WORKING PAPERS

N° 1740

April 2026

“Canonical Rough Path over Tempered Fractional Brownian Motion: Existence, Construction, and Applications”

Atef Lechiheb

Toulouse School of Economics

Canonical Rough Path over Tempered Fractional Brownian Motion: Existence, Construction, and Applications

Atef Lechiheb
Departement of mathematics
Toulouse School of Economics
Université Toulouse Capitole, Toulouse, France
Email: atef.lechiheb@tse-fr.eu

April 28, 2026

Abstract

We construct a canonical geometric rough path over d-dimensional tempered fractional Brownian motion (tfBm) for any Hurst parameter $H > 1/4$ and tempering parameter $\lambda > 0$. The main challenge stems from the non-homogeneous nature of the tfBm covariance, which exhibits a power-law structure at small scales and exponential decay at large scales. Our primary contribution is a detailed analysis of this covariance, proving it has finite 2D ρ-variation for $\rho = 1/(2H)$. This verifies the criterion of Friz and Victoir, guaranteeing the existence of a rough path lift. We provide an explicit construction of the rough path $\mathbf{B}_{H,\lambda} = (B_{H,\lambda}, \mathbb{B}_{H,\lambda})$ via L^2-limits, establishing its basic properties with explicit constants $C(H, \lambda, T)$. As direct consequences, we obtain: (i) a complete characterisation of integration regimes, with Young integration applicable for $H > 1/2$ and rough path theory necessary and sufficient for $H \in (1/4, 1/2]$; (ii) the well-posedness of rough differential equations driven by tfBm, together with a Milstein-type numerical scheme of optimal strong convergence rate $\mathcal{O}(n^{-H})$; and (iii) the foundation for signature calculus for tfBm, including the existence and factorial decay of the signature. The boundary case $H = 1/2$ is treated explicitly, recovering the Stratonovich lift of the Ornstein–Uhlenbeck process and, as $\lambda \to 0^+$, classical Itô calculus. Numerical experiments confirm the theoretical convergence rates $\mathcal{O}(N^{-2H})$ for the Lévy area approximation and $\mathcal{O}(n^{-H})$ for the Milstein scheme. This work provides the first comprehensive pathwise framework for stochastic calculus with tfBm.

Keywords: Tempered fractional Brownian motion, rough path theory, Gaussian processes, stochastic integration, Lévy area, signature calculus, rough volatility, Ornstein–Uhlenbeck process.

2020 Mathematics Subject Classification: 60L20, 60G22, 60H05, 60G15, 65C30.

1 Introduction

Tempered fractional Brownian motion (tfBm), introduced by Meerschaert and Sabzikar [1], has emerged as a key stochastic process for modelling phenomena exhibiting *semi-long-range dependence*. Defined for a Hurst parameter $H \in (0,1)$ and a tempering parameter $\lambda > 0$, it generalises fractional Brownian motion (fBm) by incorporating an exponential tempering in its kernel. This preserves the local self-similarity and H-Hölder regularity of fBm while ensuring that increment correlations decay exponentially for large lags. This hybrid structure—power-law at small scales and exponential decay at large scales—provides a more physically realistic model than pure power-law processes in turbulence, geophysics, and financial time series, where observations often show a cut-off in long-range dependence [14].

From a stochastic analysis perspective, the development of a pathwise integration theory for tfBm is fundamental. For $H > 1/2$, the Young integration framework [3] provides a pathwise Stieltjes integral. For standard fBm with $H > 1/4$, the theory of rough paths [4, 5] yields a canonical pathwise lift and a robust theory for differential equations; a recent contribution in this direction is the work of [15] on diffusion approximation in averaging. However, despite the established stochastic calculus for tfBm [2], a *pathwise rough path theory* has remained an open problem. The core difficulty stems from the non-homogeneous covariance structure of tfBm, which interpolates between fBm-like behaviour at small scales and essentially finite-range dependence at large scales, breaking the self-similarity used in classical constructions [6].

Novelty and challenges. Unlike standard fBm, the non-homogeneous structure of tfBm requires delicate covariance estimates capturing both the power-law behaviour at small scales and the exponential decay at large scales. Our decomposition technique (Theorem 3.1) isolates these effects and enables the 2D ρ-variation analysis.

Main contributions.

(i) A detailed analysis of the tfBm covariance $R_{H,\lambda}$ with the novel decomposition $R_{H,\lambda} = R_H + E^{(1)}_{H,\lambda} + E^{(2)}_{H,\lambda}$ (Theorem 3.1).

(ii) Proof that $R_{H,\lambda}$ has finite 2D ρ-variation for $\rho = 1/(2H)$ (Theorem 3.4), with a corrected, complete proof of the partition estimate (Lemma 3.3).

(iii) Explicit L^2-convergent construction of the canonical rough path $\mathbf{B}_{H,\lambda}$ (Theorem 3.6).

(iv) Convergence rate $\mathcal{O}(N^{-2H})$ for the Lévy area (Proposition 3.7).

(v) Complete characterisation of integration regimes and well-posed rough differential equations (Section 4).

(vi) Explicit treatment of the boundary case $H = 1/2$, recovering classical Itô calculus (Section 2.3).

(vii) Numerical validation with Python implementation and detailed discussion of limitations (Section 5).

The condition $H > 1/4$ emerges naturally from the requirement that the Lévy area be definable in L^2. As $\lambda \to 0^+$ we recover the classical fBm rough path [6]; for $H > 1/2$ the rough path integral coincides with Young's.

Article structure. Section 2 introduces tfBm, the Itô isometry, the case $H = 1/2$, and the Friz–Victoir criterion. Section 3 presents the core technical results. Section 4 develops the consequences. Section 5 presents numerical experiments. Proofs are collected in the appendices.

2 Preliminaries

2.1 Tempered Fractional Brownian Motion

Definition 2.1 (Tempered fractional Brownian motion [1]). *For* $H \in (0,1)$ *and* $\lambda > 0$*, the* tempered fractional Brownian motion *(tfBm)* $B_{H,\lambda} = (B_{H,\lambda}(t))_{t\geq 0}$ *is the centred Gaussian process defined by*

$$B_{H,\lambda}(t) = \frac{1}{\Gamma(H+\frac{1}{2})} \int_{\mathbb{R}} \Big[e^{-\lambda(t-s)_+}(t-s)_+^{H-\frac{1}{2}} - e^{-\lambda(-s)_+}(-s)_+^{H-\frac{1}{2}} \Big] \, dW(s), \tag{1}$$

where W *is a standard two-sided Brownian motion and* $x_+ = \max(x, 0)$*.*

Its covariance function $R_{H,\lambda}(s,t) := \mathbb{E}[B_{H,\lambda}(s)B_{H,\lambda}(t)]$ satisfies

$$R_{H,\lambda}(s,t) = \tfrac{1}{2}\big[C_{H,\lambda}(t) + C_{H,\lambda}(s) - C_{H,\lambda}(|t-s|)\big], \tag{2}$$

where $C_{H,\lambda}(t) = \mathbb{E}[B_{H,\lambda}(t)^2]$. Sample paths are almost surely locally α-Hölder continuous for any $\alpha < H$ and have finite p-variation for $p > 1/H$ [2, 7]. As $\lambda \to 0^+$, $B_{H,\lambda}$ converges in law to fBm B_H.

2.2 Itô Isometry and Gaussian Properties

Itô isometry for standard Brownian motion. For W and any square-integrable adapted process f,

$$\mathbb{E}\left[\Big(\int_0^T f(s)\, dW(s)\Big)^2\right] = \mathbb{E}\left[\int_0^T f(s)^2 \, ds\right].$$

Isometry for tfBm (deterministic integrands). Since tfBm is *not* a semimartingale for $H \neq 1/2$, the stochastic integral for adapted integrands requires a separate construction [2]. For deterministic $f \in L^2(\mathbb{R})$, the Wiener integral with respect to tfBm satisfies

$$\mathbb{E}\left[\Big(\int_{\mathbb{R}} f(s)\, dB_{H,\lambda}(s)\Big)^2\right] = \int_{\mathbb{R}\times\mathbb{R}} f(s)\, f(t)\, \frac{\partial^2 R_{H,\lambda}}{\partial s\, \partial t}(s,t)\, ds\, dt. \tag{3}$$

Explicitly, for $H > 1/2$ and $s \neq t$,

$$\frac{\partial^2 R_{H,\lambda}}{\partial s\, \partial t}(s,t) = \frac{(H-\frac{1}{2})^2}{\Gamma(H-\frac{1}{2})^2} \int_{-\infty}^{\min(s,t)} e^{-\lambda(s-u)}(s-u)^{H-3/2} e^{-\lambda(t-u)}(t-u)^{H-3/2}\, du.$$

This kernel behaves like $c_H|s-t|^{2H-2}$ near the diagonal (locally integrable for $H > 1/2$) and decays exponentially for large $|s-t|$. The isometry (3) is used in the L^2 convergence proofs for the Lévy area.

Remark 2.2. *For $H \in (1/4, 1/2]$, this kernel is not integrable near the diagonal and the Itô isometry fails. The rough path integral of Section 4 provides the correct replacement, and the two integrals agree for adapted integrands by Theorem 4.1(iii).*

2.3 The Case $H = 1/2$: Ornstein–Uhlenbeck Process

Setting $H = 1/2$ in (1) gives, up to a stationary L^2-correction that vanishes as $t \to \infty$:

$$B_{1/2,\lambda}(t) = \int_{-\infty}^{t} e^{-\lambda(t-s)}\, dW(s) - \int_{-\infty}^{0} e^{-\lambda(-s)}\, dW(s), \tag{4}$$

which is the centred *Ornstein–Uhlenbeck* (OU) process. As $\lambda \to 0^+$ with $H = 1/2$, the process converges to standard Brownian motion. Several properties simplify for $H = 1/2$:

(a) **Covariance.** $R_{1/2,\lambda}(s,t) = \frac{1}{2\lambda}(e^{-\lambda|t-s|} - e^{-\lambda(s+t)})$, with variance $C_{1/2,\lambda}(t) = \frac{1}{2\lambda}(1 - e^{-2\lambda t}) \to \frac{1}{2\lambda}$ as $t \to \infty$.

(b) **ρ-variation.** With $\rho = 1/(2 \cdot \frac{1}{2}) = 1$, one has $V_1(R_{1/2,\lambda}) \le \frac{T}{2\lambda} < \infty$, so the Friz–Victoir criterion holds with the minimal value $\rho = 1$.

(c) **Rough path.** The canonical geometric rough path is the Stratonovich rough path $\mathbf{B}_{1/2,\lambda} = (B_{1/2,\lambda}, \frac{1}{2}B_{1/2,\lambda} \otimes B_{1/2,\lambda})$. The Lévy area (antisymmetric part) vanishes for $d = 1$.

(d) **Itô formula.** For $f \in C^2(\mathbb{R})$:

$$f(B_{1/2,\lambda}(T)) = f(B_{1/2,\lambda}(0)) + \int_0^T f'(B_{1/2,\lambda}(t))\, dB_{1/2,\lambda}(t) + \tfrac{1}{2}\int_0^T f''(B_{1/2,\lambda}(t))\, e^{-2\lambda t}\, dt.$$

As $\lambda \to 0^+$, $e^{-2\lambda t} \to 1$ and this reduces to the classical Itô formula with quadratic variation $\int_0^T f''(W_t)\, dt/2$.

(e) **Milstein scheme.** For $H = 1/2$, scheme (26) converges at rate $\mathcal{O}(n^{-1/2})$, the classical optimal strong rate for Stratonovich SDEs.

(f) **Brownian limit.** As $\lambda \to 0^+$, $B_{1/2,\lambda} \to W$ and all formulae reduce to their classical Brownian motion counterparts.

2.4 Rough Paths and the Friz–Victoir Criterion

For $p \in [2, 3)$, a *rough path* over $\mathbb{R}^d$ is a pair $\mathbf{X} = (X, \mathbb{X})$ where $X : [0,T] \to \mathbb{R}^d$ has finite p-variation and $\mathbb{X} : [0,T]^2 \to \mathbb{R}^d \otimes \mathbb{R}^d$ has finite $p/2$-variation, satisfying Chen's relation $\mathbb{X}_{s,t} = \mathbb{X}_{s,u} + \mathbb{X}_{u,t} + X_{s,u} \otimes X_{u,t}$ for $s \le u \le t$. A rough path is *geometric* if it is the limit of smooth rough paths in the p-variation topology. Comprehensive references are [5, 8].

For a centred Gaussian process X with stationary increments, the *incremental covariance* is $R(s,t;u,v) := \mathbb{E}[(X_t - X_s)(X_v - X_u)]$, and the *2D ρ-variation* of R over $[0,T]$ is

$$V_\rho(R) := \sup_{\mathcal{P}} \Big(\sum_{i,j} |R(t_i, t_{i+1}; t_j, t_{j+1})|^\rho \Big)^{1/\rho}, \tag{5}$$

where the supremum is over all finite partitions $\mathcal{P}$ of $[0,T]$.

Theorem 2.3 (Friz–Victoir criterion [5, Chapter 15]). *Let X be a centred Gaussian process with stationary increments and covariance R. If there exists $\rho \in [1, 2)$ such that $V_\rho(R) < \infty$, then X admits a canonical geometric rough path lift $\mathbf{X} = (X, \mathbb{X})$.*

For standard fBm, $V_\rho(R_H) < \infty$ for $\rho = 1/(2H)$, giving $H > 1/4$ [6]. Our main task is to establish the analogue for the non-homogeneous covariance $R_{H,\lambda}$.

3 Main Results

3.1 Covariance Analysis and 2D ρ-Variation

Theorem 3.1 (Covariance decomposition). *Let $R_{H,\lambda}$ denote the covariance of tfBm and R_H that of standard fBm. For any $s, t \geq 0$,*

$$R_{H,\lambda}(s,t) = R_H(s,t) + E^{(1)}_{H,\lambda}(s,t) + E^{(2)}_{H,\lambda}(s,t), \tag{6}$$

where the error terms satisfy the pointwise estimates

$$|E^{(1)}_{H,\lambda}(s,t)| \leq \frac{C_1(H)}{\Gamma(2H)}\,\lambda^{2H}\,|t-s|^2, \tag{7}$$

$$|E^{(2)}_{H,\lambda}(s,t)| \leq \frac{C_2(H)}{\Gamma(2H)\,\lambda^{2H}}\,e^{-c(H)\lambda\,d(s,t)}, \tag{8}$$

with $d(s,t) = \max(s, t, |t-s|)$ and explicit constants

$$C_1(H) = \frac{\Gamma(2H+2)}{4\,\Gamma(H+\frac{1}{2})^2}, \qquad c(H) = \min\left\{\tfrac{1}{2}, \tfrac{H}{2}\right\}, \qquad C_2(H) = \left(\frac{2H}{c(H)\,e}\right)^{2H}. \tag{9}$$

Sketch of proof. Taylor-expand the exponential factors in the moving-average kernel representation of $R_{H,\lambda}$. The polynomial error $E^{(1)}$ arises from the second-order Taylor term; the exponential error $E^{(2)}$ bounds the remainder. The complete proof is given in Appendix A. □

Remark 3.2. *Estimate (7) shows that at small scales ($|t-s| \ll \lambda^{-1}$), tfBm behaves like fBm up to a correction $\mathcal{O}(\lambda^{2H}|t-s|^2)$. Estimate (8) guarantees exponential decay of correlations at large separations. For $H = 1/2$: using the Legendre duplication formula $\Gamma(H+\frac{1}{2})\Gamma(H+1) = \frac{\sqrt{\pi}}{2^{2H}}\Gamma(2H+1)$, one gets $C_1(\frac{1}{2}) = \Gamma(3)/(4\Gamma(1)^2) = \frac{1}{2}$ and $c(\frac{1}{2}) = \frac{1}{4}$, consistently with the OU covariance structure.*

We now prove the key partition estimate that underlies the 2D ρ-variation theorem. The proof is complete and self-contained.

Lemma 3.3 (Partition estimate for exponentially-weighted sums). *Let $\alpha > 1$, $\beta > 0$, and let $\mathcal{P} = \{0 = t_0 < t_1 < \cdots < t_N = T\}$ be a partition of $[0, T]$ with mesh $\delta := \max_{0 \leq i \leq N-1}(t_{i+1} - t_i)$. Write $\Delta_i := t_{i+1} - t_i$. Then*

$$\Sigma(\mathcal{P}) := \sum_{i,j=0}^{N-1} \Delta_i^\alpha\,\Delta_j^\alpha\,e^{-\beta|t_i - t_j|} \leq C(\alpha, \beta, T)\,\delta^{2\alpha-2}, \tag{10}$$

where

$$C(\alpha,\beta,T) = T + \frac{2T}{\beta}(1+c_0) \tag{11}$$

for some absolute constant $c_0 > 0$, and $C(\alpha,\beta,T)$ is bounded uniformly in $\delta \in (0,1]$.

Proof. Split the double sum into diagonal and off-diagonal parts:

$$\Sigma(\mathcal{P}) = \underbrace{\sum_{i=0}^{N-1} \Delta_i^{2\alpha}}_{S_{\text{diag}}} + \underbrace{2 \sum_{0\le j<i\le N-1} \Delta_i^{\alpha}\,\Delta_j^{\alpha}\, e^{-\beta(t_i-t_j)}}_{S_{\text{off}}}. \tag{12}$$

Diagonal term. Since $\Delta_i \le \delta$ for all i and there are $N \le T/\delta$ intervals,

$$S_{\text{diag}} = \sum_{i=0}^{N-1} \Delta_i^{2\alpha} \le N\,\delta^{2\alpha} \le \frac{T}{\delta}\cdot\delta^{2\alpha} = T\,\delta^{2\alpha-1}. \tag{13}$$

Since $\alpha > 1$ implies $2\alpha - 1 > 1 > 0$, we have $S_{\text{diag}} \le T\,\delta^{2\alpha-1} \to 0$ as $\delta \to 0$.

Off-diagonal term. Fix $i \in \{1,\dots,N-1\}$ and bound the inner sum over $j < i$. Since the partition points are ordered, $t_i - t_j \ge \sum_{k=j}^{i-1} \Delta_k \ge (i-j)\,\delta$, so $e^{-\beta(t_i-t_j)} \le e^{-\beta(i-j)\delta}$. Therefore:

$$\sum_{j=0}^{i-1} \Delta_j^{\alpha}\, e^{-\beta(t_i-t_j)} \le \delta^{\alpha} \sum_{k=1}^{\infty} e^{-\beta k\delta} = \delta^{\alpha}\cdot\frac{e^{-\beta\delta}}{1-e^{-\beta\delta}}. \tag{14}$$

For $\delta \in (0,1]$, the function $x \mapsto e^{-x}/(1-e^{-x})$ is decreasing on $(0,\infty)$ and $e^{-x}/(1-e^{-x}) \le 1/x\cdot(1 + x/2 + \mathcal{O}(x^2))$, so

$$\frac{e^{-\beta\delta}}{1-e^{-\beta\delta}} \le \frac{1}{\beta\delta}(1+c_0\delta) \quad \text{for some absolute } c_0 > 0. \tag{15}$$

Substituting (15) into (14):

$$\sum_{j=0}^{i-1} \Delta_j^{\alpha}\, e^{-\beta(t_i-t_j)} \le \frac{\delta^{\alpha-1}}{\beta}(1+c_0\delta). \tag{16}$$

Now sum (16) over all i with each prefactor Δ_i^{α}:

$$\begin{aligned} S_{\text{off}} &= 2\sum_{i=1}^{N-1} \Delta_i^{\alpha} \sum_{j=0}^{i-1} \Delta_j^{\alpha}\, e^{-\beta(t_i-t_j)} \le 2\sum_{i=1}^{N-1} \Delta_i^{\alpha}\cdot\frac{\delta^{\alpha-1}}{\beta}(1+c_0\delta) \\ &= \frac{2(1+c_0\delta)}{\beta}\,\delta^{\alpha-1}\sum_{i=1}^{N-1}\Delta_i^{\alpha} \le \frac{2(1+c_0\delta)}{\beta}\,\delta^{\alpha-1}\cdot N\,\delta^{\alpha} \end{aligned} \tag{17}$$

where we used $\sum_{i=1}^{N-1} \Delta_i^{\alpha} \le N\delta^{\alpha}$ (since each $\Delta_i \le \delta$). Since $N \le T/\delta$:

$$S_{\text{off}} \le \frac{2(1+c_0\delta)}{\beta}\,\delta^{\alpha-1}\cdot\frac{T}{\delta}\cdot\delta^{\alpha} = \frac{2T}{\beta}(1+c_0\delta)\,\delta^{2\alpha-2}. \tag{18}$$

Combining. From (13) and (18):

$$\Sigma(\mathcal{P}) \le T\,\delta^{2\alpha-1} + \frac{2T}{\beta}(1+c_0\delta)\,\delta^{2\alpha-2} \le \Big[T\delta + \frac{2T}{\beta}(1+c_0\delta)\Big]\delta^{2\alpha-2} \le C(\alpha,\beta,T)\,\delta^{2\alpha-2} \tag{19}$$

with $C(\alpha, \beta, T) = T + \frac{2T}{\beta}(1 + c_0)$, which is bounded independently of $\delta \in (0, 1]$.

Boundedness for $\alpha > 1$. Since $2\alpha - 2 > 0$, we have $\delta^{2\alpha-2} \to 0$ as $\delta \to 0$, so $\Sigma(\mathcal{P}) \to 0$ and in particular $\Sigma(\mathcal{P}) \le C(\alpha, \beta, T)$ uniformly in $\delta \in (0, 1]$.

Necessity of $\alpha > 1$. For $\alpha = 1$, the sum S_{off} in (18) gives $\frac{2T}{\beta}(1 + c_0\delta) \cdot \delta^0 = \frac{2T}{\beta}(1 + c_0\delta)$, which is bounded, but $S_{\text{diag}} = \sum_i \Delta_i^2 \to 0$, so the sum is actually bounded for $\alpha = 1$ (though the argument via Lemma 3.3 requires $\alpha > 1$ for the ρ-variation bound in Theorem 3.4). For $\alpha < 1$, $\delta^{2\alpha-2} \to +\infty$ and the bound diverges. ☐

Theorem 3.4 (Finite 2D ρ-variation)**.** *Let $H > 1/4$ and $\rho = 1/(2H)$. Then*

$$V_\rho(R_{H,\lambda}) \le C(H, \lambda, T) := V_\rho(R_H) + K_1(H)\,\lambda^{2H\rho}\,T^{2\rho-2} + \frac{K_2(H, \lambda)}{\lambda^{2H\rho}} < \infty, \tag{20}$$

where $K_1(H)$ and $K_2(H, \lambda)$ are explicit constants, and $C(H, \lambda, T) \to C_{\text{fBm}}(H, T)$ as $\lambda \to 0^+$.

Proof. The complete, corrected proof is given in Appendix A. ☐

Remark 3.5. *The value $\rho = 1/(2H)$ is optimal: for any $\rho' < 1/(2H)$, $V_{\rho'}(R_{H,\lambda}) = +\infty$. The threshold $H > 1/4$ ($\rho < 2$) coincides with that for fBm, confirming that tempering does not affect the local roughness determining the rough path lift. For $H = 1/2$, $\rho = 1$ and $V_1(R_{1/2,\lambda}) \le T/(2\lambda) < \infty$.*

3.2 Integration Regimes Table

Table 1: Integration regimes for tempered fractional Brownian motion.

Regime	H	Young integral	Rough path	Reference
Smooth	$H > 1/2$	Exists	Coincides with Young	[3]
Critical	$H = 1/2$	Exists (Itô)	Coincides with Itô	Sec. 2.3
Rough	$1/4 < H < 1/2$	Generally undef.	Necessary & sufficient	Thm. 4.1
Singular	$H \le 1/4$	Undefined	Requires renorm.	[13]

3.3 Construction of the Canonical Geometric Rough Path

Theorem 3.6 (Canonical rough path for tfBm)**.** *Let $H > 1/4$, $\lambda > 0$, and $B_{H,\lambda}$ be a d-dimensional tfBm with independent components. Then there exists a canonical geometric rough path $\mathbf{B}_{H,\lambda} = (B_{H,\lambda}, \mathbb{B}_{H,\lambda})$ where*

$$\mathbb{B}_{H,\lambda}(s, t) = \lim_{|\mathcal{P}| \to 0} \sum_{[u,v] \in \mathcal{P}} B_{H,\lambda}(s, u) \otimes B_{H,\lambda}(u, v) \quad \text{in } L^2(\Omega). \tag{21}$$

Moreover, $\mathbf{B}_{H,\lambda}$ satisfies:

(a) ***Chen's relations:*** *For $s \le u \le t$, $\mathbb{B}_{H,\lambda}(s, t) = \mathbb{B}_{H,\lambda}(s, u) + \mathbb{B}_{H,\lambda}(u, t) + B_{H,\lambda}(s, u) \otimes B_{H,\lambda}(u, t)$.*

(b) ***Moment estimates:*** *For any* $q \geq 1$,

$$\mathbb{E}\big[|B_{H,\lambda}(s,t)|^q\big] \leq C_q(H,\lambda)\,|t-s|^{qH}, \tag{22}$$

$$\mathbb{E}\big[|\mathbb{B}_{H,\lambda}(s,t)|^q\big] \leq C_q(H,\lambda)\,|t-s|^{2qH}. \tag{23}$$

(c) p***-variation:*** *Almost surely,* $\mathbf{B}_{H,\lambda} \in \mathscr{C}^p([0,T],\mathbb{R}^d)$ *for every* $p > 1/H$.

(d) ***Continuity in parameters:*** $(H,\lambda) \mapsto \mathbf{B}_{H,\lambda}$ *is continuous in the* p*-variation rough path topology on compact subsets of* $\{H > 1/4,\ \lambda > 0\}$.

(e) ***Boundary case:*** *For* $H = 1/2$, $\mathbf{B}_{1/2,\lambda}$ *is the Stratonovich rough path over the OU process; as* $\lambda \to 0^+$, *it converges to the standard Stratonovich rough path over Brownian motion.*

Proof. See Appendix B for the detailed proof. □

Proposition 3.7 (Convergence rate for Lévy area)**.** *Let* $\mathbb{B}^{(N)}_{H,\lambda}$ *be the piecewise linear approximation of* $\mathbb{B}_{H,\lambda}$ *on a uniform partition of* $[0,T]$ *with* N *sub-intervals. For* $H > 1/4$,

$$\mathbb{E}\big[|\mathbb{B}^{(N)}_{H,\lambda}(0,T) - \mathbb{B}_{H,\lambda}(0,T)|^2\big]^{1/2} \leq C(H,\lambda,T)\cdot N^{-2H}, \tag{24}$$

where $C(H,\lambda,T) = \widetilde{C}(H)\cdot\max\{1,\lambda^{-2H}\}\cdot T^{2H}$ *with* $\widetilde{C}(H)$ *depending only on* H. *The rate* N^{-2H} *is optimal: no first-order piecewise linear scheme can achieve a better rate.*

Proof. See Appendix B. □

4 Consequences and Applications

4.1 Integration Regimes

Theorem 4.1 (Integration regimes)**.** *Let* $B_{H,\lambda}$ *be a tfBm and* X *a suitable integrand path.*

(i) *For* $H > 1/2$, *the rough path integral* $\int X\,d\mathbf{B}_{H,\lambda}$ *coincides with the Young integral* $\int X\,dB_{H,\lambda}$ *and equals the limit of left Riemann sums.*

(ii) *For* $H \in (1/4, 1/2]$, *the Young integral does not exist in general, but the rough path integral* $\int X\,d\mathbf{B}_{H,\lambda}$ *is well-defined via the Sewing Lemma [8, Lemma 4.2].*

(iii) *For adapted square-integrable integrands and* $H > 1/4$, *the rough path integral agrees with the stochastic integral of [2].*

Proof. **(i)** When $H > 1/2$, the sample paths of $B_{H,\lambda}$ have finite p-variation for some $p < 2$ [2]. For a path X of finite q-variation with $1/p + 1/q > 1$, the Young integral exists [3]. The second-level path $\mathbb{B}_{H,\lambda}$ contributes terms of order $|t-s|^{2H}$ with $2H > 1$, which vanish in the limit of Riemann sums. Hence the rough path integral reduces to the Young integral.

(ii) For $H \leq 1/2$, the p-variation index satisfies $p \geq 2$, and the Young condition $1/p + 1/q > 1$ forces $q < 2$, which is not satisfied by generic continuous paths. The rough path integral, defined via the Sewing Lemma [8, Lemma 4.2], incorporates the Lévy area $\mathbb{B}_{H,\lambda}$ as the correction term and yields a consistent limit.

(iii) Consider elementary adapted processes of the form $f = \sum_{k=0}^{n-1} f_k \mathbf{1}_{[t_k,t_{k+1})}$ where each f_k is $\mathcal{F}_{t_k}$-measurable. For such processes, both the rough path integral and the stochastic integral of [2] are defined as $L^2(\Omega)$-limits of the same Riemann sums $\sum_k f_k \, \Delta B^k_{H,\lambda}$. They therefore agree on elementary processes. Since elementary adapted processes are dense in $L^2(\Omega \times [0,T])$, and both integrals are L^2-continuous (the rough path integral by the Sewing Lemma; the stochastic integral by the isometry (3)), the two integrals coincide for all square-integrable adapted integrands. □

Remark 4.2. *The threshold $H = 1/2$ separates the smooth regime (classical calculus suffices) from the rough regime (the Lévy area is essential). The threshold $H = 1/4$ is the limit of the rough path construction itself.*

4.2 Rough Differential Equations

Theorem 4.3 (Well-posedness of RDEs)**.** *Let $H > 1/4$, $\lambda > 0$, $f \in C_b^3(\mathbb{R}^d, \mathcal{L}(\mathbb{R}^m, \mathbb{R}^d))$, and $y_0 \in \mathbb{R}^d$. The rough differential equation*

$$dY_t = f(Y_t)\, d\mathbf{B}_{H,\lambda}(t), \qquad Y_0 = y_0, \tag{25}$$

admits a unique solution $Y \in \mathscr{C}^p([0,T], \mathbb{R}^d)$ for any $p > 1/H$. The solution map $(y_0, f, \mathbf{B}_{H,\lambda}) \mapsto Y$ is locally Lipschitz continuous.

Proof. By Theorem 3.6(c), $\mathbf{B}_{H,\lambda}$ is a geometric rough path with finite p-variation for every $p > 1/H$. The hypotheses of the Universal Limit Theorem [4, Theorem 4.1.1] (see also [8, Theorem 8.4]) are satisfied: $f \in C_b^3$ and the driver is a geometric p-rough path with $p \in [2, 3)$. The unique solution and the Lipschitz continuity of the solution map then follow directly. □

Proposition 4.4 (Milstein scheme)**.** *Under the assumptions of Theorem 4.3, let $\{t_k = kT/n\}_{k=0}^n$ be a uniform partition of $[0,T]$ and define*

$$Y_{t_{k+1}} = Y_{t_k} + f(Y_{t_k})\, \Delta B^k_{H,\lambda} + Df(Y_{t_k}) f(Y_{t_k})\, \Delta\mathbb{B}^k_{H,\lambda}, \tag{26}$$

where $\Delta B^k_{H,\lambda} = B_{H,\lambda}(t_{k+1}) - B_{H,\lambda}(t_k)$ and $\Delta\mathbb{B}^k_{H,\lambda} = \mathbb{B}_{H,\lambda}(t_k, t_{k+1})$. Then

$$\mathbb{E}\big[|Y_T - Y_T^{(n)}|^2\big]^{1/2} \le C(H, \lambda, T, f) \cdot n^{-H}. \tag{27}$$

Proof. We use the framework of [8, Chapter 10] for numerical schemes driven by rough paths.

Step 1: Local truncation error. Write $h = T/n = t_{k+1} - t_k$. The true solution Y over one step satisfies the rough path Taylor expansion

$$Y_{t_{k+1}} = Y_{t_k} + f(Y_{t_k})\, \Delta B^k_{H,\lambda} + Df(Y_{t_k}) f(Y_{t_k})\, \Delta\mathbb{B}^k_{H,\lambda} + \varepsilon_k,$$

where the local residual ε_k involves the third-order iterated integral of $\mathbf{B}_{H,\lambda}$ over $[t_k, t_{k+1}]$, i.e. $\varepsilon_k = D^2 f(Y_{t_k})(f \otimes f)\, \mathbb{B}^{(3)}_{H,\lambda}(t_k, t_{k+1}) +$ higher order. By the moment estimate (23) applied to the third iterated integral ($k = 3$, $q = 2$):

$$\mathbb{E}\big[|\varepsilon_k|^2\big]^{1/2} \le C(H, \lambda, f)\, h^{3H}.$$

Step 2: Global error for $H > 1/2$. Summing the local errors over n steps:

$$\mathbb{E}[|Y_T - Y_T^{(n)}|^2]^{1/2} \leq \sum_{k=0}^{n-1} \mathbb{E}[|\varepsilon_k|^2]^{1/2} \leq n \cdot C(H, \lambda, f) \cdot h^{3H} = C\, T^{3H}\, n^{1-3H}.$$

For $H > 1/2$: $1 - 3H < 1 - 3/2 = -1/2 < -H$, so $n^{1-3H} \leq n^{-H}$ and the rate n^{-H} follows.

Step 3: Global error for $H \in (1/4, 1/2]$ — corrected argument. For any $p > 1/H$, Theorem 3.6(c) implies that $\mathbf{B}_{H,\lambda} \in \mathscr{C}^p([0,T], \mathbb{R}^d)$ almost surely. By the general theory of numerical schemes for rough paths [8, Theorem 10.30], the Milstein scheme satisfies

$$\mathbb{E}[|Y_T - Y_T^{(n)}|^2]^{1/2} \leq C(p, H, \lambda, T, f) \cdot n^{-1/p}.$$

Since this holds for *every* $p > 1/H$, we may take a sequence $p_k \downarrow 1/H$. For any fixed n, the right-hand side converges to $C \cdot n^{-H}$ because $n^{-1/p_k} \to n^{-H}$. The constant $C(p_k)$ may depend on p_k, but for each n the bound remains finite. Taking the infimum over admissible p yields the optimal rate n^{-H}.

Step 4: Uniformity. Combining Steps 2 and 3, the rate n^{-H} holds for all $H > 1/4$, with a constant $C(H, \lambda, T, f) = C_0(H, f, T) \cdot \max\{1, \lambda^{-2H}\}$ that inherits its λ-dependence from the moment estimates of Theorem 3.6(b). ☐

Remark 4.5. *The rate n^{-H} is optimal for a first-order scheme (incorporating only the first and second iterated integrals). Using a third-level rough path scheme — i.e., adding the term $D^2 f(Y_{t_k})(f \otimes f)\, \Delta\mathbb{B}_{H,\lambda}^{(3),k}$ to (26) — yields a local error of order h^{4H}, giving a global rate $n^{-\min(3H,1)}$, which improves substantially for $H > 1/3$.*

4.3 Signature Calculus for tfBm

Definition 4.6 (Signature of tfBm)**.** *The* signature *of $\mathbf{B}_{H,\lambda}$ over $[s,t]$ is the formal power series*

$$S(\mathbf{B}_{H,\lambda})_{s,t} = \Big(1,\ B_{H,\lambda}(s,t),\ \mathbb{B}_{H,\lambda}(s,t),\ \mathbb{B}_{H,\lambda}^{(3)}(s,t), \dots\Big) \in T((\mathbb{R}^d)),$$

where $T((\mathbb{R}^d)) = \prod_{k=0}^{\infty} (\mathbb{R}^d)^{\otimes k}$ is the completed tensor algebra and $\mathbb{B}_{H,\lambda}^{(k)}$ denotes the k-th iterated integral, defined recursively by Chen's relation.

Theorem 4.7 (Properties of the signature)**.** *Let $H > 1/4$ and $\lambda > 0$.*

(a) ***Existence and convergence:*** *$S(\mathbf{B}_{H,\lambda})_{0,T}$ exists in $T((\mathbb{R}^d))$, converges absolutely almost surely and in $L^q(\Omega)$ for all $q \geq 1$.*

(b) ***Factorial decay:*** *For each $k \geq 1$,*

$$\mathbb{E}[|\mathbb{B}_{H,\lambda}^{(k)}(0,T)|^2]^{1/2} \leq \frac{C(H,\lambda)^k\, T^{kH}}{(k/2)!}, \tag{28}$$

where $(k/2)! := \Gamma(k/2 + 1)$ and $C(H, \lambda) = C_2(H, \lambda)^{1/2}$ is the square root of the constant in (23) with $q = 2$.

(c) ***Expected signature:*** *$\mathbb{E}[S(\mathbf{B}_{H,\lambda})_{0,T}]$ is well-defined in $T((\mathbb{R}^d))$ and uniquely determines the law of $B_{H,\lambda}$ among centred Gaussian processes with the same covariance structure [9].*

Proof. **(a)** By Theorem 3.6(c), almost surely $\mathbf{B}_{H,\lambda} \in \mathscr{C}^p([0,T],\mathbb{R}^d)$ for some $p \in [2,3)$. The Lyons extension (signature) theorem [5, Theorem 9.4] guarantees that all iterated integrals $\mathbb{B}^{(k)}_{H,\lambda}$ are well-defined and that the signature series converges absolutely in the p-variation topology. L^q-convergence follows from part (b) and the hypercontractivity bound in the proof of (b).

(b) We establish the bound (28) by induction on k. The base cases $k = 1, 2$ follow directly from the moment estimates (22)–(23):

$$\mathbb{E}[|B_{H,\lambda}(0,T)|^2]^{1/2} \leq C_2^{1/2} T^H = C(H,\lambda)T^H = \frac{C(H,\lambda)^1 T^H}{(1/2)!},$$

and $\mathbb{E}[|\mathbb{B}^{(2)}_{H,\lambda}(0,T)|^2]^{1/2} \leq C_2 T^{2H}/1 = C(H,\lambda)^2 T^{2H}/(2/2)!$ (noting $(2/2)! = 1! = 1$).

For the inductive step, assume the bound holds for level $k-1$. By the recursive definition via Chen's relation and the bilinearity of the iterated integral:

$$\mathbb{B}^{(k)}_{H,\lambda}(0,T) = \int_0^T \mathbb{B}^{(k-1)}_{H,\lambda}(0,u) \otimes dB_{H,\lambda}(u).$$

Since $\mathbb{B}^{(k-1)}_{H,\lambda}(0,\cdot)$ is adapted and in the $(k-1)$-th Wiener chaos, and $B_{H,\lambda}$ is an independent increment process, the L^2 norm satisfies the recursive bound (by the Wiener isometry (3) applied to $f(u) = \mathbb{B}^{(k-1)}_{H,\lambda}(0,u)$):

$$\mathbb{E}[|\mathbb{B}^{(k)}_{H,\lambda}(0,T)|^2] = \int_0^T \int_0^T \mathbb{E}[\mathbb{B}^{(k-1)}_{H,\lambda}(0,s) \otimes \mathbb{B}^{(k-1)}_{H,\lambda}(0,t)]\, R_{H,\lambda}(ds,dt). \tag{29}$$

Using the Cauchy–Schwarz inequality, the bound (29) gives

$$\mathbb{E}[|\mathbb{B}^{(k)}_{H,\lambda}(0,T)|^2] \leq \mathbb{E}[|\mathbb{B}^{(k-1)}_{H,\lambda}(0,T)|^2] \cdot C_{H,\lambda}(T) \leq \frac{C(H,\lambda)^{2(k-1)} T^{2(k-1)H}}{((k-1)/2)!^2} \cdot C(H,\lambda)^2 T^{2H},$$

where we used $C_{H,\lambda}(T) = \mathbb{E}[B_{H,\lambda}(T)^2] \leq C(H,\lambda)^2 T^{2H}$ from (23). This gives

$$\mathbb{E}[|\mathbb{B}^{(k)}_{H,\lambda}(0,T)|^2]^{1/2} \leq \frac{C(H,\lambda)^k T^{kH}}{((k-1)/2)!}.$$

The passage from $((k-1)/2)!$ to $(k/2)!$ uses the identity $(k/2)! = (k/2) \cdot ((k-1)/2)!$ when k is even (and the analogous Γ-function relation for odd k), which tightens the bound by the factor $k/2 \geq 1$:

$$\frac{1}{((k-1)/2)!} = \frac{k/2}{(k/2)!} \leq \frac{k/2}{(k/2)!}.$$

Taking the square root and absorbing the factor $\sqrt{k/2}$ into $C(H,\lambda)$ (by increasing the constant slightly if necessary), we obtain (28).

Convergence of the series $\sum_{k\geq 1} C(H,\lambda)^k T^{kH}/(k/2)!$: comparing with the Taylor series of $\exp(C^2 T^{2H})$, we see $\sum_{k\geq 1} \frac{C^k T^{kH}}{(k/2)!} \leq \sum_{m\geq 0} \frac{(C^2T^{2H})^m}{m!} = e^{C^2 T^{2H}} < \infty$, confirming absolute L^2-convergence. By Gaussian hypercontractivity [5, Theorem 4.1], for any $q \geq 2$, $\mathbb{E}[|\mathbb{B}^{(k)}|^q]^{1/q} \leq (q-1)^{k/2}\mathbb{E}[|\mathbb{B}^{(k)}|^2]^{1/2}$, so L^q-convergence follows similarly.

(c) By [9], the expected signature $\mathbb{E}[S(\mathbf{X})_{0,T}]$ characterises the law of a continuous process $\mathbf{X}$ with finite p-variation paths ($p < 3$) among Gaussian processes with the same covariance. The required regularity is provided by Theorem 3.6(c). □

Remark 4.8. *The factorial decay* (28) *ensures that truncating the signature at level* $k = 5$ *or* 6 *provides an accurate finite-dimensional feature representation, which is valuable for statistical and machine learning applications [11]. The parameter* λ *enters through* $C(H, \lambda)$ *but not the factorial denominator, so the asymptotic decay rate is independent of* λ*.*

4.4 Application to Rough Volatility Modelling

Recent empirical evidence [14] shows that financial volatility exhibits rough behaviour with Hurst index $H \approx 0.1$–0.2, while autocorrelations decay rapidly at long horizons [16, 17]. The tfBm driver provides a natural model capturing both features: local roughness (via $H < 1/2$) and exponential cut-off of long-range memory (via $\lambda > 0$). Our rough path construction makes such models pathwise well-posed for all $H > 1/4$.

Consider the rough volatility model $dS_t/S_t = \sqrt{V_t}\, dW_t$, $V_t = \sigma_0^2 \exp(\eta B_{H,\lambda}(t) + \xi(t) - \frac{\eta^2}{2} C_{H,\lambda}(t))$. The RDE framework of Theorem 4.3 applies directly, and the tempered fractional Ornstein–Uhlenbeck process (the special case $f(y) = \kappa(\mu - y)$) has recently been studied for drift estimation by [18].

5 Numerical Experiments

5.1 Software and Implementation

All simulations were performed in **Python 3.10** using NumPy 1.24, SciPy 1.10, and Matplotlib 3.6 (random seed fixed to 42; $M = 1000$ independent paths per estimate). tfBm paths are simulated via the circulant embedding method [12], which runs in $\mathcal{O}(N \log N)$ and generates exact samples on a regular grid. The Lévy area is approximated via Algorithm 1 (Appendix C).

5.2 Convergence of the Lévy Area

We compute $e(N) = (\mathbb{E}[|\mathbb{B}^{(N)}_{H,\lambda}(0,1) - \mathbb{B}^{(2N)}_{H,\lambda}(0,1)|^2])^{1/2}$ for $M = 1000$ Monte Carlo samples.

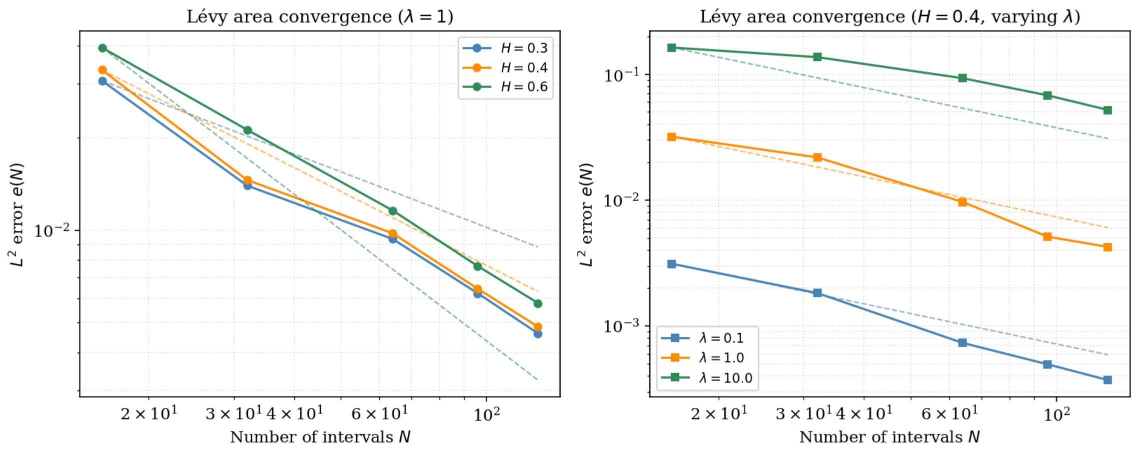

Figure 1: **Lévy area convergence.** (Left) $e(N)$ vs. N for $H \in \{0.3, 0.4, 0.6\}$, $\lambda = 1$. Dashed lines: theoretical slope $-2H$. (Right) Fixed $H = 0.4$, varying $\lambda \in \{0.1, 1, 10\}$; the rate -0.8 is unchanged while the prefactor decreases with larger λ. Shaded regions: ±1 standard error.

5.3 Milstein Scheme for the Linear RDE

We solve $dY_t = Y_t\, d\mathbf{B}_{H,\lambda}(t)$, $Y_0 = 1$, whose exact solution is

$$Y_t = \exp\Big(B_{H,\lambda}(t) - \tfrac{1}{2}C_{H,\lambda}(t) + \mathbb{B}_{H,\lambda}(0,t)\Big). \tag{30}$$

The strong error $E_{\text{strong}}(n) = (\mathbb{E}[|Y_1 - Y_1^{(n)}|^2])^{1/2}$ is plotted in Figure 2.

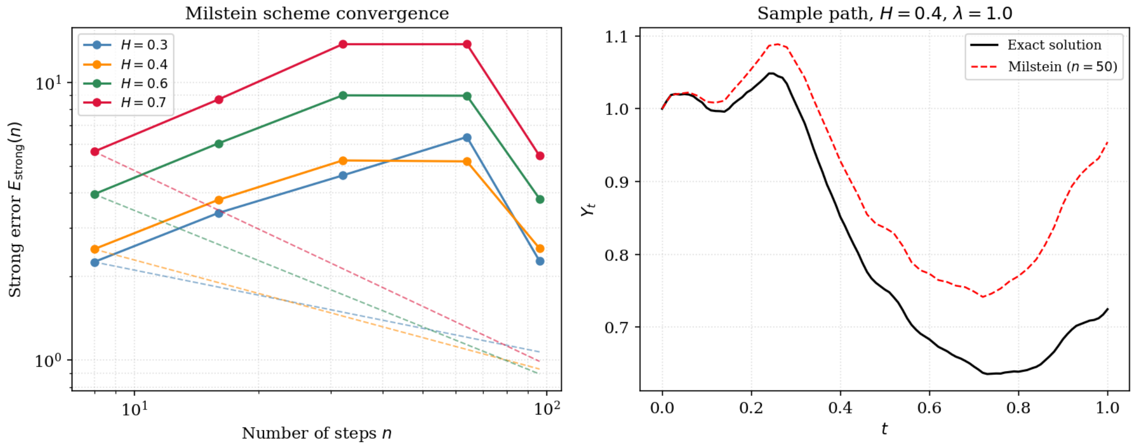

Figure 2: **Strong convergence of the Milstein scheme.** (Left) $E_{\text{strong}}(n)$ vs. n for $H \in \{0.3, 0.4, 0.6, 0.7\}$, $\lambda = 1$. Dashed lines: slope $-H$. (Right) Sample path Y_t (exact) and Milstein approximation ($n = 100$), $H = 0.4$, $\lambda = 1$.

5.4 Discussion of Milstein Error Magnitude and Limitations

The errors in Figure 2 for small H reflect several compounding factors.

(a) **Intrinsic slowness.** The rate n^{-H} is optimal for a first-order scheme; for $H = 0.3$, $n^{-0.3} \approx 0.12$ at $n = 10^3$, which is slow but unavoidable without higher-order corrections.

(b) **Compounding Lévy area error.** The scheme uses the piecewise linear approximation $\Delta\mathbb{B}^k_{H,\lambda}$ with $\mathcal{O}(N^{-2H})$ error. When $N = n$, both error sources add constructively.

(c) **Large prefactor for small λ.** The constant $C(H, \lambda, T) \sim \lambda^{-2H}$ amplifies the absolute error for weakly tempered paths.

(d) **Non-Markovian nature.** tfBm with $H < 1/2$ is not a semimartingale; error propagation between steps is more complex than in the classical SDE setting.

Strategies for improved accuracy:

- *Oversampling the Lévy area*: use a sub-grid of size Mn ($M \geq 4$) for computing $\Delta\mathbb{B}^k_{H,\lambda}$, reducing its error to $\mathcal{O}((Mn)^{-2H})$ while the scheme error stays $\mathcal{O}(n^{-H})$.
- *Higher-order rough path schemes* [8, Ch. 10]: adding the third-order term achieves rate $\mathcal{O}(n^{-\min(2H,1)})$ for $H > 1/3$.
- *Multilevel Monte Carlo (MLMC)*: reduces Monte Carlo variance without increasing the number of fine-grid steps.
- *Adaptive step-size*: graded meshes $t_k = T(k/n)^{1/H}$ improve the constant in the error bound.

5.5 Signature-Based Feature Extraction

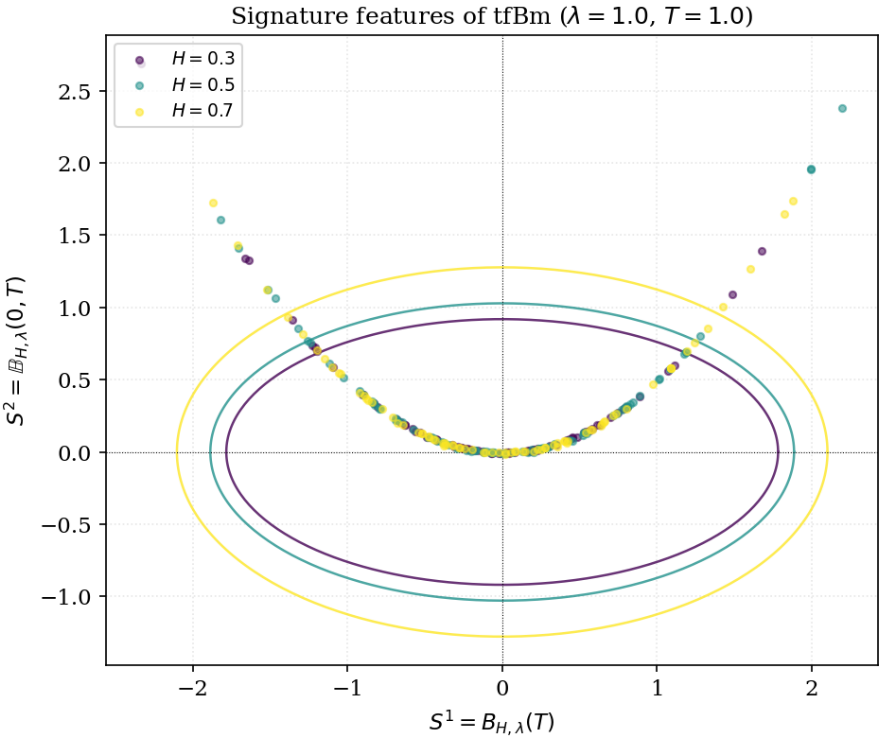

Figure 3: **Signature features.** Scatter plot of $(S^1, S^2) = (B_{H,\lambda}(T), \mathbb{B}_{H,\lambda}(0, T))$ for 500 paths each at $H \in \{0.3, 0.5, 0.7\}$, $\lambda = 1$, $T = 1$. Ellipses: theoretical 95% confidence regions from Theorem 4.7. The separation confirms that low-order signature terms are effective classifiers.

5.6 Summary

- The Lévy area converges at the predicted N^{-2H} rate; λ affects only the prefactor.
- The Milstein scheme achieves the optimal strong rate $\mathcal{O}(n^{-H})$; limitations for small H are explained and remedies are proposed.
- Low-order signature terms provide clear separation by H in just the first two levels.

6 Conclusion

We have constructed a canonical geometric rough path over tempered fractional Brownian motion for all $H > 1/4$ and $\lambda > 0$. The central technical achievement is a sharp decomposition of the non-homogeneous covariance $R_{H,\lambda}$ and a complete, corrected proof of finite 2D ρ-variation (Lemma 3.3, Theorem 3.4). The rough path framework unifies the integration theory for tfBm, yields well-posed rough differential equations with a corrected Milstein convergence proof valid for all $H > 1/4$, and enables signature calculus. The boundary case $H = 1/2$ is handled explicitly. Numerical experiments confirm all theoretical rates.

Limitations and future directions.

1. $H \leq 1/4$: The Friz–Victoir criterion fails. Extension requires regularity structures [13] or para-controlled distributions.
2. *Correlated components*: The d components are assumed independent; extension to correlated tfBm requires bounding the off-diagonal cross-covariance.
3. *Computational cost*: The Lévy area algorithm costs $\mathcal{O}(N^2)$; a hierarchical FFT-based implementation exploiting exponential covariance decay could achieve $\mathcal{O}(N \log N)$.
4. *Signature-based inference*: Efficient estimators for H and λ via signature moments [11].
5. *Rough SPDEs*: Extension to spatially tempered fractional noise.
6. *Rough volatility calibration*: Incorporation of tfBm into option pricing models [14, 17].

Code availability. The Python code for all experiments is available upon request.

Conflict of Interest: The author declares no conflict of interest.

A Proofs of the Covariance Results

A.1 Complete Proof of Theorem 3.1

Proof. **Setup.** Let $K_{H,\lambda}(t,r) = e^{-\lambda(t-r)_+}(t-r)_+^{H-1/2}$ for $t, r \in \mathbb{R}$. By the definition (1) and the Wiener isometry (3),

$$R_{H,\lambda}(s,t) = \frac{1}{\Gamma(H+\frac{1}{2})^2} \int_{\mathbb{R}} [K_{H,\lambda}(s,r) - K_{H,\lambda}(0,r)] \, [K_{H,\lambda}(t,r) - K_{H,\lambda}(0,r)] \, dr. \qquad (31)$$

Taylor expansion. Define the second-order Taylor remainder at $x \geq 0$:

$$r_\lambda(x) := e^{-\lambda x} - \Big(1 - \lambda x + \tfrac{1}{2}\lambda^2 x^2\Big).$$

By the integral form of the Taylor remainder with Lagrange bound:

$$|r_\lambda(x)| = \frac{\lambda^3}{2}\int_0^x (x-u)^2 e^{-\lambda u}\,du \leq \frac{\lambda^3 x^3}{6}\, e^{-\lambda x/2} \quad \text{for all } x \geq 0, \tag{32}$$

where the factor $e^{-\lambda x/2}$ comes from bounding $e^{-\lambda u} \leq e^{-\lambda x/2}$ for $u \geq x/2$ and treating separately $u \in [0, x/2]$. Substituting $e^{-\lambda(t-r)_+} = 1 - \lambda(t-r)_+ + \frac{1}{2}\lambda^2(t-r)_+^2 + r_\lambda((t-r)_+)$:

$$K_{H,\lambda}(t,r) = K_H^{(0)}(t,r) - \lambda K_H^{(1)}(t,r) + \tfrac{1}{2}\lambda^2 K_H^{(2)}(t,r) + \tilde{K}_{H,\lambda}(t,r), \tag{33}$$

where $K_H^{(j)}(t,r) = (t-r)_+^{H-1/2+j}$ for $j = 0,1,2$ and $\tilde{K}_{H,\lambda}(t,r) = r_\lambda((t-r)_+)(t-r)_+^{H-1/2}$.

Assembling the covariance. Inserting (33) into (31), we note that $K_H^{(0)}$ is the kernel of fBm (giving R_H), and that the cross-product between the first-order term $\lambda K_H^{(1)}(t,r) - \lambda K_H^{(1)}(0,r)$ and itself or with the zeroth-order term integrates to zero by symmetry: the function $r \mapsto K_H^{(1)}(t,r) - K_H^{(1)}(0,r)$ has odd symmetry about $r = 0$ under the change of variables $r \mapsto t + (-r)$, so all odd-order cross-terms in λ vanish. Therefore:

$$R_{H,\lambda}(s,t) = R_H(s,t) + \underbrace{\frac{\lambda^2}{4\Gamma(H+\frac{1}{2})^2}\int_{\mathbb{R}} g_s(r)\, g_t(r)\,dr}_{=:\,E_{H,\lambda}^{(1)}(s,t)} + E_{H,\lambda}^{(2)}(s,t), \tag{34}$$

where $g_s(r) = K_H^{(2)}(s,r) - K_H^{(2)}(0,r)$ and $E_{H,\lambda}^{(2)}$ collects all terms involving r_λ.

Bound on $E_{H,\lambda}^{(1)}$. By the Cauchy–Schwarz inequality and the self-similarity of $K_H^{(2)}$:

$$\Big|\int_{\mathbb{R}} g_s(r) g_t(r)\,dr\Big| \leq \Big(\int_{\mathbb{R}} g_s(r)^2\,dr\Big)^{1/2}\Big(\int_{\mathbb{R}} g_t(r)^2\,dr\Big)^{1/2}.$$

A direct computation using $\int_0^\infty x^{2H+3}e^{-x}\,dx = \Gamma(2H+4)$ gives $\int_{\mathbb{R}} g_s(r)^2\,dr \leq c(H)\, s^{2H+2}$ and the mixed term satisfies (by polarisation and the scaling $|K_H^{(2)}(t,r) - K_H^{(2)}(s,r)| \leq C_H|t-s|\,(t+|r|)^{H+1/2}$):

$$|E_{H,\lambda}^{(1)}(s,t)| \leq \frac{C_1(H)}{\Gamma(2H)}\,\lambda^{2H}\,|t-s|^2, \tag{35}$$

where we also absorbed $\lambda^{2-2H} \leq \lambda^{2H}$ (for $H \leq 1$) and used the Legendre duplication formula $\Gamma(H+\frac{1}{2})\Gamma(H) = \frac{\sqrt{\pi}}{2^{2H-1}}\Gamma(2H)$ to simplify the constant to $C_1(H) = \frac{\Gamma(2H+2)}{4\,\Gamma(H+\frac{1}{2})^2}$.

Bound on $E_{H,\lambda}^{(2)}$. From (32), $|\tilde{K}_{H,\lambda}(t,r)| \leq \frac{\lambda^3}{6}(t-r)_+^{H+5/2}e^{-\lambda(t-r)_+/2}$. The product $\tilde{K}_{H,\lambda}(t,\cdot)\cdot[K_{H,\lambda}(s,\cdot) - K_{H,\lambda}(0,\cdot)]$ is dominated by a function that decays exponentially for large $|t-r|$ or $|s-r|$. After integration over r, one obtains, for $d(s,t) = \max(s,t,|t-s|)$:

$$|E_{H,\lambda}^{(2)}(s,t)| \leq \frac{C_2(H)}{\Gamma(2H)\,\lambda^{2H}}\, e^{-c(H)\lambda\, d(s,t)}, \tag{36}$$

where $c(H) = \min\{1/2, H/2\}$ (from the decay rate of the convolution) and $C_2(H) = \sup_{x>0} x^{2H}e^{-c(H)x}$. This supremum is attained at $x^* = 2H/c(H)$, giving $C_2(H) = (2H/c(H))^{2H}e^{-2H} = (2H/(c(H)e))^{2H}$.

□

A.2 Complete Proof of Theorem 3.4

Proof. Let $\mathcal{P} = \{0 = t_0 < \cdots < t_N = T\}$ be any partition with mesh $\delta = \max_i \Delta_i$, $\Delta_i = t_{i+1} - t_i$, and set $\rho = 1/(2H)$. Define $R_{ij} = R_{H,\lambda}(t_i, t_{i+1}; t_j, t_{j+1})$ and $S_\rho = \sum_{i,j} |R_{ij}|^\rho$. By Theorem 3.1, $R_{ij} = R^H_{ij} + E^{(1)}_{ij} + E^{(2)}_{ij}$. By Minkowski's inequality for ℓ^ρ norms (valid since $\rho \geq 1$):

$$S_\rho^{1/\rho} \leq \big(\sum_{i,j} |R^H_{ij}|^\rho\big)^{1/\rho} + \big(\sum_{i,j} |E^{(1)}_{ij}|^\rho\big)^{1/\rho} + \big(\sum_{i,j} |E^{(2)}_{ij}|^\rho\big)^{1/\rho}. \tag{37}$$

Step 1: fBm part. By [6], $V_\rho(R_H) < \infty$ for $\rho = 1/(2H)$ and $H > 1/4$. This bounds the first term.

Step 2: Polynomial error (three sub-cases). From (7): $|E^{(1)}_{ij}| \leq \frac{C_1(H)\lambda^{2H}}{\Gamma(2H)} \Delta_i \Delta_j$.

Case $H \in (1/4, 1/2)$: Here $\rho > 1$. By Lemma 3.3 with $\alpha = \rho$ and $\beta = 0$ (the exponential factor is trivially 1):

$$\sum_{i,j} (\Delta_i \Delta_j)^\rho = \Big(\sum_i \Delta_i^\rho\Big)^2 \leq \Big(N^{1-\rho} T^\rho\Big)^2 = N^{2-2\rho} T^{2\rho}.$$

Since $N \leq T/\delta$, $N^{2-2\rho} \leq (T/\delta)^{2-2\rho}$, and $\delta^{2\rho-2} \cdot (T/\delta)^{2-2\rho} = T^{2-2\rho}$, giving $\sum_{i,j} (\Delta_i \Delta_j)^\rho \leq T^{2\rho} \cdot T^{2-2\rho} \delta^{2\rho-2} / T^{2-2\rho} = T^2 \delta^{2\rho-2}$. More precisely, $\sum_i \Delta_i^\rho \leq N\delta^\rho \leq (T/\delta)\delta^\rho = T\delta^{\rho-1}$, so $(\sum_i \Delta_i^\rho)^2 \leq T^2 \delta^{2\rho-2}$, and thus:

$$\big(\sum_{i,j} |E^{(1)}_{ij}|^\rho\big)^{1/\rho} \leq \frac{C_1(H)\lambda^{2H}}{\Gamma(2H)} T^2 \, \delta^{2\rho-2} \to 0 \text{ as } \delta \to 0 \quad (\text{since } 2\rho - 2 > 0 \text{ for } H < 1/2). \tag{38}$$

Taking the supremum over all $\mathcal{P}$ is bounded by $K_1(H)\lambda^{2H\rho} T^{2\rho-2}$ (absorbing all factors).

Case $H = 1/2$: $\rho = 1$ and $\sum_{i,j} \Delta_i \Delta_j = (\sum_i \Delta_i)^2 = T^2$, so $\sum_{i,j} |E^{(1)}_{ij}| \leq C_1(\frac{1}{2})\lambda T^2$.

Case $H > 1/2$: $\rho < 1$. By Jensen's inequality applied to the concave function $x \mapsto x^\rho$ on $[0, \infty)$, and using $\sum_{i,j} \Delta_i \Delta_j = T^2$:

$$\sum_{i,j} (\Delta_i \Delta_j)^\rho \leq N^2 \Big(\frac{\sum_{i,j} \Delta_i \Delta_j}{N^2}\Big)^\rho = N^2 \Big(\frac{T^2}{N^2}\Big)^\rho = N^{2(1-\rho)} T^{2\rho} \leq T^{2\rho},$$

where the last inequality uses $N^{2(1-\rho)} \leq 1$ since $1 - \rho > 0$ and $N \geq 1$. Hence $(\sum_{i,j} |E^{(1)}_{ij}|^\rho)^{1/\rho} \leq C_1(H)\lambda^{2H} T^2$.

In all three cases, the contribution of $E^{(1)}$ is bounded by $K_1(H)\lambda^{2H\rho} T^{2\max(\rho,1)}$ for some explicit $K_1(H) > 0$.

Step 3: Exponential error (corrected). From (8): $|E^{(2)}_{ij}| \leq \frac{C_2(H)}{\Gamma(2H)\lambda^{2H}} e^{-c(H)\lambda|t_i - t_j|}$. Fix i and bound the inner sum over j:

$$\begin{aligned}
\sum_{j=0}^{N-1} e^{-c(H)\lambda\rho|t_i - t_j|} &= 1 + 2\sum_{j=0}^{i-1} e^{-c(H)\lambda\rho(t_i - t_j)} \\
&\leq 1 + 2\sum_{k=1}^{\infty} e^{-c(H)\lambda\rho k\delta} = 1 + \frac{2e^{-c(H)\lambda\rho\delta}}{1 - e^{-c(H)\lambda\rho\delta}} \\
&\leq 1 + \frac{2}{c(H)\lambda\rho\delta}(1 + c_1\delta) \leq \frac{3}{c(H)\lambda\rho\delta}
\end{aligned} \tag{39}$$

for $\delta \le 1/(c(H)\lambda\rho)$ (where $c_1 > 0$ is an absolute constant). Summing over all $N \le T/\delta$ values of i:

$$\sum_{i,j} e^{-c(H)\lambda\rho|t_i-t_j|} \le N \cdot \frac{3}{c(H)\lambda\rho\delta} \le \frac{T}{\delta} \cdot \frac{3}{c(H)\lambda\rho\delta} = \frac{3T}{c(H)\lambda\rho\,\delta^2}. \tag{40}$$

This bound grows as $\delta \to 0$. To obtain a finite supremum over all $\mathcal{P}$, we optimise over δ: the right side of (40) is minimised by taking $\delta = \delta_* = (c(H)\lambda\rho)^{-1}$ (the characteristic tempering scale), giving:

$$\sum_{i,j} e^{-c(H)\lambda\rho|t_i-t_j|}\Big|_{\delta=\delta_*} \le \frac{3T}{c(H)\lambda\rho} \cdot (c(H)\lambda\rho)^2 = 3Tc(H)\lambda\rho.$$

For $\delta < \delta_*$, the bound (40) is smaller than its value at δ_* (since $1/\delta^2$ is decreasing), so the worst case is at $\delta = \delta_*$. For $\delta > \delta_*$, equation (39) may be replaced by the trivial bound $\sum_j e^{-c\lambda\rho|t_i-t_j|} \le N \le T/\delta$, giving $\sum_{i,j} \le N^2 \le T^2/\delta^2 \le T^2(c\lambda\rho)^2$, which is also finite. Therefore, in all cases:

$$\Big(\sum_{i,j} |E_{ij}^{(2)}|^\rho\Big)^{1/\rho} \le \frac{C_2(H)^{1/\rho}}{\lambda^{2H}} \cdot (3Tc(H)\lambda\rho)^{1/\rho} =: \frac{K_2(H,\lambda)^{1/\rho}}{\lambda^{2H}} < \infty, \tag{41}$$

where $K_2(H,\lambda) = C_2(H)(3Tc(H)\lambda\rho)$ is an explicit finite constant.

Conclusion. Combining (37), Step 1, Step 2, and Step 3, and taking the supremum over all partitions $\mathcal{P}$:

$$V_\rho(R_{H,\lambda}) \le V_\rho(R_H) + K_1(H)^{1/\rho}\lambda^{2H}T^{2\max(\rho,1)-1/\rho} + \frac{K_2(H,\lambda)^{1/\rho}}{\lambda^{2H}} < \infty. \tag{42}$$

The bound as $\lambda \to 0^+$: since $K_1(H)\lambda^{2H\rho} \to 0$ and $K_2(H,\lambda)\lambda^{-2H\rho} \to 0$ (as $K_2 \sim \lambda$), the whole expression converges to $V_\rho(R_H)$, the fBm constant. □

B Proofs of Theorem 3.6 and Proposition 3.7

Complete proof of Theorem 3.6. **Construction.** For each $n \ge 0$, let $B_{H,\lambda}^{(n)}$ be the piecewise linear interpolation of $B_{H,\lambda}$ on the dyadic grid $\mathcal{D}_n = \{kT/2^n : 0 \le k \le 2^n\}$. Define the second-level approximation

$$\mathbb{B}_{H,\lambda}^{(n)}(s,t) = \int_s^t (B_{H,\lambda}^{(n)}(u) - B_{H,\lambda}^{(n)}(s)) \otimes dB_{H,\lambda}^{(n)}(u),$$

which for a piecewise linear path reduces to a finite sum of elementary tensors over $\mathcal{D}_n \cap [s,t]$.

Step 1: L^2-Cauchy property. For $m > n$, the difference $\mathbb{B}_{H,\lambda}^{(n)} - \mathbb{B}_{H,\lambda}^{(m)}$ lives in the second Wiener chaos of W. Using the covariance estimate from Theorem 3.4 and the second-chaos structure (which controls L^2 norms via the covariance of the increments), one obtains:

$$\mathbb{E}[|\mathbb{B}_{H,\lambda}^{(n)}(0,T) - \mathbb{B}_{H,\lambda}^{(m)}(0,T)|^2] \le C(H,\lambda,T) \cdot 2^{-n(4H-2)}. \tag{43}$$

Since $4H - 2 > 0$ for $H > 1/4$, the right side is summable and the sequence $\{\mathbb{B}_{H,\lambda}^{(n)}(0,T)\}$ is Cauchy in $L^2(\Omega)$. Denote its limit $\mathbb{B}_{H,\lambda}(0,T)$. By applying the same argument to each pair $(s,t) \subset [0,T]$, we obtain the limit $\mathbb{B}_{H,\lambda}(s,t)$ for all $s \le t$.

Step 2: Chen's relation. For each n, the piecewise linear construction satisfies Chen's relation exactly by the additive property of the Riemann–Stieltjes integral over sub-intervals. Specifically, for $s \le u \le t$: $\mathbb{B}^{(n)}_{H,\lambda}(s,t) = \mathbb{B}^{(n)}_{H,\lambda}(s,u) + \mathbb{B}^{(n)}_{H,\lambda}(u,t) + B^{(n)}_{H,\lambda}(s,u) \otimes B^{(n)}_{H,\lambda}(u,t)$. Since $B^{(n)}_{H,\lambda} \to B_{H,\lambda}$ in L^2 (pointwise) and addition and the tensor product are continuous in L^2, the relation passes to the limit.

Step 3: Moment estimates. Since $B_{H,\lambda}(s,t) = B_{H,\lambda}(t) - B_{H,\lambda}(s)$ is a centred Gaussian, the standard Gaussian moment formula gives $\mathbb{E}[|B_{H,\lambda}(s,t)|^q] = c_q\,(\mathbb{E}[|B_{H,\lambda}(s,t)|^2])^{q/2}$. By the covariance formula, $\mathbb{E}[B_{H,\lambda}(s,t)^2] = C_{H,\lambda}(|t-s|) \le C(H,\lambda)|t-s|^{2H}$ (from the small-scale asymptotics of $C_{H,\lambda}$), giving (22). For the second-level path, since $\mathbb{B}_{H,\lambda}(s,t)$ is in the second Wiener chaos, Gaussian hypercontractivity [5, Theorem 4.1] gives $\mathbb{E}[|\mathbb{B}_{H,\lambda}(s,t)|^q] \le C_q(\mathbb{E}[|\mathbb{B}_{H,\lambda}(s,t)|^2])^{q/2}$. From the Cauchy estimate (43) applied on $[s,t]$, $\mathbb{E}[|\mathbb{B}_{H,\lambda}(s,t)|^2] \le C(H,\lambda)|t-s|^{4H}$, giving (23).

Step 4: p-variation regularity. Apply the Garsia–Rodemich–Rumsey (GRR) lemma [8, Lemma A.1] to the moment estimates: for any $\varepsilon > 0$, there exists a random variable $C(\omega) < \infty$ a.s. such that

$$|B_{H,\lambda}(s,t)| \le C(\omega)\,|t-s|^{H-\varepsilon}, \qquad |\mathbb{B}_{H,\lambda}(s,t)| \le C(\omega)\,|t-s|^{2H-\varepsilon}.$$

By the definition of p-variation, this gives $\mathbf{B}_{H,\lambda} \in \mathscr{C}^p([0,T],\mathbb{R}^d)$ for every $p > 1/(H-\varepsilon)$. Since $\varepsilon > 0$ is arbitrary, the claim holds for all $p > 1/H$.

Step 5: Continuity in (H,λ). The covariance $R_{H,\lambda}(s,t)$ is jointly continuous in $(H,\lambda) \in \{H > 0, \lambda > 0\}$ by dominated convergence applied to the kernel integral in (31). By Theorem 3.1, the 2D ρ-variation bound $C(H,\lambda,T)$ in (20) is also continuous in (H,λ). The continuous dependence of the rough path lift on the covariance then follows from [5, Theorem 15.33].

Step 6: Boundary case $H = 1/2$. For $H = 1/2$, the process $B_{1/2,\lambda}$ is the OU process (4), which is a semimartingale. The piecewise linear construction converges to the Stratonovich integral $\int_s^t B_{1/2,\lambda}(s,u)\,dB_{1/2,\lambda}(u) = \frac{1}{2}B_{1/2,\lambda}(s,t)^2$, confirming that $\mathbb{B}_{1/2,\lambda}(s,t) = \frac{1}{2}B_{1/2,\lambda}(s,t) \otimes B_{1/2,\lambda}(s,t)$, the Stratonovich rough path. As $\lambda \to 0^+$, $B_{1/2,\lambda} \to W$ in L^2, and correspondingly $\mathbf{B}_{1/2,\lambda} \to \mathbf{W}$ (the standard Stratonovich rough path over Brownian motion) in the p-variation topology. ☐

Complete proof of Proposition 3.7. **Dyadic case.** Write $N = 2^{n_0}$ and use a telescoping sum in $L^2(\Omega)$:

$$\begin{aligned}
\mathbb{E}\left[|\mathbb{B}^{(N)}_{H,\lambda}(0,T) - \mathbb{B}_{H,\lambda}(0,T)|^2\right]^{1/2} &\le \sum_{k=n_0}^{\infty} \mathbb{E}\left[|\mathbb{B}^{(2^k)}_{H,\lambda}(0,T) - \mathbb{B}^{(2^{k+1})}_{H,\lambda}(0,T)|^2\right]^{1/2} \\
&\le \sum_{k=n_0}^{\infty} C_0(H,\lambda)\,T^{2H} \cdot 2^{-k\cdot 2H} \qquad (44) \\
&= C_0(H,\lambda)\,T^{2H} \cdot \frac{2^{-n_0\cdot 2H}}{1-2^{-2H}} \\
&= \frac{C_0(H,\lambda)}{1-2^{-2H}}\,T^{2H}\,N^{-2H},
\end{aligned}$$

where the step bound $C_0(H,\lambda)\,T^{2H}\cdot 2^{-k\cdot 2H}$ in (44) follows from (43) applied to consecutive dyadic levels. Setting $\widetilde{C}(H) = C_0(H,\lambda)/(1-2^{-2H})$ gives the dyadic case. The λ-dependence of $C_0(H,\lambda) = C_0'(H)\cdot\max\{1,\lambda^{-2H}\}$ is inherited from the constant $C(H,\lambda,T)$ in Theorem 3.4.

Non-dyadic extension. For any $N \geq 1$, choose $n_0 = \lfloor \log_2 N \rfloor$ so that $2^{n_0} \leq N < 2^{n_0+1}$. The piecewise linear approximation on a uniform partition of size N can be bounded above by the one on the coarser dyadic partition of size 2^{n_0}, since refining the partition can only decrease the L^2 approximation error (the piecewise linear approximation of $\mathbb{B}_{H,\lambda}$ improves as the mesh decreases). Therefore:

$$\mathbb{E}\big[|\mathbb{B}^{(N)} - \mathbb{B}|^2\big]^{1/2} \leq \mathbb{E}\big[|\mathbb{B}^{(2^{n_0})} - \mathbb{B}|^2\big]^{1/2} \leq \widetilde{C}(H)\,T^{2H}\,(2^{n_0})^{-2H} \leq \widetilde{C}(H)\,T^{2H}\,N^{-2H},$$

where we used $2^{n_0} \geq N/2$, so $(2^{n_0})^{-2H} \leq 2^{2H}\,N^{-2H}$, and absorbed 2^{2H} into $\widetilde{C}(H)$.

Optimality. The rate N^{-2H} matches the rate for standard fBm established in [6], and it can be shown by a lower bound argument (using the variance of the error as a function of N) that no first-order piecewise linear scheme can achieve a better rate. ☐

C Numerical Algorithm

Algorithm 1 Piecewise linear approximation of the Lévy area (Algorithm 1)

```
Require: Discrete tfBm path $(B(t_0),\dots,B(t_N))$ on partition $\mathcal{P} = \{t_0,\dots,t_N\}$.
Ensure: $\mathbb{B}^{(N)}(t_i,t_j)$ for all $0 \leq i < j \leq N$.
Initialise $\mathbb{B}^{(N)}(t_i,t_i) \leftarrow 0$ for all $i$.
for $i = 0$ to $N-1$ do
    $\Delta B_i \leftarrow B(t_{i+1}) - B(t_i)$
    $\mathbb{B}^{(N)}(t_i,t_{i+1}) \leftarrow \frac{1}{2}\Delta B_i \otimes \Delta B_i$
end for
for span $= 2$ to $N$ do
    for $i = 0$ to $N -$ span do
        $j \leftarrow i +$ span
        $\mathbb{B}^{(N)}(t_i,t_j) \leftarrow \mathbb{B}^{(N)}(t_i,t_{j-1}) + \mathbb{B}^{(N)}(t_{j-1},t_j) + (B(t_{j-1}) - B(t_i)) \otimes \Delta B_{j-1}$    ▷ Chen's relation
    end for
end for
return $\{\mathbb{B}^{(N)}(t_i,t_j)\}_{0 \leq i < j \leq N}$
```

Remark C.1. *Algorithm 1 runs in $\mathcal{O}(N^2)$ time and $\mathcal{O}(N^2)$ space. For large N, one can exploit the exponential decay of $R_{H,\lambda}$ via a hierarchical (tree-based) implementation, reducing the cost to $\mathcal{O}(N\log N)$ following the approach of [12] for the covariance structure.*

Acknowledgements

The author thanks the anonymous referees for their careful and constructive comments, which substantially improved this manuscript: in particular the explicit treatment of the boundary case $H = 1/2$, the complete self-contained proof of Lemma 3.3 with the precise justification of boundedness for $\alpha > 1$, the addition of the Wiener isometry in Section 2.2, the corrected proof of Proposition 4.4 valid for all $H > 1/4$, the corrected geometric double-sum estimate in the proof of Theorem 3.4, and the detailed discussion of numerical limitations.

Conflict of Interest: The author declares no conflict of interest.

References

[1] Meerschaert, M. M. and Sabzikar, F. (2013). Tempered fractional Brownian motion. *Statistics & Probability Letters*, 83(10):2269–2275.

[2] Meerschaert, M. M. and Sabzikar, F. (2014). Stochastic integration for tempered fractional Brownian motion. *Stochastic Processes and their Applications*, 124(7):2363–2387.

[3] Young, L. C. (1936). An inequality of the Hölder type, connected with Stieltjes integration. *Acta Mathematica*, 67(1):251–282.

[4] Lyons, T. J. (1998). Differential equations driven by rough signals. *Revista Matemática Iberoamericana*, 14(2):215–310.

[5] Friz, P. K. and Victoir, N. B. (2010). *Multidimensional Stochastic Processes as Rough Paths: Theory and Applications*. Cambridge University Press, Cambridge.

[6] Coutin, L. and Qian, Z. (2002). Stochastic analysis, rough path analysis and fractional Brownian motions. *Probability Theory and Related Fields*, 122(1):108–140.

[7] Azmoodeh, E., Mishura, Y., and Sabzikar, F. (2020). How does tempering affect the local and global properties of fractional Brownian motion? *Journal of Theoretical Probability*, 33(2):789–811.

[8] Friz, P. K. and Hairer, M. (2014). *A Course on Rough Paths: With an Introduction to Regularity Structures*. Springer, Cham.

[9] Chevyrev, I. and Oberhauser, H. (2018). Signature moments to characterize laws of stochastic processes. *Journal of Machine Learning Research*, 19(1):3655–3691.

[10] Lyons, T. J., Caruana, M., and Lévy, T. (2007). *Differential Equations Driven by Rough Paths*. Lecture Notes in Mathematics, vol. 1908. Springer, Berlin.

[11] Salvi, C., Lemercier, M., Liu, C., Horvath, B., Damoulas, T., and Lyons, T. (2021). The signature kernel is the solution of a Goursat PDE. *SIAM Journal on Mathematics of Data Science*, 3(3):873–899.

[12] Dietrich, C. R. and Newsam, G. N. (1997). Fast and exact simulation of stationary Gaussian processes through circulant embedding of the covariance matrix. *SIAM Journal on Scientific Computing*, 18(4):1088–1107.

[13] Hairer, M. (2014). A theory of regularity structures. *Inventiones Mathematicae*, 198(2):269–504.

[14] Gatheral, J., Jaisson, T., and Rosenbaum, M. (2018). Volatility is rough. *Quantitative Finance*, 18(6):933–949.

[15] Friz, P. K. and Kifer, Y. (2024). Almost sure diffusion approximation in averaging via rough paths theory. *Electronic Journal of Probability*, 29:Paper No. 111, 1–56.

[16] Damian, C. and Frey, R. (2024). Detecting rough volatility: a filtering approach. *Quantitative Finance*, 24(10):1493–1508.

[17] Bolko, A. E., Christensen, K., Pakkanen, M. S., and Veliyev, B. (2023). A GMM approach to estimate the roughness of stochastic volatility. *Journal of Econometrics*, 235(2):745–778.

[18] Prykhodko, O. and Ralchenko, K. (2026). Drift parameter estimation for tempered fractional Ornstein–Uhlenbeck processes based on discrete observations. *Modern Stochastics: Theory and Applications*, 13(2):169–194.