Z(v)$, is counterintuitive. It could have been expected that a larger capacity in fossil energy conversion, because it entails a larger polluting emissions flow throughout the stabilisation phase, would result in a larger carbon price. But we have shown in the previous subsection that to a larger $\hat{K}_x$ is associated a lower level of the carbon pollution stock, Z^s, at the end of the stabilisation phase. We are going to show below the existence of another negative relationship between the maintained fossil energy conversion capacity, $\hat{K}_x$, and the carbon stock at the beginning of the stabilisation phase, $\hat{Z}$. Thus despite the increase of the pollution flow resulting from a larger $\hat{K}_x$, the carbon stock trajectory is shifted down when $\hat{K}_x$ is shifted up. The negative impact of $\hat{K}_x$ on the carbon price is the consequence of this downward shift of the pollution stock trajectory.

The negative impact of a larger $\hat{K}_y$ on the carbon price is explained in the same way. We have shown before that the carbon stock at the end of the stabilisation phase, Z^s, is reduced by a larger capacity in renewable energy conversion at least in the linear or iso-elastic energy demand case. We show below that the same applies to the pollution stock level at the beginning of the stabilisation phase, thus the pollution stock accumulation trajectory is shifted down by a larger $\hat{K}_y$, inducing a lower level of the carbon price.

Sensitivity of T^m to $\hat{K}_x$ and $\hat{K}_y$.

Since $q(t)$ is constant during the stabilisation phase, $u''(t) = \hat{u}''$, a negative constant. Thus the ratio U_ρ/I_α reads equivalently:

$$\frac{U_\rho}{I_\alpha} = -\hat{u}''\frac{\alpha}{\rho}\frac{1-e^{-\rho T^m}}{e^{\alpha T^m}-1}$$

It is possible to express T^m as a function of $(\hat{K}_x, \hat{K}_y)$ through the terminal condition on $\lambda_Z(t)$:

$$\lambda_Z^s = \hat{\lambda}_Z e^{(\rho+\alpha)T^m} \Longrightarrow d\lambda_Z^s = \left(d\hat{\lambda}_Z + (\rho+\alpha)\hat{\lambda}_Z dT^m\right) e^{(\rho+\alpha)T^m}$$

Taking account of the expressions of the partial derivatives of $\hat{\lambda}_Z$ and λ_Z^s with respect to $(\hat{K}_x, \hat{K}_y)$, we obtain:

$$\begin{aligned}
&\left[(\rho+\alpha)\hat{\lambda}_Z e^{(\rho+\alpha)t^m}\right] dT^m = \dot{\lambda}_Z(T^m)dT^m = \\
&\left[\frac{\lambda_Z^s}{\partial\hat{K}_x} - \frac{\partial\hat{\lambda}_Z}{\partial\hat{K}_x}e^{(\rho+\alpha)T^m}\right] d\hat{K}_x + \left[\frac{\lambda_Z^s}{\partial\hat{K}_y} - \frac{\partial\hat{\lambda}_Z}{\partial\hat{K}_y}e^{(\rho+\alpha)T^m}\right] d\hat{K}_y \\
&= \left[\frac{\hat{u}''}{\zeta r_x} + \frac{U_\rho}{\partial\zeta r_x I_\alpha}e^{(\rho+\alpha)T^m}\right] d\hat{K}_x + \left[\frac{\hat{u}''}{\zeta r_x} + \frac{U_\rho}{\partial\zeta r_x I_\alpha}e^{(\rho+\alpha)T^m}\right] d\hat{K}_y \\
&= \frac{\hat{u}''}{\zeta r_x}\left(d\hat{K}_x + d\hat{K}_y\right)\phi(T^m)
\end{aligned}$$

where:

$$\phi(T^m) \equiv 1 - \frac{\alpha}{\rho}\frac{e^{(\rho+\alpha)T^m} - e^{\alpha T^m}}{e^{\alpha T^m}-1} = \frac{\rho\left(e^{\alpha T^m}-1\right) - \alpha e^{(\rho+\alpha)T^m} + \alpha e^{\alpha T^m}}{\rho\left(e^{\alpha T^m}-1\right)}$$

The sign of this expression is the sign of its numerator since its denominator is positive. Let $Nu(T^m)$ be this numerator, it is immediate that $Nu(0) = 0$ and:

$$\begin{aligned}
\frac{dNu(T^m)}{dT^m} &= \alpha\rho e^{\alpha T^m} - \alpha(\rho+\alpha)e^{(\rho+\alpha)T^m} + \alpha^2 e^{\alpha T^m} \\
&= \alpha(\rho+\alpha)e^{\alpha T^m}\left(1 - e^{\rho T^m}\right) < 0
\end{aligned}$$

from which we deduce that $Nu(T^m) < 0$ and thus that $\phi(T^m) < 0$. Because $\hat{u}'' < 0$ we conclude that:

$$\frac{\partial T^m}{\partial\hat{K}_x} = \frac{\partial T^m}{\partial\hat{K}_y} = \frac{\hat{u}''\phi(T^m)}{\zeta r_x\dot{\lambda}_Z(T^m)} > 0$$

The sign of $\partial T^m/\partial\hat{K}_x$ is expected. Because the energy price is shifted down by a larger fossil energy conversion capacity, it requires a larger minimal delay to recover the investment cost on the last accumulated unit of fossil energy conversion capital. The same depressive effect on the energy price applies to an upward shift of the renewable energy conversion capacity, resulting also in a larger minimal delay before the beginning of the scraping process of fossil energy conversion capacities.

The stabilisation border in the (Z, K_x) plane.

The carbon stock balance equation over $[0, T^m)$ reads:

$$Z^s e^{\alpha T^m} = \hat{Z} + \zeta r_x\hat{K}_x I_\alpha(T^m)$$

Differentiating gets:

$$dZ^s e^{\alpha T^m} + \alpha Z^s e^{\alpha T^m} dT^m = d\hat{Z} + \zeta r_x I_\alpha d\hat{K}_x + \zeta r_x \hat{K}_x e^{\alpha T^m} dT^m$$

Remarking that $\dot{Z}(T^m) = \zeta r_x \hat{K}_x - \alpha Z^s > 0$ and expressing Z^s as a function of $(\hat{K}_x, \hat{K}_y)$, we obtain:

$$\begin{aligned} d\hat{Z} &= \left[\frac{\partial Z^s}{\partial \hat{K}_x} e^{\alpha T^m} - \dot{Z}(T^m) e^{\alpha T^m} \frac{\partial T^m}{\partial \hat{K}_x} - \zeta r_x I_\alpha\right] d\hat{K}_x \\ &+ \left[\frac{\partial Z^s}{\partial \hat{K}_y} e^{\alpha T^m} - \dot{Z}(T^m) e^{\alpha T^m} \frac{\partial T^m}{\partial \hat{K}_y}\right] d\hat{K}_y \\ &\equiv A_{Kx} d\hat{K}_x + A_{Ky} d\hat{K}_y \end{aligned}$$

where:

$$\begin{aligned} A_{Kx} &= \underbrace{\frac{\partial \hat{Z}}{\partial \hat{K}_x}}_{(<0)} e^{\alpha T^m} - \underbrace{\dot{Z}(T^m)}_{(>0)} e^{\alpha T^m} \underbrace{\frac{\partial T^m}{\partial \hat{K}_x}}_{(>0)} - \zeta r_x I_\alpha < 0 \\ A_{Ky} &= \underbrace{\frac{\partial \hat{Z}}{\partial \hat{K}_y}}_{(?)} e^{\alpha T^m} - \underbrace{\dot{Z}(T^m)}_{(>0)} e^{\alpha T^m} \underbrace{\frac{\partial T^m}{\partial \hat{K}_y}}_{(>0)} (?) \end{aligned}$$

We have shown that $\partial \hat{Z}/\partial \hat{K}_x = A_{Kx} < 0$. For a given $\hat{K}_y$, there exists a decreasing relationship between $\hat{K}_x$ and $\hat{Z}$ in the phase plane (Z, K_x). Denote by $K_x^m(Z; \hat{K}_y)$ this relationship. The relationship describes a decreasing curve in the plane, we call the *stabilisation border*. This border defines at which level the fossil energy industry should stop accumulating capacities with regard to the accumulation of the carbon stock. Taking the limits, we see that $\hat{Z} \to \bar{Z}$ when $\hat{K}_x \to \bar{K}_x$. On the other hand the carbon stock balance equation reads equivalently:

$$Z^s e^{\alpha T^m} = \hat{Z} + \zeta r_x \hat{K}_x \left(\frac{e^{\alpha T^m} - 1}{\alpha}\right) \Longrightarrow \alpha \hat{Z} = \left(\alpha Z^s - \zeta r_x \hat{K}_x\right) e^{\alpha T^m} + \zeta r_x \hat{K}_x$$

When $\hat{K}_x \to \bar{K}_x$ and $\hat{Z} \to \bar{Z}$, $\alpha \hat{Z} \to \alpha \bar{Z} = \zeta r_x \bar{K}_x$, thus the term into parenthesis converges down to 0 so that $Z^s \to \frac{\zeta r_x}{\alpha} \bar{K}_x = \bar{Z}$. The curve $K_x^m(Z; \hat{K}_y)$ cuts the curve $K_x^s(Z; K_y)$ at the point $(\bar{Z}, \bar{K}_x)$.

When $\hat{K}_y$ is increased, $\partial \hat{Z}/\partial \hat{K}_y = A_{Ky}$ is of indeterminate sign. We have shown that in the linear demand case and the iso-elastic demand case, $\partial Z^s/\partial \hat{K}_y < 0$, so that we can conclude that $\partial \hat{Z}/\partial \hat{K}_y < 0$ for these two specifications of the energy demand function. When $\hat{K}_y$ is increased, the curve $Z^s(\hat{K}_x)$ rotates counterclockwise, as the curve $\hat{Z}(\hat{K}_x)$. The Figure 8 illustrates the construction and the Proposition P.4 summarises our findings.

Proposition P. 4.

1. *There exists a decreasing relationship between the maintained fossil energy conversion capacity, $\hat{K}_x$, and the size of the carbon stock at the beginning of the stabilisation phase, $\hat{Z} = Z(t^m)$.*
2. *With a linear or iso-elastic energy demand function, the carbon stock size $\hat{Z}$ decreases with $\hat{K}_y$.*

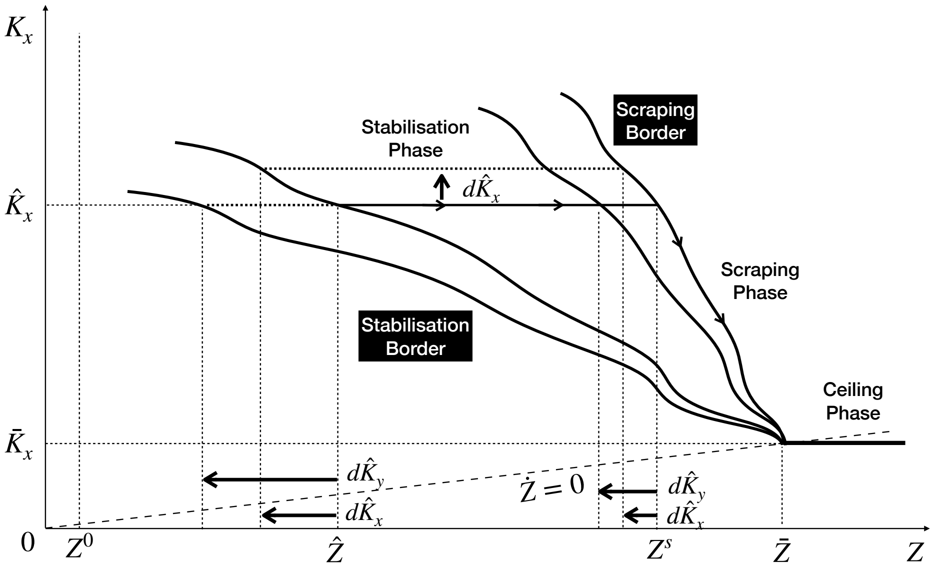

Figure 8: The phase plane (Z, K_x). The constant K_y case

3. *The carbon price at the beginning of the phase, $\hat{\lambda}_Z = \lambda_Z(t^m)$, is a decreasing function of $\hat{K}_x$, because $\hat{Z}$ decreases with $\hat{K}_x$, and a decreasing function of $\hat{K}_y$.*
4. *The energy price, $\hat{p}$ is a decreasing function of $\hat{K}_x$ and $\hat{K}_y$.*
5. *The time length of the phase, $T^m \equiv t^s - t^m$, increases with $\hat{K}_x$ because the energy price decreases with $\hat{K}_x$. It also increases with $\hat{K}_y$ for the same reason.*

7.1.3 The active investment regime

The optimal path during the initial accumulation phase, $[0, t^m)$, is determined once the vector $(\hat{K}_x, \lambda_x(0), \lambda_Z(0), t^m)$ is determined as a function of the vector or initial states, $(K_x^0, \hat{K}_y, Z^0)$. In Appendix A.2 we show how to determine this vector of variables and deliver in addition the following sensitivity results:

Proposition P. 5. *As functions of K_x^0:*

1. *The stabilisation level of the fossil energy capital stock, $\hat{K}_x$, is an increasing function of K_x^0. Thus the pollution stock at time t^m, $\hat{Z}$, is a decreasing function of K_x^0.*
2. *The fossil energy capital price trajectory, $\lambda_x(t)$ is shifted down by a positive shock on K_x^0, its initial level, $\lambda_{x0} \equiv \lambda_x(0)$ being a decreasing function of K_x^0.*
3. *The carbon price trajectory, $\lambda_Z(t)$, is shifted up by a positive shock on K_x^0, $\lambda_{Z0} \equiv \lambda_Z(0)$ being an increasing function of K_x^0.*
4. *The time length of the phase, t^m, is a decreasing function of K_x^0.*

7.1.4 Discussion

A good way to organise a discussion of the underlying arbitrages behind the construction of the optimal path is to describe the consequences of a positive shock on the fossil energy production capital stock, K_x^0, at the beginning of the planning horizon. The sensitivity results of the Proposition 5 show that a larger K_x^0 implies a larger capital stock at the beginning of the stabilisation phase, $\hat{K}_x$. As could be expected, the price of fossil energy conversion capital is reduced, $d\lambda_{x0}/dK_x^0 < 0$, and the opportunity cost of the climate constraint is increased, $d\lambda_{Z0}/dK_x^0 > 0$, meaning a positive correlation between the carbon price and the availability of fossil energy conversion capital.

The fact that a higher carbon price is associated to a lower price of the fossil energy capital could be interpreted as the effect of the carbon regulation on the profitability of the sector. However, the negative correlation between the carbon price and the price of fossil energy capital must be taken with caution. The price of capital incorporates the whole stream of profits from fossil energy generation until t^s, the time at which the capital value has been driven down to zero. The upward shift of λ_{Z0} means an upward shift of the whole carbon tax schedule, $\lambda_Z(t) = \lambda_{Z0}e^{(\rho+\alpha)t}$, thus an exponentially growing increase of the carbon price in the future impacting more and more the profits from fossil fuels exploitation. This is this compounded effect that channels the negative impact of the increase of the initial carbon tax on the initial capital value.

To the larger $\hat{K}_x$ resulting from the shift of K_x^0, is associated a lower carbon stock, $\hat{Z}$. Associated with this lower $\hat{Z}$ the carbon price at the beginning of the stabilisation phase is reduced. The time duration of the accumulation phase is shortened, $\partial t^m/\partial K_x^0 < 0$, because we know already that λ_{Z0} is increased.

In the present benchmark, because there is no accumulation of capacities by the renewable energy sector, the ceiling phase lasts at infinity because there does not exist a possibility to increase $\hat{K}_y$. The energy price is constant at the level $\bar{p} = u'(\bar{K}_x + \hat{K}_y)$ and $\beta_x = 0$, the carbon price is thus also constant at the level $\bar{\lambda}_Z = (\bar{p} - c_x - m_x)/\zeta r_x$. Hence an increase of K_x^0 meaning an increase of λ_{Z0}, the time length of the pre-ceiling phase, $\underline{t}_Z$, must be reduced.

Proposition P.4 (claim 5) states that $\partial T^m/\partial \hat{K}_x > 0$ and Proposition P.3 (claim 3) that $\partial T^s/\partial \hat{K}_x > 0$. Since $\hat{K}_x$ is being increased by a larger K_x^0, the time lengths of the stabilisation phase and the scraping phase also increase. But the time length of the whole pre-ceiling phase being reduced, the increases of T^m and T^s are of the second order with respect to the shortening of the initial accumulation phase. Say otherwise, the reduction of t^m overcompensates the increase of T^m and T^s, resulting in a more rapid convergence to the ceiling regime.

The fall of λ_{x0} and t^m following an increase of K_x^0 means a downward shift of $\lambda_x(t)$, since $\lambda_x(t^m) = \underline{n}_x$. The fossil energy industry sector invests less when benefiting from a larger initial endowment and having to cope with larger carbon price levels. Thus the increase of $\hat{K}_x$ is of the second order with respect to the increase of K_x^0, the fossil energy industry investing less over a shorter period. Say differently $0 < d\hat{K}_x < dK_x^0$.

The time effect is also crucial to understand the logic of the interplay between the capital accumulation policy of the fossil energy industry and the accumulation of the CO_2 pollution stock. Endowed initially with more capacities, the industry still choses to stabilise its capital stock at a larger level despite the climate constraint. The capital stock being constantly larger, the polluting emissions are also larger, meaning an accelerated accumulation of CO_2 in the atmosphere. However the shortening of the accumulation phase is so severe that the industry stops accumulating capacities at a smaller carbon stock size. Then despite a larger maintained capital stock, thus a larger emission flow, and a longer stabilisation period, the carbon stock size is still reduced at T^s, the end of the stabilisation phase and the beginning of the scraping phase.

In parallel, the shortening of the accumulation phase is so large that, despite its increase, the carbon price is lower at the beginning of the stabilisation phase. Even if the emissions are larger and the stabilisation phase lasts longer, the carbon price is still lower at time T^s. Thus the industry enters the scraping phase with larger capacities and enjoys a lower carbon price. The consequence is less incentives to cut down capacities, and hence emissions, during the scraping phase. The length of the scraping phase is increased, the industry having to scrap more capacities at a slower rate. In parallel, the CO_2 stock, initially lower, increases at a faster rate over a longer time period until the carbon cap is reached.

Summing up, the investment logic of the industry runs as follows. Deciding to stabilise the capacities at a larger level means a longer stay in the stabilisation regime, because the minimal delay to recover the value of the last piece of investment is enlarged, and a longer scraping regime, the industry having to scrap more capacities to comply with the climate constraint. Endowed with more capacities, the industry choses to stabilise its capital stock at a larger level but invests less over a shorter period before the stabilisation regime. That $d\hat{K}_x < dK_x^0$ means that with the modified investment policy, the peak of capacities of the industry would have been lower if it had not benefited from larger initial production capacities. This is the consequence of the increased burden of a larger minimal recovery delay and a larger stock to scrap.

The regulator adapts to the industry logic in the following way. Anticipating a larger peak of fossil energy conversion capacities, she raises the carbon price so that the industry stops investing earlier at a lower carbon stock level. Thus the main goal of the regulator is not to cap down the capacity peak but to make it happen sooner. The result is a carbon stock 'saving' at the beginning of the stabilisation regime. Then because the polluting emission flow is made larger by a larger capacity peak, the carbon 'saving' is progressively consumed until the beginning of the ceiling regime.

7.1.5 Mapping of the scenarios in the (Z, K_x) plane

The Figure 9 shows the joint optimal dynamics of the carbon pollution stock and the productive capital stock of the fossil energy conversion industry in the plane (Z, K_x), illustrating the mapping of investment/disinvestment Scenarios in the plane.

We have shown that to the pair $(\bar{K}_x, \bar{Z})$ is associated a unique capital accumulation trajectory and thus a unique pair of initial conditions, (Z^0, K_x^0), inducing the fossil industry to constantly invest in capital building during the whole pre-ceiling phase, up to $(\bar{Z}, \bar{K}_x)$.

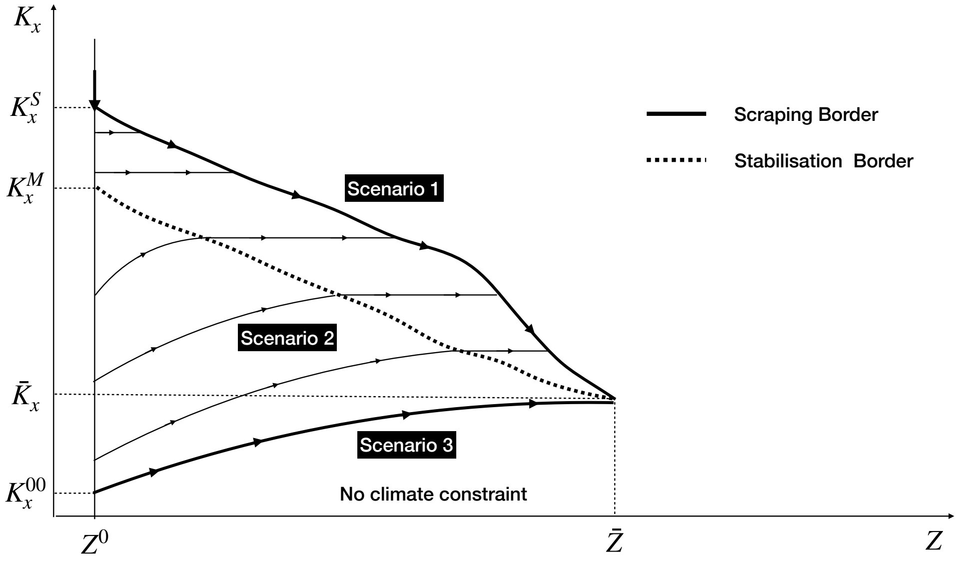

Figure 9: Investment Scenarios in the plane (Z, K_x).

This policy corresponds to the Scenario 3 previously described. Let $K_x^{00}(Z^0)$ be the initial capital stock as a function of the initial carbon pollution stock allowing for this investment scenario. By construction $K_x^{00}(Z^0)$ is an increasing function of Z^0 converging to $\bar{K}_x$ when $Z^0 \to \bar{Z}$.

If for a given Z^0, $K_x^0 < K_x^{00}$, the fossil energy industry will not accumulate capacities up to $\bar{K}_x$ and the climate constraint will never be active. In the phase plane, the zone located below the curve $K_x^{00}(Z^0)$ is the locus of accumulation paths unconstrained by the climate .

We have also shown the existence of a decreasing relationship between K_x and Z we have called the *stabilisation border* describing the capital stock levels at which the fossil energy industry should stop building new capacities with respect to the size of the carbon pollution stock. We have denoted $K_x = K_x^m(Z)$ the equation of the stabilisation border. The border cuts the vertical Z^0 at some value of K_x, we denote $K_x^M = K_x^m(Z^0)$.

If $K_x^{00} < K_x^0 < K_x^M$, then the fossil energy optimal investment policy is the Scenario 2 composed of a first phase of capital accumulation, followed by a stabilisation phase and ended by a scraping phase. We have shown that to larger initial capital stocks are associated larger maintained capital stocks but the stabilisation phase begins at a lower initial carbon stock. Increasing K_x^0, the duration of the accumulation phase declines. If $K_x^0 = K_x^M$, the accumulation phase is reduced to 0 and the industry never invests, just maintaining its capital endowment, K_x^0, before scraping it to comply with the climate constraint.

We have shown the existence of a decreasing relationship between K_x and Z we have called the *scraping border* of the fossil energy industry. We have denoted by $K_x = K_x^s(Z)$, the equation of this border. The intersection between the border and the $Z = Z^0$ vertical defines a critical capital stock level, we denote $K_x^S = K_x^s(Z^0)$. If $K_x^M < K_x^0 < K_x^S$, the optimal investment policy of the fossil energy industry is a degenerated Scenario 2, composed of a first stabilisation phase followed by a scraping phase. The difference is that now $\lambda_x(t^m)$ is no more equal to $\underline{n}_x$, but lower than the sunk cost.

To check the implications of this point, make the change of the time variable $v = t - t^m$ so that $\lambda_x(t^m) = \lambda_{x0}$, $\lambda_Z(t^m) = \lambda_{Z0}$ and $T^m = t^s - t^m$ is the time length of the stabilisation phase with this change of the time variable. The vector $(\lambda_{x0}, \lambda_{Z0}, T^m)$ characterising the stabilisation phase is the solution of the system of conditions:

$$\begin{aligned}
&\lambda_{x0} = \int_0^{T^m} \beta_x(v) e^{-\rho v} dv \\
&Z^s(K_x^0) e^{\alpha T^m} = Z^0 + \zeta r_x K_x^0 I_\alpha(0, T^m) \\
&\lambda_Z^s(K_x^0) = \lambda_{Z0} e^{(\rho+\alpha)T^m}
\end{aligned}$$

The first relation expresses the minimal delay condition but now for an initial value of λ_{x0} which is a variable to be determined. The second condition is the carbon stock balance condition but now expressed with respect to the initial carbon stock, Z^0, and $Z^s = Z^s(K_x^0)$, using the functional dependency of Z^s on K_x^0, where K_x^0 is now the maintained capital stock of the fossil energy industry during the stabilisation phase, $[0, T^m)$. The last condition is the continuity requirement on $\lambda_Z(t)$ at time T^m, taking account of the functional dependency of λ_Z^s on K_x^0. The differentiation of the first relation yields:

$$d\lambda_{x0} = \int_0^{T^s} u'' e^{-\rho v} dv dK_x^0 - \zeta r_x I_\alpha(0, T^m) d\lambda_{Z0}$$

Denoting $U_\rho = -\int_0^{T^m} u'' e^{-\rho v} dv$ and $I_\alpha = I_\alpha(0, T^m)$ and rearranging gets:

$$d\lambda_{x0} + \zeta r_x I_\alpha d\lambda_{Z0} = -U_\rho dK_x^0$$

Proceeding the same for the carbon stock balance condition, we obtain:

$$\frac{dZ^s}{dK_x^0} dK_x^0 e^{\alpha T^m} + \alpha Z^s e^{\alpha T^m} dT^m = \zeta r_x I_\alpha dK_x^0 + \zeta r_x K_x^0 e^{\alpha T^m} dT^m$$

Remembering that $\dot{Z}(T^m) = \zeta r_x K_x^0 - \alpha Z^s$ and rearranging get:

$$\dot{Z}(T^m) e^{\alpha T^m} dT^m = \left[\frac{dZ^s}{dK_x^0} e^{\alpha T^m} - \zeta r_x I_\alpha\right] dK_x^0$$

Last the differentiation of the continuity condition over λ_Z yields:

$$d\lambda_{Z0} e^{(\rho+\alpha)T^m} + \dot{\lambda}_Z(T^m) dT^m = \frac{d\lambda_Z^m}{dK_x^0} dK_x^0$$

Expressing the differentiated system in matrix form:

$$\begin{bmatrix} 1 & \zeta r_x I_\alpha & 0 \\ 0 & 0 & \dot{Z}(T^m) e^{\alpha T^m} \\ 0 & e^{(\rho+\alpha)T^s} & \dot{\lambda}_Z(T^m) \end{bmatrix} \begin{bmatrix} d\lambda_{x0} \\ d\lambda_{Z0} \\ dT^m \end{bmatrix} = \begin{bmatrix} -U_\rho \\ (dZ^s/dK_x^0) e^{\alpha T^m} - \zeta r_x I_\alpha \\ d\lambda_Z^s/dK_x^0 \end{bmatrix} dK_x^0$$

Denoting by Δ, the determinant of the system:

$$\Delta = -\dot{Z}(T^m)e^{(\rho+2\alpha)T^m} < 0$$

Applying the Cramer rule:

$$\begin{aligned}
\frac{d\lambda_{x0}}{dK_x^0} =& \frac{1}{\Delta}\begin{vmatrix} -U_\rho & \zeta r_x I_\alpha & 0 \\ (dZ^s/dK_x^0)e^{\alpha T^m} - \zeta r_x I_\alpha & 0 & \dot{Z}(T^m)e^{\alpha T^m} \\ d\lambda_Z^s/dK_x^0 & e^{(\rho+\alpha)T^m} & \dot{\lambda}_Z(T^m) \end{vmatrix} \\
=& \frac{1}{\Delta}\left\{U_\rho \dot{Z}(T^m)e^{(\rho+2\alpha)T^m} - \zeta r_x I_\alpha \left[\dot{\lambda}_Z(T^m)\left(\frac{dZ^s}{dK_x^0}e^{\alpha T^m} - \zeta r_x I_\alpha\right)\right.\right. \\
& \left.\left. -\dot{Z}(T^s)\frac{d\lambda_Z^s}{dK_x^0}e^{\alpha T^m}\right]\right\} \\
=& -U_\rho + \frac{1}{\Delta}\left[(\zeta r_x I_\alpha)^2 \dot{\lambda}_Z(T^m)\right. \\
& \left. -\zeta r_x I_\alpha e^{\alpha T^m}\left(\dot{\lambda}_Z(T^m)\frac{dZ^s}{dK_x^0} - \dot{Z}(T^s)\frac{d\lambda_Z^s}{dK_x^0}\right)\right]
\end{aligned}$$

Observe that through the chain rule:

$$\frac{d\lambda_Z^s}{dZ^s} = \frac{d\lambda_Z^s/dK_x^0}{dZ^s/dK_x^0} = \frac{\dot{\lambda}_Z(T^m)}{\dot{Z}(T^m)} \Longrightarrow \dot{Z}(T^s)\frac{d\lambda_Z^s}{dK_x^0} = \dot{\lambda}_Z(T^m)\frac{dZ^s}{dK_x^0}$$

Thus:

$$\frac{d\lambda_{x0}}{dK_x^0} = -U_\rho + \frac{(\zeta r_x I_\alpha)^2}{\Delta}\dot{\lambda}_Z(T^m) < 0$$

because $\Delta < 0$. Applying once again the Cramer rule:

$$\begin{aligned}
\frac{d\lambda_{Z0}}{dK_x^0} =& \frac{1}{\Delta}\begin{vmatrix} 1 & -U_\rho & 0 \\ 0 & (dZ^s/dK_x)e^{\alpha T^m} - \zeta r_x I_\alpha & \dot{Z}(T^m)e^{\alpha T^m} \\ 0 & d\lambda_Z^s/dK_x^0 & \dot{\lambda}_Z(T^m) \end{vmatrix} \\
=& \frac{1}{\Delta}\left[e^{\alpha T^m}\left(\dot{\lambda}_Z(T^m)\frac{dZ^s}{dK_x^0} - \dot{Z}(T^m)\frac{d\lambda_Z^s}{dK_x^0}\right) - \zeta r_x I_\alpha \dot{\lambda}_Z(T^m)\right] \\
=& -\frac{\zeta r_x I_\alpha}{\Delta}\dot{\lambda}_Z(T^m) > 0
\end{aligned}$$

Last, since $dZ^s/dK_x^0 < 0$, we get directly from the carbon stock balance equation:

$$\frac{dT^s}{dK_x^0} = \frac{1}{\dot{Z}(T^m)e^{\alpha T^m}}\left[\frac{dZ^s}{dK_x^0}e^{\alpha T^m} - \zeta r_x I_\alpha\right] < 0$$

As should be expected, λ_{x0} is a decreasing function of K_x^0, reflecting the lower value of having a larger initial endowment in fossil energy conversion capital when facing a climate constraint. On the other hand, the time duration of the stabilisation phase is shortened and the scraping phase will start earlier. Remember that when studying the stabilisation phase, we have shown that the carbon price is lowered by a larger stabilised level of K_x. This apparently counterintuitive result was the consequence of a smaller carbon stock size during the stabilisation phase. Now the rise of the carbon stock will be higher when K_x^0 is increased, resulting as expected in a larger carbon price.

When $K_x^0 > K_x^S$, the fossil industry capital stock is in excess with respect to the climate constraint. The industry should scrap immediately at time $t = 0$ the excess stock

$K_x^0 - K_x^S$ and then proceed to the progressive scraping of the capacity $K_x^S - \bar{K}_x$ during a unique scraping phase. The stabilisation phase is hence reduced to 0 and the optimal policy consists in the progressive dismantling of the fossil energy conversion equipments down to $\bar{K}_x$ when the carbon constraint begins binding.

7.2 Unconstrained optimal policies

Because of our competitiveness assumptions, even without a climate concern, fossil energy should be eliminated in the long run by renewable energy. On the other hand there is still a minimal delay between the time at which the fossil energy industry stops accumulating capacities and the time at which it begins to dismantle its production capacities. This suggests the study of the following central scenario.

The scenario is made of: (1) A first phase $[0, t^m)$ of capital accumulation in the two energy sectors until the fossil energy sector decides to stabilise its capacity at some level $\hat{K}_x$; (2) A second phase, $[t^m, t^s)$, during which the fossil energy sector keeps constant its production capacity while the renewable energy sector builds up capacities from a level $K_y^m = K_y(t^m)$ up to the level $K_y^s = K_y(t^s)$; (3) A third phase, $[t^s, t^x]$, during which the fossil energy sector scraps its capacities down to 0 at time t_x; (4) A fourth phase, $[t_x, \infty)$ of accumulation of renewable energy capacities up to $\tilde{K}_y$.

The two last phases are similar to the post-ceiling phases already described in Section 4. In the plane (K_x, K_y), we have shown the existence of a decreasing line we called the scraping border. If the maintained capital stock is $\hat{K}_x$ while $K_y^s = K_y(t^s)$, the relationship $u'(\hat{K}_x + K_y^s) = c_x + m_x = \bar{p}_x$ shows that $dK_y^s = -d\hat{K}_x$. We now focus on the stabilisation phase, $[t^m, t^s)$.

Let $T^m \equiv t^s - t^m$ denote the time length of the phase and make the change of the time variable, $v = t - t^m$, where t denotes the calendar time and v, the time elapsed since the beginning of the phase. We want to show the existence of a border in the (K_x, K_y) plane, we call the *stabilisation border*, of equation $K_x = \hat{K}_x(K_y)$, relating K_y to the maximum accumulated capital stock in the fossil energy sector. We expect the curve $\hat{K}_x(K_y)$ to be decreasing and cutting the horizontal axis at K_y^x.

To build the curve we have to determine the vector $(\hat{K}_x, \lambda_{y0}, T^m)$ as a function of a given K_y^m. In Appendix A.3 we show how to determine the vector of variables and deliver in addition the following sensitivity results:

Proposition P. 6. *As a function of K_y^m, the renewable energy production capital stock at the beginning of the stabilisation phase:*

1. *The maintained fossil energy production capital stock is a decreasing function of K_y^m, $d\hat{K}_x/dK_y^m < 0$ and in addition $|d\hat{K}_x/dK_y^m| < 1$.*
2. *The initial price of the renewable energy production capital is a decreasing function of K_y^m: $d\lambda_{y0}/dK_y^m < 0$.*
3. *The time length of the stabilisation phase, T^m, is a decreasing function of K_y^m: $dT^m/dK_y^m < 0$.*

As a function of $\hat{K}_x$, the maintained fossil energy conversion capacity during the stabilisation phase:

1. *The initial price of the renewable energy capital increases with $\hat{K}_x$: $d\lambda_{y0}/d\hat{K}_x > 0$.*
2. *The time length of the stabilisation phase increases: $dT^m/d\hat{K}_x > 0$.*

The Proposition P.6 states the existence of a decreasing relationship between $\hat{K}_x$ and K_y^m. In the plane (K_y, K_x), this relationship describes the *stabilisation border*, defining at which level the fossil energy industry should stabilise its productive capacities conditional on the capital accumulation in the renewable energy industry. Assume that $K_y^0 = 0$, i.e. the renewable energy sector has to build capacities from scratch. Then increasing K_x^0, because the trajectories cannot cross in the phase plane, the fossil energy industry stabilises its capacities at larger levels. As a result, the renewable energy sector accumulates less capacities before the beginning of the scraping phase, K_y^s being decreased by a larger $\hat{K}_x$ and hence K_x^0 along the stabilisation border. The Figure 10 illustrates the construction.

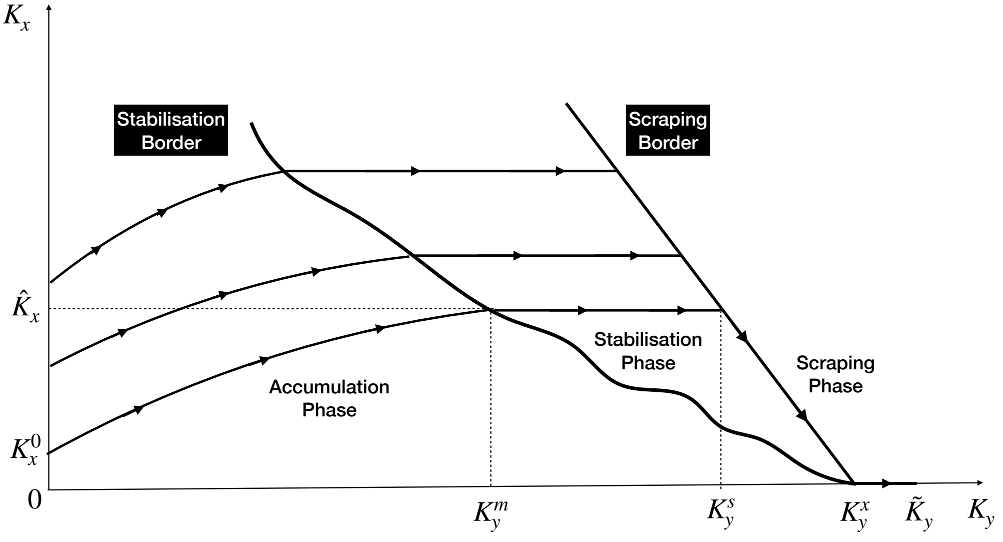

Figure 10: Unconstrained capital accumulation policies in the (K_y, K_x) plane.

Proposition P.6 (claim 2) states that $\lambda_y^m \equiv \lambda_y(t^m)$ is decreased by a larger K_y^m. Thus to a larger K_x^0 corresponds a larger λ_y^m, K_y^m being decreased. Since $\hat{K}_x$ is increased by a larger K_x^0, $K_y^s = K_y(t^s)$ is decreased along the scraping border, thus $\lambda_y^s = \lambda_y(t^s)$ is increased, $\lambda_y(K_y)$ being a decreasing function of K_y along the scraping border, as shown in Section 4. The renewable energy capital price path is thus shifted up by a larger K_x^0 during the stabilisation phase. A positive shock on K_x^0 improves the incumbent advantage of fossil energy versus renewable energy. The result is less accumulation of renewable energy capital before the stabilisation phase and thus an higher capital price in the renewable energy sector during this phase.

The time length of the stabilisation phase is submitted to two opposite pressures after a positive shock on K_x^0. On the one hand, the investment pace of the renewable energy sector is accelerated which should induce a shorter time length of the stabilisation phase. On the other hand, the fossil energy sector faces an increased minimal delay constraint with larger maintained capacities. The Proposition P.6 (claim 3) states that T^m decreases with K_y^m and thus increases with K_x^0 along the stabilisation border, showing that the minimal delay effect is of the first order with respect to the renewable energy investment acceleration effect, resulting in a net lengthening effect on the stabilisation phase after a positive shock on K_x^0.

7.2.1 Environmental consequences of the unconstrained energy transition

We now study the consequences of the investment policy of the fossil energy industry on the accumulation of CO_2 in the atmosphere. Our objective is to identify the set of initial conditions for which the optimally managed economy would not be constrained by the carbon cap, the competitiveness advantage of renewable energy being sufficiently strong for a fast energy transition and the climate constraint being sufficiently lax. Let $Z^m = Z(t^m)$ be the carbon stock size at the beginning of the stabilisation phase and $Z^s = Z(t^s)$ be its size at the end of the phase. Then recalling our change of the time variable and integrating the carbon balance condition over the stabilisation phase, $[0, T^m)$, we obtain:

$$Z^s e^{\alpha T^m} = Z^m + \frac{\zeta r_x}{\alpha} \hat{K}_x \left(e^{\alpha T^m} - 1 \right)$$

Multiplying both sides by α and rearranging yield:

$$\begin{aligned} & \alpha Z^s e^{\alpha T^m} = \alpha Z^m + \zeta r_x \hat{K}_x \left(e^{\alpha T^m} - 1 \right) \\ \iff & \zeta r_x \hat{K}_x - \alpha Z^m = \zeta r_x \hat{K}_x e^{\alpha T^m} - \alpha Z^s e^{\alpha T^m} \\ \iff & \dot{Z}(0) = \dot{Z}(T^m) e^{\alpha T^m} \end{aligned}$$

Because $\dot{Z}(0) > 0$, $\dot{Z}(T^m) > 0$. The carbon stock size is still increasing when the fossil energy sector begins to scrap its capacities. The Figure 11 shows the joint dynamics in the plane (Z, K_x).

A typical trajectory is plotted in the bottom of the Figure, labelled #1. Both $Z(t)$ and $K_x(t)$ increase up to $(Z^m, \hat{K}_x)$ after which $K_x(t)$ is stabilised at the level $\hat{K}_x$ until the beginning of the scraping phase. When this phase begins, $Z(t)$ has reached the level Z^s and goes on increasing while K_x decreases during the scraping process, The carbon stock size peaks at a level $\hat{Z}$ and next declines alongside $K_x(t)$. Observe that for the trajectory #1, $\hat{Z} < \bar{Z}$.

The unconstrained fossil energy capital paths are not capped by $\bar{K}_x$, as in the constrained case. Since we have shown that $\dot{Z}(T^m) > 0$, $(Z(T^m), \bar{K}_x)$ must be located above the $\dot{Z} = 0$ isocline and thus Z^s must be lower than $\bar{Z}$, as illustrated by the trajectory labelled #2 in the Figure 11.

The upper envelope of the unconstrained trajectories, labelled #3 on the Figure, starts from some K_x^{Z0}, uniquely determined since the trajectories are unique from a given initial vector. Then both $Z(t)$ and $K_x(t)$ increase until the fossil energy conversion capacities

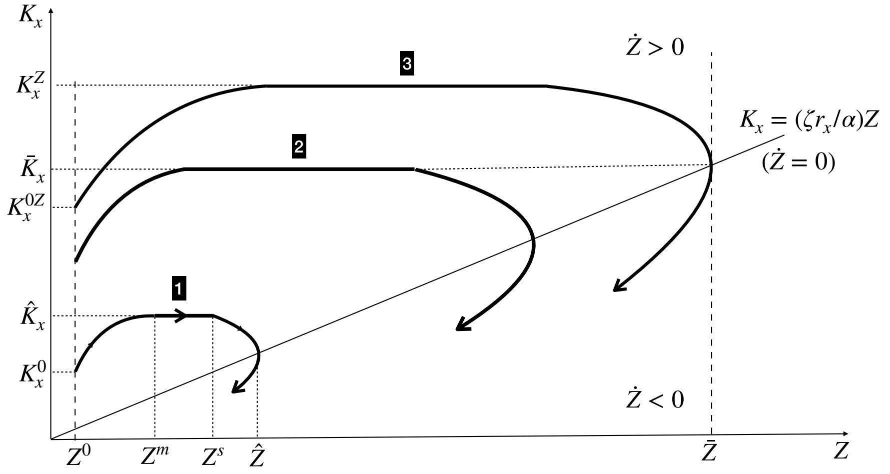

Figure 11: Pollution dynamics in the (K_x, Z) plane.

stabilise at a level $\bar{K}_x^Z$, also uniquely determined. When the scraping process begins, Z^s is still lower than $\bar{Z}$ but now $Z(t)$ peaks exactly at $\bar{Z}$, meaning that the ceiling is met at only one instant, a zero measure event.

We can now complete the mapping of optimal investment policies in the two sectors together with their consequences on the pollution accumulation in the atmosphere. Denote by $K_x^m(K_y)$ the relationship between K_y and K_x represented by the stabilisation border in the capital plane, (K_y, K_x). Denote also by K_x^{m0} the intersection of the curve with the vertical axis, that is $K_x^{0m} = K_x^m(0)$. The Figure 12 pictures the different families of optimal policies and their environmental consequences.

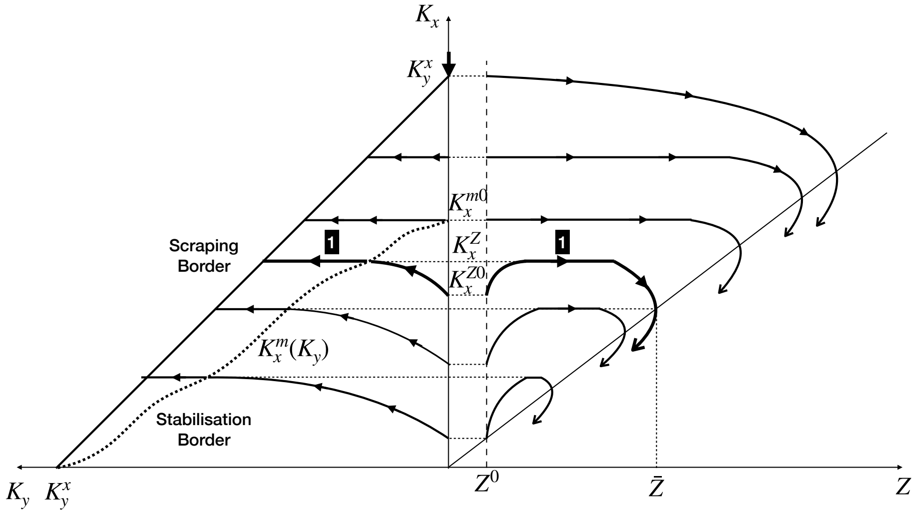

Figure 12: Optimal unconstrained policies.

Keep the idea that the renewable energy sector has to build its capacities from scratch, that is $K_y^0 = 0$. The figure shows the stabilisation border of equation $K_x^m(K_y)$ on the left panel. The curve cuts the vertical axis at K_x^{m0}. The two curves labelled #1 are the upper envelope of unconstrained trajectories. They cut the vertical axis at a level of K_x we denote K_x^Z. If $K_x^0 < K_x^Z$, the energy transition happens without a climate constraint, the economy never exhausting its carbon budget. The Figure assumes that $K_x^Z < K_x^{m0}$, so that the trajectories #1 correspond to our central scenario. If $K_x^{m0} < K_x^Z$, they would correspond to a degenerated two phases scenario, the fossil energy industry never investing in this case. When $K_x^0 > K_x^Z$, the energy transition dynamics overshoots the carbon budget. The optimal paths follow the central scenario when $K_x^0 < K_x^{0m}$, or a two phases path without accumulation of fossil energy conversion capacities when $K_x^{0m} < K_x^0 < K_x^y$. If the fossil energy sector is initially endowed with a production capacity larger than K_y^x, the capital is initially in excess and the sector should scrap immediately the excess capacity, $K_x^0 - K_y^x$, to embark on a progressive scraping path of its capacities along the scraping border.

7.3 General case

The Figure 13 pictures the mapping of the optimal investment scenarios in the two energy sectors in parallel with the accumulation of carbon in the atmosphere. Appendix A.4. provides the computation details of the construction of the figure.

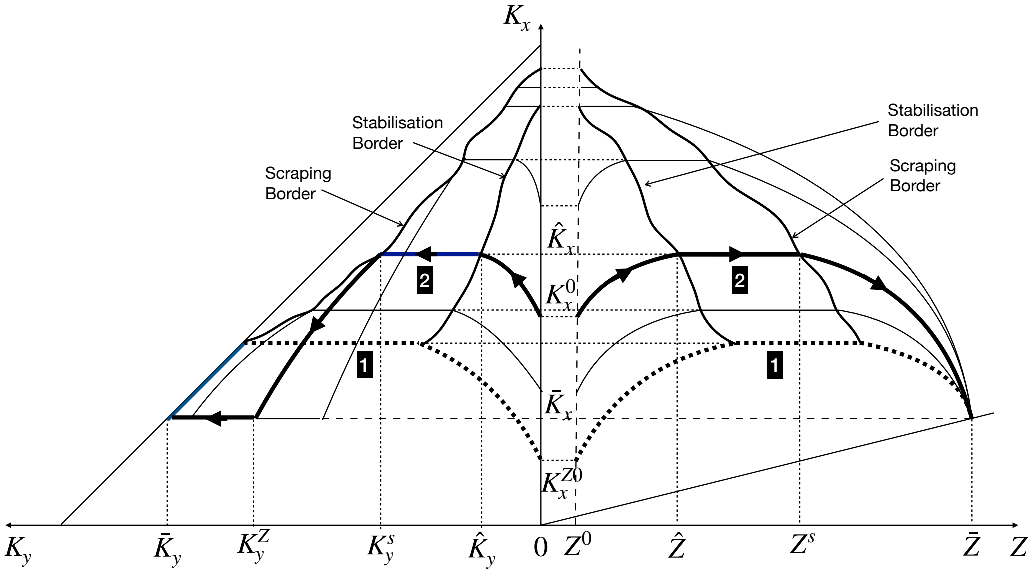

Figure 13: Scenarios 1 and 2 in the (K_x, K_y, Z) space.

The Figure is drawn assuming that the upper envelope of unconstrained trajectories corresponds to a Scenario 2, so that the only possible constrained trajectories correspond to Scenarios of type either 1 or 2, eliminating the possibility of a Scenario 3. The curves labelled #1 are the upper envelopes of unconstrained trajectories in the two phase planes (K_y, K_x) and (Z, K_x) (the left and right panels on the Figure). Assume that the renewable industry starts building capacities from scratch, i.e. $K_y(0) = 0$. When $K_x^0 < K_x^{Z0}$, the energy transition happens without exhausting the carbon budget and society never faces the climate issue. A typical climate constrained central scenario is plotted as the curves labelled #2 on the Figure. Starting from $K_x^0 > K_x^{Z0}$, the fossil energy conversion capacities peak at $\hat{K}_x$ before having to fall down to $\bar{K}_x$, to comply with the net zero constraint. The right panel shows that in parallel, the carbon stock rises first up to $\hat{Z}$ when $K_x(t)$ reaches its peak, next increases up to Z^s while $K_x(t)$ is maintained at $\hat{K}_x$, and last reaches the level $\bar{Z}$ at the end of the scraping phase of fossil energy conversion capital. The left panel illustrates the constant building of renewable energy conversion capacities, starting from a nil level up to $\tilde{K}_y$, when $K_x(t)$ reaches its peak, next increasing up to K_y^s, at the time the fossil energy industry starts scraping its capacities, and last up to K_y^Z when the ceiling regime begins.

The right panel shows that there is almost no qualitative differences in the fossil energy capital dynamics with respect to the benchmark case of a constant renewable energy conversion capital. The phase plane is structured by the scraping and stabilisation borders, both decreasing curves in the plane, separating the different optimal capital management regimes of the fossil energy industry: accumulation, stabilisation or scrapping. Observe that the capital accumulation histories obey the same logic as in the benchmark: browsing through increasing levels of K_x^0, the capital history becomes a degenerated Scenario 2 without accumulation of capacities initially with a sufficiently high initial size of the fossil energy industry, next a scrapping Scenario 1 with an even larger initial size of the fossil energy conversion capital, and last a scrapping scenario with immediate culling of some fraction of the fossil industry capital stock at the beginning of the planning horizon.

The left panel also shows strong analogies with the climate unconstrained benchmark, with a stabilisation border still being a decreasing curve in the plane but with the difference of a scraping border becoming the upper envelope of the capital dynamics trajectories during the scrapping phase. A larger initial fossil energy production capacity means a lower initial energy price. It can be shown that as a consequence, $\lambda_y(0)$ is reduced by a positive shock on K_x^0 triggering a fall of the investment rate. The renewable energy sector has less incentives to build up capacities and the accumulation phase being shortened, the accumulated renewable energy capacity at the end of the phase is smaller, thus $\hat{K}_y$ is reduced by a larger K_x^0, illustrating the initial comparative advantage of the incumbent fossil energy industry with respect to the nascent renewable energy industry. The maintained fossil energy production capital stock being at the same time increased by a larger K_x^0, a negative relationship between $\hat{K}_x$ and $\hat{K}_y$ results, illustrated by the shape of the stabilisation border.

During the stabilisation phase, a larger maintained fossil energy conversion capital stock still dampens the accumulation of renewable energy conversion capacities resulting in a lower capacity, K_y^s at the end of the phase, that is when the scraping phase begins. Because trajectories cannot cross in the phase plane, the renewable energy sector having accumulated less capacities at the beginning of the scrapping phase ends the phase with less capacities, thus K_y^Z is reduced by a larger K_x^0. The negative relationship between the initial fossil energy industry production capacity and the finally accumulated capacity in the renewable energy industry is an important consequence of the climate constraint. Faced with an initially larger incumbent fossil industry and thus a more severe climate burden, society has to make a trade-off between a longer stay under the climate ceiling constraint and a more aggressive pricing of carbon emissions, impeding more burden on early energy consumption by the households. The optimal outcome of this trade-off is to accept less accumulation of capital in the renewable energy sector at the end of the pre-ceiling phase and thus a longer stay under the ceiling constraint since we have shown in Section 5 a negative correlation between the time length of the ceiling regime and the availability of the renewable energy capital at the beginning of the regime. The burden of a larger incumbent fossil energy industry must be split over time between the pre-ceiling phase, the carbon price having to be larger during this phase, and the climate constrained era, through a longer stay under the constraint.

8 Concluding remarks

Throughout its history, the energy industry has continuously taken advantage of technical advances to expand the range of mobilised energy sources under the pressure of an ever increasing energy demand. This move has mostly benefited fossil fuels exploitation with harmful consequences on the planet climate regime. The climate economics literature has devoted much attention to the policies promoting clean energy alternatives but much less to the downside implications of the dismantling of the fossil energy conversion capital. The present paper intends to fill this gap providing a comprehensive study of the joint dynamics of renewable energy conversion capital building and fossil energy conversion capital scraping under a climate constraint in the form of a ceiling on the admissible atmospheric carbon concentrations.

To make explicit the time to build dimension of the energy transition we introduce convex adjustment costs in the investment decision problem of the energy industry. Therefore the issue of the relative competitiveness of the fossil and renewable energy sources is no more reduced to an operating variable cost comparison exercise but has to include the whole profitability patterns of investment and disinvestments plans in energy production from the two crude energy sources.

Under a mild assumption implying a complete transition in finite time to renewable energy, the climate history is made of three episodes: a first one during which the atmospheric carbon stock increases up to the carbon cap, a second one during which the economy is constrained by the cap, waiting for the renewable energy capacity to accumulate up to a critical level allowing an escape from the climate constraint, and last a third climate unconstrained phase over which the fossil energy exploitation industry dismantles its remaining capacity until the complete transition to renewable energy. During the ceiling phase, either the optimal carbon tax is determined and follows a decreasing patten, either it is indeterminate, the only constraint on the regulator being to prevent through the carbon pricing any capacity accumulation by the fossil energy conversion industry. The energy price decreases over the ceiling and post-ceiling phases.

What happens before the climate constrained episode is the most interesting part of the study. The energy price is constantly increasing in the limit case where the fossil energy industry would have to scrap constantly its excess capacity with respect to the climate constraint. This happens because scraping should occur in a more than one to one substitution between fossil energy conversion capacities and renewable energy conversion capacities, inducing an aggregate energy supply contraction sustaining higher energy prices. In most cases, the energy price should first decline before increasing until the beginning of the climate constrained episode. The consequence of this non monotonous patten of the energy price is the possibility of investment waves in the renewable energy industry, investment in this industry accelerating or decelerating with the fluctuations of the energy price.

The capital history of the fossil energy industry is typically made of three phases, first an accumulation phase, followed by a stabilisation phase, ended by a scraping phase down to a production capital stock compatible with the assimilative capacities of the carbon cycle. It is thus generally socially optimal that the fossil energy sector overshoots the sustainable capacity corresponding to the net zero target before the climate constrained episode. The overshooting of fossil energy conversion capacities combined with the progressive deployment of renewable energy conversion capacities fuels a decline of the energy price until the last scrapping phase when energy scarcity increases.

The analysis can be extended in several directions. An empirical assessment of the implications of the theory can be made through numerical simulations. Casual observation reveals that the energy transition pace is faster in developed countries than in developing countries inducing an increasing divergence of the countries energy systems. This could be accommodated with a simple North-South model showing the relocation of clean renewable and fossil energy production capacities throughout the world and their respective dynamics. The present analysis postulates a stationary energy demand function at the global scale and a stationary energy delivery cost structure. The demand and cost structures could be allowed to vary over time, thus taking into account the cost reduction effects of technical progress in the energy provision and the world energy demand proper dynamics. Most climate policies proposals take the form of agendas prescribing carbon emissions cuts to be achieved before a given year. The present study imposes a given stabilisation target on the atmospheric carbon concentrations but leaves free the time at which society should face the constraint. Incorporating explicitly time limits on the achievement of climate objectives would bring the analysis closer to the current state-of-the-art in climate governance. We leave these issues for future research.

References

Acemoglu D., Aghion P., Bursztyn L., and D. Hemous, (2012), The Environment and Directed Technical Change, *American Economic Review*, 102 (1), 131-66.

Acemoglu D., Aghion P., Barrage L. and D. Hemous, (2023), Climate change, directed innovation, and energy transition: the long-run consequences of the shale gas revolution, NBER, WP 31657.

Amigues, J. P., and G. Lafforgue, (2025), Optimal adaptation policies under a carbon budget constraint, *Resource and Energy Economics*,Volume 82, 101481.

Amigues, J. P., Ayong Le Kama, A. and M. Moreaux, (2015), Equilibrium transitions from non-renewable energy to renewable energy under capacity constraints, *Journal of Economic Dynamics and Control*, 55, issue C, 89-112.

Arrow,K.,1968. Optimal capital policy with irreversible investment. In: Wolfe, J.N. (Ed.), *Value, Capital and Growth*, Essays in Honour of Sir John Hicks. Edinburgh University Press, Edinburgh.

Atkeson, A. and Kehoe, P. J. (1999). Models of energy use: Putty-putty versus putty-clay., *American Economic Review*, 89(4), 1028–1043.

Baldwin E., Cai Y., and Kuralbayeva K., (2020), To build or not to build? Capital stocks and climate policy, *Journal of Environmental Economics and Management*, Volume 100, 2020, 102235.

Chakravorty U., Magné B., and M. Moreaux, (2006), A Hotelling model with a ceiling on the stock of pollution, *Journal of Economic Dynamics and Control*, 30(12), 2875-2904.

Coulomb R., Lecuyer O., and A. Vogt-Schilb, (2019), Optimal Transition from Coal to Gas and Renewable Power Under Capacity Constraints and Adjustment Costs, *Environmental and Resource Economics*, 73, issue 2, 557-590.

Curtin J., McInerney C., B. Gallachóir O., Hickey C., Deane P., and P. Deeney, (2019), Quantifying stranding risk for fossil fuel assets and implications for renewable energy investment: A review of the literature, *Renewable and Sustainable Energy Reviews*, Volume 116, 2019, 109402.

Eichner, T., Kollenbach G., and M. Schopf, (2023), Demand- Versus Supply-Side Climate Policies with a Carbon Dioxide Ceiling, *The Economic Journal*, 133(652), 1371–1406.

Fischer, C., Withagen, C. and Toman, M., (2004), Optimal Investment in Clean Production Capacity. *Environmental and Resource Economics* 28, 325–345

Fowlie, M., Reguant, M., and Ryan, S. P. (2016). Market-based emissions regulation and industry dynamics. *Journal of Political Economy*, 124(1):249–302.

Hassler J., Krusell P., and C. Olovsson, (2021), Directed Technical Change as a Response to Natural Resource Scarcity *Journal of Political Economy* 2021, 129, 11, 3039-3072.

Hawkins-Pierot J. T. and K. R. H. Wagner, (2024), Technology Lock-In and Costs of Delayed Climate Policy, mimeo, September 3, 2024.

Johnson N., Krey V., David L. McCollum D. L., Rao S., Riahi K., and J.Rogelj, (2015), Stranded on a low-carbon planet: Implications of climate policy for the phase-out of coal-based power plants, *Technological Forecasting and Social Change*, Volume 90, 2015, Part A, 89-102.

Kumar Saha C., and A. V. Carter, (2022), Phase-out or lock-in fossil fuels? Least developed countries' burning dilemma, *The Extractive Industries and Society*, Volume 11, 2022, 101140.

Kollenbach, G., (2017). On the optimal accumulation of renewable energy generation capacity. *Journal of Economic Dynamics and Control*, 77, 157-179.

Lennox J. A., Witajewski-Baltvilks J., (2017), Directed technical change with capital-embodied technologies: Implications for climate policy, *Energy Economics*, Volume 67, 2017, 400-409.

Moreaux M., Amigues J. P., van der Meijden G., and C. Withagen, (2024), Carbon capture: Storage vs. Utilization, *Journal of Environmental Economics and Management*, Volume 125, 102976.

Pindyck, R. S. and Rotemberg, J. J. (1983). Dynamic factor demands and the effects of energy price shocks. *American Economic Review*, 73(5):1066–1079.

Pommeret A., Schubert K., (2022), Optimal energy transition with variable and intermittent renewable electricity generation, *Journal of Economic Dynamics and Control*, Volume 134, 2022, 104273,

Rozenberg J., A. Vogt-Schilb A., and S. Hallegatte, (2020), Instrument choice and stranded assets in the transition to clean capital, *Journal of Environmental Economics and Management*, Vol 100, 102183,

Seto K. C., Davis S. J., Mitchell R. B., Stokes E. C., Unruh G., and D. Ürge-Vorsatz, (2016), Carbon Lock-In: Types, Causes, and Policy Implications, *Annual Review of Environment and Resources* Volume 41, October 2016, 425-452.

Unruh G.,(2000), Understanding carbon lock-in, *Energy Policy*, Volume 28, Issue 12, 817-830,

van der Ploeg F., and Armon Rezai, (2017), Cumulative emissions, unburnable fossil fuel, and the optimal carbon tax, *Technological Forecasting and Social Change*, Volume 116, 2017, 216-222.

van der Ploeg F., (2020), Race to burn the last ton of carbon and the risk of stranded assets, *European Journal of Political Economy*, Volume 64, 2020, 101915.

van Sluisveld M. A. E., Harmsen J. H. M., Bauer N., David L. McCollum L., Riahi K., Tavoni M., Detlef P., van Vuuren D. P., Wilson C., van der Zwaan B., (2015), Comparing future patterns of energy system change in 2°C scenarios with historically observed rates of change, *Global Environmental Change*, Volume 35, 2015, 436-449.

Wirl, F. ,(1991), Monopolistic Resource Extraction and Limit Pricing: the Market Penetration of Competitively Produced Synfuels, *Environmental and Resource Economics* 1, 157–178.

Wirl, F., Withagen, C., (2000), Complexities due to sluggish expansion of backstop technologies. *Zeitschr. f. Nationalökonomie, 'Journal of Economics'* 72, 153–174.

Appendix

A.1 Closed form solutions

We develop in the Appendix A., algorithmic arguments able to compute the optimal paths in the different scenarios. In the next section we examine the domain of validity of the scenarios, thus the algorithmic procedures outlined below are relevant only under initial state restrictions to be determined later on.

A.1.1 The Scenario 1

Consider first the Scenario 1. This scenario can be possibly optimal only when $K_x^0 > \bar{K}_x$. The fossil energy sector constantly scraping capacities before $\underline{t}_Z$, $\lambda_x(t) = 0$ and $\beta_x(t) = 0$. Then β_y is implicitly determined as a function of (K_y, λ_Z) by the relationship:

$$\beta_y = c_x + \zeta r_x \lambda_Z - a_y - b_y'(K_y) - m_y \tag{A.1.1}$$

On the other hand, the fossil energy industry capacity is defined as a function of (K_y, λ_Z) by the relationship:

$$u'(K_x + K_y) = c_x + \zeta r_x \lambda_Z \tag{A.1.2}$$

Invert the time order by setting: $v = \underline{t}_Z - t$. Then β_y being defined as a function of (K_y, λ_Z) by (A.1.1), the optimal path $\{K_y(v), \lambda_y(v), \lambda_Z(v)\}$ is the general solution of the autonomous differential system:

$$\begin{aligned}
\dot{K}_y(v) &= -k_y(\lambda_y(v)) \\
\dot{\lambda}_y(v) &= -\rho\lambda_y(v) + \beta_y(K_y(v), \lambda_Z(v)) \\
\dot{\lambda}_Z(v) &= -(\rho + \alpha)\lambda_Z(v)
\end{aligned}$$

In addition $K_x(v) = K_x(K_y(v), \lambda_Z(v))$ defined by (A.1.2).

Pick some vector $(K_y^Z, \bar{\lambda}_Z)$. Remember that the study of the ceiling phase has shown the existence of a decreasing relationship between λ_y and K_y, we denoted, $\lambda_y^*(K_y)$. At the 'initial' time, $v = 0$, $\lambda_y(0) = \lambda_y^*(K_y(0))$ and to $(K_y^Z, \bar{\lambda}_Z)$, is associated the particular solution of the differential system at this time: $(K_y(0), \lambda_y(0), \lambda_Z(0)) = (K_y^Z, \lambda_y^*(K_y^Z), \bar{\lambda}_Z)$. The system admits a unique solution for this particular solution. Let $\{K_y(v; K_y^Z, \bar{\lambda}_Z)$, $\lambda_y(v; K_y^Z, \bar{\lambda}_Z), \lambda_Z(v; K_y^Z, \bar{\lambda}_Z)\}$ denote the general solution as a function of the given vector $(K_y^Z, \bar{\lambda}_Z)$. Last the vector $(K_y^Z, \bar{\lambda}_Z)$ and $\underline{t}_Z$ are determined by the system of three conditions:

1. The initial condition on K_x.

$$K_x(K_y(\underline{t}_Z; K_y^Z, \bar{\lambda}_Z), \lambda_Z(\underline{t}_Z; K_y^Z, \bar{\lambda}_Z)) = K_x^0$$

2. The initial condition on K_y:

$$K_y(\underline{t}_Z; K_y^Z, \bar{\lambda}_Z) = K_y^0$$

3. The carbon ceiling attainment condition.

$$\bar{Z} = Z^0 e^{-\alpha \underline{t}_Z} + \zeta r_x \int_0^{\underline{t}_Z} K_x(K_y(v; K_y^Z, \bar{\lambda}_Z), \lambda_Z(v; K_y^Z, \bar{\lambda}_Z)) e^{-\alpha v} dv$$

A.1.2 Scenario 2

The Scenario 2 is composed of 3 phases. During the first phase, $[0, t^m)$, $k_x(t) > 0$. During the second phase, $[t^m, T^s)$, $K_x(t)$ is maintained at a constant level $\hat{K}_x > \bar{K}_x$ so that $k_x(t) = 0$. During the third phase, $[T^s, \underline{t}_Z)$, the fossil energy conversion sector scraps the extra capacity, $\hat{K}_x - \bar{K}_x$. Assume to simplify that the conditions are satisfied for $\beta_y(t)$ to be larger than $\rho \underline{n}_y$, so that $k_y(t) > 0$, $t \in [0, \underline{t}_Z)$, as shown before. It is possible to compute the optimal path through the following argument.

Invert the time order by setting $v = t^m - t$. During the first time period, $[0, t^m)$, β_x and β_y are defined by:

$$\beta_x = \beta_x(K_x, K_y, \lambda_Z) = u'(K_x + K_y) - c_x - m_x - \zeta r_x \lambda_Z \tag{A.1.3}$$

$$\beta_y = \beta_y(K_x, K_y) = u'(K_x + K_y) - a_y - m_y - b_y'(K_y) \tag{A.1.4}$$

and $k_x(v) > 0$, together with $k_y(v) > 0$. Pick a vector $\hat{\omega} \equiv (\hat{K}^x, \hat{K}_y, \hat{Z}, \underline{n}_x, \hat{\lambda}_y, \hat{\lambda}_Z)$. Then the optimal path is the general solution of the 6 dimensions autonomous differential system:

$$\begin{aligned}
\dot{K}_x(v) &= -k_x(\lambda_x(v)) \\
\dot{K}_y(v) &= -k_y(\lambda_y(v)) \\
\dot{Z}(v) &= -\zeta r_x K_x(v) + \alpha Z(v) \\
\dot{\lambda}_x(v) &= -\rho \lambda_x(v) + \beta_x(K_x(v), K_y(v), \lambda_Z(v)) \\
\dot{\lambda}_y(v) &= -\rho \lambda_y(v) + \beta_y(K_x(v), K_y(v)) \\
\dot{\lambda}_Z(v) &= -(\rho + \alpha) \lambda_Z(v)
\end{aligned}$$

With the particular solution at the 'initial' time, $v = 0$, $\hat{\omega}$, this system admits a unique general solution. Then the vector $(\hat{\omega}, t^m)$ is constrained by the set of conditions:

$$\begin{aligned}
K_x(t^m; \hat{\omega}) &= K_x^0 \\
K_y(t^m; \hat{\omega}) &= K_y^0 \\
Z(t^m; \hat{\omega}) &= Z^0
\end{aligned}$$

The restriction set allows expressing t^m and 2 elements of $\hat{\omega}$ as functions of 3 elements of $\hat{\omega}$, the last element of $\hat{\omega}$ being $\underline{n}_x$, a parametric constant. Let $\hat{\omega}^M$ be the given vector of the state variables at the 'initial' time, $v = 0$: $\hat{\omega}^M = (\hat{K}_x, \hat{K}_y, \hat{Z})$. Then the restrictions imply that $(\hat{\lambda}_y, \hat{\lambda}_Z, t^m)$ can be expressed as functions of $\hat{\omega}^M$ and denote by a slight abuse of notations, $\hat{\lambda}_y = \lambda_y^M(\hat{\omega}^M)$, $\hat{\lambda}_Z = \lambda_Z^M(\hat{\omega}^M)$ and $t^m = t_M^M(\hat{\omega}^M)$ these functions.

Invert the time order by setting $v = T^s - t$ and denote by $\hat{T} = T^s - t^m$, the time length of the stabilisation phase. During the second time phase, $[t^m, T^s)$, the industry maintains at a constant level the fossil energy conversion capacity. Pick some vector $\omega^S = (K_y^S, Z^S, 0, \lambda_y^S, \lambda_Z^S)$. At the 'initial' time, $v = 0$, $\beta_x(0) = 0$ because $\lambda_x(0) = 0$ and $\dot{\lambda}_x(0) = 0$. Thus $\hat{K}_x$ can be expressed as a function of K_y^S and λ_Z^S through the optimality condition: $u'(\hat{K}_x + K_y^S) = c_x + \zeta r_x \lambda_Z^S + m_x$. Let $\hat{K}_x(K_y^S, \lambda_Z^S)$ denote this relationship. β_x and β_y are still defined by (A.1.3), (A.1.4).

Then the optimal path during the stabilisation phase is the general solution of the differential system:

$$\begin{aligned}
&\dot{K}_y(v) = -k_y(\lambda_y(v)) \\
&\dot{Z}(v) = -\zeta \hat{K}_x(K_y^S, \lambda_Z^S) + \alpha Z(v) \\
&\dot{\lambda}_x(v) = -\rho \lambda_x(v) + \beta_x(\hat{K}_x(K_y^S, \lambda_Z^S), K_y(v), \lambda_Z(v)) \\
&\dot{\lambda}_y(v) = -\rho \lambda_y(v) + \beta_y(\hat{K}_x(K_y^S, \lambda_Z^S), K_y(v)) \\
&\dot{\lambda}_Z(v) = -(\rho + \alpha) \lambda_Z(v)
\end{aligned}$$

With the particular solution at the 'initial' time, $v = 0$, ω^S, this system admits a unique solution. The vectors ω^S and ω^M must comply to the following set of restrictions:

$$\begin{aligned}
&K_y(\hat{T}; \omega^S) = \hat{K}_y \\
&Z(\hat{T}; \omega^S) = \hat{Z} \\
&\lambda_x(\hat{T}; \omega^S) = \underline{n}_x \\
&\lambda_y(\hat{T}; \omega^S) = \hat{\lambda}_y = \lambda_y^M(\omega^M) = \lambda_y^M(\hat{K}_x(K_y^S, \lambda_Z^S), \hat{K}_y, \hat{Z}) \\
&\lambda_Z(\hat{T}; \omega^S) = \hat{\lambda}_Z = \lambda_Z^M(\omega^M) = \lambda_Z^M(\hat{K}_x(K_y^S, \lambda_Z^S), \hat{K}_y, \hat{Z}) \\
&\hat{T} = T^s - t_M^M(\omega^M) = T^s - t_M^M(\hat{K}_x(K_y^S, \lambda_Z^S), \hat{K}_y, \hat{Z})
\end{aligned}$$

Denote by θ, the vector $\theta = (\hat{K}_x, \hat{K}_y, \hat{Z}, K_y^S, Z^S, 0, \lambda_y^S, \lambda_Z^S, T^s)$. Equivalently, $\theta = (\omega^M, \omega^S, T^s)$. Through the 6 restrictions, it is possible to express 6 components of the vector θ as functions of the 2 other components. Let $\theta^S \equiv (K_y^S, \bar{Z}^S)$. Then $(\hat{K}_x, \hat{K}_y, \hat{Z}, \lambda_y^S, \lambda_Z^S, T^s)$, that is $(\omega^M, \lambda_y^S, \lambda_Z^S, T^s)$ can be expressed as functions of θ^S, and let by a slight abuse of notations, $\hat{K}_x = K_x^S(\theta^S)$, $\hat{K}_y = K_y^S(\theta^S)$, $\hat{Z} = Z^S(\theta^S)$, equivalently $\omega^M = \omega^M(\theta^S)$, $\lambda_y^S = \lambda_y^S(\theta^S)$, $\lambda_Z^S = \lambda_Z^S(\theta^S)$, $T^s = T^s(\theta^S)$, denote these functions.

Last, during the third time phase, $[T^s, \underline{t}_Z)$, the fossil industry sector scraps its productive capacities down to $\bar{K}_x$, to comply with the carbon ceiling, while the renewable energy sector continues to build up capacities. $\beta_x = 0$ and β_y is given by (A.1.1). The fossil energy conversion capacity at time t is defined by: $u'(K_x + K_y) = c_x + \zeta r_x \lambda_Z + m_x$ as a function of K_y and λ_Z, let $K_x(K_y, \lambda_Z)$ denote this relationship.

Pick some vector $(K_y^Z, \bar{\lambda}_Z)$ and some time $\underline{t}_Z$. Invert the time order and take $\underline{t}_Z$ as the time origin by setting $v = \underline{t}_Z - t$.

The optimal path is the solution of the autonomous differential system:

$$\begin{aligned}
&\dot{K}_y(v) = -k_y(\lambda_y(v))\\
&\dot{Z}(v) = -\zeta K_x(K_y(v), \lambda_Z(v)) + \alpha Z(v)\\
&\dot{\lambda}_y(v) = -\rho\lambda_y(v) + \beta_y(K_y(v), \lambda_Z(v))\\
&\dot{\lambda}_Z(v) = -(\rho+\alpha)\lambda_Z(v)
\end{aligned}$$

With the particular solution at the 'initial' time, $v = 0$, $\bar{\omega} \equiv (K_y^Z, \bar{Z}, \lambda_y^*(K_y^Z), \bar{\lambda}_Z)$, this system admits a unique solution. The vector $\bar{\theta} \equiv (K_y^Z, \bar{\lambda}_Z, \underline{t}_Z)$ must comply with the set of restrictions:

$$\begin{aligned}
&K_y(\underline{t}_Z - T^s(\theta^S); \bar{\theta}) = K_y^S\\
&Z(\underline{t}_Z - T^s(\theta^S); \bar{\theta}) = Z^S\\
&K_x\left(K_y(\underline{t}_Z - T^s(\theta^S); \bar{\theta}), \lambda_Z(\underline{t}_Z - T^s(\theta^S); \bar{\theta})\right)\\
&= \hat{K}_x = K_x^S(\theta^S)\\
&K_x(K_y^Z, \bar{\lambda}_Z) = \bar{K}_x\\
&\lambda_y(\underline{t}_Z - T^s(\theta^S); \bar{\theta}) = \hat{\lambda}_y = \lambda_y^S(\theta^S)
\end{aligned}$$

This system of 5 restrictions determines the vector $\bar{\theta}^Z = (K_y^Z, K_y^S, Z^S, \bar{\lambda}_Z, \underline{t}_Z) = (\theta^S, \bar{\theta})$.

A.1.3 The Scenario 3

Consider now the scenario 3, where the two energy conversion sectors accumulate capacities up to $(\bar{K}_x, K_y^Z)$. Both $\beta_x(t)$ and $\beta_y(t)$ decrease before $\underline{t}_Z$ and the same applies to $\lambda_x(t)$ and $\lambda_y(t)$. The corresponding investment scenario is one of an initial positive impulse in investments initially, $k_x(0^+) > 0$ and $k_y(0^+) > 0$, followed by a constant decline of the investment efforts in both sectors, $\dot{k}_x(t) < 0$ and $\dot{k}_y(t) < 0$, $t \in [0, \underline{t}_Z)$.

The optimal policy can be identified as follows. Let $\beta_x(K_x, K_y, \lambda_Z)$ be the function implicitly defined by (A.1.3) and $\beta_y(K_x, K_y)$ the function implicitly defined by (A.1.4). Invert the time order by setting $v = \underline{t}_Z - t$. The optimal path $\{K_x(v), K_y(v), \lambda_x(v), \lambda_y(v), \lambda_Z(v)\}$ is the general solution of the autonomous differential system:

$$\begin{aligned}
&\dot{K}_x(v) = -k_x(\lambda_x(v))\\
&\dot{K}_y(v) = -k_y(\lambda_y(v))\\
&\dot{\lambda}_x(v) = -\rho\lambda_x(v) + \beta_x(K_x(v), K_y(v), \lambda_Z(v))\\
&\dot{\lambda}_y(v) = -\rho\lambda_y(v) + \beta_y(K_x(v), K_y(v))\\
&\dot{\lambda}_Z(v) = -(\rho+\alpha)\lambda_Z(v)
\end{aligned}$$

Pick some given vector $(K_y^Z, \bar{\lambda}_Z)$. The vector determines a particular solution of the differential system at the 'initial' time, $v = 0$, with $K_x(0) = \bar{K}_x$, $K_y(0) = K_y^Z$, $\lambda_x(0) = \underline{n}_x$, $\lambda_y(0) = \lambda_y^*(K_y^Z)$, $\lambda_Z(0) = \bar{\lambda}_Z$. We denote by $\lambda_y^*(K_y)$ the implicit relationship between

λ_y and K_y along the optimal path during the ceiling phase. We know that $\lambda_y^*(K_y)$ is a strictly decreasing function of K_y, so that $\lambda_y(0)$ is well defined as a function of K_y^Z.

The Lipschitz condition being satisfied, the differential system admits a unique solution given the particular solution $(\bar{K}_x, K_y^Z, \underline{n}_x, \lambda_y^*(K_y^Z), \bar{\lambda}_Z)$. The optimal vector $(K_y^Z, \bar{\lambda}_Z, \underline{t}_Z)$ is determined by the system of 3 conditions:

1. The initial condition on K_x.
$$K_x(\underline{t}_Z) = K_x^0$$
2. The initial condition on K_y:
$$K_y(\underline{t}_Z) = K_y^0$$
3. The carbon ceiling attainment condition.
$$\bar{Z} = Z^0 e^{-\alpha \underline{t}_Z} + \zeta r_x \int_0^{\underline{t}_Z} K_x(v) e^{-\alpha v} dv$$

A.2 Proof of the Proposition P.5

A.2.1 Expressions of $d\lambda_x(t)$ and $dK_x(t)$

We first compute expressions of $d\lambda_x(t)$ and $dK_x(t)$ as functions of the initial vector $(dK_x^0, d\hat{K}_y, d\lambda_{x0}, d\lambda_{Z0})$ for further use. The capital accumulation equation reads:

$$K_x(t) = K_x^0 + \int_0^t k_x(\lambda_x(\tau)) d\tau$$

Differentiating this equation, we obtain:

$$dK_x(t) = dK_x^0 + \int_0^t k_x'(\tau) d\lambda_x(\tau) d\tau \tag{A.2.1}$$

Integrating over a given time interval, $[t_0, t) \subset [0, t^m)$, $\dot{\lambda}_x = \rho\lambda_x - \beta_x$, yields:

$$\lambda_x(t) e^{-\rho t} = \lambda_x(t_0) e^{-\rho t_0} - \int_{t_0}^t \beta_x(\tau) e^{-\rho\tau} d\tau$$

Differentiating, we obtain:

$$\begin{aligned} d\lambda_x(t) e^{-\rho t} &= d\lambda_x(t_0) e^{-\rho t_0} - \int_{t_0}^t d\beta_x(\tau) e^{-\rho\tau} d\tau = \\ d\lambda_x(t_0) e^{-\rho t_0} &- \int_{t_0}^t u''(\tau) e^{-\rho\tau} dK_x(\tau) d\tau + \left(\int_{t0}^t e^{\alpha\tau} d\tau \right) \zeta r_x d\lambda_{Z0} \end{aligned}$$

Denote:

$$U_\rho(h_0, h_1) \equiv -\int_{h_0}^{h_1} u''(\tau) e^{-\rho\tau} d\tau > 0 \quad \text{and} \quad I_\alpha(h_0, h_1) \equiv \int_{h_0}^{h_1} e^{\alpha\tau} d\tau$$

then the previous relation writes in the more compact form:

$$d\lambda_x(t)e^{-\rho t} = d\lambda_x(t_0)e^{-\rho t_0} + I_\alpha(t_0,t)\zeta r_x d\lambda_{Z0} + \int_{t_0}^{t} |u''(\tau)|e^{-\rho\tau} dK_x(\tau)d\tau$$

Using the expression (A.2.1) of $dK_x(\tau)$:

$$\begin{aligned} d\lambda_x(t)e^{-\rho t} &= d\lambda_x(t_0)e^{-\rho t_0} + I_\alpha(t_0,t)\zeta r_x d\lambda_{Z0} \\ &+ U_\rho(t_0,t)dK_x(t_0) + \int_{t_0}^{t} |u''(\tau)|e^{-\rho\tau} \int_{t_0}^{\tau} k_x'(s)d\lambda_x(s)dsd\tau \end{aligned}$$

Integrating over $[t_0, t^m)$ gets:

$$\begin{aligned} \int_{t_0}^{t^m} d\lambda_x(t)e^{-\rho t}dt &= d\lambda_x(t_0)e^{-\rho t_0}(t^m - t_0) + \left(\int_{t_0}^{t^m}\int_{t_0}^{t} e^{\alpha\tau}d\tau dt\right)\zeta r_x d\lambda_{Z0} \\ &+ \left(\int_{t_0}^{t^m}\int_{t_0}^{t} |u''(\tau)|e^{-\rho\tau}d\tau dt\right) dK_x(t_0) + \int_{t_0}^{t^m}\int_{t_0}^{t} |u''(\tau)|e^{-\rho\tau}\int_{t_0}^{\tau} k_x'(s)d\lambda_x(s)dsd\tau dt \end{aligned}$$

Inverting the integration order of the three double integrals in the r.h.s.:

$$\begin{aligned} &\int_{t_0}^{t^m} d\lambda_x(t)e^{-\rho t}dt = \\ &d\lambda_x(t_0)e^{-\rho t_0}(t^m - t_0) + \left(\int_{t_0}^{t^m} e^{\alpha t}(t^m - t)dt\right)\zeta r_x d\lambda_{Z0} \\ &+ \left(\int_{t_0}^{t^m} |u''(t)|e^{-\rho t}(t^m - t)dt\right) dK_x(t_0) + \int_{t_0}^{t^m} |u''(t)|e^{-\rho t}(t^m - t)\int_{t_0}^{t} k_x'(\tau)d\lambda_x(\tau)d\tau dt \end{aligned}$$

Inverting the integration order in the last double integral yields:

$$\begin{aligned} &\int_{t_0}^{t^m} d\lambda_x(t)e^{-\rho t}dt = \\ &d\lambda_x(t_0)e^{-\rho t_0}(t^m - t_0) + \left(\int_{t_0}^{t^m} e^{\alpha t}(t^m - t)dt\right)\zeta r_x d\lambda_{Z0} \\ &+ \left(\int_{t_0}^{t^m} |u''(t)|e^{-\rho t}(t^m - t)dt\right) dK_x(t_0) + \int_{t_0}^{t^m} k_x'(t)d\lambda_x(t)\int_{t}^{t^m} |u''(\tau)|e^{-\rho\tau}(t^m - \tau)d\tau dt \end{aligned}$$

Rearranging, we obtain:

$$\begin{aligned} &\int_{t_0}^{t^m} d\lambda_x(t)\left[e^{-\rho t} - k_x'(t)\int_{t}^{t^m} |u''(\tau)|e^{-\rho\tau}(t^m - \tau)d\tau\right]dt = \\ &d\lambda_x(t_0)e^{-\rho t_0}(t^m - t_0) + \left(\int_{t_0}^{t^m} e^{\alpha t}(t^m - t)dt\right)\zeta r_x d\lambda_{Z0} \\ &+ \left(\int_{t_0}^{t^m} |u''(t)|e^{-\rho t}(t^m - t)dt\right) dK_x(t_0) \end{aligned}$$

Differentiating with respect to t_0 for given $dK_x(t_0) \equiv dK_x^0$ and t^m yields:

$$\begin{aligned}
&- d\lambda_x(t_0)\left[e^{-\rho t_0} - k_x'(t_0)\int_{t_0}^{t^m} |u''(\tau)|e^{-\rho\tau}(t^m-\tau)d\tau\right] = \\
&\frac{d}{dt_0}d\lambda_x(t_0)e^{-\rho t_0}(t^m - t_0) - \rho d\lambda_x(t_0)e^{-\rho t_0}(t^m-t_0) - d\lambda_x(t_0)e^{-\rho t_0} \\
&- |u''(t_0)|e^{-\rho t_0}(t^m-t_0)dK_x^0 - e^{\alpha t_0}(t^m-t_0)\zeta r_x d\lambda_{Z0}
\end{aligned}$$

Simplifying the $-d\lambda_x(t_0)e^{-\rho t_0}$ term and dividing both sides by $(t^m - t_0)e^{-\rho t_0}$ gets:

$$\begin{aligned}
&d\lambda_x(t_0)\frac{k_x'(t_0)e^{\rho t_0}}{t^m - t_0}\int_{t_0}^{t^m} |u''(\tau)|e^{-\rho\tau}(t^m-\tau)d\tau = \\
&\frac{d}{dt_0}d\lambda_x(t_0) - \rho d\lambda_x(t_0) - |u''(t_0)|dK_x^0 - e^{(\rho+\alpha)t_0}\zeta r_x d\lambda_{Z0}
\end{aligned}$$

Let $\theta_x \equiv d\lambda_x$. The above expression is a differential equation of the form:

$$\begin{aligned}
\dot{\theta}_x(t_0) = \theta_x(t_0)&\left[\rho + \frac{k_x'(t_0)e^{\rho t_0}}{t^m - t_0}\int_{t_0}^{t^m} |u''(\tau)|e^{-\rho\tau}(t^m-\tau)d\tau\right] \\
&+ |u''(t_0)|dK_x^0 + e^{(\rho+\alpha)t_0}\zeta r_x d\lambda_{Z0}
\end{aligned}$$

Let:

$$a(t) \equiv \rho + \frac{k_x'(t)e^{\rho t}}{t^m - t}\int_{t}^{t^m} |u''(\tau)|e^{-\rho\tau}(t^m-\tau)d\tau \text{ and } A(t_0,t_1) \equiv \int_{t_0}^{t_1} a(\tau)d\tau$$

Because $\lambda_x(t^m) = \underline{n}_x$, we get the particular solution $\theta_x(t^m) = 0$. Hence integrating over $[t, t^m)$ yields:

$$\begin{aligned}
&\theta_x(t^m) = 0 = e^{A(t,t^m)} \\
&\left[\theta_x(t) + \left(\int_t^{t^m} |u''(\tau - t)|e^{-A(t,\tau)}d\tau\right)dK_x^0 + \zeta r_x\left(\int_t^{t^m} e^{(\rho+\alpha)(\tau-t)-A(t,\tau)}d\tau\right)d\lambda_{Z0}\right] \\
&\implies \theta_x(t) = -\left(\int_t^{t^m} |u''(\tau - t)|e^{-A(t,\tau)}d\tau\right)dK_x^0 - \zeta r_x\left(\int_t^{t^m} e^{(\rho+\alpha)(\tau-t)-A(t,\tau)}d\tau\right)d\lambda_{Z0}
\end{aligned}$$

Let:

$$U^A(t,t^m) \equiv \int_t^{t^m} |u''(\tau - t)|e^{-A(t,\tau)}d\tau \text{ and } I_\alpha^A(t,t^m) \equiv \int_t^{t^m} e^{(\rho+\alpha)(\tau-t)-A(t,\tau)}d\tau$$

then:

$$d\lambda_x(t) = -U^A(t,t^m)dK_x^0 - \zeta r_x I_\alpha^A(t,t^m)d\lambda_{Z0}$$

This expression gets the following expression of $dK_x(t)$:

$$\begin{aligned}
dK_x(t) &= dK_x^0 + \int_0^t k_x'(\tau)d\lambda_x(\tau)d\tau \\
&= dK_x^0\left[1 - \int_0^t k_x'(\tau)U^A(\tau,t^m)d\tau\right] - \zeta r_x\left(\int_0^t k_x'(\tau)I_\alpha^A(\tau,t^m)d\tau\right)d\lambda_{Z0}
\end{aligned}$$

Let:

$$B_K(t) \equiv \int_0^t k_x'(\tau)U^A(\tau,t^m)d\tau \quad\text{and}\quad B_{\lambda Z}(t) \equiv \int_0^t k_x'(\tau)I_\alpha^A(\tau,t^m)d\tau$$

give the compact expression:

$$dK_x(t) = \left(1 - B_K(t)\right)dK_x^0 - \zeta r_x B_{\lambda Z}(t)d\lambda_{Z0}$$

System determining the optimum. Existence and unicity of the solution

The triplet $(d\hat{K}_x, d\lambda_{Z0}, dt^m)$ is the solution of the system of three conditions:

1. The continuity condition on $K_x(t)$ at t^m: $\hat{K}_x = K_x(t^m)$. Differentiating yields $d\hat{K}_x = dK_x(t^m)$ since $k_x(t^m) = 0$, thus denoting $B_K = B_K(t^m)$ and $B_{\lambda Z} = B_{\lambda Z}(t^m)$:
$$d\hat{K}_x + \zeta r_x B_{\lambda Z} d\lambda_{Z0} = (1 - B_K)\, dK_x^0$$

2. The carbon stock balance condition:
$$\hat{Z}(\hat{K}_x, \hat{K}_y)e^{\alpha t^m} = Z^0 + \zeta r_x \int_0^{t^m} K_x(t)e^{\alpha t}dt$$
Differentiating this condition for a given Z^0 we obtain:
$$\begin{aligned} &d\hat{Z}e^{\alpha t^m} + \alpha\hat{Z}e^{\alpha t^m}dt^m = \zeta r_x \hat{K}_x e^{\alpha t^m}dt^m + \zeta r_x \int_0^{t^m} dK_x(t)e^{\alpha t}dt \\ \Longleftrightarrow\ &d\hat{Z}e^{\alpha t^m} = \dot{Z}^m e^{\alpha t^m}dt^m + \zeta r_x \int_0^{t^m} dK_x(t)e^{\alpha t}dt \end{aligned}$$
denoting $\dot{Z}^m \equiv \dot{Z}(t^m) = \zeta r_x \hat{K}_x - \alpha\hat{Z}$. Taking account of the expression of $dK_x(t)$, we obtain:
$$\begin{aligned} d\hat{Z}e^{\alpha t^m} &= \dot{Z}^m e^{\alpha t^m}dt^m + \zeta r_x \int_0^{t^m} dK_x(t)e^{\alpha t}dt \\ &= \dot{Z}^m e^{\alpha t^m}dt^m + \zeta r_x \int_0^{t^m} \Big\{(1 - B_K(t))\, dK_x^0 - \zeta r_x B_{\lambda Z}(t)d\lambda_{Z0}\Big\}\, e^{\alpha t}dt. \\ &= \dot{Z}^m e^{\alpha t^m}dt^m \\ &+ \zeta r_x \left(\int_0^{t^m} (1 - B_K(t))e^{\alpha t}dt\right) dK_x^0 - (\zeta r_x)^2 \left(\int_0^{t^m} B_{\lambda Z}(t)e^{\alpha t}dt\right) d\lambda_{Z0} \end{aligned}$$
Let:
$$B_K^\alpha \equiv \int_0^{t^m} (1 - B_K(t))e^{\alpha t}dt \ \text{ and } \ B_{\lambda Z}^\alpha \equiv \int_0^{t^m} B_{\lambda Z}(t)e^{\alpha t}dt$$
then a compact expression of the differentiated condition reads:
$$d\hat{Z}e^{\alpha t^m} = \dot{Z}^m e^{\alpha t^m}dt^m + \zeta r_x B_K^\alpha dK_x^0 - (\zeta r_x)^2 B_{\lambda Z}^\alpha d\lambda_{Z0}$$
Since $\hat{Z}$ is a function of $\hat{K}_x$ along the stabilisation border, we get:
$$\frac{\partial\hat{Z}}{\partial\hat{K}_x}e^{\alpha t^m}d\hat{K}_x + (\zeta r_x)^2 B_{\lambda Z}^\alpha d\lambda_{Z0} - \dot{Z}^m e^{\alpha t^m}dt^m = \zeta r_x B_K^\alpha dK_x^0$$

3. The continuity condition on λ_Z at time t^m. Denoting $\dot{\lambda}_Z^m \equiv \dot{\lambda}_Z(t^m) = (\rho + \alpha)\lambda_{Z0}e^{(\rho+\alpha)t^m}$, the differentiation of the continuity condition: $\hat{\lambda}_Z = \lambda_{Z0}e^{(\rho+\alpha)t^m}$ leads to:
$$\begin{aligned} &d\hat{\lambda}_Z = d\lambda_{Z0}e^{(\rho+\alpha)t^m} + (\rho + \alpha)\lambda_{Z0}e^{(\rho+\alpha)t^m}dt^m \\ \Longrightarrow &\frac{\partial\hat{\lambda}_Z}{\partial\hat{K}_x}d\hat{K}_x = d\lambda_{Z0}e^{(\rho+\alpha)t^m} + \dot{\lambda}_Z^m dt^m \\ \Longrightarrow &- \frac{\partial\hat{\lambda}_Z}{\partial\hat{K}_x}d\hat{K}_x + d\lambda_{Z0}e^{(\rho+\alpha)t^m} + \dot{\lambda}_Z^m dt^m = 0 \end{aligned}$$

The triplet $(d\hat{K}_x, d\lambda_{Z0}, dt^m)$ is thus the solution of the system of equations:

$$\begin{aligned}
& d\hat{K}_x + \zeta r_x B_{\lambda Z} d\lambda_{Z0} = (1 - B_K)\, dK_x^0 \\
& \frac{\partial \hat{Z}}{\partial \hat{K}_x} e^{\alpha t^m} d\hat{K}_x + (\zeta r_x)^2 B^{\alpha}_{\lambda Z} d\lambda_{Z0} - \dot{Z}^m e^{\alpha t^m} dt^m = \zeta r_x B^{\alpha}_K dK^0_x \\
& -\frac{\partial \hat{\lambda}_Z}{\partial \hat{K}_x} d\hat{K}_x + d\lambda_{Z0} e^{(\rho+\alpha)t^m} + \dot{\lambda}^m_Z dt^m = 0
\end{aligned}$$

This system reads in matrix form:

$$\begin{bmatrix} 1 & \zeta r_x B_{\lambda Z} & 0 \\ \frac{\partial \hat{Z}}{\partial \hat{K}_x} e^{\alpha t^m} & (\zeta r_x)^2 B^{\alpha}_{\lambda Z} & -\dot{Z}^m e^{\alpha t^m} \\ -\frac{\partial \hat{\lambda}_Z}{\partial \hat{K}_x} & e^{(\rho+\alpha)t^m} & \dot{\lambda}^m_Z \end{bmatrix} \begin{bmatrix} d\hat{K}_x \\ d\lambda_{Z0} \\ dt^m \end{bmatrix} = \begin{bmatrix} 1 - B_K \\ \zeta r_x B^{\alpha}_K \\ 0 \end{bmatrix} dK^0_x$$

Let Δ denote the determinant of the system. Remark that through the chain rule:

$$\frac{\partial \hat{\lambda}_Z}{\partial \hat{Z}} = \frac{\partial \hat{\lambda}_Z}{\partial \hat{K}_x}\frac{\partial \hat{K}_x}{\partial \hat{Z}} = \frac{\partial \hat{\lambda}_Z / \partial \hat{K}_x}{\partial \hat{Z} / \partial \hat{K}_x} = \frac{\dot{\lambda}^m_Z}{\dot{Z}^m} \Longrightarrow \frac{\partial \hat{Z}}{\partial \hat{K}_x} = \frac{\partial \hat{\lambda}_Z}{\partial \hat{K}_x}\frac{\dot{Z}^m}{\dot{\lambda}^m_Z}$$

Substituting for $\partial \hat{Z}/\partial \hat{K}_x$ in the expression of Δ, we obtain:

$$\Delta = \begin{vmatrix} 1 & \zeta r_x B_{\lambda Z} & 0 \\ \frac{\partial \hat{\lambda}_Z}{\partial \hat{K}_x}\frac{\dot{Z}^m}{\dot{\lambda}^m_Z} e^{\alpha t^m} & (\zeta r_x)^2 B^{\alpha}_{\lambda Z} & -\dot{Z}^m e^{\alpha t^m} \\ -\frac{\partial \hat{\lambda}_Z}{\partial \hat{K}_x} & e^{(\rho+\alpha)t^m} & \dot{\lambda}^m_Z \end{vmatrix}$$

Multiplying the first column by $\dot{\lambda}^m_Z/(\partial \hat{\lambda}_Z/\partial \hat{K}_x)$ and adding to the third column, we obtain:

$$\Delta = \begin{vmatrix} 1 & \zeta r_x B_{\lambda Z} & \frac{\dot{\lambda}^m_Z}{\partial \hat{\lambda}_Z / \partial \hat{K}_x} \\ \frac{\partial \hat{\lambda}_Z}{\partial \hat{K}_x}\frac{\dot{Z}^m}{\dot{\lambda}^m_Z} e^{\alpha t^m} & (\zeta r_x)^2 B^{\alpha}_{\lambda Z} & 0 \\ -\frac{\partial \hat{\lambda}_Z}{\partial \hat{K}_x} & e^{(\rho+\alpha)t^m} & 0 \end{vmatrix}$$

Developing with respect to the third column:

$$\Delta = \dot{\lambda}^m_Z \left[\frac{\dot{Z}^m}{\dot{\lambda}^m_Z} e^{(\rho+2\alpha)t^m} + (\zeta r_x)^2 B^{\alpha}_{\lambda Z} \right]$$

which is > 0, proving the existence of a unique solution to this system.

Next, applying the Cramer rule, we obtain:

$$\frac{\partial \hat{K}_x}{\partial K^0_x} = \frac{1}{\Delta} \begin{vmatrix} 1 - B_K & \zeta r_x B_{\lambda Z} & 0 \\ \zeta r_x B^{\alpha}_K & (\zeta r_x)^2 B^{\alpha}_{\lambda Z} & -\dot{Z}^m e^{\alpha t^m} \\ 0 & e^{(\rho+\alpha)t^m} & \dot{\lambda}^m_Z \end{vmatrix}$$

Developing with respect to the third line yields:

$$\frac{\partial \hat{K}_x}{\partial K^0_x} = \frac{1}{\Delta} \left\{ (1 - B_K)\dot{Z}^m e^{(\rho+2\alpha)t^m} + (\zeta r_x)^2 \dot{\lambda}^m_Z \left[(1 - B_K) B^{\alpha}_{\lambda Z} - B^{\alpha}_K B_{\lambda Z} \right] \right\}$$

Using the expressions of $B^{\alpha}_{\lambda Z}$ and B^{α}_{B}, the term into square brackets reads:

$$(1 - B_K)B^{\alpha}_{\lambda Z} - B^{\alpha}_{K}B_{\lambda Z} = \int_0^{t^m} \left[(1 - B_K)B_{\lambda Z}(t) - B_{\lambda Z}(1 - B_K(t)\right] e^{\alpha t}dt \equiv \int_0^{t^m} \phi(t)e^{\alpha t}dt$$

Because $B_{\lambda Z}(0) = B_K(0) = 0$, we get $\phi(0) = -B_{\lambda Z} < 0$, and $B_{\lambda Z}(t^m) = B_{\lambda Z}$ together with $B_K(t^m) = B_K$ imply that $\phi(t^m) = 0$. Differentiating using the expressions of $B_{\lambda Z}(t)$ and $B_K(t)$, we obtain:

$$\phi'(t) = (1-B_K)k'_x(t)I^A_{\alpha}(t,t^m)+B_{\lambda Z}k'_x(t)U^A(t,t^m) = k'_x(t)\left[(1 - B_K)I^A_{\alpha}(t,t^m) + B_{\lambda Z}U^A(t,t^m)\right]$$

Using the expressions of B_K and $B_{\lambda Z}$:

$$\begin{aligned}
\phi'(t) =& k'_x(t)\left[\left(1 - \int_0^{t^m} k'_x(t)U_A(t,t^m)dt\right) I^A_{\alpha}(t,t^m) + \left(\int_0^{t^m} k'_x(t)I^A_{\alpha}(t,t^m)dt\right) U^A(t,t^m)\right] \\
=& k'_x(t)\left[I^A_{\alpha}(t,t^m) + \int_0^{t^m} k'_x(t)\left[I^A_{\alpha}(t,t^m)U^A(t,t^m) - U_A(t,t^m)I^A_{\alpha}(t,t^m)\right] dt\right] \\
=& k'_x(t)I^A_{\alpha}(t,t^m) > 0
\end{aligned}$$

Since $\phi(0) < 0$, $\phi(t^m) = 0$ and $\phi'(t) > 0$, we conclude that $\phi(t) < 0$. Because the trajectories cannot cross in the phase plane, we must have $\partial\hat{K}_x/\partial K^0_x > 0$, which is only possible if $1 - B_K > 0$, implying in turn that $1 - B_K(t) > 0$, because $B_K(t) < B_K$, and hence that $B^{\alpha}_K > 0$.

Applying once again the Cramer rule, we obtain:

$$\frac{\partial\lambda_{Z0}}{\partial K^0_x} = \frac{1}{\Delta}\begin{vmatrix} 1 & 1 - B_K & 0 \\ \frac{\partial\hat{Z}}{\partial\hat{K}_x}e^{\alpha t^m} & B^{\alpha}_K & -\dot{Z}^m e^{\alpha t^m} \\ -\frac{\partial\hat{\lambda}_Z}{\partial\hat{K}_x} & 0 & \dot{\lambda}^m_Z \end{vmatrix}$$

Using the same manipulations as done for the computation of Δ, we get:

$$\frac{\partial\lambda_{Z0}}{\partial K^0_x} = \frac{1}{\Delta}\begin{vmatrix} 1 & 1 - B_K & \frac{\dot{\lambda}^m_Z}{\partial\hat{\lambda}_Z/\partial\hat{K}_x} \\ \frac{\partial\hat{\lambda}_Z}{\partial\hat{K}_x}\frac{\dot{Z}^m}{\dot{\lambda}^m_Z}e^{\alpha t^m} & B^{\alpha}_K & 0 \\ -\frac{\partial\hat{\lambda}_Z}{\partial\hat{K}_x} & 0 & 0 \end{vmatrix}$$

Developing with respect to the third column yields:

$$\frac{\partial\lambda_{Z0}}{\partial K^0_x} = \frac{\dot{\lambda}^m_Z}{\Delta}B^{\alpha}_K$$

which is > 0 since we have shown already that $B^{\alpha}_K > 0$.

This result has two interesting implications. First concerning $d\lambda_x(t)$:

$$\frac{\partial\lambda_x(t)}{\partial K^0_x} = -U^A(t,t^m) - \zeta r_x I^A_{\alpha}(t,t^m)\frac{\partial\lambda_{Z0}}{\partial K^0_x}$$

shows that $\partial\lambda_x/\partial K^0_x < 0$, a positive shock on K_x slows down the investment pace in fossil energy conversion capacities. Second:

$$dK_x(t) = (1 - B_K(t))\,dK^0_x - \zeta r_x B_{\lambda Z}(t)d\lambda_{Z0} = dK^0_x - \left[B_K(t)dK^0_x + \zeta r_x B_{\lambda_Z}(t)d\lambda_{Z0}\right] < dK^0_x$$

shows that the slow down of the investment rate induces a dampening effect of the initial shock on K_x on the accumulation trajectory of $K_x(t)$.

Last using the differentiation of the continuity condition over λ_Z:

$$\dot{\lambda}_Z^m \frac{\partial t^m}{\partial K_x^0} = \frac{\partial \hat{\lambda}_Z}{\partial \hat{K}_x} \frac{\partial \hat{K}_x}{\partial K_x^0} - \frac{\partial \lambda_{Z0}}{\partial K_x^0} e^{(\rho+\alpha)t^m}$$

shows that $\partial t^m / \partial K_x^0 < 0$, because we have previously shown that $\partial \hat{\lambda}_Z / \partial \hat{K}_x < 0$, and on the other hand that $\partial \hat{K}_x / \partial K_x^0 > 0$ together with $\partial \lambda_{Z0} / \partial K_x^0 > 0$.

A.2.2 Proof of the Proposition P.5

The claims of the Proposition P.5 are easily checked. Because trajectories do not cross in the (Z, K_x) plane and $\hat{K}_x^m(Z)$ is a decreasing curve in the plane, an increase of K_x^0 induces an upward shift of $\hat{K}_x$, the claim (i) of the Proposition. Next since we have shown that $\partial \lambda_x(t)/\partial K_x^0 < 0$, we get $\partial \lambda_{x0}/\partial K_x^0 < 0$, the claim (ii) of the Proposition. We have also already shown that $\partial \lambda_{Z0}/\partial K_x^0 > 0$, the claim (iii) of the Proposition, and $\partial t^m/\partial K_x^0 < 0$, the claim (iv), completing the proof of the Proposition.

A.3 Proof of the Proposition P.6

To prove the claims of the Proposition we follow a parallel method as used in the proof of the Proposition P.5. For starters we compute expressions of $d\lambda_y(t)$ and $dK_y(t)$ as functions of $(d\lambda_y(0), dK_y(0))$. We next describe the system determining the optimum and use the expressions of $d\lambda_y(t)$ and $dK_y(t)$ to linearize this system and prove the existence and unicity of its solution. We last perform a sensitivity analysis with respect to the vector of initial conditions to show the claims of the Proposition P.6.

Expressions of $d\lambda_y(t)$ and $dK_y(t)$

Denote $d\lambda_{y0} \equiv d\lambda_y(0)$ and $dK_y^m \equiv dK_y(0)$. Integrating over a time interval, $[t_0, t)$:

$$K_y(t) = K_y(t_0) + \int_{t_0}^{t} k_y(\lambda_y(\tau)) d\tau$$
$$\lambda_y(t)e^{-\rho t} = \lambda_y(t_0)e^{-\rho t_0} - \int_{t_0}^{t} \left[u'(\hat{K}_x + K_y(\tau)) - a_y - m_y - b_y'(K_y(\tau)) \right] e^{-\rho\tau} d\tau$$

Differentiating yields:

$$dK_y(t) = dK_y(t_0) + \int_{t_0}^{t} k_y'(\tau) d\lambda_y(\tau) d\tau$$
$$d\lambda_y(t)e^{-\rho t} = d\lambda_y(t_0)e^{-\rho t_0} + \int_{t_0}^{t} \left[|u''(\tau)| d\hat{K}_x + \left(b_y''(\tau) - u''(\tau) \right) dK_y(\tau) \right] e^{-\rho\tau} d\tau$$

Substituting for $dK_y(\tau)$ its expression inside the expression of $d\lambda_y(t)$, we obtain:

$$
\begin{aligned}
&d\lambda_y(t)e^{-\rho t} = d\lambda_y(t_0)e^{-\rho t_0} + \left(\int_{t_0}^{t} |u''(\tau)|e^{-\rho\tau}d\tau\right) d\hat{K}_x \\
&+ \left(\int_{t_0}^{t}(b_y''(\tau) - u''(\tau))e^{-\rho\tau}d\tau\right) dK_y(t_0) \\
&+ \int_{t_0}^{t}(b_y''(\tau) - u''(\tau))e^{-\rho\tau}\int_{t_0}^{\tau} k_y'(s)d\lambda_y(s)dsd\tau
\end{aligned}
$$

Integrating over $[t_0, t^m)$:

$$
\begin{aligned}
&\int_{t_0}^{t^m} d\lambda_y(t)e^{-\rho t}dt = d\lambda_y(t_0)e^{-\rho t_0}(t^m - t_0) \\
&+ \left(\int_{t_0}^{t^m}\int_{t_0}^{t} |u''(\tau)|e^{-\rho\tau}d\tau dt\right) d\hat{K}_x \\
&+ \left(\int_{t_0}^{t^m}\int_{t_0}^{t}(b_y''(\tau) - u''(\tau))e^{-\rho\tau}d\tau dt\right) dK_y(t_0) \\
&+ \int_{t_0}^{t^m}\int_{t_0}^{t}(b_y''(\tau) - u''(\tau))e^{-\rho\tau}\int_{t_0}^{\tau} k_y'(s)d\lambda_y(s)dsd\tau dt
\end{aligned}
$$

Inverting the integration order:

$$
\begin{aligned}
&\int_{t_0}^{t^m}\int_{t_0}^{t} |u''(\tau)|e^{-\rho\tau}d\tau dt = \int_{t_0}^{t^m} |u''(t)|e^{-\rho t}(t^m - t)dt \\
&\int_{t_0}^{t^m}\int_{t_0}^{t}(b_y''(\tau) - u''(\tau))e^{-\rho\tau}d\tau dt = \int_{t_0}^{t^m} (b_y''(t) - u''(t))e^{-\rho t}(t^m - t)dt \\
&\int_{t_0}^{t^m}\int_{t_0}^{t}(b_y''(\tau) - u''(\tau))e^{-\rho\tau}\int_{t_0}^{\tau} k_y'(s)d\lambda_y(s)dsd\tau dt \\
&= \int_{t_0}^{t^m} (b_y''(t) - u''(t))e^{-\rho t}(t^m - t)\int_{t_0}^{t} k_y'(s)d\lambda_y(s)dsdt \\
&= \int_{t_0}^{t^m} k_y'(t)d\lambda_y(t)\int_{t}^{t^m} (b_y''(\tau) - u''(\tau))e^{-\rho\tau}(t^m - \tau)d\tau dt
\end{aligned}
$$

Rearranging we obtain:

$$
\begin{aligned}
&\int_{t_0}^{t^m} d\lambda_y(t)\left[e^{-\rho t} - k_y'(t)\int_{t}^{t^m} (b_y''(\tau) - u''(\tau))e^{-\rho\tau}(t^m - \tau)d\tau\right] dt = \\
&d\lambda_y(t_0)e^{-\rho t_0}(t^m - t_0) + \left(\int_{t_0}^{t^m} |u''(t)|e^{-\rho t}(t^m - t)dt\right) d\hat{K}_x \\
&+ \left(\int_{t_0}^{t^m} (b_y''(t) - u''(t))e^{-\rho t}(t^m - t)dt\right) dK_y(t_0)
\end{aligned}
$$

Differentiating with respect to t_0 for a given $dK_y(t_0)$ yields:

$$
\begin{aligned}
&- d\lambda_y(t_0)e^{-\rho t_0} + d\lambda_y(t_0)k_y'(t_0)\int_{t_0}^{t^m} (b_y''(\tau) - u''(\tau))e^{-\rho\tau}(t^m - \tau)d\tau = \\
&\frac{d}{dt_0}d\lambda_y(t_0)e^{-\rho t_0}(t^m - t_0) - \rho d\lambda_y(t_0)e^{-\rho t_0}(t^m - t_0) - d\lambda_y(t_0)e^{-\rho t_0} \\
&- |u''(t_0)|e^{-\rho t_0}(t^m - t_0)d\hat{K}_x - (b_y''(t_0) - u''(t_0))e^{-\rho t_0}(t^m - t_0)dK_y(t_0)
\end{aligned}
$$

Simplifying and dividing both sides by $e^{-\rho t_0}(t^m - t_0)$ gets:

$$d\lambda_y(t_0)\left[\rho + \frac{k'_y(t_0)e^{\rho t_0}}{t^m - t_0}\int_{t_0}^{t^m}(b''_y(\tau) - u''(\tau))e^{-\rho\tau}(t^m - \tau)d\tau\right] =$$
$$\frac{d}{dt_0}d\lambda_y(t_0) - |u''(t_0)|d\hat{K}_x - (b''_y(t_0) - u''(t_0))dK_y(t_0)$$

Let $\theta_y \equiv d\lambda_y$, the above expression is a differential equation of the form:

$$\dot{\theta}_y(t_0) = \left[\rho + \frac{k'_y(t_0)e^{\rho t_0}}{t^m - t_0}\int_{t_0}^{t^m}(b''_y(\tau) - u''(\tau))e^{-\rho\tau}(t^m - \tau)d\tau\right]\theta_y(t_0)$$
$$+ |u''(t_0)|d\hat{K}_x + (b''_y(t_0) - u''(t_0))dK_y(t_0)$$

Let us denote:

$$f(t) = \rho + \frac{k'_y(t)e^{\rho t}}{t^m - t}\int_{t}^{t^m}(b''_y(\tau) - u''(\tau))e^{-\rho\tau}(t^m - \tau)d\tau \text{ and } F(t) = \int_0^t f(\tau)d\tau$$

Then with the particular solution, (K^m_y, λ_{y0}), the general solution of the differential equation reads:

$$d\lambda_y(t) = e^{F(t)}\left\{d\lambda_{y0} + \left(\int_0^t |u''(\tau)|e^{-F(\tau)}d\tau\right)d\hat{K}_x\right.$$
$$\left. + \left(\int_0^t (b''_y(\tau) - u''(\tau))e^{-F(\tau)}d\tau\right)dK^m_y\right\}$$

Denoting:

$$U^F(t) \equiv \int_0^t |u''(\tau)|e^{-F(\tau)}d\tau \text{ and } B^F(t) \equiv \int_0^t (b''_y(\tau) - u''(\tau))e^{-F(\tau)}d\tau$$

a compact expression of $d\lambda_y(t)$ reads:

$$d\lambda_y(t) = e^{F(t)}\left[d\lambda_{y0} + U^F(t)d\hat{K}_x + B^F(t)dK^m_y\right] \tag{A.3.1}$$

Inserting this expression into the expression of $dK_y(t)$ we obtain:

$$\begin{aligned} dK_y(t) &= dK^m_y + \int_0^t k'_y(t)e^{F(\tau)}\left[d\lambda_{y0} + U^F(\tau)d\hat{K}_x + B^F(\tau)dK^m_y\right]d\tau \\ &= dK^m_y\left[1 + \int_0^t k'_y(t)e^{F(\tau)}B^F(\tau)d\tau\right] + \left(\int_0^t k'_y(t)e^{F(\tau)}d\tau\right)d\lambda_{y0} \\ &+ \left(\int_0^t k'_y(t)e^{F(\tau)}U^F(\tau)d\tau\right)d\hat{K}_x \end{aligned}$$

Denoting:

$$\begin{aligned} H_{Kx}(t) &\equiv \int_0^t k'_y(\tau)e^{F(\tau)}U^F(\tau)d\tau \\ H_{Ky}(t) &\equiv \int_0^t k'_y(\tau)e^{F(\tau)}B^F(\tau)d\tau \\ H_{\lambda y}(t) &\equiv \int_0^t k'_y(\tau)e^{F(\tau)}d\tau \end{aligned}$$

gets the compact expression:

$$dK_y(t) = \left[1 + H_{Ky}(t)\right]dK^m_y + H_{\lambda_y}(t)d\lambda_{y0} + H_{Kx}(t)d\hat{K}_x \tag{A.3.2}$$

System determining the optimum

The vector $(\hat{K}_x, \lambda_{y0}, T^m)$ is determined as a function of K_y^m by the system:

1. *The renewable energy capital accumulation condition.* Since K_y^s is a function of $\hat{K}_x$, denote it by $K_y^s(\hat{K}_x)$ by a slight abuse of notations. Then:

$$K_y^s(\hat{K}_x) = K_y^m + \int_0^{T^m} k_y(\lambda_y(t))dt \tag{A.3.3}$$

2. *The continuity condition over λ_y at time T^m.* Since $\lambda_y(T^m)$ is a decreasing function of K_y^m and thus an increasing function of $\hat{K}_x$, the integration of the motion of $\lambda_y(t)$ over the time interval, $[0, T^m)$, leads to:

$$\begin{aligned}&\lambda_y^s(\hat{K}_x)e^{-\rho T^m} = \\ &\lambda_{y0} - \int_0^{T^m} \left[u'(\hat{K}_x + K_y(t)) - a_y - m_y - b_y'(K_y(t))\right] e^{-\rho t}dt\end{aligned} \tag{A.3.4}$$

3. *The minimal delay condition during the stabilisation phase.* This condition reads:

$$\underline{n}_x = \int_0^{T^m} \left[u'(\hat{K}_x + K_y(t)) - c_x - m_x\right] e^{-\rho t}dt \tag{A.3.5}$$

We linearize by differentiation the system. Because $dK_y^s = -d\hat{K}_x$ along the scraping border, the differentiation of (A.3.3) leads to:

$$-d\hat{K}_x = dK_y^m + k_y^s dT^m + \int_0^{T^m} k_y'(t)d\lambda_y(t)dt$$

where we denote $k_y^s \equiv k_y(T^m)$, the investment rate in renewable energy conversion capacities building at time T^m, that is at the beginning of the scraping phase. Taking account of the expression (A.3.1) of $d\lambda_y(t)$, we obtain:

$$\begin{aligned}-d\hat{K}_x &= dK_y^m + k_y^s dT^m + \int_0^{T^m} k_y'(t)d\lambda_y(t)dt \\ &= [1 + H_{Ky}(T^m)]\, dK_y^m + H_{\lambda y}(T^m)d\lambda_{y0} + H_{Kx}(T^m)d\hat{K}_x + k_y^s dT^m\end{aligned}$$

Denote $H_{\lambda y} \equiv H_{\lambda y}(T^m)$, $H_{Kx} \equiv H_{Kx}(T^m)$ and $H_{Ky} \equiv H_{Ky}(T^m)$. Then rearranging gets:

$$[1 + H_{Kx}]\, d\hat{K}_x + H_{\lambda y}d\lambda_{y0} + k_y^s dT^m = -\left[1 + H_{Ky}\right] dK_y^m$$

Let $\lambda_y^{s'} \equiv d\lambda_y(t)/dK_y(t)|_{t=T^m} = \dot{\lambda}_y^s/k_y^s$ denote the derivative of λ_y with respect to K_y, evaluated at the end of the stabilisation phase. Denote also by $\lambda_y^s \equiv \lambda_y(T^m)$, the value of λ_y at the beginning of the scraping phase, and $\dot{\lambda}_y^s$, its time derivative at the same time. Differentiating the second condition using $dK_y^s = -d\hat{K}_x$, we obtain:

$$-\lambda_y^{s'} e^{-\rho T^m} d\hat{K}_x = \dot{\lambda}_y^s e^{-\rho T^m} dT^m + d\lambda_{y0} - \int_0^{T^m} d\beta_y(t)e^{-\rho t}dt$$

Because $d\lambda_{y0} - \int_0^{T^m} d\beta_y(t)e^{-\rho t}dt = d\lambda_y(T^m)$ and (A.3.1):

$$d\lambda_y(T^m)e^{-\rho T^m} = \left[e^{F(T^m)}\left(d\lambda_{y0} + U^F(T^m)d\hat{K}_x + B^F(T^m)dK_y^m\right)\right]e^{-\rho T^m}$$

the above relation is equivalent to:

$$-\lambda_y^{s'}e^{-\rho T^m}d\hat{K}_x = \dot{\lambda}_y^s e^{-\rho T^m}dT^m + e^{F(T^m)}\left(d\lambda_{y0} + U^F(T^m)d\hat{K}_x + B^F(T^m)dK_y^m\right)e^{-\rho T^m}$$

Denoting $F \equiv F(T^m)$, $U^F \equiv U^F(T^m)$, $B^F \equiv B^F(T^m)$ and simplifying the $e^{-\rho T^m}$ term on both sides yields:

$$-\lambda_y^{s'}d\hat{K}_x = \dot{\lambda}_y^s dT^m + e^F\left(d\lambda_{y0} + U^F d\hat{K}_x + B^F dK_y^m\right)$$

Rearranging we obtain:

$$\left(\lambda_y^{s'} + U^F e^F\right)d\hat{K}_x + e^F d\lambda_{y0} + \dot{\lambda}_y^s dT^m = -e^F B^F dK_y^m$$

Last differentiating the third condition, remembering that $\beta_x(T^m) = 0$, yields:

$$\int_0^{T^m} u''(t)e^{-\rho t}(d\hat{K}_x + dK_y(t))dt = 0$$

Taking account of the expression (A.3.2) of $dK_y(t)$, this is equivalent to:

$$\int_0^{T^m} u''(t)e^{-\rho t}\left[\left(1 + H_{Ky}(t)\right)dK_y^m + H_{\lambda_y}(t)d\lambda_{y0} + \left(1 + H_{Kx}(t)\right)d\hat{K}_x\right]dt$$

Denoting:

$$G_{\lambda y} \equiv \int_0^{T^m} |u''(t)|e^{-\rho t}H_{\lambda y}(t)dt \ ; \ G_{Kx} \equiv \int_0^{T^m} |u''(t)|e^{-\rho t}(1 + H_{Kx}(t))dt$$
$$\text{and } G_{Ky} \equiv \int_0^{T^m} |u''(t)|e^{-\rho t}(1 + H_{Ky}(t))dt$$

gets the compact expression:

$$G_{Kx}d\hat{K}_x + G_{\lambda y}d\lambda_{y0} = -G_{Ky}dK_y^m$$

The vector $(d\hat{K}_x, d\lambda_{y0}, dT^m)$ is the solution of the system of 3 conditions:

$$\left[1 + H_{Kx}\right]d\hat{K}_x + H_{\lambda y}d\lambda_{y0} + k_y^s dT^m = -\left[1 + H_{Ky}\right]dK_y^m \tag{A.3.6}$$

$$\left(\lambda_y^{s'} + U^F e^F\right)d\hat{K}_x + e^F d\lambda_{y0} + \dot{\lambda}_y^s dT^m = -e^F B^F dK_y^m \tag{A.3.7}$$

$$G_{Kx}d\hat{K}_x + G_{\lambda y}d\lambda_{y0} = -G_{Ky}dK_y^m \tag{A.3.8}$$

Implications of the core dynamics of $\lambda_y(t)$ and $K_y(t)$

We know that the system determining the joint dynamics of $K_y(t)$ and $\lambda_y(t)$ during the stabilisation phase is time autonomous and parametrized by $\hat{K}_x$. For a given $\hat{K}_x$, the terminal pair, (K_y^s, λ_y^s), is determined, giving a particular solution of the differential system and hence its unique general solution. Because λ_y should be a decreasing function of K_y, it must be the case that $\partial\lambda_{y0}/\partial K_y^m < 0$ at the beginning of the stabilisation phase. Moreover, the system being time autonomous, it takes less time to move from an increased initial position to the same final position, implying that $\partial T^m/\partial K_y^m < 0$. The two dimensional differentiated system reads in matrix form:

$$\begin{bmatrix} H_{\lambda y} & k_y^s \\ e^F & \dot{\lambda}_y^s \end{bmatrix} \begin{bmatrix} d\lambda_{y0} \\ dT^m \end{bmatrix} = \begin{bmatrix} -(1+H_{Kx}) \\ -\left(\lambda_y^{s'} + U^F e^F\right) \end{bmatrix} d\hat{K}_x + \begin{bmatrix} -(1+H_{Ky}) \\ -B^F e^F \end{bmatrix} dK_y^m$$

Let D denote the determinant of this system. Because $\dot{\lambda}_y^s < 0$, it is immediate that:

$$D = H_{\lambda y}\dot{\lambda}_y^s - k_y^s e^F < 0$$

Fixing $\hat{K}_x$, thus setting $d\hat{K}_x = 0$, and applying the Cramer rule, we get:

$$\frac{\partial\lambda_{y0}}{\partial K_y^m} = \frac{1}{D}\begin{vmatrix} -(1+H_{Ky}) & k_y^s \\ -e^F B^F & \dot{\lambda}_y^s \end{vmatrix} = \frac{1}{D}\left[-(1+H_{Ky})\dot{\lambda}_y^s + k_y^s B^F e^F\right]$$

which must be < 0 because $D < 0$ and $\dot{\lambda}_y^s < 0$., confirming that λ_y^0 is a decreasing function of K_y^m for a given $\hat{K}_x$. Applying once again the Cramer rule:

$$\frac{\partial T^m}{\partial K_y^m} = \frac{1}{D}\begin{vmatrix} H_{\lambda y} & -(1+H_{Ky}) \\ e^F & -B^F e^F \end{vmatrix} = \frac{e^F}{D}\left[-H_{\lambda y}B^F + (1+H_{Ky})\right]$$

which is of indeterminate sign at first sight. Because $\partial T^m/\partial K_y^m < 0$, it must then be the case that $1 + H_{Ky} > H_{\lambda y}B^F$. Using the expressions of H_{Ky} and $H_{\lambda y}$, this is equivalent to:

$$1 + \int_0^{T^m} k_y'(t)e^{F(t)}B^F(t)dt - B^F\int_0^{T^m} k_y'(t)e^{F(t)}dt = 1 + \int_0^{T^m} k_y'(t)e^{F(t)}(B^F(t) - B^F)dt > 0$$

$$\implies \quad 0 < \int_0^{T^m} k_y'(t)e^{F(t)}(B^F - B^F(t))dt < 1$$

Now fixing K_y^m, that is setting $dK_y^m = 0$ and applying the Cramer rule:

$$\frac{\partial\lambda_{y0}}{\partial\hat{K}_x} = \frac{1}{D}\begin{vmatrix} -(1+H_{Kx}) & k_y^s \\ -\left(\lambda_y^{s'} + U^F e^F\right) & \dot{\lambda}_y^s \end{vmatrix} = \frac{1}{D}\left[-(1+H_{Kx})\dot{\lambda}_y^s + k_y^s\left(\lambda_y^{s'} + U^F e^F\right)\right]$$

Because $\lambda_y^{s'} = \dot{\lambda}_y^s/k_y^s$, an equivalent expression of $\partial\lambda_{y0}/\partial\hat{K}_x$ reads:

$$\frac{\partial\lambda_{y0}}{\partial\hat{K}_x} = \frac{1}{D}\left[-\dot{\lambda}_y^s - H_{Kx}\dot{\lambda}_y^s + k_y^s\left(\frac{\dot{\lambda}_y^s}{k_y^s} + k_y^s U^F e^F\right)\right] = \frac{1}{D}\left[-H_{Kx}\dot{\lambda}_y^s + k_y^s U^F e^F\right]$$

which is < 0 because $D < 0$ and $\dot{\lambda}_y^s < 0$. We know that K_y^s is a decreasing function of $\hat{K}_x$ along the scraping border, thus λ_y^s being a decreasing function of K_y^s, it is an increasing function of $\hat{K}_x$. A positive shock on $\hat{K}_x$ has a pivotal effect on the trajectory of $\lambda_y(t)$, lowering it initially before raising it at the end of the stabilisation path. Applying once again the Cramer rule, we get also:

$$\frac{\partial T^m}{\partial\hat{K}_x} = \frac{1}{D}\begin{vmatrix} H_{\lambda y} & -(1+H_{Kx}) \\ e^F & -\left(\lambda_y^{s'} + U^F e^F\right) \end{vmatrix} = \frac{1}{D}\left[-H_{\lambda y}\left(\lambda_y^{s'} + U^F e^F\right) + (1+H_{Kx})e^F\right]$$

which is of indeterminate sign at first sight. $\partial T^m/\partial \hat{K}_x$ reads equivalently:

$$\frac{\partial T^m}{\partial \hat{K}_x} = \frac{1}{D}\left[-H_{\lambda y}\lambda_y^{s'} + \left(1 + H_{Kx} - U^F H_{\lambda y}\right)e^F\right]$$

Using the expressions of H_{Kx} and $H_{\lambda y}$:

$$1 + H_{Kx} - U^F H_{\lambda y} = 1 + \int_0^{T^m} k_y'(t)e^{F(t)}(U^F(t) - U^F)dt$$

It is immediate that:

$$\begin{aligned}
B^F(t) = &\int_0^t (b_y''(\tau) - u''(\tau))e^{-F(\tau)}d\tau = \int_0^t b_y''(\tau)d\tau + \int_0^t |u''(\tau)|e^{F(\tau)}d\tau \\
> &\int_0^t |u''(\tau)|e^{F(\tau)}d\tau = U^F(t) \\
\Longrightarrow &\int_0^{T^m} k_y'(t)e^{F(t)}(B^F - B^F(t))dt > \int_0^{T^m} k_y'(t)e^{F(t)}(U^F - U^F(t))dt \\
\Longrightarrow & 1 > B^F H_{\lambda_y} - H_{Ky} > U^F H_{\lambda y} - H_{Kx} \Longrightarrow 1 + H_{Kx} - U^F H_{\lambda y} > 0
\end{aligned}$$

This implies that $\partial T^m/\partial \hat{K}_x < 0$ because $D < 0$ and $\lambda_y^{s'} < 0$. The length of the stabilisation phase is shortened by an increase of the stabilised level of K_x.

The stabilisation border

We are now in position to determine the qualitative shape of the stabilisation border. The condition (A.3.8) reads equivalently:

$$\left[G_{Kx} + G_{\lambda y}\frac{\partial \lambda_{y0}}{\partial \hat{K}_x}\right]d\hat{K}_x + \left[G_{Ky} + G_{\lambda y}\frac{\partial \lambda_{y0}}{\partial K_y^m}\right]dK_y^m = 0 \Longleftrightarrow \Omega_x d\hat{K}_x + \Omega_y dK_y^m = 0$$

where:

$$\begin{aligned}
\Omega_x &= DG_{Kx} + G_{\lambda y}\left[k_y^s U^F e^F - H_{Kx}\dot{\lambda}_y^s\right] \\
&= \left(H_{\lambda y}\dot{\lambda}_y^s - k_y^s e^F\right)G_{Kx} + G_{\lambda y}\left[k_y^s U^F e^F - H_{Kx}\dot{\lambda}_y^s\right] \\
&= \dot{\lambda}_y^s\left[H_{\lambda y}G_{Kx} - H_{Kx}G_{\lambda y}\right] - k_y^s e^F\left[G_{Kx} - U^F G_{\lambda y}\right] \\
&\equiv \dot{\lambda}_y^s\Theta_{x1} - k_y^s e^F\Theta_{x2}
\end{aligned}$$

and:

$$\begin{aligned}
\Omega_y &= DG_{Ky} + G_{\lambda y}\left[k_y^s B^F e^F - (1 + H_{Ky})\dot{\lambda}_y^s\right] \\
&= \left(H_{\lambda y}\dot{\lambda}_y^s - k_y^s e^F\right)G_{Ky} + G_{\lambda y}\left[k_y^s B^F e^F - (1 + H_{Ky})\dot{\lambda}_y^s\right] \\
&= \dot{\lambda}_y^s\left[H_{\lambda y}G_{Ky} - (1 + H_{Ky})G_{\lambda y}\right] - k_y^s e^F\left[G_{Ky} - B^F G_{\lambda y}\right] \\
&\equiv \dot{\lambda}_y^s\Theta_{y1} - k_y^s e^F\Theta_{y2}
\end{aligned}$$

Taking account of the expressions of $H_{\lambda y}$, G_{Kx}, H_{Kx}, $G_{\lambda y}$ and applying the Fubini theorem, we obtain:

$$\begin{aligned}\Theta_{x1} &= \int_0^{T^m} \int_0^{T^m} |u''(t)| k_y'(t) e^{F(t)-\rho t} \Big[(1 + H_{Kx}(t)) - U^F(t) H_{\lambda y}(t) \Big] \\ &\equiv \int_0^{T^m} \int_0^{T^m} |u''(t)| k_y'(t) e^{F(t)-\rho t} \phi(t) dt\end{aligned}$$

and $\phi(0) = 1$, $\phi(T^m) = 1 + H_K - U^F H_{\lambda y} > 0$, as shown before, and:

$$\phi'(t) = k_y'(t) e^{F(t)} U^F(t) - \dot{U}^F(t) H_{\lambda y}(t) - U^F(t) k_y'(t) e^{F(t)} = -\dot{U}^F(t) H_{\lambda y}(t) < 0$$

show that $\phi(t) > 0$ and hence that $\Theta_{x1} > 0$.

Taking account of the expressions of G_{Kx} and $G_{\lambda y}$ yields the following expression of Θ_{x2}:

$$\Theta_{x2} = \int_0^{T^m} |u''(t)| e^{-\rho t} \Big[(1 + H_{Kx}(t)) - U^F H_{\lambda y}(t) \Big] dt \equiv \int_0^{T^m} |u''(t)| e^{-\rho t} \phi(t) dt$$

and $\phi(0) = 1$, $\phi(T^m) = 1 + H_{Kx} - U^F H_{\lambda y} > 0$, as seen before, and using the expressions of $H_{Kx}(t)$ and $H_{\lambda y}(t)$:

$$\phi'(t) = k_y'(t) e^{F(t)} \Big(U^F(t) - U^F \Big) < 0$$

show that $\phi(t) > 0$, $t \in [0, T^m)$, and thus that $\Theta_{x2} > 0$. Because $\dot{\lambda}_y^s < 0$, $\Theta_{x1} > 0$ and $\Theta_{x2} > 0$ imply that $\Omega_x < 0$.

Taking account of the expressions of G_{Ky} and $G_{\lambda y}$, we obtain:

$$\Theta_{y1} = \int_0^{T^m} |u''(t)| e^{-\rho t} \left[H_{\lambda y}(1 + H_{Ky}(t)) - (1 + H_{Ky}) H_{\lambda} y(t) \right] dt \equiv \int_0^{T^m} |u''(t)| e^{-\rho t} \phi(t) dt$$

and $\phi(0) = H_{\lambda y}$, $\phi(T^m) = H_{\lambda y}(1 + H_{Ky}) - (1 + H_{Ky}) H_{\lambda y} = 0$, and:

$$\phi'(t) = k_y'(t) e^{F(t)} \Big[H_{\lambda y} B^F(t) - (1 + H_{Ky}) \Big] < 0$$

because $1 + H_{Ky} > B H_{\lambda y} > B^F(t) H_{\lambda y}$, show that $\phi(t) > 0$ and hence that $\Theta_{y1} > 0$.

Taking account of the expressions of G_{Ky} and $G_{\lambda y}$ yields the following expression of Θ_{y2}:

$$\Theta_{y2} = \int_0^{T^m} |u''(t)| e^{-\rho t} \Big[(1 + H_{Ky}(t)) - B^F H_{\lambda y}(t) \Big] dt \equiv \int_0^{T^m} |u''(t)| e^{-\rho t} \phi(t) dt$$

and $\phi(0) = 1$, $\phi(T^m) = 1 + H_{Ky} - B^F H_{\lambda y} > 0$, as seen before, and using the expressions of $H_{Ky}(t)$ and $H_{\lambda y}(t)$:

$$\phi'(t) = k_y'(t) e^{F(t)} \Big(B^F(t) - B^F \Big) < 0$$

show that $\phi(t) > 0$, $t \in [0, T^m)$, and thus that $\Theta_{y2} > 0$. Because $\dot{\lambda}_y^s < 0$, $\Theta_{y1} > 0$ and $\Theta_{y2} > 0$ imply that $\Omega_y < 0$. We thus conclude that $d\hat{K}_x / dK_y^m = -\Omega_y / \Omega_x < 0$.

We want to show in addition that the slope of the stabilisation border is lower than 1 in absolute value.

$$\left|\frac{d\hat{K}_x}{dK_y^m}\right| = \frac{\Omega_y}{\Omega_x} = \frac{|\Omega_y|}{|\Omega_x|} < 1 \Longleftrightarrow |\dot{\lambda}_y^s|\Theta_{y1} + k_y^s e^F \Theta_{y2} < |\dot{\lambda}_y^s|\Theta_{x1} + k_y^s e^F \Theta_{x2}$$

Let us first show that $\Theta_{x1} > \Theta_{y1}$:

$$\Theta_{x1} - \Theta_{y1} =$$
$$H_{\lambda y}G_{Kx} - H_{Kx}G_{\lambda y} - H_{\lambda y}G_{Ky} + (1 + H_{\lambda y})G_{\lambda y} =$$
$$\int_0^{T^m} |u''(t)|e^{-\rho t}\left[H_{\lambda y}(1 + H_{Kx}(t)) - H_{Kx}H_{\lambda_y}(t) - H_{\lambda y}(1 + H_{Ky}(t)) + (1 + H_{Ky})H_{\lambda y}(t)\right] dt$$
$$\equiv \int_0^{T^m} |u''(t)|e^{-\rho t}\phi(t)dt$$

and $\phi(0) = 0$, $\phi(T^m) = H_{\lambda y}$ and $\phi'(t) > 0$ show that $\phi(t) > 0$, thus $\Theta_{x1} > \Theta_{y1}$. Next we show that $\Theta_{x2} > \Theta_{y2}$:

$$\Theta_{x2} - \Theta_{y2} = G_{Kx} - U^F G_{\lambda y} - G_{Ky} + B^F G_{\lambda y} =$$
$$\int_0^{T^m} |u''(t)|e^{-\rho t}\left[(1 + H_{Kx}(t)) - U^F H_{\lambda y}(t) - (1 + H_{Ky}(t)) + B^F H_{\lambda_{y(t)}}\right] dt$$
$$\equiv \int_0^{T^m} |u''(t)|e^{-\rho t}\phi(t)dt$$

and:

$$\begin{aligned}\phi(t) &= H_{Kx}(t)) - U^F H_{\lambda y}(t) - H_{Ky}(t)) + B^F H_{\lambda_{y(t)}} \\ &= \int_0^t k_y'(\tau)e^{F(\tau)}\left[U^F(\tau) - U^F - B^F(\tau) + B^F\right] dt \\ &= \int_0^t k_y'(\tau)e^{F(\tau)}\left[B^F - U^F - (B^F(\tau) - U^F(\tau))\right] dt > 0\end{aligned}$$

because $B^F(t) > U^F(t)$. Hence $\Theta_{x2} > \Theta_{y2}$.

$\Theta_{x1} > \Theta_{y1}$ and $\Theta_{x2} > \Theta_{y2}$ in turn imply that $|\dot{\lambda}_y^s|\Theta_{y1} + k_y^s e^F \Theta_{y2} < |\dot{\lambda}_y^s|\Theta_{x1} + k_y^s e^F \Theta_{x2}$ and thus that $|\Omega_y| < |\Omega_x|$, showing that $|d\hat{K}_x/dK_y^m| < 1$.

Further sensitivity results along the stabilisation border

Sensitivity of λ_{y0} along the stabilisation border

Observe that $B^F(t) > U^F(t)$ implies that $H_{Ky} > H_{Kx}$ and $B^F > U^F$, so that:

$$\begin{aligned}&\left|\frac{\partial\lambda_{y0}}{\partial K_y^m}\right| = \frac{1}{|D|}\left[(1 + H_{Ky})|\dot{\lambda}_y^s| + k_y^s B^F e^F\right] \\ >&\frac{1}{|D|}\left[(1 + H_{Kx})|\dot{\lambda}_y^s| + k_y^s U^F e^F\right] \\ >&\frac{1}{|D|}\left[(1 + H_{Ky})|\dot{\lambda}_y^s| + k_y^s B^F e^F\right] + \frac{k_y^s}{|D|}\lambda_y^{s'} = \left|\frac{\partial\lambda_{y0}}{\partial\hat{K}_x}\right|\end{aligned}$$

because $\lambda_y^{s'} < 0$. Thus we conclude from $|d\hat{K}_x/dK_y^m| < 1$, $\partial\lambda_{y0}/\partial K_y^m < 0$ and $\partial\lambda_{y0}/\partial\hat{K}_x < 0$ that:

$$\frac{d\lambda_{y0}}{dK_y^m} = \frac{\partial\lambda_{y0}}{\partial\hat{K}_x}\frac{d\hat{K}_x}{dK_y^m} + \frac{\partial\lambda_{y0}}{\partial K_y^m} < 0$$

Conversely we get from (A.3.8)

$$-G_{\lambda y}\frac{d\lambda_{y0}}{d\hat{K}_x} = G_{Kx} - G_{Ky}\left|\frac{dK_y^m}{d\hat{K}_x}\right| < G_{Kx} - G_{Ky}$$

because $\left|d\hat{K}_x/dK_y^m\right| < 1$ implies that $\left|dK_y^m/d\hat{K}_x\right| > 1$. And

$$\begin{aligned}
G_{Kx} - G_{Ky} &= \int_0^{T^m} |u''(t)|e^{-\rho t}\left[(1 + H_{Kx}(t)) - (1 + H_{Ky}(t)\right] dt \\
&= \int_0^{T^m} |u''(t)|e^{-\rho t}\left[H_{Kx}(t)) - H_{Ky}(t)\right] dt \\
&= \int_0^{T^m} |u''(t)|e^{-\rho t}\int_0^t k_y'(\tau)e^{F(\tau)}\left[U^F(\tau) - B^F(\tau)\right] d\tau dt < 0
\end{aligned}$$

because $U^F(\tau) < B^F(\tau)$, implies that:

$$-G_{\lambda y}\frac{d\lambda_{y0}}{d\hat{K}_x} < G_{Kx} - G_{Ky} < 0 \implies \frac{d\lambda_{y0}}{d\hat{K}_x} > 0$$

Sensitivity of T^m along the stabilisation border

Expressing the differentiation of T^m:

$$\frac{dT^m}{dK_y^m} = \frac{\partial T^m}{\partial\hat{K}_x}\frac{d\hat{K}_x}{dK_y^m} + \frac{\partial T^m}{\partial K_y^m}$$

We want to show that $\partial T^m/\partial K_y^m > \partial T^m/\partial\hat{K}_x$.

$$\begin{aligned}
&\frac{\partial T^m}{\partial K_y^m} - \frac{\partial T^m}{\partial\hat{K}_x} = \\
&\frac{1}{D}\left[(1 + H_{Ky})e^F - H_{\lambda y}B^Fe^F - (1 + H_{Kx})e^F + H_{\lambda y}(U^Fe^F + \lambda_y^{s'})\right] \\
&= H_{\lambda y}\frac{\lambda_y^{s'}}{D} + \frac{e^F}{D}\left[H_{Ky} - H_{\lambda y}B^F - H_{Kx} + H_{\lambda y}U^F\right] \\
&= H_{\lambda y}\frac{\lambda_y^{s'}}{D} + \frac{e^F}{D}\phi(t)
\end{aligned}$$

The first term of the sum on the r.h.s. is > 0 because $D < 0$ and $\lambda_y^{s'} < 0$. Using the expressions of $G_{\lambda y}$ and G_{Ky}:

$$\phi(t) = \int_0^{T^m} k_y'(t)e^{F(t)}\left[(B^F(t) - U^F(t)) - (B^F - U^F)\right] dt < 0$$

because $B^F - U^F > B^F(t) - U^F(t) > 0$, thus the second term is also > 0 because $D < 0$, from which we conclude that $\partial T^m/\partial K_y^m > \partial T^m/\partial\hat{K}_x$.

Since we have already shown that $|d\hat{K}_x/dK_y^m| < 1$, we conclude that the direct term $\partial T^m/\partial K_y^m$ dominates the indirect term $(\partial T^m/\partial\hat{K}_x)(\partial\hat{K}_x/\partial K_y^m)$, implying that $dT^m/dK_y^m < 0$. Conversely $dT^m/d\hat{K}_x > 0$ because $dT^m/dK_y^m < 0$ and $dK_y^m/d\hat{K}_x < 0$.

Proof of the Proposition P.6

Our previous results show the claims of the Proposition. We have shown that $d\hat{K}_x/K_y^m < 0$, the claim (i) of the Proposition. We have shown that $d\lambda_{y0}/dK_y^m < 0$ and $d\lambda_{y0}/d\hat{K}_x$, the claims (ii) and $(iv$ of the Proposition. Last we have shown that $dT^m/dK_y^m < 0$ and $dT^m/d\hat{K}_x > 0$, the claims (iii) and (v) of the Proposition.

A.4 The general case

We proceed recursively. In the next subsection we study the pure scraping policies initiated from an arbitrary vector (K_x^0, K_y^0, Z^0). We show the existence of restrictions on the initial conditions implying decreasing relationships between K_x^0 and Z^0, and between K_x^0 and K_y^0, taking the form of scraping borders in the phase planes (Z, K_x) and (K_y, K_x). These policies correspond to the scraping phase of the central scenario but now inherited from a vector (K_x^s, K_y^s, Z^s) at the end of the stabilisation phase. The follow up subsection studies the stabilisation phase using the implications of the relationships between the state variables at the end of the phase. It shows that these restrictions imply the existence of other relationships between the elements of the state vector, $(\hat{K}_x, \hat{K}_y, \hat{Z})$ at the beginning of the phase, taking the form of decreasing stabilisation borders in the planes (Z, K_x) and (K_y, K_x).

A.4.1 Pure scraping policies

Our first objective is to assess the qualitative properties of a relationship between K_x^0 and Z^0 implied by a pure scraping policy under a carbon stock constraint, the context of a Scenario 1. We show that this relationship itself implies the existence of another relation between K_x^0 and K_y^0. The two relations describe what we call the *scraping borders* in the planes (Z, K_x) and (K_y, K_x).

We go through several steps. We first study the system defining the joint dynamics of $K_y(t)$ and $\lambda_y(t)$ and determine the sensitivity of the optimal trajectory $\{K_y(t), \lambda_y(t)\}$ to the vector $(K_x^0, K_y^0, \lambda_{y0}, \lambda_{Z0})$ where $\lambda_{y0} = \lambda_y(0)$ and $\lambda_{Z0} = \lambda_Z(0)$ are the starting values of λ_y and λ_Z respectively. We denote by T^s, the time length of the scraping phase. In the context of a Scenario 1, made of a unique scraping phase, note that $T^s = \underline{t}_Z$. We also denote by $K_y^Z \equiv K_y(T^s)$, the accumulated capital stock in the renewable energy sector at the end of the scraping phase, that is at the beginning of the ceiling phase. Next we exploit our findings to identify the vector $(K_y^Z, \lambda_{y0}, \lambda_{Z0}, T^s)$ as a function of (K_x^0, K_y^0), the initial capital vector.

The differentiation of λ_y and K_y

Let $\zeta r_x \lambda_{Z0} \equiv \zeta r_x \lambda_Z(0)$, be the initial level of the carbon price. Differentiating the renewable energy capital accumulation equation over a time interval, $[t_0, t)$, yields:

$$dK_y(t) = dK_y(t_0) + \int_{t_0}^{t} k'_y(\tau) d\lambda_y(\tau) d\tau$$

Integrating the motion of $\lambda_y(t)$ over the time interval, $[t_0, t)$ gets:

$$\lambda_y(t) e^{-\rho(t-t_0)} = \lambda_y(t_0) - \int_{t_0}^{t} \left[c_x + m_x + \zeta r_x \lambda_Z(\tau) - a_y - m_{Ky} - b'_y(K_y(\tau)) \right] e^{-\rho(\tau - t_0)} d\tau$$

Differentiating after multiplying both sides by $e^{-\rho t_0}$, we obtain:

$$d\lambda_y(t) e^{-\rho t} = d\lambda_y(t_0) e^{-\rho t_0} - \zeta r_x d\lambda_{Z0} \left(\int_{t_0}^{t} e^{\alpha \tau} d\tau \right) + \int_{t_0}^{t} b''_y(\tau) e^{-\rho \tau} dK_y(\tau) d\tau$$

Inserting the expression of dK_y:

$$\begin{aligned} d\lambda_y(t) e^{-\rho t} =& d\lambda_y(t_0) e^{-\rho t_0} - \zeta r_x d\lambda_{Z0} \left(\int_{t_0}^{t} e^{\alpha \tau} d\tau \right) \\ &+ \left(\int_{t_0}^{t} b''_y(\tau) e^{-\rho \tau} d\tau \right) dK_y(t_0) + \int_{t_0}^{t} b''_y(\tau) e^{-\rho \tau} \int_{t_0}^{\tau} k'_y(s) d\lambda_y(s) ds d\tau \end{aligned}$$

Integrating both sides over, $[t_0, T^s)$:

$$\begin{aligned} \int_{t_0}^{T^s} d\lambda_y(t) e^{-\rho t} dt =& d\lambda_y(t_0) e^{-\rho t_0} (T^s - t_0) \\ & - \zeta r_x d\lambda_{Z0} \left(\int_{t_0}^{T^s} \int_{t_0}^{t} e^{\alpha \tau} d\tau dt \right) \\ & + \left(\int_{t_0}^{T^s} \int_{t_0}^{t} b''_y(\tau) e^{-\rho \tau} d\tau dt \right) dK_y(t_0) \\ & + \int_{t_0}^{T^s} \int_{t_0}^{t} b''_y(\tau) e^{-\rho \tau} \int_{t_0}^{\tau} k'_y(s) d\lambda_y(s) ds d\tau dt \end{aligned}$$

Inverting the integration order:

$$\begin{aligned} & \int_{t_0}^{T^s} \int_{t_0}^{t} e^{\alpha \tau} d\tau dt = \int_{t_0}^{T^s} e^{\alpha t} (T^s - t) dt \\ & \int_{t_0}^{T^s} \int_{t_0}^{t} b''_y(\tau) e^{-\rho \tau} d\tau dt = \int_{t_0}^{T^s} b''_y(t) e^{-\rho t} (T^s - t) dt \\ & \int_{t_0}^{T^s} \int_{t_0}^{t} b''_y(\tau) e^{-\rho \tau} \int_{t_0}^{\tau} k'_y(s) d\lambda_y(s) ds d\tau dt = \\ & \int_{t_0}^{T^s} b''_y(t) e^{-\rho t} (T^s - t) \int_{t_0}^{t} k'_y(\tau) d\lambda_y(\tau) d\tau dt = \\ & \int_{t_0}^{T^s} k'_y(t) d\lambda_y(t) \int_{t}^{T^s} b''_y(\tau) e^{-\rho \tau} (T^s - \tau) d\tau dt \end{aligned}$$

Thus:

$$\begin{aligned}\int_{t_0}^{T^s} d\lambda_y(t)e^{-\rho t}dt =& d\lambda_y(t_0)e^{-\rho t_0}(T^s-t_0) - \zeta r_x d\lambda_{Z0}\left(\int_{t_0}^{T^s} e^{\alpha t}(T^s-t)dt\right)\\ &+\left(\int_{t_0}^{T^s} b_y''(t)e^{-\rho t}(T^s-t)dt\right)dK_y(t_0)\\ &+\int_{t_0}^{T^s} k_y'(t)d\lambda_y(t)\int_t^{T^s} b_y''(\tau)e^{-\rho\tau}(T^s-\tau)d\tau dt\end{aligned}$$

Rearranging:

$$\begin{aligned}&\int_{t_0}^{T^s} d\lambda_y(t)\left[e^{-\rho t}-k_y'(t)\int_t^{T^s} b_y''(\tau)e^{-\rho\tau}(T^s-\tau)d\tau\right]dt\\ &= d\lambda_y(t_0)e^{-\rho t_0}(T^s-t_0) - \zeta r_x d\lambda_{Z0}\left(\int_{t_0}^{T^s} e^{\alpha t}(T^s-t)dt\right)\\ &+\left(\int_{t_0}^{T^s} b_y''(t)e^{-\rho t}(T^s-t)dt\right)dK_y(t_0)\end{aligned}$$

Differentiating with respect to t_0 for a given $dK_y(t_0)$ we obtain:

$$\begin{aligned}&-d\lambda_y(t_0)\left[e^{-\rho t_0}-k_y'(t_0)\int_{t_0}^{T^s} b_y''(\tau)e^{-\rho\tau}(T^s-\tau)d\tau\right]=\\ &\frac{d}{dt_0}(d\lambda_y(t_0))e^{-\rho t_0}(T^s-t_0) - \rho d\lambda_y(t_0)e^{-\rho t_0}(T^s-t_0) - d\lambda_y(t_0)e^{-\rho t_0}\\ &-\zeta r_x d\lambda_{Z0}\left(-e^{\alpha t_0}(T^s-t_0)\right) - b_y''(t_0)e^{-\rho t_0}(T^s-t_0)dK_y(t_0)\end{aligned}$$

Simplifying and multiplying both sides by $e^{\rho t_0}/(T^s-t_0)$ gets:

$$\begin{aligned}&d\lambda_y(t_0)\frac{k_y'(t_0)e^{\rho t_0}}{T^s-t_0}\int_{t_0}^{T^s} b_y''(\tau)e^{-\rho\tau}(T^s-\tau)d\tau =\\ &\frac{d}{dt_0}(d\lambda_y(t_0)) - \rho d\lambda_y(t_0) + \zeta r_x d\lambda_{Z0}e^{(\rho+\alpha)t_0} - b_y''(t_0)dK_y(t_0)\end{aligned}$$

Denoting $\theta_y \equiv d\lambda_y$, the above expression is a differential equation of the form:

$$\begin{aligned}\dot{\theta}_y(t_0) =& \left[\rho+\frac{k_y'(t_0)e^{\rho t_0}}{T^s-t_0}\int_{t_0}^{T^s} b_y''(\tau)e^{-\rho\tau}(T^s-\tau)d\tau\right]\theta_y(t_0)\\ &-\zeta r_x d\lambda_{Z0}e^{(\rho+\alpha)t_0} + b_y''(t_0)dK_y(t_0)\end{aligned}$$

Let:

$$f(t) \equiv \rho + \frac{k_y'(t)e^{\rho t}}{T^s-t}\int_t^{T^s} b_y''(\tau)e^{-\rho\tau}(T^s-\tau)d\tau \quad \text{and} \quad F(t)=\int_0^t f(\tau d\tau$$

Then the general solution of the differential equation reads:

$$d\lambda_y(t) = e^{F(t)}\left\{d\lambda_{y0} - \zeta r_x J_\alpha(t)d\lambda_{Z0} + B_y(t)dK_y^0\right\} \tag{A.4.1}$$

where:

$$\begin{aligned}J_\alpha(t) &\equiv \int_0^t e^{(\rho+\alpha)\tau-F(\tau)}d\tau\\ B_y(t) &\equiv \int_0^t b_y''(\tau)e^{-F(\tau)}d\tau\end{aligned}$$

Then $dK_y(t)$ is easily deduced from (A.4.1):

$$\begin{aligned}
dK_y(t) &= dK_y^0 + \int_0^t k_y'(\tau) d\lambda_y(\tau) d\tau \\
&= dK_y^0 + \int_0^t k_y'(\tau) e^{F(\tau)} \left\{ d\lambda_{y0} - \zeta r_x J_\alpha(\tau) d\lambda_{Z0} + J_b(\tau) dK_y^0 \right\} d\tau \\
&= \left(1 + \int_0^t k_y'(\tau) e^{F(\tau)} J_b(\tau) d\tau\right) dK_y^0 + \left(\int_0^t k_y'(\tau) e^{F(\tau)} d\tau\right) d\lambda_{y0} \\
&- \zeta r_x \left(\int_0^t k_y'(\tau) e^{F(\tau)} J_\alpha(\tau) d\tau\right) d\lambda_{Z0}
\end{aligned}$$

Denoting:

$$\begin{aligned}
C_{\lambda y}(t) &\equiv \int_0^t k_y'(\tau) e^{F(\tau)} d\tau \\
C_{\lambda Z} &\equiv \int_0^t k_y'(\tau) e^{F(\tau)} J_\alpha(\tau) d\tau \\
C_{K y}(t) &\equiv \int_0^t k_y'(\tau) e^{F(\tau)} B_y(\tau) d\tau
\end{aligned}$$

we get in compact form:

$$dK_y(t) = C_{\lambda y}(t) d\lambda_{y0} - \zeta r_x C_{\lambda Z}(t) d\lambda_{Z0} + (1 + C_{Ky}(t))\, dK_y^0 \tag{A.4.2}$$

System identifying the optimum and existence of the solution

Once the vector $(K_y^Z, \lambda_{y0}, \lambda_{Z0}, T^s)$ is determined, the optimal path is determined. Let $K_x^s \equiv K_x(0)$ and $K_y^s \equiv K_y(0)$, denote the starting values at the beginning of the scraping phase. The vector is the solution of the following system of 4 conditions.

1. The renewable energy capital accumulation condition:

$$K_y^Z = K_y^s + \int_0^{T^s} k_y(t) dt$$

2. The continuity condition over $\lambda_y(t)$ at time T^s:

$$\lambda_y^Z(K_y^Z) = \lambda_y(T^s)$$

   where $\lambda_y^Z(K_y^Z)$ expresses the decreasing relationship between K_y and λ_y at the beginning of the ceiling phase.

3. The energy price continuity condition at time 0:

$$u'(K_x^s + K_y^s) = c_x + m_x + \zeta r_x \lambda_{Z0}$$

4. The continuity condition over $\lambda_Z(t)$ at time T^s:

$$\lambda_Z^Z(K_y^Z) = \lambda_{Z0} e^{(\rho+\alpha)T^s}$$

   where $\lambda_Z^Z(K_y^Z)$ is defined by: $u'(\bar{K}_x + K_y^Z) = c_x + m_x + \zeta r_x \lambda_Z^Z$ through the energy price continuity condition at time T^s.

We now differentiate the system to show the existence of a unique vector solution of this system for given K_x^s and K_y^s.

Differentiation of the capital accumulation condition

Differentiating and using the expression (A.4.2) of $dK_y(t)$, denoting $k_y^Z \equiv k_y(T^s)$:

$$dK_y^Z = k_y^Z dT^s + C_{\lambda y}(T^s) d\lambda_{y0} - \zeta r_x C_{\lambda Z}(T^s) d\lambda_{Z0} + \left(1 + C_{Ky}(T^s)\right) dK_y^s$$

To simplify the notations let $C_{\lambda y} \equiv C_{\lambda y}(T^s)$, $C_{\lambda Z} \equiv C_{\lambda Z}(T^s)$ and $C_{Ky} \equiv C_K(T^s)$, we obtain after rearranging:

$$dK_y^Z - C_{\lambda y} d\lambda_{y0} + \zeta r_x C_{\lambda Z} d\lambda_{Z0} - k_y^Z dT^s = \left[1 + C_{Ky}\right] dK_y^s \tag{A.4.3}$$

Differentiation of the continuity condition over λ_y at time T^s

Differentiating at time T^s yields:

$$\lambda_y^{Z\prime} dK_y^Z e^{-\rho T^s} = \dot{\lambda}_y^Z e^{-\rho T^s} dT^s + d\lambda_{y0} - \int_0^{T^s} d\beta_y(t) dt$$

Taking account of (A.4.1) evaluated at time T^s:

$$d\lambda_{y0} - \int_0^{T^s} d\beta_y(t) dt = e^{-\rho T^s} d\lambda_y(T^s) = e^{-\rho T^s + F(T^s)} \left[d\lambda_{y0} - \zeta r_x J_\alpha(T^s) d\lambda_{Z0} + B_y(T^s) dK_y^s\right]$$

Then we obtain:

$$\lambda_y^{Z\prime} dK_y^Z e^{-\rho T^s} = \dot{\lambda}_y^Z e^{-\rho T^s} dT^s + e^{-\rho T^s + F(T^s)} \left[d\lambda_{y0} - \zeta r_x J_\alpha(T^s) d\lambda_{Z0} + B_y(T^s) dK_y^s\right]$$

Denoting $F \equiv F(T^s)$, $J_\alpha \equiv J_\alpha(T^s)$, $B_y \equiv B_y(T^s)$, simplifying and rearranging, this is equivalent to:

$$-\lambda_y^{Z\prime} dK_y^Z + e^F d\lambda_{y0} - \zeta r_x J_\alpha e^F d\lambda_{Z0} + \dot{\lambda}_y^Z dT^s = -B_y e^F dK_y^s \tag{A.4.4}$$

Differentiation of the price continuity condition at time 0

Differentiating yields:

$$-dK_x^s - dK_y^s = \frac{\zeta r_x}{|u''(0)|} d\lambda_{Z0} \tag{A.4.5}$$

Differentiation of the continuity condition over $\lambda_Z(t)$ at time T^s

Differentiating yields:

$$dK_y^Z = \frac{\zeta r_x}{u''(T^s)} d\lambda_{Z0} e^{(\rho+\alpha)T^s} + \frac{\zeta r_x}{u''(T^s)} \dot{\lambda}_Z(t^{S-}) dT^s$$

Denote $k^Z = \dot{K}(t^{S-}) = \dot{K}_x(t^{S-}) + k_y(T^s)$, then $k^Z = (\zeta r_x / u''(T^s)) \dot{\lambda}_Z(t^{S-})$ implies that:

$$dK_y^Z + \frac{\zeta r_x}{|u''(T^s)|} d\lambda_{Z0} e^{(\rho+\alpha)T^s} - k^Z dT^s = 0 \tag{A.4.6}$$

Existence and unicity of the solution

The differentiated system (A.4.3)-(A.4.6) writes in matrix form:

$$\begin{bmatrix} 1 & -C_{\lambda_y} & \zeta r_x C_{\lambda_Z} & -k_y^Z \\ -\lambda_y^{Z'} & e^F & -\zeta r_x J_\alpha e^F & \dot{\lambda}_y^Z \\ 0 & 0 & \frac{\zeta r_x}{|u''(0)|} & 0 \\ 1 & 0 & \frac{\zeta r_x}{|u''(T^s)|} e^{(\rho+\alpha)T^s} & -k^Z \end{bmatrix} \begin{bmatrix} dK_y^Z \\ d\lambda_{y0} \\ d\lambda_{Z0} \\ dT^s \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} dK_x^s + \begin{bmatrix} 1 + C_{Ky} \\ -B_y e^F \\ -1 \\ 0 \end{bmatrix} dK_y^s$$

Let Δ be the determinant of the system. Then developing with respect to the third line:

$$\Delta = \frac{\zeta r_x}{|u''(0)|} \begin{vmatrix} 1 & -C_{\lambda_y} & -k_y^Z \\ -\lambda_y^{Z'} & e^F & \dot{\lambda}_y^Z \\ 1 & 0 & -k^Z \end{vmatrix}$$

Remarking that $\lambda_y^{Z'} = \dot{\lambda}_y^Z / k_y^Z$, denoting $k_x^Z = \dot{K}_x(T^{s-})$, so that $k^Z = k_x^Z + k_y^Z$, multiplying the first column by k_y^Z and adding to the third column yield:

$$\Delta = \frac{\zeta r_x}{|u''(0)|} \begin{vmatrix} 1 & -C_{\lambda_y} & 0 \\ -\lambda_y^{Z'} & e^F & 0 \\ 1 & 0 & -k_x^Z \end{vmatrix} = -\frac{\zeta r_x k_x^Z}{|u''(0)|} \left(e^F - C_{\lambda y} \lambda_y^{Z'} \right) > 0$$

because $k_x^Z < 0$ and $\lambda_y^{Z'} < 0$. Hence the system admits a unique solution.

Sensitivity analysis of λ_{y0} and λ_{Z0}

For further use in the construction of the scraping borders we perform a sensitivity analysis for λ_{y0} and λ_{Z0}. Applying the Cramer rule and developing the resulting determinant with respect to the second column, we obtain:

$$\frac{\partial \lambda_{y0}}{\partial K_x^s} = \frac{1}{\Delta} \begin{vmatrix} 1 & 0 & \zeta r_x C_{\lambda_Z} & -k_y^Z \\ -\lambda_y^{Z'} & 0 & -\zeta r_x J_\alpha e^F & \dot{\lambda}_y^Z \\ 0 & -1 & -\frac{\zeta r_x}{|u''(0)|} & 0 \\ 1 & 0 & \frac{\zeta r_x}{|u''(T^s)|} e^{(\rho+\alpha)T^s} & -k^Z \end{vmatrix} = \frac{1}{\Delta} \begin{vmatrix} 1 & \zeta r_x C_{\lambda_Z} & -k_y^Z \\ -\lambda_y^{Z'} & -\zeta r_x J_\alpha e^F & \dot{\lambda}_y^Z \\ 1 & \frac{\zeta r_x}{|u''(T^s)|} e^{(\rho+\alpha)T^s} & -k^Z \end{vmatrix}$$

Multiplying the first column by k_y^Z and adding to the third column yield:

$$\frac{\partial \lambda_{y0}}{\partial K_x^s} = \frac{1}{\Delta} \begin{vmatrix} 1 & \zeta r_x C_{\lambda_Z} & 0 \\ -\lambda_y^{Z'} & -\zeta r_x J_\alpha e^F & 0 \\ 1 & \frac{\zeta r_x}{|u''(T^s)|} e^{(\rho+\alpha)T^s} & -k_x^Z \end{vmatrix} = \frac{\zeta r_x k_x^Z}{\Delta} \left(J_\alpha e^F - C_{\lambda Z} \lambda_y^{Z'} \right) < 0$$

because $k_x^Z < 0$, $\Delta > 0$ and $\lambda_y^{Z'} < 0$.

Applying once again the Cramer rule:

$$\frac{\partial \lambda_{y0}}{\partial K_y^s} = \frac{1}{\Delta} \begin{vmatrix} 1 & 1 + C_{Ky} & \zeta r_x C_{\lambda_Z} & -k_y^Z \\ -\lambda_y^{Z'} & -B_y e^F & -\zeta r_x J_\alpha e^F & \dot{\lambda}_y^Z \\ 0 & -1 & \frac{\zeta r_x}{|u''(0)|} & 0 \\ 1 & 0 & \frac{\zeta r_x}{|u''(T^s)|} e^{(\rho+\alpha)T^s} & -k^Z \end{vmatrix}$$

Multiplying the first column by k_y^Z and adding to the third column yield:

$$\frac{\partial \lambda_{y0}}{\partial K_y^s} = \frac{1}{\Delta} \begin{vmatrix} 1 & 1 + C_{Ky} & \zeta r_x C_{\lambda_Z} & 0 \\ -\lambda_y^{Z'} & -B_y e^F & -\zeta r_x J_\alpha e^F & 0 \\ 0 & -1 & \frac{\zeta r_x}{|u''(0)|} & 0 \\ 1 & 0 & \frac{\zeta r_x}{|u''(T^s)|} e^{(\rho+\alpha)T^s} & -k_x^Z \end{vmatrix} = -\frac{k_x^Z}{\Delta} \begin{vmatrix} 1 & 1 + C_{Ky} & \zeta r_x C_{\lambda_Z} \\ -\lambda_y^{Z'} & -B^y e^F & -\zeta r_x J_\alpha e^F \\ 0 & -1 & \frac{\zeta r_x}{|u''(0)|} \end{vmatrix}$$

Developing with respect to the third line:

$$\frac{\partial \lambda_{y0}}{\partial K_y^s} = -\frac{k_x^Z}{\Delta} \left\{ \begin{vmatrix} 1 & \zeta r_x C_{\lambda_Z} \\ -\lambda_y^{Z'} & -\zeta r_x J_\alpha e^F \end{vmatrix} + \frac{\zeta r_x}{|u''(0)|} \begin{vmatrix} 1 & 1 + C_{Ky} \\ -\lambda_y^{Z'} & -B^y e^F \end{vmatrix} \right\}$$

Hence:

$$\begin{aligned} \frac{\partial \lambda_{y0}}{\partial K_y^s} &= -\frac{\zeta r_x k_x^Z}{\Delta} \left\{ -\left(J_\alpha e^F - C_{\lambda Z} \lambda_y^{Z'} \right) + \frac{1}{|u''(0)|} \left(-B_y e^F + (1 + C_{Ky}) \lambda_y^{Z'} \right) \right\} \\ &= \frac{\partial \lambda_{y0}}{\partial K_x^s} - \frac{\zeta r_x k_x^Z}{\Delta |u''(0)|} \left(-B_y e^F + (1 + C_{Ky}) \lambda_y^{Z'} \right) \equiv \frac{\partial \lambda_{y0}}{\partial K_x^s} - h < 0 \end{aligned}$$

because $h > 0$ since $k_x^Z < 0$ and $\lambda_y^{Z'} < 0$. We conclude that $\partial \lambda_{y0} / \partial K_y^s < 0$. For further reference we list below the expressions of $\partial \lambda_0 / \partial K_x^s$ and $\partial \lambda_{y0} / \partial K_y^s$:

$$\frac{\partial \lambda_{y0}}{\partial K_x^s} = \frac{\zeta r_x k_x^Z}{\Delta} \left(J_\alpha e^F - C_{\lambda Z} \lambda_y^{Z'} \right) \tag{A.4.7}$$

$$\frac{\partial \lambda_{y0}}{\partial K_y^s} = \frac{\zeta r_x k_x^Z}{\Delta} \left(J_\alpha e^F - C_{\lambda Z} \lambda_y^{Z'} \right) - \frac{\zeta r_x k_x^Z}{\Delta |u''(0)|} \left(-B^y e^F + (1 + C_{Ky}) \lambda_y^{Z'} \right) \tag{A.4.8}$$

Last, we get directly from (A.4.5):

$$\frac{\partial \lambda_{Z0}}{\partial K_x^s} = \frac{\partial \lambda_{Z0}}{\partial K_y^s} = -\frac{|u''(0)|}{\zeta r_x} < 0 \tag{A.4.9}$$

Implication of the core dynamics of $\lambda_y(t)$ and $K_y(t)$

For a given λ_{Z0}, the joint dynamics of $\{K_y(t), \lambda_y(t)\}$ is determined by a semi-autonomous differential system. Observe that through (A.4.6), $dK_y^Z = k^Z dT^s$ for a fixed λ_{Z0}. If $dT^s / dK_y^s > 0$, then $dK_y^Z < 0$ because $k^Z < 0$ would imply that the system should take more time to evolve from a larger initial position K_y^s to a smaller final position K_y^Z, an impossibility. Thus it must be the case that $dT^s / dK_y^s < 0$ for a fixed λ_{Z0}. Taking account of $dK_y^Z = k^Z dT^s$, the system (A.4.3)-(A.4.4) reduces for a fixed λ_{Z0} to:

$$\begin{aligned} -C_{\lambda y} d\lambda_{y0} + k_x^Z dT^s &= (1 + C_{Ky}) dK_y^s \\ e^F d\lambda_{y0} - \lambda_y^{Z'} k_x^Z dT^s &= -B_y e^F dK_y^s \end{aligned}$$

The determinant of this system reads $D = k_x^Z \left(C_{\lambda y} \lambda_y^{Z'} - e^F \right)$ which is > 0 because $k_x^Z < 0$ and $\lambda_y^{Z'} < 0$. Then we get through the Cramer rule:

$$\frac{\partial T^s}{\partial K_y^s} = \frac{1}{D} \begin{vmatrix} -C_{\lambda y} & 1 + C_{Ky} \\ e^F & -B_y e^F \end{vmatrix} = \frac{e^F}{D} \left[C_{\lambda y} B_y - (1 + C_{Ky}) \right]$$

and $\partial T^s / dK_y^s < 0$ implies that:

$$m_0 \equiv 1 + C_{Ky} - C_{\lambda y} B_y > 0$$

Sensitivity of K_y^Z and T^s

We get through the Cramer rule:

$$\frac{\partial K_y^Z}{\partial K_x^s} = \frac{1}{\Delta}\begin{vmatrix} 0 & -C_{\lambda_y} & \zeta r_x C_{\lambda_Z} & -k_y^Z \\ 0 & e^F & -\zeta r_x J_\alpha e^F & \dot{\lambda}_y^Z \\ -1 & 0 & \frac{\zeta r_x}{|u''(0)|} & 0 \\ 0 & 0 & \frac{\zeta r_x}{|u''(T^s)|} e^{(\rho+\alpha)T^s} & -k^Z \end{vmatrix} = -\frac{1}{\Delta}\begin{vmatrix} -C_{\lambda_y} & \zeta r_x C_{\lambda_Z} & -k_y^Z \\ e^F & -\zeta r_x J_\alpha e^F & \dot{\lambda}_y^Z \\ 0 & \frac{\zeta r_x}{|u''(T^s)|} e^{(\rho+\alpha)T^s} & -k^Z \end{vmatrix}$$

Remembering that $\dot{\lambda}_y^Z = \lambda_y^{Z\prime} k_y^Z$ and developing with respect to the third line we obtain:

$$\frac{\partial K_y^Z}{\partial K_x^s} = \frac{\zeta r_x}{\Delta}\left\{ \frac{e^{(\rho+\alpha)T^s}}{|u''(T^s)|} k_y^Z \left(e^F - C_{\lambda y} \lambda_y^{Z\prime} \right) + k^Z e^F \left(C_{\lambda y} J_\alpha - C_{\lambda Z} \right) \right\}$$

The first term of the sum inside curly brackets is > 0 while the second term is < 0 because $k^Z < 0$ and $C_{\lambda y} J_\alpha - C_{\lambda Z} > 0$ in view of the expressions of $C_{\lambda y}$ and $C_{\lambda Z}$, thus the sign of $\partial K_y^Z / \partial K_x^s$ is indeterminate. Applying the Cramer rule we get also:

$$\frac{\partial K_y^Z}{\partial K_y^s} = \frac{1}{\Delta}\begin{vmatrix} 1 + C_{Ky} & -C_{\lambda_y} & \zeta r_x C_{\lambda_Z} & -k_y^Z \\ -B_y e^F & e^F & -\zeta r_x J_\alpha e^F & \dot{\lambda}_y^Z \\ -1 & 0 & \frac{\zeta r_x}{|u''(0)|} & 0 \\ 0 & 0 & \frac{\zeta r_x}{|u''(T^s)|} e^{(\rho+\alpha)T^s} & -k^Z \end{vmatrix}$$

Developing with respect to the third line, remarking that the first determinant is in fact $\partial K_y^Z / \partial K_x^s$, yields:

$$\begin{aligned} \frac{\partial K_y^Z}{\partial K_y^s} &= \frac{\partial K_y^Z}{\partial K_x^s} + \frac{1}{\Delta}\frac{\zeta r_x}{|u''(0)|}\begin{vmatrix} 1 + C_{Ky} & -C_{\lambda_y} & -k_y^Z \\ -B_y e^F & e^F & \dot{\lambda}_y^Z \\ 0 & 0 & -k^Z \end{vmatrix} \\ &= \frac{\partial K_y^Z}{\partial K_x^s} + \frac{1}{\Delta}\frac{\zeta r_x}{|u''(0)|}(-k^Z) e^F \left[(1 + C_{Ky}) - B_y C_{\lambda y} \right] \\ &= \frac{\partial K_y^Z}{\partial K_x^s} + \frac{1}{\Delta}\frac{\zeta r_x}{|u''(0)|}(-k^Z) e^F m_0 \end{aligned}$$

The first term is of indeterminate sign while the second term is > 0 because $k^Z < 0$ and $m_0 > 0$, as shown before.

Turning to T^s, we get through the Cramer rule:

$$\frac{\partial T^s}{\partial K_x^s} = \frac{1}{\Delta}\begin{vmatrix} 1 & -C_{\lambda_y} & \zeta r_x C_{\lambda_Z} & 0 \\ -\lambda_y^{Z\prime} & e^F & -\zeta r_x J_\alpha e^F & 0 \\ 0 & 0 & \frac{\zeta r_x}{|u''(0)|} & -1 \\ 1 & 0 & \frac{\zeta r_x}{|u''(T^s)|} e^{(\rho+\alpha)T^s} & 0 \end{vmatrix} = \frac{1}{\Delta}\begin{vmatrix} 1 & -C_{\lambda_y} & \zeta r_x C_{\lambda_Z} \\ -\lambda_y^{Z\prime} & e^F & -\zeta r_x J_\alpha e^F \\ 1 & 0 & \frac{\zeta r_x}{|u''(T^s)|} e^{(\rho+\alpha)T^s} \end{vmatrix}$$

$$= \frac{\zeta r_x}{\Delta}\left[e^F \left(C_{\lambda y} J_\alpha - C_{\lambda Z} \right) + \frac{e^{(\rho+\alpha)T^s}}{|u''(T^s)|}\left(e^F - \lambda_y^{Z\prime} C_{\lambda y} \right) \right]$$

Because $C_{\lambda y} J_\alpha - C_{\lambda Z} > 0$ and $\lambda_y^{Z\prime} < 0$, the term into square brackets is > 0 and thus

$\partial T^s/\partial K_x^s > 0$. Applying once again the Cramer rule:

$$\begin{aligned}
\frac{\partial T^s}{\partial K_y^s} &= \frac{1}{\Delta}\begin{vmatrix} 1 & -C_{\lambda_y} & \zeta r_x C_{\lambda_Z} & 1+C_{Ky} \\ -\lambda_y^{Z'} & e^F & -\zeta r_x J_\alpha e^F & -B_y e^F \\ 0 & 0 & \frac{\zeta r_x}{|u''(0)|} & -1 \\ 1 & 0 & \frac{\zeta r_x}{|u''(T^s)|}e^{(\rho+\alpha)T^s} & 0 \end{vmatrix} \\
&= \frac{\partial T^s}{\partial K_x^s} + \frac{\zeta r_x}{\Delta|u''(0)|}\begin{vmatrix} 1 & -C_{\lambda_y} & 1+C_{Ky} \\ -\lambda_y^{Z'} & e^F & -B_y e^F \\ 1 & 0 & 0 \end{vmatrix} \\
&= \frac{\partial T^s}{\partial K_x^s} + \frac{\zeta r_x}{\Delta|u''(0)|}\left[C_{\lambda y}B_y - (1+C_{Ky})\right] = \frac{\partial T^s}{\partial K_x^s} - \frac{\zeta r_x}{\Delta|u''(0)|}m_0
\end{aligned}$$

and $\partial T^s/\partial K_x^s > 0$ and $m_0 > 0$ imply that the sign of $\partial T^s/\partial K_y^s$ is indeterminate.

A.4.2 The scraping borders

We now introduce the carbon stock accumulation condition to show the existence of a relationship between Z^s, the initial carbon stock size at the beginning of the scraping phase, and the initial capital vector (K_x^s, K_y^s). Let $K(t) = K_x(t) + K_y(t)$, be the aggregate capital stock level at time t. We get from the optimality condition:

$$u'(K(t)) = c_x + m_x + \zeta r_x \lambda_{Z0} e^{(\rho+\alpha)t}$$

Differentiating yields:

$$dK(t) = \frac{\zeta r_x}{u''(t)} d\lambda_{Z0} e^{(\rho+\alpha)t} \tag{A.4.10}$$

Because $u''(t) < 0$, we conclude to the existence of a decreasing relationship between the carbon price and the aggregate capital stock, irrespective of the composition of this capital. Integrating over $[0, T^s)$, the motion of the carbon stock, we obtain:

$$\bar{Z}e^{\alpha T^s} = Z^s + \zeta r_x \int_0^{T^s} K_x(t)e^{\alpha t}dt = Z^s + \zeta r_x \int_0^{T^s} \left[K(t) - K_y(t)\right] e^{\alpha t}dt$$

Differentiating yields:

$$\alpha \bar{Z} e^{\alpha T^s} dT^s = dZ^s + \zeta r_x \bar{K}_x e^{\alpha T^s} dT^s + \zeta r_x \int_0^{T^s} \left[dK(t) - dK_y(t)\right] e^{\alpha t} dt$$

Since $\alpha \bar{Z} = \zeta r_x \bar{K}_x$, the dT^s terms cancel out and using (A.4.10), the above expression simplifies to:

$$dZ^s + (\zeta r_x)^2 \left[\int_0^{T^s} \frac{e^{(\rho+2\alpha)t}}{u''(t)} dt\right] d\lambda_{Z0} - \zeta r_x \int_0^{T^s} dK_y(t) e^{\alpha t} dt = 0$$

Denote:

$$U_{2\alpha}(t_0, t_1) \equiv -\left[\int_{t_0}^{t_1} \frac{e^{(\rho+2\alpha)t}}{u''(t)} dt\right] > 0$$

and $U_{2\alpha} \equiv U_{2\alpha}(0, T^s)$, then the above relationship reads:

$$dZ^s - (\zeta r_x)^2 U_{2\alpha} d\lambda_{Z0} - \zeta r_x \int_0^{T^s} dK_y(t) e^{\alpha t} dt = 0$$

Expressing the integral using the expression (A.4.2) of $dK_y(t)$:

$$\begin{aligned}
&\int_0^{T^s} dK_y(t)e^{\alpha t}dt \\
=&\int_0^{T^s} \left[C_{\lambda y}(t)d\lambda_{y0} - \zeta r_x C_{\lambda Z}(t)d\lambda_{Z0} + (1 + C_{Ky}(t))\, dK_y^s\right] e^{\alpha t}dt \\
=&\left(\int_0^{T^s} C_{\lambda y}(t)e^{\alpha t}dt\right) d\lambda_{y0} - \zeta r_x \left(\int_0^{T^s} C_{\lambda Z}(t)e^{\alpha t}dt\right) d\lambda_{Z0} \\
&+ \left(\int_0^{T^s} (1 + C_{Ky}(t))\, e^{\alpha t}dt\right) dK_y^s
\end{aligned}$$

Denote:

$$\begin{aligned}
C_{\lambda y}^{\alpha} &\equiv \int_0^{T^s} e^{\alpha t}C_{\lambda y}(t)dt \\
C_{\lambda Z}^{\alpha} &\equiv \int_0^{T^s} e^{\alpha t}C_{\lambda Z}(t)dt \\
C_{Ky}^{\alpha} &\equiv \int_0^{T^s} e^{\alpha t}\left[1 + C_{Ky}(t)\right] dt
\end{aligned}$$

then:

$$\int_0^{T^s} dK_y(t)e^{\alpha t}dt = C_{\lambda y}^{\alpha}d\lambda_{y0} - \zeta r_x C_{\lambda Z}^{\alpha}d\lambda_{Z0} + C_{Ky}^{\alpha}dK_y^s$$

We thus have shown that:

$$dZ^s - \zeta r_x C_{\lambda y}^{\alpha}d\lambda_{y0} - (\zeta r_x)^2 \left(U_{2\alpha} - C_{\lambda Z}^{\alpha}\right) d\lambda_{Z0} - \zeta r_x C_{Ky}^{\alpha}dK_y^s = 0$$

In the previous subsection we have also shown that λ_{y0} and λ_{Z0} are uniquely determined by the vector (K_x^s, K_y^s), thus an equivalent writing of the above equation is:

$$\begin{aligned}
&dZ^s - \zeta r_x \left[C_{\lambda y}^{\alpha}\frac{\partial \lambda_{y0}}{\partial K_x^s} + \zeta r_x \left(U_{2\alpha} - C_{\lambda Z}^{\alpha}\right)\frac{\partial \lambda_{Z0}}{\partial K_x^s}\right] dK_x^s \\
&- \zeta r_x \left[C_{\lambda y}^{\alpha}\frac{\partial \lambda_{y0}}{\partial K_y^s} + \zeta r_x \left(U_{2\alpha} - C_{\lambda Z}^{\alpha}\right)\frac{\partial \lambda_{Z0}}{\partial K_y^s} + C_{Ky}^{\alpha}\right] dK_y^s = 0
\end{aligned} \tag{A.4.11}$$

The relationship (A.4.11) is the implicit equation of a plane in the space (K_x^s, K_y^s, Z^s). Let:

$$\begin{aligned}
R_{Kx} &\equiv \zeta r_x \left[C_{\lambda y}^{\alpha}\frac{\partial \lambda_{y0}}{\partial K_x^s} + \zeta r_x \left(U_{2\alpha} - C_{\lambda Z}^{\alpha}\right)\frac{\partial \lambda_{Z0}}{\partial K_x^s}\right] \\
R_{Ky} &\equiv \zeta r_x \left[C_{\lambda y}^{\alpha}\frac{\partial \lambda_{y0}}{\partial K_y^s} + \zeta r_x \left(U_{2\alpha} - C_{\lambda Z}^{\alpha}\right)\frac{\partial \lambda_{Z0}}{\partial K_y^s} + C_{Ky}^{\alpha}\right]
\end{aligned}$$

then (A.4.11) reads in compact form:

$$dZ^s - R_{Kx}dK_x^s - R_{Ky}dK_y^s = 0$$

Remark that $\partial\lambda_{y0}/\partial K_y^s = \partial\lambda_{y0}/\partial K_x^s - h$ and $\partial\lambda_{Z0}/\partial K_y^s = \partial\lambda_{Z0}/\partial K_x^s$ imply that:

$$R_{Ky} = R_{Kx} + \zeta r_x \left[C_{Ky}^{\alpha} - C_{\lambda y}^{\alpha}h\right] \equiv R_{Kx} + H$$

Using the expressions of C^{α}_{Ky}, $C^{\alpha}_{\lambda y}$:

$$H = \zeta r_x \int_0^{T^s} e^{\alpha t}\left[(1 + C_{Ky}(t)) - C_{\lambda y}(t)h\right]dt \equiv \zeta r_x \int_0^{T^s} e^{\alpha t}\phi(t)dt$$

Using the expression of Δ, an equivalent expression of h reads:

$$\begin{aligned} h &= \frac{\zeta r_x k_x^Z}{\Delta|u''(0)|}\left(-B^y e^F + (1 + C_{Ky})\lambda_y^{Z'}\right) = \frac{\zeta r_x k_x^Z}{|u''(0)|\left(-\frac{\zeta r_x k_x^Z}{|u''(0)|}\right)} \frac{-B^y e^F + (1 + C_{Ky})\lambda_y^{Z'}}{e^F - C_{\lambda y}\lambda_y^{Z'}} \\ &= \frac{B^y e^F + (1 + C_{Ky})|\lambda_y^{Z'}|}{e^F + C_{\lambda y}|\lambda_y^{Z'}|} \end{aligned}$$

Evaluating $\phi(t)$ at the bounds, we get $\phi(0) = 1$ and:

$$\begin{aligned} \phi(T^s) &= (1 + C_{Ky}) - C_{\lambda y}h \\ &= \frac{(1 + C_{Ky})\left(e^F + C_{\lambda y}|\lambda_y^{Z'}|\right) - C_{\lambda y}\left(B^y e^F + (1 + C_{Ky})|\lambda_y^{Z'}|\right)}{e^F + C_{\lambda y}|\lambda_y^{Z'}|} \\ &= \frac{e^F\left[(1 + C_{Ky}) - C_{\lambda y}B_y\right] + |\lambda_y^{Z'}|\left[(1 + C_{Ky})C_{\lambda y} - C_{\lambda y}(1 + C_{Ky})\right]}{e^F + C_{\lambda y}|\lambda_y^{Z'}|} \\ &= \frac{e^F\left[(1 + C_{Ky}) - C_{\lambda y}B_y\right]}{e^F + C_{\lambda y}|\lambda_y^{Z'}|} = \frac{e^F m_0}{e^F + C_{\lambda y}|\lambda_y^{Z'}|} \end{aligned}$$

which is > 0 because $m_0 > 0$. Furthermore:

$$\begin{aligned} \dot{\phi}(t) &= \dot{C}_{Ky}(t) - \dot{C}_{\lambda y}(t)h = k_y'(t)e^{F(t)}\left[B_y(t) - h\right] \\ &= \frac{k_y'(t)e^{F(t)}}{e^F + C_{\lambda y}|\lambda_y^{Z'}|}\left[B_y(t)\left(e^F + C_{\lambda y}|\lambda_y^{Z'}|\right) - \left(B_y e^F + (1 + C_{Ky})|\lambda_y^{Z'}|\right)\right] \\ &= \frac{k_y'(t)e^{F(t)}}{e^F + C_{\lambda y}|\lambda_y^{Z'}|}\left[e^F\left(B_y(t) - B_y\right) + |\lambda_y^{Z'}|\left(C_{\lambda y}B_y(t) - (1 + C_{Ky})\right)\right] \end{aligned}$$

and $B_y(t) < B_y$, $C_{\lambda y}B_y(t) - (1 + C_{Ky}) < C_{\lambda y}B_y - (1 + C_{Ky}) < 0$ imply that $\dot{\phi}(t) < 0$. Thus $\phi(T^s) > 0$ implies that $\phi(t) > 0$, and hence that:

$$0 < \zeta r_x \phi(T^s) I_\alpha < H = \zeta r_x \int_0^{T^s} \phi(t)e^{\alpha t}dt < \zeta r_x I_\alpha$$

The scraping border in the (K_x, Z) plane

For $dK_y^s = 0$, (A.4.11) gives:

$$\frac{\partial Z^s}{\partial K_x^s} = R_{Kx}$$

Substituting for $\partial\lambda_{y0}/\partial K_x^s$ and $\partial\lambda_{Z0}/\partial K_x^s$ their respective expressions (A.4.7) and (A.4.9), we obtain:

$$\begin{aligned} R_{Kx} &= \zeta r_x C^{\alpha}_{\lambda y}\left[\frac{\zeta r_x k_x^Z}{\Delta}\left(J_\alpha e^F - C_{\lambda Z}\lambda_y^{Z'}\right)\right] + (\zeta r_x)^2\left[U_{2\alpha} - C^{\alpha}_{\lambda Z}\right]\left[-\frac{|u''(0)|}{\zeta r_x}\right] \\ &= \frac{(\zeta r_x)^2}{\Delta}\left\{C^{\alpha}_{\lambda y}k_x^Z\left(J_\alpha e^F - C_{\lambda Z}\lambda_y^{Z'}\right) - \frac{|u''(0)|}{\zeta r_x}\Delta\left(U_{2\alpha} - C^{\alpha}_{\lambda Z}\right)\right\} \end{aligned}$$

Inserting the expression of Δ and rearranging get:

$$\begin{aligned}
R_{Kx} &= \frac{(\zeta r_x)^2}{\Delta}\Big\{C^{\alpha}_{\lambda y}k^{Z}_{x}\left(J_\alpha e^F - C_{\lambda Z}\lambda^{Z'}_{y}\right)\\
&\quad -\frac{|u''(0)|}{\zeta r_x}\left[-\frac{\zeta r_x k^Z_x}{|u''(0)|}\left(e^F - C_{\lambda y}\lambda^{Z'}_{y}\right)\right]\left(U_{2\alpha} - C^{\alpha}_{\lambda Z}\right)\Big\}\\
&= \frac{(\zeta r_x)^2 k^Z_x}{\Delta}\left\{C^{\alpha}_{\lambda y}\left(J_\alpha e^F - C_{\lambda Z}\lambda^{Z'}_{y}\right) + \left(e^F - C_{\lambda y}\lambda^{Z'}_{y}\right)\left(U_{2\alpha} - C^{\alpha}_{\lambda Z}\right)\right\}\\
&= \frac{(\zeta r_x)^2 k^Z_x}{\Delta}\left\{\left[C^{\alpha}_{\lambda y}J_\alpha + \left(U_{2\alpha} - C^{\alpha}_{\lambda Z}\right)\right]e^F - \lambda^{Z'}_{y}\left[C^{\alpha}_{\lambda y}C_{\lambda Z} + C_{\lambda y}\left(U^{\alpha} - C^{\alpha}_{\lambda Z}\right)\right]\right\}\\
&\equiv \frac{(\zeta r_x)^2 k^Z_x}{\Delta}\left\{R_{Kx1}e^F - \lambda^{Z'}_{y}R_{Kx2}\right\}
\end{aligned}$$

Using the expressions of $C^{\alpha}_{\lambda y}$ and $C^{\alpha}_{\lambda Z}$:

$$R_{Kx1} = U_{2\alpha} + \int_0^{T^s} e^{\alpha t}\int_0^t k'_y(\tau)e^{F(\tau)}\left[J_\alpha - J_\alpha(\tau)\right]d\tau dt > 0$$

and:

$$\begin{aligned}
R_{Kx2} &= C_{\lambda y}U^{\alpha} + C^{\alpha}_{\lambda y}C_{\lambda Z} - C_{\lambda y}C^{\alpha}_{\lambda Z}\\
&= C_{\lambda y}U^{\alpha} + \int_0^{T^s} e^{\alpha t}\left[C_{\lambda y}(t)C_{\lambda Z} - C_{\lambda y}C_{\lambda Z}(t)\right]dt\\
&\equiv C_{\lambda y}U^{\alpha} + \int_0^{T^s} e^{\alpha t}\phi(t)dt
\end{aligned}$$

and $\phi(0) = \phi(T^s) = 0$ together with:

$$\phi'(t) = k'_y(t)e^{F(t)}\int_0^{T^s} k'_y(\tau)e^{F(\tau)}\left(J_\alpha(\tau) - J_\alpha(t)\right)d\tau \begin{cases}\to \phi'_0 > 0 & \text{if } t \to 0\\ \to \phi'_1 < 0 & \text{if } t \to T^s\end{cases}$$

show that $\phi(t) > 0$, and thus that $R_{Kx2} > 0$. Hence:

$$\frac{\partial Z^s}{\partial K^s_x} = R_{Kx} = \frac{(\zeta r_x)^2 k^Z_x}{\Delta}\left\{R_{Kx1}e^F - \lambda^{Z'}_{y}R_{Kx2}\right\} < 0$$

because $k^Z_x < 0$, $\Delta > 0$, $R_{Kx1} > 0$, $\lambda^{Z'}_{y} < 0$ and $R_{Kx2} > 0$.

We have shown the existence of a decreasing curve in the (Z, K_x) plane describing what we have called the scraping border in this plane. The curve intersects the $\dot{Z} = 0$ isocline, that is the line of equation: $K_x = (\alpha/\zeta r_x)Z$ at the point $(\bar{Z}, \bar{K}_x)$.

The scraping border in the (K_x, K_y) plane

Fixing Z^s, the carbon stock accumulation equation determines a relationship between K^s_x and K^s_y. Setting $dZ^s = 0$ in (A.4.11), we obtain:

$$\frac{\partial K^s_x}{\partial K^s_y} = -\frac{R_{Ky}}{R_{Kx}}$$

We have already shown that $R_{Kx} < 0$, we have to assess the sign of R_{Ky}. Because we have also shown that $R_{Ky} = R_{Kx} + H$ and H is > 0, the sign of R_{Ky} is indeterminate. Factoring the $U_{2\alpha}$ terms in the expression of R_{Kx}, we obtain:

$$\begin{aligned} R_{Kx} = & \frac{(\zeta r_x)^2.k_x^Z}{\Delta} \left\{ e^F \int_0^{T^s} e^{\alpha t} \int_0^t k'_y(\tau) e^{F(\tau)} \left[J_\alpha - J_\alpha(\tau) \right] d\tau dt \right. \\ & \left. -\lambda_y^{Z'} \left[C^\alpha_{\lambda y} C_{\lambda Z} - C_{\lambda y} C^\alpha_{\lambda Z} \right] + \left(e^F - \lambda_y^{Z'} C_{\lambda_y} \right) U_{2\alpha} \right\} \\ & \equiv R^1_{Kx} + \frac{(\zeta r_x)^2.k_x^Z}{\Delta} \left(e^F - \lambda_y^{Z'} C_{\lambda_y} \right) U_{2\alpha} \end{aligned}$$

We have shown that $R^1_{Kx} < 0$. Inserting the expression of Δ yields $R_{Kx} = R^1_{Kx} - \zeta r_x |u''(0)| U_{2\alpha}$, thus:

$$R_{Ky} = R^1_{Kx} - \zeta r_x |u''(0)| U_{2\alpha} + H = R^1_{Kx} - \zeta r_x |u''(0)| \int_0^{T^s} \frac{e^{(\rho+2\alpha)t}}{|u''(t)|} dt + H$$

For a linear energy demand function, $|u''(0)| = |u''(t)|$, and we have shown previously that $H < \zeta r_x I_\alpha$, thus:

$$\begin{aligned} & -\zeta r_x |u''(0)| U_{2\alpha} + H = -\zeta r_x \int_0^{T^s} e^{(\rho+2\alpha)t} dt + H < \zeta r_x \left[- \int_0^{T^s} e^{(\rho+2\alpha)t} dt + I_\alpha \right] \\ & = \zeta r_x \left[- \int_0^{T^s} e^{(\rho+2\alpha)t} dt + \int_0^{T^s} e^{\alpha t} dt \right] < 0 \end{aligned}$$

We conclude that $R_{Ky} < 0$ with a linear energy demand function. For an iso-elastic energy demand function, we have shown when studying the benchmark of a constant K_y, that:

$$J \equiv \int_0^{T^s} \left[- \frac{u''(0)}{u''(t)} e^{(\rho+2\alpha)t} + e^{\alpha v} \right] dt = -|u''(0)| U_{2\alpha} + I_\alpha$$

is < 0, implying that $-\zeta r_x |u''(0)| U_{2\alpha} + H < -\zeta r_x |u''(0)| U_{2\alpha} + \zeta r_x I_\alpha = \zeta r_x J < 0$. Thus $R_{Ky} < 0$ for an iso-elastic energy demand function.

We have shown the existence of a negative relationship between K_x and K_y in the (K_y, K_x) plane corresponding to the scraping border with a linear or an iso-elastic energy demand function. For now on we focus on a model configuration leading to such a negative relationship, thus including as particular cases the linear and iso-elastic energy demand cases. Since:

$$1 + \frac{\partial K_y^s}{\partial K_x^s} = 1 - \frac{R_{Kx}}{R_{Ky}} = \frac{1}{R_{Ky}} \left[R_{Ky} - R_{Kx} \right] = \frac{1}{R_{Ky}} \left[R_{Kx} + H - R_{Kx} \right] = \frac{H}{R_{Ky}} < 0$$

we observe that the slope of the border is lower than 1 in absolute value and since the border initiates from the point $(\bar{K}_y, \bar{K}_x)$, the border is located below the line of slope -1 describing the scraping border for the post-ceiling phase.

A.4.3 The stabilisation phase

The previous subsection has described the last scraping phase of the Scenario 2. We have shown the existence of decreasing borders in the (K_x, Z) and (K_y, K_x) subspaces of the state variables we called the scraping borders. We want to show the existence of two other borders in the two subspaces, defining the maximum size of the accumulated capacities as a decreasing function of the carbon stock size and the accumulated renewable energy conversion capital stock, what we call the *stabilisation borders.* We proceed through the following steps. First we use the restrictions implied by the plane equation at the beginning of the scraping phase to express the derivatives of λ_Z and λ_y at the end of the stabilisation phase with respect to $\hat{K}_x$, the maintained level of the fossil energy conversion capital stock and Z^s, the carbon stock level at the end of the stabilisation phase. Second we compute expressions of $d\lambda_y(t)$ and $dK_y(t)$ over the stabilisation phase. Third we show that the vector composed of the starting values of the dual variable, the starting value of the renewable energy production capital stock, the final level of this stock and the time duration of the stabilisation phase are uniquely determined by a 5 dimensional system as functions of the starting values of the fossil energy production capital stock and the carbon stock size. Last we exploit these findings to perform a sensitivity analysis establishing the qualitative properties of the stabilisation borders in the (Z, K_x) plane and the (K_y, K_x) plane.

Sensitivity of λ_Z and λ_y

Denote by $\hat{K}_x$, the maintained fossil energy capital stock during the stabilisation phase and Z^s, the corresponding size of the carbon stock at the end time of the phase and thus at the beginning of the scraping phase. Similarly denote by K_y^s, the renewable energy conversion capital, λ_y^s, the value of the renewable energy capital and λ_Z^s, the carbon price level at the same time. The previous analysis has shown that the triplet $(d\hat{K}_x, dK_y^s, dZ^s)$ lies along a plane of equation:

$$R_{Kx} d\hat{K}_x + R_{Ky} dK_y^s - dZ^s = 0$$

Because the plane equation implies that $dK_y^s = -(R_{Kx}/R_{Ky})d\hat{K}_x + (1/R_{Ky})dZ^s$ and denoting $\Lambda_Z \equiv |u''(0)|/(\zeta r_x) > 0$, we get from our previous computations:

$$\begin{aligned} d\lambda_Z^s &= -\Lambda_Z \left[d\hat{K}_x + dK_y^s \right] = -\Lambda_Z \left[\left(1 - \frac{R_{K_x}}{R_{Ky}} \right) d\hat{K}_x + \frac{dZ^s}{R_{Ky}} \right] \\ &= -\Lambda_Z \frac{H}{R_{Ky}} d\hat{K}_x - \frac{\Lambda_Z}{R_{Ky}} dZ^s \equiv M_{\lambda Z}^K d\hat{K}_x + M_{\lambda Z}^Z dZ^s \end{aligned}$$

with $M_{\lambda Z}^K > 0$ and $M_{\lambda Z}^Z > 0$, showing that $\lambda_Z^s(\hat{K}_x, Z^s)$ is an increasing function of its arguments.

We have also shown that λ_y^s is a function of $\hat{K}_x$ and K_y^s and that $\partial\lambda_y^s/\partial K_y^s = \partial\lambda_y^s/\partial\hat{K}_x - h$, with $h > 0$, so that:

$$d\lambda_y^s = \frac{\partial\lambda_y^s}{\partial\hat{K}_x} d\hat{K}_x + \frac{\partial\lambda_y^s}{\partial K_y^s} dK_y^s = \frac{\partial\lambda_y^s}{\partial\hat{K}_x} d\hat{K}_x + \left[\frac{\partial\lambda_y^s}{\partial\hat{K}_x} - h \right] dK_y^s$$

Because the plane equation implies that $dK_y^s = -(R_{Kx}/R_{Ky})d\hat{K}_x + (1/R_{Ky})dZ^s$, and denoting $\Lambda_y \equiv -\partial\lambda_y^s/\partial\hat{K}_x > 0$, this is equivalent to:

$$\begin{aligned}
d\lambda_y^s &= -\Lambda_y d\hat{K}_x - [\Lambda_y + h]\left[-\frac{R_{K_x}}{R_{Ky}}d\hat{K}_x + \frac{1}{R_{Ky}}dZ^s\right] \\
&= -\left[\Lambda_y - (\Lambda_y + h)\frac{R_{K_x}}{R_{Ky}}\right] - \frac{1}{R_{Ky}}(\Lambda_y + h)\, dZ^s \\
&= \frac{hR_{Kx} - \Lambda_y H}{R_{Ky}}d\hat{K}_x - \frac{\Lambda_y + h}{R_{Ky}}dZ^s \equiv M_{\lambda y}^K d\hat{K}_x + M_{\lambda y}^Z dZ^s
\end{aligned}$$

with $M_{\lambda y}^K > 0$ and $M_{\lambda y}^Z > 0$, showing that $\lambda_y^s(\hat{K}_x, Z^s)$ is an increasing function of it arguments.

We have shown the possibility of investment waves in the renewable energy industry. Such waves imply that at the beginning of the scraping phase it is possible that $\dot{\lambda}_y = \rho\lambda_y^s - \beta_y$ be either > 0 of < 0. The critical condition $\dot{\lambda}_y^s = 0$ defines an implicit relationship in the space (K_x, Z). Expressing this condition:

$$\dot{\lambda}_y^s = \rho\lambda_y^s(\hat{K}_x, Z^s) - \left[u'(\hat{K}_x + K_y^s(\hat{K}_x, Z^s)) - a_y - m_y - b_y'(K_y^s(\hat{K}_x, Z^s))\right] = 0$$

The equation defines an implicit relationship between $\hat{K}_x$ and Z^s. Differentiating we obtain:

$$\left[\rho\frac{\partial\lambda_y^s}{\partial\hat{K}_x} - u''(t^s) + (b_y''(t^s) - u''(t^s))\frac{\partial K_y^s}{\partial\hat{K}_x}\right]d\hat{K}_x + \left[\rho\frac{\partial\lambda_y^s}{\partial Z^s} + (b_y''(t^s) - u''(t^s))\frac{\partial\hat{K}_x}{\partial Z^s}\right]dZ^s = 0$$

Computation of $d\lambda_y(t)$ and $dK_y(t)$

Let T^m denote the time length of the phase. Perform the change of the time variable $v = t - t^m$ so at time $v = 0$, $K_x(0) = \hat{K}_x$, the maintained fossil energy conversion capacity during the phase, and let $Z(0) = \hat{Z}$ and $K_y(0) = \hat{K}_y$. Let similarly $\hat{\lambda}_y = \lambda_y(0)$ and $\hat{\lambda}_Z = \lambda_Z(0)$. We use the same method as before to compute an expression of $d\lambda_y(t)$ and $dK_y(t)$ over the stabilisation phase. Integrating over a time interval $[t_0, t) \subset [0, T^m)$ the motion of $K_y(t)$ and differentiating yields:

$$dK_y(t) = dK_y(t_0) + \int_{t_0}^{t} k_y'(\tau)d\lambda_y(\tau)d\tau$$

Integrating the motion of $\lambda_y(t)$ over the same time interval $[t_0, t)$, gets:

$$\lambda_y(t)e^{-\rho t} = \lambda_y(t_0)e^{-\rho t_0} - \int_{t_0}^{t}\left[u'(\hat{K}_x + K_y(\tau)) - a_y - m_{Ky} - b'(K_y(\tau))\right]e^{-\rho\tau}d\tau$$

Differentiating for a a given t_0, we obtain:

$$\begin{aligned}
d\lambda_y(t)e^{-\rho t} =& d\lambda_y(t_0)e^{-\rho t_0} - \left(\int_{t_0}^{t} u''(\tau)e^{-\rho\tau}d\tau\right)d\hat{K}_x \\
&- \int_{t_0}^{t}\left[u''(\tau) - b_y''(\tau)\right]e^{-\rho\tau}dK_y(\tau)d\tau
\end{aligned}$$

Substituting for $dK_y(\tau)$ gets:

$$\begin{aligned}
d\lambda_y(t)e^{-\rho t} =& d\lambda_y(t_0)e^{-\rho t_0} - \left(\int_{t_0}^{t} u''(\tau)e^{-\rho\tau}d\tau\right) d\hat{K}_x \\
& - \left(\int_{t_0}^{t}\left[u''(\tau)-b_y''(\tau)\right]e^{-\rho\tau}d\tau\right) dK_y(t_0) \\
& - \int_{t_0}^{t}\left[u''(\tau)-b_y''(\tau)\right]e^{-\rho\tau}\int_{t_0}^{\tau}k_y'(s)d\lambda_y(s)dsd\tau
\end{aligned}$$

Denoting:

$$\begin{aligned}
U_\rho(t) &\equiv -\int_{t_0}^{t} u''(\tau)e^{-\rho\tau}d\tau > 0 \\
B_\rho(t) &\equiv -\int_{t_0}^{t}\left[u''(\tau)-b_y''(\tau)\right]e^{-\rho\tau}d\tau
\end{aligned}$$

and integrating over $[t_0, T^m)$, we obtain:

$$\begin{aligned}
\int_{t_0}^{T^m} d\lambda_y(t)e^{-\rho t}dt =& d\lambda_y(t_0)e^{-\rho t_0}(T^m - t_0) + \int_{t_0}^{T^m} U_\rho(t)dtd\hat{K}_x \\
& + \int_{t_0}^{T^m} B_\rho(t)dtd\hat{K}_y(t_0) \\
& + \int_{0}^{T^m}\int_{t_0}^{t}\left|u''(\tau)-b_y''(\tau)\right|e^{-\rho\tau}\int_{t_0}^{\tau}k_y'(s)d\lambda_y(s)dsd\tau dt
\end{aligned}$$

inverting the integration order in the double integrals:

$$\begin{aligned}
&\int_{t_0}^{T^m} U_\rho(t)dt = \int_{t_0}^{T^m} |u''(t)|(T^m - t)e^{-\rho t}dt \\
&\int_{t_0}^{T^m} B_\rho(t)dt = \int_{t_0}^{T^m} |u''(t) - b_y''(t)|(T^m - t)e^{-\rho t}dt \\
&\int_{t_0}^{T^m}\int_{t_0}^{t}\left|u''(\tau)-b_y''(\tau)\right|e^{-\rho\tau}\int_{t_0}^{\tau}k_y'(s)d\lambda_y(s)dsd\tau dt \\
&= \int_{t_0}^{T^m}\left|u''(t)-b_y''(t)\right|e^{-\rho t}\int_{t_0}^{t}k_y'(\tau)d\lambda_y(\tau)d\tau(T^m - t)dt \\
&= \int_{t_0}^{T^m} k_y'(t)d\lambda_y(t)\int_{t}^{Tm}\left|u''(\tau)-b_y''(\tau)\right|e^{-\rho\tau}(T^m - \tau)d\tau dt
\end{aligned}$$

Rearranging:

$$\begin{aligned}
&\int_{t_0}^{T^m} d\lambda_y(t)\left[e^{-\rho t} - k_y'(t)\int_{t}^{T^m}\left|u''(\tau)-b_y''(\tau)\right|e^{-\rho\tau}(T^m - \tau)d\tau\right]dt \\
=& d\lambda_y(t_0)e^{-\rho t_0}(T^m - t_0) + \int_{t_0}^{T^m}|u''(t)|(T^m - t)e^{-\rho t}dtd\hat{K}_x \\
& + \int_{t_0}^{T^m}|u''(t) - b_y''(t)|e^{-\rho t}(T^m - t)dtdK_y(t_0)
\end{aligned}$$

Differentiating with respect to t_0 for a given $dK_y(t_0)$ gets:

$$\begin{aligned}
& - d\lambda_y(t_0)\left[e^{-\rho t_0} - k_y'(t_0)\int_{t_0}^{T^m}\left|u''(\tau)-b_y''(\tau)\right|e^{-\rho\tau}(T^m - \tau)d\tau\right] \\
=& \frac{d}{dt}d\lambda_y(t_0)e^{-\rho t_0}(T^m - t_0) - \rho d\lambda_y(t_0)e^{-\rho t_0}(T^m - t_0) - d\lambda_y(t_0)e^{-\rho t_0} \\
& - |u''(t_0)|(T^m - t_0)e^{-\rho t_0}d\hat{K}_x - |u''(t_0) - b_y''(t_0)|e^{-\rho t_0}(T^m - t_0)dK_y(t_0)
\end{aligned}$$

Simplifying the term $-d\lambda_y(t_0)e^{-\rho t_0}$ and dividing both sides by $(T^m - t_0)e^{\rho t_0}$, we obtain:

$$\begin{aligned}
&d\lambda_y(t_0)\frac{k_y'(t_0)e^{\rho t_0}}{T^m - t_0}\int_{t_0}^{T^m}\left|u''(\tau) - b_y''(\tau)\right|e^{-\rho\tau}(T^m - \tau)d\tau \\
=&\frac{d}{dt}d\lambda_y(t_0) - \rho d\lambda_y(t_0) - |u''(t_0)|d\hat{K}_x - |u''(t_0) - b_y''(t_0)|dK_y(t_0)
\end{aligned}$$

Denoting $\theta_y(t) \equiv d\lambda_y(t)$, we have obtained a differential equation of the form:

$$\begin{aligned}
\dot{\theta}_y(t_0) = &\left[\rho + \frac{k_y'(t_0)e^{\rho t_0}}{T^m - t_0}\int_{t_0}^{T^m}\left|u''(\tau) - b_y''(\tau)\right|e^{-\rho\tau}(T^m - \tau)d\tau\right]\theta_y(t) \\
&+ |u''(t_0)|d\hat{K}_x + |u''(t_0) - b_y''(t_0)|dK_y(t_0)
\end{aligned}$$

Let:

$$a(t) \equiv \rho + \frac{k_y'(t)e^{\rho t}}{T^m - t}\int_t^{Tm}\left|u''(\tau) - b_y''(\tau)\right|e^{-\rho\tau}(T^m - \tau)d\tau \text{ and } A(t) \equiv \int_0^t a(\tau)d\tau$$

Then the general solution of the differential equation reads:

$$d\lambda_y(t) = e^{A(t)}\left\{d\lambda_{y0} + \left(\int_0^t |u''(\tau)|e^{-A(\tau)}d\tau\right)d\hat{K}_x + \left(\int_0^t |u''(\tau) - b_y''(\tau)|e^{-A(\tau)}d\tau\right)d\hat{K}_y\right\}$$

Denote:

$$\begin{aligned}
U^A(t) &\equiv \int_0^t |u''(\tau)|e^{-A(\tau)}d\tau > 0 \\
B^A(t) &\equiv \int_0^t |u''(\tau) - b_y''(\tau)|e^{-A(\tau)}d\tau = \int_0^t \left(b_y''(\tau) + |u''(\tau)|\right)e^{-A(\tau)}d\tau > 0
\end{aligned}$$

We get in a more compact form:

$$d\lambda_y(t) = e^{A(t)}\left\{d\lambda_{y0} + U^A(t)d\hat{K}_x + B^A(t)d\hat{K}_y\right\} \tag{A.4.12}$$

The following expression of $dK_y(t)$ results from (A.4.12)

$$\begin{aligned}
dK_y(t) =&d\hat{K}_y + \left(\int_0^t k_y'(\tau)e^{A(\tau)}d\tau\right)d\lambda_{y0} + \left(\int_0^t k_y'(\tau)e^{A(\tau)}U^A(\tau)d\tau\right)d\hat{K}_x \\
&+ \left(\int_0^t k_y'(\tau)e^{A(\tau)}B^A(\tau)d\tau\right)d\hat{K}_y
\end{aligned}$$

Let:

$$\begin{aligned}
E_{\lambda y}(t) &\equiv \int_0^t k_y'(\tau)e^{A(\tau)}d\tau \\
E_{Kx}(t) &\equiv \int_0^t k_y'(\tau)e^{A(\tau)}U^A(\tau)d\tau \\
E_{Ky}(t) &\equiv \int_0^t k_y'(\tau)e^{A(\tau)}B^A(\tau)d\tau
\end{aligned}$$

then we obtain in compact form:

$$dK_y(t) = E_{\lambda y}(t)d\lambda_{y0} + E_{Kx}(t)d\hat{K}_x + (1 + E_{Ky}(t))\,d\hat{K}_y \tag{A.4.13}$$

The core dynamics of $K_y(t)$ and $\lambda_y(t)$

We have shown that λ_y^s and K_y^s are functions of $(\hat{K}_x, Z^s)$. Thus for a given pair, $(\hat{K}_x, Z^s)$, the trajectory of $\lambda_y(t)$ and $K_y(t)$ is the general solution of the elementary autonomous dynamical system, $\dot{K}_y = k_y(\lambda_y)$ and $\dot{\lambda}_y = \rho\lambda_y - \beta_y(K_y)$. Differentiating the capital accumulation condition:

$$K_y^s = \hat{K}_y + \int_0^{T^m} k_y((\lambda_y(t))dt$$

we obtain taking account of (A.4.13):

$$dK_y^s = k_y^s dT^m + E_{\lambda y} d\hat{\lambda}_y + E_{Kx} d\hat{K}_x + (1 + E_{Ky})\, d\hat{K}_y$$

where we denote $k_y^s = k_y(T^m)$, $E_{\lambda y} \equiv E_y(T^m)$, $E_{Kx} \equiv E_{Kx}(T^m)$, $E_{Ky} \equiv E_{Ky}(T^m)$. Using the expression of dK_y^s as a function of $d\hat{K}_x$ and dZ^s and rearranging yield:

$$E_{\lambda y} d\hat{\lambda}_y + k_y^s dT^m = -\left(E_{Kx} + \frac{R_{Kx}}{R_{Ky}}\right) d\hat{K}_x - (1 + E_{Ky})\, d\hat{K}_y + \frac{1}{R_{Ky}} dZ^s \tag{A.4.14}$$

Integrating the motion of $\lambda_y(t)$ and taking account of (A.4.12):

$$d\lambda_y^s = \dot{\lambda}_y^s dT^m + d\lambda_y(T^m) = \dot{\lambda}_y^s dT^m + e^A \left\{ d\hat{\lambda}_y + U^A d\hat{K}_x + B^A d\hat{K}_y \right\}$$

denoting $\dot{\lambda}_y^s \equiv d\lambda_y(t)/dt|_{t=T^{m}-}$, $A = A(T^m)$, $U^A = U^A(T^m)$, $B^A = B^A(T^m)$. Using the expression of $d\lambda_y^s$ as a function of $\hat{K}_x$ and Z^s and rearranging, we obtain:

$$e^A d\hat{\lambda}_y + \dot{\lambda}_y^s dT^m = \left(M_{\lambda y}^K - U^A e^A\right) d\hat{K}_x - B^A e^A d\hat{K}_y + M_{\lambda y}^Z dZ^s \tag{A.4.15}$$

The system (A.4.14)-(A.4.15) reads in matrix form:

$$\begin{bmatrix} E_{\lambda y} & k_y^s \\ e^A & \dot{\lambda}_y^s \end{bmatrix} \begin{bmatrix} d\hat{\lambda}_y \\ dT^m \end{bmatrix} = \begin{bmatrix} -\left(E_{Kx} + \frac{R_{Kx}}{R_{Ky}}\right) \\ M_{\lambda y}^K - U^A e^A \end{bmatrix} d\hat{K}_x + \begin{bmatrix} -(1 + E_{Ky}) \\ -B^A e^A \end{bmatrix} d\hat{K}_y + \begin{bmatrix} \frac{1}{R_{Ky}} \\ M_{\lambda y}^Z \end{bmatrix} dZ^s$$

The determinant, D, of the system is:

$$D = E_{\lambda y}\dot{\lambda}_y^s - k_y^s e^A < 0$$

For a fixed pair $(\hat{K}_x, Z^s)$, the dynamical system behaves as an autonomous system with a fixed terminal position $(K_y^s(\hat{K}_x, Z^s), \lambda_y^s(\hat{K}_x, Z^s))$ and we have shown that $\lambda_y(t)$ should be a decreasing function of $K_y(t)$, thus a decreasing time function, while T^m should be a decreasing function of $\hat{K}_y$ because the terminal position is fixed. Applying the Cramer rule yields:

$$\frac{\partial\hat{\lambda}_y}{\partial\hat{K}_y} = \frac{1}{D} \begin{vmatrix} -(1 + E_{Ky}) & k_y^s \\ -B^A e^A & \dot{\lambda}_y^s \end{vmatrix} = \frac{1}{D}\left[-(1 + E_{Ky})\dot{\lambda}_y^s + k_y^s B^A e^A \right]$$

The term into brackets is > 0 because $\dot{\lambda}_y^s < 0$, thus $D < 0$ implies that $\partial\hat{\lambda}_y/\partial\hat{K}_y < 0$. Applying once again the Cramer rule:

$$\frac{\partial T^m}{\partial\hat{K}_y} = \frac{1}{D} \begin{vmatrix} E_{\lambda y} & -(1 + E_{Ky}) \\ e^A & -B^A e^A \end{vmatrix} = \frac{e^A}{D}\left[-E_{\lambda_y} B^A + (1 + E_{Ky}) \right]$$

Since we know that $\partial T^m/\partial \hat{K}_x$ must be < 0 and $D < 0$, we conclude that: $1 + E_{Ky} - E_{\lambda_y} B^A > 0$.

We get also through the Cramer rule for fixed $\hat{K}_y$ and Z^s:

$$\frac{\partial \hat{\lambda}_y}{\partial \hat{K}_x} = \frac{1}{D}\begin{vmatrix} -\left(E_{Kx} + \frac{R_{Kx}}{R_{Ky}}\right) & k_y^s \\ M^K_{\lambda y} - U^A e^A & \dot{\lambda}_y^s \end{vmatrix} = \frac{1}{D}\left[-\left(E_{Kx} + \frac{R_{Kx}}{R_{Ky}}\right)\dot{\lambda}_y^s - k_y^s\left(M^K_{\lambda y} - U^A e^A\right)\right]$$

Because:

$$\frac{\dot{\lambda}_y^s}{k_y^s} = \frac{d\lambda_y^s}{dK_y^s} = \frac{d\lambda_y^s}{d\hat{K}_x}\frac{d\hat{K}_x}{dK_y^s} = -M^K_{\lambda y}\frac{R_{Ky}}{R_{Kx}} \Longrightarrow -k_y^s M^K_{\lambda_y} = \dot{\lambda}_y^s\frac{R_{Kx}}{R_{Ky}}$$

the above expression simplifies to:

$$\frac{\partial \hat{\lambda}_y}{\partial \hat{K}_x} = \frac{1}{D}\left[-E_{Kx}\dot{\lambda}_y^s + k_y^s U^A e^A\right]$$

which is < 0 because $D < 0$ and $\dot{\lambda}_y^s < 0$. Next:

$$\frac{\partial T^m}{\partial \hat{K}_x} = \frac{1}{D}\begin{vmatrix} E_{\lambda y} & -\left(E_{Kx} + \frac{R_{Kx}}{R_{Ky}}\right) \\ e^A & M^K_{\lambda y} - U^A e^A \end{vmatrix} = \frac{1}{D}\left[E_{\lambda y}\left(M^K_{\lambda y} - U^A e^A\right) + \left(E_{Kx} + \frac{R_{Kx}}{R_{Ky}}\right)e^A\right]$$

Observe that because $B^A > U^A$

$$1 + E_{Kx} - B^A E_{\lambda y} > 0 \implies 1 + E_{Kx} - U^A E_{\lambda y} > 0$$

thus $R_{Kx}/R_{Ky} > 1$, because $R_{Kx} = R_{Ky} - H$, $H > 0$, implies that:

$$E_{Kx} + \frac{R_{Kx}}{R_{Ky}} - U^A E_{\lambda y} > E_{Kx} + 1 - U^A E_{\lambda y} > 0$$

We conclude that the term into square brackets in the expression of $\partial T^m/\partial \hat{K}_x$ is > 0, since $M^K_{\lambda y} > 0$, and hence that $\partial T^m/\partial \hat{K}_x < 0$, D being < 0.

Last, fixing $\hat{K}_x$ and $\hat{K}_y$:

$$\frac{\partial \hat{\lambda}_y}{\partial Z^s} = \frac{1}{D}\begin{vmatrix} \frac{1}{R_{Ky}} & k_y^s \\ M^Z_{\lambda y} & \dot{\lambda}_y^s \end{vmatrix} = \frac{1}{D}\left[\frac{\dot{\lambda}_y^s}{R_{Ky}} - M^Z_{\lambda y}k_y^s\right] = -\frac{k_y^s}{D}\left[\frac{M^K_{\lambda y}}{R_{Kx}} + M^Z_{\lambda y}\right]$$

System determining the optimal path

For given $(\hat{K}_x, \hat{Z})$, the vector $(K_y^s, \hat{K}_y, \hat{\lambda}_y, \hat{\lambda}_Z, T^m)$ is determined through the set of conditions:

1. The capital accumulation condition:

$$K_y^s = \hat{K}_y + \int_0^{T^m} k_y((\lambda_y(t))dt$$

2. The continuity condition on $\lambda_y(t)$ at time T^m, recalling that $\lambda_y(T^m)$, we denote λ_y^s, is itself a decreasing function of both $(\hat{K}_x, K_y^s)$:
$$\lambda_y^s(\hat{K}_x, K_y^s) = \lambda_y(T^m)$$
3. The minimal delay condition:
$$\underline{n}_x = \int_0^{T^m} \left[u'(\hat{K}_x + K_y(t)) - c_x - m_x - \zeta r_x \lambda_Z(t)\right] e^{-\rho t} dt$$
4. The continuity condition on λ_Z at time T^m remembering that $\lambda_Z(T^m) \equiv \lambda_Z^s$ is a decreasing function of both $\hat{K}_x$ and K_y^s:
$$\lambda_Z^s(\hat{K}_x, K_y^s) = \hat{\lambda}_Z e^{(\rho+\alpha)T^m}$$
5. The carbon stock balance condition between time 0 and time T^m:
$$Z^s e^{\alpha T^m} = \hat{Z} + \zeta r_x \hat{K}_x I_\alpha$$
denoting $I_\alpha \equiv \int_0^{T^m} e^{\alpha t} dt$

Once the differentiated vector $(dK_y^s, d\hat{K}_y, d\hat{\lambda}_y, d\hat{\lambda}_Z, dT^m)$ is expressed as a function of the vector $(d\hat{K}_x, d\hat{Z})$, it becomes possible to compute the signs of $\partial\hat{K}_y/\partial\hat{Z}$ and $\partial\hat{K}_y/\partial\hat{K}_x$, giving the sign of the slopes of the stabilisation borders in the (K_x, Z) and (K_x, K_y)planes.

Existence and unicity

To prove the existence and unicity of the optimal vector $(K_y^s, \hat{K}_y, \hat{\lambda}_y, \hat{\lambda}_Z, T^m)$, we differentiate the system. Denoting $k_y^s = k_y(T^m)$, $E_{\lambda y} \equiv E_y(T^m)$, $E_{Kx} \equiv E_{Kx}(T^m)$, $E_{Ky} \equiv E_{Ky}(T^m)$ and using (A.4.13), we obtain:
$$dK_y^s - (1 + E_{Ky})\, d\hat{K}_y - E_{\lambda y} d\hat{\lambda}_y - k_y^s dT^m = E_{Kx} d\hat{K}_x$$
Next rearranging (A.4.15), we obtain:
$$(\Lambda_y + h) dK_y^s + B^A e^A d\hat{K}_y + e^A d\hat{\lambda}_y + \dot{\lambda}_y^s dT^m = -\left(\Lambda_y + e^A U^A\right) d\hat{K}_x$$
Differentiating the minimal delay relation, remembering that $\beta_x(T^m) = 0$, gets:
$$0 = \left(\int_0^{T^m} u''(t) e^{-\rho t} dt\right) d\hat{K}_x + \int_0^{T^m} u''(t) e^{-\rho t} dK_y(t) dt - \zeta r_x \left(\int_0^{T^m} e^{\alpha t} dt\right) d\hat{\lambda}_Z$$
With our compact notations U_ρ and I_α, the previous relation reads equivalently:
$$U_\rho d\hat{K}_x + \zeta r_x I_\alpha d\hat{\lambda}_Z = \int_0^{T^m} u''(t) e^{-\rho t} dK_y(t) dt$$
Substituting for $dK_y(t)$ its expression (A.4.13) yields:
$$\begin{aligned} &U_\rho d\hat{K}_x + \zeta r_x I_\alpha d\hat{\lambda}_Z \\ &= -\int_0^{T^m} |u''(t)| e^{-\rho t} \left[E_{\lambda y}(t) d\hat{\lambda}_y + E_{Kx}(t) d\hat{K}_x + (1 + E_{Ky}(t)) d\hat{K}_y\right] dt \end{aligned}$$

Let:

$$F_{\lambda y} \equiv \int_0^{T^m} |u''(t)| e^{-\rho t} E_{\lambda y}(t) dt$$
$$F_{Kx} \equiv \int_0^{T^m} |u''(t)| e^{-\rho t} (1 + E_{Kx}(t)) dt$$
$$F_{Ky} \equiv \int_0^{T^m} |u''(t)| e^{-\rho t} (1 + E_{Ky}(t)) dt$$

then:

$$F_{Ky} d\hat{K}_y + F_{\lambda y} d\hat{\lambda}_y + \zeta r_x I_\alpha d\hat{\lambda}_Z = -F_{Kx} d\hat{K}_x$$

Differentiating this continuity condition over λ_Z and denoting $\dot{\lambda}_Z^m \equiv \dot{\lambda}_Z(T^{m-})$ yields:

$$\Lambda_Z dK_y^s + e^{(\rho+\alpha)T^m} d\hat{\lambda}_Z + \dot{\lambda}_Z^m dT^m = -\Lambda_Z d\hat{K}_x$$

Express the carbon stock balance condition during the stabilisation phase using our I_α notation:

$$Z^s e^{\alpha T^m} = \hat{Z} + \zeta r_x \hat{K}_x I_\alpha$$

Differentiating, we obtain:

$$dZ^s e^{\alpha T^m} - \dot{Z}(T^m) e^{\alpha T^m} dT^m = d\hat{Z} + \zeta r_x I_\alpha d\hat{K}_x$$

Denote $\dot{Z}^m \equiv \dot{Z}(T^m)$. We have shown in the previous section that Z^s is a function of $(\hat{K}_x, K_y^s)$, with $\partial Z^s / \partial \hat{K}_x = R_{Kx}$ and $\partial Z^s / \partial K_y^s = R_{Ky}$, thus the previous relation can be transformed as:

$$R_{Ky} e^{\alpha T^m} dK_y^s - \dot{Z}^m e^{\alpha T^m} dT^m = d\hat{Z} + \left[\zeta r_x I_\alpha - R_{Kx} e^{\alpha T^m}\right] d\hat{K}_x$$

The system reads in matrix form:

$$\begin{bmatrix} 1 & -(1+E_{Ky}) & -E_{\lambda y} & 0 & -k_y^s \\ \Lambda_y + h & B^A e^A & e^A & 0 & \dot{\lambda}_y^s \\ 0 & F_{Ky} & F_{\lambda y} & \zeta r_x I_\alpha & 0 \\ \Lambda_Z & 0 & 0 & e^{(\rho+\alpha)T^m} & \dot{\lambda}_Z^m \\ R_{Ky} e^{\alpha T^m} & 0 & 0 & 0 & -\dot{Z}^m e^{\alpha T^m} \end{bmatrix} \begin{bmatrix} dK_y^s \\ d\hat{K}_y \\ d\hat{\lambda}_y \\ d\hat{\lambda}_Z \\ dT^m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} d\hat{Z}$$
$$+ \begin{bmatrix} E_{Kx} \\ -\left(\Lambda_y + e^A U^A\right) \\ -F_{Kx} \\ -\Lambda_Z \\ \zeta r_x I_\alpha - R_{Kx} e^{\alpha T^m} \end{bmatrix} d\hat{K}_x$$

Let Δ denote the determinant of this system. Remark that:

$$\lambda_y^s = \lambda_y^s(\hat{K}_x, K_y^s) \Longrightarrow \dot{\lambda}_y^s = \frac{\partial \lambda_y^s}{\partial \hat{K}_x} k_x^s + \frac{\partial \lambda_y^s}{\partial \hat{K}_y^s} k_y^s = -\Lambda_y k_x^s - (\Lambda_y + h) k_y^s$$

$$\lambda_Z^s = \lambda_Z^s(\hat{K}_x, K_y^s) \Longrightarrow \dot{\lambda}_Z^m = \frac{\partial \lambda_Z^s}{\partial \hat{K}_x} k_x^s + \frac{\partial \lambda_Z^s}{\partial \hat{K}_y^s} k_y^s = -\Lambda_Z k_x^s - \Lambda_Z k_y^s$$

$$Z^s = Z^s(\hat{K}_x, K_y^s) \Longrightarrow \dot{Z}^m = \frac{\partial Z^s}{\partial \hat{K}_x} k_x^s + \frac{\partial Z^s}{\partial \hat{K}_y^s} k_y^s = R_{Kx} k_x^s + R_{Ky} k_y^Z$$

Multiplying the first column by k_y^s and adding to the fifth column while using the computed expressions of $\dot{\lambda}_y^s$, $\dot{\lambda}_Z^m$ and $\dot{Z}^m$, we get the following expression of Δ:

$$\Delta = \begin{vmatrix} 1 & -(1+E_{Ky}) & -E_{\lambda y} & 0 & 0 \\ \Lambda_y + h & B^A e^A & e^A & 0 & -\Lambda_y k_x^s \\ 0 & F_{Ky} & F_{\lambda y} & \zeta r_x I_\alpha & 0 \\ \Lambda_Z & 0 & 0 & e^{(\rho+\alpha)T^m} & -\Lambda_Z k_x^s \\ R_{Ky} e^{\alpha T^m} & 0 & 0 & 0 & -R_{Kx} k_x^s e^{\alpha T^m} \end{vmatrix}$$

Developing with respect to the fifth line gets:

$$\Delta e^{-\alpha T^m} = R_{Ky}\Delta_1 - R_{Kx} k_x^s \Delta_2$$

with:

$$\Delta_1 = \begin{vmatrix} -(1+E_{Ky}) & -E_{\lambda y} & 0 & 0 \\ B^A e^A & e^A & 0 & -\Lambda_y k_x^s \\ F_{Ky} & F_{\lambda y} & \zeta r_x I_\alpha & 0 \\ 0 & 0 & e^{(\rho+\alpha)T^m} & -\Lambda_Z k_x^s \end{vmatrix}$$

and:

$$\Delta_2 = \begin{vmatrix} 1 & -(1+E_{Ky}) & -E_{\lambda y} & 0 \\ \Lambda_y + h & B^A e^A & e^A & 0 \\ 0 & F_{Ky} & F_{\lambda y} & \zeta r_x I_\alpha \\ \Lambda_Z & 0 & 0 & e^{(\rho+\alpha)T^m} \end{vmatrix}$$

Developing Δ_1 with respect to its third column yields:

$$\begin{aligned} \Delta_1 &= \zeta r_x I_\alpha \begin{vmatrix} -(1+E_{Ky}) & -E_{\lambda y} & 0 \\ B^A e^A & e^A & -\Lambda_y k_x^s \\ 0 & 0 & -\Lambda_Z k_x^s \end{vmatrix} - e^{(\rho+\alpha)T^m} \begin{vmatrix} -(1+E_{Ky}) & -E_{\lambda y} & 0 \\ B^A e^A & e^A & -\Lambda_y k_x^s \\ F_{Ky} & F_{\lambda y} & 0 \end{vmatrix} \\ &= \zeta r_x I_\alpha \left(-\Lambda_Z k_x^s\right) e^A(-m_0) - e^{(\rho+\alpha)T^m}\left(\Lambda_y k_x^s\right)\left[E_{\lambda y}F_{Ky} - (1+E_{Ky})F_{\lambda y}\right] \\ &= k_x^s \left[\zeta r_x I_\alpha \Lambda_Z e^A m_0 - e^{(\rho+\alpha)T^m}\Lambda_y\left(E_{\lambda y}F_{Ky} - (1+E_{Ky})F_{\lambda y}\right)\right] \end{aligned}$$

Let:

$$\begin{aligned} m_1 &\equiv E_{\lambda y}F_{Ky} - (1+E_{Ky})F_{\lambda y} \\ &= \int_0^{T^m} |u''(t)| e^{-\rho t}\left[E_{\lambda y}(1+E_{Ky}(t)) - (1+E_{Ky})E_{\lambda_y}(t)\right]dt \\ &\equiv \int_0^{T^m} |u''(t)|\phi(t)dt \end{aligned}$$

We get $\phi(0) = E_{\lambda y}$, $\phi(T^s) = 0$ and is is easily checked that $\dot{\phi}(t) < 0$, so that $m_1 > 0$ and:

$$0 < m_1 < E_{\lambda y}\int_0^{T^m} |u''(t)| e^{-\rho t}dt$$

Thus:

$$\Delta_1 = k_x^s\left[\zeta r_x I_\alpha \Lambda_Z e^A m_0 - e^{(\rho+\alpha)T^m}\Lambda_y m_1\right]$$

is of indeterminate sign.

Developing Δ_2 with respect to its fourth column yields:

$$\begin{aligned}
\Delta_2 &= -\zeta r_x I_\alpha \begin{vmatrix} 1 & -(1+E_{Ky}) & -E_{\lambda y} \\ \Lambda_y + h & B^A e^A & e^A \\ \Lambda_Z & 0 & 0 \end{vmatrix} + e^{(\rho+\alpha)T^m} \begin{vmatrix} 1 & -(1+E_{Ky}) & -E_{\lambda y} \\ \Lambda_y + h & B^A e^A & e^A \\ 0 & F_{Ky} & F_{\lambda y} \end{vmatrix} \\
&= -\zeta r_x I_\alpha \Lambda_Z e^A(-m_0) + e^{(\rho+\alpha)T^m} \left[e^A \left(B^A F_{\lambda y} - F_{Ky} \right) - (\Lambda_y + h)\, m_1 \right] \\
&= \zeta r_x I_\alpha \Lambda_Z e^A m_0 - e^{(\rho+\alpha)T^m} \left[e^A \left(F_{Ky} - B^A F_{\lambda y} \right) + (\Lambda_y + h)\, m_1 \right]
\end{aligned}$$

Let:

$$\begin{aligned}
m_2 \equiv F_{Ky} - B^A F_{\lambda y} &= \int_0^{T^m} |u''(t)| e^{-\rho t} \left[(1 + E_{Ky}(t)) - B^A E_{\lambda y}(t) \right] dt \\
&\equiv \int_0^{T^m} |u''(t)| e^{-\rho t} \phi(t) dt
\end{aligned}$$

and $\phi(0) = 1$, $\phi(T^m) = (1 + E_{Ky}) - B^A E_{\lambda y} = m_0 > 0$, $\dot{\phi}(t) < 0$ show that $\phi(t) > 0$, and hence $m_2 > 0$ with:

$$m_0 \int_0^{T^m} |u''(t)| e^{-\rho t} dt < m_2 < \int_0^{T^m} |u''(t)| e^{-\rho t} dt$$

Thus:

$$\Delta_2 = \zeta r_x I_\alpha \Lambda_Z e^A m_0 - e^{(\rho+\alpha)T^m} \left[e^A m_2 + (\Lambda_y + h)\, m_1 \right]$$

is of indeterminate sign. Collecting the expressions of Δ_1 and Δ_2, we obtain:

$$\begin{aligned}
\Delta \frac{e^{-\alpha T^m}}{k_x^s} =& R_{Ky} \left[\zeta r_x I_\alpha \Lambda_Z e^A m_0 - e^{(\rho+\alpha)T^m} \Lambda_y m_1 \right] \\
& - R_{Kx} \left[\zeta r_x I_\alpha \Lambda_Z e^A m_0 - e^{(\rho+\alpha)T^m} \left(e^A m_2 + (\Lambda_y + h)\, m_1 \right) \right] \\
=& \left(R_{Ky} - R_{Kx} \right) \left[\zeta r_x I_\alpha \Lambda_Z e^A m_0 - e^{(\rho+\alpha)T^m} \Lambda_y m_1 \right] \\
& + R_{Kx} e^{(\rho+\alpha)T^m} \left(e^A m_2 + h m_1 \right) \\
=& H \left[\zeta r_x I_\alpha \Lambda_Z e^A m_0 - e^{(\rho+\alpha)T^m} \Lambda_y m_1 \right] + R_{Kx} e^{(\rho+\alpha)T^m} \left(e^A m_2 + h m_1 \right)
\end{aligned}$$

where we use $R_{Ky} = R_{Kx} + H$. Because $R_{Ky} < 0$, $H < -R_{Kx}$, thus denoting by G the expression on the r.h.s.:

$$\begin{aligned}
G <& R_{Kx} \left[e^{(\rho+\alpha)T^m} \left(e^A m_2 + h m_1 \right) - \left(\zeta r_x I_\alpha \Lambda_Z e^A m_0 - e^{(\rho+\alpha)T^m} \Lambda_y m_1 \right) \right] \\
=& R_{Kx} \left[e^{(\rho+\alpha)T^m} e^A m_2 - \zeta r_x I_\alpha \Lambda_Z e^A m_0 + e^{(\rho+\alpha)T^m} h m_1 + e^{(\rho+\alpha)T^m} \Lambda_y m_1 \right] \\
=& R_{Kx} e^A \left[e^{(\rho+\alpha)T^m} m_2 - \zeta r_x I_\alpha \Lambda_Z m_0 \right] + R_{Kx} e^{(\rho+\alpha)T^m} m_1 \left(\Lambda_y + h \right)
\end{aligned}$$

We check that $L \equiv e^{(\rho+\alpha)T^m} m_2 - \zeta r_x I_\alpha \Lambda_Z m_0 > 0$. Using the lower bound on m_2 and the expression of Λ_Z:

$$\begin{aligned}
L = e^{(\rho+\alpha)T^m} m_2 - \zeta r_x I_\alpha \Lambda_Z m_0 &> e^{(\rho+\alpha)T^m} m_0 \int_0^{T^m} |u''(t)| e^{-\rho t} dt - \zeta r_x I_\alpha \Lambda_Z m_0 \\
&= m_0 \left[e^{(\rho+\alpha)T^m} \int_0^{T^m} |u''(t)| e^{-\rho t} dt - I_\alpha |u''(T^m)| \right] \\
&= m_0 \int_0^{T^m} e^{-\rho t} \left[e^{(\rho+\alpha)T^m} |u''(t)| - e^{(\rho+\alpha)t} |u''(T^m)| \right] dt
\end{aligned}$$

In the case of a linear energy demand where $|u''(t)| = |u''(T^m)|$, it is immediate that the integrand is > 0, thus $L > 0$, implying in turn that $G < 0$, because $R_{Kx} < 0$. In the case of an iso-elastic demand, we have shown that:

$$\frac{e^{(\rho+\alpha)T^m}/|u''(T^m)|}{e^{(\rho+\alpha)t}/|u''(t)|} = \frac{k(T^m)}{k(t)} > 1$$

hence the integrand is also > 0, so that $G < 0$. We conclude that at least in the case of a linear or an iso-elastic energy demand function, $G < 0$ implies that $\Delta > 0$ because $k_x^s < 0$.

The stabilisation borders

Because of the restrictions implied by the scraping borders and the accumulation paths of the capital stocks and the atmospheric carbon stock, $\hat{K}_x$, $\hat{K}_y$ and $\hat{Z}$ must lie on a plane in the state variables phase space, (K_x, K_y, Z). In differentiated terms the corresponding plane equation reads:

$$d\hat{Z} - S_{Kx} d\hat{K}_x - S_{Ky} d\hat{K}_y = 0$$

This implies that:

$$\frac{\partial \hat{K}_y}{\partial \hat{Z}} = \frac{1}{S_{Ky}} \; ; \; \frac{\partial \hat{K}_y}{\partial \hat{K}_x} = -\frac{S_{Kx}}{S_{Ky}} \text{ and } \frac{\partial \hat{K}_x}{\partial \hat{Z}} = \frac{1}{S_{Kx}}$$

We are going to show that at least with a linear or iso-elastic energy demand function:

$$\frac{\partial \hat{K}_y}{\partial \hat{Z}} < 0 \Longleftrightarrow S_{Ky} < 0$$

$$\frac{\partial \hat{K}_y}{\partial \hat{K}_x} < 0 \Longleftrightarrow \frac{S_{Kx}}{S_{Ky}} > 0 \Longrightarrow S_{Kx} < 0 \text{ because } S_{Ky} < 0$$

We then deduce from $S_{Kx} < 0$ that $\partial \hat{K}_x / \partial \hat{Z} = 1/S_{Kx} < 0$, giving the claimed shape of the stabilisation border in the (Z, K_x) plane while $\partial \hat{K}_y / \partial \hat{K}_x < 0$ gives the claimed shape of the stabilisation border in the (K_y, K_x) plane.

Computation of $\partial \hat{K}_y / \partial \hat{Z}$.

Applying the Cramer rule, we obtain:

$$\frac{\partial \hat{K}_y}{\partial \hat{Z}} = \frac{1}{\Delta} \begin{vmatrix} 1 & 0 & -E_{\lambda y} & 0 & 0 \\ \Lambda_y + h & 0 & e^A & 0 & -\Lambda_y k_x^s \\ 0 & 0 & F_{\lambda y} & \zeta r_x I_\alpha & 0 \\ \Lambda_Z & 0 & 0 & e^{(\rho+\alpha)T^m} & -\Lambda_Z k_x^s \\ R_{Ky} e^{\alpha T^m} & 1 & 0 & 0 & -R_{Kx} k_x^s e^{\alpha T^m} \end{vmatrix}$$

Developing with respect to the second column and next with respect to the third column the resulting determinant yields:

$$\begin{aligned} \frac{\partial \hat{K}_y}{\partial \hat{Z}} &= -\frac{1}{\Delta} \begin{vmatrix} 1 & -E_{\lambda y} & 0 & 0 \\ \Lambda_y + h & e^A & 0 & -\Lambda_y k_x^s \\ 0 & F_{\lambda y} & \zeta r_x I_\alpha & 0 \\ \Lambda_Z & 0 & e^{(\rho+\alpha)T^m} & -\Lambda_Z k_x^s \end{vmatrix} \\ &= -\frac{1}{\Delta} \left[\zeta r_x I_\alpha \Delta_1 - e^{(\rho+\alpha)T^m} \Delta_2 \right] \end{aligned}$$

with:

$$\begin{aligned}\Delta_1 &= \begin{vmatrix} 1 & -E_{\lambda y} & 0 \\ \Lambda_y + h & e^A & -\Lambda_y k_x^s \\ \Lambda_Z & 0 & -\Lambda_Z k_x^s \end{vmatrix} \\ &= \Lambda_Z \Lambda_y E_{\lambda y} k_x^s - \Lambda_Z k_x^s \left[e^A + (\Lambda_y + h) E_{\lambda y} \right] \\ &= \Lambda_Z \Lambda_y E_{\lambda y} k_x^s - \Lambda_Z k_x^s e^A - \Lambda_Z k_x^s \Lambda_y E_{\lambda y} - \Lambda_Z k_x^s h E_{\lambda y} \\ &= -k_x^s \Lambda_Z \left[e^A + h E_{\lambda y} \right]\end{aligned}$$

and $k_x^s < 0$ implies that $\Delta_1 > 0$. Furthermore:

$$\Delta_2 = \begin{vmatrix} 1 & -E_{\lambda y} & 0 \\ \Lambda_y + h & e^A & -\Lambda_y k_x^s \\ 0 & F_{\lambda y} & 0 \end{vmatrix} = \Lambda_y k_x^s F_{\lambda y} < 0$$

because $k_x^s < 0$. We thus conclude that:

$$\frac{\partial \hat{K}_y}{\partial \hat{Z}} = -\frac{1}{\underbrace{\Delta}_{(>0)}} \left[\zeta r_x I_\alpha \underbrace{\Delta_1}_{(>0)} - e^{(\rho+\alpha)T^m} \underbrace{\Delta_2}_{(<0)} \right] < 0$$

because we have already shown that $\Delta > 0$, at least for a linear or an iso-elastic energy demand function. We thus conclude that $S_{Ky} < 0$.

Computation of $\partial \hat{K}_y / \partial \hat{K}_x$.

Applying once again the Cramer rule and developing with respect to the fourth column the resulting determinant:

$$\begin{aligned}\frac{\partial \hat{K}_y}{\partial \hat{K}_x} &= \frac{1}{\Delta} \begin{vmatrix} 1 & E_{Kx} & -E_{\lambda y} & 0 & 0 \\ \Lambda_y + h & -(\Lambda y + e^A U^A) & e^A & 0 & -\Lambda_y k_x^s \\ 0 & F_{Ky} & F_{\lambda y} & \zeta r_x I_\alpha & 0 \\ \Lambda_Z & -\Lambda_Z & 0 & e^{(\rho+\alpha)T^m} & -\Lambda_Z k_x^s \\ R_{Ky} e^{\alpha T^m} & \zeta r_x I_\alpha - R_{Kx} e^{\alpha T^m} & 0 & 0 & -R_{Kx} k_x^s e^{\alpha T^m} \end{vmatrix} \\ &= \frac{1}{\Delta} \left[-\zeta r_x I_\alpha \Delta_1 + e^{(\rho+\alpha)T^m} \Delta_2 \right]\end{aligned}$$

with:

$$\Delta_1 = \begin{vmatrix} 1 & E_{Kx} & -E_{\lambda y} & 0 \\ \Lambda_y + h & -(\Lambda y + e^A U^A) & e^A & -\Lambda_y k_x^s \\ \Lambda_Z & -\Lambda_Z & 0 & -\Lambda_Z k_x^s \\ R_{Ky} e^{\alpha T^m} & \zeta r_x I_\alpha - R_{Kx} e^{\alpha T^m} & 0 & -R_{Kx} k_x^s e^{\alpha T^m} \end{vmatrix}$$

and:

$$\Delta_2 = \begin{vmatrix} 1 & E_{Kx} & -E_{\lambda y} & 0 \\ \Lambda_y + h & -(\Lambda y + e^A U^A) & e^A & -\Lambda_y k_x^s \\ 0 & F_{Ky} & F_{\lambda y} & 0 \\ R_{Ky} e^{\alpha T^m} & \zeta r_x I_\alpha - R_{Kx} e^{\alpha T^m} & 0 & -R_{Kx} k_x^s e^{\alpha T^m} \end{vmatrix}$$

Adding the first column to the second one in the Δ_1 determinant, remembering that $R_{Ky} = R_{Kx} + H$, and developing the result with respect to the third line we obtain:

$$\begin{aligned}
\Delta_1 &= \begin{vmatrix} 1 & 1 + E_{Kx} & -E_{\lambda y} & 0 \\ \Lambda_y + h & h - e^A U^A & e^A & -\Lambda_y k_x^s \\ \Lambda_Z & 0 & 0 & -\Lambda_Z k_x^s \\ R_{Ky} e^{\alpha T^m} & \zeta r_x I_\alpha + H e^{\alpha T^m} & 0 & -R_{Kx} k_x^s e^{\alpha T^m} \end{vmatrix} \\
&= \Lambda_Z k_x^s \left[\Delta_{11} + \Delta_{12}\right]
\end{aligned}$$

with:

$$\begin{aligned}
\Delta_{11} &= \begin{vmatrix} 1 + E_{Kx} & -E_{\lambda y} & 0 \\ h - e^A U^A & e^A & -\Lambda_y \\ \zeta r_x I_\alpha + H e^{\alpha T^m} & 0 & -R_{Kx} e^{\alpha T^m} \end{vmatrix} \\
&= \left(\zeta r_x I_\alpha + H e^{\alpha T^m}\right) \Lambda_y E_{\lambda y} - R_{Kx} e^{\alpha T^m} m_3
\end{aligned}$$

where:

$$m_3 = e^A \left[(1 + E_{Kx}) - U^A E_{\lambda y}\right] + h E_{\lambda y} > 0$$

because we have shown earlier that $1 + E_{Kx} - U^A E_{\lambda y} > 0$. Next:

$$\begin{aligned}
\Delta_{12} &= \begin{vmatrix} 1 & 1 + E_{Kx} & -E_{\lambda y} \\ \Lambda_y + h & h - e^A U^A & e^A \\ R_{Ky} e^{\alpha T^m} & \zeta r_x I_\alpha + H e^{\alpha T^m} & 0 \end{vmatrix} \\
&= R_{Ky} e^{\alpha T^m} \left[e^A (1 + E_{Kx} - U^A E_{\lambda y}) + h E_{\lambda y}\right] \\
&- \left(\zeta r_x I_\alpha + H e^{\alpha T^m}\right) \left[e^A + E_{\lambda y}(\Lambda_y + h)\right] \\
&= R_{Ky} e^{\alpha T^m} m_3 - \left(\zeta r_x I_\alpha + H e^{\alpha T^m}\right) \left[e^A + E_{\lambda y}(\Lambda_y + h)\right]
\end{aligned}$$

Hence using once again $R_{Ky} = R_{Kx} + H$:

$$\begin{aligned}
\Delta_{11} + \Delta_{12} &= \left(-R_{Kx} + R_{Ky}\right) m_3 e^{\alpha T^m} \\
&+ \left(\zeta r_x I_\alpha + H e^{\alpha T^m}\right) \left[\Lambda_y E_{\lambda y} - e^A - E_{\lambda y} \Lambda_y - E_{\lambda y} h\right] \\
&= H m_3 e^{\alpha T^m} - \left(\zeta r_x I_\alpha + H e^{\alpha T^m}\right) \left[e^A + E_{\lambda y} h\right] \\
& H e^{\alpha T^m} \left[e^A + e^A \left(E_{Kx} - U^A E_{\lambda y}\right) + h E_{\lambda y} - e^A - E_{\lambda y} h\right] \\
&- \zeta r_x I_\alpha \left[e^A + E_{\lambda y} h\right] \\
&= H e^{\alpha T^m} e^A \left(E_{Kx} - U^A E_{\lambda y}\right) - \zeta r_x I_\alpha \left[e^A + E_{\lambda y} h\right]
\end{aligned}$$

Using the expressions of E_{Kx} and $E_{\lambda y}$:

$$E_{Kx} - U^A E_{\lambda y} = \int_0^{T^m} k_y'(t) e^A(t) \left(U^A(t) - U^A\right) dt < 0$$

which implies that $\Delta_{11} + \Delta_{12} < 0$, thus:

$$\Delta_1 = \Lambda_Z k_x^s \left[\Delta_{11} + \Delta_{12}\right] > 0$$

because $k_x^s < 0$.

Adding the first column to the second one in the expression of Δ_2 yields:

$$\Delta_2 = \begin{vmatrix} 1 & 1+E_{Kx} & -E_{\lambda y} & 0 \\ \Lambda_y + h & h - e^A U^A & e^A & -\Lambda_y k_x^s \\ 0 & -F_{Kx} & F_{\lambda y} & 0 \\ R_{Ky} e^{\alpha T^m} & \zeta r_x I_\alpha + H e^{\alpha T^m} & 0 & -R_{Kx} k_x^s e^{\alpha T^m} \end{vmatrix}$$

Developing with respect to the fourth line:

$$\Delta_2 = -R_{Ky} e^{\alpha T^m} \Delta_{21} + \left(\zeta r_x I_\alpha + H e^{\alpha T^m}\right) \Delta_{22} - R_{Kx} k_x^s e^{\alpha T^m} \Delta_{23}$$

Recalling the notation $m_1 = E_{\lambda y} F_{Kx} - (1 + E_{Kx}) F_{\lambda y} > 0$,

$$\begin{aligned} \Delta_{21} &= \begin{vmatrix} 1+E_{Kx} & -E_{\lambda y} & 0 \\ h - e^A U^A & e^A & -\Lambda_y k_x^s \\ -F_{Kx} & F_{\lambda y} & 0 \end{vmatrix} = \Lambda_y k_x^s \left[(1+E_{Kx}) F_{\lambda y} - E_{\lambda y} F_{Kx}\right] \\ &= -\Lambda_y k_x^s m_1 \end{aligned}$$

Next:

$$\Delta_{22} = \begin{vmatrix} 1 & -E_{\lambda y} & 0 \\ \Lambda_y + h & e^A & -\Lambda_y k_x^s \\ 0 & F_{\lambda y} & 0 \end{vmatrix} = \Lambda_y k_x^s F_{\lambda y}$$

Last:

$$\begin{aligned} \Delta_{23} &= \begin{vmatrix} 1 & 1+E_{Kx} & -E_{\lambda y} \\ \Lambda_y + h & h - e^A U^A & e^A \\ 0 & -F_{Kx} & F_{\lambda y} \end{vmatrix} \\ &= (1)\left[(h - e^A U^A) F_{\lambda y} + e^A F_{Kx}\right] - (\Lambda_y + h)\left[(1+E_{Kx}) F_{\lambda y} - E_{\lambda y} F_{Kx}\right] \\ &= h F_{\lambda y} + e^A \left[F_{Kx} - U^A F_{\lambda y}\right] + m_1 (\Lambda_y + h) \end{aligned}$$

Let:

$$m_4 \equiv F_{Kx} - U^A F_{\lambda y} = \int_0^{T^m} |u''(t)| e^{-\rho t} \left[(1 + E_{Kx}(t)) - U^A E_{\lambda y}(t)\right] dt \equiv \int_0^{T^m} |u''(t)| e^{-\rho t} \phi(t) dt$$

and $\phi(0) = 1$, $\phi(T^m) = (1 + E_{Kx}) - U^A E_{\lambda y} > 0$, as shown before, and $\dot{\phi}(t) < 0$ show that $\phi(t) > 0$, and thus that $m_4 > 0$ in the expression of Δ_{23}:

$$\Delta_{23} = h F_{\lambda y} + e^A m_4 + m_1 (\Lambda_y + h)$$

Collecting the expressions of Δ_{21}, Δ_{22} and Δ_{23} yields the following expression of Δ_2:

$$\begin{aligned} \Delta_2 = {} & -R_{Ky} \left[-\Lambda_y k_x^s m_1\right] + \left(\zeta r_x I_\alpha + H e^{\alpha T^m}\right) \left[\Lambda_y k_x^s F_{\lambda y}\right] \\ & - R_{Kx} k_x^s e^{\alpha T^m} \left[h F_{\lambda y} + e^A m_4 + m_1 (\Lambda_y + h)\right] \end{aligned}$$

Rearranging and using $R_{Ky} = R_{Kx} + H$:

$$\begin{aligned} \frac{\Delta_2}{k_x^s} &= (R_{Ky} - R_{Kx}) \Lambda_y m_1 + \left(\zeta r_x I_\alpha + H e^{\alpha T^m}\right) \Lambda_y F_{\lambda y} \\ &\quad - R_{Kx} e^{\alpha T^m} \left[h F_{\lambda y} + e^A m_4 + m_1 h\right] \\ &= H \Lambda_y m_1 + \left(\zeta r_x I_\alpha + H e^{\alpha T^m}\right) \Lambda_y F_{\lambda y} - R_{Kx} e^{\alpha T^m} \left[h(F_{\lambda y} + m_1) + e^A m_4\right] \end{aligned}$$

The r.h.s. of the above relation is > 0 because $R_{Kx} < 0$ and all the other terms are > 0, thus $\Delta_2 < 0$ because $k_x^s < 0$. Hence:

$$\frac{\partial \hat{K}_y}{\partial \hat{K}_x} = \frac{1}{\underbrace{\Delta}_{(>0)}} \left[-\zeta r_x I_\alpha \underbrace{\Delta_1}_{(>0)} + e^{(\rho+\alpha)T^m} \underbrace{\Delta_2}_{(<0)} \right] < 0$$

at least in the linear or iso-elastic energy demand cases where we have shown that $\Delta > 0$. We thus conclude that $S_{Kx} < 0$, having previously shown that $S_{Ky} < 0$, hence getting the claimed shapes of the stabilisation borders, $\partial \hat{K}_x / \partial \hat{Z} < 0$ and $\partial \hat{K}_x / \partial \hat{K}_y < 0$.